# Paper Evaluation 1: Part B

**LaTeX Source:** `https://www.overleaf.com/read/ffdgxhhfthsf#bf3a46`

**Sponsored Search and GSP.** In the sponsored search auction problem, there are $k$ slots, the $j$th slot has a click-through rate (CTR) of $\alpha_j$ (non-increasing in $j$), and the utility of bidder $i$ in slot $j$ is $\alpha_j(v_i - p_j)$, where $v_i$ is the value-per-click of the bidder and $p_j$ is the price charged per-click in slot $j$. In class, we showed that the following auction (lets call it the *VCG auction*) is dominant-strategy proof.

The *Vickrey-Clarke-Groves* (VCG) auction for sponsored-search is defined below.

**Vickrey-Clarke-Groves (VCG) auction.**

1. Rank the advertisers from highest to lowest bid-per-click $b_i$; assume without loss of generality that $b_1 \geq b_2 \geq \ldots \geq b_n$.
2. For $i = 1, 2, \ldots, k$, assign the $i$th bidder to the $i$th slot.
3. For $i = 1, 2, \ldots, k$, charge the $i$th bidder a price-per-click given by Myerson's formula:

$$p_i = \sum_{j=i}^{k} b_{j+1} \left( \frac{\alpha_j - \alpha_{j+1}}{\alpha_i} \right)$$

The *generalized-Second-Price* (GSP) auction is defined below:

**Gneralized Second Price (GSP) Auction.**

1. Rank the advertisers from highest to lowest bid; assume without loss of generality that $b_1 \geq b_2 \geq \ldots \geq b_n$.
2. For $i = 1, 2, \ldots, k$, assign the $i$th bidder to the $i$th slot.
3. For $i = 1, 2, \ldots, k$, charge the $i$th bidder a price of $b_{i+1}$ per click.

**Understanding Edelman et al. (2007).** Edelman et al. (2007) analyze this GSP auction formally and show that it has a canonical equilibrium that is equivalent to the dominant strategyproof outcome of the VCG auction.

Their analysis can be broken down and formalized in several parts.

1. Prove that for every $k \geq 2$ and sequence $\alpha_1 \geq \ldots \alpha_k > 0$ of CTRs, the GSP auction is not dominant strategyproof (that is, truthful bidding is not a dominant strategy).

2. Fix CTRs for slots and valuers-per-click for bidders. We can assume that $k = n$ by adding dummy slots with zero CTRs (if $k < n$) or dummy bidders with zero value-per-click (if $k > n$). A bid profile $\mathbf{b}$ is a Nash equilibrium of GSP if no bidder can increase her utility by unilaterally changing her bid. Verify that this condition translates to the following inequalities, under our standing assumption that $b_1 \geq b_2 \ldots \leq b_n$ for every $i$:

$$\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_j) \text{ for every higher slot } j < i \tag{1}$$
$$\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_{j+1}) \text{ for every lower slot } j > i \tag{2}$$

3. A bid profile $\mathbf{b}$ with $b_1 \geq \ldots \geq b_n$ is *envy-free* if for every bidder $i$ and slot $j \neq i$:

$$\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_{j+1}). \tag{3}$$

   (a) Verify that every envy-free bid profile is a Nash equilibrium.[1]

   (b) Next, a bid profile is *locally envy-free* if the inequality 3 holds for every pair of adjacent slots—for every $i$ and $j \in \{i-1, i+1\}$. By definition, an envy-free bid profile is also locally envy-free. Prove that, for strictly decreasing CTRs, every locally envy-free bid profile is also envy-free.

4. Prove that, for every value-per-click and strictly decreasing CTRs, there is a locally envy-free equilibrium of the GSP auction in which the assignment of bidders to slots and all payments-per-click equal those in the truthful outcome of the corresponding dominant-strategyproof VCG sponsored-search auction. *Note. This exactly what Theorem 1 in Edelman et al. (2007) is proving.*

5. Prove that the equilibrium in Part (4) is the lowest-revenue envy-free bid profile.

[1] Why "envy free"? Setting $p_j = b_{j+1}$ for the current price-per-click of slot $j$, then these inequalities translate to: "every bidder $i$ is as happy getting her current slot at the current price as she would be getting any other slot at that slot's current price.