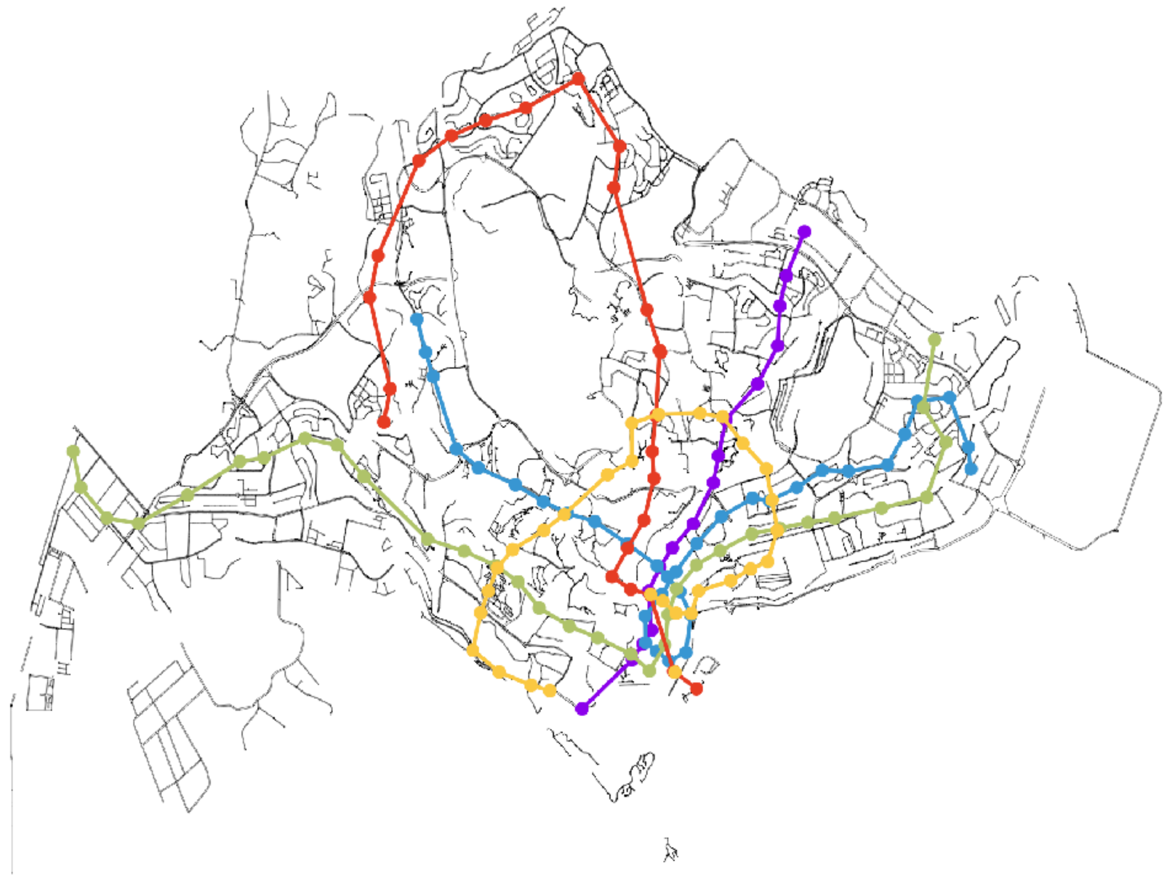
CSCI 357: Algorithmic Game Theory

Lecture 15: Decentralized Markets

Shikha Singh



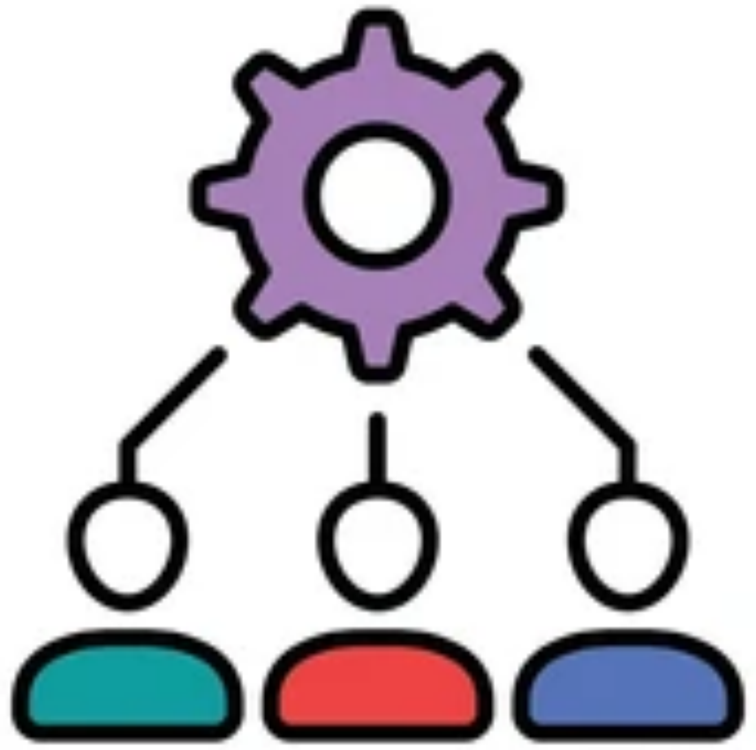
Announcements and Logistics

- Paper evaluation # 3 was due on Gradescope at noon
- Working on grading Assignment # 3
- Assignment # 4 will be released today, due next Friday at noon
- Midterm # 2 will be on April 28 in class
    - Mostly focused on everything covered after Midterm 1
    - From markets with money: Bayes Nash and revenue equivalence will be included
    - Need to remember and know how to use fundamental definitions (dominant-strategyproof, Nash equilibrium, Condorcet consistency, etc.)
    - Similar to Exam 1: can bring up to 5 pages of notes

Questions?

Story So Far: Centralized Markets

Centralized algorithm
Centralized algorithm
coordinates



Centralized Markets | with Money | without Money

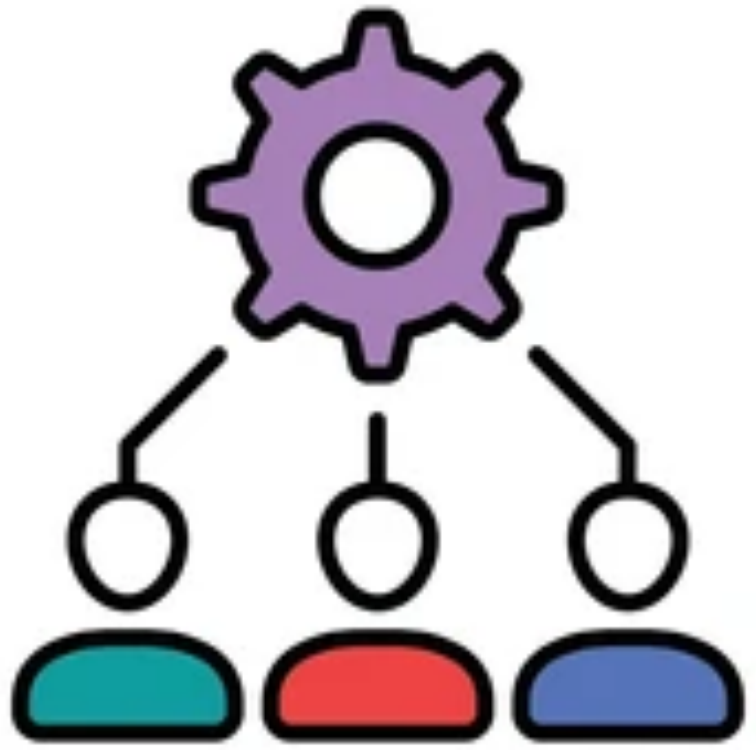
Goal. Elicit preferences from individuals so as to make a collective decision on allocation in a way that maximizes global objectives with participant objective (maximize utility).

Story So Far: Centralized Markets

Centralized algorithm coordinates

Centralized Markets

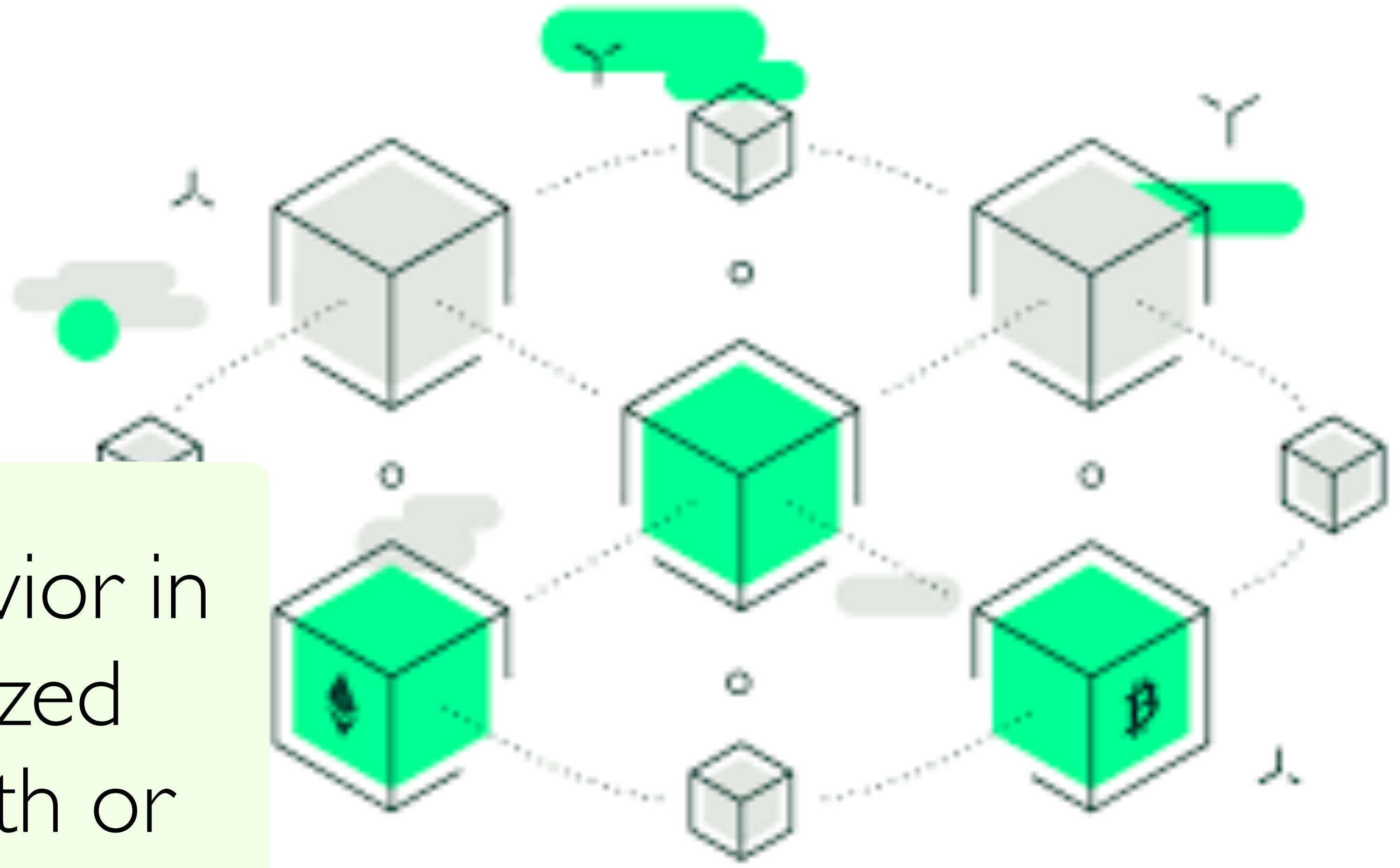
with Money



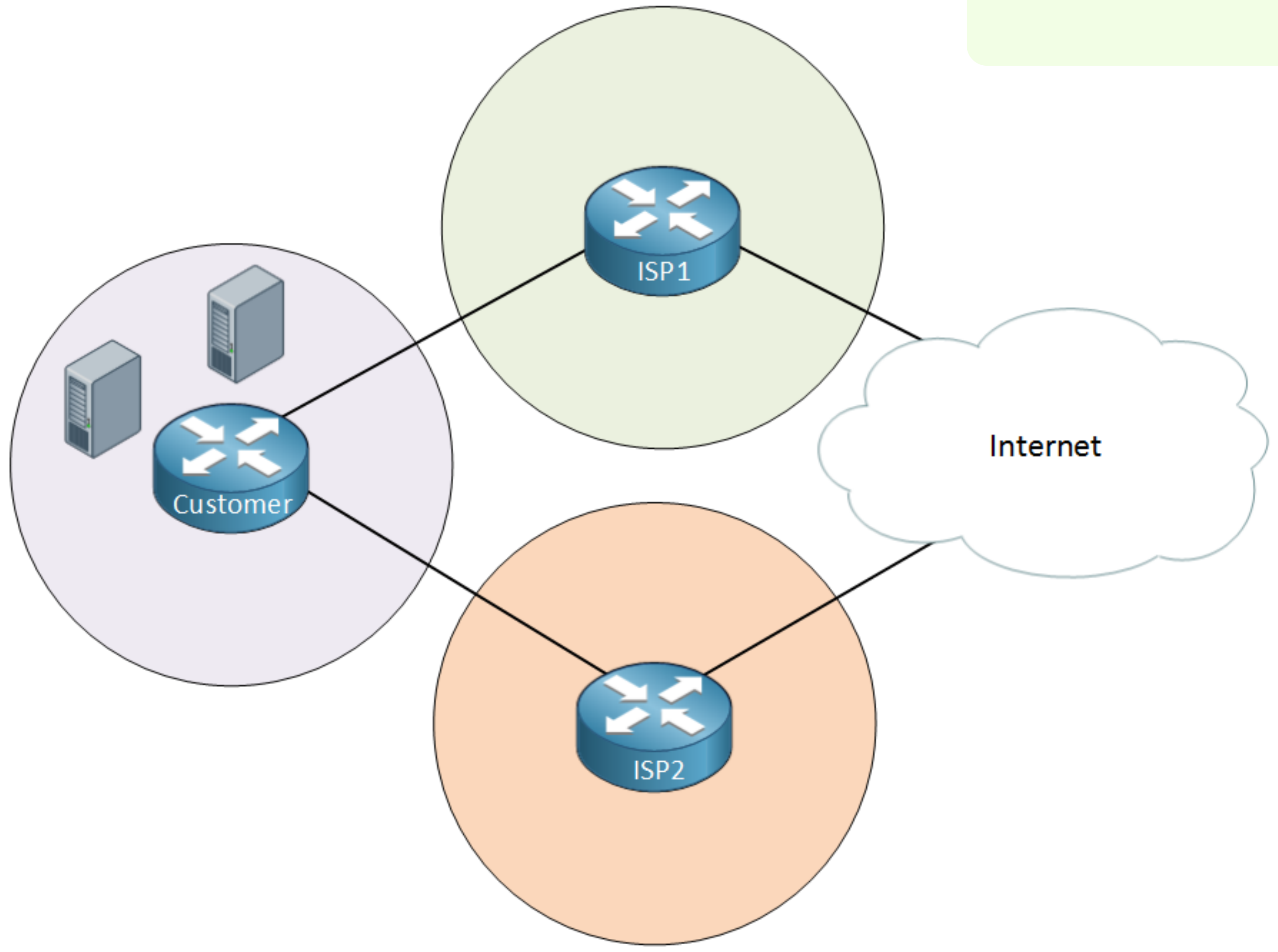
without Money

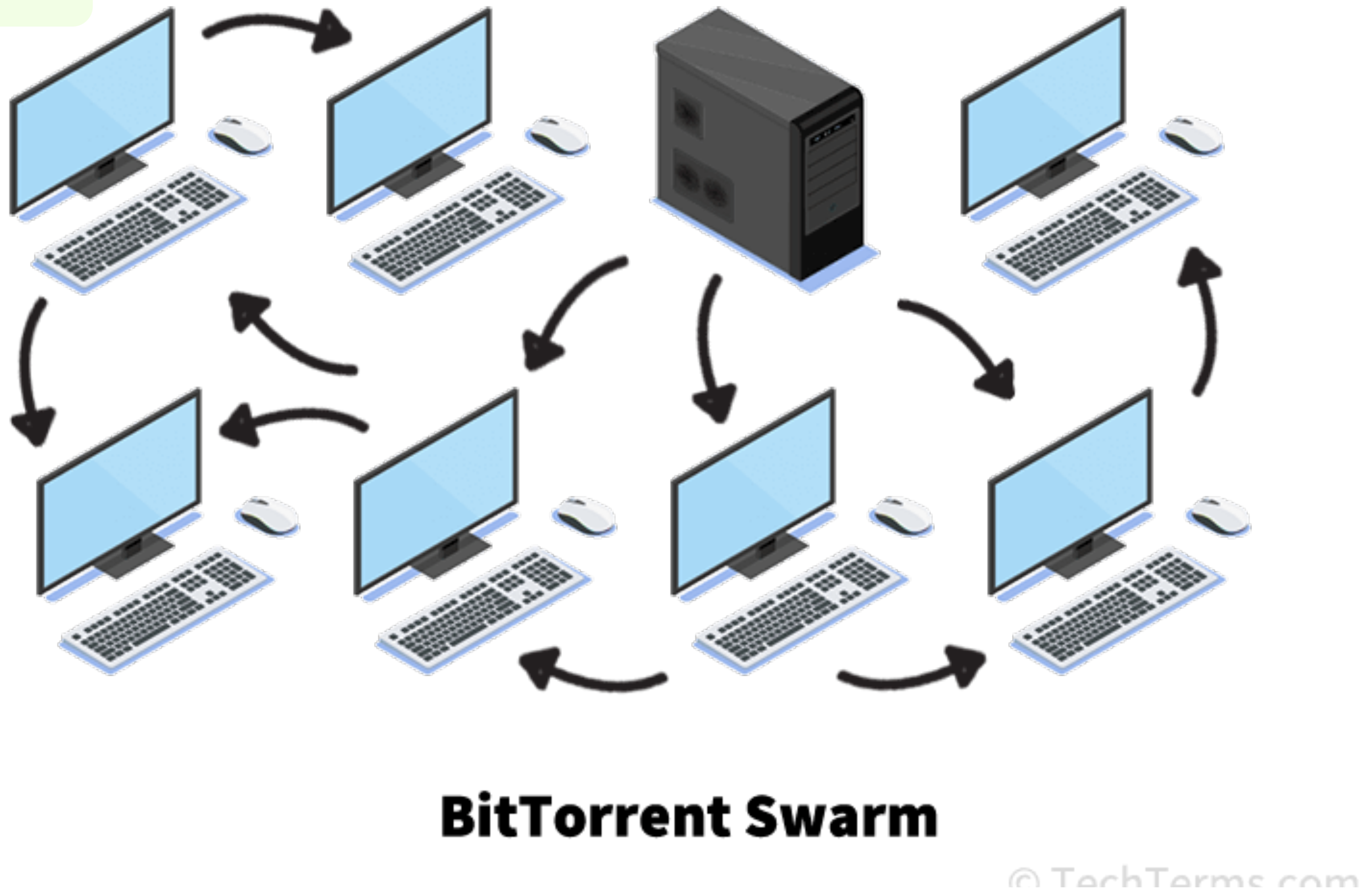
Downsides?

Later: Decentralized Systems



Selfish behavior in decentralized markets with or without money





Decentralized Markets

Decentralized Market

- A market is **decentralized** if participants are free to transact directly with each other, without any central coordination
  - College admissions in the US and most job markets are decentralized
- Even decentralized markets may have some central coordination
  - ride-sharing markets are decentralized with some coordination mechanism

Multiple Item Matching Market

- We will discuss a **decentralized asynchronous** matching market where buyers are free to buy the item they wish
- Each buyer wants only **one** item & each item can be given to at most one buyer
- More formally, we have n potential buyers, m different items
- Assume $m \geq n$ (if this is not true, we can always create dummy items that everyone values at 0)
- Each buyer i has a **private valuation** $v_{ij} \geq 0$ for each item j
- **Examples**: matching houses to buyers, renters to Airbnb rooms, or any idiosyncratic items to buyers

airbnb

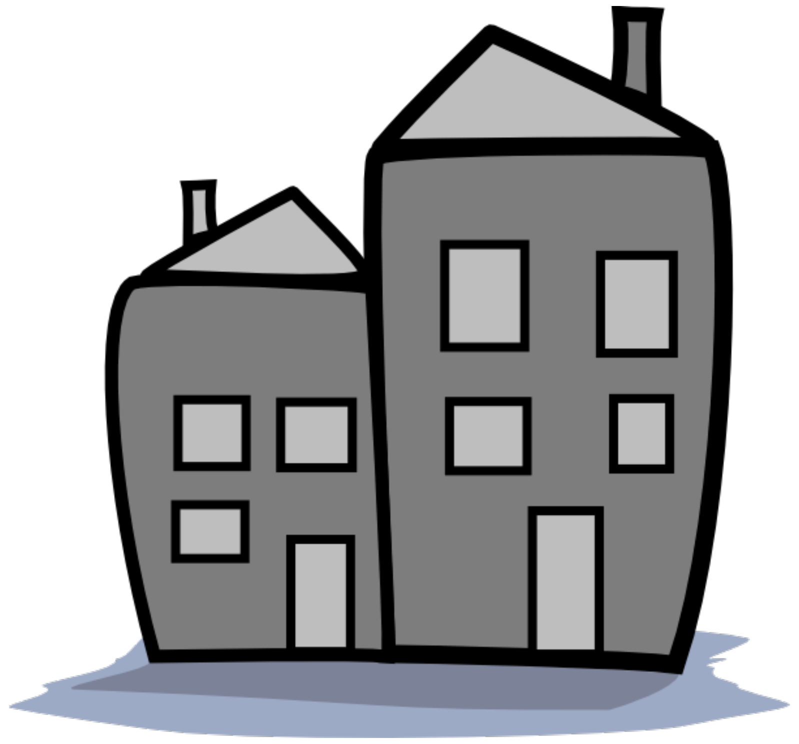
Zillow®

Example Market

Prices

p_1?

p_2?



p_3?

Zoe

Valuations

12, 2, 4

Ch

8, 7, 6

Valuations listed are in **order** of houses **(top down)**

Jing

7, 5, 2

Prices and Utility

- If the price of item j is p_j, the utility that person i receives from getting item j is

$$u_{ij} = v_{ij} - p_j$$

- **Goal of buyers:** choose items individually to maximize their utility

Questions

- What prices do we expect to see in a market?
- Does there exists **prices** and **a way to match buyers to items** (find a matching) such that each buyer gets an item that maximizes their utility?
- Do these prices **"clear the market"**: sell all items that have any demand
- Imagine an ascending price clock and bidders dropping out of contention

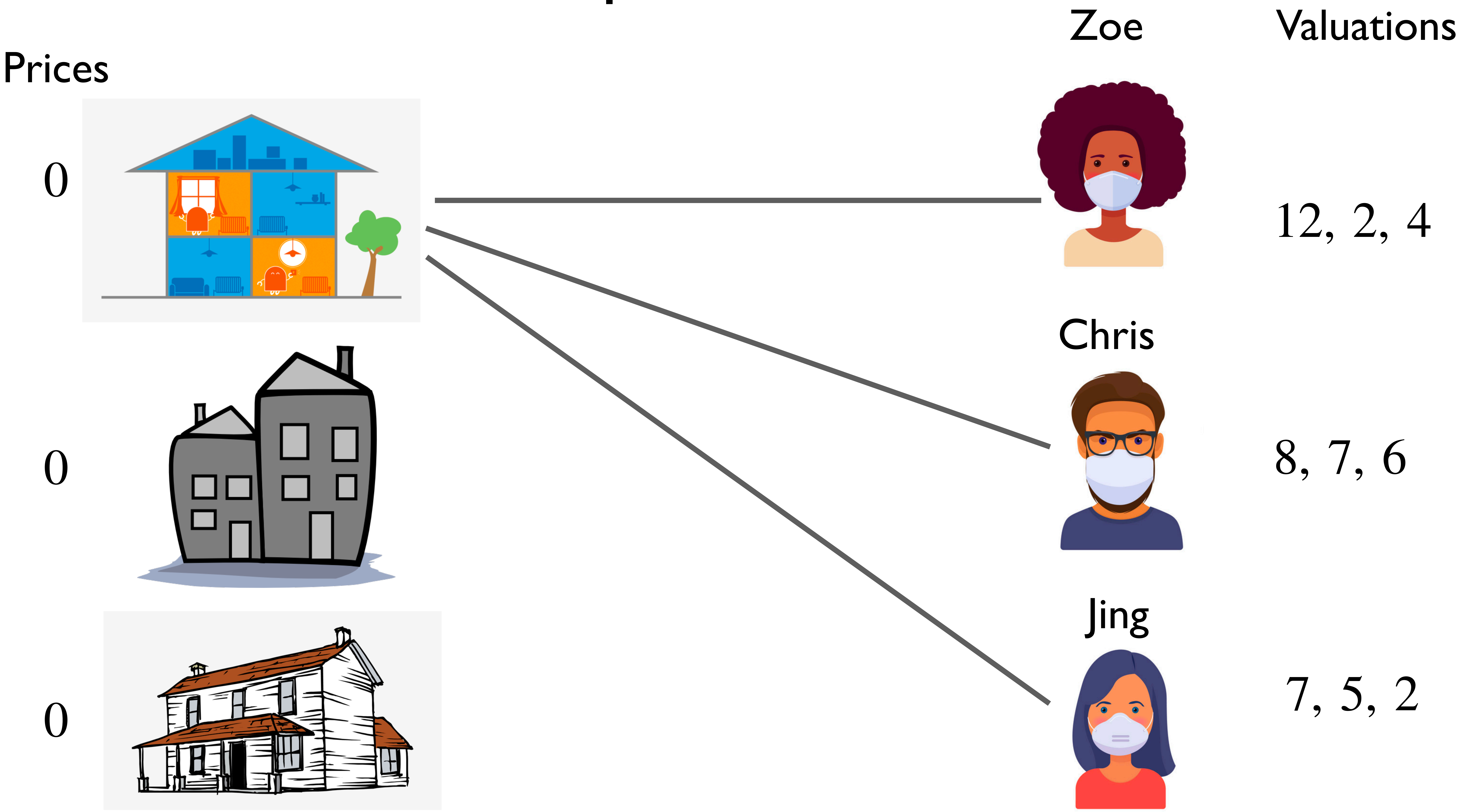
Social Welfare

- Let $M(i)$ denote the item matched with buyer i or $\varnothing$ if none
- Our goal has been to design mechanisms that maximize social welfare, that is, find a matching M of buyers to items that maximizes $\sum_{i=1}^{n} v_{iM(i)}$
- **Goal:** an outcome that achieves good guarantees but we have no control over it
- **Question.** If we let the market run its course what prices and allocation do we see?
    - How good is the social welfare of such an outcome?

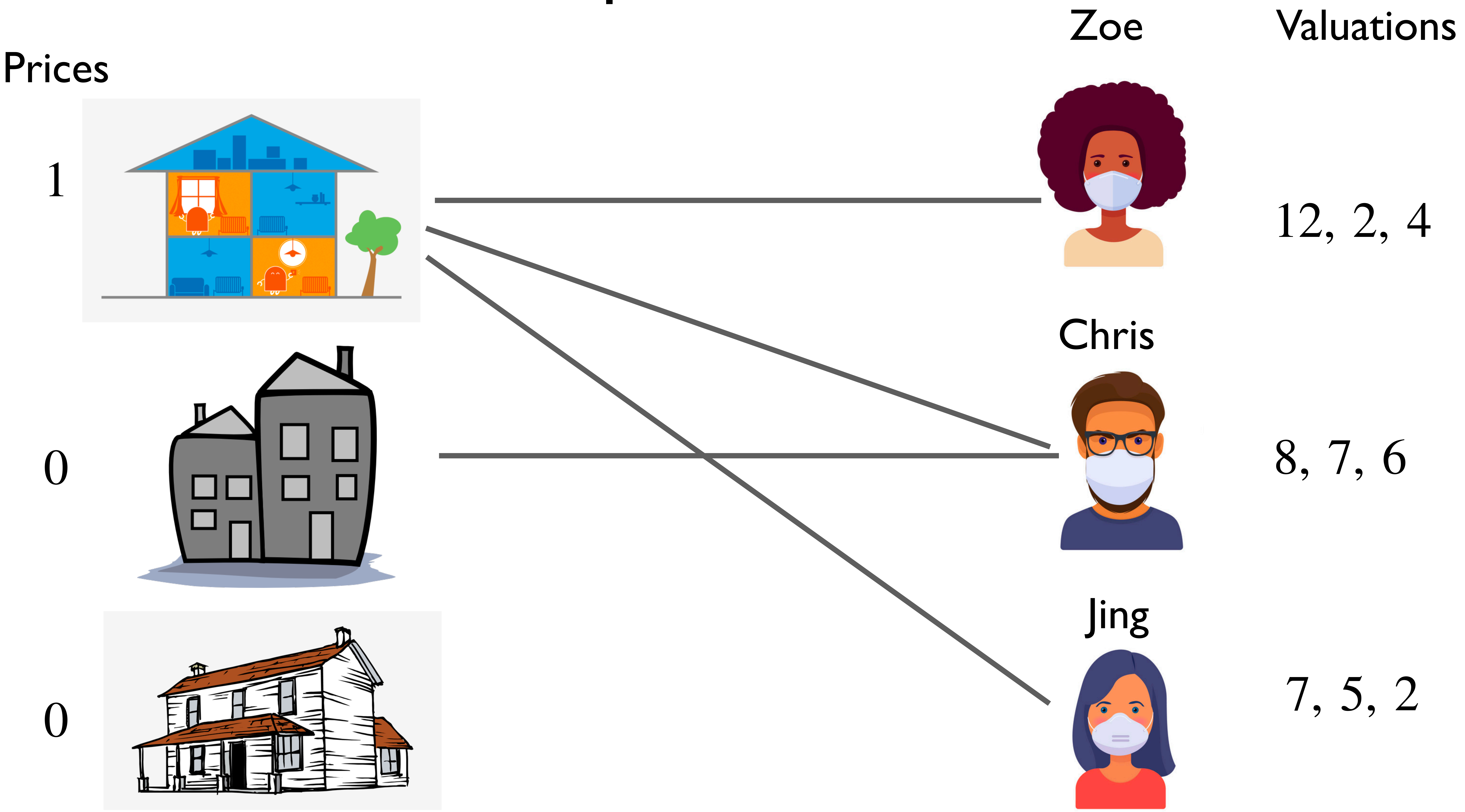
Preferred Items & Graph

- Given prices $\mathbf{p} = (p_1, \ldots, p_m)$ for the items, the preferred items for buyer i are all the items that maximize its utility
- Let $u_i^* = \text{max}_{\text{all items } j}\ (v_{ij} - p_j)$ be the maximum utility i can obtain given $\mathbf{p}$
- Let the set of preferred items P_j of buyer i given the prices $\mathbf{p}$ be all items that maximize its utility: $P_j = \{j \mid v_{ij} - p_j = u_i^*\}$ assuming $u_i^* \geq 0$
    - If $u_i^* < 0$ then $P_j = \emptyset$
- Create a **preferred-item graph** (given prices $\mathbf{p}$) where nodes are items and buyers and there is an edge between buyers and their preferred items

Preferred-Item Graph



Preferred-Item Graph



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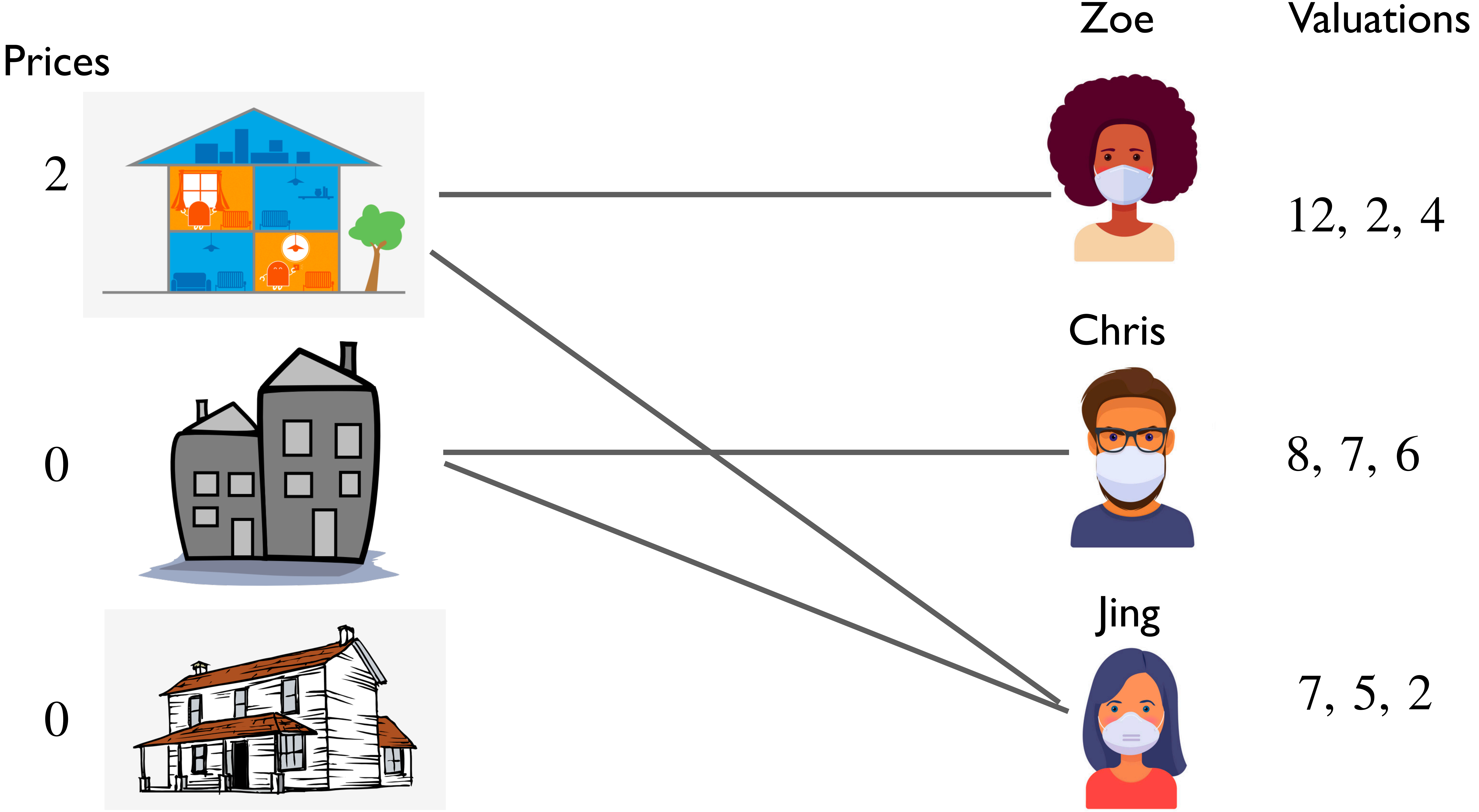

Market-Clearing Prices

- A selection of prices $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ is **market-clearing** if:
  - **Condition 1.** There is a **matching** in the preferred-item graph such that **all buyers are matched** to an item
  - **Condition 2.** If an item j is not matched to any buyer, then its price $p_j = 0$, in other words, every item with non-zero price $p_j > 0$ must get sold
- This means that at market-clearing prices, each buyer can come by and pick some item that maximizes its utility
  - Assume tie-breaks can occur in a coordinated way
- Furthermore, at these **prices the market will "clear"**
  - Only items left behind are those that are not desirable (price 0)

Matching. A subset of edges M form a matching if no two edges in M is incident on the same node

Market-Clearing Prices

What does this condition remind you off?

- **Condition 1** says that, given the prices, all buyers:
  - Get a utility maximizing item

$$\underbrace{v_{ij} - p_j}_{\text{Utility from receiving } j \text{ at price } p_j} \geq \underbrace{v_{ij'} - p_{j'}}_{\text{Utility from receiving } j' \text{ at price } p_{j'}}$$

- Why do we need **Condition 2?**
- **Condition 2** says that the outcome is "market clearing" in the sense that every good that is desired is sold
  - Only good that is allowed to be not sold are those with $p_j = 0$

Market-Clearing Prices

Outcome must be **envy free**!

- **Condition 1** says that, given the prices, all buyers:
  - Get a utility maximizing item

$$\underbrace{v_{ij} - p_j}_{\text{Utility from receiving } j \text{ at price } p_j} \geq \underbrace{v_{ij'} - p_{j'}}_{\text{Utility from receiving } j' \text{ at price } p_{j'}}$$

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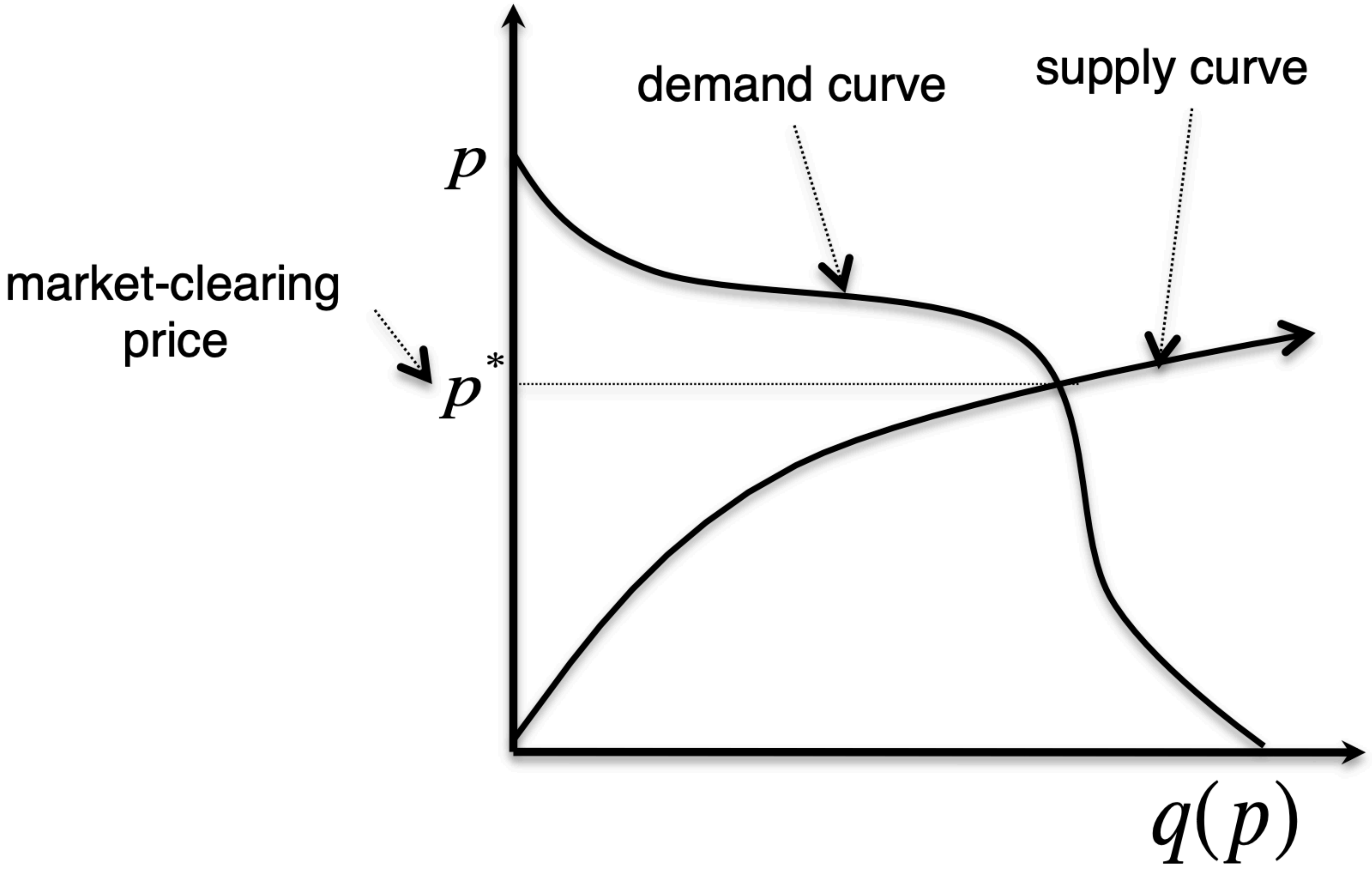
Market-Clearing Prices

- **Condition 2** says that the outcome is "market clearing" in the sense that every good that is desired is sold
  - Only good that is allowed to be not sold are those with $p_j = 0$
- Why is this condition important?
  - Notice that we can trivially satisfy Condition 1 by setting all prices to be ∞
  - At that price, no buyer wants any item
- But is this a good outcome?
  - No one gets anything: no welfare/surplus generated!
- Need prices to **clear market** and to **optimize surplus** generated

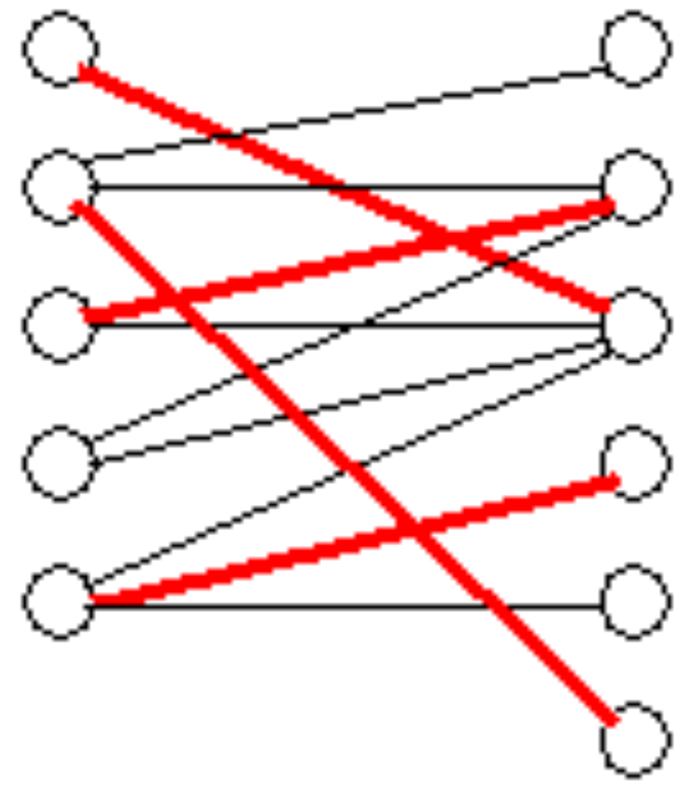
Economics Point of View

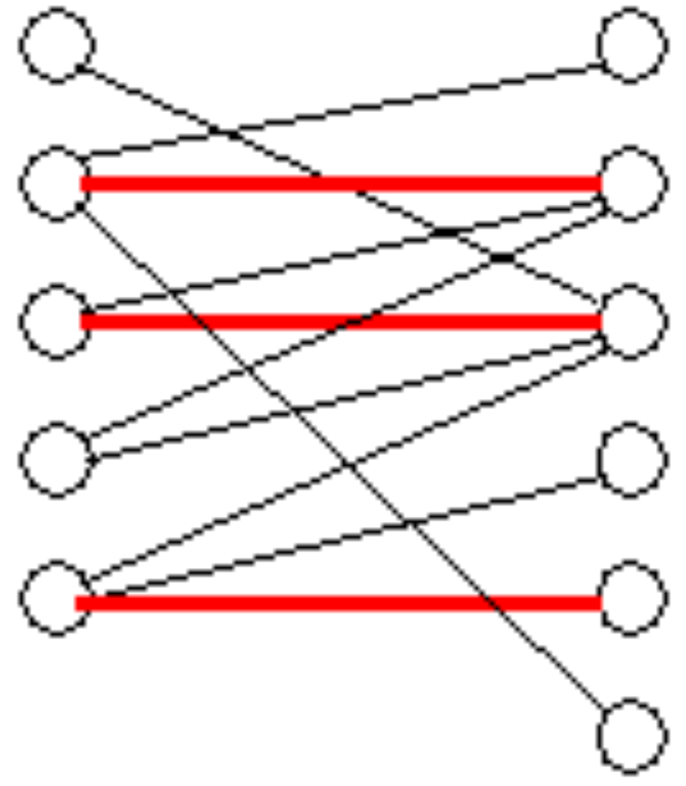
- Market clearing prices in economics are prices at which supply is equal to demand
- **Demand curve:** as price increases, typically demand goes down
- **Supply curve:** As price increases, typically supply $s(p)$ increases
- Price where they meet: market clearing



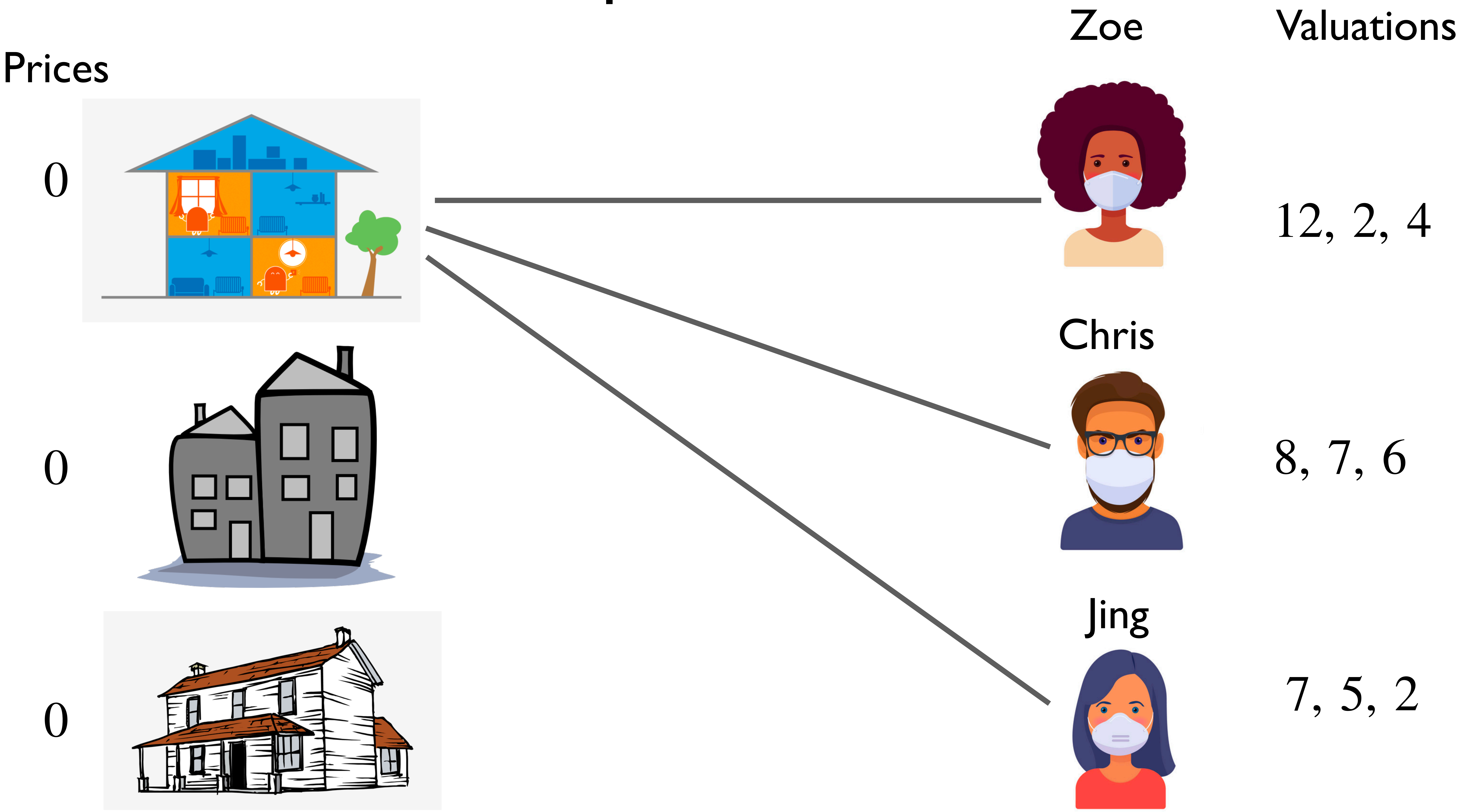
Market-Clearing Prices

- **Reminder (matching definition).** A subset of edges M form a matching if no two edges in M is incident on the same node
  - An independent set of edges
- Looking for a **buyer-perfect matching** (a matching that "covers" all buyer nodes)
- Since the edges in the preferred-item graph depend on prices of items, the question is,
  - What prices cause a buyer-perfect matching to exist?
- **Question.** How do we know when a buyer-perfect matching is not possible?





Preferred-Item Graph



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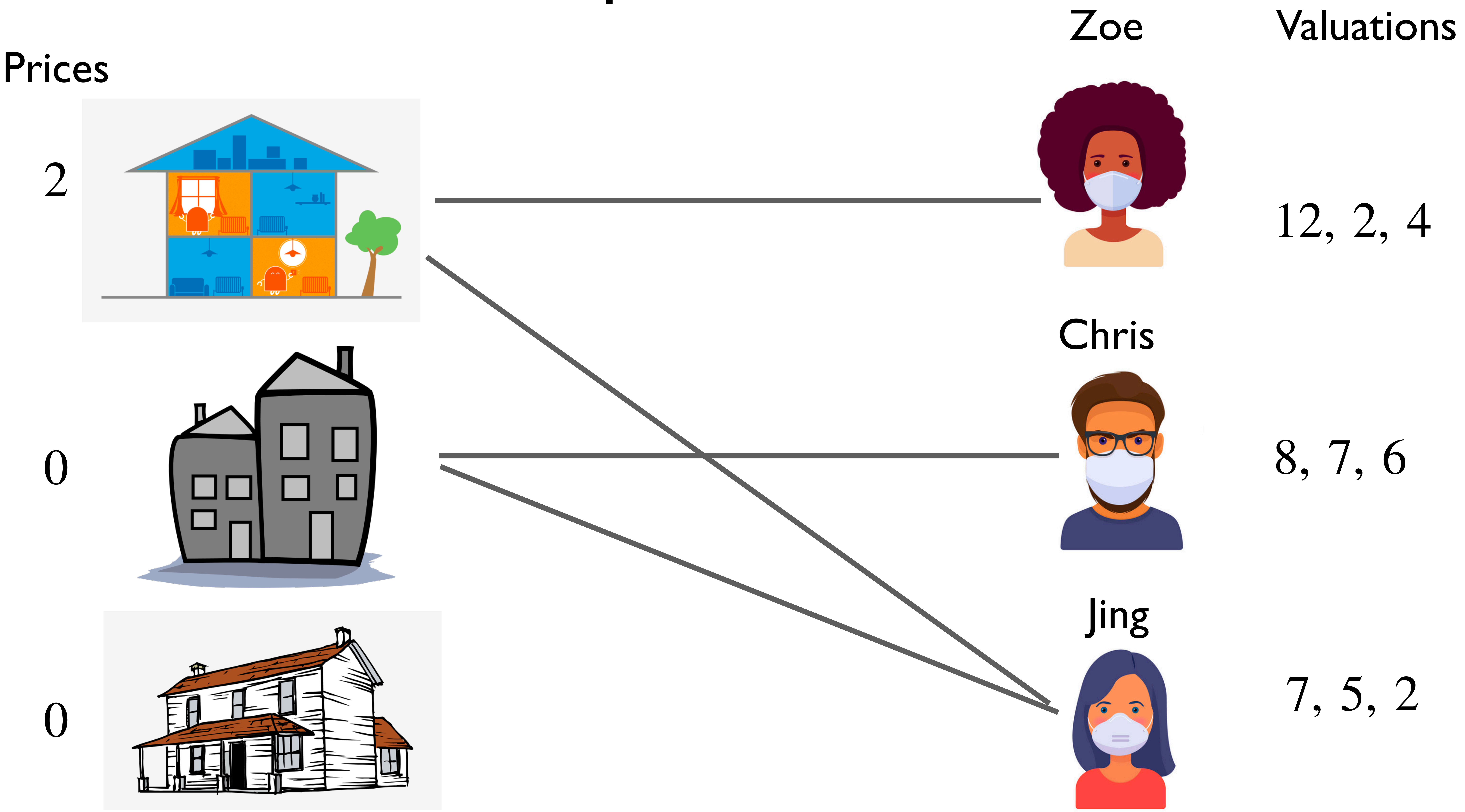
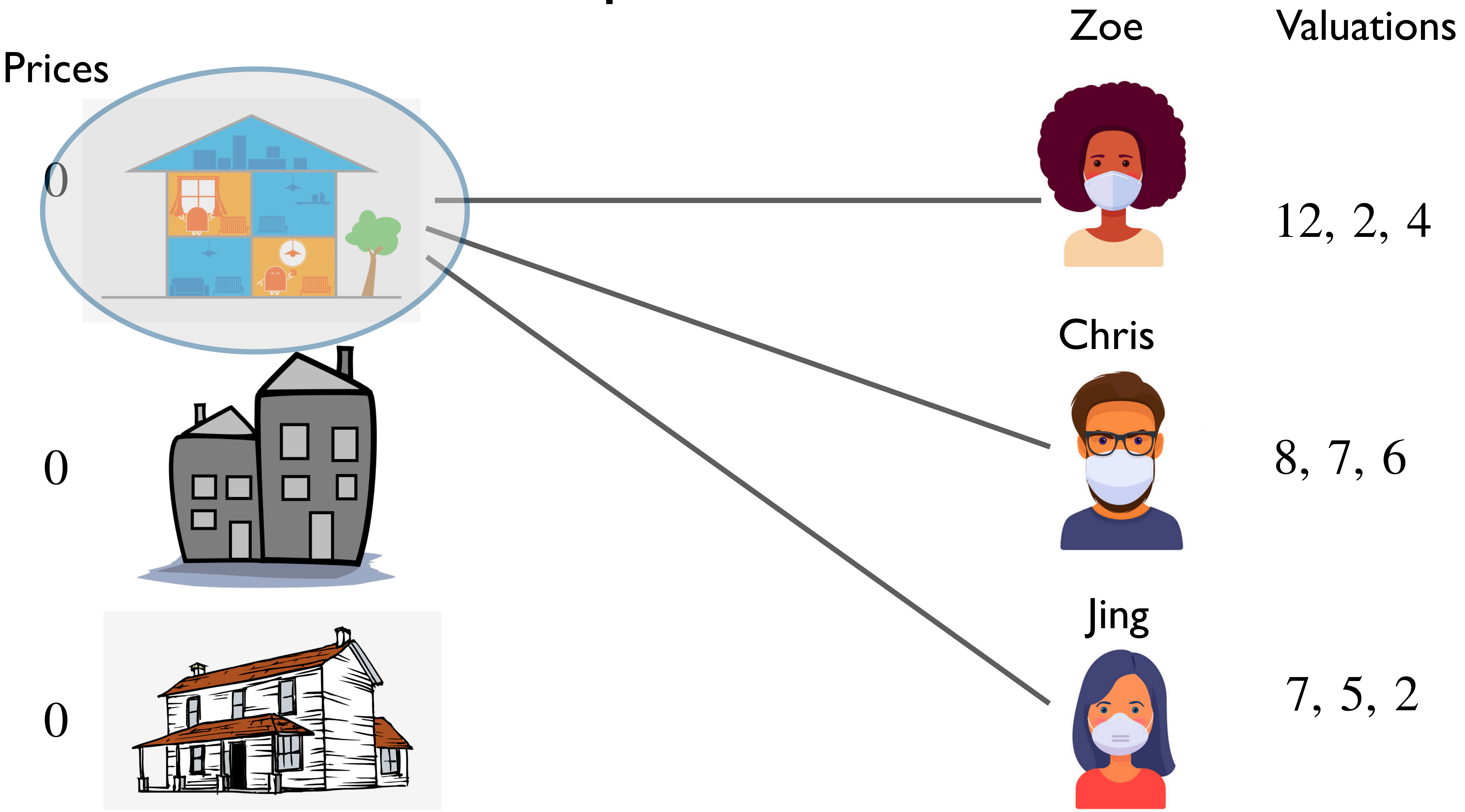


Hall's Theorem

- Let S be a subset of nodes, then the **neighborhood** $N(S)$ is the set of all nodes that adjacent to nodes in S
- In a bipartite graph (X, Y) has a Y perfect matching **if and only if** for every $S \subseteq Y$ with neighborhood $N(S)$ the following holds:

  $|N(S)| \geq |S|$ (neighborhood is at least as large)
- Thus, when a Y-perfect matching is not possible, there exists a subset $T \subseteq Y$ that violates the above condition, that is,
  - Such a set $N(T)$ is called a **constricted set** set
- If there is no buyer-perfect matching: can always find a constricted set
  - "Over-demanded" items at current price

Preferred-Item Graph



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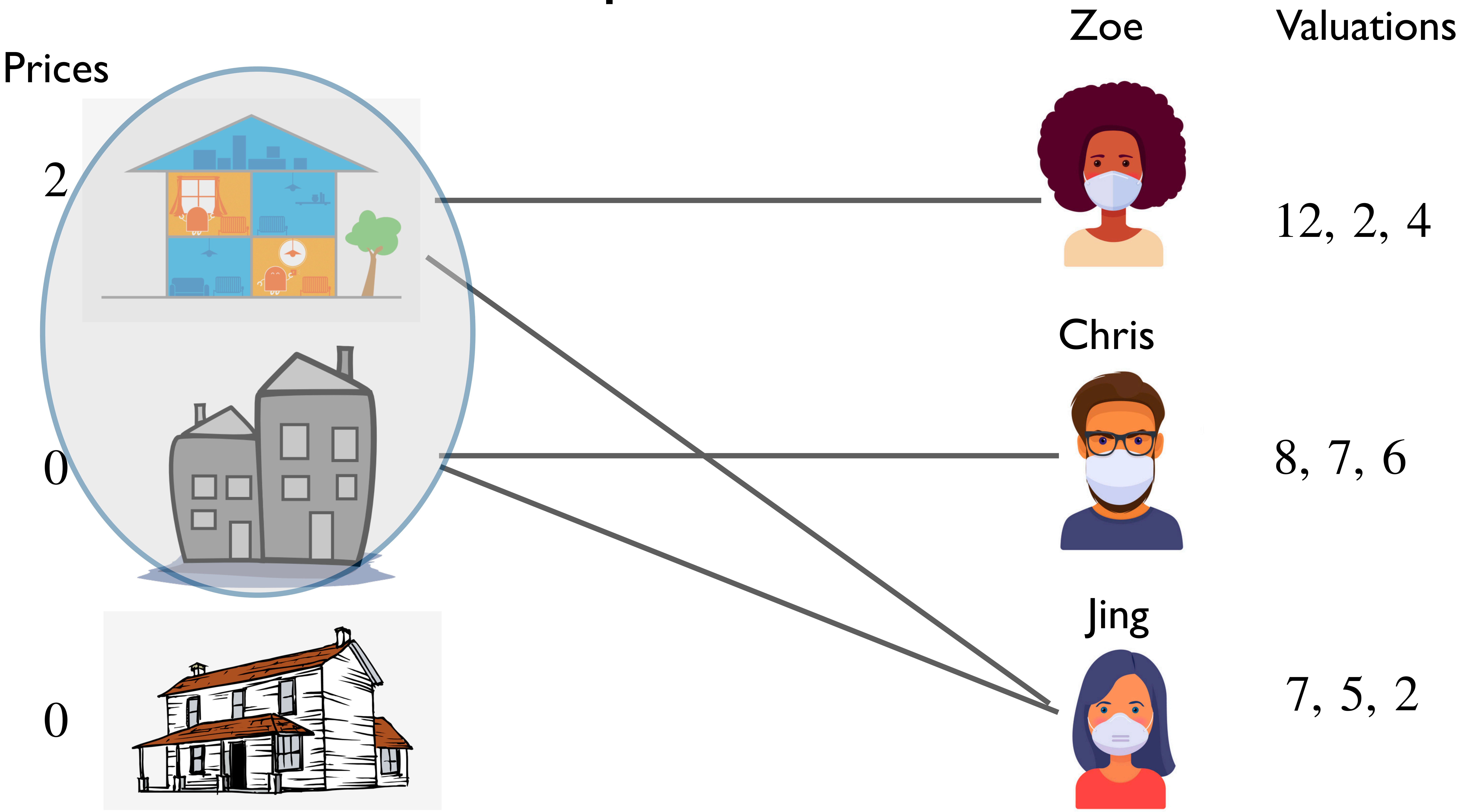

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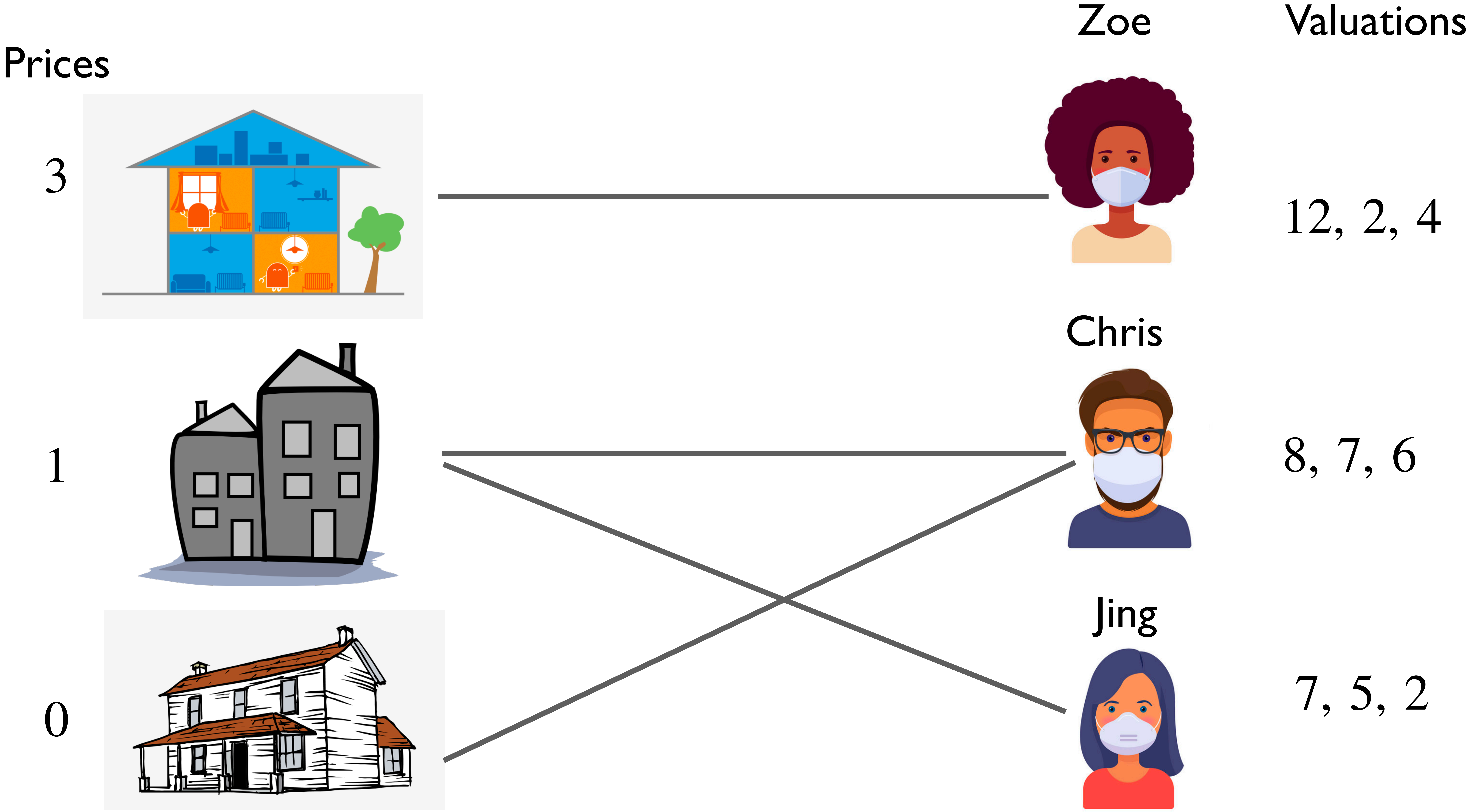

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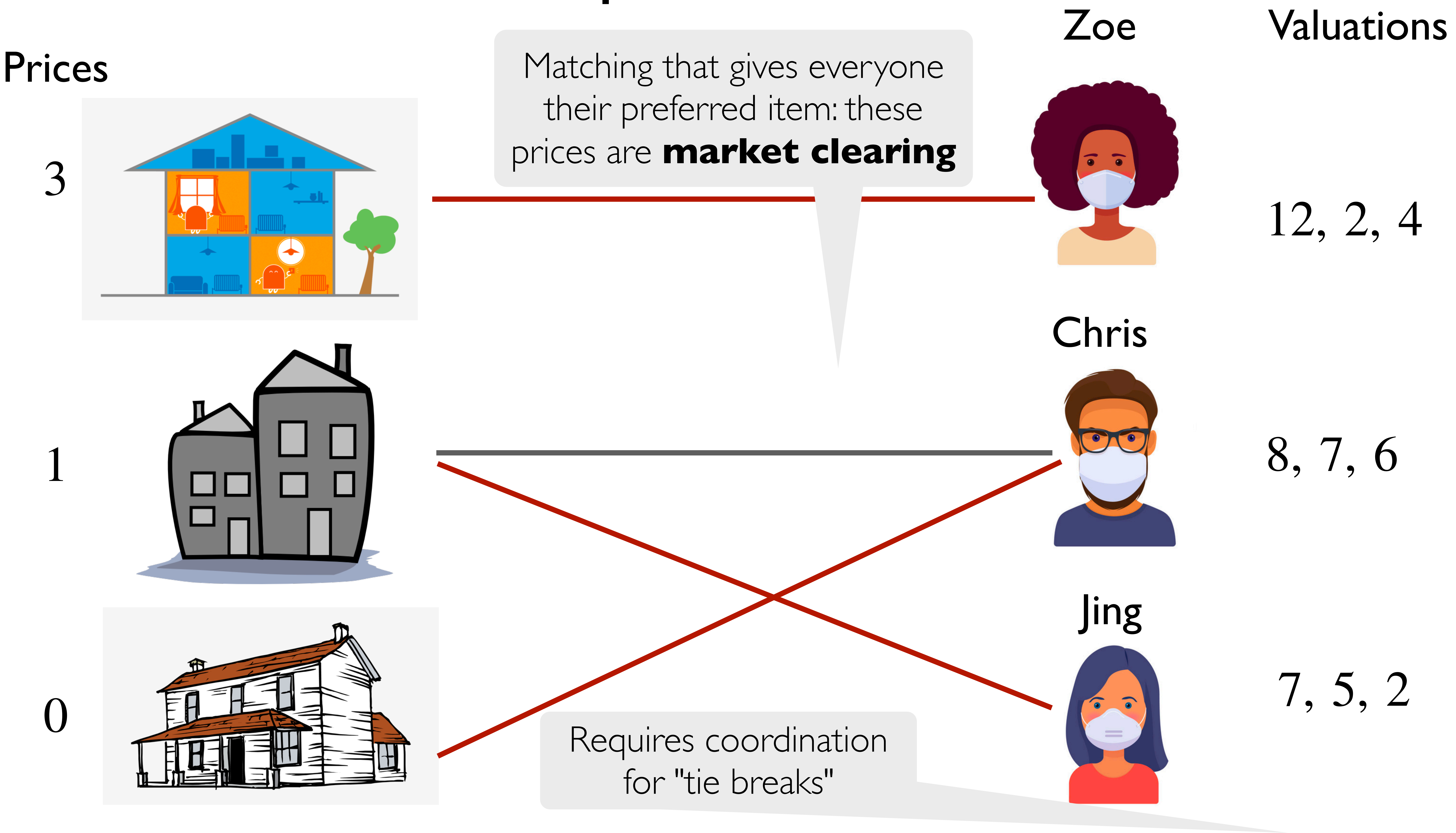
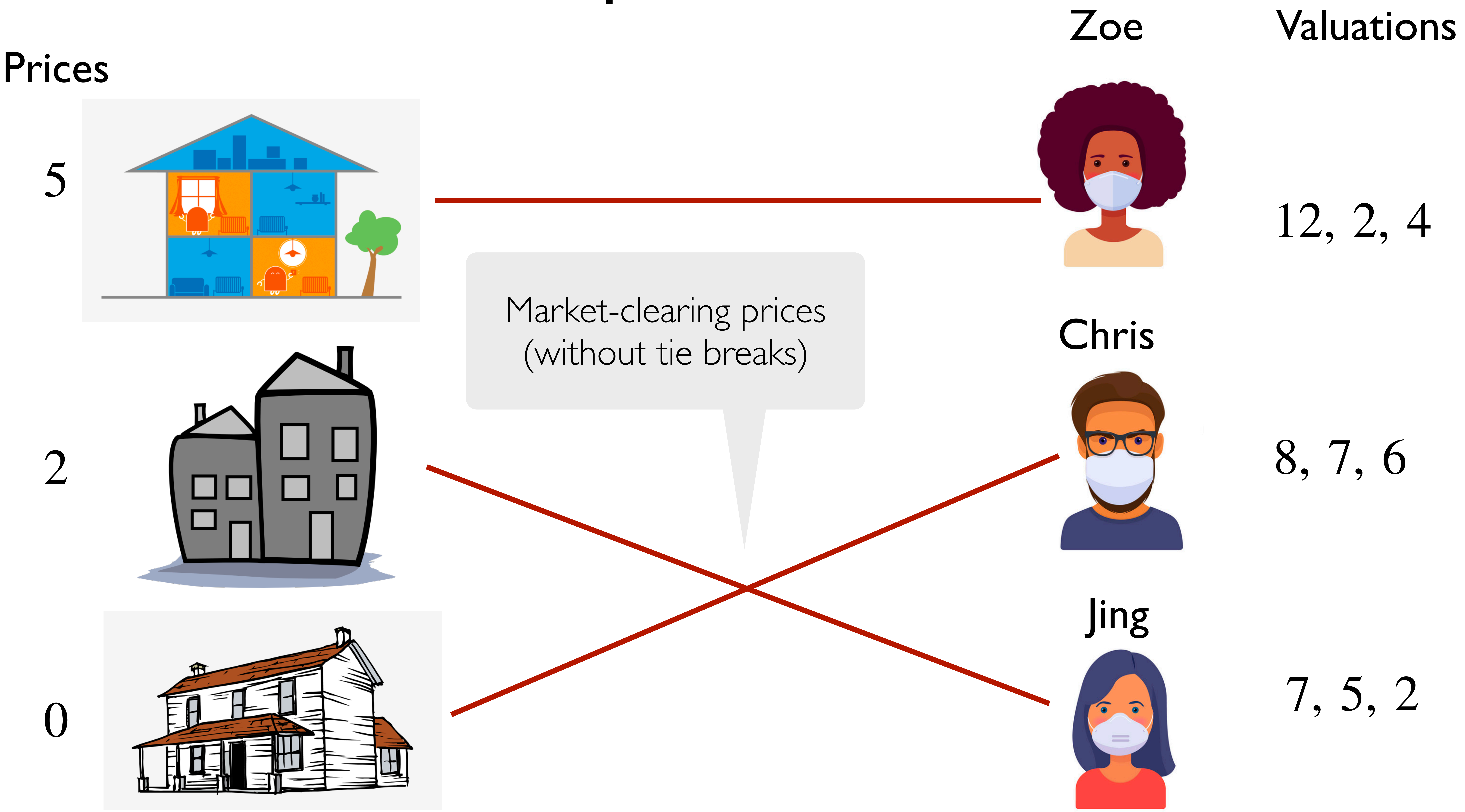


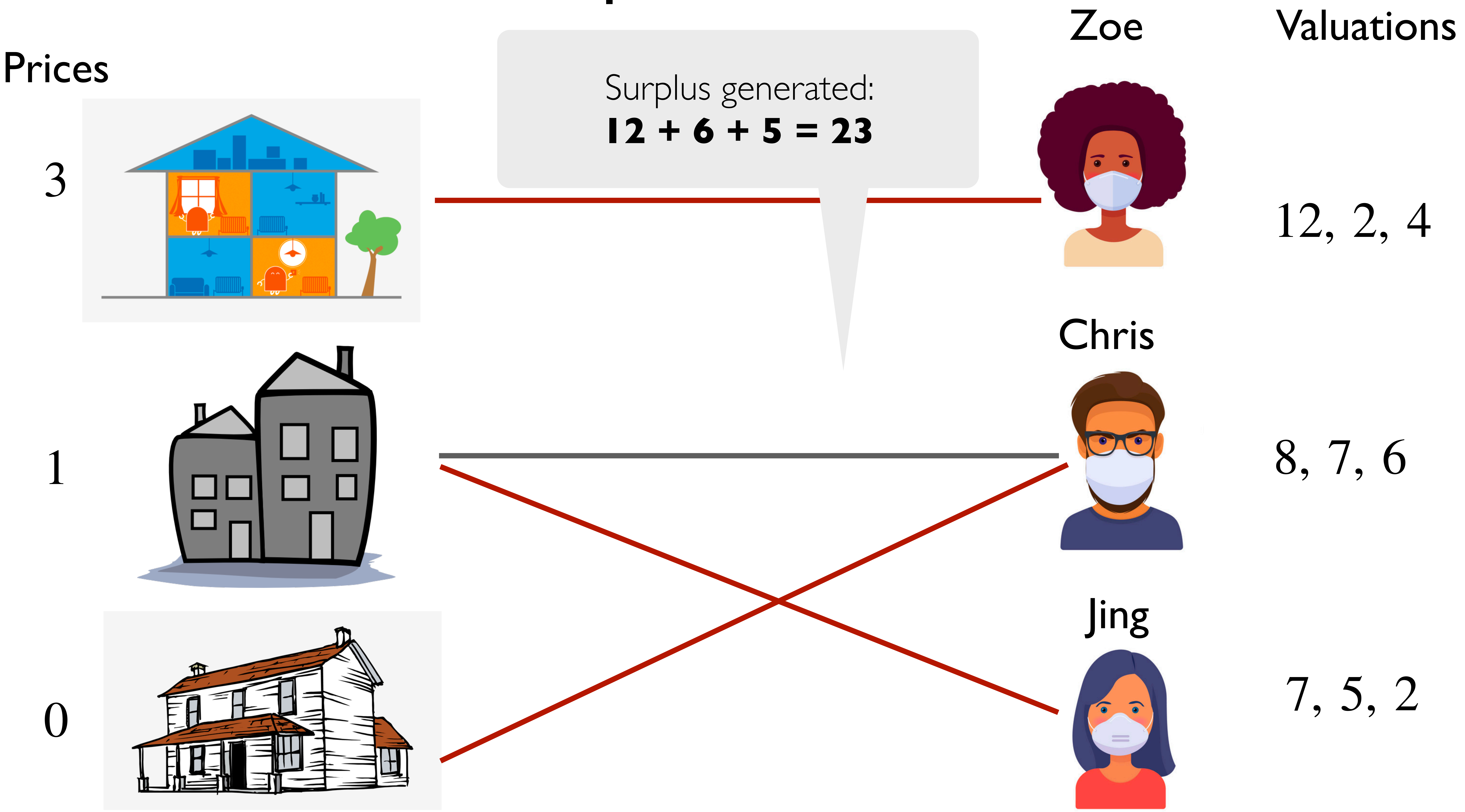
Preferred-Item Graph



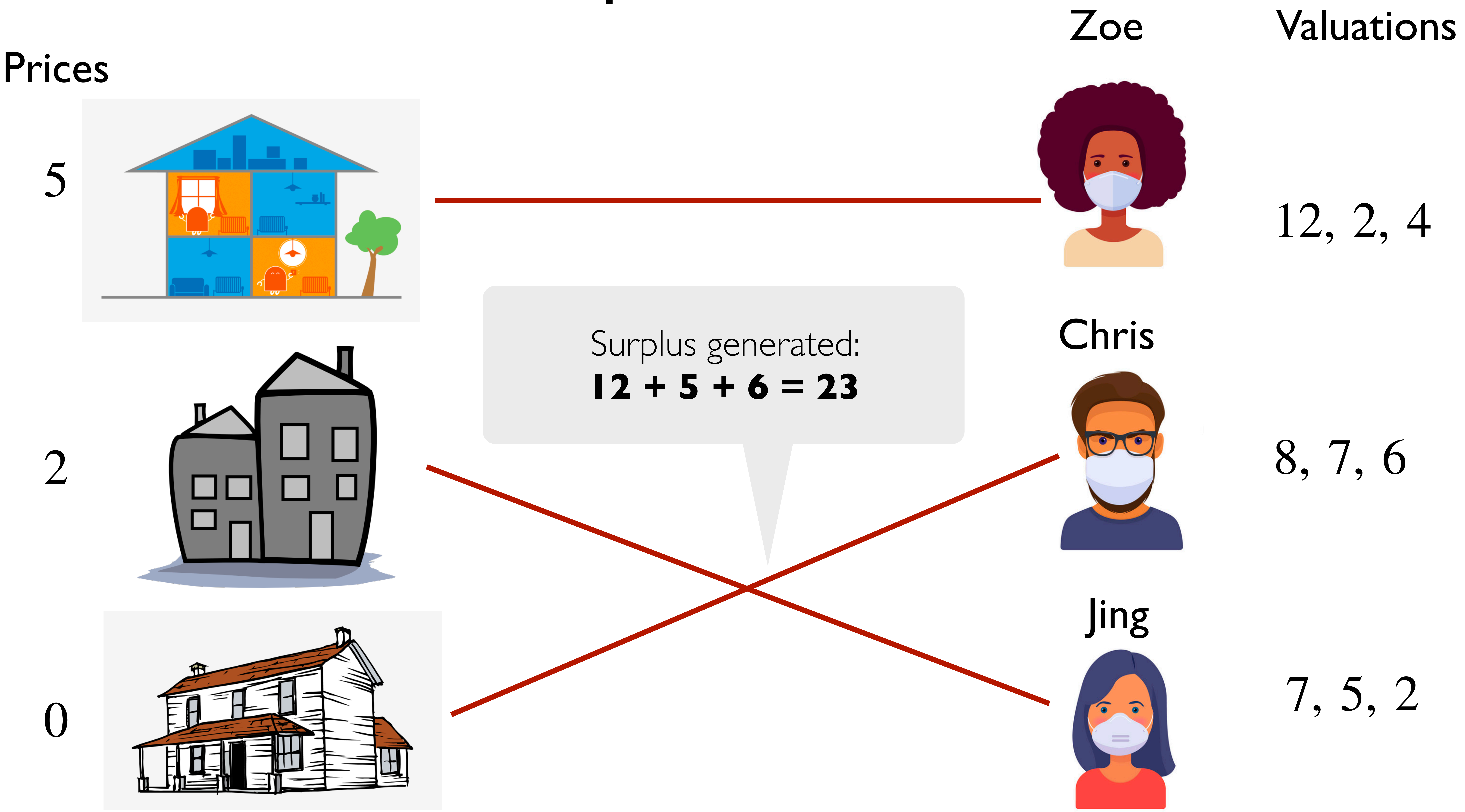
Competitive Equilibrium

- Market-clearing prices $\mathbf{p}$ along with the matching M from buyers to their preferred item is called a **competitive** or **Walrasian equilibrium**
- Requirements of competitive equilibrium are strong
  - Put a price tag p_j on each good
  - Let each buyer i independently pick whichever good they want
- Magically, there are no conflicts and each buyer gets what they want
  - (Allowing ties to be broken in a coordinated way)
- **Question.** Seems too good to be true, does it always exist?
- **Question.** Should we be happy with the outcome of a competitive equilibrium?

Preferred-Item Graph



Preferred-Item Graph



First Welfare Theorem

- Matchings in a competitive equilibrium are exactly the matching with maximum possible value!
- **First Welfare Theorem (Max-weight matching).** If $(M, \mathbf{p})$ is a competitive equilibrium, then M is a matching with maximum total value, that is,
    - $\sum_{i=1}^{n} v_{iM(i)} \geq \sum_{i=1}^{n} v_{iM'(i)}$ for every matching M'
- In particular, among all possible ways of allocating items such that each buyer is matched to at most one item good and each item is matched to at most one buyer, the allocation achieved at a competitive equilibrium maximizes welfare

First Welfare Theorem Proof

- **Proof.** Consider some matching M^* with the maximum-possible total value
- What we know: $(M, \mathbf{p})$ is a competitive equilibrium
- Using envy-free condition to compare M and M^* at price $\mathbf{p}$:

$$v_{iM(i)} - p_{M(i)} \geq v_{iM^*(i)} - p_{M^*(i)} \quad \text{for every bidder } i$$

- Let the sum of prices $\sum_{j=1}^{m} p_j = P$
- Summing up the inequality in blue over all bidders

M^* can assign each bidder at most one item

$$\underbrace{\sum_{i=1}^{n} v_{iM(i)}}_{\text{total value of } M} - \underbrace{\sum_{i=1}^{n} p_{M(i)}}_{= P \text{ by CE property (b)}} \geq \underbrace{\sum_{i=1}^{n} v_{iM^*(i)}}_{\text{total value of } M^*} - \underbrace{\sum_{i=1}^{n} p_{M^*(i)}}_{\leq P},$$

First Welfare Theorem Proof

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- What we know: $(M, \mathbf{p})$ is a competitive equilibrium
- Using envy-free condition to compare M and M^* at price $\mathbf{p}$:

  $v_{iM(i)} - p_{M(i)} \geq v_{iM^*(i)} - p_{M^*(i)}$ for every bidder i

- Let the sum of prices $\sum_{j=1}^{m} p_j = P$
- Reorganizing this inequality, we get that value of $M \geq$ value of M^* ∎

M^* can assign each bidder at most one item

$$\underbrace{\sum_{i=1}^{n} v_{iM(i)}}_{\text{total value of } M} - \underbrace{\sum_{i=1}^{n} p_{M(i)}}_{= P \text{ by CE property (b)}} \geq \underbrace{\sum_{i=1}^{n} v_{iM^*(i)}}_{\text{total value of } M^*} - \underbrace{\sum_{i=1}^{n} p_{M^*(i)}}_{\leq P},$$

Takeaways

- Competitive equilibrium automatically solves a non-trivial computational problem: **computing a maximum weight matching in a bipartite graph!**
  - Polynomial-time solvable but the algorithm is quite nontrivial
- Individually selfish agents reach a globally efficient outcome
- When economists say "markets are efficient", they are referring to a phenomenon like competitive equilibrium
- **Question.** Given their strong requirements, is a competitive equilibrium even guaranteed to exist?

Competitive Eq: Existence

- **Theorem.** In every market where at most one good is assigned to each buyer, there is at least one competitive equilibrium
    - Equivalently, market-clearing prices are guaranteed to exist
- We prove this constructively through an mechanism that shows how such prices might emerge organically in a market
- Intuition idea behind our **"ascending-price auction"**
    - If a set of k items is preferred by more than k buyers at its current price, then the prices of these items should rise
    - Keep identifying such "constricted sets" and increasing prices until the market clears

Ascending-Price Mechanism

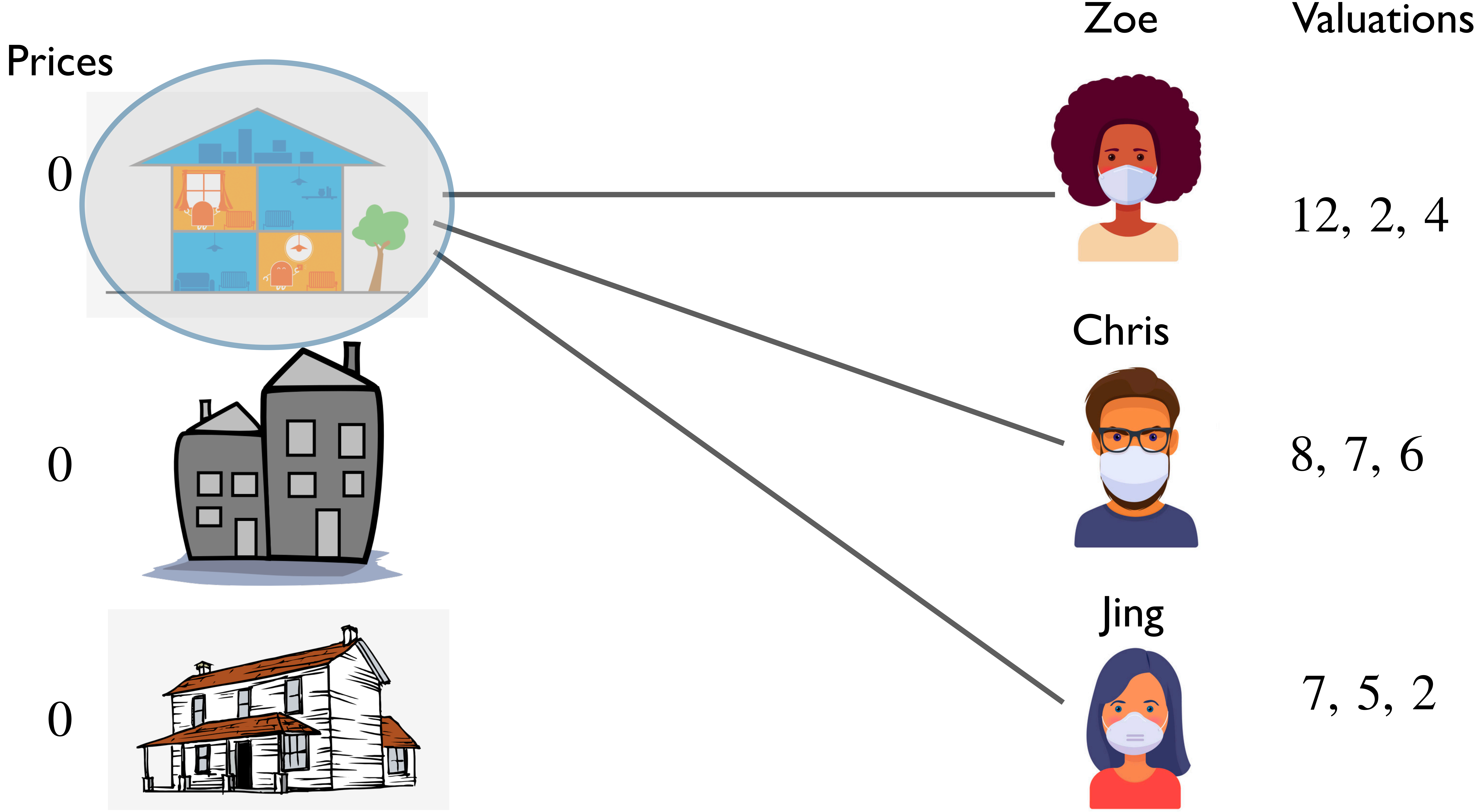
- Start with prices of all items $p_j = 0$
- Assume all valuations are integers $v_{ji} \in \mathbb{Z}$ (simplifying assumption)
- **Step 1.** Check if the current prices are market clearing, if so we are done
  - build the preferred graph, check if there is a buyer-perfect matching
- **Step 2.** Else, there must a constricted set:
  - There exists $S \subseteq \{1, \ldots, n\}$ such that $|S| > |N(S)|$
  - $N(S)$ are items that are **over-demanded**
  - If there are multiple such sets, choose the **minimal set** $N(S)$
    - Increase $p_j \leftarrow p_j + 1$ for all items in the set $j \in N(S)$
  - Go back to **Step 1.**

Single Item Case

- A single item (labelled 1) for which each buyer has a value $v_i > 0$
- Add $n - 1$ dummy items $(2, \ldots, n)$ that everyone values at 0
- At the beginning preferred-item graph has edges from each buyer to item 1
- Thus, $\{1\}$ is our minimal constricted set
- We need to keep raises the price of item 1 until all except one buyer has a preferred edge to at least one item in $\{2, 3, \ldots, n\}$
- At what price does this happen?
    - Exactly when $p_1 =$ second-highest valuation
    - The person with the highest valuation is matched to item 1

We have recreated the **second-price auction** outcome!

Preferred-Item Graph



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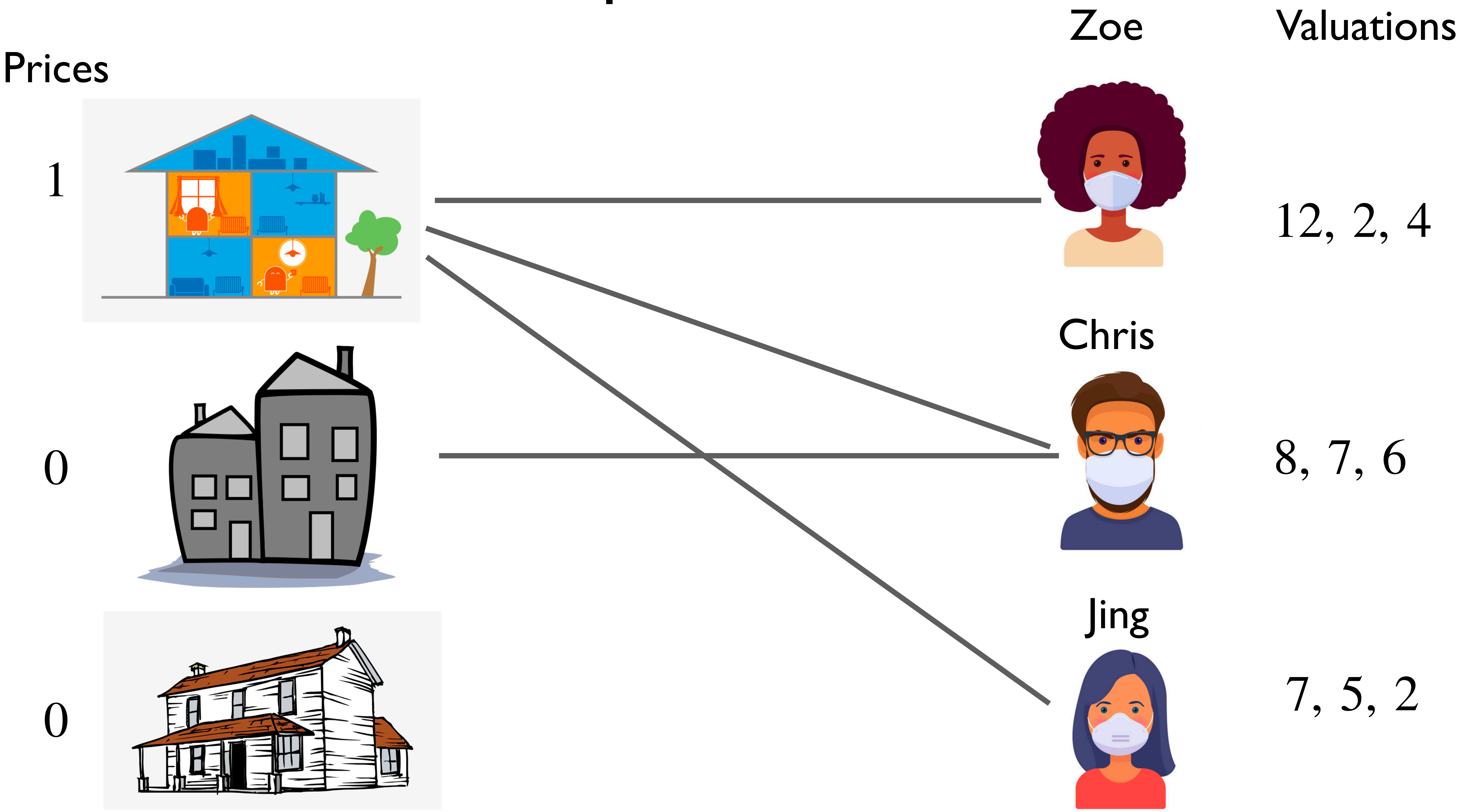

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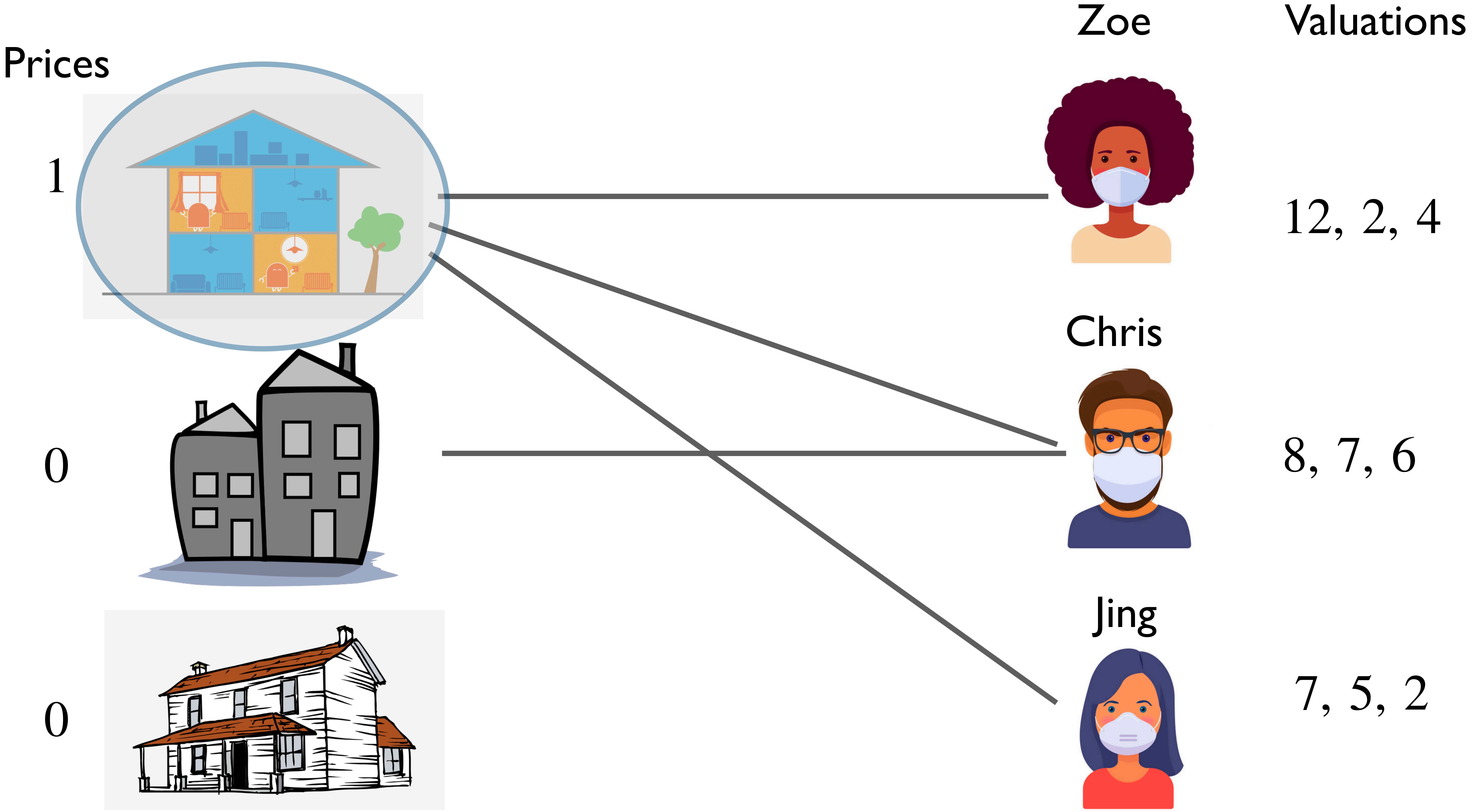

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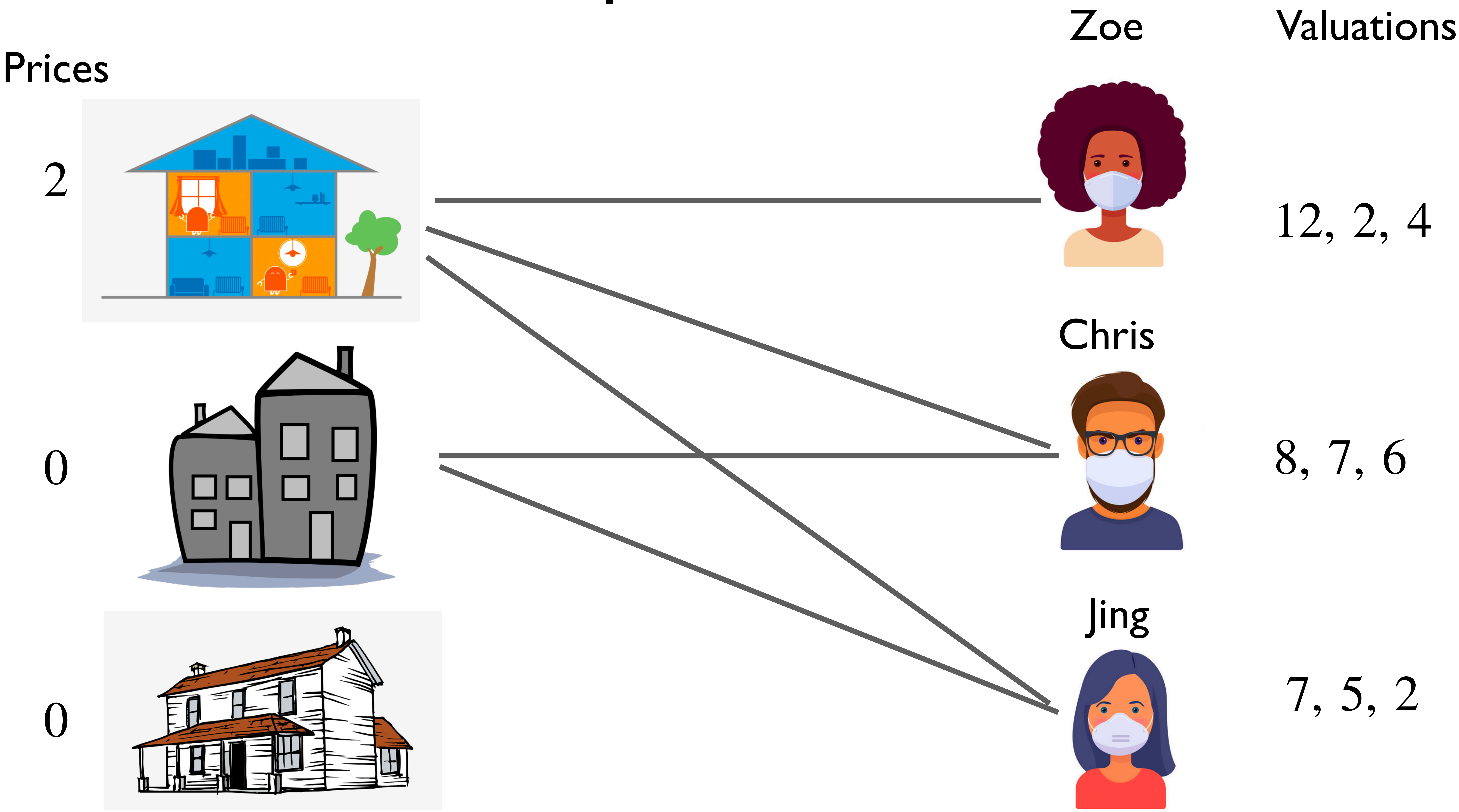

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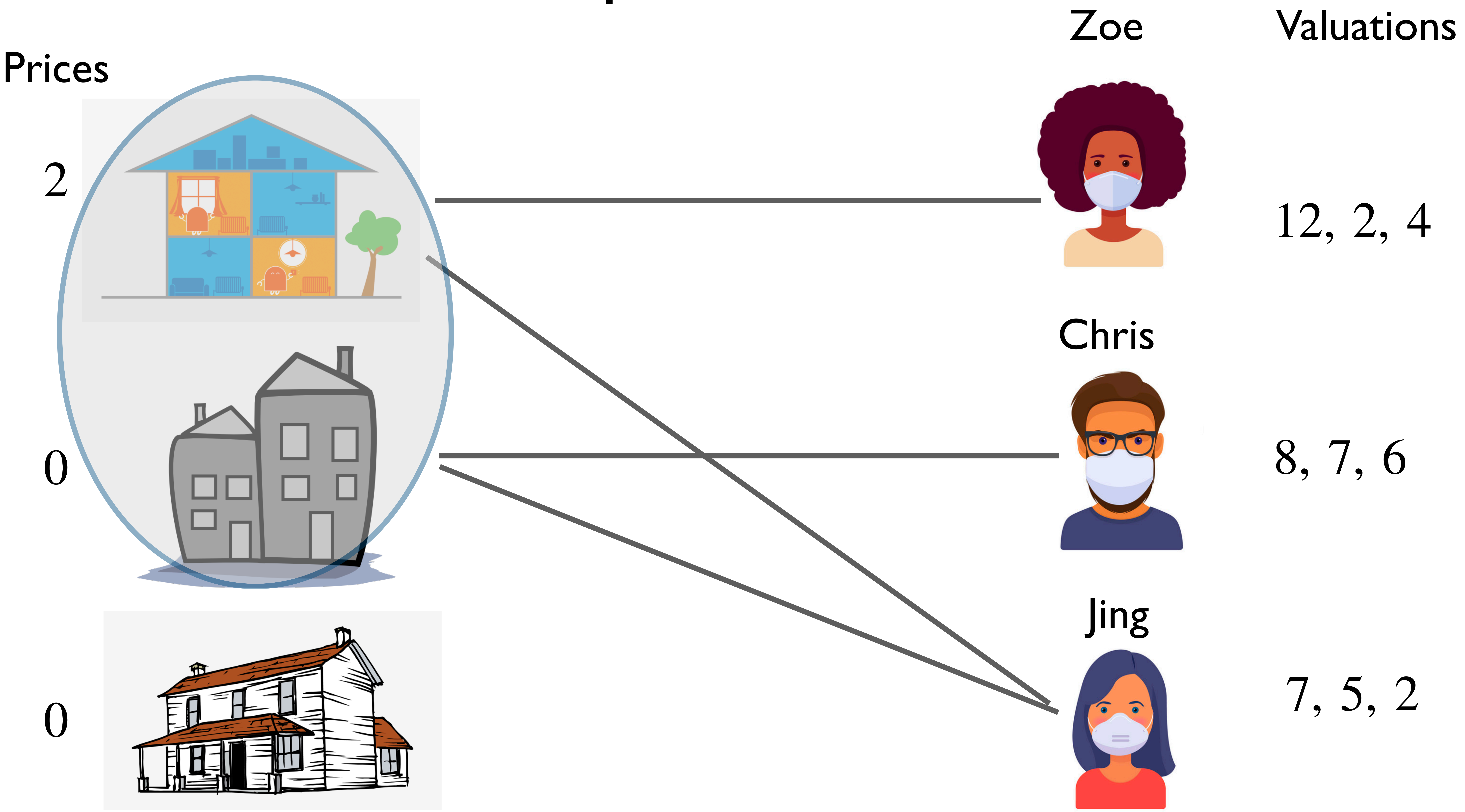

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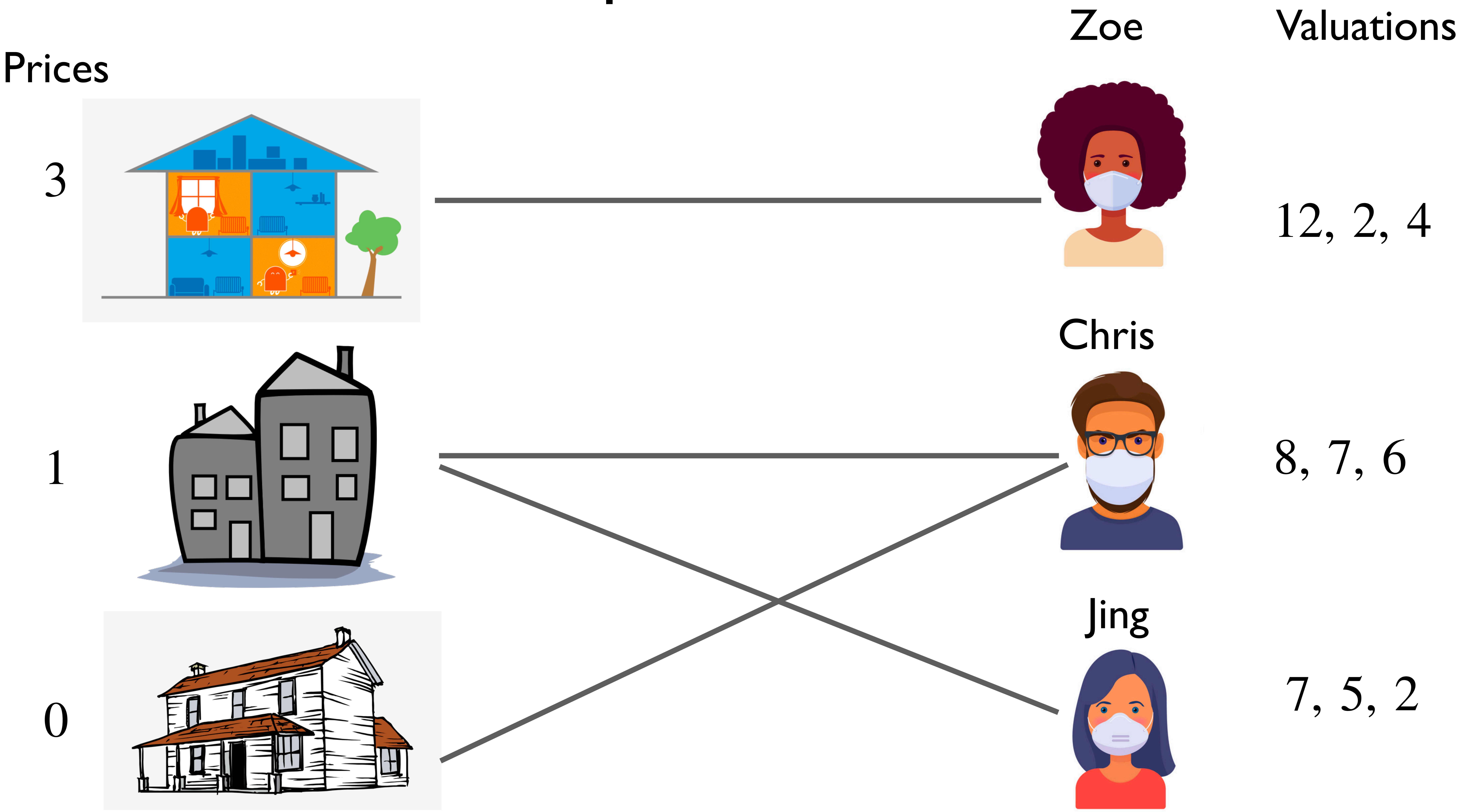

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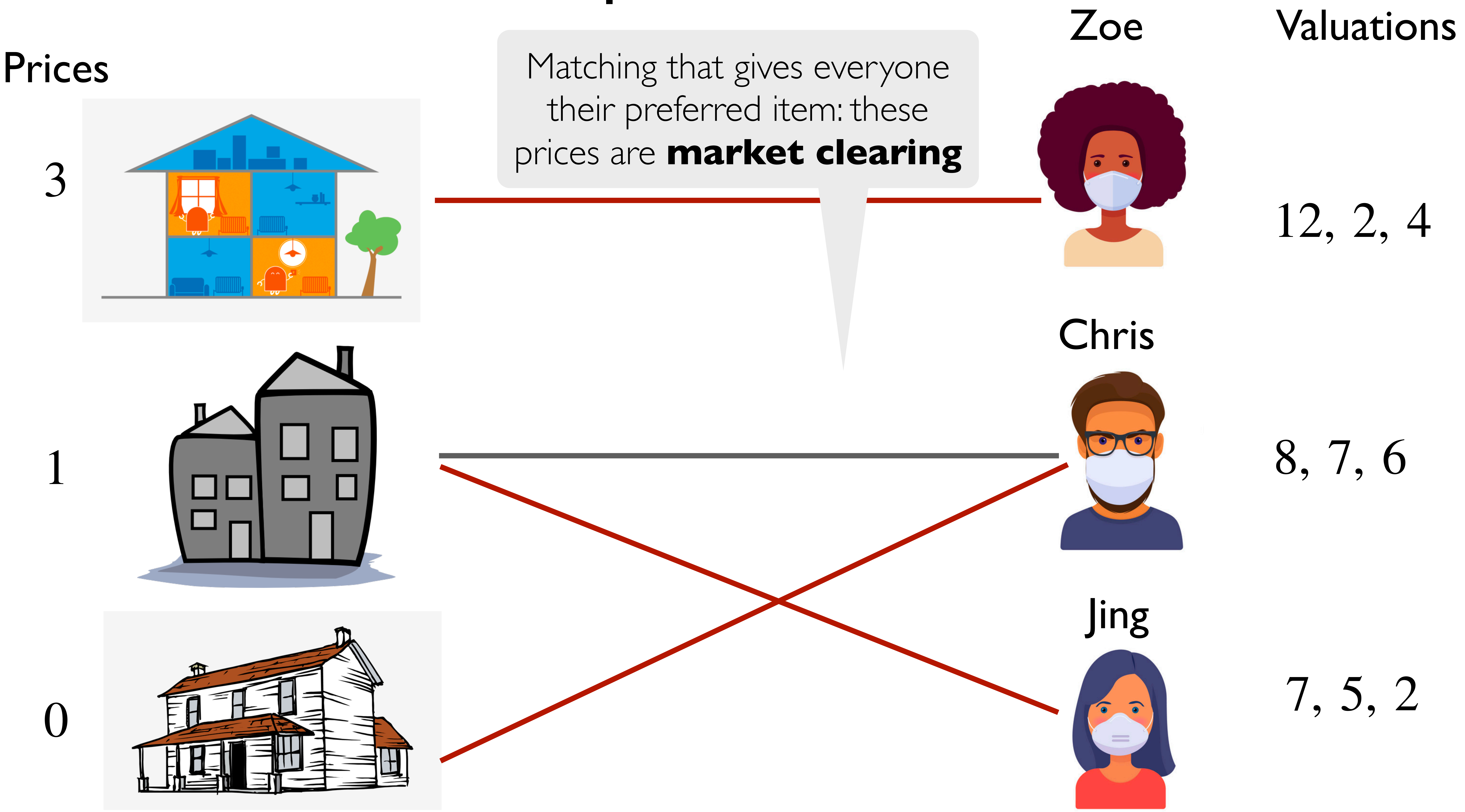


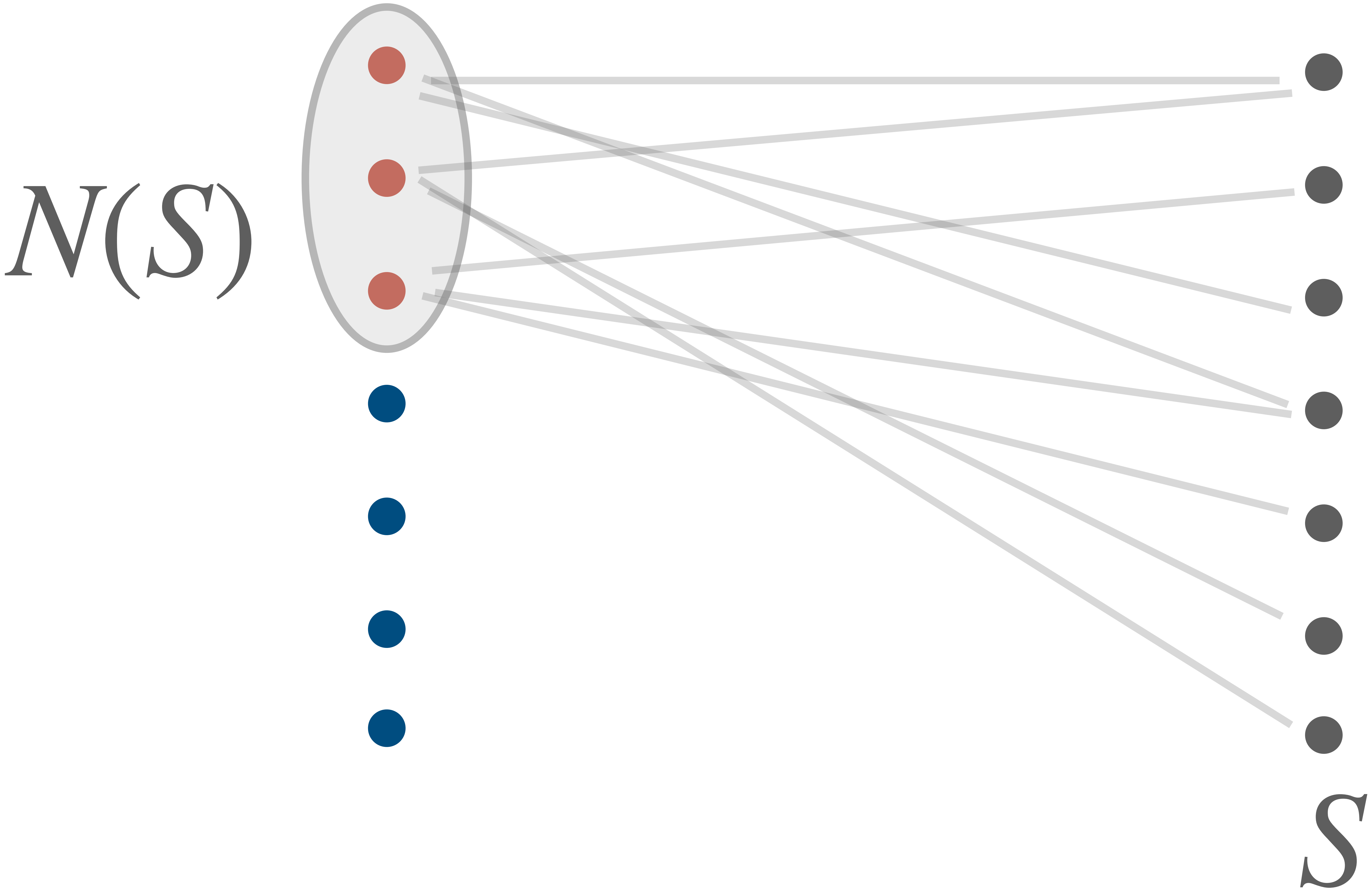
Ascending-Price Algorithm

- Start with prices of all items $p_j = 0$, assume all valuations $v_{ji} \in \mathbb{Z}$
- **Step 1.** Check if there is a **buyer-perfect matching** in preferred item graph
- **Step 2.** Else, there must a constricted set:
  - There exists $S \subseteq \{1,\ldots,n\}$ such that $|S| > |N(S)|$
  - $N(S)$ are items that are **over-demanded**
  - If there are multiple such sets, choose the **minimal set** $N(S)$
    - Increase $p_j \leftarrow p_j + 1$ for all items in the set $j \in N(S)$
  - Go back to **Step 1.**
- **Invariant:** if an item has non-zero cost, that item is tentatively matched to some buyer: $p_j > 0 \implies \exists i : (j, i) \in M$

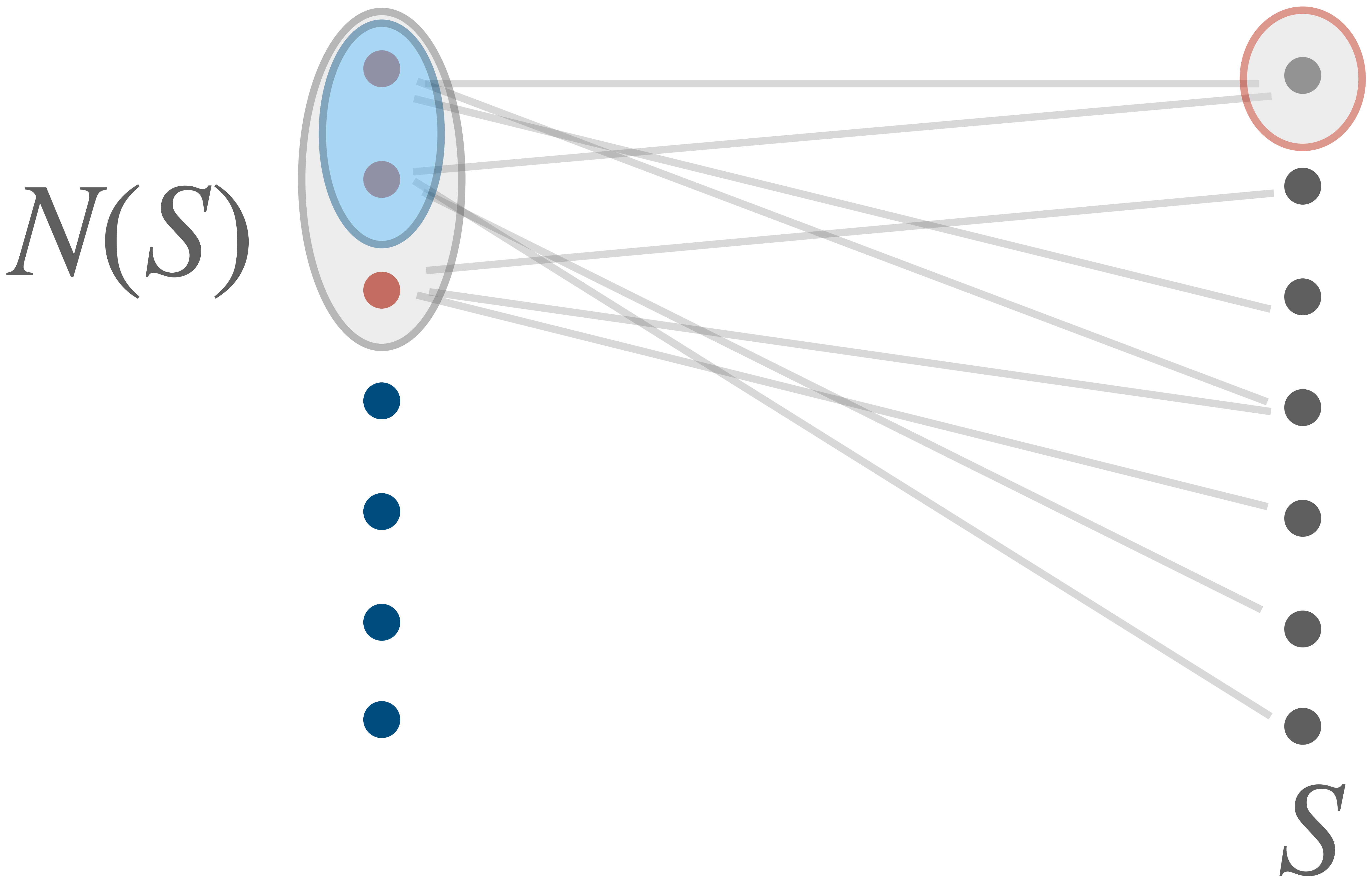
Analyzing Our Auction

- **Maintain invariant:** if an item has non-zero cost, that item is tentatively matched to some buyer: $p_j > 0 \implies \exists i : (j, i) \in M$
- Suppose until step t you have invariant maintained and we identify minimal constricted set $N(S)$ whose prices increase by 1 in this step
- At the new price, all edges between S to $N(S)$ still exist (buyers in S may have more edges to items outside that are now just as good)
- Tentatively match items in $N(S)$ to buyers in S (if these items were matched to other buyers, or buyers to other items, remove those edges from the matching)
  - Why is this matching possible?
  - We use Hall's theorem on items in $T = N(S)$

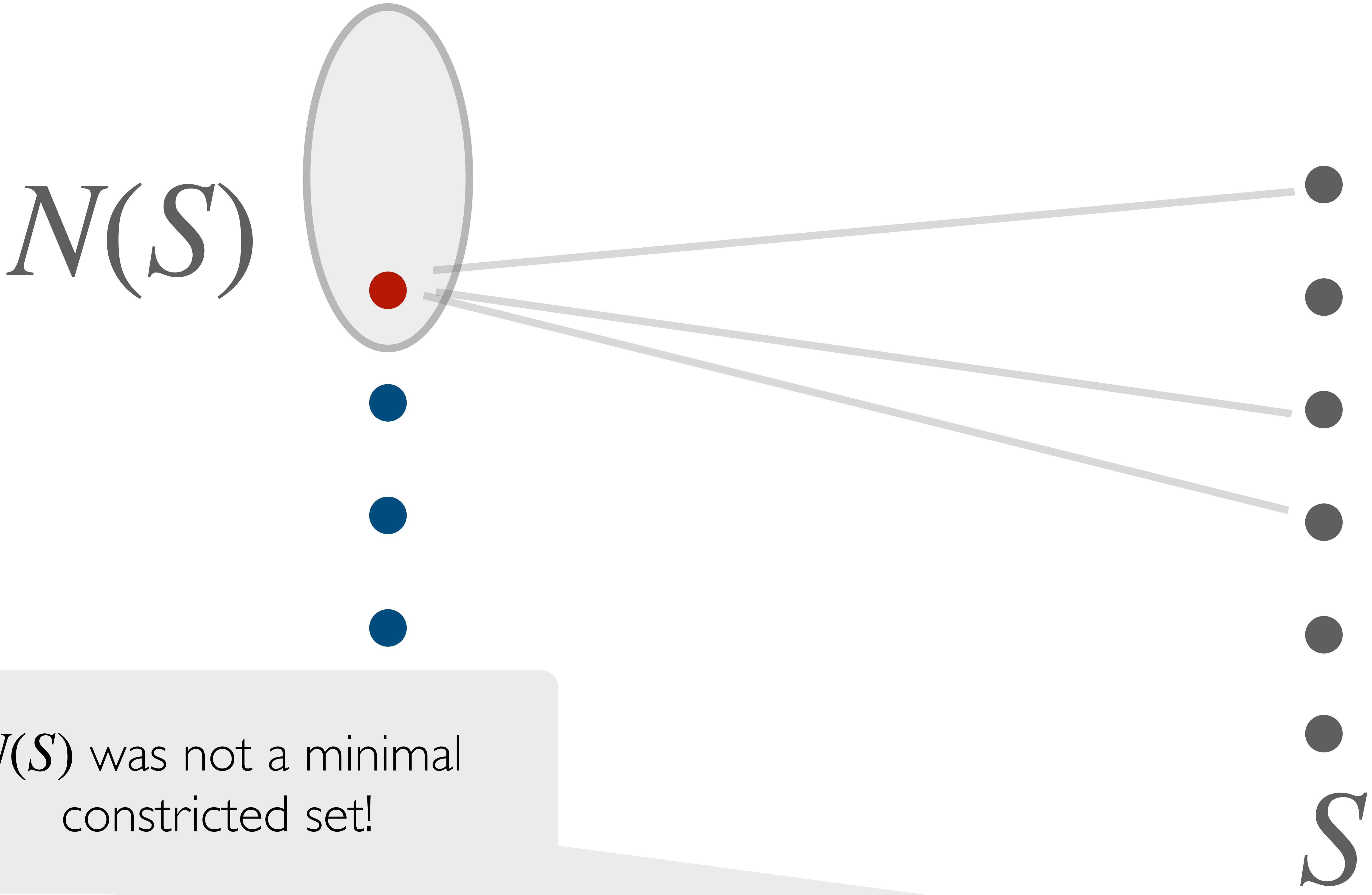
Why Such a Matching Exists



Why Such a Matching Exists



Why Such a Matching Exists



$N(S)$ was not a minimal constricted set!

Ascending-Price Algorithm

- **Invariant:** if an item has non-zero cost, that item is tentatively matched to some buyer: $p_j > 0 \implies \exists i : (j, i) \in M$
- Final question:
  - Does this algorithm ever terminate?
- **Intuition:** Since items are always tentatively matched, prices cannot rise for forever, why?
  - At some point, no buyer would want the items!

Proving Our Algorithm Terminates

- **Theorem.** The ascending price auction terminates.
- **Proof.** Show that algorithm starts with a certain amount of **"potential energy"** which goes down by at least 1 in each iteration
- Let the potential of any round be defined as:

$$E = \sum_{\text{items } j} p_j + \sum_{\text{buyers } i} u_i^*$$

- where p_j is the price of item j in that round and u_i^* is the maximum utility i can obtain given prices $\mathbf{p}$ in that round

Proving Our Algorithm Terminates

$$E = \sum_{\text{items } j} p_j + \sum_{\text{buyers } i} u_i^*$$

- **Theorem.** The ascending price auction terminates.
- **Proof.**
- At the the beginning, all prices are zero and $u_i^* = \max_j v_{ij}$
- Thus, before the auctions starts $E_0 = \sum_i \max_j v_{ij}$
- To wrap up proof, we show
  - Potential can never be negative $E \geq 0$
  - Potential at each step goes down by at least 1
- Thus, in E_0 steps the algorithm terminates. ∎

Proving Our Algorithm Terminates

$$E = \sum_{\text{items } j} p_j + \sum_{\text{buyers } i} u_j^*$$

- **Lemma:** Potential energy E is always non-negative.
- **Proof.**
- If there is at least one item with price 0 then each $u_i^* \geq 0$
  - Why is this true? Use our invariant!
  - Every non-zero priced item is matched, thus when $n - 1$ items are matched, no need to raise the price of nth item
- Since prices are always are always nonnegative $E \geq 0$

Proving Our Algorithm Terminates

$$E = \sum_{\text{items } j} p_j + \sum_{\text{buyers } i} u_j^*$$

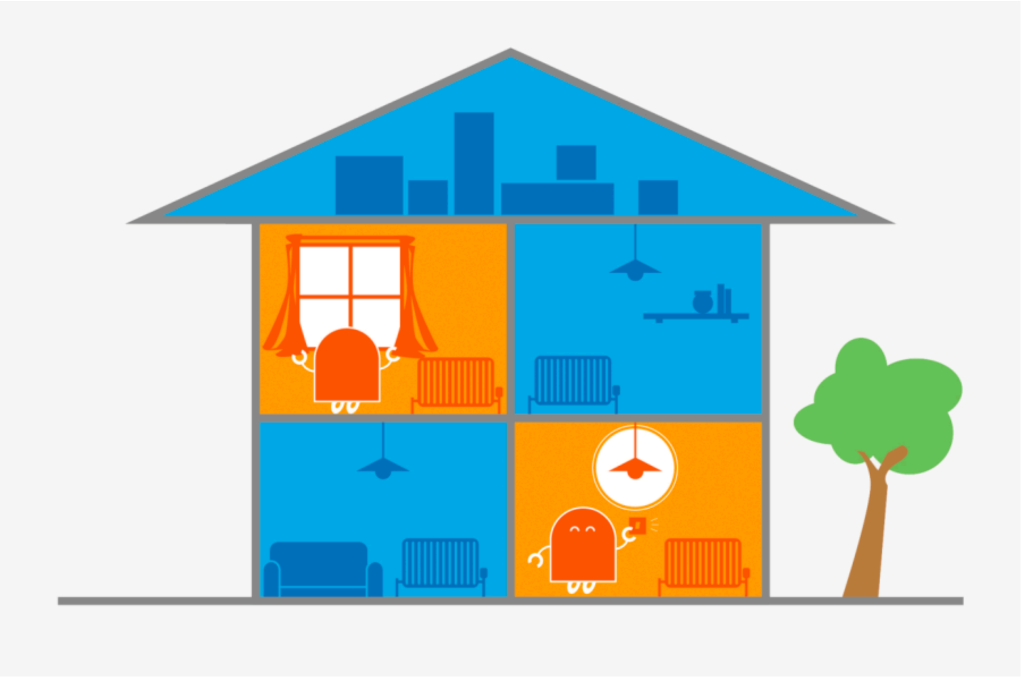
- **Claim.** Potential E goes down by at least one each step.
- **Proof**. At each step, we raise the price of all items in $N(S)$, how much does it increase the first term in E ?
  - $|N(S)|$
- However, the value of u_i^* goes down by one for each node in S, how much does this decrease the second term in E?
  - $|S|$
- Since $|N(S)| < |S|$, then potential decreases by at least 1
- Thus, the algorithm must terminate in E_0 steps ∎
- Our ascending auction terminates at market clearing prices!

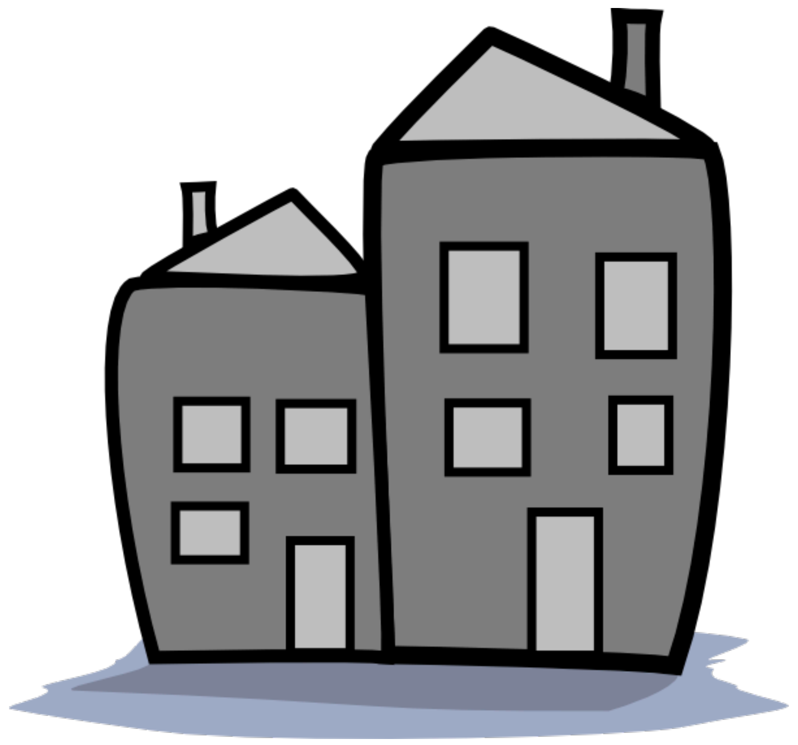
VCG Prices vs Market-Clearing

- VCG prices set **centrally**: ask each buyer to report their valuation and charge each buyer a **"personalized price"** for their allocation
- VCG prices are only set after a matching has been determined (the matching that maximizes total valuation of the buyers)
  - Not just about the item itself, but who gets the item
- Market-clearing prices are **"posted prices"** at which buyers are free to pick whatever item they like
  - Prices are chosen first and posted on the item
  - Prices cause certain buyers to select certain items leading to a matching

Applying VCG

Prices





VCG. Need to find surplus maximizing allocation first

		Valuations
Zoe		12, 2, 4
Chris		8, 7, 6
Jing		7, 5, 2

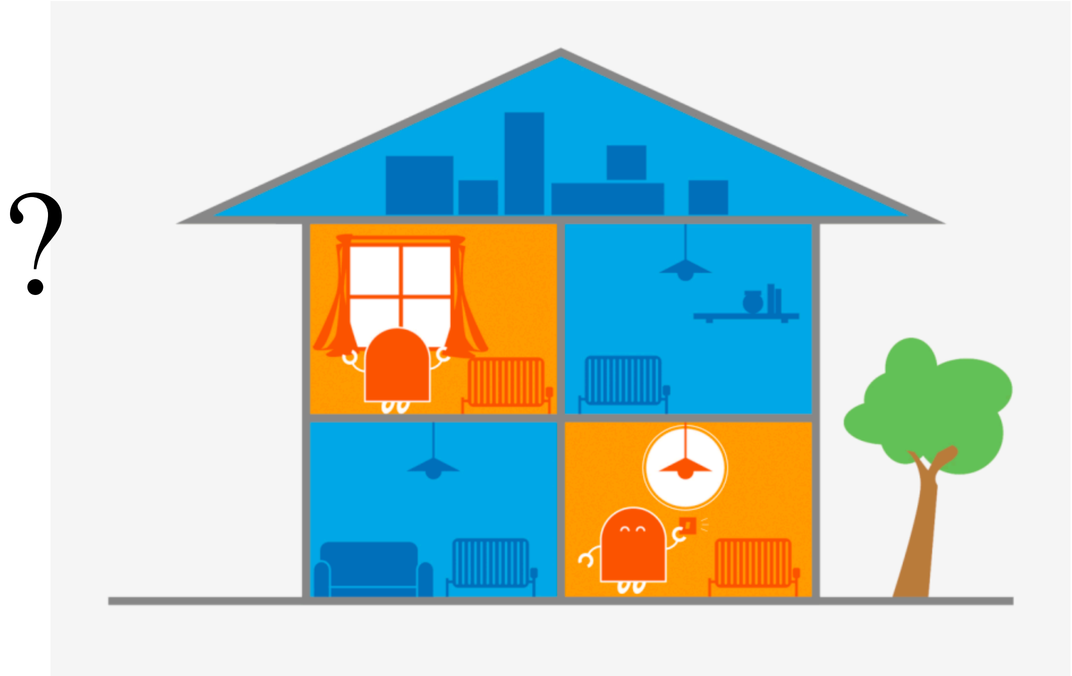
Applying VCG

Prices

Zoe

Valuations

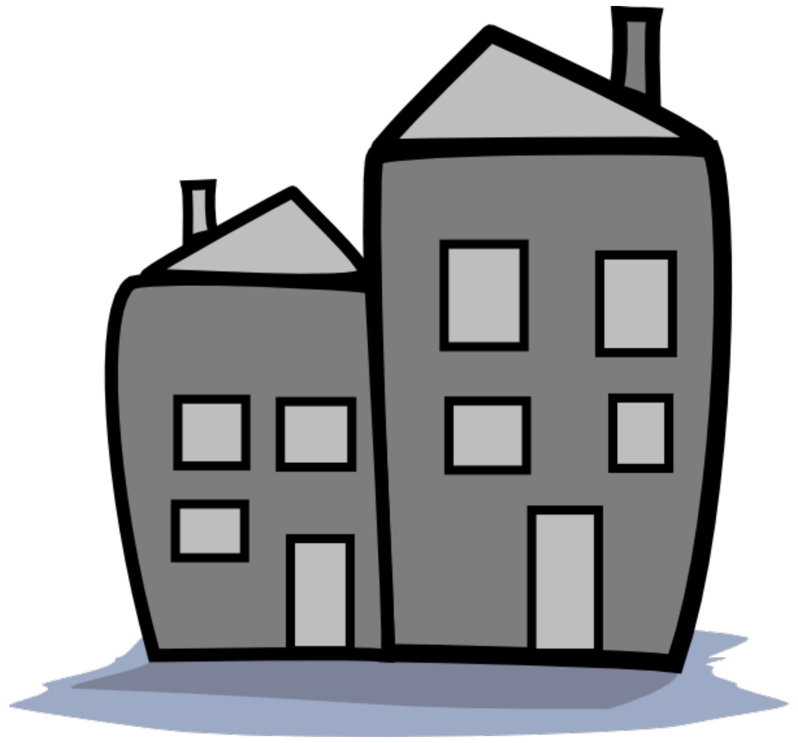
p_1?



12, 2, 4

Chris

p_2?



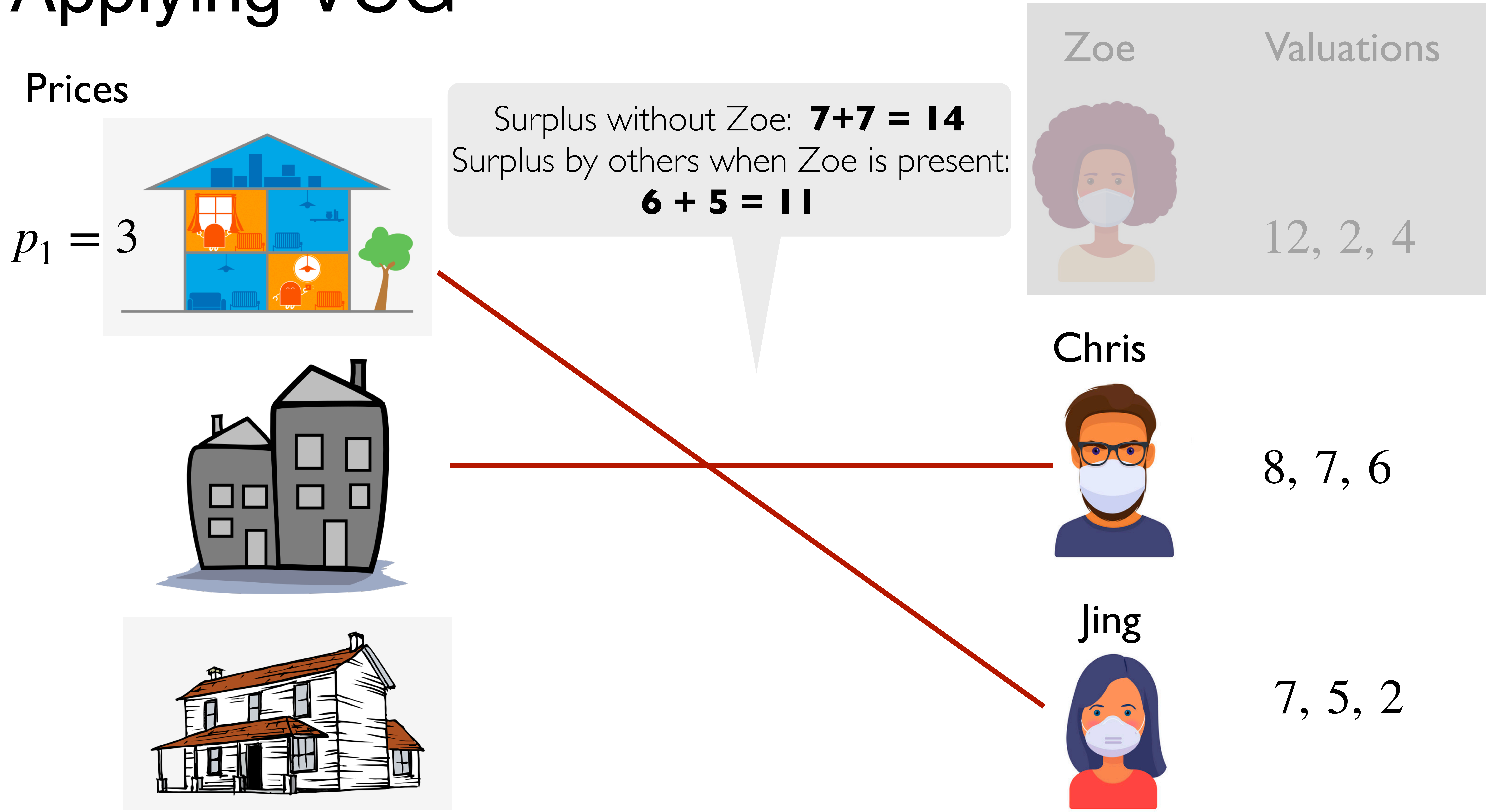
8, 7, 6

Jing

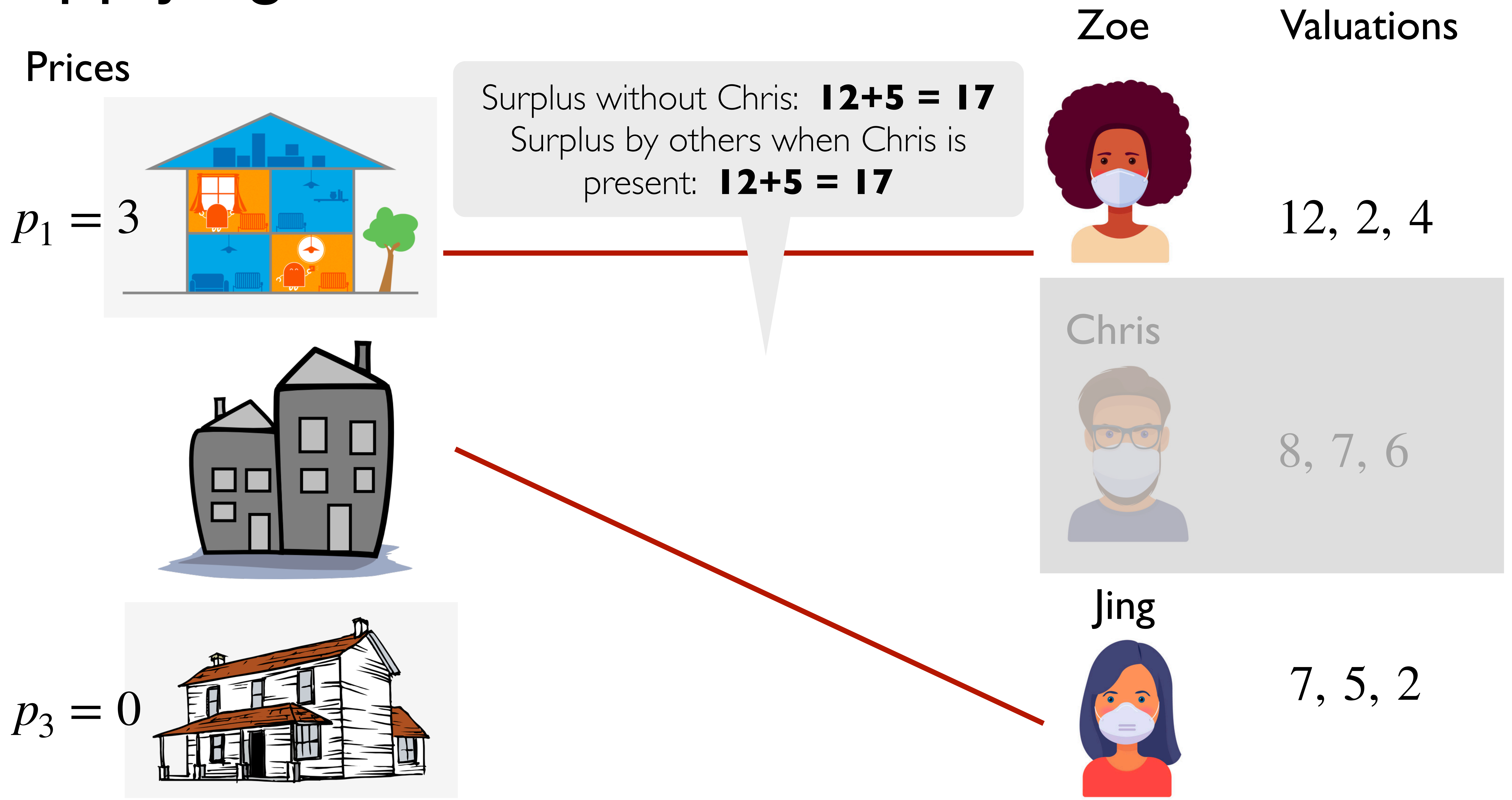
p_3?

7, 5, 2

Applying VCG



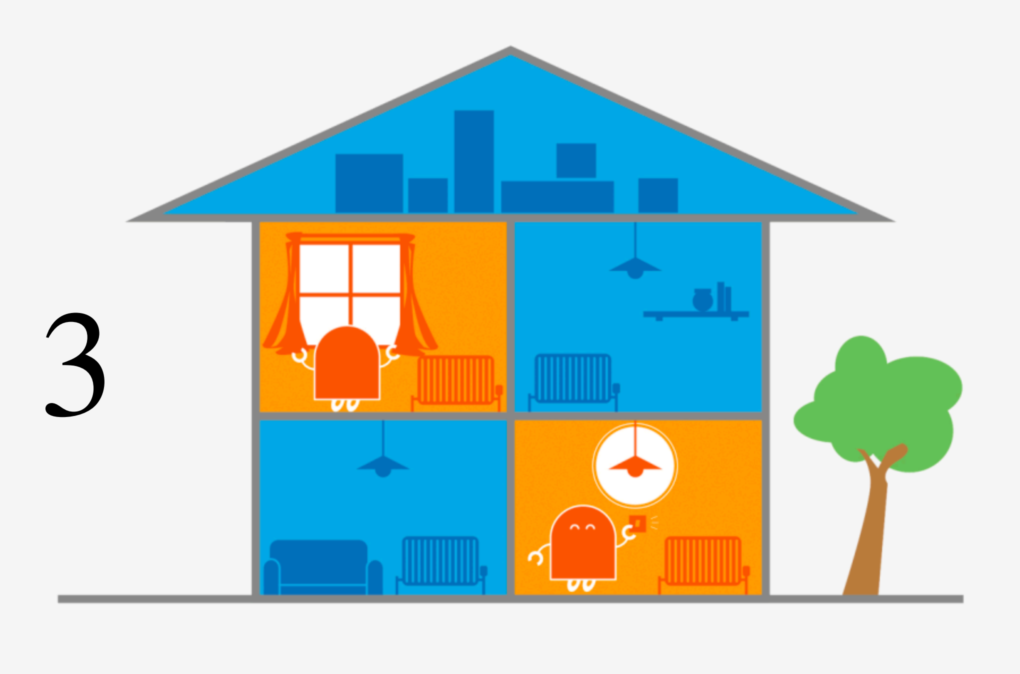
Applying VCG



Applying VCG

Prices

$p_1 = 3$



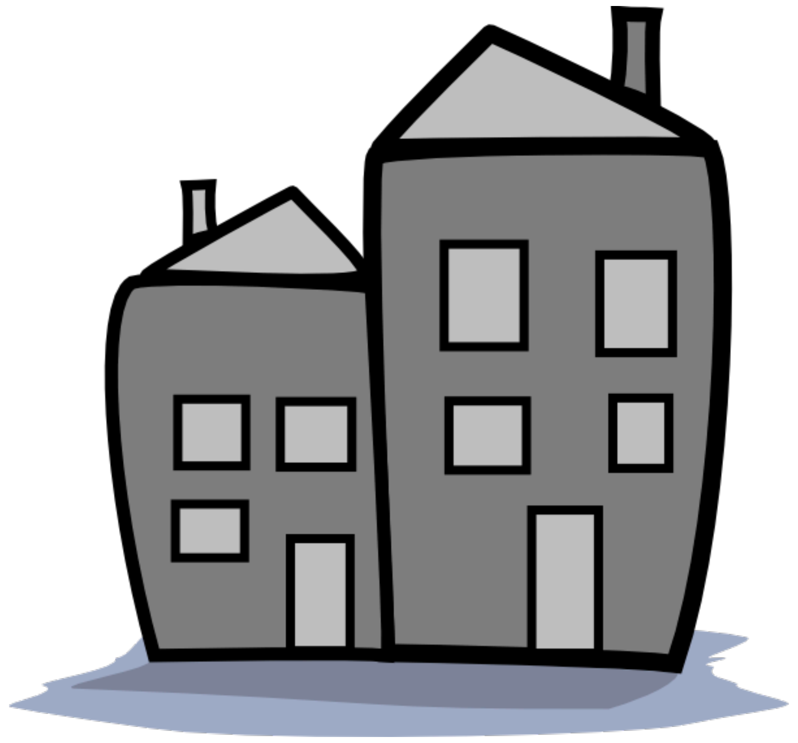
Surplus without Jing: **12+7 = 19**
Surplus by others when Jing is present: **12+6 = 18**

Zoe

Valuations

12, 2, 4

$p_2 = 1$



Chris

8, 7, 6

$p_3 = 0$

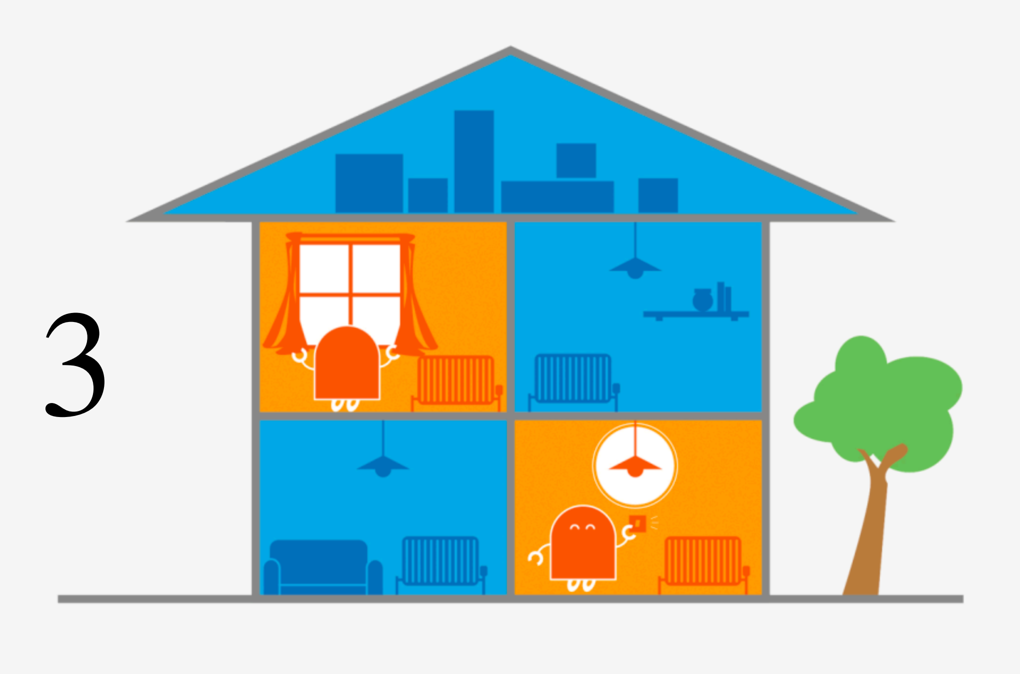
Jing

7, 5, 2

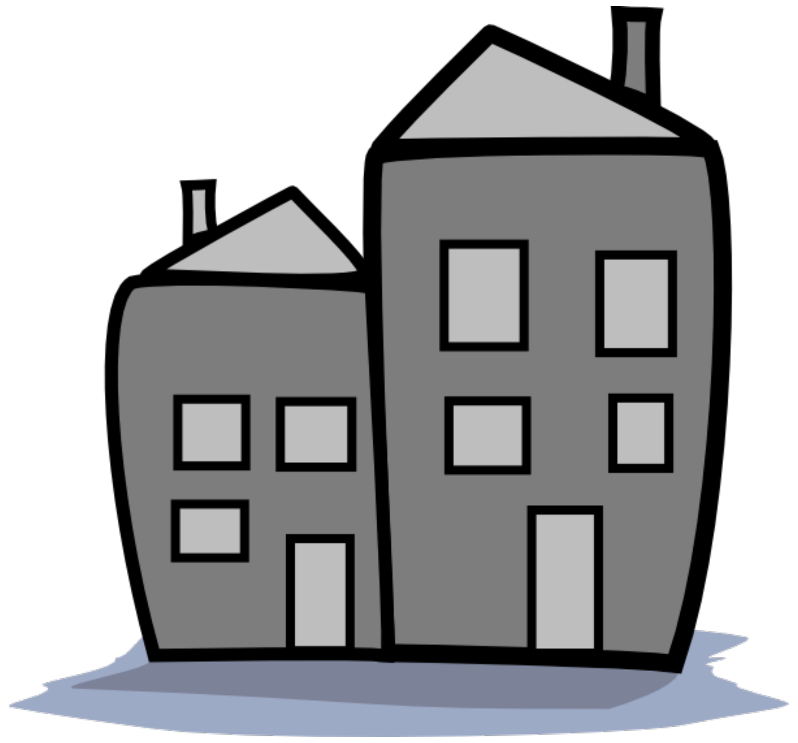
Applying VCG

Prices

$p_1 = 3$



$p_2 = 1$



$p_3 = 0$

We got the same **prices** & **matching** as our **competitive equilibrium**

Zoe

Chris

Jing

Valuations

Zoe: 12, 2, 4

Chris: 8, 7, 6

Jing: 7, 5, 2

VCG Prices are Market Clearing

- Despite their definition as personalized prices, VCG prices are always market clearing (for the case when each buyer wants a single item)
- Suppose we computed VCG prices for a given matching market
- Then, instead of assigning the VCG allocation and charging the price, we post the prices publicly
    - Without requiring buyers to follow the VCG match
- Despite this freedom, each buyer will in fact achieve the highest utility by selecting the item that was allocated by the VCG mechanism!
- **Theorem.** In any matching market (where each buyer can receive a single item) the VCG prices form the unique set of **market clearing prices of minimum total sum**.

This is a generalization of the VCG/GSP result (where valuations are constrained). The general proof is beyond the scope of this course.

General Demand

- Market clearing prices **may not exist in combinatorial markets**
- **Example**, suppose our market has two items $\{L, R\}$
- Two buyers Alice and Maya
- Alice wants both $v_a(\{L, R\}) = 5$, $v_a(\{L\}) = v_s(\{R\}) = 0$
- Maya wants either, $v_p(\{L\}) = v_p(\{R\}) = v_p(\{L, R\}) = 3$
- What's the welfare-maximizing allocation?
  - Give both to Alice
- What must the price of each be so that Maya doesn't want it?
  - $p(\{L\}) \geq 3, p(\{R\}) \geq 3$
- At a price of ≥ 6 does Alice want it?

Summary

- Ascending price auction is also called Hungarian algorithm in matching literature
- Hungarian algorithm is used to find max-weight bipartite matching
  - Prices are just a conceptual interpretation of "dual" variables
- Caveats:
  - No sales occur until prices have settled at their equilibrium point
  - Coordination required for tie breaks
  - Running time to convergence can be very slow

Competitive Equilibrium Research

- [Left] 2016 Article argues that competitive equilibrium's tie breaking requirement can be fairly strong
  - Use **learning theory** to predict buyer's behavior and demand and show convergence under such some mild assumptions
- [Right 2021]. Algorithms with predictions paper predicts "prices" for faster runtime

Do Prices Coordinate Markets?

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Faster Matchings via Learned Duals

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Abstract

A recent line of research investigates how algorithms can be augmented with machine-learned predictions to overcome worst case lower bounds. This area has revealed interesting algorithmic insights into problems, with particular success in the design of competitive online algorithms. However, the question of improving algorithm running times with predictions has largely been unexplored.

We take a first step in this direction by combining the idea of machine-learned predictions with the idea of "warm-starting" primal-dual algorithms. We consider one of the most important primitives in combinatorial optimization: weighted bipartite matching and its generalization to b-matching. We identify three key challenges when using learned dual variables in a primal-dual algorithm. First, predicted duals may be infeasible, so we give an algorithm that efficiently maps predicted infeasible duals to nearby feasible solutions. Second, once the duals are feasible, they may not be optimal, so we show that they can be used to quickly find an optimal solution. Finally, such predictions are useful only if they can be learned, so we show that the problem of learning duals for matching has low sample complexity. We validate our theoretical findings through experiments on both real and synthetic data. As a result we give a rigorous, practical, and empirically effective method to compute bipartite matchings.

Fluctuations in Practice: Research

- In practice, one might imagine that sales are actually happening concurrently with price adjustment
- It turns out, the way buyers and sellers respond to prices in the short-run can dramatically influence prices
- **Example**. Surge pricing on ride-sharing platforms can be viewed as an attempt to find market-clearing prices
- However, if passengers and drivers respond to prices myopically, the resulting behavior can be erratic
- Recent research in AGT studies **dynamic (online) resource allocation problems** that take these factors into account