MATH23A Ch.12 Notes

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1 Introduction

Welcome to Chapter 12. We will learn about Paths and Curves in space, which in turn allow us to model different situations in real life.

2 Paths and Curves

We usually think of a curve as line drawn on a paper, yet, it is useful to think of a curve C mathematically as a set of values of a function that maps an interval of real numbers into the plane or space. Thus, we shall call such a map a *path*. We usually denote a path by c. The image of C of the path then corresponds to the curve we see on paper. Often, we write t for the independent varaible and imagine it to be *time*, so that $c(t)$ is the position at time t of a moving particle, which *traces out* a curve as t varies. Additionally, we say *parametrizes* C. Thus, we should distinguish between $c(t)$ as a *point* in space and as a *vector* based at the origin.

A path in $\mathbb{R}^n$ is a map $c : [a, b] \to \mathbb{R}^n$; it is a *path in the plane* if n = 2 and a *path in space* if n = 3. The collection C of points $c(t)$ as t varies in [a, b] is called a *curve*, and c(a) and c(b) are its *endpoints*. The path c is said to *parametrize* the curve C. We also say c(t) *traces out* C as t varies.

If c is a path in $\mathbb{R}^3$, we can write $c(t) = (x(t), y(t), z(t))$, and we call $x(t), y(t),$ and $z(t)$ the *component functions* of c. We form components functions similarly in $\mathbb{R}^2$ or, generally, in $\mathbb{R}^n$. We also consider paths whose domain is the whole real line as in the next example.

The curve when the wheel rolls on a circle is called an *epicycle*. When the wheel is outside the circle and the point is on the rim, the curve is called an *epicycloid*. When the wheel is inside the circle, it is a *hypocycloid*.

Projections are a way to visualize space curves. The projection of a path $c(t) = (x(t), y(t), z(t))$ onto the xy-plane is $p(t) = (x(t), y(t), 0)$. Basically, set the variable that is not part of the plane to 0. That means the projections onto the yz- and xz-planes are the paths $(0, y(t), z(t))$ and $(x(t), 0, z(t))$, respectively.

To parametrize a curve obtained as the intersection of two surfaces,

1. Solve the given equations for y and z in terms of x. First, solve for y.
2. Substitute y into the other equation and solve for x.
3. Set x = t.
4. Then, the new vector(s) are the functions to parametrize the entire curve.

Alternatively,

1. Find the trigonometric parametrization of one of the equations.
2. Plug values into the second equation and solve for z.
3. Plug into c(t) = (x(t), y(t), z(t)).

Path c(t) approaches the limit u (a vector) as t approaches t_0 if $\lim_{t \to t_0} ||c(t) - u|| = 0$. In this case, we write

$$\lim_{t \to t_0} c(t) = u$$

That means a path $c(t) = (x(t), y(t), z(t))$ approaches a limit as $t \to t_0$ if and only if each component approaches a limit, and in this case,

$$\lim_{t \to t_0} c(t) = (\lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t))$$

Thus, c(t) is differentiable at t if the following limit exists.

$$c^{'}(t) = \frac{d}{dt} c(t) = \lim_{h \to 0} \frac{c(t+h) - c(t)}{h}$$

Basically, just find the derivative of each component within c(t) to differentiate it. Remember that the derivative is defined as

$$f^{'}(x) = \frac{d}{dx} f(x) = \frac{d}{dx}[x^n] = nx^{n-1}$$

Thinking of c(t) as the curve traced out by a particle and t as time, it is reasonable to define the velocity vector as follows. If c is a path and it is differentiable, we say c is a *differentiable path*. The velocity of c at time t is defined by

$$c^{'}(t) = \text{limt}_{h \to 0} \frac{c(t+h) - c(t)}{h}$$

1. The vector $c^{'}(t)$ is usually drawn with its tail at $c(t)$.
2. The *speed* of the path c(t) is $s = ||c^{'}(t)||$, which is the length of the velocity vector.
3. If $c(t) = (x(t), y(t))$ in $\mathbb{R}^2$, then
$$c^{'}(t) = (x^{'}(t), y^{'}(t)) = x^{'}(t)i + y^{'}(t)j$$
4. Similarly, if $c(t) = (x(t), y(t), z(t))$ in $\mathbb{R}^3$, then
$$c^{'}(t) = (x^{'}(t), y^{'}(t), z^{'}(t)) = x^{'}(t)i + y^{'}(t)j + z^{'}(t)k$$

The velocity $c^{'}(t)$ is a vector *tangent* to the path c(t) at time t. If C is a curve traced out by c and if $c^{'}(t)$ is not equal to 0, then $c^{'}(t)$ is a vector tangent to the curve C at the point c(t).

1. Find the derivative of the path to obtain the tangent vector in terms of t.
2. Place in the form $l(t) = c(t_0) + (t - t_0)(c^{'}(t_0))$ where
   (a) $c(t) =$ the path.
   (b) $t_0 =$ the value of t at a certain point.
   (c) $c^{'}(t) =$ the tangent vector.
3. Plug in t to obtain the equation of the tangent line l(t) to the path.

3 Acceleration

Same thing as before, except we now learn about more topics including acceleration.

A path in $\mathbb{R}^n$ is a map c of $\mathbb{R}$ in $\mathbb{R}$ to $\mathbb{R}^n$. The derivative at each time t is a vector with n components. Baiscally, the derivative of $c(t)$, $c^{'}(t)$ is given by

$$c^{'}(t) = (dx_1/dt_1, ..., dx_n/dt) \text{ or } c^{'}(t) = (x_1^{'}(t), ..., x_n^{'}(t))$$

Also recall that if c represents the path of a moving particle, the velocity vector is

$$v = c^{'}(t)$$

and its speed is $s = ||v||$.

Let b(t) and c(t) be differentiable paths in $\mathbb{R}^3$ and p(t) and q(t) be differentiable scalar functions. The following are the differentation rules.

- Sum Rule: $\frac{d}{dx}[b(t) + c(t)] = b^{'}(t) + c^{'}(t)$
- Scalar Multiplication Rule: $\frac{d}{dt}[p(t)c(t)] = p^{'}(t)c(t) + p(t)c^{'}(t)$
- Dot Product Rule: $\frac{d}{dt}[b(t) \cdot c(t)] = b^{'}(t) \cdot c(t) + b(t) \cdot c^{'}(t)$
- Cross Product Rule: $\frac{d}{dt}[b(t) \times c(t)] = b^{'}(t) \times c(t) + b(t) \times c^{'}(t)$
- Chain Rule: $\frac{d}{dt}[c(q(t))] = q^{'}(t)c^{'}(q(t))$

For a path describing uniform rectilinear motion, the velocity vector is constant. Generally, the velocity vector is a vector fuunction $v = c^{'}(t)$ that depends on t. The derivative $a = dv/dt = c^{''}(t)$, if it exists, is called the acceleration of the curve. If a curve is $(x(t), y(t), z(t))$, then the acceleration at time t is given by

$$a(t) = x^{''}(t)i + y^{''}(t)j + z^{''}(t)k$$

4 Arc Length

Sometimes we need to find the length of a path c(t). How do we do this? Well, simply, with represent to th etime over the interval $[t_0, t_1]$ of travel time, the length of the path, also known as *arc length*, in

$$L(c) = \int_{t_0}^{t_1} ||c^{'}(t)||dt.$$

Alternatively, the length of the path $c(t) = (x(t), y(t), z(t))$ for $t_0 \leq t \leq t_1$ is

$$L(c) = \int_{t_0}^{t_1} \sqrt{[x^{'}(t)]^2 + [y^{'}(t)]^2 + [z^{'}(t)]^2}$$

An *infinitesimal displacement* of a particle following a path $c(t) = x(t)i + y(t)j + z(t)k$ in

$$ds = dxi + dyj + dzk = (\frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k)dt$$

and its length

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}dt$$

To more easily remember the arc length formula, we can just remember it as

$$\text{arc length} = \int_{t_0}^{t_1} ds$$