

數學分析 · MAT2050

Mathematical Analysis · MAT2050

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Abstract

This is the *lecture notes* for the *course MAT2050: course Mathematical Analysis*. Nearly all content is based on the course materials of Professor Fengyi Yuan at The Chinese University of Hong Kong, Shenzhen (CUHK (SZ))[1].

We primarily reference materials from course MAT2050, including *lecture notes*, *homework (HW)*, and *exam papers* by Professor Fengyi Yuan, as well as *tutorial slides* from the course id teaching faculty[1].

這是課程 **MAT2050: 數學分析 (Mathematical Analysis)** 的課程講義與整理歸納。幾乎所有內容都是基於香港中文大學（深圳）Fengyi Yuan 教授的課程材料 [1]。

我們主要參考了 CSC3001 的材料，包括 Fengyi Yuan 教授的講義、作業和試卷，以及 CSC3001 教學團隊的輔導幻燈片。[1]。

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MAT2050 before Midterm

1 Sup and Inf

Definition

Definition of Infimum

Let $A \subseteq \mathbb{R}$ be a nonempty set that is **bounded below**. A real number $\inf A$ is the **infimum** (greatest lower bound) of A if and only if:

1. $\inf A$ is a lower bound for A (i.e., $a \geq \inf A$ for all $a \in A$);
2. For **every** $\varepsilon > 0$, there exists $a_\varepsilon \in A$ such that $a_\varepsilon < \inf A + \varepsilon$.

Definition of Supremum

Let $A \subseteq \mathbb{R}$ be a nonempty set that is **bounded above**. A real number $\sup A$ is the **supremum** (least upper bound) of A if and only if:

1. $\sup A$ is an upper bound for A (i.e., $a \leq \sup A$ for all $a \in A$);
2. For **every** $\varepsilon > 0$, there exists $a_\varepsilon \in A$ such that $a_\varepsilon > \sup A - \varepsilon$.

For example:

Example

Let $A = (1, 2]$. Then $\inf A = 1$ and $\sup A = 2$.

Fact and proposition

Since $\sup A$ is the smallest upper bound. Thus, there must exist $a_i \in A$ such that $\sup A - \epsilon < a_i \leq \sup A$.

If $a_i = \sup A + \epsilon$ (where $\epsilon > 0$), then $\sup A + \epsilon > \sup A$, contradicting the fact that $\sup A$ is an upper bound. Thus, such an a_i cannot exist for a valid $\sup A$

Exercise

Question: To prove $\sup A$ exists and $\sup A = \sqrt{2}$ for $A = \{x \in \mathbb{R} : x^2 < 2\}$:

证明. proof:

□

i Exercise

Question: Let $A, B \subset \mathbb{R}$ be nonempty and bounded. Prove: $\sup(A \cup B) = \max\{\sup A, \sup B\}$ and $\inf(A \cup B) = \min\{\inf A, \inf B\}$.

证明. proof:

□

2 Density in R

Property

(Archimedean Property): If $x, y \in \mathbb{R}$ and $x > 0$, then $\exists n \in \mathbb{N}$ such that $nx > y$.

证明. proof:

□

so we yield the corollaries below:

Corollary & Secondary Conclusion

C1: For all $y \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $n > y$.

C2: For all $a \in \mathbb{R}$, there exists a **unique** $m \in \mathbb{Z}$ such that $m \leq a < m + 1$. We denote $m = \lfloor a \rfloor$ (the integer part of a).

证明. proof:

□

From Claim2, we can yield :

Property

(Density of \mathbb{Q}) For all $a, b \in \mathbb{R}$ with $a < b$, $\exists p \in \mathbb{Q}$ such that $a < p < b$.

证明. proof:

□

3 Limit, monotone convergence theorem

we will show 4 important rules about limit:

Theorem

- **Addition:** If $\lim u_n = u$ and $\lim v_n = v$, then $\lim(u_n + v_n) = u + v$.
- **Multiplication:** If $\lim u_n = u$ and $\lim v_n = v$, then $\lim(u_n v_n) = (\lim u_n)(\lim v_n)$.
- **Reciprocal:** If $\lim u_n = u$ and $u \neq 0$, then $\lim \frac{1}{u_n} = \frac{1}{\lim u_n}$.

证明. proof:

□

Theorem

If (u_n) converges, then it is bounded.

证明. proof:

□

【Remark】:

If (u_n) is bounded, then it may not converges. Example: $u_n = (-1)^n$

And now we arrive the last Theorem in this lecture:

Theorem

Monotone Convergence Theorem: If (u_n) is **increasing and bounded from above (so bounded)**, then

$$\lim_{n \rightarrow \infty} u_n = \sup\{u_n : n \in \mathbb{N}\}.$$

If (u_n) is **decreasing and bounded from below**, then

$$\lim_{n \rightarrow \infty} u_n = \inf\{u_n : n \in \mathbb{N}\}.$$

证明. proof:

□

【Remark】:

A monotone sequence converges *iff* it is bounded

i Exercise

Question: Given $a_1 \in \mathbb{R}$, define

$$a_{n+1} = \frac{a_n + 4}{5} \quad (n \geq 1).$$

Show that (a_n) converges and find its limit (hint: solve $L = \frac{L+4}{5}$ and prove $|a_{n+1} - L| = \frac{1}{5}|a_n - L|$).

证明. proof:

□

Order preserving:

Lemma

If for two sequences (u_n) and (v_n) we have $u_n \leq v_n$ for all n , then

$$\lim_{n \rightarrow \infty} u_n \leq \lim_{n \rightarrow \infty} v_n$$

whenever both limits exist.

we will argue by contradiction

证明. proof:

□

Theorem

(Nested Interval Theorem). Given closed bounded intervals $I_1 \supset I_2 \supset I_3 \supset \dots$, $I_n = [a_n, b_n]$, then

$$\bigcap_{k=1}^{\infty} I_k \neq \emptyset.$$

证明. proof:

□

4 Liminf and Limsup

Recall: A sequence (u_n) has $\lim_{n \rightarrow \infty} u_n = u^*$ if

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall n \geq N_\varepsilon, |u_n - u^*| < \varepsilon.$$

We generalize this to discuss "limits" for *any* sequence (as not all sequences converge in the usual sense).

There are three types of divergence: divergence to $+\infty$, divergence to $-\infty$, or fluctuation/oscillation (no single limit, but bounded variation). For oscillating sequences, we use "*upper/lower bounds*": choose $A \leq B$ and require $A - \varepsilon < u_n < B + \varepsilon$ for large n , with A, B chosen "optimally".

Definition

For any sequence (u_n) :

$$\liminf_{n \rightarrow \infty} u_n = \sup_{k \geq 1} \inf_{n \geq k} u_n, \quad \limsup_{n \rightarrow \infty} u_n = \inf_{k \geq 1} \sup_{n \geq k} u_n.$$

- Inner inf / sup control lower/upper bounds for u_n when n is sufficiently large.
- Outer sup / inf select the "optimal" A, B (tightest bounds).
- \limsup and \liminf always exist (may be $\pm\infty$) by supremum / infimum axioms.

Example

- Let $u_n = (-1)^n$. For any n , $\{u_k : k \geq n\} = \{-1, 1\}$, so $\sup\{u_k : k \geq n\} = 1$ and $\inf\{u_k : k \geq n\} = -1$. Thus $\limsup_{n \rightarrow \infty} u_n = 1$, $\liminf_{n \rightarrow \infty} u_n = -1$.
- Let $u_n = n$. Then $\limsup_{n \rightarrow \infty} u_n = \liminf_{n \rightarrow \infty} u_n = +\infty$.

we can see that:

$\limsup u_n$ is the *largest* limit of any convergent subsequence; $\liminf u_n$ is the *smallest* limit of any convergent subsequence.

Now let us check Liminf/Limsup for Bounded Sequences:

Let (u_n) be bounded. Define:

Definition

$$a_n := \inf_{k \geq n} u_k$$

$$b_n := \sup_{k \geq n} u_k.$$

Then we have these **propositions**:

Fact and proposition

- $a_{n+1} \geq a_n$ (so (a_n) is *increasing*),
- $b_{n+1} \leq b_n$ (so (b_n) is *decreasing*),
- Thus (a_n) and (b_n) converge, with $\liminf_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} a_n$ and $\limsup_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} b_n$.

证明. proof:

□

So we can have another version of the definition of limsup(or liminf):

Definition

For a sequence (a_n) , $\limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (\sup_{k \geq n} a_k) = \sup_{k \geq 1} \inf_{n \geq k} u_n$, $\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (\inf_{k \geq n} a_k) = \inf_{k \geq 1} \sup_{n \geq k} u_n$.

Now let's check the *Characterization of liminf / limsup* in bounded sequences

Property

Let $A = \liminf_{n \rightarrow \infty} u_n$ and $B = \limsup_{n \rightarrow \infty} u_n$. Then:

1. $A \leq B$.
2. $\forall \varepsilon > 0$, $\exists N_1$ such that $\forall n \geq N_1$, $u_n > A - \varepsilon$.
3. $\forall \varepsilon > 0$, $\exists N_2$ such that $\forall n \geq N_2$, $u_n < B + \varepsilon$.
4. If $A = B$, then (u_n) converges (to $A = B$).

证明. proof:

□

Compared to the *proposition of inf and sup*, we can find the **difference** that:

In the proposition of inf and sup instead of liminf or limsup, we notice that:

- For every $\varepsilon > 0$, there exists $a_\varepsilon \in A$ such that $a_\varepsilon > \sup A - \varepsilon$;
- For every $\varepsilon > 0$, there exists $a_\varepsilon \in A$ such that $a_\varepsilon < \inf A + \varepsilon$.

The idea of "Constant": try to Construct a constant with respect to k :

Fact and proposition

$$\sup_{k \geq n} (x_k + y_k) \leq \sup_{k \geq n} x_k + \sup_{k \geq n} y_k.$$

So we can have these properties:

Property

Consider two sequences (x_n) and (y_n)

$$\limsup_{n \rightarrow \infty} (x_n + y_n) \leq \limsup_{n \rightarrow \infty} x_n + \limsup_{n \rightarrow \infty} y_n$$

$$\liminf_{n \rightarrow \infty} (x_n + y_n) \geq \liminf_{n \rightarrow \infty} x_n + \liminf_{n \rightarrow \infty} y_n$$

Hint: Think about the definition of the limsup and liminf to construct these structure.

证明. proof:

□

i Exercise

(limit and limsup preserve the order): If there exists N_1 such that $u_n \leq B'$ for all $n \geq N_1$, then $\limsup u_n \leq B'$. If there exists N_2 such that $u_n \geq A'$ for all $n \geq N_2$, then $\liminf u_n \geq A'$.

i Exercise

Question: Consider two sequences (x_n) and (y_n) . Show that, if $\lim_{n \rightarrow \infty} y_n = y$, then

$$\limsup_{n \rightarrow \infty} (x_n + y_n) = \limsup_{n \rightarrow \infty} x_n + y.$$

Subsequences

Definition

For a sequence (u_n) , a *subsequence* is a sequence (u_{k_n}) where $k_1 < k_2 < \dots$ (indices are strictly increasing).

Easy to get that: If $n_1 < n_2 < \dots$ are positive integers, then $n_k \geq k$ for all k .

Lemma

(Convergent Subsequences from Liminf/Limsup): Let (u_n) be bounded, with $A = \liminf_{n \rightarrow \infty} u_n$ and $B = \limsup_{n \rightarrow \infty} u_n$. Then:

- There exist subsequences $(u_{k_j}) \rightarrow A$ and $(u_{p_j}) \rightarrow B$.
- For each $j \in \mathbb{N}$, these subsequences satisfy:

$$A - \frac{1}{j} < u_{k_j} < A + \frac{1}{j}, \quad B - \frac{1}{j} < u_{p_j} < B + \frac{1}{j},$$

with $k_1 < k_2 < \dots$ and $p_1 < p_2 < \dots$

【Remark】:

To solve this problem, we must completely understand the concept of limsup and liminf and their two forms.

证明. proof:

□

Now we can get our Main Theorem:

Theorem

Bolzano–Weierstrass Theorem: For any bounded numerical sequence (u_n) , there exists a convergent subsequence.

证明. proof:

□

Exercise

Question: Consider a bounded sequence (x_n) and let $A = \liminf_{n \rightarrow \infty} x_n$, $B = \limsup_{n \rightarrow \infty} x_n$. Show that, if (x_{k_n}) is a convergent subsequence of (x_n) , then

$$A \leq \lim_{n \rightarrow \infty} x_{k_n} \leq B.$$

5 Cauchy Sequence

Definition

(Cauchy sequence). (u_n) is said to be a Cauchy sequence if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall m, n \geq N, |u_m - u_n| < \varepsilon.$$

Lemma

A Cauchy sequence is bounded.

证明. proof:

□

Theorem

The sequence (u_n) converges, **iff** it is Cauchy.

【Remark】:

For the converse, there are two ways to prove it!

证明. proof 1:Bolzano-Weierstrass:

proof 2:Constant thinking, fixed technique:

□

Iteration for $\sqrt{2}$

Consider $g(t) = \frac{t}{2} + \frac{1}{t}$. This gives a sequence: $x_1 = 1$, $x_{n+1} = g(x_n)$. Clearly $x_n \in \mathbb{Q}$ for all n .

Claim

(x_n) is Cauchy; the limit is $\sqrt{2}$.

证明. proof:

□

6 Convergence of Series

Today's topic: series — infinite summation, a special type of sequence. In other words, we compute

$$\sum_{n=1}^{\infty} a_n \equiv a_1 + a_2 + a_3 + \cdots$$

Two possible scenarios for (*):

1. We can assign a real value to $(*) \Rightarrow$ **convergence**.
2. We cannot assign a real value to $(*)$; sometimes we set it to $\pm\infty$. But in other cases, we cannot meaningfully define any value to $(*) \Rightarrow$ **divergence**.

Consider a numerical sequence (a_n) .

Definition

Consider the **partial-sum sequence**

$$S_n := \sum_{k=1}^n a_k.$$

(S_n) is called the *partial sums* of the series $\sum_{k=1}^{\infty} a_k$.

If $S_n \rightarrow S$ (as $n \rightarrow \infty$), the series is **convergent**, and we write $S = \sum_{k=1}^{\infty} a_k$. If S_n does not converge, the series is **divergent**.

Property

Cauchy's Criterion. The series $\sum_{n=1}^{\infty} a_n$ converges **if and only if**

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall m > n > N, \left| \sum_{k=n+1}^m a_k \right| < \varepsilon.$$

Proof. A series $\sum a_n$ converges \Leftrightarrow its partial-sum sequence (S_n) converges $\Leftrightarrow (S_n)$ is a Cauchy sequence. But for $m < n$,

$$|S_n - S_m| = \left| \sum_{k=m+1}^n a_k \right|.$$

Property

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \rightarrow 0$ (as $n \rightarrow \infty$).

Importantly, if a_n does not converge, or converges to a limit other than 0, then $\sum a_n$ diverges.

Exercise

Prove Harmonic series: $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges (even though $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$).

证明. proof:

□

Definition

$\sum a_n$ is said to be **absolutely convergent** if $\sum |a_n|$ converges.

If $\sum_{n=1}^{\infty} |a_n|$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

Absolute convergence usually induces better properties (rearrangement: next lecture).

Test

Method

(Comparison Test). Consider two numerical series $\sum a_n$ and $\sum b_n$ with $0 \leq a_n \leq b_n$.

Then:

1. If $\sum b_n$ converges, so does $\sum a_n$.
2. If $\sum a_n$ diverges, so does $\sum b_n$.

Proof. Let (S_n) and (T_n) be sequences of partial sums. Then $\forall n, 0 \leq S_n \leq T_n$. Thus, if (T_n) is bounded, (S_n) is bounded; if (S_n) is unbounded, (T_n) is unbounded. \square

Method

(Root Test). Consider (a_n) and let $\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$.

1. If $0 \leq \alpha < 1$, then $\sum |a_n|$ converges (hence $\sum a_n$ converges absolutely).
2. If $\alpha > 1$, then $a_n \not\rightarrow 0$ and $\sum a_n$ diverges.

Proof. (1) If $0 \leq \alpha < 1$, choose $\alpha < b < 1$. By lim sup characterization, $\exists N$ such that $\forall n \geq N, \sqrt[n]{|a_n|} < b < 1$. Thus $|a_n| \leq b^n$, and $\sum a_n$ converges (by comparison with $\sum b^n$, a convergent geometric series).

(2) If $\alpha > 1$, lim sup implies a subsequence a_{n_k} with $\sqrt[n_k]{|a_{n_k}|} > 1$, so $|a_{n_k}| \geq 1$ for large k . Thus $a_n \not\rightarrow 0$. \square

【Remark】:

The test is useless when $\alpha = 1$.

Method

(Ratio Test). Consider (a_n) and let $\alpha = \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$.

If $0 \leq \alpha < 1$, then $\sum |a_n|$ converges (so $\sum a_n$ converges absolutely).

Proof. If $\alpha < 1$, choose $\alpha < r < 1$ and let $\varepsilon = \frac{r-\alpha}{2}$. By lim sup characterization, $\exists N$

such that $\forall n \geq N$,

$$\left| \frac{a_{n+1}}{a_n} \right| < \alpha + \varepsilon < r.$$

By iteration, for $n \geq N$:

$$|a_{n+1}| < r|a_n| < r^2|a_{n-1}| < \cdots < r^{n-N+1}|a_N| \leq Cr^n$$

where $C = r^{-N+1}|a_N|$. Thus $\sum |a_n| \leq \sum_{n < N} |a_n| + C \sum_{n \geq N} r^n < \infty$ (since $\sum r^n$ converges). \square

【Remark】:

The test is also useless when $\alpha = 1$.

Then, we will introduce a special series: Power series.

Definition

For a numerical sequence (c_n) and real number z , the series

$$\sum_{n=0}^{\infty} c_n z^n = c_0 + c_1 z + c_2 z^2 + \cdots$$

is a **power series** in z .

Example

Example: $\sum_{n \geq 1} x^n$ is a power series and converges if $|x| < 1$.

Definition

Let $\ell = \limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|}$ and define

$$R = \begin{cases} \frac{1}{\ell}, & \ell > 0, \\ +\infty, & \ell = 0, \\ 0, & \ell = +\infty. \end{cases}$$

Then $\sum c_n z^n$ converges absolutely for $|z| < R$ and diverges for $|z| \geq R$.

And we denote R as the **radius of convergence**.

Proof. Let $a_n = c_n z^n$. Then

$$\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|} |z| = \ell |z|.$$

The conclusion follows from the Root Test. \square

Example

Example 1 (Geometric series). $\sum_{n=1}^{\infty} z^n$ (where $c_n \equiv 1$) has radius of convergence $R = 1$.

Example 2 (Exponential series). $\sum_{n=1}^{\infty} \frac{z^n}{n!}$ has radius of convergence $R = +\infty$.

From the example above, we have:

Definition

(Exponential). We define

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad (\text{note we start from } n = 0).$$

Lemma

Let $S_n = \sum_{k=0}^n \frac{1}{k!}$. Then $\forall n \geq 1$, then:

For all $n \geq 1$,

$$0 \leq e - S_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} \leq \frac{1}{n \cdot n!}.$$

证明. proof:

□

And using the lemma above, we can gain:

Corollary & Secondary Conclusion

e is irrational.

证明. proof:

□

Theorem

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

证明. proof:

□

And then we will discuss a new topic: Abel' spartial summation.

Theorem

Consider two numerical sequences $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$. Let $A_n = \sum_{k=0}^n a_k$. For integers $p \leq q$,

$$\sum_{n=p}^q a_n b_n = \sum_{n=p}^q A_n (b_n - b_{n+1}) + A_q b_{q+1} - A_{p-1} b_p. \quad (1)$$

use the Table below for further understanding:

Discrete Scenario (Series)	Continuous Scenario (Integral)	Core Role
Sum of product of sequences $\sum a_n b_n$	Integral of product of functions $\int a(x)b(x)dx$	“Product-type object” to be handled
Partial sum $A_n = \sum_{k=0}^n a_k$	Primitive function $A(x) = \int a(t)dt$	Intermediate object from “accumulation/integration”
Difference $b_n - b_{n+1}$	Differential $db(x) = b'(x)dx$	“Discrete differentiation/differential” of b
Boundary term $A_q b_{q+1} - A_{p-1} b_p$	Boundary value $A(x)b(x) _a^b$	“Endpoint term” left after splitting

表 1: Analogy between discrete and continuous scenarios

证明. proof:

□

Theorem

Dirichlet Test: Consider (a_n) and (b_n) , and let $A_n = \sum_{k=0}^n a_k$. Suppose

1. A_n is bounded;
2. $b_0 \geq b_1 \geq b_2 \geq \dots$ (i.e. (b_n) is decreasing);
3. $b_n \rightarrow 0$ ($n \rightarrow \infty$).

Then $\sum_{n=0}^{\infty} a_n b_n$ converges.

证明. proof:

□

Example

Example 1: Violates Condition 1 (A_n is not bounded)

- Let $a_n = 1$ for all n (so $A_n = \sum_{k=0}^n 1 = n + 1$, which is **unbounded**).

- Let $b_n = \frac{1}{n+1}$ (**decreasing** and $b_n \rightarrow 0$ as $n \rightarrow \infty$).

Series: $\sum_{n=0}^{\infty} a_n b_n = \sum_{n=0}^{\infty} \frac{1}{n+1}$.

Conclusion: This is the harmonic series, which **diverges**.

Example 2: Violates Condition 2 (b_n is not decreasing)

- Let $a_n = (-1)^n$ (so $A_n = \sum_{k=0}^n (-1)^k$, which is **bounded** by 1).

- Define b_n as:

$$b_n = \begin{cases} \frac{1}{n+1}, & \text{if } n \text{ is even,} \\ \frac{2}{n+1}, & \text{if } n \text{ is odd.} \end{cases}$$

Here, $b_n \rightarrow 0$ (as $n \rightarrow \infty$), but b_n is **not decreasing** (e.g., $b_1 = 1$, $b_2 = \frac{1}{3} < 1$, but $b_3 = \frac{1}{2} > \frac{1}{3}$).

Series: $\sum_{n=0}^{\infty} a_n b_n = 1 - 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{3} + \frac{1}{7} - \frac{1}{4} + \dots$.

Conclusion: The series **diverges** (partial sums grow negatively without bound).

Example 3: Violates Condition 3 ($b_n \not\rightarrow 0$)

- Let $a_n = (-1)^n$ (so $A_n = \sum_{k=0}^n (-1)^k$, which is **bounded** by 1).

- Let $b_n = 1$ for all n (**decreasing**, since $1 \geq 1 \geq 1 \geq \dots$, but $b_n \not\rightarrow 0$).

Series: $\sum_{n=0}^{\infty} a_n b_n = \sum_{n=0}^{\infty} (-1)^n$.

Conclusion: This alternating series oscillates ($1 - 1 + 1 - 1 + \dots$) and **diverges**.

Corollary & Secondary Conclusion

Suppose $u_1 \geq u_2 \geq \dots$ and $u_n \rightarrow 0$. Then $\sum_{n=1}^{\infty} (-1)^n u_n$ converges. In particular, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges, but not absolutely.

证明. proof:

□

Definition

Consider two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$. Put

$$c_n = \sum_{k=0}^n a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0.$$

The series $\sum_{n=0}^{\infty} c_n$ is said to be the **product** (Cauchy product) of $\sum a_n$ and $\sum b_n$.

For finite sums, $\left(\sum_{k=0}^N a_k\right) \left(\sum_{k=0}^N b_k\right)$ expands into the diagonal sums that form c_0, c_1, \dots, c_{2N} . If both series converge, one expects the product rule.

Theorem

Product rule. If $\sum a_n$ converges absolutely and $\sum b_n$ converges, then the product $\sum c_n$ converges, and

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right) \left(\sum_{n=0}^{\infty} b_n\right).$$

Hint: We let $B_n = \sum_{k=0}^n b_k$ and $B = \sum_{k=0}^{\infty} b_k$ (so $B_n \rightarrow B$). Then

证明. proof:

□

7 Rearrangements of series

Definition

Consider a bijection $\varphi : \mathbb{N} \rightarrow \mathbb{N}$. For $n \in \mathbb{N}$ define $\tilde{a}_n = a_{\varphi(n)}$. The series $\sum \tilde{a}_n$ is a **rearrangement** of $\sum a_n$.

Example

$$\varphi(n) = \begin{cases} n+1, & n \text{ odd}, \\ n-1, & n \text{ even}, \end{cases} \text{ maps } 1, 2, 3, 4, 5, \dots \text{ to } 2, 1, 4, 3, 6, 5, \dots$$

Before we talk about our main theorem in this section, we first start a claim:

Claim

Absolute Convergence can control "Tails"

By definition, absolute convergence means $\sum |a_n|$ converges. For any $\varepsilon > 0$, the Cauchy criterion for convergent series guarantees there exists $N \in \mathbb{N}$ such that: $\sum_{k=N+1}^{\infty} |a_k| < \varepsilon$.

This says the "tail" of $\sum |a_n|$ (terms after N) is arbitrarily small.

Theorem

Theorem. Suppose $\sum a_n$ converges absolutely and let $A = \sum a_n$. Then for any rearrangement $\sum \tilde{a}_n$, it converges to A .

证明. proof:

□

【Remark】:

If a series converges but not absolutely, then its rearrangements may give any numbers as the limit!

8 Finite and Countable sets

Definition

Finite sets: Given $n \in \mathbb{N}$, let $J_n = \{1, 2, \dots, n\}$.

Consider a nonempty set X . Then X is said to be **finite** if there exists $n \in \mathbb{N}$ and a bijection $f : J_n \rightarrow X$.

By convention, \emptyset is finite. If X is not finite, it is said to be infinite.

Definition

Countable sets: Consider a nonempty set X . X is said to be countable if either X is finite, or if there exists a bijection $f : \mathbb{N} \rightarrow X$.

Alternatively, X is countable if there exists a sequence (u_n) such that $X = \{u_1, u_2, \dots\}$ (all elements of X can be listed).

Example

1. \mathbb{N} is infinite and countable.
2. \mathbb{Z} is infinite and countable. One enumeration is given by

$$f(n) = \begin{cases} \frac{n-1}{2}, & n \text{ odd}, \\ -\frac{n}{2}, & n \text{ even}, \end{cases}$$

and $\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$.

3. \mathbb{Q} is countable. One listing is

$$\mathbb{Q} = \left\{ 0, \frac{1}{1}, -\frac{1}{1}, \frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1}, \frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}, \frac{3}{1}, -\frac{3}{1}, \dots \right\}.$$

4. The set \mathbb{R} (real numbers) is uncountable.

and we have these 2 properties:

Property

- **Proposition 1.** If A is infinite and $A \subset X$, then X is infinite.
- **Proposition 2.** If X is countable and $A \subset X$, then A is countable.

证明. proof:

□

Definition

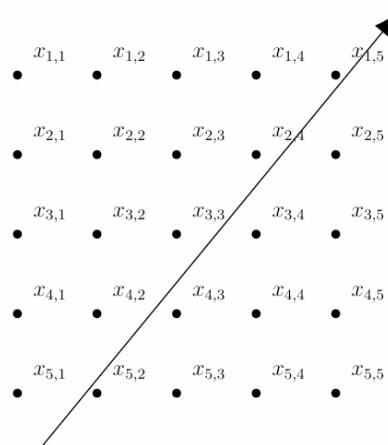
Let X be a nonempty set. Suppose for each $n \in \mathbb{N}$ we have a subset $E_n \subset X$. Denote this family by $(E_n)_{n \in \mathbb{N}}$. Define

$$\bigcap_{n=1}^{\infty} E_n = \{x \in X : \forall n \in \mathbb{N}, x \in E_n\}, \quad \bigcup_{n=1}^{\infty} E_n = \{x \in X : \exists n \in \mathbb{N}, x \in E_n\}.$$

Lemma

If for all n , E_n is countable, then $\bigcup_{n=1}^{\infty} E_n$ is countable.

证明. proof:



Traverse the Array Along Diagonals

To list all elements of $\bigcup_{n=1}^{\infty} E_n$ in a single sequence, traverse the array along diagonals defined by $n + k = \text{constant}$ (where $n = \text{row index}$, $k = \text{column index}$):

Diagonal 1 ($n + k = 2$): Only $(n, k) = (1, 1)$, so element $x_{1,1}$.

Diagonal 2 ($n + k = 3$): $(n, k) = (1, 2), (2, 1)$, so elements $x_{1,2}, x_{2,1}$.

Diagonal 3 ($n + k = 4$): $(n, k) = (1, 3), (2, 2), (3, 1)$, so elements $x_{1,3}, x_{2,2}, x_{3,1}$.

Diagonal m ($n+k = m+1$): $(n, k) = (1, m), (2, m-1), \dots, (m, 1)$, so m elements: $x_{1,m}, x_{2,m-1}, \dots$

□

The superposition of countably infinitely many "countable infinities" still results in a countable infinity.

Theorem

The set \mathbb{R} (real numbers) is uncountable.

Hint: If \mathbb{R} were countable, we could list all real numbers as a sequence: $\mathbb{R} = \{r_1, r_2, r_3, \dots\} = \{r_k\}_{k \geq 1}$. Our goal is to show this assumption leads to a contradiction.

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & b_3 & b_2 & b_1 \end{array}$$

证明. proof:

□

We will discuss a brand new topic: Fundamentals of Topology

9 Metric Space

Instead of treating a set X merely as a collection of elements, we often equip X with additional structure.

Definition

A function $d : X \times X \rightarrow \mathbb{R}$ is a distance (or metric) on X if

1. (*positive-definiteness*) $d(p, q) > 0$ for $p \neq q$ and $d(p, p) = 0$;
2. (*symmetry*) $d(p, q) = d(q, p)$;
3. (*triangle inequality*) $d(p, q) \leq d(p, r) + d(r, q)$ for all $p, q, r \in X$.

Example

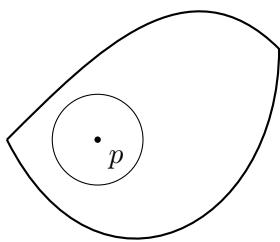
1. If $X = \mathbb{R}$, then $d(x, y) = |x - y|$.
2. If $X = \mathbb{R}^k$, then $d(x, y) = \sqrt{\sum_{i=1}^k (x_i - y_i)^2}$ (Euclidean metric).
3. If $X = \mathbb{C}$, then $d(z_1, z_2) = |z_1 - z_2|$ where for $z = a + bi$, $|z| = \sqrt{a^2 + b^2}$.

10 Interior, adherent, exterior points

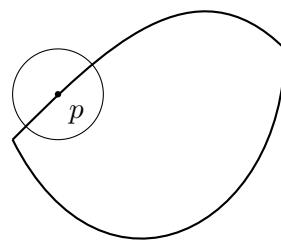
Let (X, d) be a metric space and $A \subset X$ nonempty.

Definition

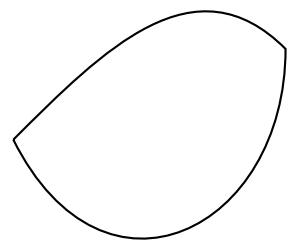
1. A point p is called an **interior point** of A if there exists $r > 0$ such that **all** points q satisfying $d(p, q) < r$ belong to the set A .
2. A point p is an **adherent point** of A if $\forall \varepsilon > 0, \exists q \in A$ such that $d(p, q) < \varepsilon$.
3. A point p is called an **exterior point** of A if there exists $r > 0$ such that **all** points q satisfying $d(p, q) < r$ do not belong to the set A .



interior



adherent



exterior

【Remark】:

An **interior point** is a *special case* of an **adherent point**.

Notations.

$$\text{int}(A) = \mathring{A} = \{p \in X : p \text{ is an interior point of } A\};$$

$$\text{Closure: } \overline{A} = \{p \in X : p \text{ is an adherent point of } A\};$$

$$\text{ext}(A) = \{p \in X : p \text{ is an exterior point of } A\}.$$

And we can gain this property below:

Property

$$\mathring{A} \subset A \subset \overline{A}, \quad \text{ext}(A) = \text{int}(A^c).$$

证明. proof:

□

For $p \in X$ and $r > 0$, we can define the (open) neighborhood.(we will prove it is open later.)

Definition

(Open) Neighborhood: $N_r(p) = \{q \in X : d(p, q) < r\}$.

For $p \in X$ and $A \subset X$,

$$p \in \mathring{A} \iff \exists r > 0, N_r(p) \subset A,$$

$$p \in \overline{A} \iff \forall r > 0, N_r(p) \cap A \neq \emptyset,$$

$$p \in \text{ext}(A) \iff \exists r > 0, N_r(p) \subset A^c. (\text{or } N_r(p) \cap A = \emptyset)$$

Definition

Let (X, d) be a metric space and nonempty $A \subset X$.

1. A is open if $\mathring{A} = A$.
2. A is closed if $\overline{A} = A$.

Mathematical Description:

A is **open** if $\mathring{A} = A$ means: Every point in A is an interior point.

This is equivalent to: For *every* $x \in A$, there exists $\rho > 0$ such that the open ball $U_\rho(x) = \{y \in X \mid d(y, x) < \rho\}$ is entirely contained in A (the "**open ball**" definition).

The **closure** $\overline{A} = A$ means: Every adherent point of A lies in A : a set $A \subset X$ is closed if: For every $x \in X$, if $\forall \rho > 0$, $U_\rho(x) \cap A \neq \emptyset$, then $x \in A$.

Lemma

If there exists $r > 0$ with $N_r(p) \subset A^c$, it is equal to the proposition: existing an $\varepsilon > 0$, $N_\varepsilon(p) \cap A = \emptyset$.

Theorem

For $A \subset X$, A is closed $\iff A^c$ is open.

证明. proof:

□

And we can also consider some useful properties of open and closed sets:

1. For any family of open sets $\{G_\alpha\}_{\alpha \in J}$, $\bigcup_{\alpha \in J} G_\alpha$ is open.

2. For finitely many open sets G_i , $i = 1, 2, \dots, N$, $\bigcap_{i=1}^N G_i$ is open. (Hint: You can prove it when $n=2$ and then use mathematical induction.)

【Remark】:

Let $X = \mathbb{R}$, $d(x, y) = |x - y|$, consider $I_n = (-1/n, 1/n)$. I_n is open for any n , but $\bigcap_{n=1}^{\infty} I_n$ is not open. This shows that (b) is not true for **infinite** intersection.

【Remark】:

A **subset of an open set** is **not** necessarily **open** in general.

for example:

Example

Let $U = (0, 2)$, which is open.

Take the subset $V = (0, 1] \subseteq U$. $V = (0, 1]$ is not open in the standard topology

11 Limit Isolated Boundary Points

Definition

Let (X, d) be a metric space, $p \in X$ and $A \subset X$.

1. p is a **limit point** of A if $\forall \varepsilon > 0$, $\exists q \in A \setminus \{p\}$ with $d(p, q) < \varepsilon$.
2. p is an **isolated point** of A if $p \in A$ and p is not a limit point of A (equivalently, $\exists r > 0$ s.t. $N_r(p) \cap (A \setminus \{p\}) = \emptyset$).
3. p is a **boundary point** of A if $p \in \overline{A}$ and $p \notin \overset{\circ}{A}$.

We can understand Boundary points in this way:

A boundary point is a point that neither completely belongs to A nor completely belongs to the exterior of A . **In any neighborhood** of such a point, there are both points of A and points not of A .

Example

1. **Limit point:** Let $X = \mathbb{R}$ (with the usual metric $d(x, y) = |x - y|$) and $A = (0, 1)$. The point $p = 0$ is a limit point of A : for any $\varepsilon > 0$, there exists $q = \varepsilon/2 \in A \setminus \{0\}$ with $d(0, q) < \varepsilon$.
2. **Isolated point:** Let $X = \mathbb{R}$ and $A = \{1\} \cup (2, 3)$. The point $p = 1$ is an isolated point of A : take $r = 0.5$, then $N_{0.5}(1) = (0.5, 1.5)$, and $N_{0.5}(1) \cap (A \setminus \{1\}) = \emptyset$, so $p = 1$ is not a limit point of A .
3. **Boundary point:** Let $X = \mathbb{R}$ and $A = [0, 1]$. The point $p = 0$ is a boundary point of A : $p \in \overline{A} = [0, 1]$, but $p \notin \mathring{A} = (0, 1)$.

And we Denote :

Definition

Boundary: $\partial A = \{\text{boundary points of } A\}$

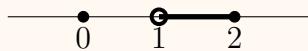
Derived set: $A' = \{\text{limit points of } A\}$.

And you can understand these items in one example:

Example

Let $X = \mathbb{R}$, $A = \{0\} \cup (1, 2]$. Then $0 \in A$ but $0 \notin A'$ (isolated). $1 \notin A$ but 1 is a limit point. $2 \in A$ and 2 is a limit point. Hence

$$\mathring{A} = (1, 2), \quad A' = [1, 2], \quad \overline{A} = \{0\} \cup [1, 2], \quad \partial A = \{0, 1, 2\}.$$



For $A \subset X$, we have:

A closure is A itself unions all points that are "infinitely close to A " (limit points), in short: $\overline{A} = A \cup A'$. (So the closure not only contains the limit points of A , but also all the points of isolated points.)

So we yield: $A' \subseteq \overline{A}$

12 Boundedness and Convergence

Definition

Bounded set: A is said to be **bounded** if $\exists M > 0$ and $p \in X$ such that $d(p, q) \leq M$ for every $q \in A$. Equivalently, $A \subset U_M(p)$

So if the set A can be covered by a ball with finite radius, then we say A is bounded.

Claim

Claim1: The union of bounded sets is bounded.

Claim2: Trivially, any subset of a bounded set is bounded.

Hint: for claim1, only prove the two-set case is ok.

证明. proof:

□

Then, let's talk about the **dense set**:

Definition

Let (X, d) be a metric space and $A \subset X$. We say that A is a **dense subset** of X if the closure of A equals X , i.e., $\overline{A} = X$.

【Remark】:

Denseness depends on both the **ambient space** X and the **metric** d defined on it.

Example

The set of rational numbers \mathbb{Q} is dense in \mathbb{R} with the usual metric.

证明.

□

Easy to gain:

Example

The set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is dense in \mathbb{R} with the usual metric.

Sequential convergence in X

Recall for a real sequence: $u_n \rightarrow u^*$ iff $\forall \varepsilon > 0, \exists N$ s.t. $\forall n \geq N, |u_n - u^*| < \varepsilon$. The concept generalizes to metric spaces by replacing absolute value with distance.

Definition

Convergence A sequence $(p_n) \subset X$ is said to converge to $p \in X$ iff $d(p_n, p) \rightarrow 0$; i.e. $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \geq N, d(p_n, p) < \varepsilon$.

And we need to prove an important property:

Theorem

Limit characterization of closure: $p \in \overline{A} \iff$ there exists a sequence $(p_n) \subset A$ such that $p_n \rightarrow p$ in X .

proof:

Corollary & Secondary Conclusion

A is dense in $X \iff$ for every $x \in X$ there exists a sequence $(p_n) \subset A$ with $p_n \rightarrow x$.

And ultimately in this section, we will consider 2 types of "Convergence":

Euclidean space \mathbb{R}^k . Write $x = (x_1, \dots, x_k)$ and $d(x, y) = \sqrt{\sum_{i=1}^k (x_i - y_i)^2}$. For a sequence $x_n = (x_{n,1}, \dots, x_{n,k})$,

$$x_n \rightarrow x \iff (\forall i) \lim_{n \rightarrow \infty} x_{n,i} = x_i.$$

The convergence can also be characterized by the metric

$$d_\infty(x, y) := \max_{1 \leq i \leq k} |x_i - y_i|,$$

which satisfies $d_\infty(x, y) \leq d(x, y) \leq \sqrt{k} d_\infty(x, y)$ for all $x, y \in \mathbb{R}^k$.

证明. proof: □

The inequalities $d_\infty \leq d \leq \sqrt{k} d_\infty$ imply:

If $x_n \rightarrow x$ in the Euclidean metric ($d(x_n, x) \rightarrow 0$), then $d_\infty(x_n, x) \leq d(x_n, x) \rightarrow 0$, so $x_n \rightarrow x$ in d_∞ .

If $x_n \rightarrow x$ in d_∞ ($d_\infty(x_n, x) \rightarrow 0$), then $d(x_n, x) \leq \sqrt{k} d_\infty(x_n, x) \rightarrow 0$, so $x_n \rightarrow x$ in the Euclidean metric.

Thus, a sequence converges in d if and only if it converges in d_∞ (to the same limit). So the two metrics "describe convergence in the same way" for \mathbb{R}^k .

13 Compactness

Definition

A collection of open sets $\{O_\alpha\}_{\alpha \in I}$ is an **open cover** of a set K if: $K \subset \bigcup_{\alpha \in I} O_\alpha$.

In other words, every point $p \in K$ lies in at least one open set O_α from the collection.

From the concept of Open Covering, we can have:

Definition

Compact set: Let (X, d) be a metric space and $A \subset X$. A is compact **if every open cover** of A has a **finite** subcover;

i.e., Whenever $\{E_\alpha\}_{\alpha \in J}$ is a family of open sets with $A \subset \bigcup_{\alpha \in J} E_\alpha$, there exist $\alpha_1, \dots, \alpha_N$ such that $A \subset \bigcup_{i=1}^N E_{\alpha_i}$.

Fact and proposition

Why compactness matters: it provides a way to *find limits*.

Compactness turns many problems about **infinite coverings** into problems about **finite coverings**.

【Remark】:

Compactness **doesn't just mean existing a finite open cover** for set A.

Example

Example: Finite Set in \mathbb{R} . Let $X = \mathbb{R}$ (with the usual distance $|x - y|$) and $A = \{1, 2, 3\}$.

Why compact?

If the open cover is very "large" (for example, a single open set $(-1, 10)$ can cover A), then its finite subcover is itself, which certainly holds.

If the open cover is very "fine" (for example, covering A with three open sets $U_1 = (0.5, 1.5)$ (covering 1), $U_2 = (1.5, 2.5)$ (covering 2), $U_3 = (2.5, 3.5)$ (covering 3)), since A has only 3 points, no matter how the open cover is constructed, it is sufficient to select "the open set corresponding to each point" from the cover (at most 3), which can cover the entire A.

Therefore, any open cover can find a **finite** subcover, so A is compact.

Example

Counter-Example: Open Interval $(0, 1)$ in \mathbb{R} . Let $X = \mathbb{R}$ (with the usual metric $d(x, y) = |x - y|$) and $A = (0, 1)$ (the open interval from 0 to 1).

To show $(0, 1)$ is not compact, we can construct an open cover of $(0, 1)$ • **with no finite subcover**. While some open covers of $(0, 1)$ (like $(-1, 0.5) \cup (0.4, 2)$) do have finite subcovers, not all open covers do.

Consider open sets $U_n = (\frac{1}{n}, 1 - \frac{1}{n})$ for all integers $n \geq 3$. The family $\{U_n \mid n \geq 3\}$ covers $(0, 1)$ because for any $x \in (0, 1)$, there exists some $n \geq 3$ such that $\frac{1}{n} < x < 1 - \frac{1}{n}$.

However, no finite subcollection of $\{U_n \mid n \geq 3\}$ can cover $(0, 1)$. Suppose we pick finitely many U_n , say U_{n_1}, \dots, U_{n_k} , and let $N = \max\{n_1, \dots, n_k\}$. Their union is $(\frac{1}{N}, 1 - \frac{1}{N})$, which misses points like $\frac{1}{2N}$ (since $0 < \frac{1}{2N} < \frac{1}{N}$, so $\frac{1}{2N} \notin (\frac{1}{N}, 1 - \frac{1}{N})$).

Thus, this open cover has no finite subcover, so $(0, 1)$ is not compact.

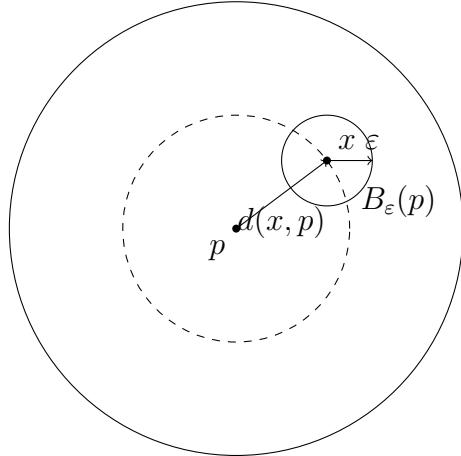
Before this section's main theorem, we first prove a useful lemma.

Lemma

Open balls are open: Let (X, d) be a metric space. For any $p \in X$ and $r > 0$, the open ball

$$U_r(p) = \{x \in X : d(x, p) < r\}$$

is an **open** set in (X, d) .



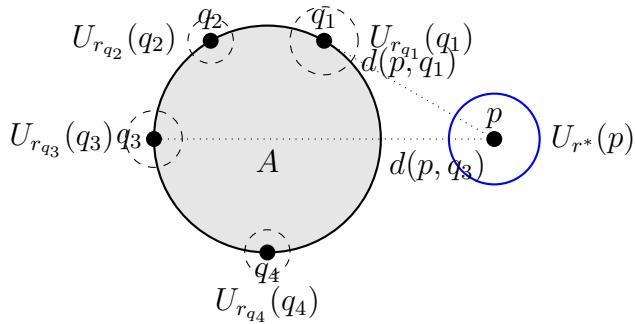
证明. proof:

□

Theorem

A **compact** set is **bounded** and **closed**.

Hint: For closed part, it is sufficient to show that its complement A^c is an open set.



And we noticed that the only variable is the distance between q (q belongs to A) and p (p belongs to A^c) ; and the important thing we need to deal with is to :

1. find a suitable radius r s.t. fulfills the condition;

2. Constructing a suitable open cover (we always use open ball) to help us find a good r .

证明. proof:

□

Theorem

If $A \subset X$ is compact and $K \subset A$ is closed, then K is compact.

Hint : we notice that K^c is open and $K \cap K^c = \emptyset$. And what we need to do is How to use the condition of A 's compactness?

证明. proof:

□

And we will introduce our main theorem:

Theorem

Heine–Borel in \mathbb{R} : For $X = \mathbb{R}$ with $d(x, y) = |x - y|$, a set $A \subset \mathbb{R}$ is compact \iff it is bounded and closed.

证明. proof:

□

And we can extend the theorem to:

Corollary & Secondary Conclusion

For $X = \mathbb{R}^k$, and $d(x, y)$ being the Euclidean distance, a set $A \subset X$ is compact \iff A is bounded and closed.

【Remark】:

Heine-Borel is a special property of \mathbb{R}^k (and similar spaces).

In general metric spaces (or even **subspaces** of \mathbb{R}), “closed + bounded” does not imply compactness.

We must notice that:

In metric spaces, compact sets are always closed and bounded, but the **converse fails**: a closed and bounded set in a metric space is not necessarily compact.

Compactness depends on the global structure of the ambient space, while “closed/bounded in a subspace” are local properties.

we can give a counter-example:

Example

The open interval $(0, 1)$ as a subspace of \mathbb{R}

Consider $X_0 = (0, 1)$ with the standard metric $d(x, y) = |x - y|$.

And we consider the subset: $K = (0, 1) \subseteq X_0$

The set $K = (0, 1) \subseteq X_0$ is closed in X_0 : Its complement in X_0 is \emptyset , which is open in any topology.

And the set $K = (0, 1) \subseteq X_0$ is Bounded: All points lie within distance 1 of each other.

But $K = (0, 1)$ is not compact: the open cover $\left\{ \left(\frac{1}{n}, 1 \right) \mid n \geq 2 \right\}$ has no finite sub-cover

And we will discuss about some facts about the compact sets:

Property

For **finitely** many compact sets K_i , $i = 1, 2, \dots, N$, $\bigcup_{i=1}^N K_i$ is compact.

证明. proof:

□

On $X = \mathbb{R}$, $d(x, y) = |x - y|$, consider $I_n = [0, 1 - 1/n]$. we can show that I_n is compact for any n but $\bigcup_{n=1}^{\infty} I_n$ is not compact. This shows the conclusion above is not true for **infinite** union.

Property

For any family of compact set $\{K_\alpha\}_{\alpha \in J}$, $\bigcap_{\alpha \in J} K_\alpha$ is compact.

(Hint: show that the intersection is a closed subset of some compact set.)

证明. proof:

□

14 Sequentially Compactness

Definition

Sequentially Compact: Let (X, d) be a metric space. A set $K \subset X$ is said to be *sequentially compact* if every sequence $\{u_n\}_{n \geq 1} \subset K$ admits a subsequence $\{u_{n_k}\}_{k \geq 1}$ converging to some $u \in K$.

Then we pick an **arbitrary** sequence $\{u_n\}_{n \geq 1} \subset K$ and denote the (countable) set as:

$$A := \{u_n : n \in \mathbb{N}\} \subset K.$$

Easy to notice that:

【Remark】:

The set A is a **countable** set (since it's indexed by natural numbers \mathbb{N}).

And set A is also a **subset** of K : $A \subseteq K$.

Our target is to show : Compact \iff sequentially compact, but we first depart this main theorem into two parts(two lemmas below):

Lemma

(Compact \implies sequentially compact) Let $K \subset X$ be compact. Then K is sequentially compact.

(Sequentially compact \implies compact): Let $K \subset X$ be sequentially compact. Then K is compact.

We first start with the Compact \implies sequentially compact side:

We first start with two claims:

Claim

If a set A is contained in a union $\bigcup S_i$, then $A \cap \bigcup S_i = A$.

easy to prove it.

Claim

Set A has a limit point $u \in K$.

Otherwise, for every $p \in K$ there exists $r_p > 0$ such that

$$U_{r_p}(p) \cap A = \begin{cases} \{p\}, & p \in A, \\ \emptyset, & p \notin A. \end{cases}$$

证明. proof:

□

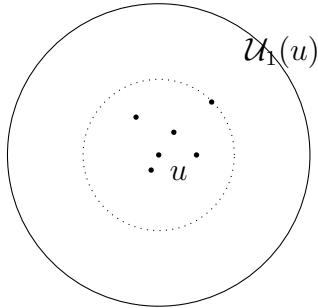
Using the claim above, we can yield this important lemma:

Lemma

(Compact \implies sequentially compact) Let $K \subset X$ be compact. Then K is sequentially compact.

To prove this, we construct a convergent subsequence for an arbitrary sequence $\{u_n\}_{n \geq 1} \subset K$ using induction:

K



- **Base Case:** Since u is a limit point of set A , the open ball $U_1(u)$ (centered at u with radius 1) must contain points from A . So we can pick an index n_1 such that the term u_{n_1} satisfies $d(u_{n_1}, u) < 1$.

- **Inductive Step:** Suppose we've already chosen an index n_k . Again, because u is a limit point, the open ball $U_{1/(k+1)}(u)$ (centered at u with radius $\frac{1}{k+1}$) contains infinitely many points of A . So we can pick an index $n_{k+1} > n_k$ such that $d(u_{n_{k+1}}, u) < \frac{1}{k+1}$.

By repeating this process, we obtain a subsequence $\{u_{n_k}\}$. As $k \rightarrow \infty$, $\frac{1}{k+1} \rightarrow 0$, so $d(u_{n_k}, u) \rightarrow 0$, meaning $u_{n_k} \rightarrow u$. This shows K is sequentially compact.

IDEA: It's a familiar picture on: $(\mathbb{R}, |x - y|)$. By Bolzano–Weierstrass, any sequence in K has a convergent subsequence. By closedness, the limit stays in K .

And now we wanna prove the other side: **Sequentially compact \implies compact**: Let $K \subset X$ be sequentially compact, then K should be compact.

To prove this we use two claims derived from sequential compactness.

Claim

Claim 1: Let $K \subset X$ be sequentially compact. For every $\varepsilon_0 > 0$, there exist **finitely** many points $x_1, \dots, x_N \in K$ such that

$$K \subset \bigcup_{i=1}^N U_{\varepsilon_0}(x_i).$$

证明. proof:

□

Claim

Claim 2: Let $K \subset X$ be sequentially compact and let $\{G_\alpha\}_{\alpha \in J}$ be an open cover of K ($K \subset \bigcup_{\alpha \in J} G_\alpha$).

Then there exists a **uniform** radius $\delta_0 > 0$ such that for every $x \in K$ there is an index $\alpha(x)$ with

$$U_{\delta_0}(x) \subset G_{\alpha(x)}.$$

Prove by contradiction: in other words, just prove: $\forall \delta > 0 \left(\exists x \in K \forall \alpha \in J (U_\delta(x) \not\subseteq G_\alpha) \right)$ is totally wrong.

证明. proof:

□

Example

Let $K = [0, 1]$. Take a proper open cover that includes $\frac{1}{4}$ and $\frac{3}{4}$, say: $\left\{ \left(-\frac{1}{8}, \frac{3}{8} \right), \left(\frac{1}{8}, \frac{5}{8} \right), \left(\frac{3}{8}, \frac{9}{8} \right) \right\}$
 Now, apply claim with $\delta_0 = \frac{1}{8}$.

For $x = \frac{1}{4}$: $U_{\frac{1}{8}}\left(\frac{1}{4}\right) = \left(\frac{1}{4} - \frac{1}{8}, \frac{1}{4} + \frac{1}{8}\right) = \left(\frac{1}{8}, \frac{3}{8}\right) \subset \left(-\frac{1}{8}, \frac{3}{8}\right)$ (first set in the cover);

For $x = \frac{3}{4}$: $U_{\frac{1}{8}}\left(\frac{3}{4}\right) = \left(\frac{3}{4} - \frac{1}{8}, \frac{3}{4} + \frac{1}{8}\right) = \left(\frac{5}{8}, \frac{7}{8}\right) \subset \left(\frac{1}{8}, \frac{5}{8}\right)$ (second set in the cover).

So using these two claims, we can prove the lemma below:

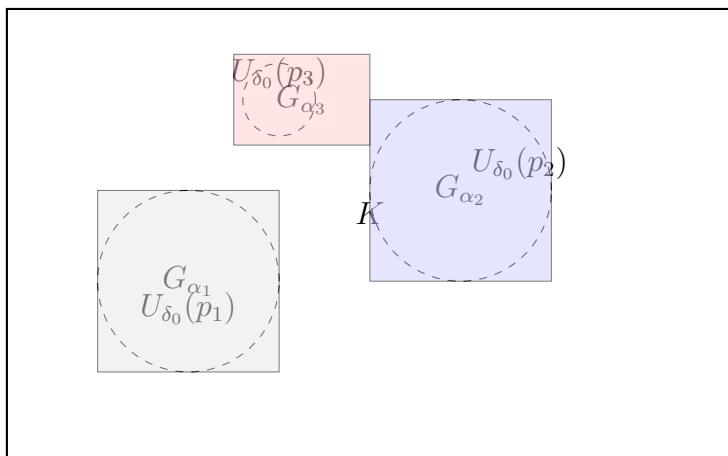
Lemma

(Sequentially compact \implies compact): Let $K \subset X$ be sequentially compact. Then K is compact.

证明. proof:

□

Finite covering of K by $\bigcup_{i=1}^N U_{\delta_0}(p_i)$ (Claim 1)



Open cover $\{G_\alpha\}_{\alpha \in J}$: $U_{\delta_0}(p_i) \subset G_{\alpha_i}$ by Claim 2

And we now conclude the core results in metric space:

Compact \iff sequentially compact.

15 Connectedness & Subspace Topology

Definition

Connected metric space: A metric space (X, d) is **connected** if there **do not exist** two **non-empty** and **disjoint open** sets $E, F \subset X$ such that

$$X = E \cup F.$$

Otherwise X is **disconnected**.

Intuitive imagination: Think of a metric space as a "geometric figure". If it is a single whole without breakpoints, it is connected; if it can be "split into two separate open blocks", it is disconnected.

Example

- **Connected example:** The real line \mathbb{R} (with the usual metric $|\cdot|$).

Suppose we want to find two non-empty disjoint open sets E, F covering \mathbb{R} . For instance, let $E = (-\infty, a)$ and $F = (a, +\infty)$, but they **do not contain** a , so they cannot cover \mathbb{R} .

If we forcefully put a into E , then $E = (-\infty, a]$, but $(-\infty, a]$ is not an open set (because the neighborhood of a is not entirely in E). Therefore, \mathbb{R} is connected.

In fact, you cannot find such E and F .

- **Disconnected example:** The space $X = (-\infty, 0) \cup (1, +\infty)$ (with the usual metric).

Here, $E = (-\infty, 0)$ and $F = (1, +\infty)$ are both open sets in X (because in the "subspace topology" of X , we will discuss later). Moreover, $E \cap F = \emptyset$ and $E \cup F = X$, so X is disconnected.

If (X, d) is disconnected, then one may even find a "**disconnected**" ball $U_r(p)$ (its intersection with the two components splits), so a whole space being disconnected is unusual but it may happen.

Intuitive understanding: The entire space can be split into two open blocks, and naturally, the local area will also be "split" by these two open blocks.

For example, in the above example $X = (-\infty, 0) \cup (1, +\infty)$, take the ball $U_2(0) = (-2, 2)$. Its intersection with X is $U_X = (-2, 0) \cup (1, 2)$, which is obviously two "split"

open sets.

And we consider another example:

Example

To determine whether the set $[-1, 2] \cup [5, 10]$ is connected.

Denote the set as $X = [-1, 2] \cup [5, 10]$, which is a subspace of the real number set \mathbb{R} (inheriting the usual metric of \mathbb{R}).

And of course, this set is disconnected. However if you are in the perspective of the WHOLE space \mathbb{R} , it's literally hard to depart this set as two open sets. (e.g. $E = [-1, 2]$ and $F = [5, 10]$, but E and F are closed in the WHOLE space \mathbb{R})

So, we need to introduce the concept of **Relatively open of subspace topology**. That is in the perspective of X itself, E and F are both "open".

And we call it relatively open.

Example

$X = \mathbb{R}$, $A = [0, 1)$, $U = [0, \frac{1}{2})$:

In \mathbb{R} , $[0, \frac{1}{2})$ is not open. Because $0 \in [0, 1/2)$, and any neighborhood of 0 in \mathbb{R} (such as $(-\epsilon, \epsilon)$) contains negative numbers, which are not in $[0, 1/2)$. Therefore, 0 is not an interior point of $[0, 1/2)$, so $[0, 1/2)$ is not an open set in \mathbb{R} .

In A , $[0, \frac{1}{2})$ is open: $U_{\frac{1}{2}}^A(0) = \{q \in [0, 1) \mid |0 - q| < \frac{1}{2}\} = [0, \frac{1}{2})$, so U is open in A .

But it's too early to give the precise definition of relatively open right now, so we can first introduce an important theorem:

Theorem

If space A is a subspace of X . For any $U \subset A$ we have: U is **open** in A **iff** there exists open set V in X such that $U = A \cap V$.

证明. proof:

□

This proposition characterizes the **"relativity" of the subspace topology:**

An open set in the subspace A is essentially the **intersection of an open set in the original space X and A** ;

conversely, the intersection of an **open set in the original space and A** must be an open set **within A** .

Example

In $(\mathbb{R}, |\cdot|)$, take $A = [0, 1]$. The set $[0, 1/2]$ is open in A , but not open in $X = \mathbb{R}$.

Open in A : In the subspace topology, an "open set" requires that there exists an open set V in X such that $U = V \cap A$. For $[0, 1/2]$, take the open set $V = (-1, 1/2)$ in X , then $V \cap A = [0, 1/2]$, so $[0, 1/2]$ is an open set in A .

So we can gain the an essential way in the field of subspace topology to **Relatively Open Subset**. And it's time to get the definition of Relatively open set:

Definition

Relatively open subset: A subset $U \subset A$ is said to be **relatively open** if there exists an open set $V \subset X$ such that $U = A \cap V$.

And if A itself is open in X , we can gain more properties:

Corollary & Secondary Conclusion

If $A \subset X$ is open, then $E \subset A$ is relatively open in A if and only if E is open in X .

证明. proof:

□

Above our discussion we can have the def. of the connected subset

Definition

Connected subsets: $A \subset X$ is said to be connected if (A, d) is connected as a metric space.

【Remark】:

The open sets we will discuss will be all relative open.

So that means a subset $A \subset X$ is connected *iff* it cannot be decomposed as the disjoint union of two non-empty sets that are relatively open.

we will show an example:

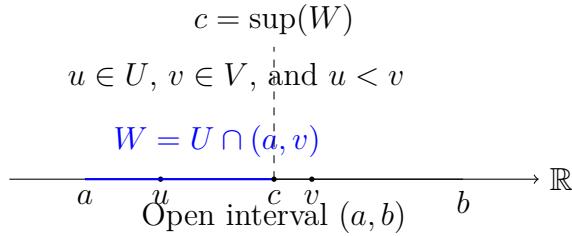
Example

- To determine whether the set $[-1, 2] \cup [5, 10]$ is connected.
- Analyzing the structure of $[-1, 2] \cup [5, 10]$
 - Denote the set as $X = [-1, 2] \cup [5, 10]$, which is a subspace of the real number set \mathbb{R} (inheriting the usual metric of \mathbb{R}).
 - According to the definition of subspace topology: An open set in the subspace X is the intersection of an open set in \mathbb{R} with X .
- Constructing a decomposition into disjoint open sets
 - Take:
 - * $U = [-1, 2]$,
 - * $V = [5, 10]$.
 - Verify that U and V are open sets in X :
 - * For U : Take the open set $(-2, 3)$ in \mathbb{R} , then $(-2, 3) \cap X = [-1, 2] = U$, so U is an open set in X .
 - * For V : Take the open set $(4, 11)$ in \mathbb{R} , then $(4, 11) \cap X = [5, 10] = V$, so V is an open set in X .
- Verifying the conditions for the decomposition
 - Both U and V are non-empty;
 - $U \cap V = \emptyset$ (because $[-1, 2]$ and $[5, 10]$ have no intersection);
 - $U \cup V = X$ (which is exactly the original set).

Theorem

(Open intervals are connected) For $X = \mathbb{R}$, $d(x, y) = |x - y|$, and $a < b$, we will show that $A = (a, b)$ is connected.

Suppose otherwise, i.e., $(a, b) = U \cup V$, where U and V are nonempty relative open sets and $U \cap V = \emptyset$. Pick $u \in U$, $v \in V$ and assume $u < v$ (swap if needed). Let $W = U \cap (a, v)$ and define $c = \sup(W)$.



证明. proof:

□

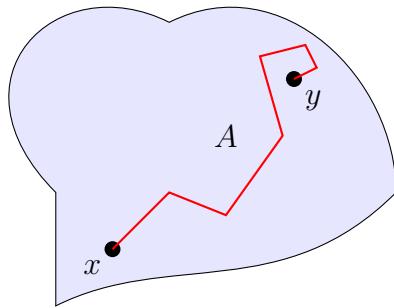
By a segment we mean $[x, y] = \{tx + (1 - t)y : 0 \leq t \leq 1\}$.

A polygonal chain from x to y is a finite sequence of segments $[x_1, y_1], \dots, [x_m, y_m]$ with

$$x_1 = x, \quad y_i = x_{i+1} \quad (1 \leq i < m), \quad y_m = y.$$

Theorem

If $A \subset \mathbb{R}^k$ is **open and connected**, then for any $x, y \in A$ there exists a polygonal chain joining x and y whose **every segment lies in A** .



证明. proof:

□

And we will check two questions in subspace topology:

i Exercise

- (a) If $K \subset X_0$, show that K is compact in (X_0, d) if and only if it is compact in (X, d) .
- (b) Suppose X_0 is itself closed in X . Show that for any compact set $K \subset X$, $K \cap X_0$ is compact in X_0 .

证明. proof:

proof:

□

16 Continuity

Key Points in this section: Continuous functions (mappings) **preserve topological structure** (open/closed /compact/connected).

After equipping special structures on abstract sets (e.g. metric), it is natural to study mappings which are consistent with the structure.

And the concept of limit is the fundamental of the concept of continuity.

We start with $f : I \rightarrow \mathbb{R}$, where $I = [a, b]$, $[a, b)$, $(a, b]$ or (a, b) . For now assume $-\infty < a < b < +\infty$; the endpoint cases $a = -\infty$ or $b = +\infty$ can be incorporated similarly. In all cases $I \subset \mathbb{R}$.

Definition

$f : I \rightarrow \mathbb{R}$ has limit ℓ at $p \in \bar{I}$ if

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that if } x \in I \text{ and } |x - p| < \delta, \text{ then } |f(x) - \ell| < \varepsilon.$$

We write $\lim_{x \rightarrow p} f(x) = \ell$.

【Remark】:

- (1) No matter how close $f(x)$ is required to be to ℓ , as long as x and p are sufficiently close this can be achieved.
- (2) Since p need not belong to the domain of f , it may happen that $\ell = \pm\infty$ (e.g. $f(x) = 1/x$ near $p = 0$).
- (3) For $b = +\infty$ we define $\lim_{x \rightarrow \infty} f(x) = \ell$ (e.g. $\lim_{x \rightarrow \infty} 1/x = 0$).

Definition

For $p \in \bar{I}$, f is *continuous at p* if $\lim_{x \rightarrow p} f(x) = f(p)$.

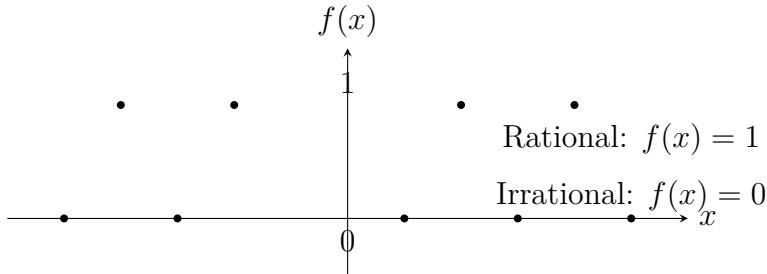
It is *continuous on I* if it is *continuous at every $p \in I$* .

Single-variable facts extend to general metric spaces. This includes $f : \mathbb{R}^k \rightarrow \mathbb{R}^m$ (vector-valued) and $f : \mathbb{C} \rightarrow \mathbb{C}$ (complex variable). For continuity alone we only need a metric.

Dirichlet function. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Then f is discontinuous at every $p \in \mathbb{R}$, and each discontinuity is of the second kind.



Now we will introduce the concept of one-side limit:

Let $I \subset \mathbb{R}$ be an interval and $f : I \rightarrow \mathbb{R}$. Fix $p \in I$.

(Left/right limits at p). We say f has *left limit ℓ* at p (written $f(p-) = \ell$) if

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } 0 < p - x < \delta, x \in I \implies |f(x) - \ell| < \varepsilon.$$

We say f has *right limit r* at p (written $f(p+) = r$) if

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } 0 < x - p < \delta, x \in I \implies |f(x) - r| < \varepsilon.$$

Corollary & Secondary Conclusion

f is continuous at p if and only if both one-sided limits exist and

$$f(p-) = f(p+) = f(p).$$

Explanation. (\Rightarrow) If f is continuous at p , then the ε - δ definition with $x \rightarrow p$ through $x < p$ (resp. $x > p$) gives $f(p-) = f(p)$ and $f(p+) = f(p)$.

(\Leftarrow) If the one-sided limits exist and equal $f(p)$, then for any $\varepsilon > 0$ there are $\delta_-, \delta_+ > 0$ such that $|f(x) - f(p)| < \varepsilon$ whenever $p - \delta_- < x < p$ and whenever $p < x < p + \delta_+$.

With $\delta = \min\{\delta_-, \delta_+\}$ we have $|f(x) - f(p)| < \varepsilon$ for all $|x - p| < \delta$, so f is continuous at p . \square

Definition

Discontinuities of first/second kind. Let $p \in I$ and suppose f is *not* continuous at p .

- If both $f(p-)$ and $f(p+)$ exist (finite), we say f has a *discontinuity of the first kind* (also called a *jump* or *simple discontinuity*) at p .
- Otherwise we say f has a *discontinuity of the second kind* at p .

17 Continuity in Metric Space

Then we will discuss the continuity in metric space:

Let (X, d_X) and (Y, d_Y) be metric spaces and $f : E \subset X \rightarrow Y$.

Definition

(1) [Limit] For $p \in \overline{E}$, f has *limit ℓ at p* if

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } x \in E, d_X(x, p) < \delta \Rightarrow d_Y(f(x), \ell) < \varepsilon.$$

(2) [Continuity] For $p \in E$, f is *continuous at p* if $\lim_{x \rightarrow p} f(x) = f(p)$.

And f is *continuous on E* if it is continuous at every $p \in E$.

Theorem

Let $f : E \rightarrow Y$ and $p \in \overline{E}$.

(Sequential characterization of limits).

$\lim_{x \rightarrow p} f(x) = \ell$ iff for every sequence $(p_n) \subset E$ with $p_n \rightarrow p$ in X , we have $f(p_n) \rightarrow \ell$ in Y .

(Sequential characterization of continuity).

The mapping f is *continuous at p* iff for every sequence $(p_n) \subset E$ with $p_n \rightarrow p$, we have $f(p_n) \rightarrow f(p)$.

Example

(Vector-valued functions). Let $Y = \mathbb{R}^k$ with Euclidean distance $\|\cdot\|$. Writing $f(x) = (f_1(x), \dots, f_k(x))$, the following are equivalent:

- (1) $\lim_{x \rightarrow p} f(x) = \ell = (\ell_1, \dots, \ell_k)$;
- (2) for each $i = 1, \dots, k$, $\lim_{x \rightarrow p} f_i(x) = \ell_i$.

$f = (f_1, \dots, f_k)$ is continuous iff each component f_i is continuous.

[Q]: Consider a continuous function $f : \mathbb{R}^k \rightarrow \mathbb{R}$. Fix a $x = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k$. For $i = 1, 2, \dots, k$, define a function $f_i : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f_i(t) = f(x_1, x_2, \dots, x_{i-1}, t, x_{i+1}, \dots, x_k).$$

Show that, if f is continuous at x , then f_i is continuous at x_i for each $i = 1, 2, \dots, k$.

Let $f, g : E \rightarrow \mathbb{R}$ with $\lim_{x \rightarrow p} f(x) = A$ and $\lim_{x \rightarrow p} g(x) = B$. Then

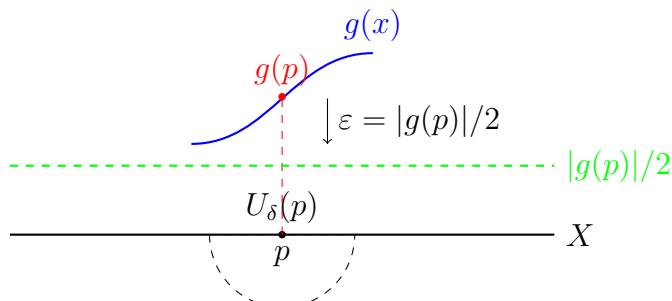
$$\lim_{x \rightarrow p} (f + g)(x) = A + B, \quad \lim_{x \rightarrow p} (fg)(x) = AB, \quad B \neq 0 \Rightarrow \lim_{x \rightarrow p} \frac{f}{g}(x) = \frac{A}{B}.$$

Claim If f, g are continuous at p , then $f + g$ and fg are continuous at p . If, in addition, $g \neq 0$ in a neighborhood of p , then f/g is continuous at p .

Corollary & Secondary Conclusion

If g is continuous at p and $g(p) \neq 0$, then $g \neq 0$ in some neighborhood of p .

Put $\varepsilon = |g(p)|/2 > 0$



证明. proof:

□

18 Topological characterization of continuity

Theorem

$f : X \rightarrow Y$ is **continuous** iff for every open $G \subset Y$, the **preimage** $f^{-1}(G) \subset X$ is open.

Use $f^{-1}(G^c) = (f^{-1}(G))^c$ and the theorem above, we gain:

Corollary & Secondary Conclusion

$f : X \rightarrow Y$ is continuous iff for every closed $G \subset Y$, the preimage $f^{-1}(G) \subset X$ is closed.

【Remark】:

To clarify: A continuous function **does not** directly "keep" (preserve) open/closed sets in the forward direction (i.e., f (open set in X) is not necessarily open in Y , and f (closed set in X) is not necessarily closed in Y).

Instead, continuous functions preserve open/closed sets in the **reverse direction** (via preimages):

f is continuous \iff preimages of open sets in Y are open in X .

f is continuous \iff preimages of closed sets in Y are closed in X .

In short: Continuity guarantees that **pulling back** open/closed sets from the codomain Y to the domain X preserves openness/closedness —not that pushing forward sets from X to Y does.

Before we go to the relationship between compactness and continuous function, we will first mention three properties regarding the preimage:

For any function (mapping) $f : X \rightarrow Y$, and $U \subset X$, $V \subset Y$, define

$$f^{-1}(V) = \{x \in X : f(x) \in V\},$$

$$f(U) = \{y \in Y : y = f(x) \text{ for some } x \in U\}.$$

we have:

Lemma

- (a) If $V \subset V' \subset Y$, then $f^{-1}(V) \subset f^{-1}(V')$; if $U \subset U' \subset X$, then $f(U) \subset f(U')$.
- (b) For any $V \subset Y$, $f(f^{-1}(V)) \subset V$.
- (c) If $V_\alpha \subset Y$ for each $\alpha \in J$, then

$$f^{-1} \left(\bigcup_{\alpha \in J} V_\alpha \right) = \bigcup_{\alpha \in J} f^{-1}(V_\alpha).$$

【Remark】:

For lemma (b), we should notice that $f(f^{-1}(V)) \subset V$ instead of $f(f^{-1}(V)) = V$.

Example

Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$, defined as $f(x) = x^2$ (not surjective), and take the set $V = [-1, +\infty)$.

Step 1: Find $f^{-1}(V)$ $f^{-1}(V) = \{x \in \mathbb{R} \mid x^2 \in [-1, +\infty)\}$. Since for all real numbers x , $x^2 \geq 0$, $x^2 \in [-1, +\infty)$ holds for all $x \in \mathbb{R}$. Therefore: $f^{-1}(V) = \mathbb{R}$.

Step 2: Find $f(f^{-1}(V))$ Substitute the elements in \mathbb{R} into $f(x) = x^2$, and the resulting set is $[0, +\infty)$ (due to the non-negativity of square numbers).

We also need to notice a common mistake:

【Remark】:

If $f : D \rightarrow I$ is continuous, then its inverse mapping $f^{-1} : I \rightarrow D$ may not be continuous.

Compactness and Continuous function

Theorem

If $K \subseteq X$ is compact and $f : K \rightarrow Y$ is continuous, then $f(K)$ is compact in Y .

And we can use two kinds of proofs: open-cover proof and sequential compactness proof in metric spaces.

open-cover proof:

证明. proof:

□

sequential compactness proof in metric spaces:

证明. proof:

□

Definition

Bounded function: f is *bounded* on E if there exists $C > 0$ such that $|f(x)| \leq C$ for all $x \in E$.

Equivalently, f is bounded on E iff the image set $f(E) \subset \mathbb{R}$ is bounded.

(1) (Bounded $\not\Rightarrow$ continuous) With $E = X = \mathbb{R}$,

$$f(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & \text{otherwise,} \end{cases}$$

is bounded but discontinuous.

(2) (Continuous $\not\Rightarrow$ bounded) On $E = (0, \infty)$, $f(x) = 1/x$ is continuous but unbounded.

Theorem

If E is **compact** and $f : E \rightarrow \mathbb{R}$ is **continuous**, then f is **bounded** on E .

Theorem

Extreme value theorem: If E is **compact** and $f : E \rightarrow \mathbb{R}$ is **continuous**, then there exist $p, q \in E$ such that $f(p) = \max$ and $f(q) = \min$.

Example

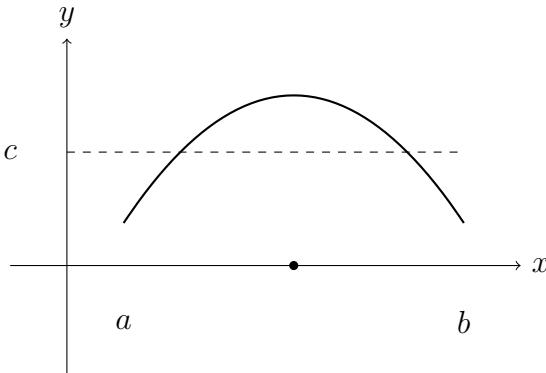
Take $E = (0, 1)$ (**not compact**) and $f(x) = \frac{1}{x}$. f is continuous on $(0, 1)$, but f is unbounded above (approaches $+\infty$ as $x \rightarrow 0^+$) and has no maximum value.

Lemma

If $E \subset X$ is **connected** and $f : E \rightarrow Y$ is **continuous**, then $f(E)$ is **connected**.

Theorem

(Intermediate Value Theorem). Let $E \subset X$ be **connected** and $f : E \rightarrow \mathbb{R}$ **continuous**. Then $f(E)$ is connected in \mathbb{R} , hence an interval. Consequently, if $a, b \in E$ and c lies between $f(a)$ and $f(b)$, there **exists** $p \in E$ with $f(p) = c$.



The only connected subsets of \mathbb{R} are intervals. A common application: if $f : [a, b] \rightarrow \mathbb{R}$ is continuous and $f(a)f(b) < 0$, there exists $\xi \in (a, b)$ with $f(\xi) = 0$.

[Q]: Suppose $A \subset \mathbb{R}^k$ is an open set. Assume that for any $x, y \in A$, there exists a continuous function $f_{x,y} : [0, 1] \rightarrow \mathbb{R}^k$ such that $f_{x,y}(0) = x$, $f_{x,y}(1) = y$, and $f_{x,y}(t) \in A$ for any $t \in [0, 1]$. Show that A is connected.

19 Uniform continuity

Definition

Uniform continuity. $f : E \rightarrow Y$ is *uniformly continuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in E$, $d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$.

Ordinary continuity is local: the permissible δ may depend on the point;

Uniform continuity requires a single δ that works for all points simultaneously. This is especially important near the *boundary* of the domain (e.g. $f(x) = 1/x$ on $(0, \infty)$).

Example

$f(x) = 1/x$ on $(0, \infty)$ is not uniformly continuous. Take $\varepsilon_0 = 1$. For any $\delta \in (0, 1)$ choose $x = \delta/2$, $y = \delta/3$. Then $|x - y| = \delta/6 < \delta$ while $|f(x) - f(y)| = \left|\frac{2}{\delta} - \frac{3}{\delta}\right| = \frac{1}{\delta} > 1$.

Lemma

(Uniform continuity preserves Cauchy sequences). If $f : E \rightarrow Y$ is uniformly continuous and $(x_n) \subset E$ is Cauchy in (X, d_X) , then $(f(x_n))$ is Cauchy in (Y, d_Y) .

This fact lets us **extend uniformly continuous functions to the boundary of intervals** by defining the value at an endpoint as the limit along any sequence approaching that endpoint.

Corollary & Secondary Conclusion

(Uniformly continuous extension on an open interval). Let $I = (a, b) \subset \mathbb{R}$ (allow $a = -\infty$ or $b = +\infty$). If $f : I \rightarrow \mathbb{R}$ is **uniformly continuous**, there exists a **continuous** $\bar{f} : \bar{I} \rightarrow \mathbb{R}$ with $\bar{f}(x) = f(x)$ for $x \in I$.

This construction above works in any *complete* metric target (because Cauchy sequences converge). This is a central tool in analysis (e.g. in complex analysis).

If a continuous function is already defined on a compact set, then it must be uniformly continuous (Heine–Cantor below). Together with the previous remark, this provides a route to extend continuous functions from open sets to compact sets.

Theorem

Heine–Cantor: If $E \subset X$ is compact and $f : E \rightarrow Y$ is continuous, then f is uniformly continuous on E .

证明. proof:

□

參考文献

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