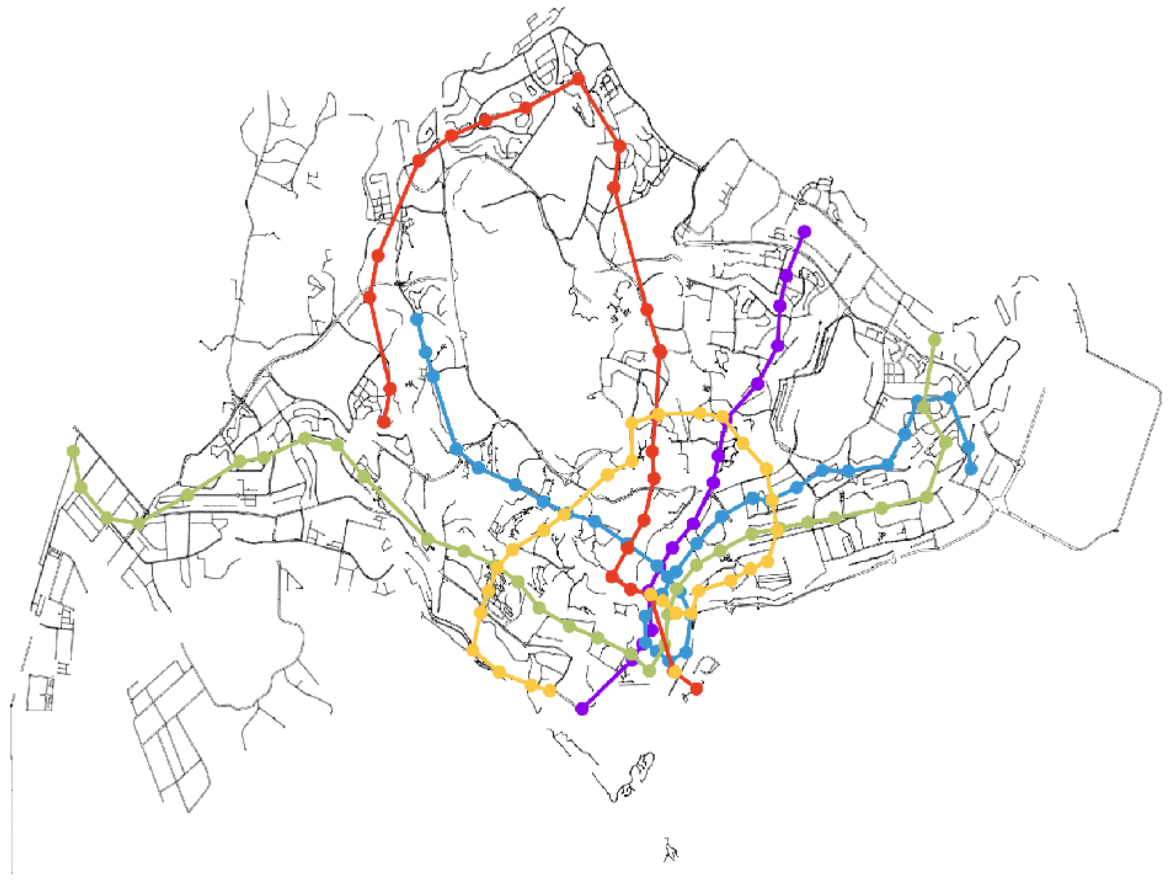
CSCI 357: Algorithmic Game Theory

Lecture 15: Decentralized Markets

Shikha Singh

Announcements and Logistics

- Paper evaluation # 3 was due on Gradescope at noon
- Working on grading Assignment # 3
- Assignment # 4 will be released today, due next Friday at noon
- Midterm # 2 will be on April 28 in class
    - Mostly focused on everything covered after Midterm 1
    - From markets with money: Bayes Nash and revenue equivalence will be included
    - Need to remember and know how to use fundamental definitions (dominant-strategyproof, Nash equilibrium, Condorcet consistency, etc.)
    - Similar to Exam 1: can bring up to 5 pages of notes

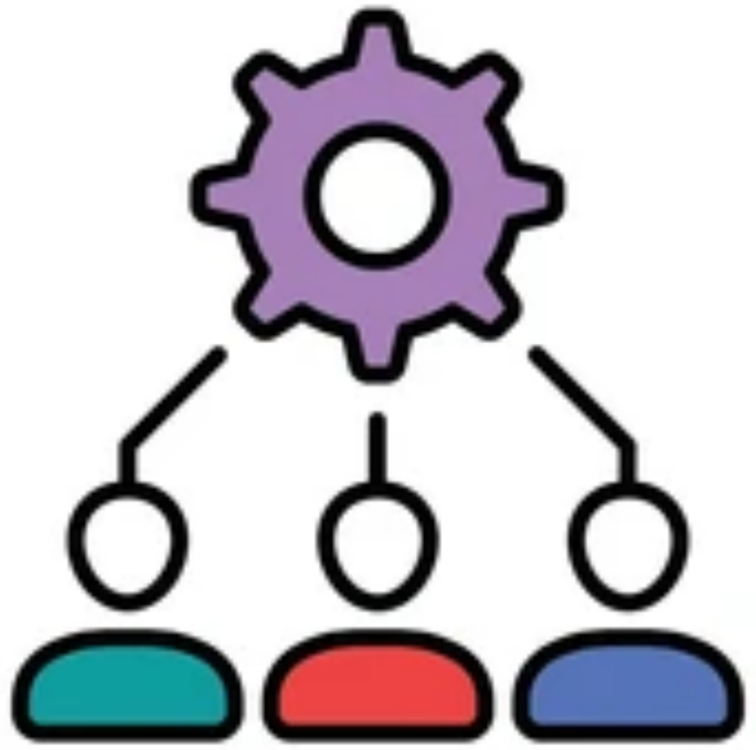
Questions?

Story So Far: Centralized Markets

Centralized algorithm
Centralized algorithm
coordinates

Centralized Markets

with Money



without Money

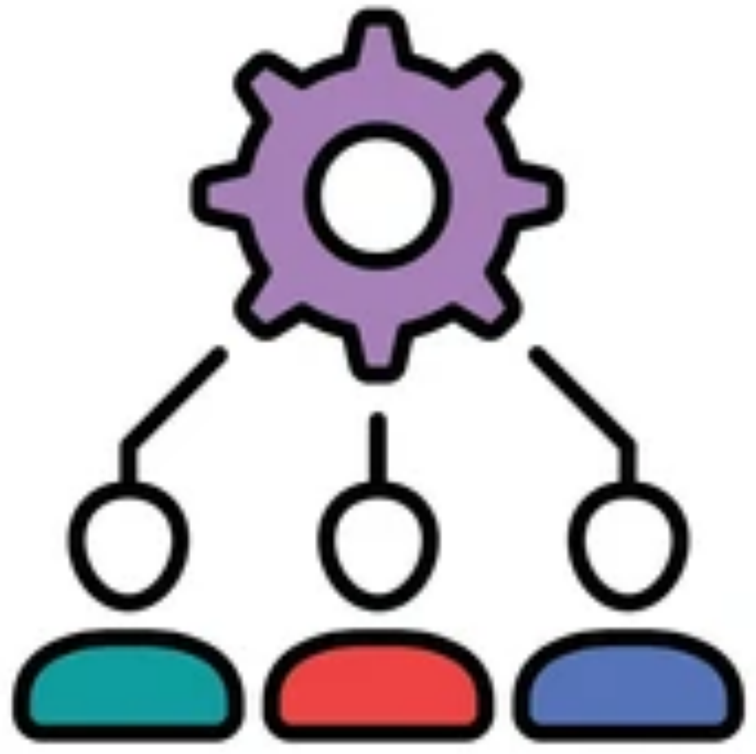
Goal. Elicit preferences from individuals so as to make a collective decision on allocation in a way that maximizes global objectives with participant objective (maximize utility).

Story So Far: Centralized Markets

Centralized algorithm coordinates

Centralized Markets

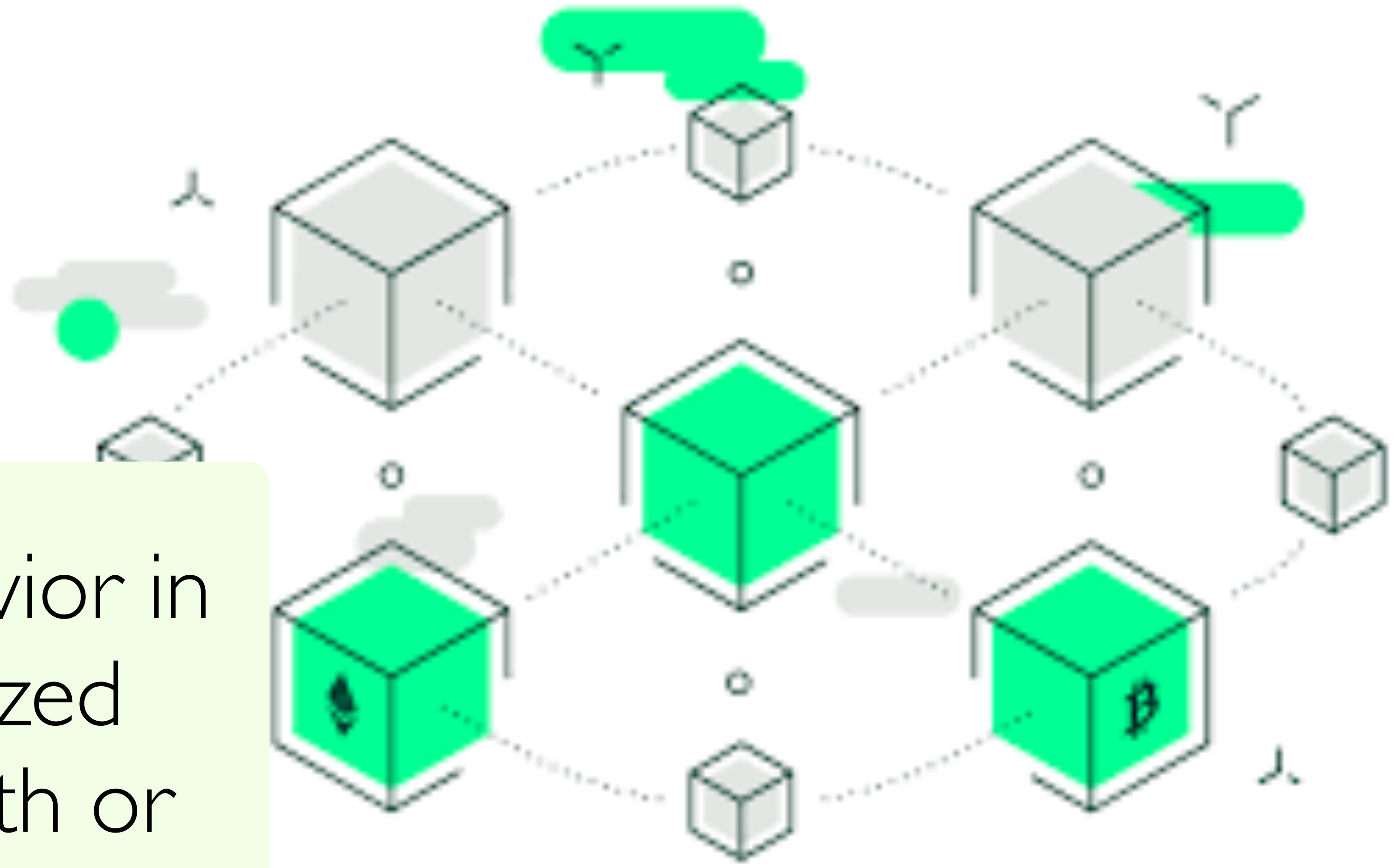
with Money



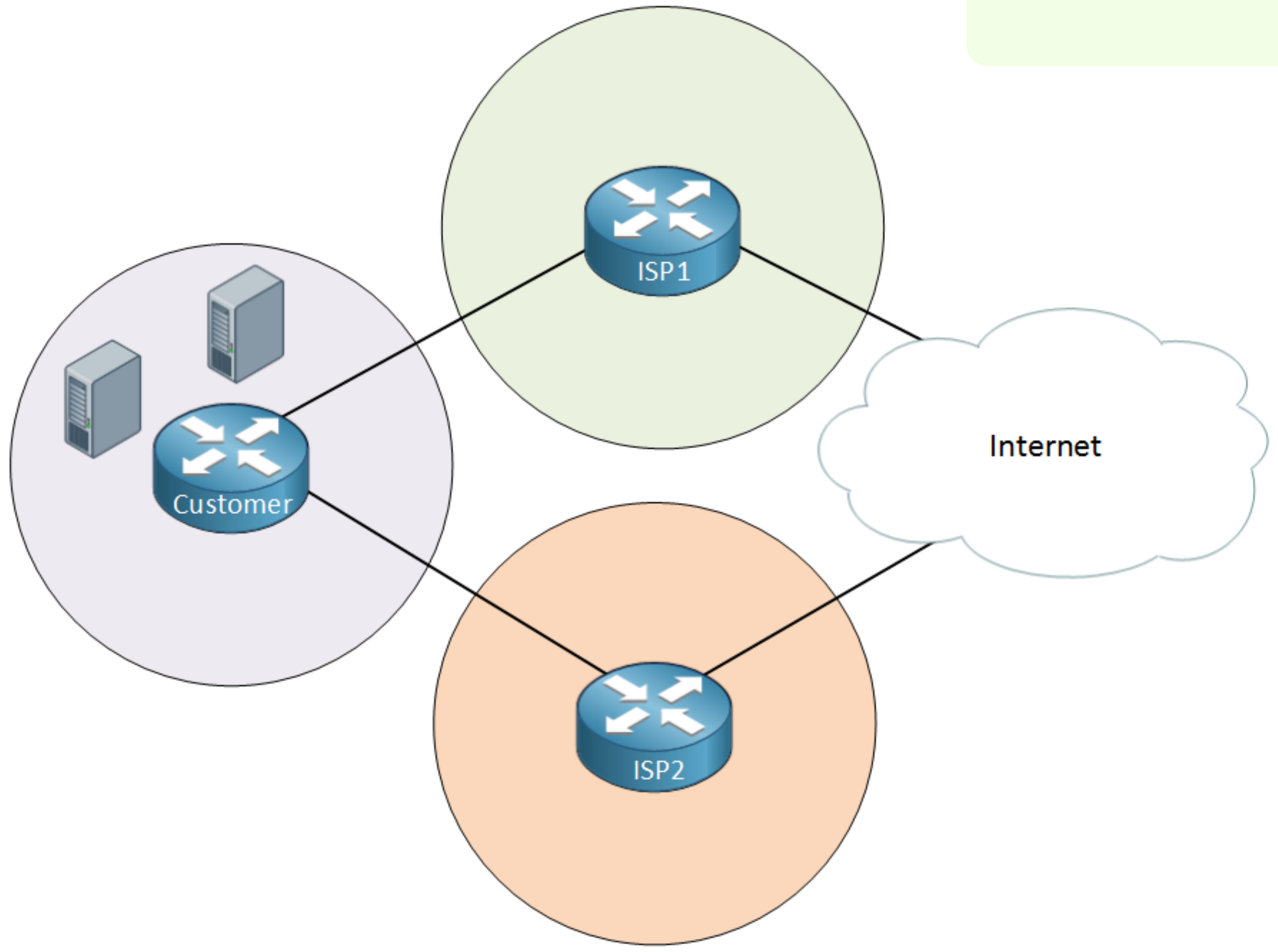
without Money

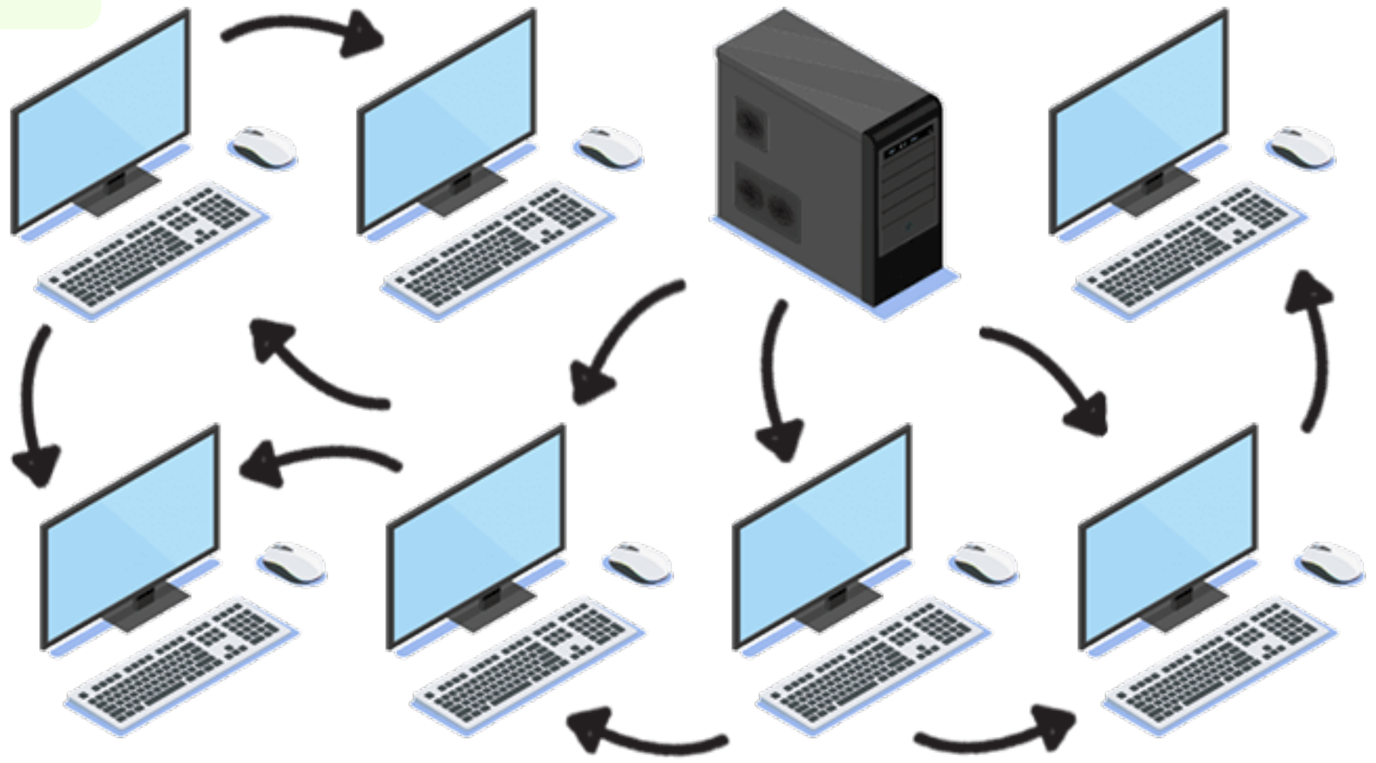
Downsides?

Later: Decentralized Systems



Selfish behavior in decentralized markets with or without money





Decentralized Markets

Decentralized Market

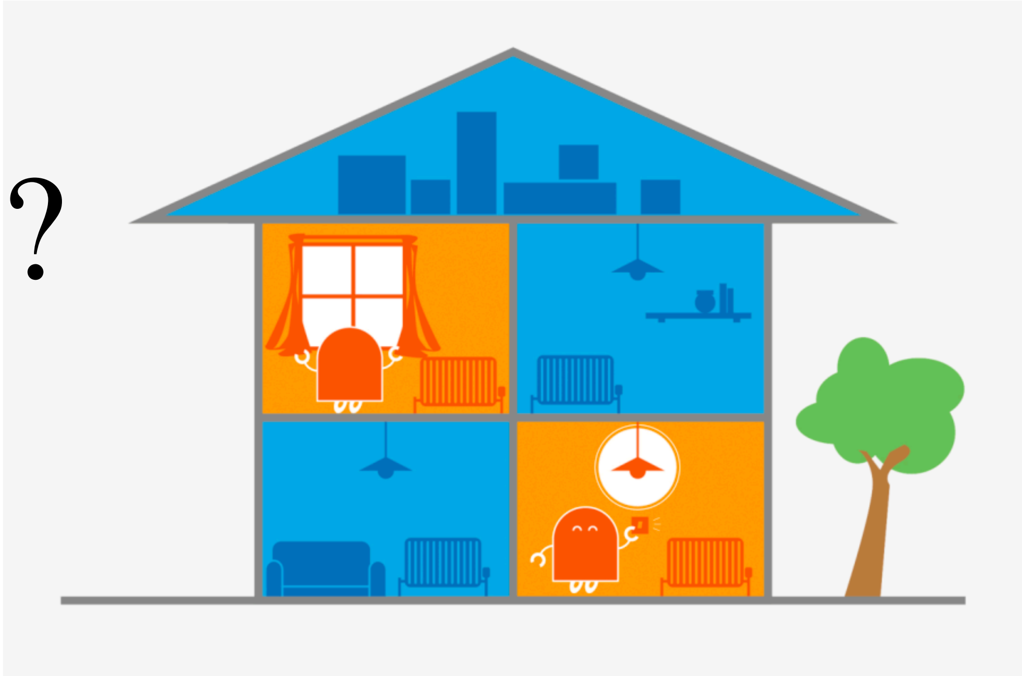
- A market is **decentralized** if participants are free to transact directly with each other, without any central coordination
    - College admissions in the US and most job markets are decentralized
- Even decentralized markets may have some central coordination
    - ride-sharing markets are decentralized with some coordination mechanism

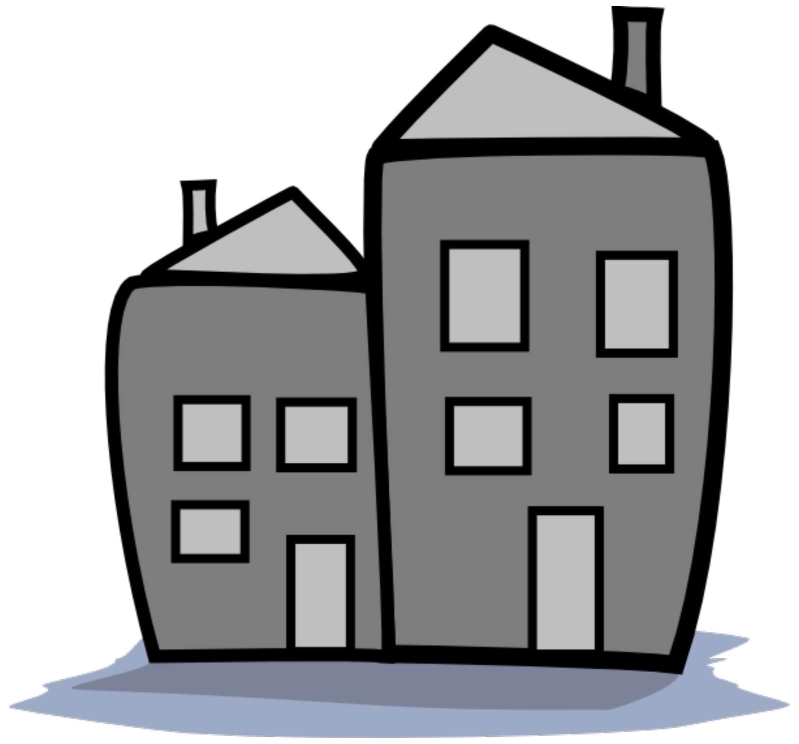
Multiple Item Matching Market

- We will discuss a **decentralized asynchronous** matching market where buyers are free to buy the item they wish
- Each buyer wants only **one** item & each item can be given to at most one buyer
- More formally, we have n potential buyers, m different items
- Assume $m \geq n$ (if this is not true, we can always create dummy items that everyone values at 0)
- Each buyer i has a **private valuation** $v_{ij} \geq 0$ for each item j
- **Examples**: matching houses to buyers, renters to Airbnb rooms, or any idiosyncratic items to buyers

Example Market

Prices

p_1? 

p_2? 

p_3? 

Valuations listed are in **order** of houses **(top down)**

	Valuations
Zoe 	12, 2, 4
Chris 	8, 7, 6
Jing 	7, 5, 2

Prices and Utility

- If the price of item j is p_j, the utility that person i receives from getting item j is

$$u_{ij} = v_{ij} - p_j$$

- **Goal of buyers:** choose items individually to maximize their utility

Questions

- What prices do we expect to see in a market?
- Does there exists **prices** and **a way to match buyers to items** (find a matching) such that each buyer gets an item that maximizes their utility?
- Do these prices **"clear the market"**: sell all items that have any demand
- Imagine an ascending price clock and bidders dropping out of contention

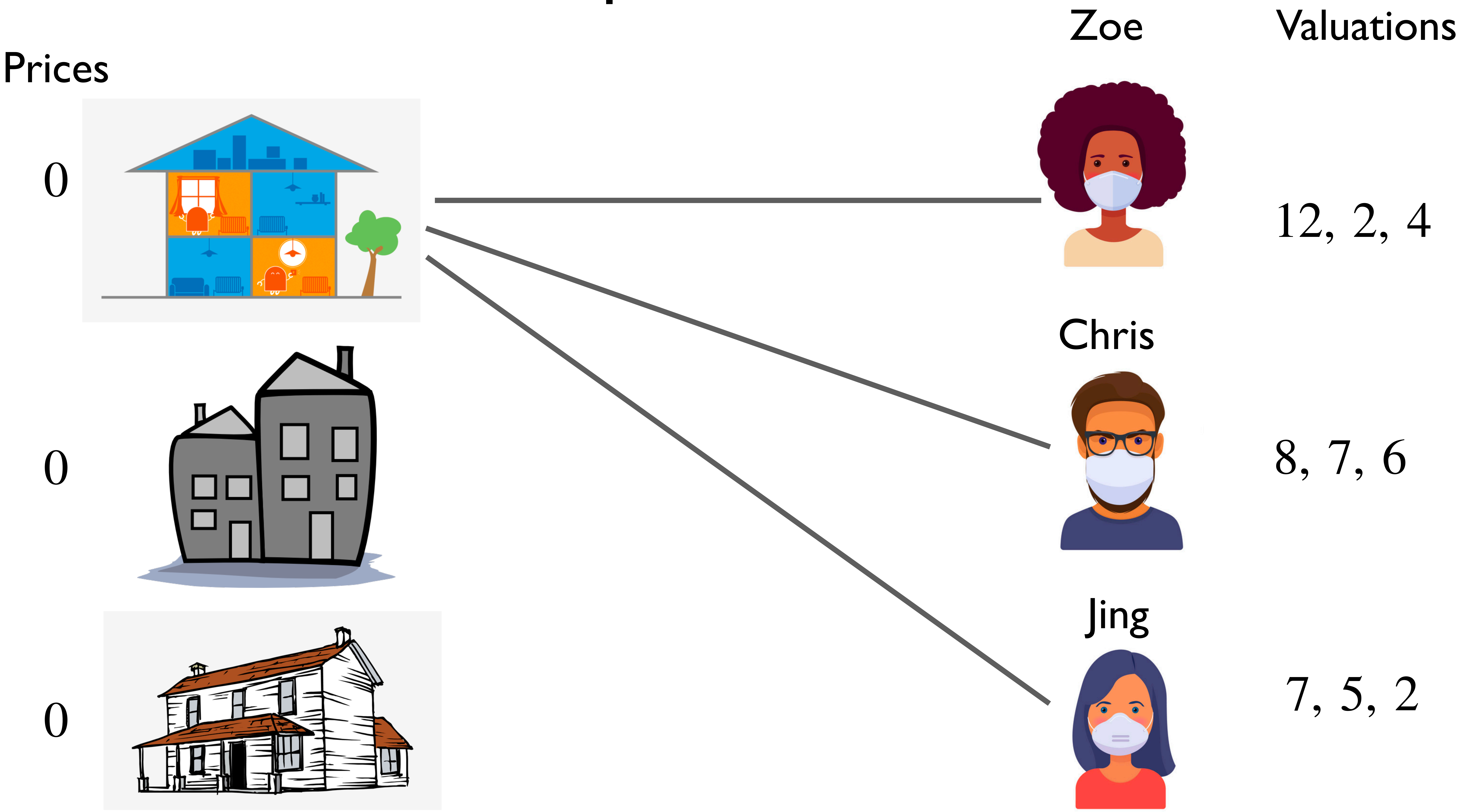
Social Welfare

- Let $M(i)$ denote the item matched with buyer i or $\varnothing$ if none
- Our goal has been to design mechanisms that maximize social welfare, that is, find a matching M of buyers to items that maximizes $\sum_{i=1}^{n} v_{iM(i)}$
- **Goal:** an outcome that achieves good guarantees but we have no control over it
- **Question.** If we let the market run its course what prices and allocation do we see?
  - How good is the social welfare of such an outcome?

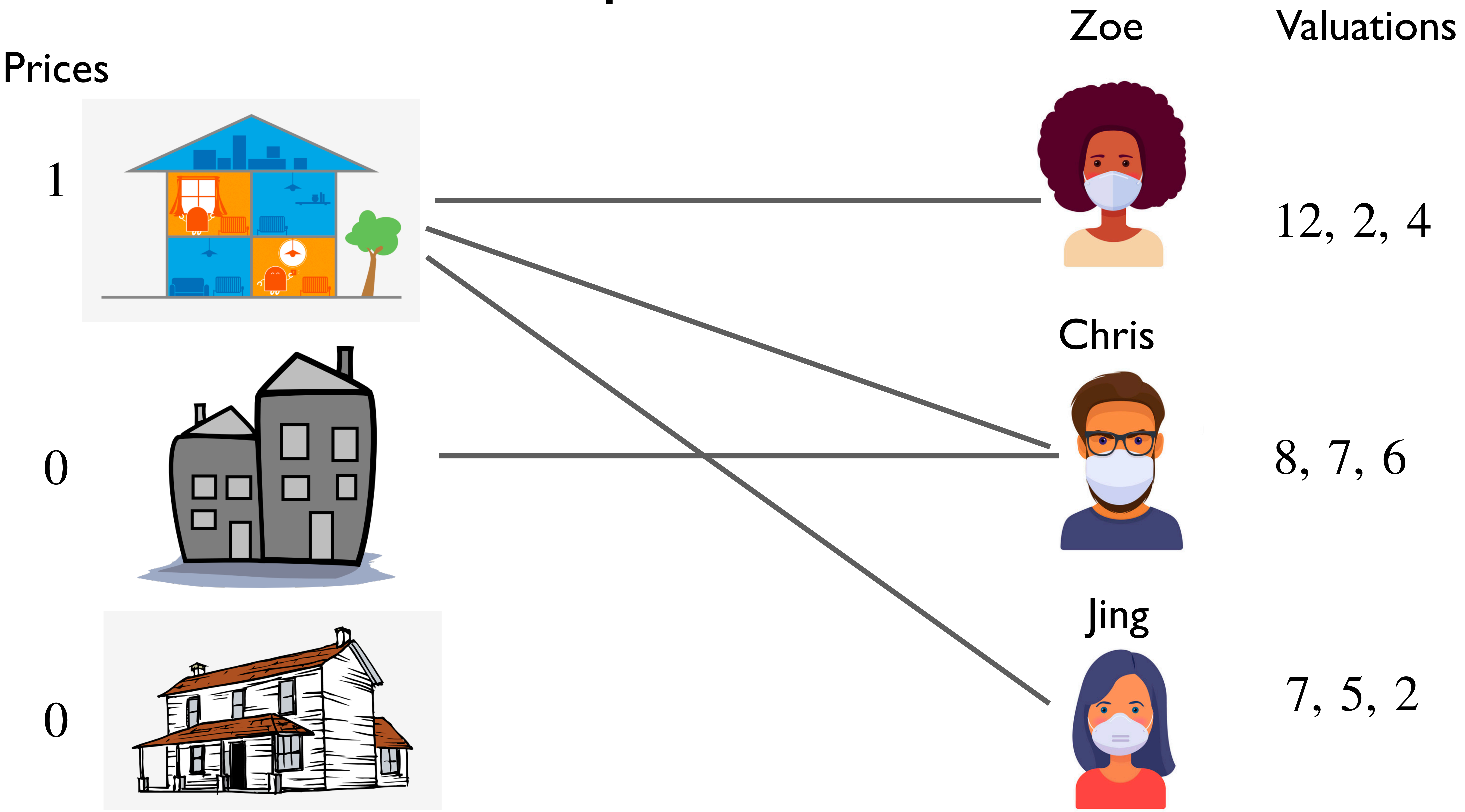
Preferred Items & Graph

- Given prices $\mathbf{p} = (p_1, \ldots, p_m)$ for the items, the preferred items for buyer i are all the items that maximize its utility
- Let $u_i^* = \max_{\text{all items } j} (v_{ij} - p_j)$ be the maximum utility i can obtain given $\mathbf{p}$
- Let the set of preferred items P_j of buyer i given the prices $\mathbf{p}$ be all items that maximize its utility: $P_j = \{j \mid v_{ij} - p_j = u_i^*\}$ assuming $u_i^* \geq 0$
  - If $u_i^* < 0$ then $P_j = \varnothing$
- Create a **preferred-item graph** (given prices $\mathbf{p}$) where nodes are items and buyers and there is an edge between buyers and their preferred items

Preferred-Item Graph



Preferred-Item Graph



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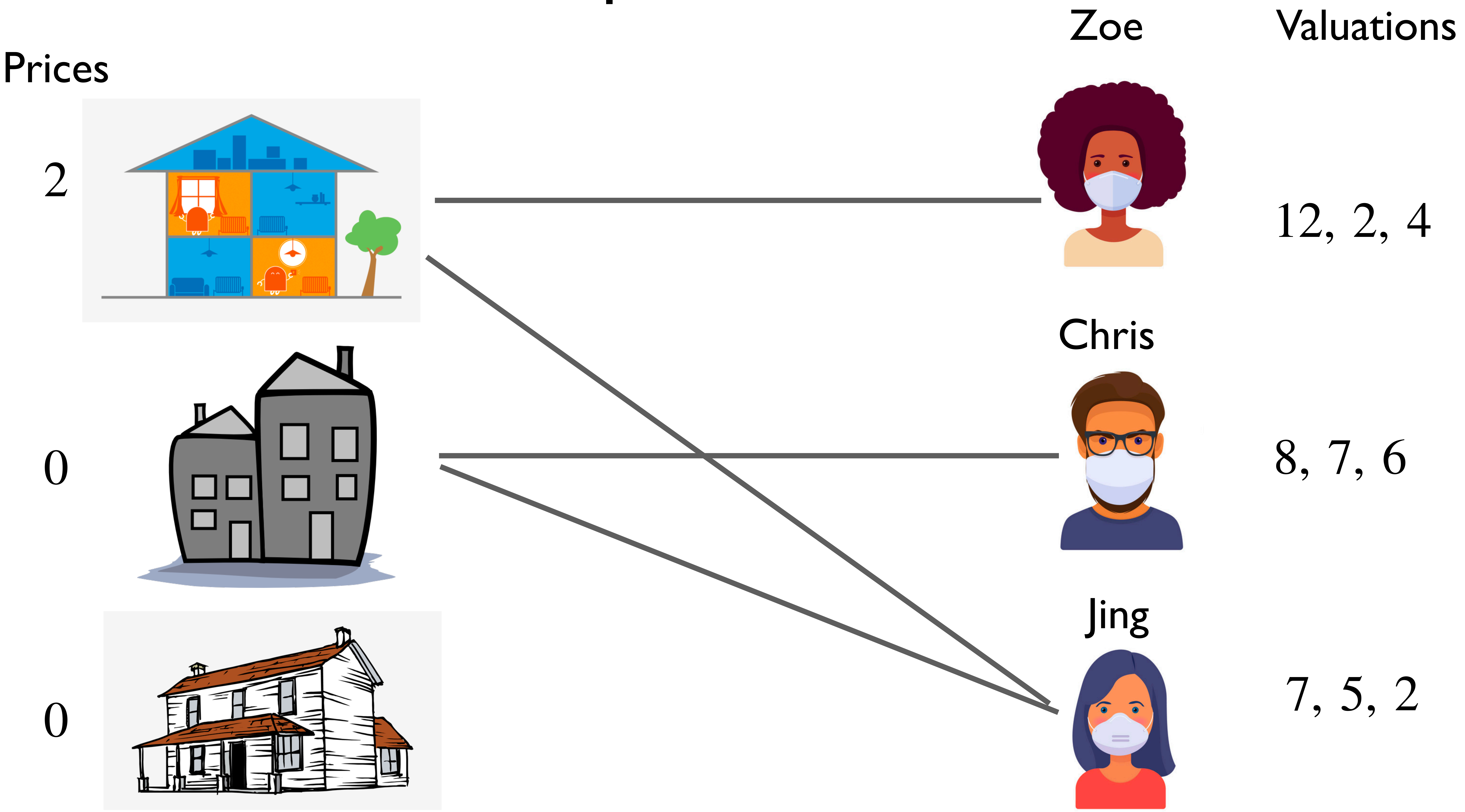

Market-Clearing Prices

- A selection of prices $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ is **market-clearing** if:
  - **Condition 1.** There is a **matching** in the preferred-item graph such that **all buyers are matched** to an item
  - **Condition 2.** If an item j is not matched to any buyer, then its price $p_j = 0$, in other words, every item with non-zero price $p_j > 0$ must get sold
- This means that at market-clearing prices, each buyer can come by and pick some item that maximizes its utility
  - Assume tie-breaks can occur in a coordinated way
- Furthermore, at these **prices the market will "clear"**
  - Only items left behind are those that are not desirable (price 0)

Matching. A subset of edges M form a matching if no two edges in M is incident on the same node

Market-Clearing Prices

What does this condition remind you off?

- **Condition 1** says that, given the prices, all buyers:
    - Get a utility maximizing item

$$\underbrace{v_{ij} - p_j}_{\text{Utility from receiving } j \text{ at price } p_j} \geq \underbrace{v_{ij'} - p_{j'}}_{\text{Utility from receiving } j' \text{ at price } p_{j'}}$$

- Why do we need **Condition 2?**
- **Condition 2** says that the outcome is "market clearing" in the sense that every good that is desired is sold
    - Only good that is allowed to be not sold are those with $p_j = 0$

Market-Clearing Prices

Outcome must be **envy free**!

- **Condition 1** says that, given the prices, all buyers:
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$$\underbrace{v_{ij} - p_j}_{\text{Utility from receiving } j \text{ at price } p_j} \geq \underbrace{v_{ij'} - p_{j'}}_{\text{Utility from receiving } j' \text{ at price } p_{j'}}$$

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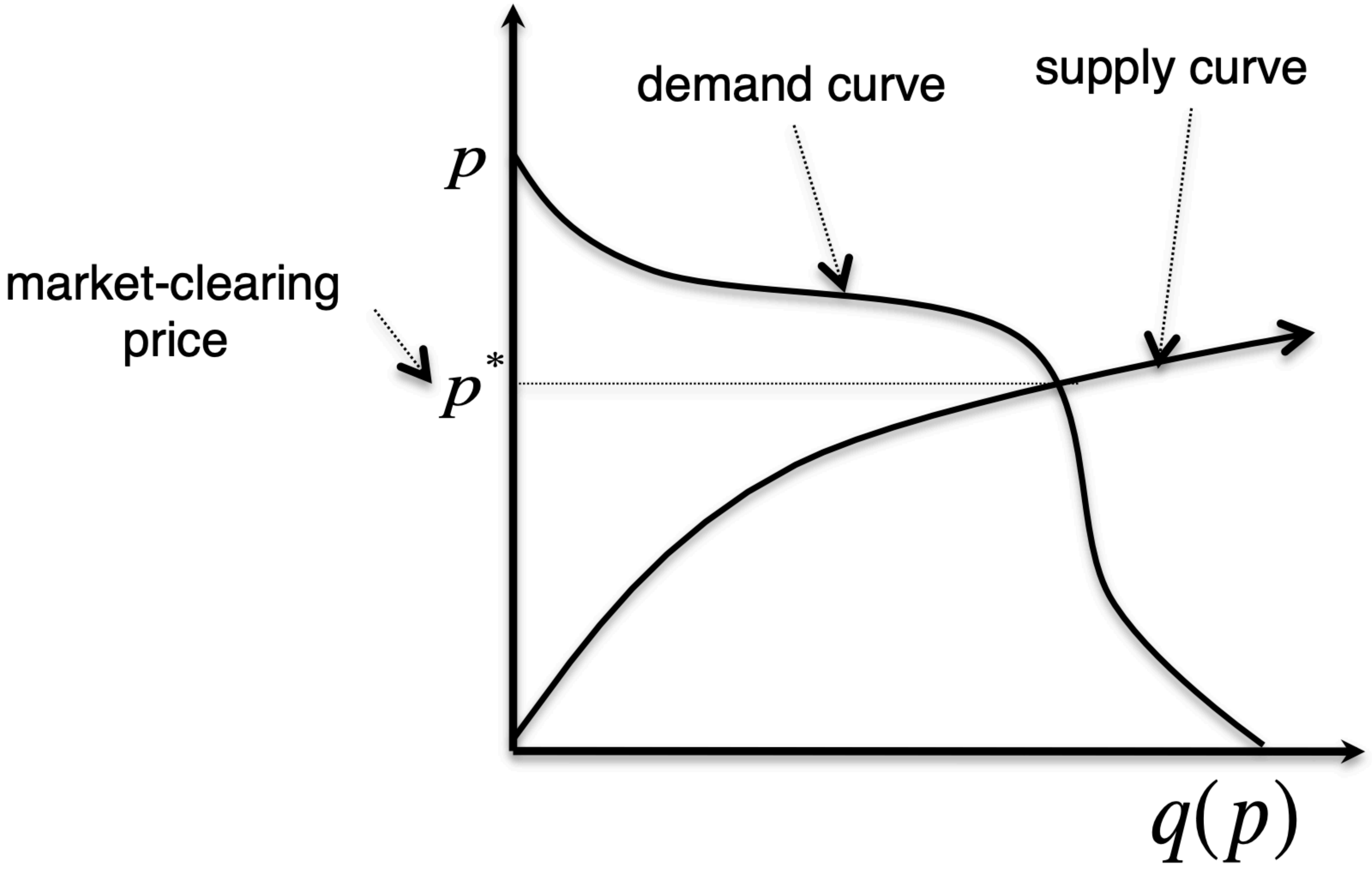
Market-Clearing Prices

- **Condition 2** says that the outcome is "market clearing" in the sense that every good that is desired is sold
  - Only good that is allowed to be not sold are those with $p_j = 0$
- Why is this condition important?
  - Notice that we can trivially satisfy Condition 1 by setting all prices to be ∞
  - At that price, no buyer wants any item
- But is this a good outcome?
  - No one gets anything: no welfare/surplus generated!
- Need prices to **clear market** and to **optimize surplus** generated

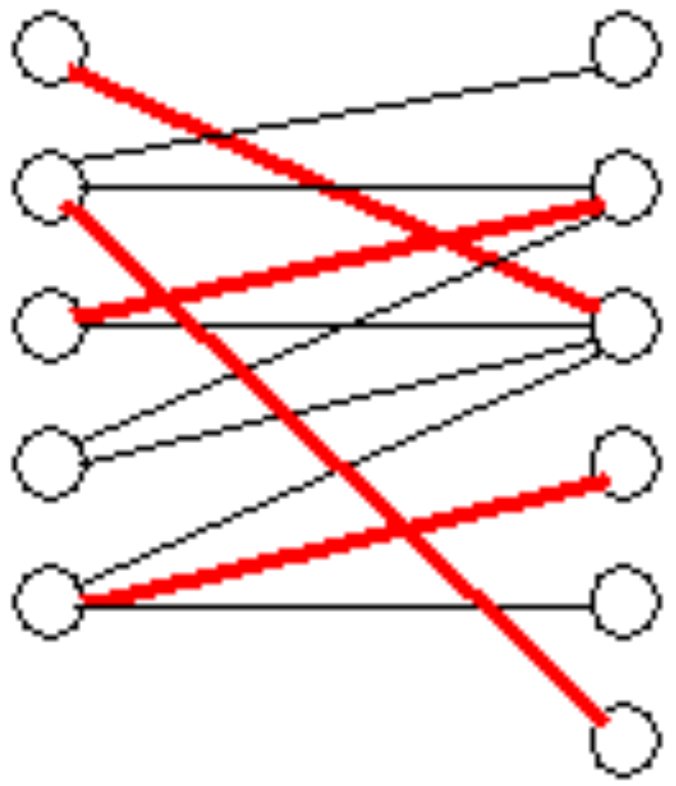
Economics Point of View

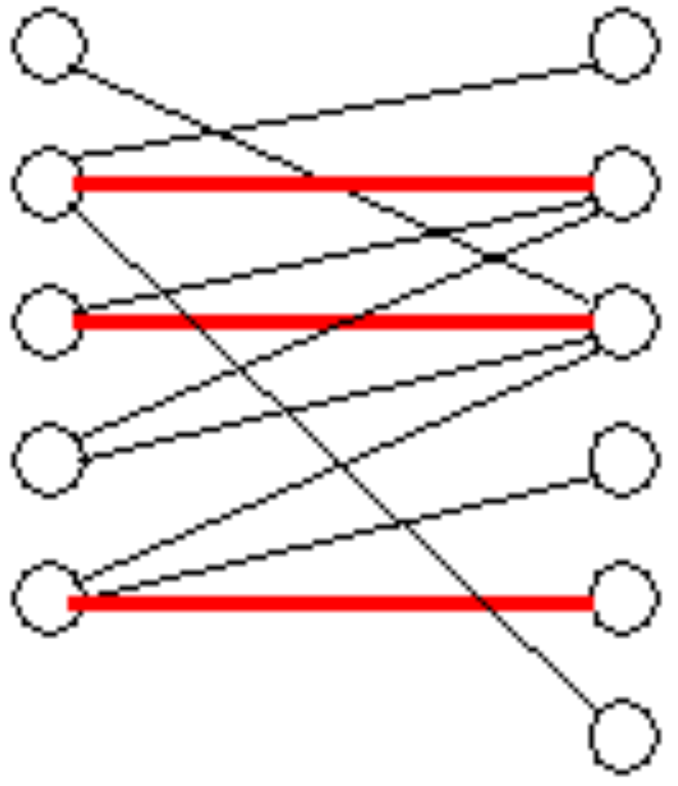
- Market clearing prices in economics are prices at which supply is equal to demand
- **Demand curve:** as price increases, typically demand goes down
- **Supply curve:** As price increases, typically supply $s(p)$ increases
- Price where they meet: market clearing



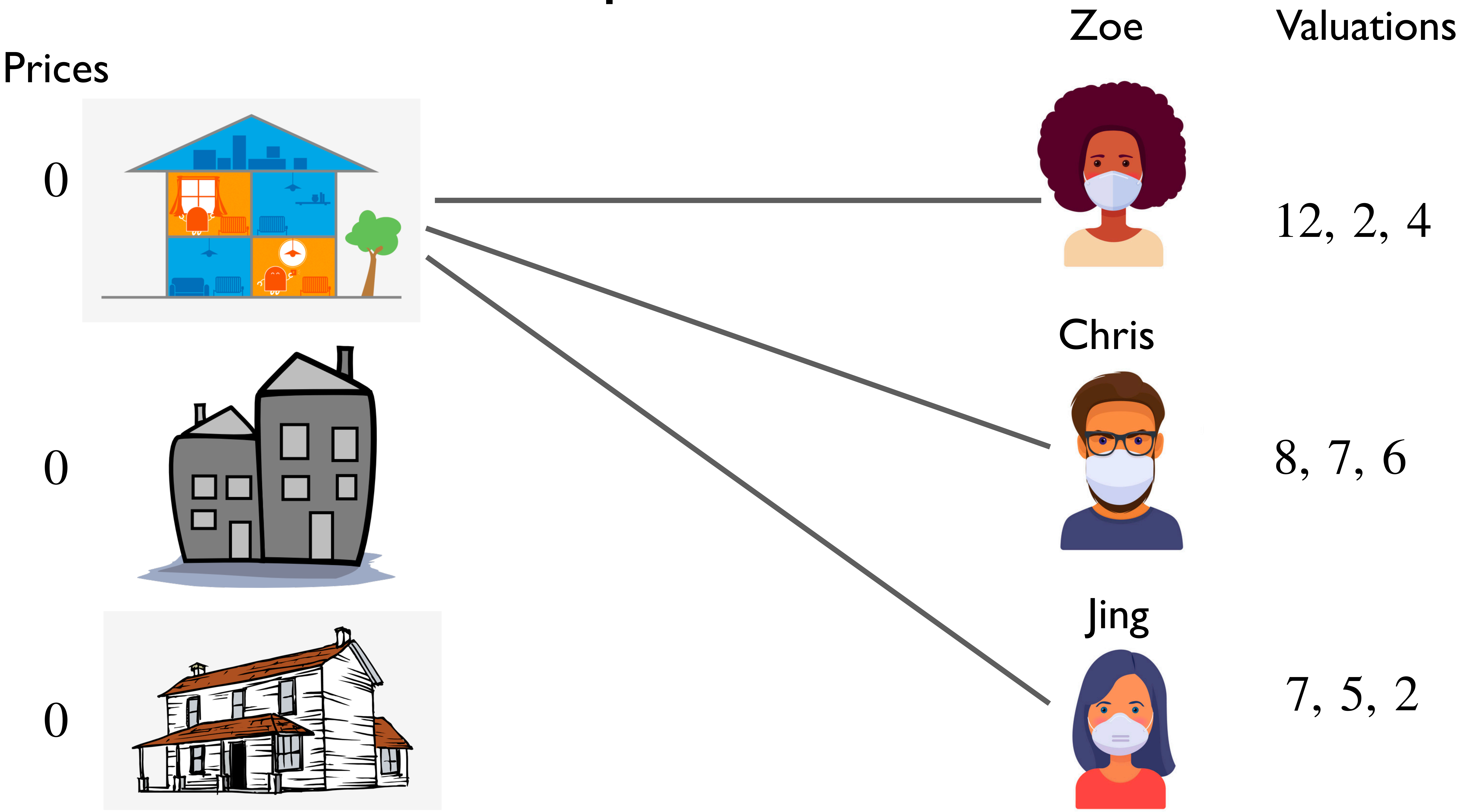
Market-Clearing Prices

- **Reminder (matching definition).** A subset of edges M form a matching if no two edges in M is incident on the same node
  - An independent set of edges
- Looking for a **buyer-perfect matching** (a matching that "covers" all buyer nodes)
- Since the edges in the preferred-item graph depend on prices of items, the question is,
  - What prices cause a buyer-perfect matching to exist?
- **Question.** How do we know when a buyer-perfect matching is not possible?





Preferred-Item Graph



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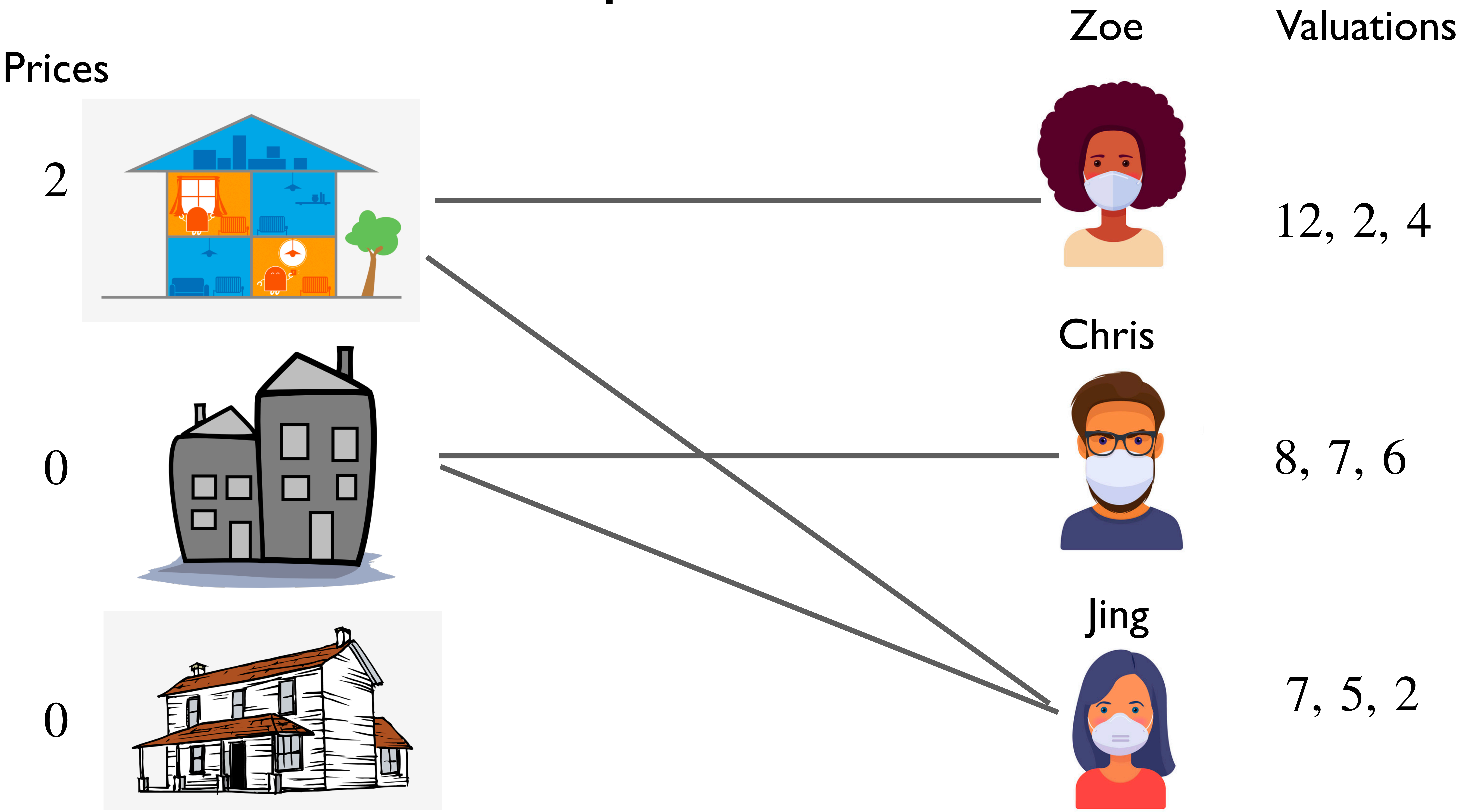
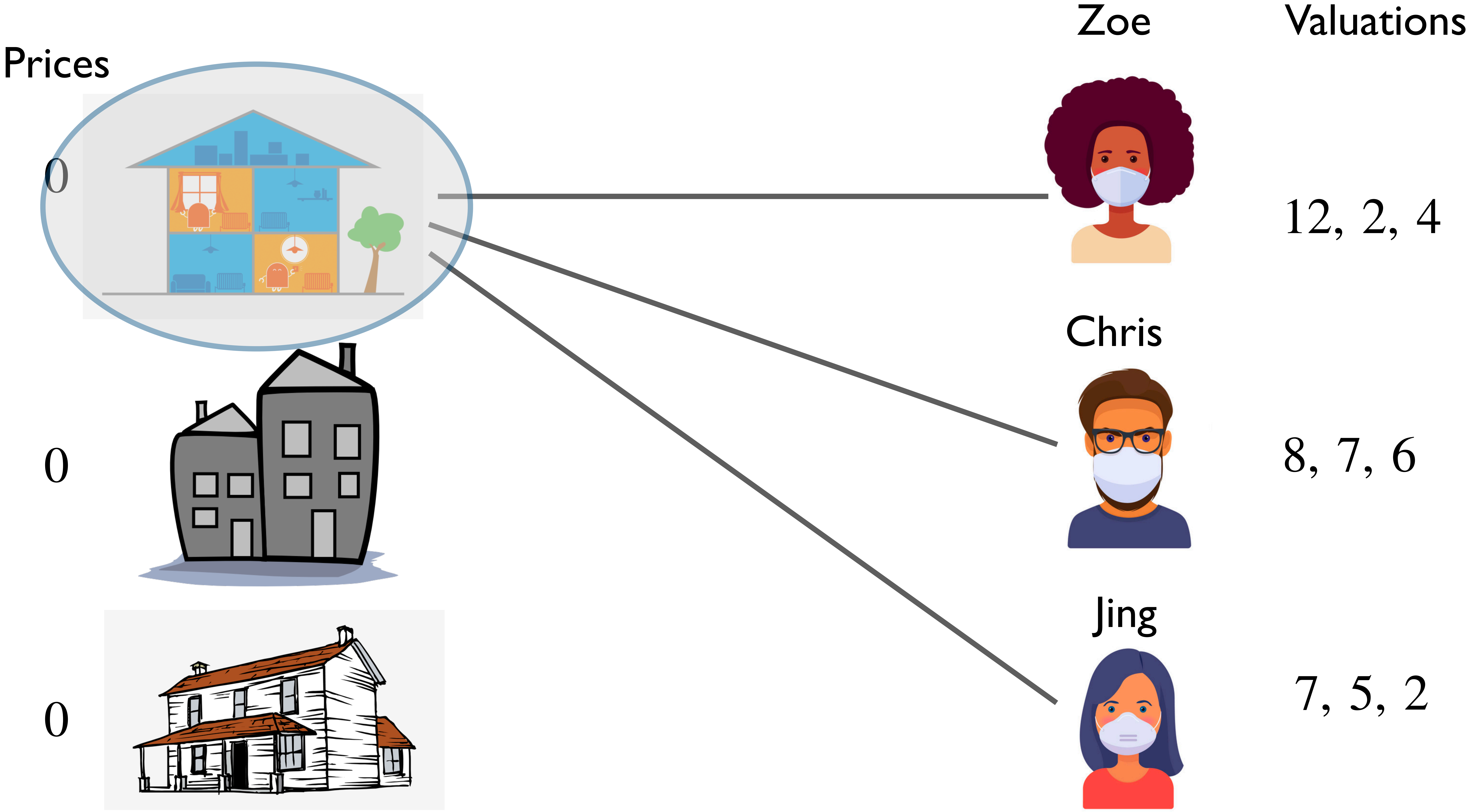


Hall's Theorem

- Let S be a subset of nodes, then the **neighborhood** $N(S)$ is the set of all nodes that adjacent to nodes in S
- In a bipartite graph (X, Y) has a Y perfect matching **if and only if** for every $S \subseteq Y$ with neighborhood $N(S)$ the following holds:

  $|N(S)| \geq |S|$ (neighborhood is at least as large)
- Thus, when a Y-perfect matching is not possible, there exists a subset $T \subseteq Y$ that violates the above condition, that is,
  - Such a set $N(T)$ is called a **constricted set** set
- If there is no buyer-perfect matching: can always find a constricted set
  - "Over-demanded" items at current price

Preferred-Item Graph



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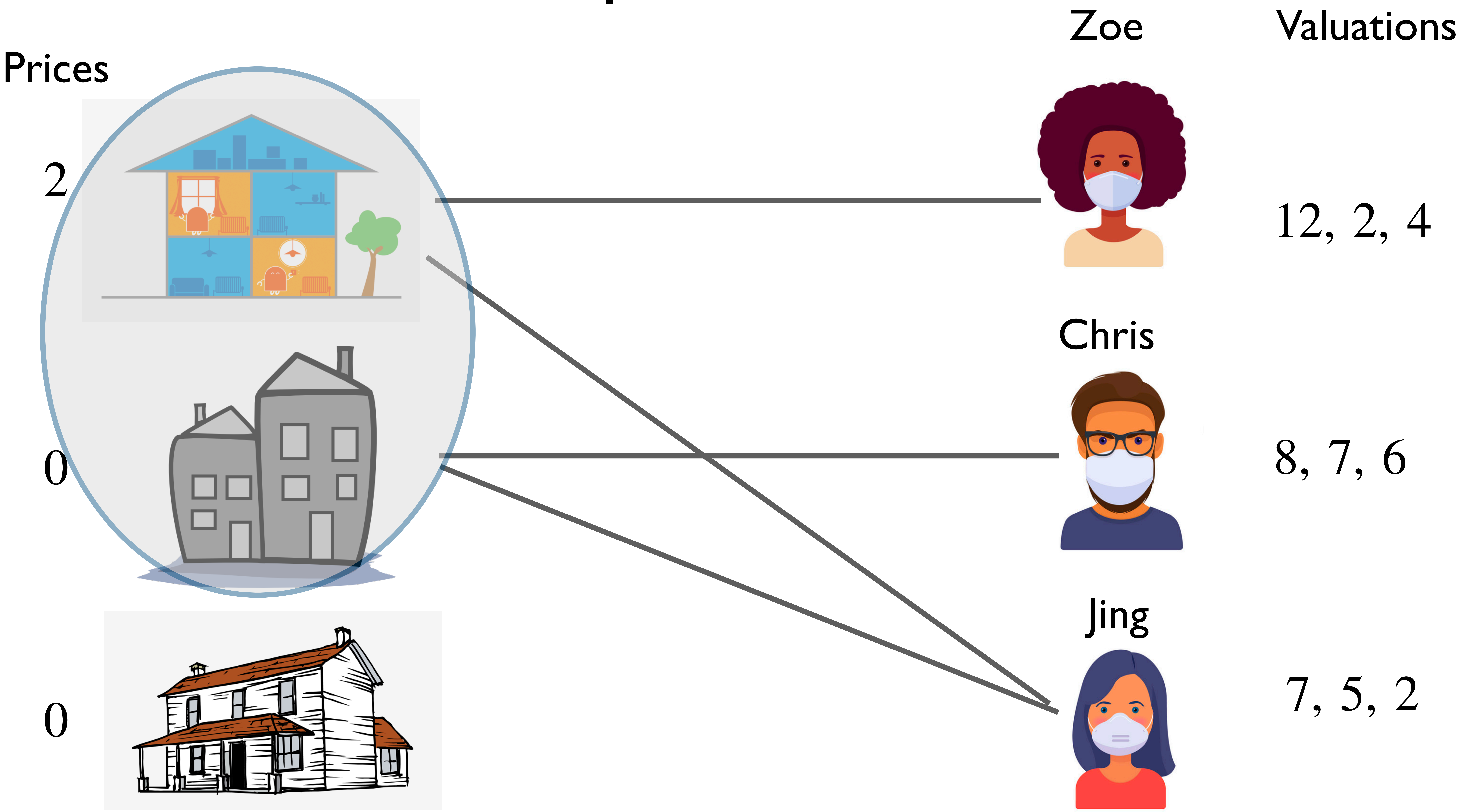

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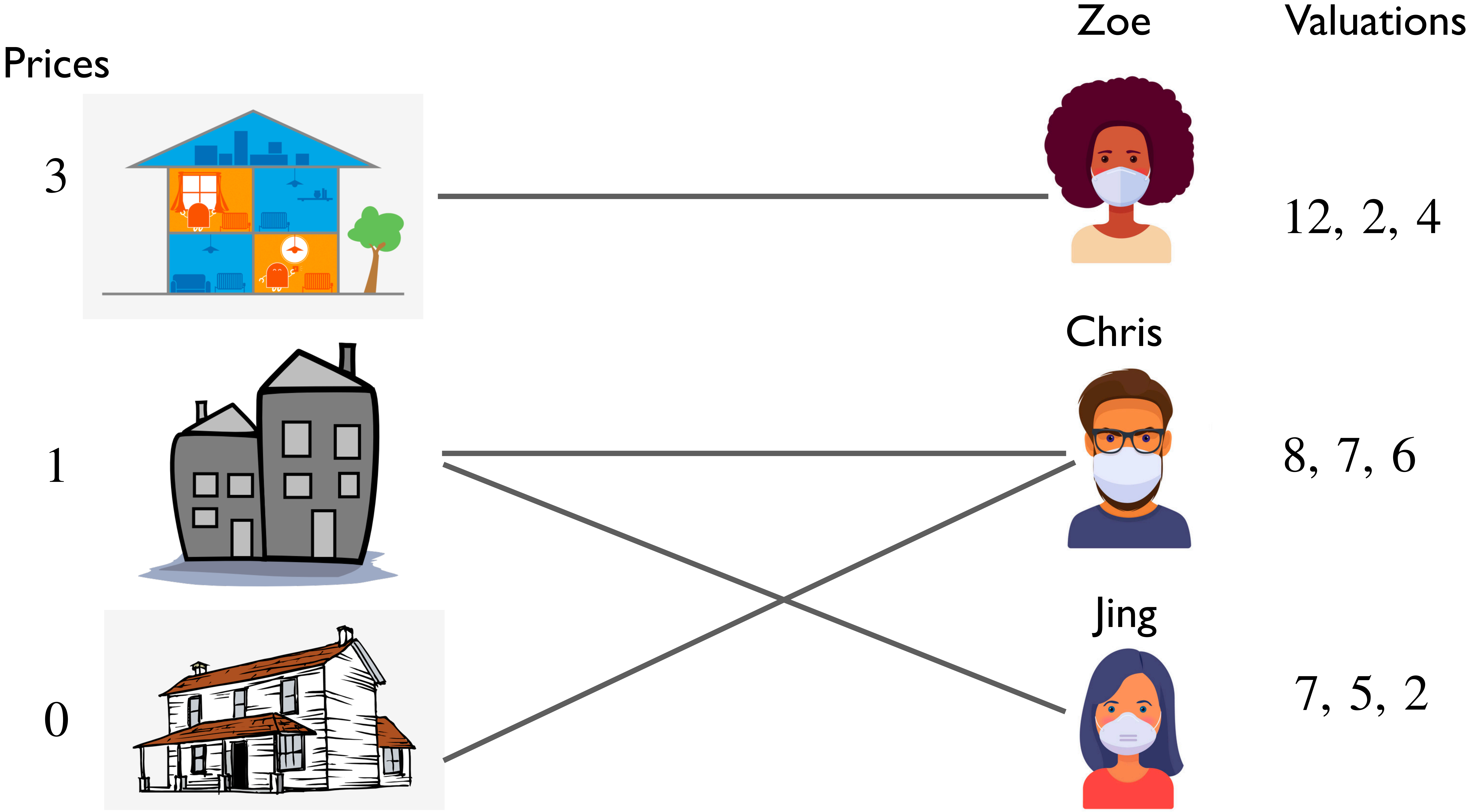

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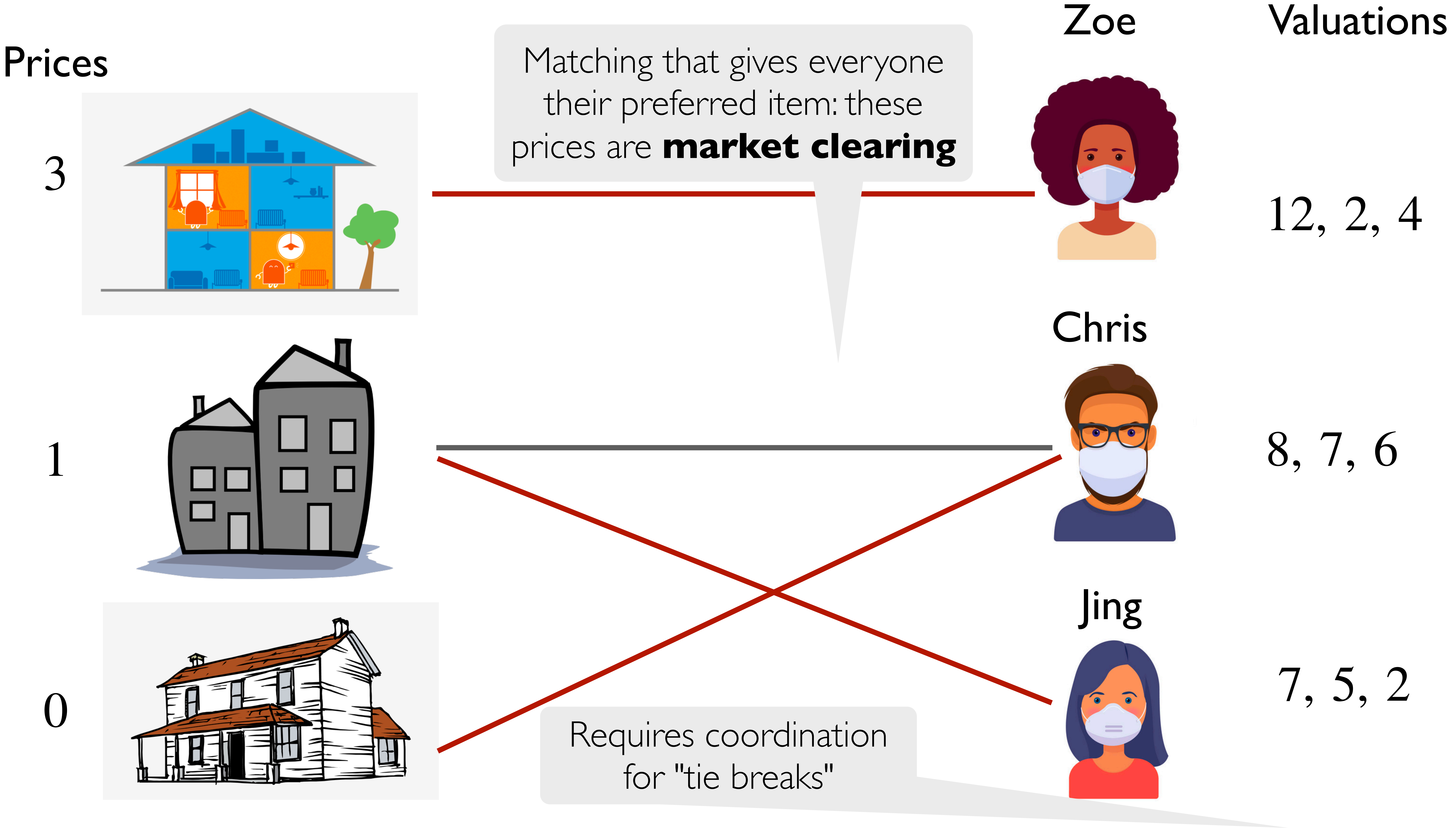

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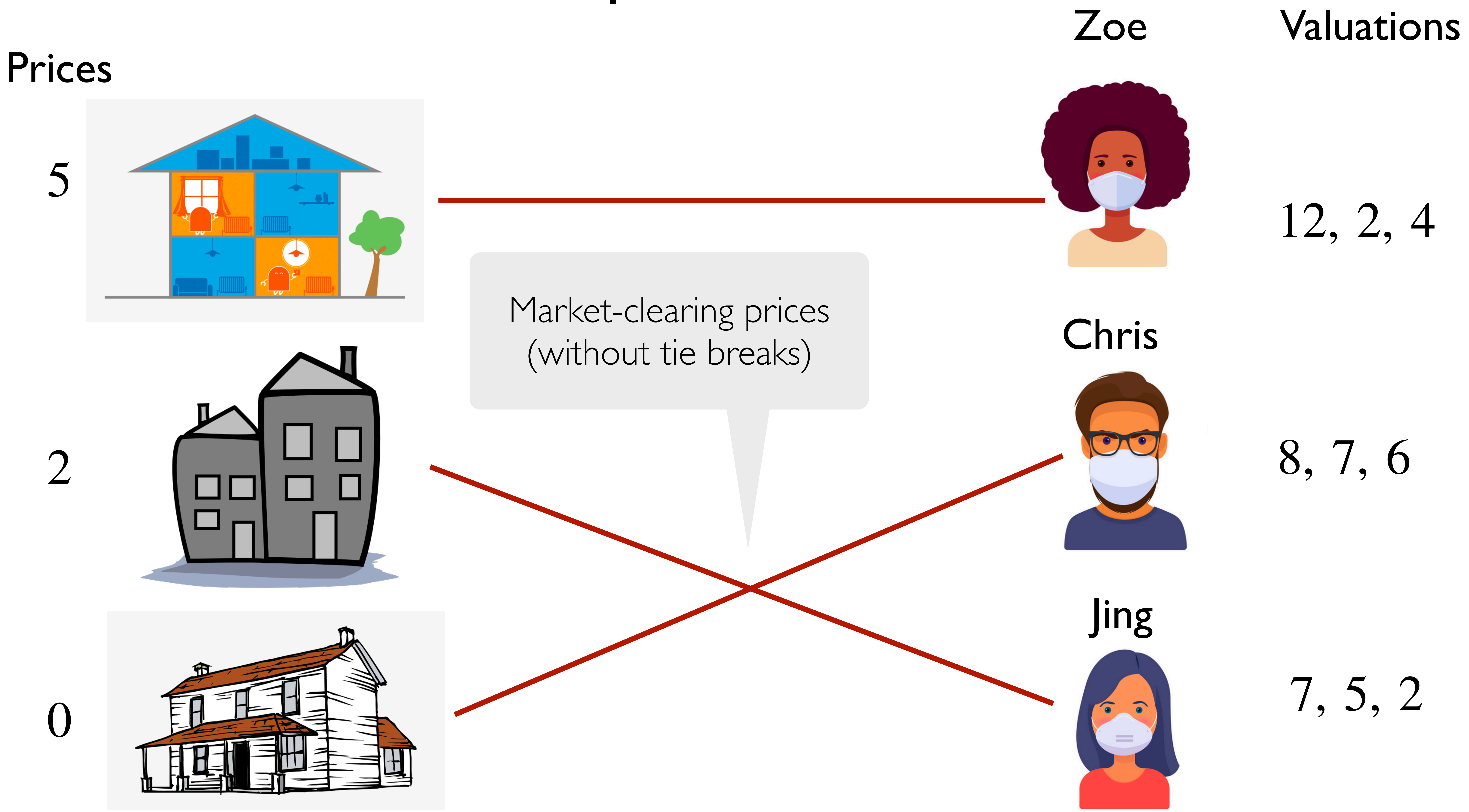
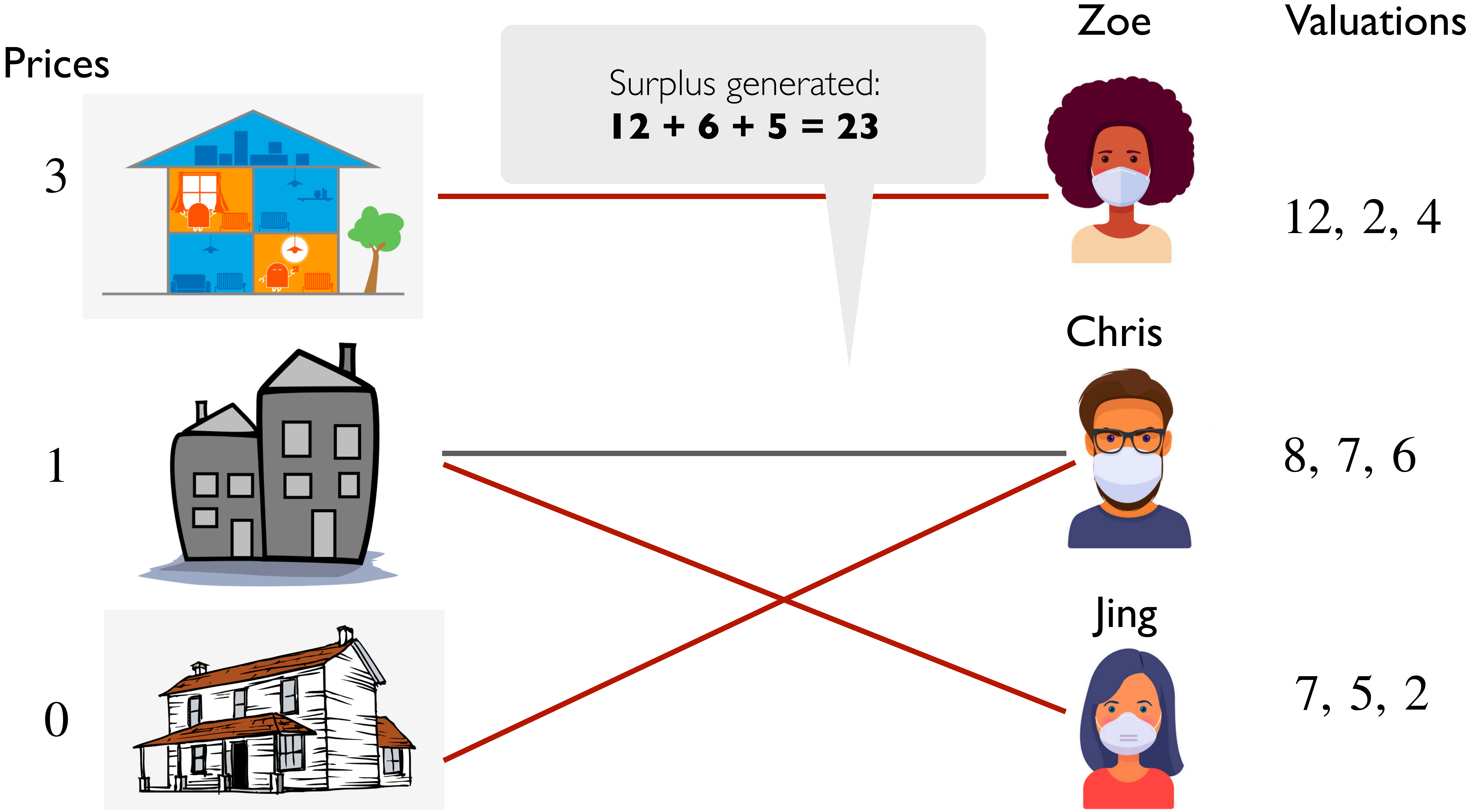


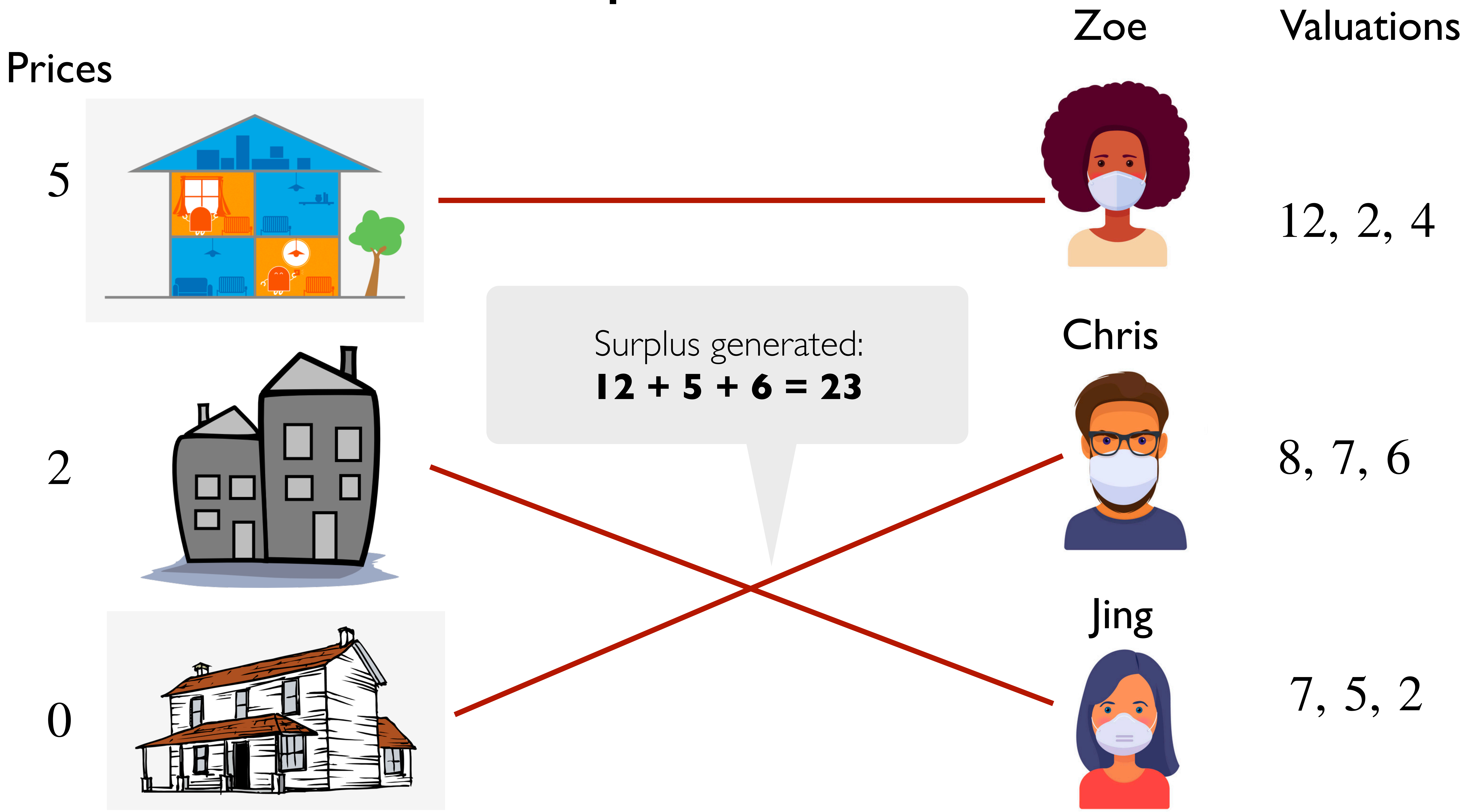
Competitive Equilibrium

- Market-clearing prices $\mathbf{p}$ along with the matching M from buyers to their preferred item is called a **competitive** or **Walrasian equilibrium**
- Requirements of competitive equilibrium are strong
  - Put a price tag p_j on each good
  - Let each buyer i independently pick whichever good they want
- Magically, there are no conflicts and each buyer gets what they want
  - (Allowing ties to be broken in a coordinated way)
- **Question.** Seems too good to be true, does it always exist?
- **Question.** Should we be happy with the outcome of a competitive equilibrium?

Preferred-Item Graph



Preferred-Item Graph



First Welfare Theorem

- Matchings in a competitive equilibrium are exactly the matching with maximum possible value!
- **First Welfare Theorem (Max-weight matching).** If $(M, \mathbf{p})$ is a competitive equilibrium, then M is a matching with maximum total value, that is,
  - $\sum_{i=1}^{n} v_{iM(i)} \geq \sum_{i=1}^{n} v_{iM'(i)}$ for every matching M'
- In particular, among all possible ways of allocating items such that each buyer is matched to at most one item good and each item is matched to at most one buyer, the allocation achieved at a competitive equilibrium maximizes welfare

First Welfare Theorem Proof

- **Proof.** Consider some matching M^* with the maximum-possible total value
- What we know: $(M, \mathbf{p})$ is a competitive equilibrium
- Using envy-free condition to compare M and M^* at price $\mathbf{p}$:

$$v_{iM(i)} - p_{M(i)} \geq v_{iM^*(i)} - p_{M^*(i)}$$ for every bidder i

- Let the sum of prices $\sum_{j=1}^{m} p_j = P$
- Summing up the inequality in blue over all bidders

M^* can assign each bidder at most one item

$$\underbrace{\sum_{i=1}^{n} v_{iM(i)}}_{\text{total value of } M} - \underbrace{\sum_{i=1}^{n} p_{M(i)}}_{= P \text{ by CE property (b)}} \geq \underbrace{\sum_{i=1}^{n} v_{iM^*(i)}}_{\text{total value of } M^*} - \underbrace{\sum_{i=1}^{n} p_{M^*(i)}}_{\leq P},$$

First Welfare Theorem Proof

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- What we know: $(M, \mathbf{p})$ is a competitive equilibrium
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$$v_{iM(i)} - p_{M(i)} \geq v_{iM^*(i)} - p_{M^*(i)} \quad \text{for every bidder } i$$

- Let the sum of prices $\sum_{j=1}^{m} p_j = P$
- Reorganizing this inequality, we get that value of $M \geq$ value of M^* ∎

M^* can assign each bidder at most one item

$$\underbrace{\sum_{i=1}^{n} v_{iM(i)}}_{\text{total value of } M} - \underbrace{\sum_{i=1}^{n} p_{M(i)}}_{= P \text{ by CE property (b)}} \geq \underbrace{\sum_{i=1}^{n} v_{iM^*(i)}}_{\text{total value of } M^*} - \underbrace{\sum_{i=1}^{n} p_{M^*(i)}}_{\leq P},$$

Takeaways

- Competitive equilibrium automatically solves a non-trivial computational problem: **computing a maximum weight matching in a bipartite graph!**
  - Polynomial-time solvable but the algorithm is quite nontrivial
- Individually selfish agents reach a globally efficient outcome
- When economists say "markets are efficient", they are referring to a phenomenon like competitive equilibrium
- **Question.** Given their strong requirements, is a competitive equilibrium even guaranteed to exist?

Competitive Eq: Existence

- **Theorem.** In every market where at most one good is assigned to each buyer, there is at least one competitive equilibrium
  - Equivalently, market-clearing prices are guaranteed to exist
- We prove this constructively through an mechanism that shows how such prices might emerge organically in a market
- Intuition idea behind our **"ascending-price auction"**
  - If a set of k items is preferred by more than k buyers at its current price, then the prices of these items should rise
  - Keep identifying such "constricted sets" and increasing prices until the market clears

Ascending-Price Mechanism

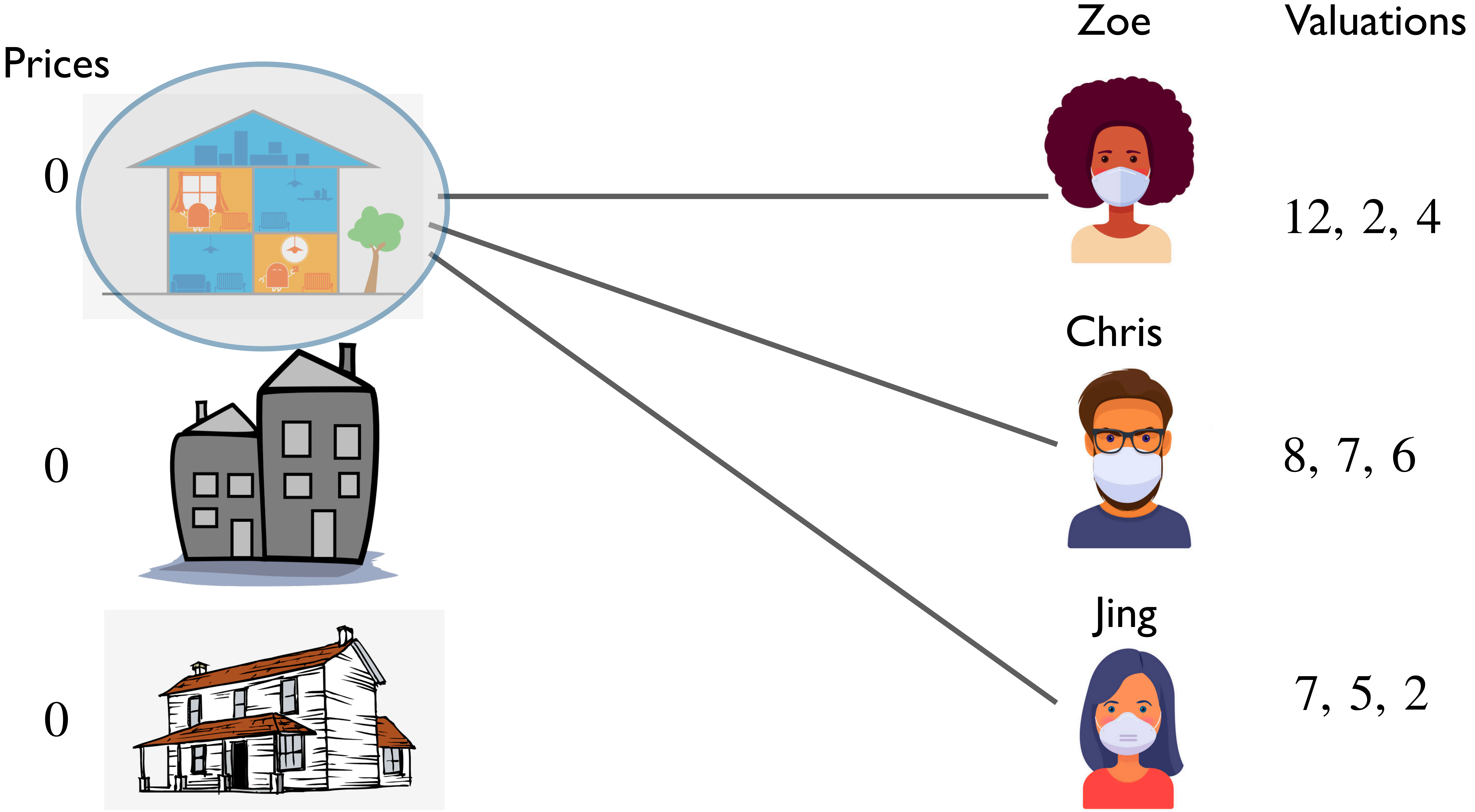
- Start with prices of all items $p_j = 0$
- Assume all valuations are integers $v_{ji} \in \mathbb{Z}$ (simplifying assumption)
- **Step 1.** Check if the current prices are market clearing, if so we are done
  - build the preferred graph, check if there is a buyer-perfect matching
- **Step 2.** Else, there must a constricted set:
  - There exists $S \subseteq \{1,\ldots,n\}$ such that $|S| > |N(S)|$
  - $N(S)$ are items that are **over-demanded**
  - If there are multiple such sets, choose the **minimal set** $N(S)$
    - Increase $p_j \leftarrow p_j + 1$ for all items in the set $j \in N(S)$
  - Go back to **Step 1.**

Single Item Case

- A single item (labelled 1) for which each buyer has a value $v_i > 0$
- Add $n - 1$ dummy items $(2, \ldots, n)$ that everyone values at 0
- At the beginning preferred-item graph has edges from each buyer to item 1
- Thus, $\{1\}$ is our minimal constricted set
- We need to keep raises the price of item 1 until all except one buyer has a preferred edge to at least one item in $\{2, 3, \ldots, n\}$
- At what price does this happen?
  - Exactly when $p_1 =$ second-highest valuation
  - The person with the highest valuation is matched to item 1

We have recreated the **second-price auction** outcome!

Preferred-Item Graph



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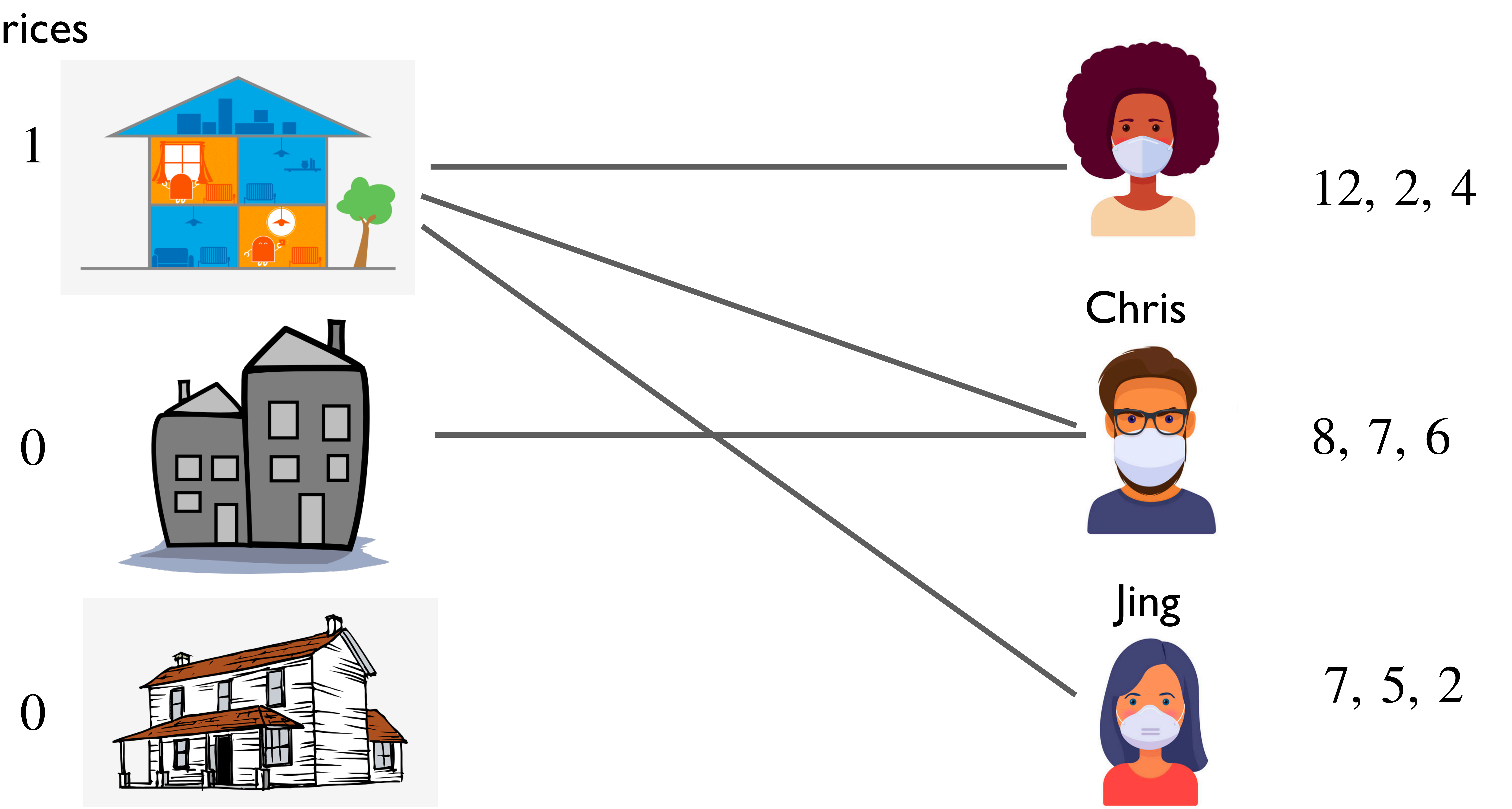

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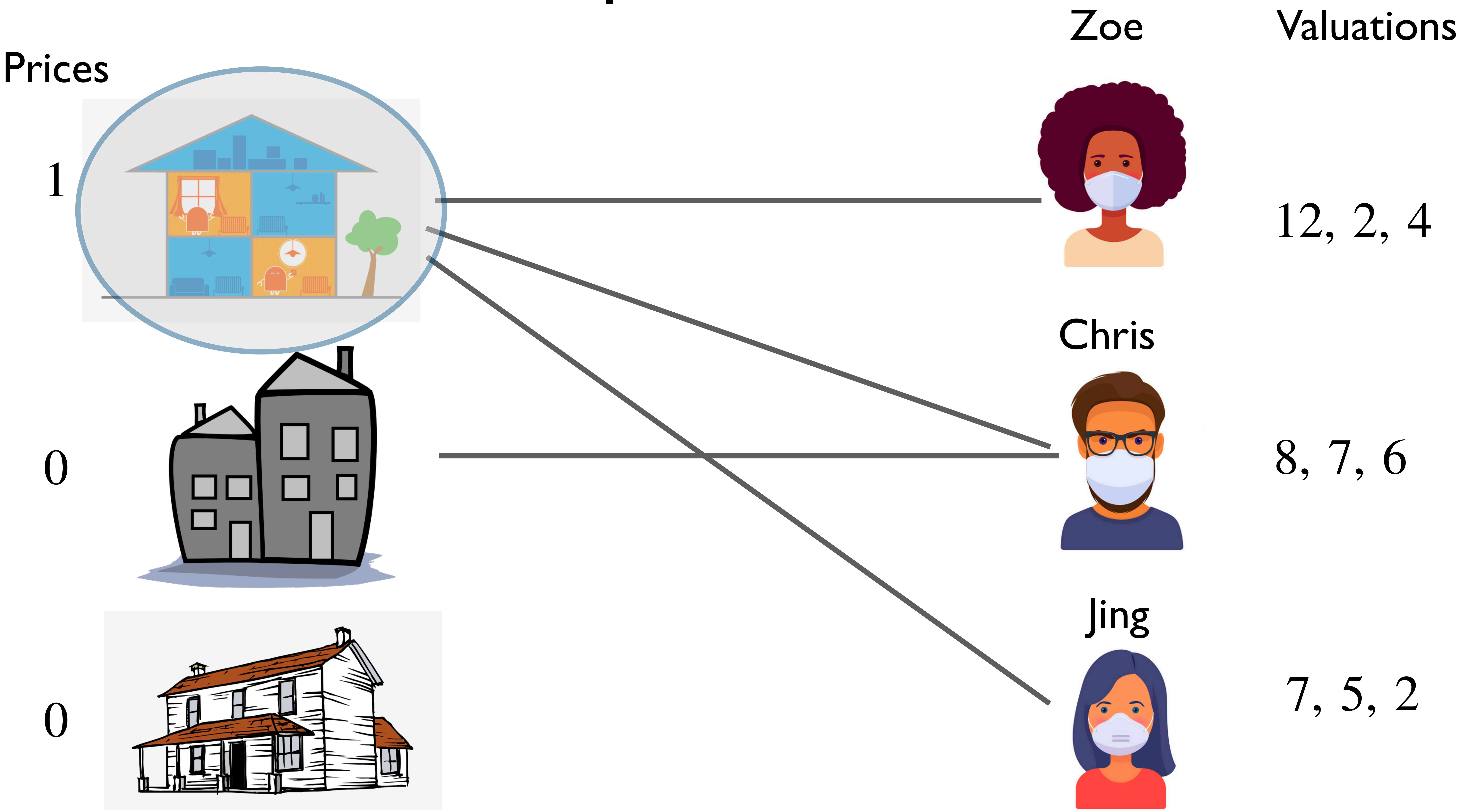

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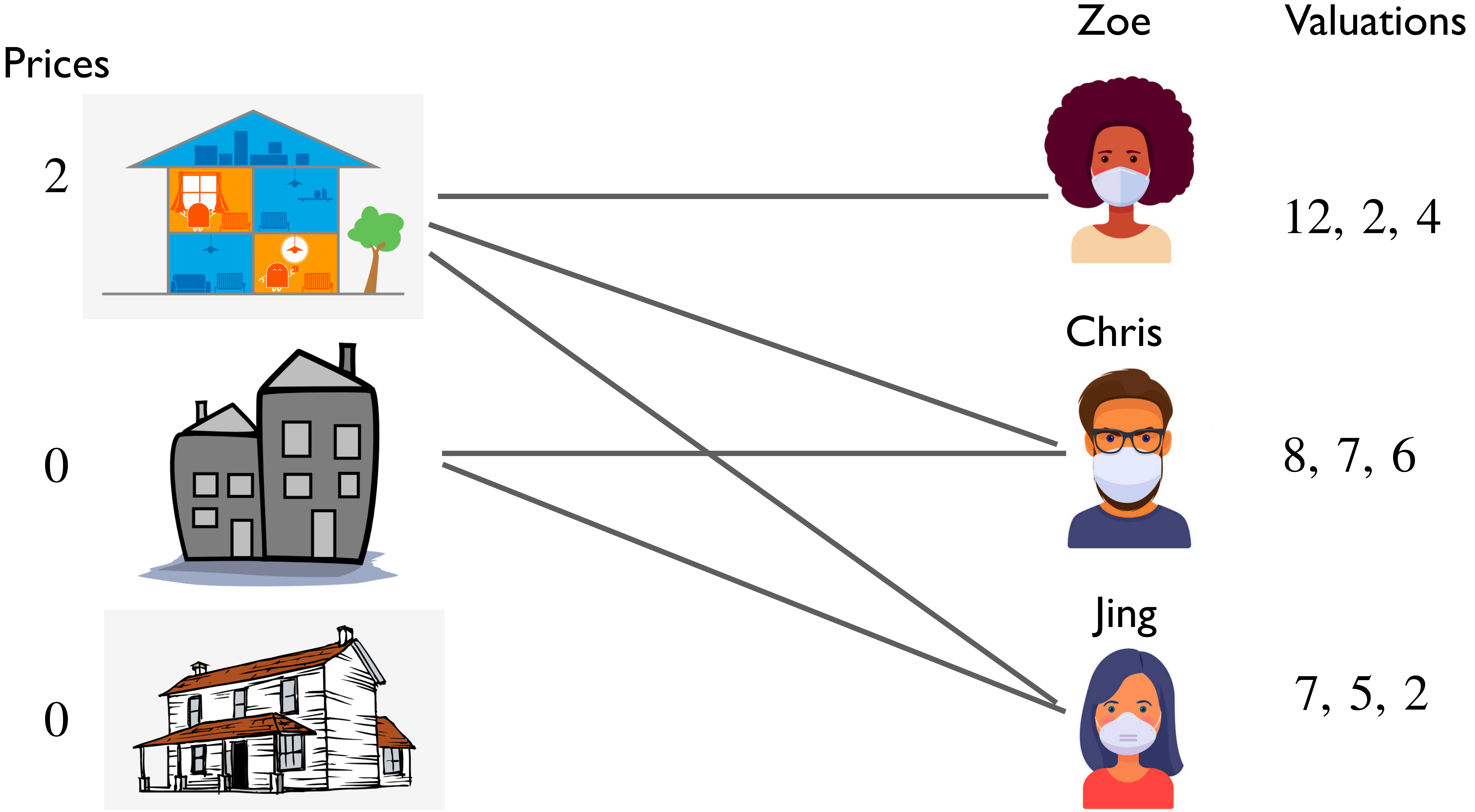

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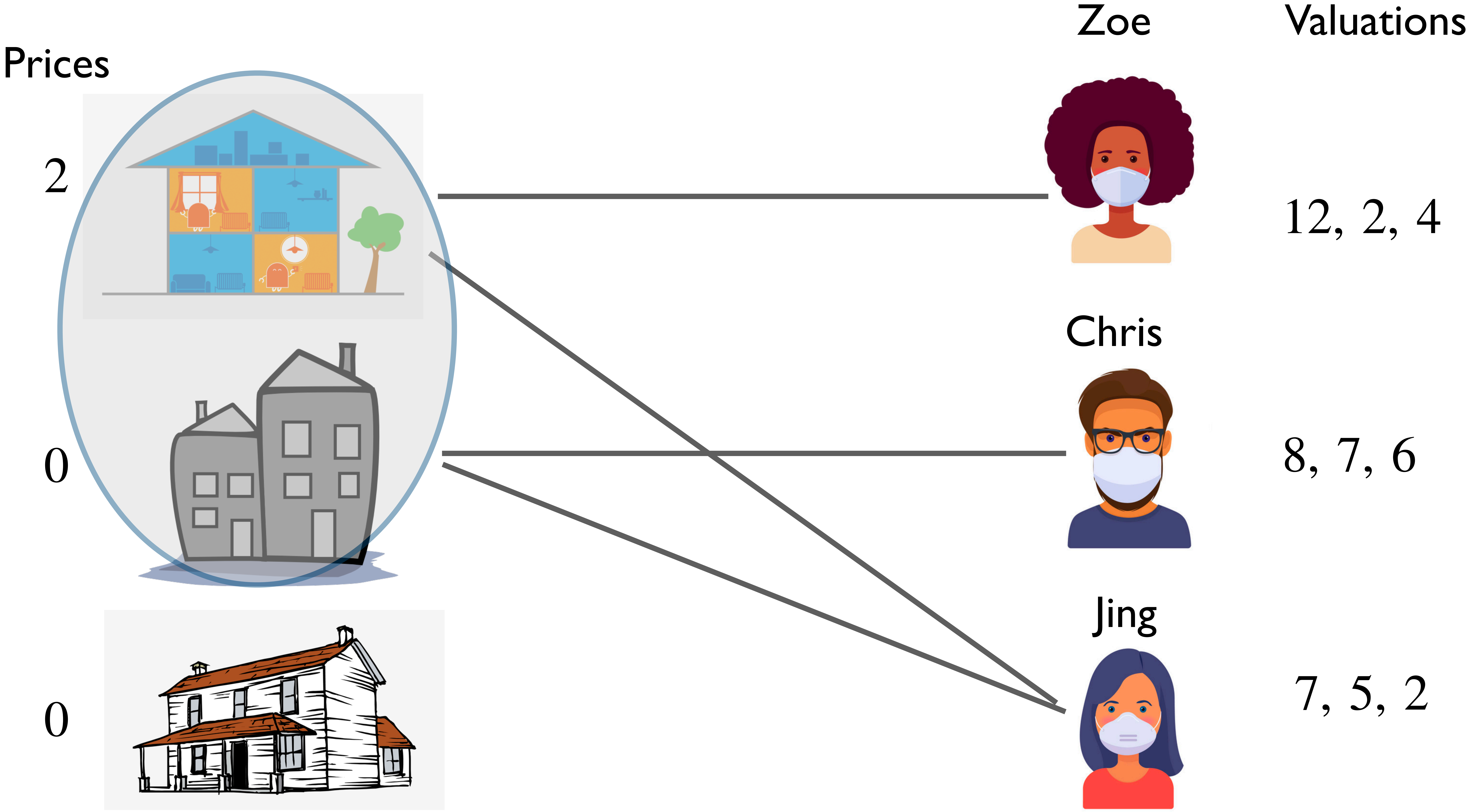

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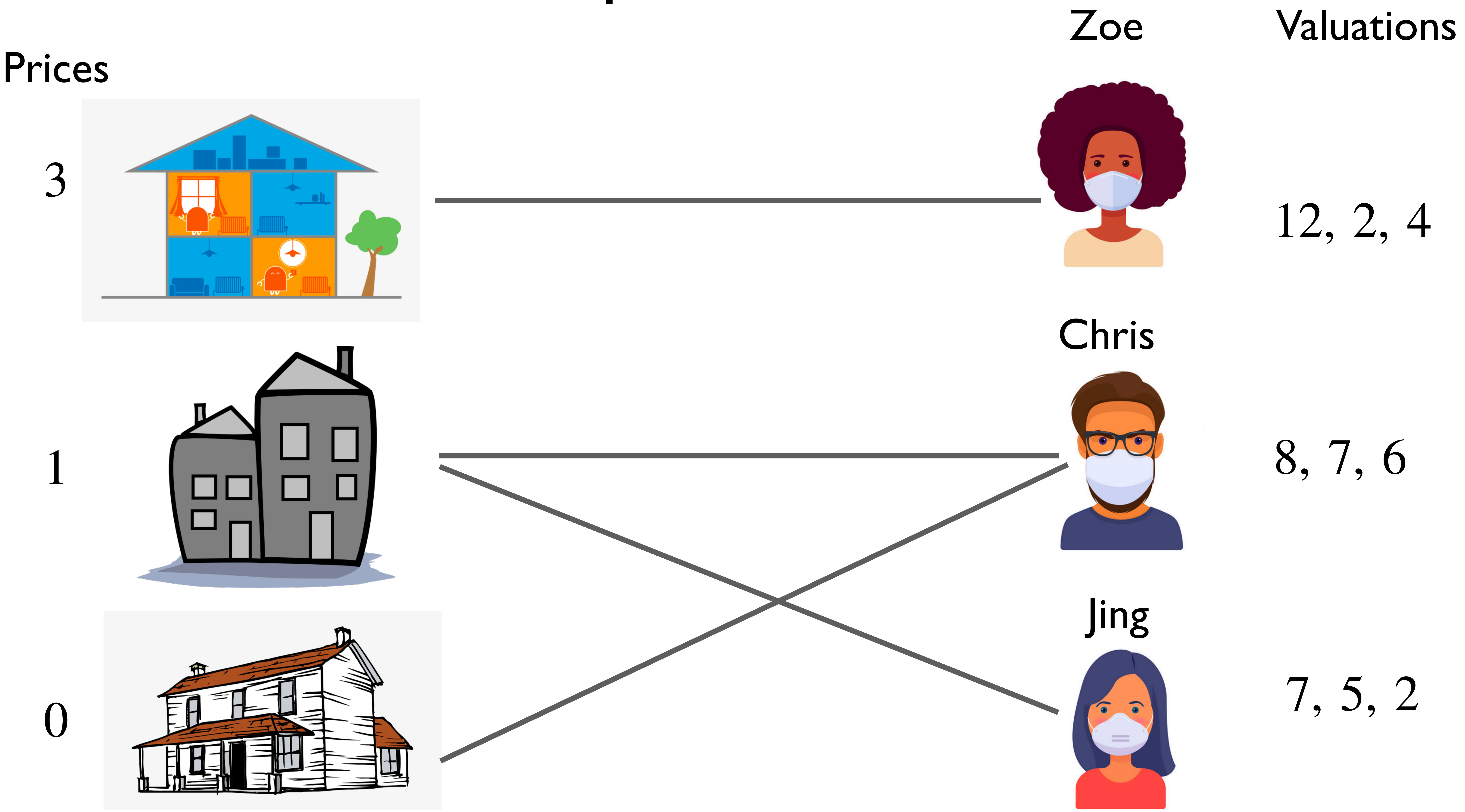

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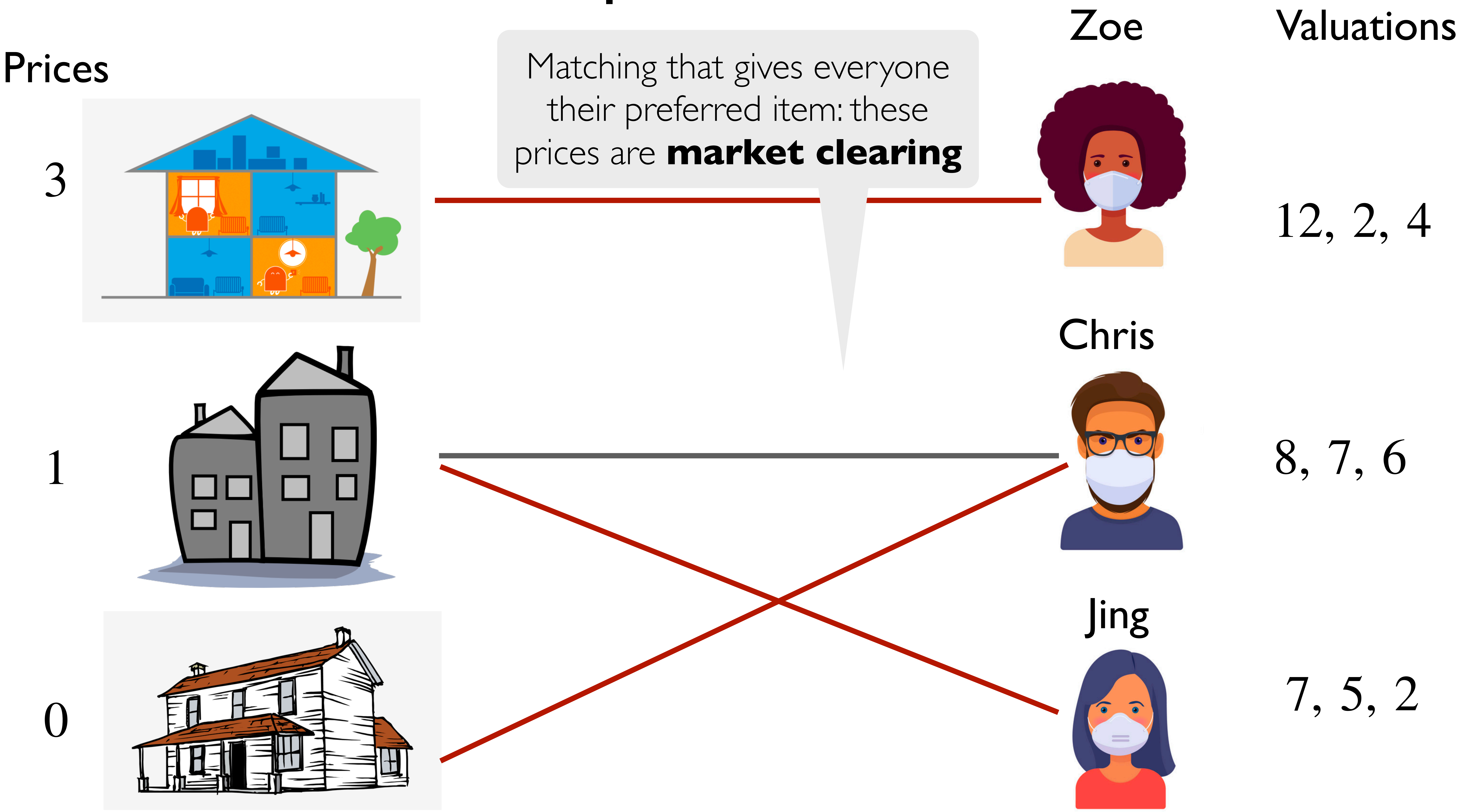


Towards a Proof

- Does this auction ever end?
    - If it ends, we know we have reached market clearing prices
- Can the prices keep rising forever?
    - At some point, the prices are too high for everyone
- **Proof idea**.
  - Maintain invariant: items with nonzero prices are always tentatively matched
  - Show that every buyer has at least one preferred item
  - Show that the price rising process must eventually come to an end by analyzing the "potential energy of the auction"

Analyzing Our Auction

- **Maintain invariant:** if an item has non-zero cost, that item is tentatively matched to some buyer: $p_j > 0 \implies \exists i : (j, i) \in M$
  - Initially M is empty and all prices are zero: invariant satisfied

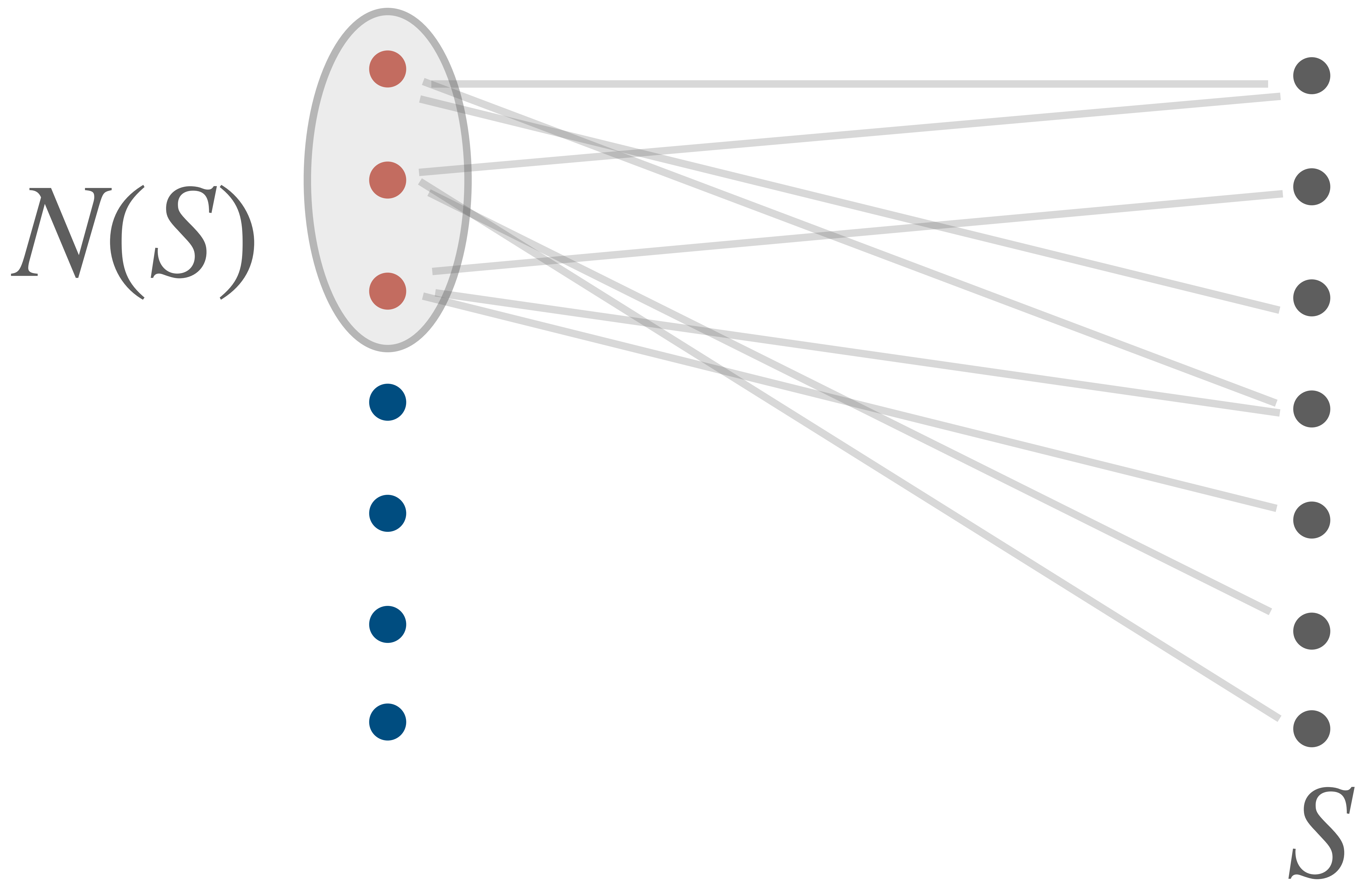
Analyzing Our Auction

- **Maintain invariant:** if an item has non-zero cost, that item is tentatively matched to some buyer: $p_j > 0 \implies \exists i : (j, i) \in M$
- Suppose until step t you have invariant maintained and we identify minimal constricted set $N(S)$ whose prices increase by 1 in this step
- At the new price, all edges between S to $N(S)$ still exist (buyers in S may have more edges to items outside that are now just as good)
- Tentatively match items in $N(S)$ to buyers in S (if these items were matched to other buyers, or buyers to other items, remove those edges from the matching)
  - Why is this matching possible?
  - We use Hall's theorem on items in $T = N(S)$

Why Such a Matching Exists

- Let $T = N(S)$ be the minimal constricted set at this step
    - That is no other constricted set is a subset of T
- Hall's theorem says we can match all items in $T = N(S)$ to some buyers in S, as long as there is no subset $T' \subseteq T$ such that
    - $|N(T')| < |T'|$
- We can show that such a subset cannot exist if T is a minimal constricted set
    - By contradiction, suppose such a subset exists
    - Can remove T' from T and $N(T')$ from S and end up with a constricted set that is a subset of T $\Rightarrow \Leftarrow$

Why Such a Matching Exists



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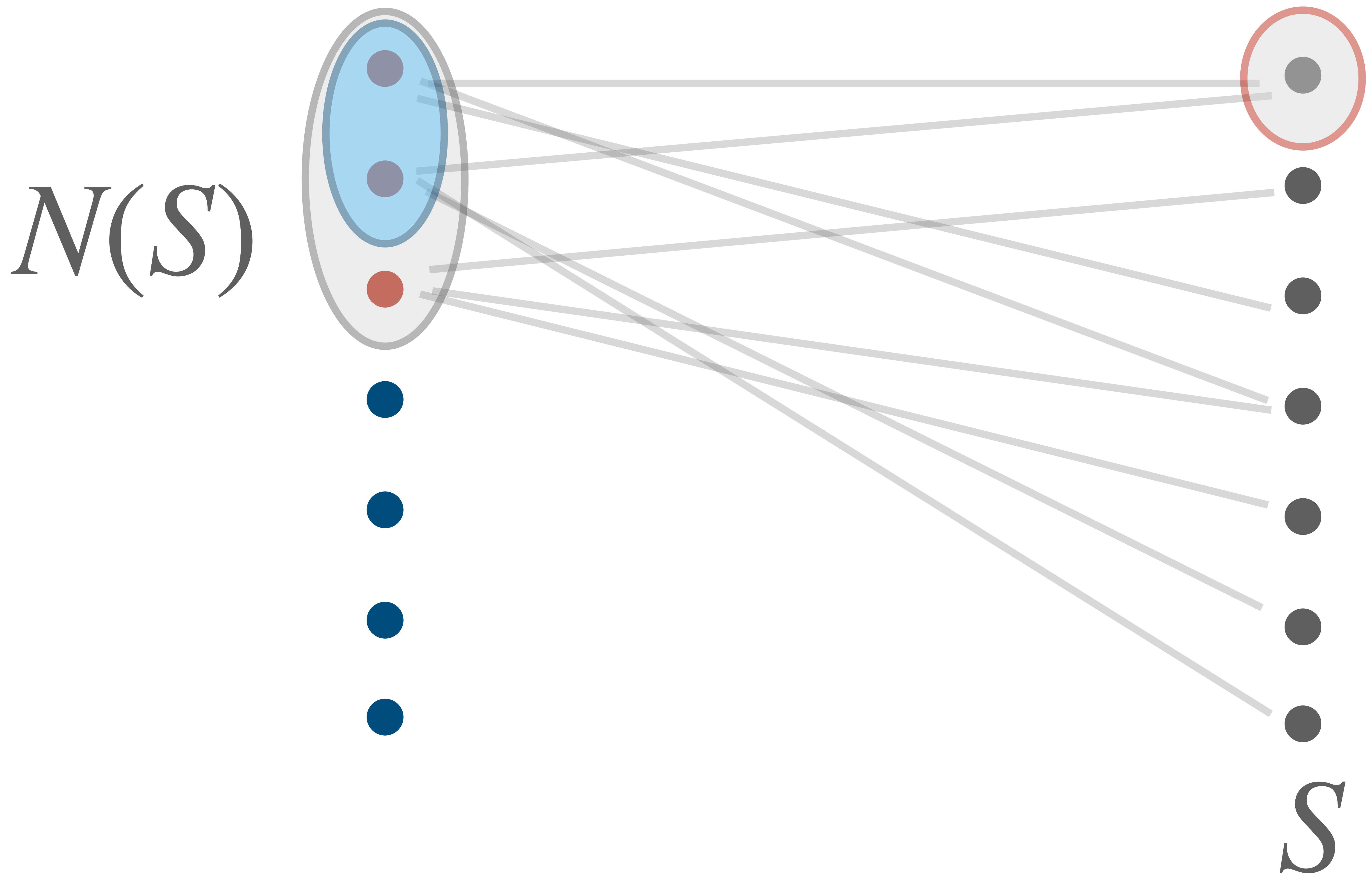
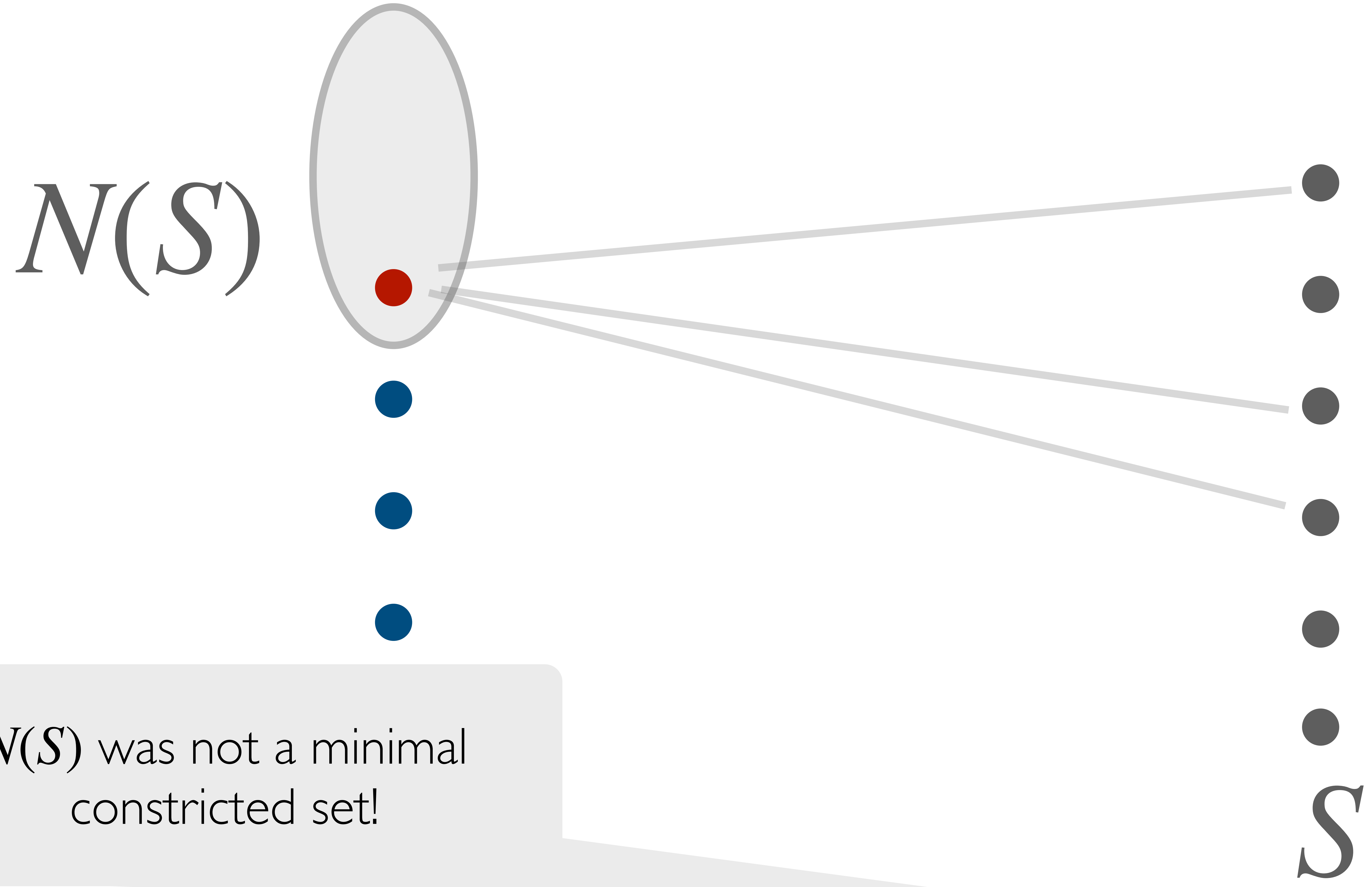


Why Such a Matching Exists



$N(S)$ was not a minimal constricted set!

Maintaining a Matching

- **Maintain invariant:** if an item has non-zero cost, that item is tentatively matched to some buyer: $p_j > 0 \implies \exists i : (j, i) \in M$
- Suppose until step t you have invariant maintained and we identify minimal constricted set $N(S)$ whose prices increase by 1 in this step
- Notice that at this price, all edges between S to $N(S)$ still exist (buyers in S may not have more edges to items outside of edge that are now just as good)
- Tentatively match items in $N(S)$ to buyers in S (if these items were matched to other buyers, or buyers to other items, drop those edges)
- Notice that items outside of $N(S)$ must still be tentatively matched to buyers outside of S (since all neighbors of S are in $N(S)$)
- Thus the invariant is maintained at time t as well $\blacksquare$