CSCI 361 Lecture 2: Countability and Automata

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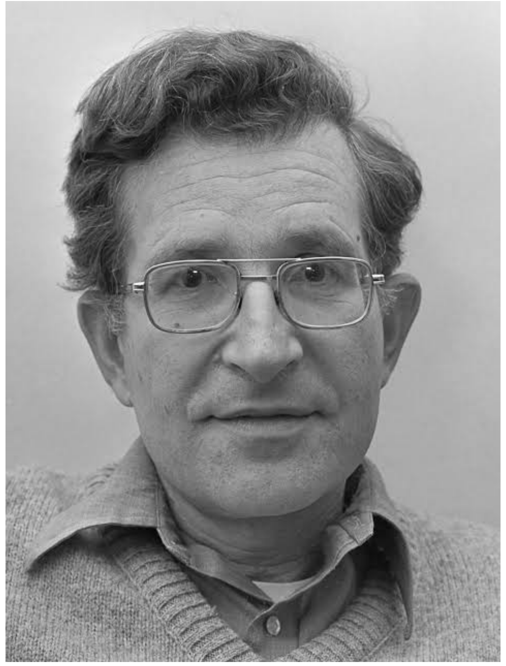
Announcements & Logistics

- **HW 1 released**: due Sept 17 (next Tuesday)
  - Office hours and TA hours posted on course calendar
- Hand in reading assignment #1
- Pick up reading assignment #2, due at the start of Thurs lecture
- Looking ahead:
  - I will be out next Thurs (Sept 18, at Tapia conference in San Diego)
  - Lecture will be held as usual, covered by another CS prof
- **Questions?**

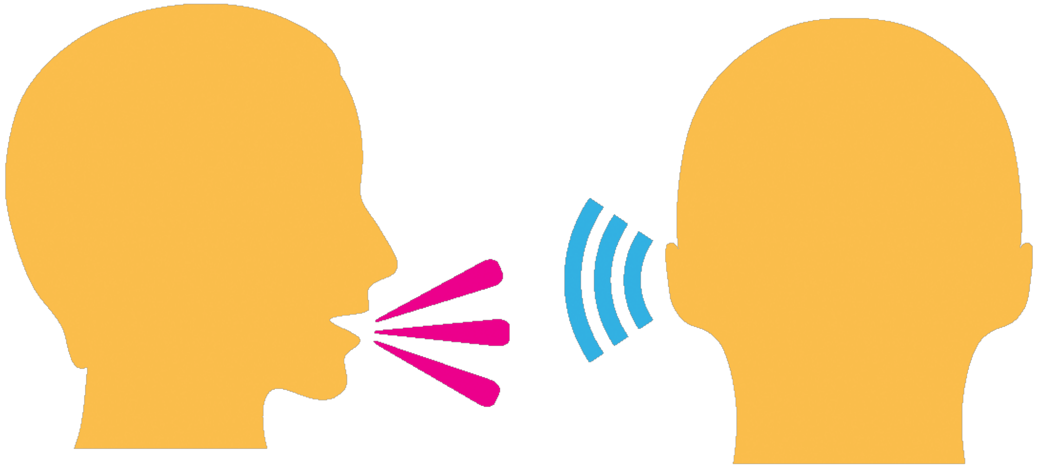
Last Time

- Introduced history and overview of theory of computation
- Discussed course logistics and reviewed syllabus
- Defined fundamentals of input/output representation
  - **Alphabet** Σ and set of all strings Σ
  - **Language**: any subset of strings from alphabet, i.e., $L \subseteq \Sigma^*$
  - **Length** of string s (# of symbols)
- All input/output in this course will be **binary strings**, that is, $\Sigma = \{0,1\}$
- Function problem vs decision problem:
  - A function problem is given by $f : \Sigma^* \to \Sigma^*$
  - A decision problem is given by $f : \Sigma^* \to \{0,1\}$

Influence of Chomsky



- We will study computation through the lens of languages
- Influence of linguist Chomsky on computation
- A **grammars** ***generates*** a language (akin to speaking)
  - Any string in the language can be generated using the rules of the grammar
- A **machine** ***recognizes*** a language (akin to listening)
  - If a given input string is in a language, the machine will "accept" (output true), otherwise "reject" (output false)



Background

Review Definitions

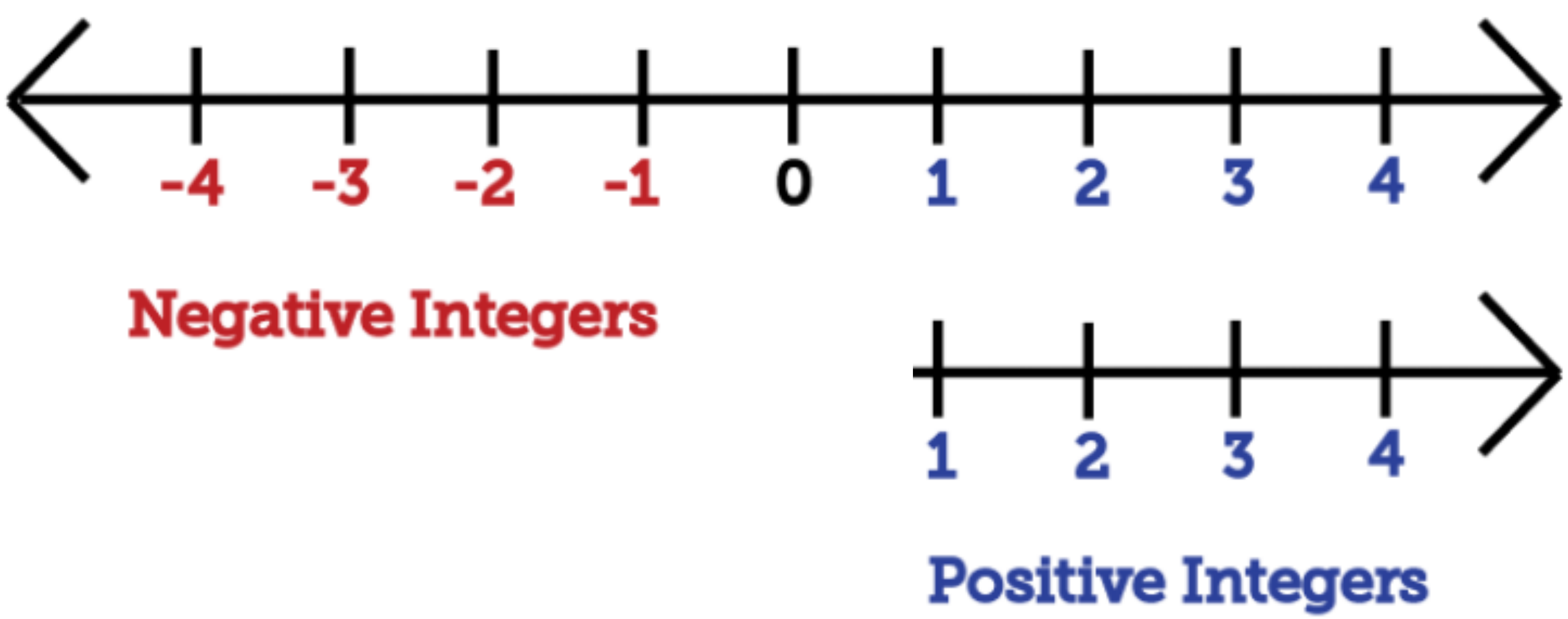
- **Sets:**
  - Subsets, empty set, equivalence of sets, power set
  - Union, intersection, complement, difference, etc
  - Cartesian product of sets: $A \times B = \{(a, b) \mid a \in A, b \in B\}$
- **Relations:** subset of Cartesian product of sets
  - Definition of a binary relation being reflexive, symmetric, transitive
  - Equivalence relation
- **Function:** mapping from an input to an output
  - $f : D \to R$ is a relation on $D \times R$ such that for all $x \in D$, there is exactly one $y \in R$ such that $(x, y) \in F$
  - $(x, y) \in f$ written as $f(x) = y$
  - Review definitions of one-to-one, onto and bijective functions

Cardinality

- Sets can be finite or infinite
  - S is **finite** if there is a bijection function $f : S \to \{1,2,\ldots,n\}$ for some $n \in \mathbb{N}$
  - Denote the size or cardinality as $|S| = n$
  - S is **infinite** if no such bijection exists
- **Question.** Are all ***infinite sets*** the same size?
- S is a **countably infinite set** if there is a bijection function $f : S \to \mathbb{N}$
- S is a **countable** if it is finite or countably infinite
- S is a **uncountable** if it is not countable

Proof by Construction Example

- **Theorem**: The set of integers $\mathbb{Z}$ is countable.
- Proof Outline:
  - Need to construct a bijective function f from $\mathbb{Z}$ to $\mathbb{N}$
  - Show that it is one-one and onto
  - Any ideas?



Proof by Construction Example

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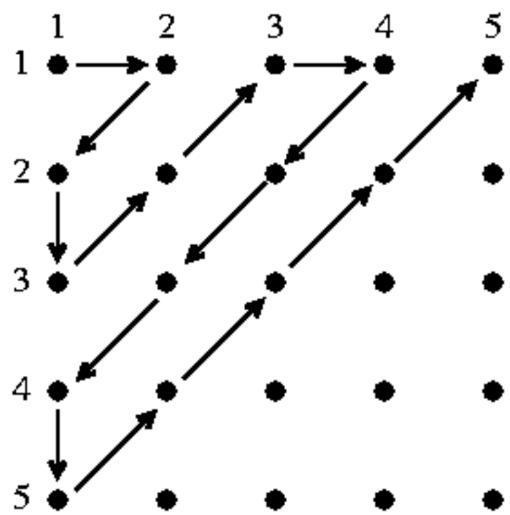
$$f(x) = \begin{cases} -2x & \text{if } x < 0 \\ 2x + 1 & \text{if } x \geq 0 \end{cases}$$

Countability

- Any set that can be "enumerated" using natural numbers is countable
  - Can talk about 1st item, 2nd item and so on
  - All items of the set appear in this on this list
- The set of all strings over a finite alphabet Σ is countable
  - Recall $\Sigma^* = \cup_{n \in \mathbb{N} \cup \{0\}} \Sigma^n$
  - Here Σ^n is a set of all strings of length exactly n over Σ
  - Each Σ^n is finite and contains $\ell = |\Sigma|^k$ elements, can list them in lexicographic order at indices $1, \ldots, \ell$
- **Theorem**: Σ^* is countable.

Countability

- **Theorem**: $\mathbb{N} \times \mathbb{N}$ is countable.



- **Theorem**: The countable union of countable sets is countable.
  - Proof by picture (on the board)

Uncountability: Proof by Contradiction

- **Theorem**: The power set $\mathscr{P}(\mathbb{N})$ of natural numbers is **not** countable.
- **Proof Outline:**
  - Assume that $\mathscr{P}(\mathbb{N})$ is countable: a bijection to $\mathbb{N}$ exists
  - That is, can enumerate all sets in $\mathscr{P}(\mathbb{N})$ as $S_1, S_2, \ldots, S_k, \ldots$
  - Reach an absurdity/contradiction $\Rightarrow\!\Leftarrow$
- Consider set of indices s.t.
  - (i is in corresponding S_i): $D = \{i \in \mathbb{N} \mid i \in S_i\}$
  - (i is not in corresponding S_i): $\overline{D} = \{i \in \mathbb{N} \mid i \notin S_i\}$
  - $\overline{D} = \mathbb{N} - D$, that is, $\overline{D}$ is a valid subset of $\mathbb{N}$: $\exists k$ such that $S_k = \overline{D}$
  - What is the contradiction?

Cantor's Diagnolizational Argument

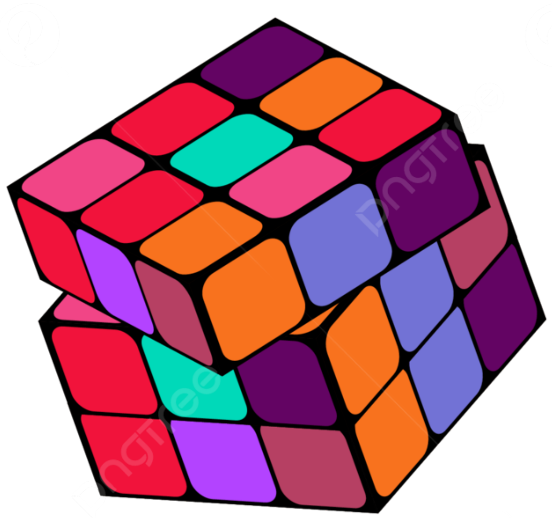
- This is an example of diagnolizational argument
- Visualize where S_k appears on the diagonal
- ***Fun soviet version***:
  - https://algorithmsoup.wordpress.com/2018/09/18/soviet-version-of-cantors-diagonalization-argument/

Countability and Languages

- Recall, we said that a set A is ***encodable*** if there is a one-to-one function $\text{Enc} : A \rightarrow \Sigma^*$ for some finite alphabet Σ
  - Thus, the size of any encodable set is *at most* the size of Σ^*
- As we can encode any CS program/Turing machine using a finite alphabet, the set of such programs/Turing machines is countable
- However, the power set $\mathscr{P}(\Sigma^*)$ is uncountable
  - Similar argument as to power set of natural numbers
  - That is, there are uncountably many languages over any alphabet
- **Takeaway**: can only encode countable things, but uncountably many "decidable problems" to solve $\Longrightarrow$ existence of undecidable problems
  - What do these problems look like?

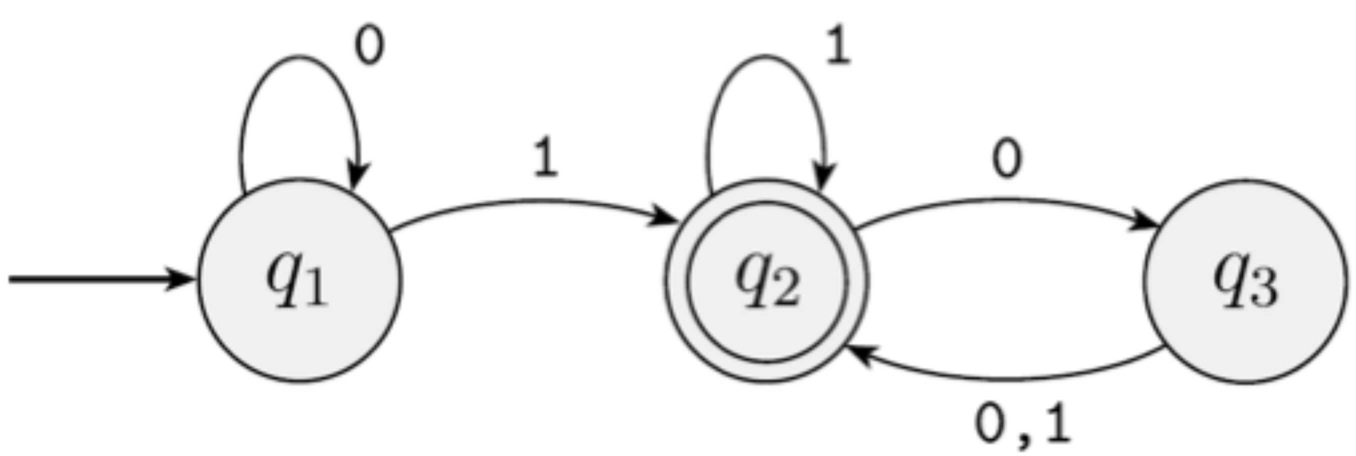
Finite State Automata

Simplest Form of Computation



Example: Deterministic Finite Automata

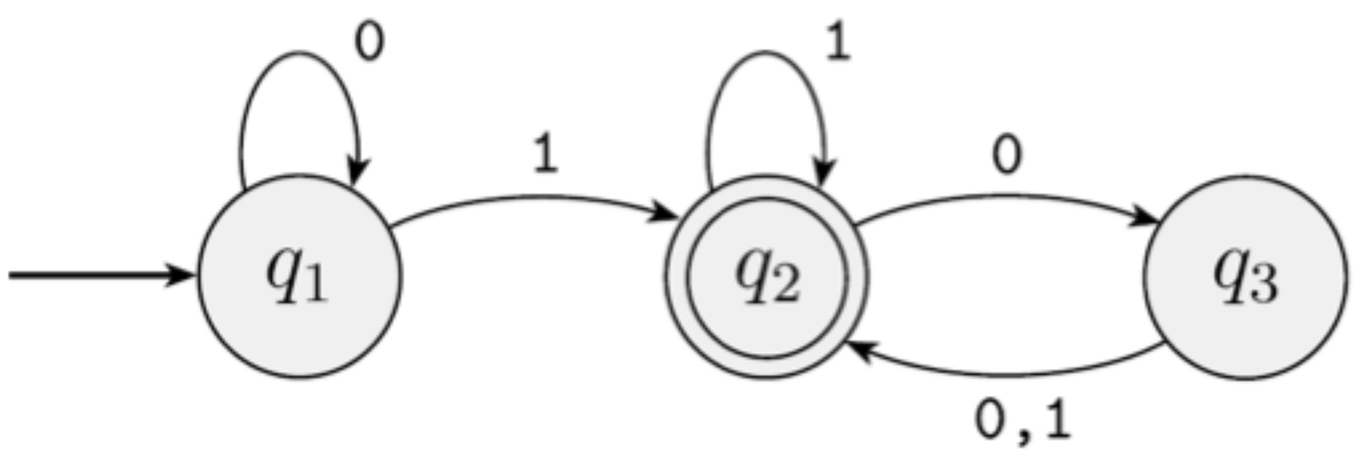
- We will study computation through the lens of languages
- A **machine** ***recognizes*** a language (akin to listening)
  - If a given input string is in a language, the machine will "accept" (output true), otherwise "reject" (output false)
- **Question.** What language is recognized by this machine?
  - Try some example strings



Definition of a Finite Automaton

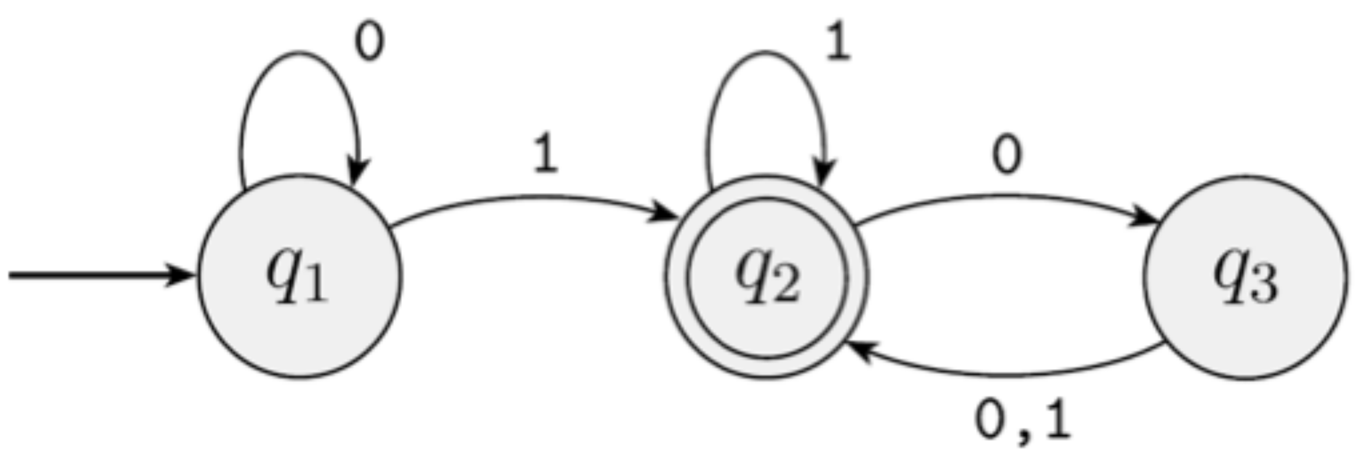
A finite automaton is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

- Q is a finite set called the states,
- Σ is a finite set called the alphabet,
- $\delta : Q \times \Sigma \rightarrow Q$ is the transition function,
- $q_o \in Q$ is the start state and $F \subseteq Q$ is the set of accept states.

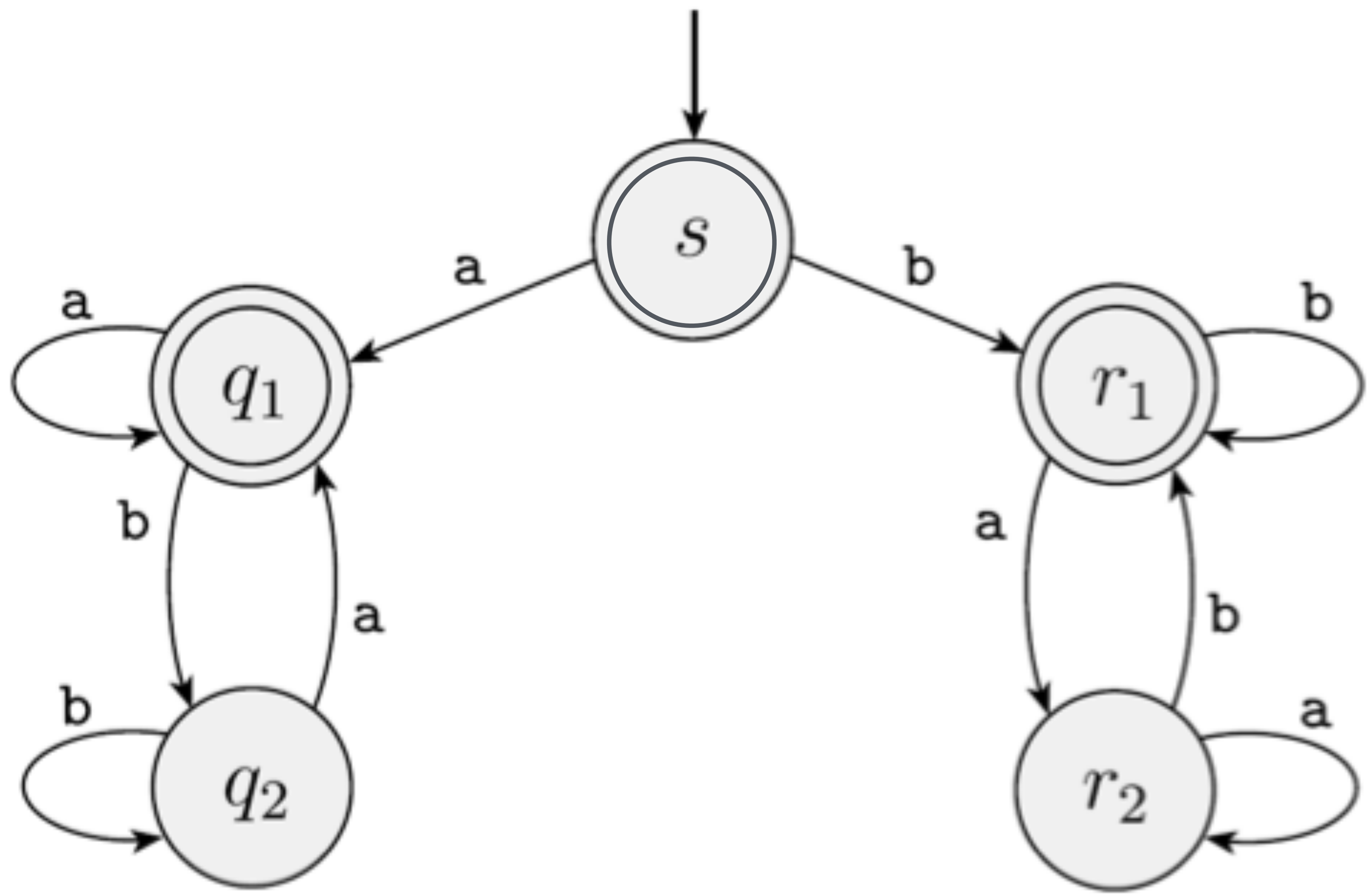


Language of a Machine

- The set of all strings accepted by a finite automaton M is called the language of machine M, and is written $L(M)$.
    - Say M **recognizes** language $L(M)$
- We will define M accepts w more formally
- Intuitive it is the strings on which it reaches an accepting state



What Language?



Automaton Computation

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be a finite automaton and let $w = w_1 w_2 \cdots w_n$ be a string where each $w_i \in \Sigma$. Then M ***accepts*** w if there is a sequence of $r_0, r_1, \ldots, r_n$ in Q such that
  - $r_0 = q_0$
  - $\delta(r_i, w_{i+1}) = r_{i+1}$ for $i = 0, 1, \ldots, n-1$ and
  - $r_n \in F$

Extended Transition Function

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA
- Transition function $\delta : Q \times \Sigma \to Q$ is often extended to $\delta^* : Q \times \Sigma^* \to Q$ where $\delta^*(q, w)$ is defined as the state the DFA ends up in if it starts at q and reads the string w
- Alternate definition of M accepts $w \iff \delta^*(q_0, w) \in F$

Regular Languages

- A language is called a regular language if some finite automaton recognizes it.
- Seen examples of the following regular languages today:
- $L(M_1) = \{w \mid w$ contains at least one 1 and an even number of zeroes follow the last $1\}$
- $L(M_2) = \{w \mid w \in \{a, b\}^*$ that starts and ends with the same symbol$\}$