CS 357: Algorithmic Game Theory

Lecture 2: Game Theory Background

Shikha Singh

Announcements

- Fill out course survey if you have not already
- Pick up **Homework 1**: due **in class on Tues (next lecture)**
  - Pen and paper practice of definitions from class
- **Assignment 1** will be released today and due **Fri Feb 21 (Fri) noon**
  - Shorter because of winter carnival
  - Must do in pairs; will send out a google form to coordinate partners (will assign if you cannot find one)
  - Type in LaTeX, submit on Gradescope
- Office hours, tentatively:
  - Monday 10-11.30, Wed 2.30-4 pm and Fri 9.30 -10.30 am
  - Will post on course calendar

Review: Course Logistics

Syllabus and Grade Breakdown

- Posted on course webpage
  - https://williams-cs.github.io/cs357-s25/handouts/syllabus.pdf
- Grading breakdown (many components):
  - Homework (10%): *weekly, pen-and-paper, solo*
  - Paper evaluations (10%): *~ four, discuss in class in groups*
  - Attendance and Class Participation (5%): *most imp!*
  - Exams (15 + 15%): *two, in class, test understanding of concepts*
  - Assignments (20%): *LaTeX, longer, in pairs, every 1.5 weeks*
  - Final Project (25%): *last 3 weeks*

Tentative Outline of the Semester

https://docs.google.com/document/d/1uzpMsk7D3gn0-92Hpd9XkoaqUK19YFl-uXXrq2nKQ4w/edit?tab=t.0

Submit to change, but sharing it in case it is helpful for planning purposes

Classroom Culture

- A good learning experience for all requires you to engage
    - Be a good teammate, come prepared and contribute
- Help and support each other; build a positive classroom community
- There are no wrong answers in my class!
- Class participation does not mean dominating classroom discussions or interrupting your peers
    - Be respectful and kind to each other

Bottom line. *Help create a vibrant, positive, and inclusive classroom environment!*

Honor Code

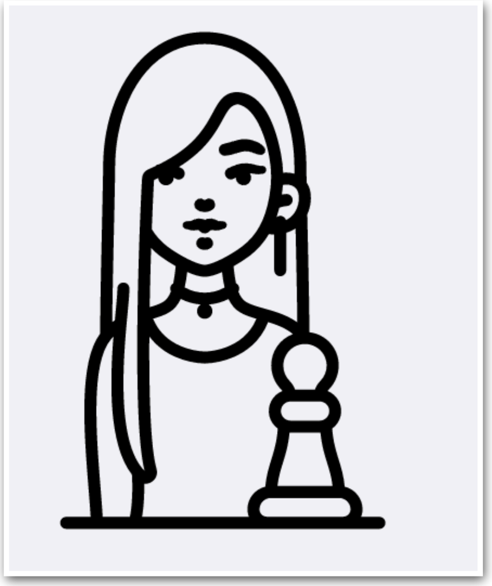
- Read: academic honesty section of the syllabus
- Gist:
  - No collaboration on HW
  - Only collaborate with your partner on assignment
  - Can help others find resources/ answer clarifying questions
  - No internet/ ChatGPT for work assigned as part of the course **unless explicitly stated differently**
- Important: You must understand the work you submit

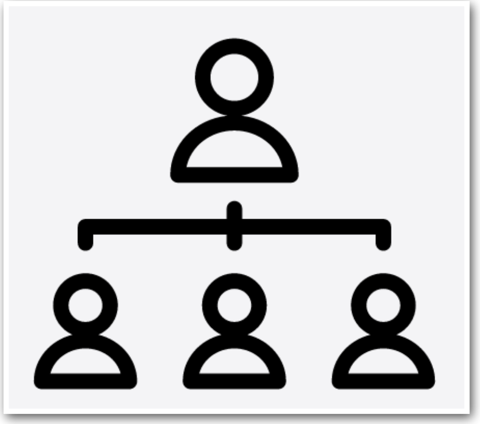
Bottom line. *Any work that is not your own is a violation of the Honor Code.*

Game Theory Basics

Defining a Game

- **Players**: the decision makers
  - People, governments, companies
- **Actions**: what can the players do
  - Enter a bid in an auction
  - Decide when to sell stock
  - Decide who to vote for
  - Outcome
- **Payoffs/Utility** of each outcome to players
  - Represented a number (cardinal)
  - Or ordering over outcomes (ordinal)






Towards a Game Representation

- To start, consider the simplest games
- Simultaneous move, single-action games
    - Eg. Rock, paper, scissors
- How many players?
- What are the actions available to players?
- What are the outcomes?
- What are the payoffs to players of the outcomes?
    - How can we represent this?

Normal-Form Games

- Normal form/ Matrix Form/ Strategic Form:
  - List payoffs of players as a function of their actions
  - Assume players move simultaneously
- Conventions:
  - Row player is usually player 1
  - Column player is player 2
  - Payoffs for each outcome are written in each cell as a tuple, where first is player 1's payoff, then player 2

	Rock	Paper	Scissors
Rock	$0, 0$	$-1, 1$	$1, -1$
Paper	$1, -1$	$0, 0$	$-1, 1$
Scissors	$-1, 1$	$1, -1$	$0, 0$

Prisoner's Dilemma

- Two alleged criminals questioned in separate rooms
- Each player has two actions:
    - Cooperate (C): stay silent and not admit to anything
    - Defect (D): testify agains the other person
- (C, C): each serves 1 year in prison for minor offense
- (C, D) or (D, C): confessor goes free while other person gets a long prison sentence
- (D, D): each serve 3 years in prison
- Also write preferences as an ordering or assign values

Prisoner's Dilemma

	C	D
C	$-1, -1$	$-5, 0$
D	$0, -5$	$-3, -3$

	C	D
C	a, a	b, c
D	c, b	d, d

$$c > a > d > b$$

	C	D
C	$4, 4$	$0, 5$
D	$5, 0$	$2, 2$

Reasoning about Play

- Suppose you are player 1
- If Player 2 plays C
  - Best move?

	C	*D*
C	4, 4	0, 5
D	5, 0	2, 2

Reasoning about Play

- Suppose you are player 1
- If Player 2 plays D
    - Best move?

	C	D
C	4, 4	0, 5
D	5, 0	2, 2

Reasoning about Play

- Suppose you are player 2
- Similar reasoning

	C	*D*
C	4, 4	0, 5
D	5, 0	2, 2

Dominant Strategy Equilibrium

- Strongest guarantee a game can have of player behavior
- **Intuition:** regardless of the actions other players chooses, each player has a "dominant action"
  - Dominates other actions they can play, *for every combination of actions other players can play*
  - No brainer to always play it, no matter what others are doing
- For a formal definition, we need to define notation

Definitions: Normal Form

- Finite, n-person normal form game (N, A, u)
  - **Players:** $N = \{1,\ldots,n\}$
  - **Action set:** for player i, set of actions A_i available
  - **Action profile:** $a = (a_1, \ldots, a_n) \in A = A_1 \times \cdots A_n$
  - **Outcome:** Action profile played
  - **Utility function:** (Payoff function) $u_i : A \to \mathbb{R}$ for player i
- **Rationality assumption.** Players will always act to maximize their utility
- **Common knowledge assumption.** Player rationality is common knowledge
  - Each players knows that everyone else knows that everyone else is rational.....

Dominant Strategy Equilibrium

- **Notation.** Suppose other players use actions a_{-i}, where $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ (standard notation for everyone's actions except i)
- Overall action profile is $a = (a_i, a_{-i})$
- **Domination:** For a player i, an action a_i (weakly) dominates action $a_i' \in A$ if it is always beneficial to play a over a'
    - That is, for all $a_{-i} \in A_{-i}$: $u_i\ (a_i,\ a_{-i}) \ \geq \ u_i\ (a_i',\ a_{-i})$ and the inequality is strict for some $a_{-i} \in A_{-i}$
- What actions are dominated in Prisoner's dilemma?
    - C is dominated by D for both players

Dominant Strategy Equilibrium

- An action $a \in A_i$ is **dominant** for player $\boldsymbol{i}$ if it weakly dominates all actions $a_i' \in A_i, \;\; a_i' \neq a_i$
- **Dominant strategy equilibrium (DSE):** An equilibrium where each player plays their dominant action, that is,
  - Action profile $a^* = (a_1^*, \ldots, a_n^*)$ is a dominant- strategy equilibrium if and only if a_i^* is the dominant action for each player $\boldsymbol{i}$
- In a DSE, when players choose their action (it is independent of the choice a_{-i} of others)

Best Response

- **Best response definition:** Let BR (a_{-i}) denote the set of actions that form i's best response given a_{-i} then,

  $a_i^* \in$ BR (a_{-i}) if only if

  $$u_i\ (a_i^*, a_{-i}) \ \geq \ u_i\ (a_i, a_{-i}) \quad \forall a_i \in A_i$$

- A dominant action is always a best response (regardless of others actions)

Dominant Strategy Equilibrium

- DSE is a strong guarantee on player behavior
  - In AGT, goal is to shoot for mechanisms that admit a DSE
- **Question.** Will a DSE always exist in a game?
    - We will see an example soon
- For games that do not have a DSE, we need a different notion of equilibrium (expected outcome)
- What is the DSE in Prisoner's dilemma?
  - (D, D)
  - Is this a good outcome?

Pareto Optimality

- **Question.** Which outcome is better as a whole (to a neutral observer)?
- An outcome o is at least as good for every player, as another outcome o', and there is some agent who strictly prefers o to o'
  - Reasonable to say o is better than o'
  - We say o Pareto dominates o'
- **Definition (Pareto Optimality).**
  - An outcome o^* is Pareto-optimal if there is no other outcome that Pareto dominates it
- **Question.** Can you point out Pareto optimal outcomes in Prisoner's dilemma?

Pareto Optimality

- **Question.** Can you point out Pareto optimal outcomes in Prisoner's dilemma?
- What is good for one is not good for the group!

All but (D, D), the unique DSE, are Pareto optimal!

	C	D
C	4, 4	0, 5
D	5, 0	2, 2

Does DSE Always Exist?

- Dominant strategy equilibria is a very strong guarantee, but may not exist for most games
- Does the given game have a DSE? Why?

	L	M	R
T	1, 0	1, 2	0, 1
B	0, 3	0, 1	2, 0

Does DSE Always Exist?

- Even when DSE does not exist, there still may exist a dominated strategy for some player
- Does player 2 have a dominated strategy?
  - R is strictly dominated by M
  - Can be eliminated
- In the reduced game, does player 1 have a dominated strategy?
  - B is strictly dominated by T
- Finally, L is dominated by M

Only outcome that survives (T, M)

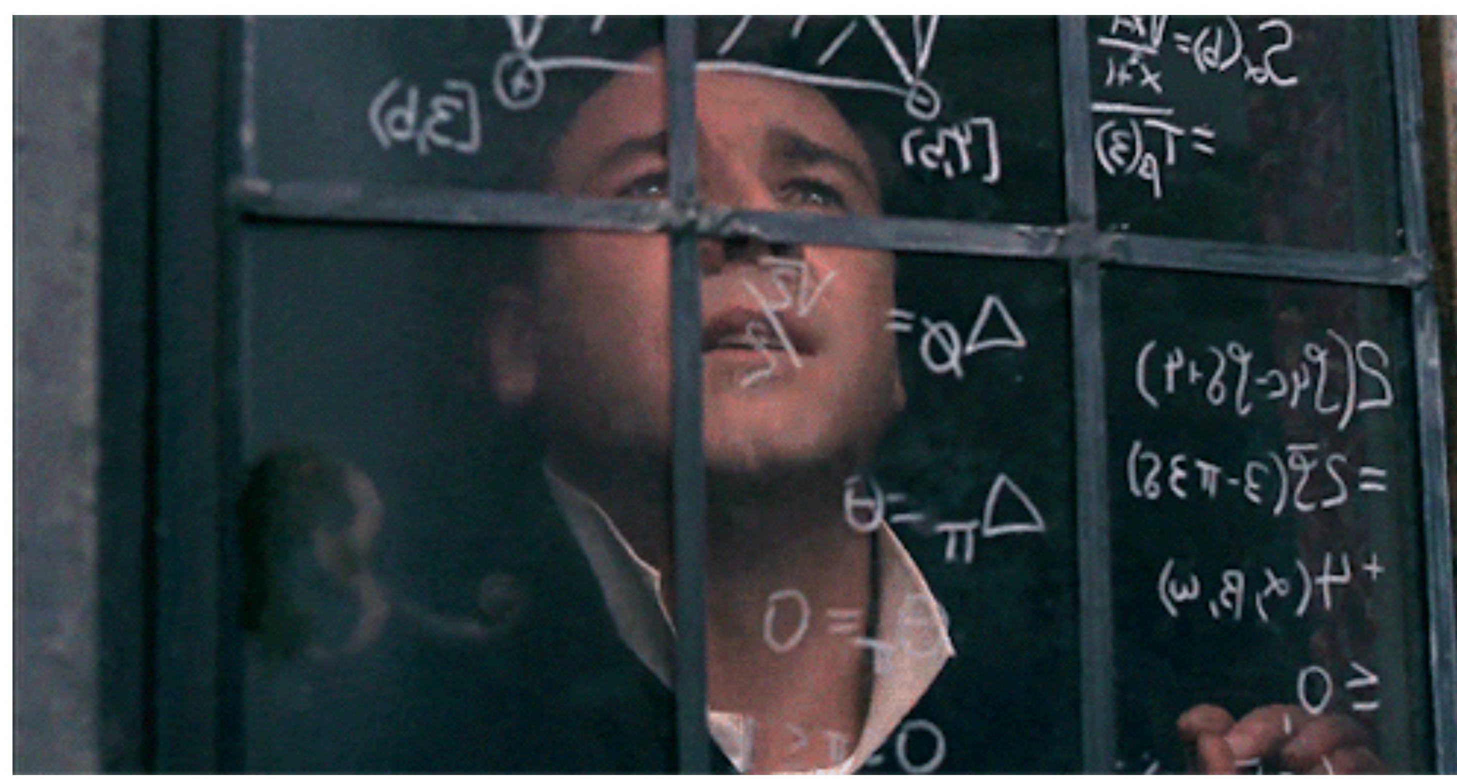
	L	M	R
T	1, 0	1, 2	0, 1
B	0, 3	0, 1	2, 0

Towards Nash

- This process is called **iterated elimination of dominated strategies**
- **Idea:** A dominated action can never be a best response
- **Towards Nash**: In a "stable" outcome, each rational players must play their best response to others
    - If they can unilaterally change their action to improve their utility, they would: outcome is not stable
- Our goal is to find out when are there stable solutions

	L	M	R
T	1, 0	1, 2	0, 1
B	0, 3	0, 1	2, 0

Nash Equilibrium



https://www.youtube.com/watch?v=2d_dtTZQyUM

Nash Equilibrium

- In a Nash equilibrium, every agent plays a best response to the actions of others
  - No agent has any incentive to deviate unilaterally
- **Pure-strategy Nash equilibrium:** Action profile $a^* = (a_1^*, \ldots, a_n^*)$ is a pure-strategy Nash equilibrium of a simultaneous-move game if and only if. for each player i

  $u_i(a_i, a_{-i}) \;\geq\; u_i(a_i', a_{-i})$ for all actions $a_i' \in A_i$:
- Whenever we need to verify an action profile is a Nash: fix a_{-i} and check if anyone has an incentive to unilaterally deviate

Nash Equilibrium

- What is the pure-strategy Nash equilibrium of this game?
  - (T, M)
- **Claim.** If iterated elimination of dominated strategies results in a unique solution, then it is a pure strategy Nash equilibrium **(Assignment 1)**
- Is it always possible to find actions to eliminate?
  - HW 1

	L	M	R
T	1, 0	1, 2	0, 1
B	0, 3	0, 1	2, 0

Let's Play a Game

- Each of you must choose an integer between 1 and 100
- Player(s) who name the integer closest to ***two thirds of the average*** wins a prize, the other players get nothing
  - If average is X, you want to name an integer close to $2X/3$
- (Ties will lead to multiple winners)
- Click on the google form link on zoom to play:
  - https://tinyurl.com/357numbers

Let's Play a Game

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Reasoning About the Game

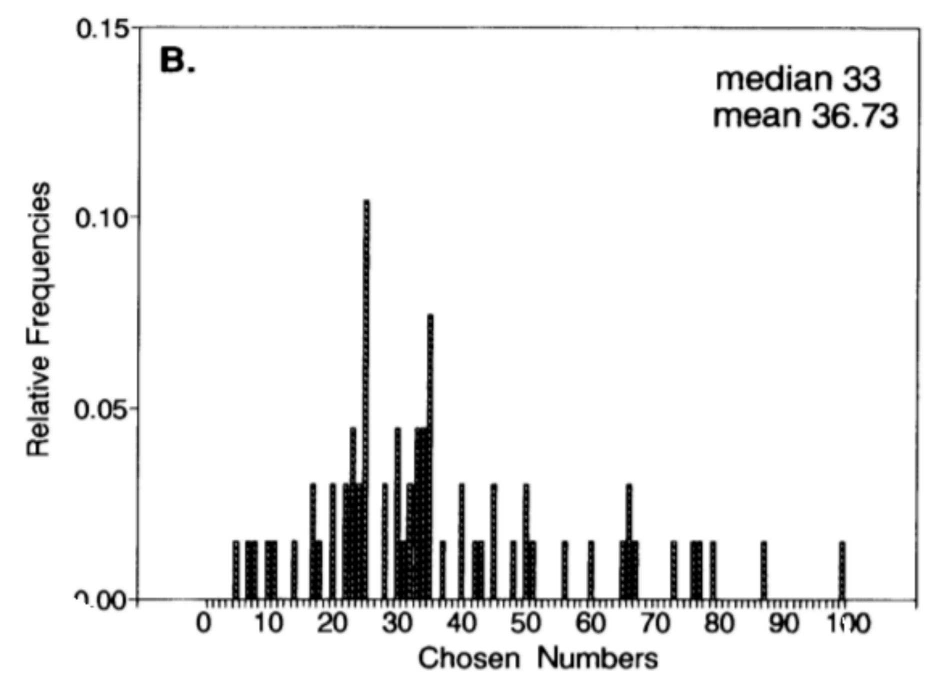
- What you should do, depends on what other players do
    - What makes game theory interesting and also challenging!
- Suppose you believe that the average play will be X (including your own guess)
- What should do you play in response?
    - Closest integer to $\frac{2X}{3}$
- Some back of envelope calculation
    - $X \leq 100$, that means $\frac{2X}{3} \leq 67$
- That is, number greater than 67 cannot win, regardless of the play

Reasoning About the Game

- So no guess above 67 can win and everyone can reason like this
- If everyone plays $X \leq 67$, what is your best response?
  - Play a number $\leq (2/3) \cdot 67$
- Everyone else is reasoning exactly like you
- Thus each player will play a number $\leq (2/3) \cdot 67$
  - What is your best response?
  - Play a number $\leq (2/3)^2 \cdot 67$
- When does this end?

Nash Play

- If everyone is "perfectly rational" then playing 1 is the unique pure Nash equilibrium of the game
- Empirical analysis
    - People may not be perfectly rational; usually peaks around 10/20
- Does that mean Nash is not a good predictor of behavior in practice?
- What happens if we play the game again?
    - Players learn and "converge" to a Nash



Rosemarie Nagel. Unraveling in guessing games: An experimental study (1995)

Keynes "Beauty" Contest (1936)

- John M. Keynes came up with the concept of a "newspaper beauty contest" to explain price fluctuations in stock markets
- Relation to stock market: "*People pricing shares not based on what they think their fundamental value is, but rather on what they think everyone else thinks their value is, or what everybody else would predict the average assessment of value to be.*"
- Stylized version: Guess the 2/3rd of average
    - **21.6** was the winning value in a large online competition organized by the Danish newspaper Politiken.
    - 19,196 people participated and the prize was 5000 Danish kroner
- Difference between **rationality** and **common knowledge of rationality**

Common Knowledge

- Common knowledge of rationality means
  - a players knows that each player is rational
  - knows that each player knows that each player is rational
  - knows that each player knows that each player knows that each player is rational
  - and so on, ad infinitum

Nash Equilibrium

Bach or Stravinsky

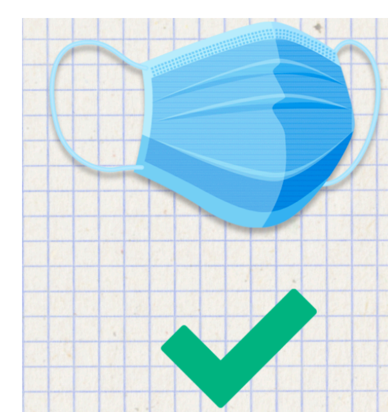
- Classic coordination game called Battle of the Sexes or the less problematic variant BoS (Bach or Stravinsky):
    - Two people wish to go out together to a concert of music by either Bach or Stravinsky
    - They both want to go out together, but one person prefers Bach and the other person prefers Stravinsky
    - Assume no cellphones to coordinate ahead of time
- What are the pure Nash equilibria of this game?
    - (B, B) and (S, S)

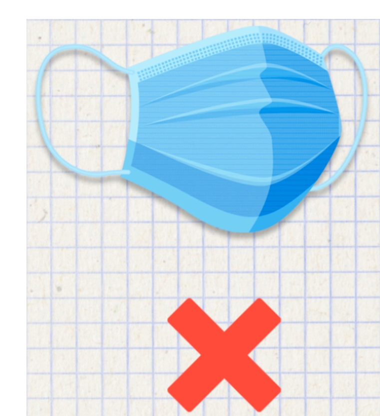
	Bach	*Stravinsky*
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

Mask or No Mask

- Two players, symmetric game
- Actions are to wear or not wear a mask
- When both wear masks, both are protected; if neither wear masks, neither are protected
- If only one wears masks, the other person is protected
- Pareto optimal outcome? Nash outcomes?

	M	$\overline{M}$
M	$1,\ 1$	$-1,\ 1$
$\overline{M}$	$1,\ -1$	$-1,\ -1$



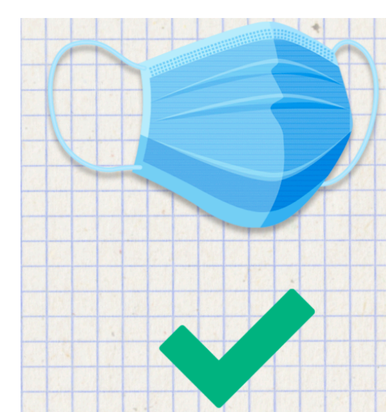


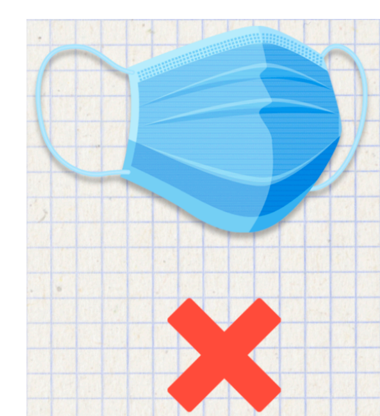
Mask or No Mask

- Two players, symmetric game
- Actions are to wear or not wear a mask
- When … neither …
- If only one wears masks, the other person is protected
- Pareto optimal outcome? Nash outcomes?

Problem of equilibrium selection: **Reasonable to assume that Pareto-optimal outcomes are more likely to be played**

	M	$\overline{M}$
M	$1, 1$	$-1, 1$
$\overline{M}$	$1, -1$	$-1, -1$





Dealing with Multiple Equilibrium

- When you have multiple equilibria, this creates an equilibrium selection problem
- How do players know which equilibrium to play?
- Reasonable to assume "good" equilibria" are better
  - Social welfare (overall sum of utility is same)
  - Pareto optimal
- When designing our own mechanism, it is important to strive for "good" equilibrium
  - Unique/ Social welfare maximizer, Pareto optimal etc