Generell Topologi

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23 Lectures 23–27

23.1 $\triangle$-complexes

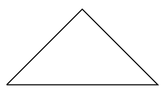
Remark 23.1. In the remaining lectures we'll introduce ideas around the classification of surfaces. We'll focus on the essence of this beautiful story, and not be completely precise. Rest assured that everything can be made entirely rigorous!

Terminology 23.2. A 0-*simplex* is a point in $\mathbb{R}^2$. A 1-*simplex* is a closed line segment in $\mathbb{R}^2$. A 2-*simplex* is a closed filled in triangle in $\mathbb{R}^2$.

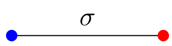
By a *simplex* we shall mean a 0-simplex, a 1-simplex, or a 2-simplex.

Remark 23.3. We will often regard a simplex as equipped with its subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$, but will omit to mention this from now on.

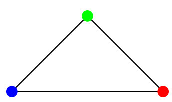
Remark 23.4. From now on all triangles in our pictures are to be regarded as filled in, or in other words as 2-simplices. For example, the following is to be regarded as a picture of a 2-simplex.



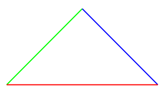
Terminology 23.5. A *vertex* of a 1-simplex is one of the following two 0-simplices.



A *vertex* of a 2-simplex is one of the following three 0-simplices.

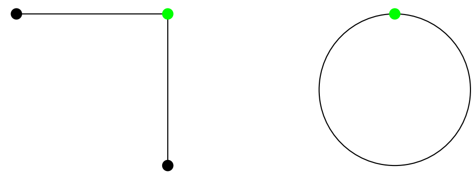


An *edge* of a 2-simplex is one of the following three 1-simplices.

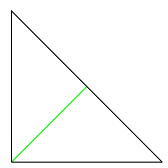


Definition 23.6. A Δ-*complex structure* on a topological space $(X, \mathcal{O}_X)$ is a recipe for constructing $(X, \mathcal{O}_X)$ by glueing together simplices in one of the following ways.

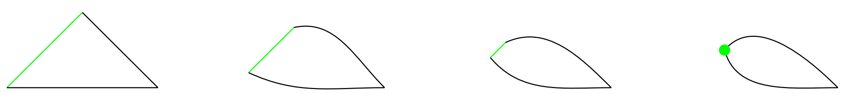
(1) A vertex of a 1-simplex may be glued to a vertex of a 1-simplex. These two 1-simplices may be the same or different.



(2) An edge of a 2-simplex may be glued to an edge of a 2-simplex. These two 2-simplices may be the same or different. We may glue with or without a twist.

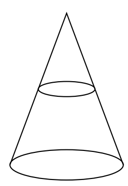


(3) An edge of a 2-simplex may be glued to a 0-simplex. This means that we identify all points on an edge of a 2-simplex together, and can be thought of as shrinking the edge to a point.

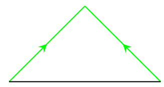


Examples 23.7.

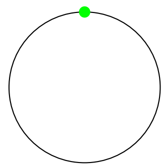
(1) A Δ-complex structure on a hollow cone



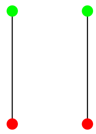
is given by glueing two edges of a single 2-simplex together.



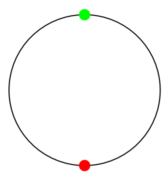
(2) A Δ-complex structure on a circle is given by glueing the two vertices of a single 1-simplex together.



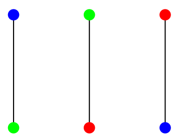
There are many other ways to equip a circle with a Δ-complex structure. For example we can glue two 1-simplices



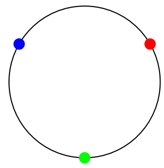
together by identifying the green vertices and identifying the red vertices.



We could glue three 1-simplices

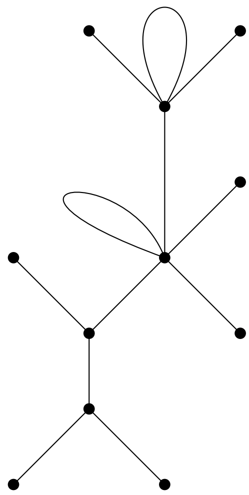


together, identifying each pair of vertices with the same colour.

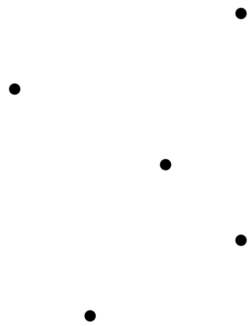


And so on!

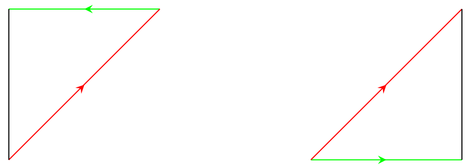
(3) Glueing together vertices of lots of 1-simplices we can equip a tree — possibly with loops — with the structure of a Δ-complex.



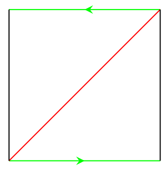
(4) A collection of points has the structure of a Δ-complex.



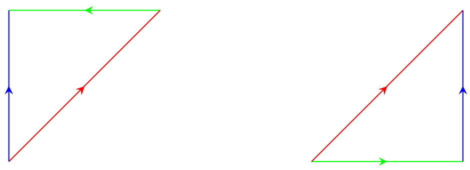
(6) A Δ-complex structure on the Möbius band $(M^2, \mathcal{O}_{M^2})$ is given by glueing together the green edges and glueing together the red edges of two 2-simplices as follows.



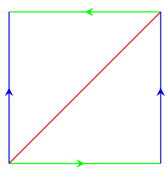
We often depict this in the following manner.



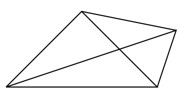
(7) A Δ-complex structure on the torus $(T^2, \mathcal{O}_{T^2})$ is given by glueing together the green edges, glueing together the blue edges, and glueing together the red edges of two 2-simplices as follows.



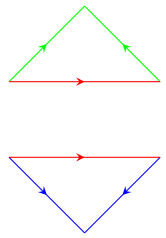
We often depict this in the following manner.



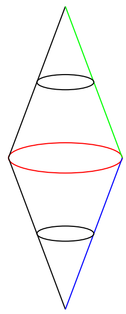
(8) A Δ-complex structure on the 2-sphere $(S^2, \mathcal{O}_{S^2})$ is given by glueing together edges of four 2-simplices as follows.



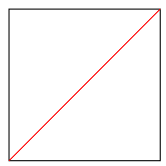
There are many other ways to equip $(S^2, \mathcal{O}_{S^2})$ with a Δ-complex structure. For example, we can glue together edges of two 2-simplices as follows.



This can be thought of as glueing the hollow cone from (1) to an upside hollow cone.

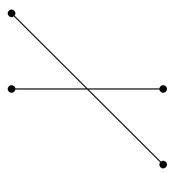


A third Δ-complex structure on $(S^2, \mathcal{O}_{S^2})$ is given by glueing two 2-simplices together to obtain a square



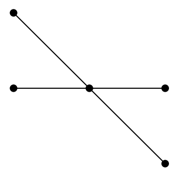
and moreover glueing all four of the remaining edges to a 0-simplex. This is the same idea as in the construction of $(S^2, \mathcal{O}_{S^2})$ in Examples 3.9 (6).

(9) Glueing two 1-simplices as follows does not define a Δ-complex structure.

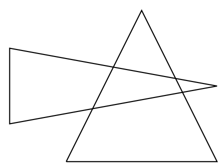


We are only permitted to glue in the three ways prescribed in Definition 23.6. Here we have glued the two 1-simplices in the middle, rather than glueing a vertex to a vertex.

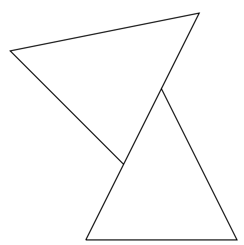
Nevertheless we can certainly equip this topological space with the structure of a Δ-complex by glueing together more 1-simplices.



(9) Glueing two 2-simplices as follows does not define a Δ-complex structure.



Nor does glueing two 2-simplices as follows.



We are only allowed to glue edges to edges.

23.2 Surfaces

Terminology 23.8. Let $(X, \mathcal{O}_X)$ be a topological space. Then $(X, \mathcal{O}_X)$ is *locally homeomorphic to an open disc* if for every $x \in X$ there is a neighbourhood U of x in $(X, \mathcal{O}_X)$ such that U equipped with its subspace topology with respect to $(X, \mathcal{O}_X)$ is homeomorphic to an open disc.

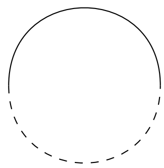
Definition 23.9. A topological space $(X, \mathcal{O}_X)$ is a *surface* if it is compact, connected, Hausdorff and is locally homeomorphic to an open disc.

Remark 23.10. A surface in the sense of Definition 23.9 is also known as a *closed surface.*

Terminology 23.11. We refer to the set

$$\left\{(x,y) \in \mathbb{R}^2 \mid 0 \leq \|(x,y)\| \leq 1 \text{ and } y \geq 0\right\} \cup \left\{(x,y) \in \mathbb{R}^2 \mid 0 \leq \|(x,y)\| < 1 \text{ and } y < 0\right\}$$

equipped with its subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ as a *half open disc.*

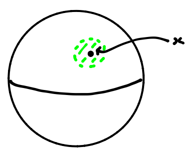


Remark 23.12. When deciding whether or not a given topological space $(X, \mathcal{O}_X)$ is a surface, we frequently encounter the situation that for some point x in X we can find a neighbourhood which equipped with its subspace topology with respect to $(X, \mathcal{O}_X)$ is homeomorphic to a half open disc.

This can be shown to imply that there does not exist a neighbourhood of x which is homeomorphic to an open disc. One needs techniques a little more sophisticated than those we have studied to prove this, which you will meet if you take Algebraic Topology I in the autumn. We shall take it on faith.

Examples 23.13.

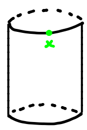
(1) $(S^2, \mathcal{O}_{S^2})$ is a surface. A point x on S^2 and a neighbourhood of x which equipped with its subspace topology is homeomorphic to an open disc is depicted below.



(2) $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is not a surface. It is connected, Hausdorff, and locally homeomorphic to an open disc, but is not compact.

(3) The cylinder $(S^1 \times I, \mathcal{O}_{S^1 \times I})$ is not a surface. It is compact, connected, and Hausdorff, but is not locally homeomorphic to an open disc.

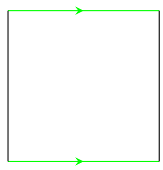
To see this, let x be a point on one of the boundary circles.



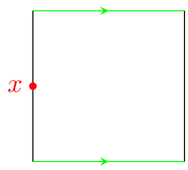
Then x admits a neighbourhood which is homeomorphic to a half open disc.

By Remark 23.12 we conclude that x does not admit a neighbourhood which is homeomorphic to an open disc.

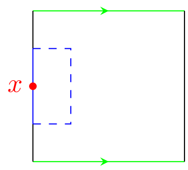
Let us carry out this argument if we instead view the cylinder as the quotient of I^2 by the equivalence relation indicated below.



We let x be a point on one of the black boundary edges.



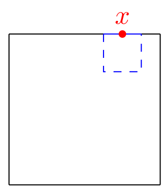
Then x admits a neighbourhood which is homeomorphic to a half open disc.



By Remark 23.12 we conclude that x does not admit a neighbourhood which is homeomorphic to an open disc.

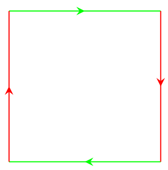
(3) $(I^2, \mathcal{O}_{I^2})$ is not a surface. It is compact, connected, and Hausdorff, but is not locally homeomorphic to an open disc.

Every point on its boundary admits a neighbourhood which is homeomorphic to a half open disc, and hence cannot admit a neighbourhood which is homeomorphic to an open disc.

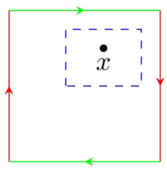


(5) $(T^2, \mathcal{O}_{T^2})$ and $(K^2, \mathcal{O}_{K^2})$ are surfaces. Let us explain why $(K^2, \mathcal{O}_{K^2})$ is locally homeomorphic to an open disc.

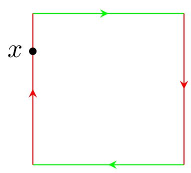
We view $(K^2, \mathcal{O}_{K^2})$ as the quotient of I^2 by the equivalence relation indicated below.



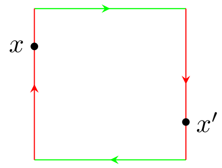
We can clearly find a neighbourhood homeomorphic to an open disc of any $[x] \in K^2$ such that x does not belong to $\partial_{\mathbb{R}^2} I^2$.



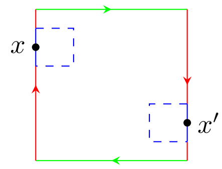
This was also true for the cylinder in (3). The difference with the cylinder is that we can also find a neighbourhood homeomorphic to an open disc of $[x] \in K^2$ for any $x \in \partial_{\mathbb{R}^2} I^2$.



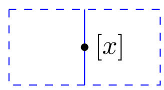
Let us explore this. For such an x there is a point x' on the opposite edge such that $[x'] = [x]$.



We can take a neighbourhood of each point in I^2 as indicated below.

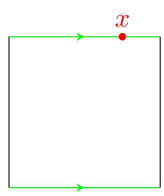


Each neighbourhood is homeomorphic to a half open disc in $(I^2, \mathcal{O}_{I^2})$, but in $(K^2, \mathcal{O}_{K^2})$ they become glued together to give a neighbourhood of $[x] = [x']$ which is homeomorphic to an open disc.



A similar argument proves that $(T^2, \mathcal{O}_{T^2})$ is locally homeomorphic to an open disc.

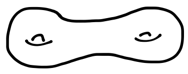
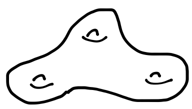
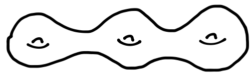
Moreover, let us view the cylinder as a quotient $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ of I^2 as in (3). A similar argument proves that $[x] \in I^2/\sim$ has a neighbourhood which is homeomorphic to an open disc for every point x belonging to a green edge.



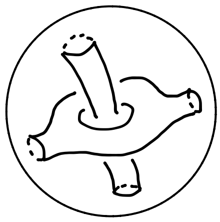
It is exactly the points on the black edges that do not admit a neighbourhood which is homeomorphic to an open disc.

(6) Here are a few, more exotic, examples of surfaces!

Gadgets similar to $(T^2, \mathcal{O}_{T^2})$ except with two or more holes.

A sphere with two intertwining tunnels.



A kind of knotted torus-like gadget.