MATH23A Ch.11 Notes

William Santosa

Winter 2022 Quarter

1 The Geometry of Euclidian Space

This document covers Chapter 11 in MATH 23A, University of California, Santa Cruz, online textbook from macmillianhighered.com. Here, we learn to build our mathematical models of space. This document will contain information on how to search for and represent these vectors and objects, as well as help us build a better understanding of how mathematicians see the world around us. A summary of terms from the textbook is within this document.

1.1 Vectors in Two and Three Dimensional Space

Points P in the plane are represented by ordered pairs of real numbers (a_1, a_2); the numbers a_1 and a_2 are the *Cartesian coordinates of P*. Drawing two perpendicular lines (x and y) as the axes, dropping perpendiculars from P to those axes, allows us to create two signed distances a_1 and a_2. a_1 is the **x component** of P and a_2 is the **y component** of P.

Similarly, points in space could be represented as an ordered triple with three axis. They are represented by the *x axis, y axis, and z axis*. The point where they intersect is called the origin.

1. The triple (0, 0, 0) is the origin of the coordinate system

2. We label axes x, y, and z by convention.

   (a) **Note:** This satisfies the right-hand rule, where your right hand is positioned so that fingers curl from the positive x-axis to the positive y-axis, and the thumb is the positive z-direction.

(b) Real number line is denoted as $\mathbb{R}^1$ or $\mathbb{R}$

(c) Set of all ordered pairs (x, y) of real numbers is denoted $\mathbb{R}^2$

(d) Set of all ordered triples (x, y, z) of real numbers is denoted $\mathbb{R}^3$

Addition is extended from $\mathbb{R}$ to $\mathbb{R}^2$ to $\mathbb{R}^3$

1. $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

2. The *zero element* of addition is (0, 0, 0)

3. The *additive inverse* (*negative*) of (a_1, a_2, a_3) is $(-a_1, -a_2, -a_3)$

Multiplication by a *scalar multiple* is done by multiplying each element of the tuple by the scalar

1. $s(a_1, a_2, a_3) = (sa_1, sa_2, sa_3)$

Vectors are directed line segments in a plane or space represented by directed line segments with a tail and head. When obtained from one another through parallel translation (not rotation) represent the same vector.

1. Written as $a = (a_1, a_2, a_3)$

2. To express a vector whose components are (a_1, a_2, a_3) in the standard basis:

$$(a_1 i, a_2 j, a_3 k)$$

Points P with (x, y, z) and Q with (a, b, c), vector $\overrightarrow{PQ} = (a - x, b - y, c - z)$. The equation of a line l through the tip of a and pointing in the direction of the vector v is $l(t) = a + tv$, where parameter t takes on all real values. In coordinate form, the equations are

$$x = x_1 + at,$$
$$y = y_1 + bt,$$
$$z = z_1 + ct,$$

As such, the equation of a line can be represented in point-direction form

$$l(t) = (x, y, z) + t(a, b, c) = (x + ta, y + tb, z + tc)$$

2

To find where two lines intersect (or if they do not intersect), set the lines equal to one another and solve for t_1 and t_2 Solve and set equal to one another. Check if true, if they do, they intersect.

$$(t_1, -6t_1 + 1, 2t_1 - 8) = (3t_2 + 1, 2t_2, 0)$$

$$t_1 = 3t_2 + 1$$

$$-6t_1 + 1 = 2t_2$$

$$2t_1 - 8 = 0$$

The equation of a line passing through the endpoints of two given vectors, **a** and **b**, is

$$l(t) = a + t(b - a) = (1 - t)a + tb$$

The Point-Point Form of a line through points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ are

$$x = x_1 + (x_2 - x_1)t$$

$$y = y_1 + (y_2 - y_1)t$$

$$z = z_1 + (z_2 - z_1)t$$

where (x,y,z) is the general point of l, and the parameter t takes on all real values.

1.2   The Inner/Dot Product, Length, and Distance

The inner/dot product, length, and distance are all important concepts in order to understand and manipulate vectors, as they allow us to gain better understanding of these difficult concepts and make it easier to use them to our advantage.

The dot product, defined as $a \cdot b$ by two vectors a and b, is the real number $a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$. It is also represented as $< a, b >$.

The *length* or *modulus* of a vector $a = a_1 i + a_2 j + a_3 k$ is $\sqrt[2]{a_1^2 + a_2^2 + a_3^2}$. The length of a vector a is denoted as $||a||$.

Vectors with norm 1 are *unit vectors*. For any nonzero vector a, $\frac{a}{||a||}$ is a unit vector; when we divide a by $||a||$, we say that we have *normalized* a.

The *distance between* the endpoints of a and b is $||a - b||$, and the distance between P and Q is $||\overrightarrow{PQ}||$

The measure the angle between two vectors a and b, solve the equation

$$a \cdot b = ||a||||b||cos(\theta)$$

Equivalently, you can manipulate the equation to get

$$\theta = cos^{-1}(\frac{a \cdot b}{||a||||b||})$$

The *Corollary Cauchy-Schwarz Inequality* states that for any two vectors a and b, we have

$$|a \cdot b| \leq ||a||||b||$$

where they're equaivalent only if a is a scalar multiple of b or one of them is zero.

- **Note:** If a and b are nonzero vectors and $a \cdot b = 0$, then we know $cos(\theta) = 0$, and that the two vectors are perpendicular. Thus, they are known as *orthogonal.*

- The triangle inequality, a derivative of the Cauchy-Schwarz Inequallity, states that $||a + b|| \leq ||a|| + ||b||$

The *orthogonal projection* of v on a is the vector

$$p = \frac{a \cdot v}{||a||^2}a$$

1.3   Matrices, Determinants, and the Cross Product

The dot product is a product of vectors that results in a scalar. The cross product, instead, gives a product of vectors that is a vector. Thus, we will show how to produce a third vector $a \times b$ given vectors a and b.

Matrices are arrays with $r \times c$, where r is the amount of rows and c is the amount of columns. A 2×2 matrix is the array

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The *determinant* is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

A 3×3 matrix is a little bit different. You can search that up online or go to this youtube video.

The cross product of two vectors, $a = a_1 i + a_2 j + a_3 k$ and $b = b_1 i + b_2 j + b_3 k$ is

$$a \cdot b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

1. Similarly, $||a \times b|| = ||a|| ||b|| sin(\theta)$

2. $a \times b$ is perpendicular to a and b, and the triple $(a, b, a \times b)$ obeys the right hand rule.

To find the area of a parallelogram spanned by two vectors a and b, find the magnitude of the cross product of a and b, $||a \times b||$
IMPORTANT NOTE: If a and b are perpendicular, then $a \cdot b = 0$. If a and b are parallel, then $a \times b = 0$.

The equation of a plane, P, through (x_0, y_0, z_0) with a normal vector $n = Ai + Bj + Ck$ is $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$. Therefore, $(x, y, z) \in P$ iff $Ax + By + Cz + D = 0$.

1. To find the equation of the plane containing the points P, Q, and R,

   (a) Select any endpoint (we use P), and find $\overrightarrow{QP}$ and $\overrightarrow{RP}$.

   (b) Find the cross product of $\overrightarrow{QP}$ and $\overrightarrow{RP}$, $\overrightarrow{QP} \times \overrightarrow{RP}$.

   (c) Replace i, j, and k with x, y, and z and subtract the endpoint P from the resulting equation, set equal to zero.

2. The distance from a point (x_1, y_1, z_1) to a plane $Ax + By + Cz + D = 0$ is equal to

$$\text{Distance} = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt[2]{A^2 + B^2 + C^2}}$$

The intersection of a plane P with a coordinate plane or plane parallel to a coordinate plane is called a *trace*. The trace is a line unless P is a parallel to the coordinate plane.

1. To find the trace in the xy-plane, set z $= 0$ in the plane's equation. Similarly, the trace in the xz-plane is obtained setting y $= 0$ in the equation.