Introduction to Zeta Functions and Applications

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Abstract

We introduce the Riemann and Dedekind zeta functions and their basic properties. Among applications, which are the main purpose of this paper, we mention the prime number theorem, splitting behavior of primes, the class number formula and Dirichlet's theorem on arithmetic progressions. Most results are mentioned but not proved. The material is learned from Neukirch's "Algebraic Number Theory" and Kowalski's notes on "Elementary Theory of L-functions."

1 Riemann zeta function and its properties.

1.1 The Riemann zeta function and the prime number theorem

The Riemann zeta function defined on $\Re s > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is one of central important in number theory. With it comes the celebrated unsolved Riemann Hypothesis: all nontrivial zeroes of ζ lies on the critical line $\Re s = 1/2$. One reason for the importance of ζ is that it encodes a surprising amount of arithmetic information within itself. Consider, for example, the following

Proposition 1. *Explicit Formula:*

$$\begin{aligned}\sum_p \sum_{k\geq 1} \log p\,(\varphi(p^k) + \psi(p^k)) \;&=\; \int_0^{+\infty} \varphi(x)dx - \sum_{\zeta(\rho)=0, 0<\Re\rho<1} \hat{\varphi}(\rho)\\ &+\frac{1}{2\pi i}\int_{-1/2}\left(\frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{1-s}{2}\Big)\right)\widehat{\varphi(s)}ds\end{aligned}$$

where φ is a smooth function with compact support, $\hat{\varphi}$ is its Mellin transform and $\psi(x)$: $=\frac{1}{x}\varphi\big(\frac{1}{x}\big)$. By the Mellin transform, we mean

$$\hat{\varphi}(s) = \int_0^{\infty} (\varphi(0) - \varphi(\infty))y^s \frac{dy}{y}.$$

This is the formula that captures the strange "duality" between prime numbers and zeroes of the ζ (notice how they come up in the second summand on the right hand side). When we take φ to be test functions converging (in the sense of distributions) to the characteristic function of $[0, X]$, the Explicit Formula yields

$$\sum_{p^k \leq X} \log p = X + O(X^{\beta})$$

iff $\beta > \sup \Re\rho$. In this light, the Riemann Hypothesis means that primes are distributed in the best possible way.

1.2 Properties of the Riemann zeta function

The Riemann zeta function enjoys many good analytic properties that make it pleasant to work with. The defining series converges absolutely and uniformly on $\Re s > 1$. In this region, we also have the Euler product

$$\zeta(s) = \sum_{p\text{ prime}} \frac{1}{1-p^s}$$

due to unique factorization in $\mathbb{Z}$. Much more important is the fact that ζ has an analytic continuation to $\mathbb{C} - \{0,1\}$ (in fact it has a simple pole at 1 and a removable singularity at 0) and that it has a functional equation

$$\xi(s) = \xi(1-s)$$

where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the normalized zeta function. This can be proved in various ways, Riemann himself gave two proofs. One proof uses integration along the Hankel contour, the other uses Poisson summation and theta functions. We give a sketch of the second proof, since the same idea is used in section 2.

The idea comes from the following identity

$$\int_0^\infty e^{-\pi t n^2} t^{s/2}\frac{dt}{t} = \pi^{s/2}\Gamma(s/2)n^{-s},$$

which follows by substituting $y \mapsto \pi n^2 y$ in the definition of the Γ function. Then for $\Re s > 1$,

$$\begin{aligned}
\xi(s) &= \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\sum_{n=1}^\infty \frac{1}{n^s} \\
&= \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 t} t^{s/2}\frac{dt}{t} \\
&= \int_0^\infty t^{s/2}\sum_{n=1}^\infty e^{-\pi n^2 t}\frac{dt}{t} \\
&= \int_0^\infty t^{s/2}\left(\theta(t) - \frac{1}{2}\right)\frac{dt}{t}
\end{aligned}$$

where $\theta(t)$ is the theta series $\frac{1}{2} + \sum_{n=1}^\infty e^{-\pi n^2 t} = \frac{1}{2}\sum_{n=-\infty}^\infty e^{-\pi n^2 t}$. On the second row, equality is due to absolute convergence of ζ in $\Re s > 1$. One nice feature of the theta series $\theta(t)$ is that the Fourier transform of $e^{-\pi n^2 t}$ is $\frac{1}{\sqrt{t}}e^{-\pi n^2/t}$. Thus under the Poisson summation formula, we have

$$\begin{aligned}
\theta(t) &= \frac{1}{2}\sum_{n=-\infty}^\infty e^{-\pi n^2 t} \\
&= \frac{1}{2}\sum_{n=-\infty}^\infty \frac{1}{\sqrt{t}}e^{-\pi n^2/t} \\
&= \frac{1}{\sqrt{t}}\theta\left(\frac{1}{t}\right).
\end{aligned}$$

Then substituting this into the formula for $\xi(s)$ yields at the same time analytic continuation and the desired functional equation.

One consequence is the "convexity bound"[1] of ζ on the critical strip:

$$|\zeta(1/2+it)| \leq C|t|^{1/4+\varepsilon}.$$

Essentially, from the Stirling approximation of Γ and that ζ is bounded on $\Re s = 1+\varepsilon$, the functional equation shows that ζ is $O\big(t^{1/2}\big)$ on $\Re s = -\varepsilon$. Then Hadamard's three line theorem implies the statement immediately. The conjecture that $1/4$ can replaced by 0 is the Lindelof Hypothesis, which turns out to be a consequence of the Riemann Hypothesis, remains an unsolved problem.

2 The Dedekind zeta function and its properties

2.1 Definition and simple properties

There are many different simple ways to generalize of the Riemann zeta function. One can replace the coefficients by Dirichlet characters and thus get the Dirichlet L-functions, or by the coefficients of Ramanujan's discriminant function. On the other hand, one can work over a larger field than $\mathbb{Q}$, and keeping the coefficients 1. In this case, one gets the Dedekind zeta function, and we will focus on this particular generalization.

Definition 2. *Let K be a number field, that is, an extension of finite degree d over $\mathbb{Q}$. The Dedekind zeta function on K is defined by the series*

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s},$$

where $\mathfrak{a}$ varies over the nonzero integral ideals of K, and $\mathfrak{N}(\mathfrak{a})$ denotes their absolute value norm.

The Dedekind zeta function enjoys many nice properties of the Riemann zeta function. First of all, ζ_K has the Euler product

$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1-\mathfrak{N}(\mathfrak{p})^{-s}},$$

where $\mathfrak{p}$ runs over the prime ideals of K. This is true formally due to unique factorization, as $\mathcal{O}_K$ is a Dedekind domain. Of course we still have to prove convergence of the product in $\Re s > 1$. Formally taking logarithm of the right hand side yields

$$\sum_{\mathfrak{p}} \sum_{n=1}^{\infty} \frac{1}{n\mathfrak{N}(\mathfrak{p})^{ns}}.$$

1. The name comes from the proof using Hadamard's three line theorem, which asserts convexity of the maximal function $M(z) = \sup_{x \text{ fixed}} (f(x+iy))$.

Notice that for $\Re s \geq 1+\delta$, $|\mathfrak{N}(\mathfrak{p})^s| = |\mathfrak{N}(\mathfrak{p})|^{\Re s} \geq p^{f_{\mathfrak{p}}(1+\delta)} \geq p^{1+\delta}$ where $f_{\mathfrak{p}}$ is the inertia degree of $\mathfrak{p}$. Furthermore, $\#(p/\mathfrak{p}) \leq d = [K\colon \mathbb{Q}]$, so we have the following dominating series

$$\sum_{p,n} \frac{d}{n p^{n(1+\delta)}} = d \log \zeta(1+\delta).$$

Since ζ converges absolutely and uniformly on $\Re s > 1$, the same must then be true for the Euler product. Then by a simple comparison, we find that the defining series for ζ_K also converges absolutely and uniformly over $\Re s > 1$.

2.2 Functional equation

The Dedekind zeta function also satisfies a functional equation. It can be proven for partial zeta functions

$$\zeta(\mathfrak{K}, s) = \sum_{\mathfrak{a} \in \mathfrak{K}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s}$$

where $\mathfrak{a}$ runs through the integral ideals in a class $\mathfrak{K}$. Since

$$\zeta_K(s) = \sum_{\mathfrak{K}} \zeta(\mathfrak{K}, s),$$

the functional equation for $\zeta_K(s)$ follows immediately.

PROOF. (sketch) The argument for partial zeta functions mimicks the one for the Riemann zeta function but relies on Minkowski theory. In the following sketch of the proof, we will not shy from using the jargons and will not try to explain by building the necessary theory. It is possible to choose an ideal $\mathfrak{a}^{-1}$ in the class $\mathfrak{K}$ such $\mathfrak{a}$ parametrizes integral ideals in $\mathfrak{K}$, and we fix such a choice. The strategy is to show that $\zeta(\mathfrak{K}, s)$ is the Mellin transform of a generalized theta series. More precisely, if we denote

$$Z(\mathfrak{K}, s) := |d_K|^{s/2} \pi^{-ns/2} \Gamma_K(s/2) \zeta(\mathfrak{K}, s)$$

where Γ_K is a higher dimensional Gamma function[2], then $Z(\mathfrak{K}, s)$ is the Mellin transform of

$$f_F(\mathfrak{a}, t) = \frac{1}{w} \int_F \theta\big(\mathfrak{a}, i x t^{1/n}\big) d^* x$$

where w is the number of roots of unity in K and F is the fundamental domain of the norm-one hyper surface $S = \{x \in R_+^* |\, N(x) = 1\}$ under the action of $\{|\varepsilon|^2 |\, \varepsilon \in \mathcal{O}^*\}$. The measure $d^* x$ here is the unique Haar measure on S such that the canonical Haar measure on R_+^* becomes $d^* x \times \frac{dt}{t}$. The function $f_F(\mathfrak{a}, t)$ satisfies a transformation formula

$$f_F(\mathfrak{a}, 1/t) = t^{1/2} f_{F^{-1}}((\mathfrak{a}\mathfrak{d})^{-1}, t)$$

2. It is defined by $\Gamma_K(s) = \Gamma_{\mathrm{Hom}(K,\mathbb{C})}(s) = \int_{R_+^*} N(e^{-y} y^s) \frac{dy}{y}$, where R_+^* denotes the "positive reals" in the Minkowski space $K \otimes_{\mathbb{Q}} \mathbb{R}$, and $N(z)$ denotes the norm, i.e. the determinant of the endomorphism by z of the $K \otimes_{\mathbb{Q}} \mathbb{C}$.

where $\mathfrak{d}$ is the different of $K/\mathbb{Q}$. By the Mellin principle[3], this implies the functional equation for $Z(\mathfrak{K}, s)$, and the proof is complete. $\square$

3 Applications of the Dedekind zeta function

3.1 Prime splitting in $\mathcal{O}_K$

The first reason why Dedekind zeta functions are important is the following. If we write $\zeta_K(s)$ as an ordinary Dirichlet series with coefficients $r_K(n)$, i.e.

$$\zeta_K(s) = \sum_{n \geq 1} \frac{r_K(n)}{n^s},$$

then information about how prime numbers split in $\mathcal{O}_K$ can be read off from ζ_K. Recall that in K, we have $(p) = \prod_i \mathfrak{p}_i^{e_i}$ and $\sum_i e_i f_i = d$, where f_i is the inertia degree of $\mathfrak{p}_i$ and $d = [K:\mathbb{Q}]$.

Notice now that $r_K(p)$ is the number of ideals with norm p. Only primes above p can have such a norm. We then have the following dictionary:

- $r_K(p) = d$ iff p splits completely in K. This is because each prime over p contributes at least 1 to the left hand side of $\sum_i e_i f_i = d$.
- $r_K(p) = 1$ iff p is totally ramified. This is because there is only one prime over p.
- In general, $r_K(p)$ gives the number of distinct primes above p.

In this light, the Riemann zeta function is “trivial” because it does not tell us anything we do not already know about splitting behavior of primes.

3.2 Class number formula

We can write

$$\begin{aligned} f_F(\mathfrak{a}, t) &= \frac{1}{w}\int_F d^*x + \frac{1}{w}\int_F \left(\theta\left(\mathfrak{a}, ixt^{1/n}\right) - 1\right) \\ &= \frac{\mathrm{vol}(F)}{w} + r(t) \end{aligned}$$

It turns out that $r(t)$ is $O\left(e^{-ct^{1/n}}\right)$, so the residue of $Z(\mathfrak{K}, s)$ at 1 is $2\frac{\mathrm{vol}(F)}{w} = \frac{2^r}{w}R$, where R is the regulator of K. As $Z_K(s)$ is obtained by summing $Z(\mathfrak{K}, s)$ over all classes, its residue is $\frac{2^r h}{w}R$, where h is the class number of K. Thus, the residue of $\zeta_K(s)$ at $s = 1$ is given by

$$\mathrm{Res}_{s=1}\zeta_K(s) = \frac{2^r (2\pi)^s h R}{w\sqrt{|d_K|}}$$

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3. (Mellin principle) Let $f, g\colon \mathbb{R}_+^* \longrightarrow \mathbb{C}$ be continuous functions such that $f(y) = a_0 + O(e^{-cy^\alpha})$, $g(y) = b_0 + O(e^{-cy^\alpha})$ for $y \to \infty$, positive constants c, α. If these functions satisfy the equation $f\left(\frac{1}{y}\right) = Cy^k g(y)$ for some real $k > 0$, and complex $C \neq 0$, then we have the functional equation

$$L(f, s) = CL(g, k - s).$$

This is essentially the step where we substitute $\theta(1/t)/\sqrt{t}$ in ξ, but with more generality.

where r is the number of real embeddings and s the number of conjugate pairs of complex embeddings. By rearranging, we get a formula for the class number if we know the residue at 1:

$$h = \frac{\mathrm{Res}_{s=1}\zeta_K(s) w \sqrt{|d_K|}}{2^r (2\pi)^s R}.$$

This is an instance of a number of formulae expressing the special values of L-functions in terms of global invariants. A more famous instance is the Birch and Swinnerton-Dyer conjecture, which asserts that the rank of the abelian group $E(K)$ of points of an elliptic curve E over a number field K is the order of the zero of $L(E, s)$ at $s = 1$.

3.3 Dirichlet's theorem on arithmetic progressions

If $K = \mathbb{Q}(\mu_m)$ is a cyclotomic field, then

$$\zeta_K(s) = \prod_{\mathfrak{p}|m} (1 - \mathfrak{N}(\mathfrak{p})^{-s})^{-1} \prod_{\chi} L(\chi, s)$$

where $L(\chi, s)$ is the Dirichlet series $\sum_{n=1}^{\infty} \chi(n) n^{-s}$ for a Dirichlet character χ, and χ varies among all such character mod m. Among all characters, the trivial one yields the Riemann zeta function. Since both ζ_K and ζ have a simple pole at 1, $L(\chi, 1) \neq 0$ for all non-trivial χ.

Now fix a Dirichlet character χ mod m. For $\Re s > 1$,

$$\begin{aligned} \log L(\chi, s) &= -\sum_p \log(1 - \chi(p) p^{-s}) \\ &= \sum_p \sum_{r=1}^{\infty} \frac{\chi(p^r)}{r p^{rs}} \\ &= \sum_p \frac{\chi(p)}{p^s} + g_{\chi}(s) \end{aligned}$$

where $g_{\chi}(s)$ is the remaining terms. Summing over all χ, letting $g = \sum_{\chi} \chi(a^{-1}) g_{\chi}$, we have

$$\begin{aligned} \sum_{\chi} \chi(a^{-1}) \log L(\chi, s) &= \sum_{\chi} \sum_p \frac{\chi(a^{-1}p)}{p^s} + g(s) \\ &= \sum_{b=1}^{m} \sum_{\chi} \chi(a^{-1}b) \sum_{p \equiv b (\mathrm{mod}\, m)} \frac{1}{p^s} + g(s) \\ &= \sum_{p \equiv a\, (\mathrm{mod}\, m)} \frac{\varphi(m)}{p^s} + g(s) \end{aligned}$$

When $s \to 1$, except for the term with the trivial character, all terms on the left hand side are bounded because $L(\chi, 1) \neq 0$. The term with trivial character tends to ∞ because the Riemann zeta function has a pole at 1. On the right hand side, $g(s)$ is holomorphic, so $\sum_{p \equiv a\, (\mathrm{mod}\, m)} \frac{\varphi(m)}{p^s}$ diverges. Thus there has to be infinitely many terms. We have thus proved

Theorem 3. *(Dirichlet) Every arithmetic progression* $a x + b$ *with* $\gcd(a, b) = 1$ *contains infinitely many prime numbers.*

References

J. Neukirch, Algebraic Number Theory, Chapter 7, Springer.

E. Kowalski, Elementary Theory of L-functions I, Introduction to the Langlands Program.