Questions about the package of L-functions of the symmetric n-th powers of a given automorphic form

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Rough Notes for a talk for the Tatefest

1 Brief Introduction

Specifically, let f be a newform of weight k and *assume* that, for all $n = 1, 2, 3, \dots$ the L function $L(symm^n f, s)$ extends to a meromorphic (meromorphic is enough) function on the entire complex plane and satisfies the expected functional equation. We normalize things so that $Re(s) = 1/2$ is the central line. Form

$$r_f(n) := \text{ the order of vanishing of } L(symm^n f, s) \text{ at } s = 1/2.$$

What can we say about the asymptotics of the integer-valued function

$$n \ \mapsto \ r_f(n)?$$

It is timely to ask this type of question since we have seen some progress in the problem of showing meromorphic continuation of various L-functions recently (—in the proof by Clozel, Harris, Shepherd-Baron, and Taylor of the Sato-Tate Conjecture for elliptic curves possessing some place of multiplicative reduction). This breakthrough in our understanding of Sato-Tate, plus ideas of Sarnak, Rubenstein, and computations of William Stein, comprise the motivation for the present lecture.

2 Bias

One of the many questions that seems natural to ask, given the recent work on Sato-Tate is what I'll call *bias questions,* a typical one being:

Given an elliptic curve over the rational numbers, and letting p range through prime numbers, how often is $p+1$ an over-count or an under-count for the number of rational points on the curve modulo p?

This question, of course, bears on Birch's and Swinnerton-Dyer's initial "hunch" that the statistical preponderance of solutions modulo p of an elliptic curve is a predictor of whether or not the elliptic curve has infinitely many rational points.

Grossly, the answer is expected to be *equally often* in the sense that, putting

$$N_E(p) = 1 + p - a_E(p) := \text{ the number of rational points on } E \text{ over } \mathbf{F}_p,$$

the ratio

$$\frac{\#\{p < X \mid N_E(p) < p+1\}}{\#\{p > X \mid N_E(p) > p+1\}} = \frac{\#\{p < X \mid a_E(p) > 0\}}{\#\{p > X \mid a_E(p) < 0\}}$$

tends to 1 as X goes to infinity. And this follows for a large class of elliptic curves, as a consequence of recent work on the Sato-Tate Conjecture.

But—given that this gross question is resolved in lots of cases—we can ask finer questions, not about the ratio, but about, say, the difference. I say "say, the difference" because there is a variety of "smoother" clever ways to measure the preponderance for the $a_E(p)$'s to be positive or negative (and to view this preponderance in graphs). To give some ad hoc terms for this, let us refer—in this lecture only— to (slightly doctored version of) the straight difference,

$$D_E(X) := \frac{\log\ X}{\sqrt{X}} \#\{p < X \mid a_E(p) > 0\} \ - \ \#\{p < X \mid a_E(p) < 0\},$$

as the **raw data,** to

$$\mathcal{D}_E(X) := \frac{\log\ X}{\sqrt{X}} \sum_{p \le X} \frac{a_E(p)}{\sqrt{p}}$$

as the **medium-rare data**, and

$$\Delta_E(X) := \frac{1}{\log\ X} \sum_{p \le X} \frac{a_E(p) \log p}{p}$$

as the **well-done data**.

Not to build up too much suspense here, the reason for selecting these three formats for the "data" and for the specific normalizations chosen (i.e., the factor $\frac{\log\ X}{\sqrt{X}}$ occurring in the first two, and the factor $\frac{1}{\log\ X}$ in the third) is that—if a *load* of conjectures hold—then all three formats will have finite *means* (relative to the measure dx/x on $\mathbf{R}$), and what distinguishes these formats is that

- the raw data will have *infinite* variance,

- the medium-rare data will have *finite variance*, and
- the well-done data will actually achieve its mean as a limiting value.

The fun here is that the conjecture for the values of *means* in the three formats is as follows:

- **The well-done data:** the mean is (conjecturally) $r :=$ the *analytic rank* of E.
- **The medium-rare data:** the mean is (conjecturally) $1 - 2r$ and
- **The raw data:** the mean is (conjecturally)

  $$\frac{2}{\pi} - \frac{16}{3\pi}r \quad + \quad \frac{4}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\Big[\frac{1}{2k+1} + \frac{1}{2k+3}\Big]r(2k+1).$$

  where

  $$r(n) := \; r_{f_E}(n) \; = \; \text{the order of vanishing of } L(symm^n f_E, s) \text{ at } s = 1/2,$$

  with $f_E :=$ the newform of weight two corresponding to the elliptic curve E,

which leads us to our initial question:

> What is the behavior of the function
>
> $$n \mapsto r_f(n)$$
>
> for fixed f and varying n?

3 Distributions on R; mean and variance

Recall that if $X \mapsto \delta(X)$ **is a (continuous) function of a real variable, to say that** $\delta(X)$ **possesses a limiting distribution** μ_δ **with respect to the multiplicative measure** dx/x means that for continuous bounded functions f on $\mathbf{R}$ we have:

$$\lim_{X\to\infty} \frac{1}{\log X}\int_0^X f(\delta(x))dx/x \; = \; \int_{\mathbf{R}} f(x)d\mu_\delta(x).$$

Recall that the **mean** of the function $\delta(X)$ (again, relative to dx/x) is defined by the limit

$$\mathcal{E}(\delta) := \lim_{X\to\infty} \frac{1}{\log X}\int_0^X \delta(x)dx/x \; = \; \int_{\mathbf{R}} d\mu_\delta(x).$$

The depressing thing here is that if you take a function $\delta(X)$ that is anything you want up to $X = 4,000,000$ and equal to 5 for $X > 4,000,000$ then the mean of δ is equal to 5, so what in the world can it mean[1] to compute data up to $4,000,000$? But we press on.

[1] poor pun intended

4 The well-done data

(Graphs of $X \mapsto \Delta_E(X) = \frac{1}{\log \ X} \sum_{p \le X} \frac{a_E(p) \log p}{p}$**)**

Rank $r = 0$**:** $\mathcal{E} =$**11A.**

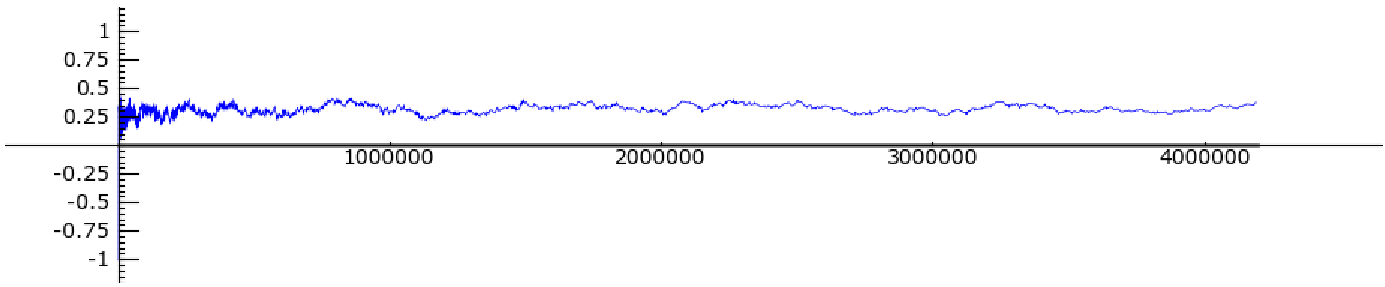

Figure 4.1:

Rank $r = 1$**:** $\mathcal{E} =$**37A.**

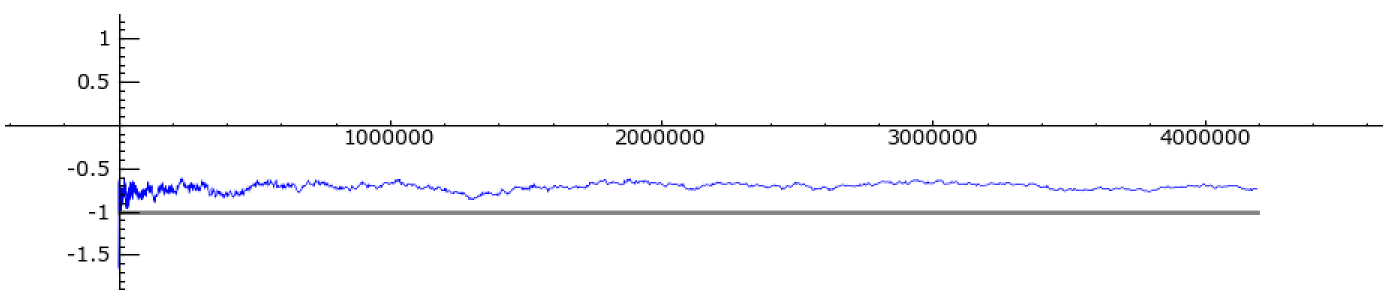

Figure 4.2:

Rank $r = 2$: $\mathcal{E}$ **=389A.**

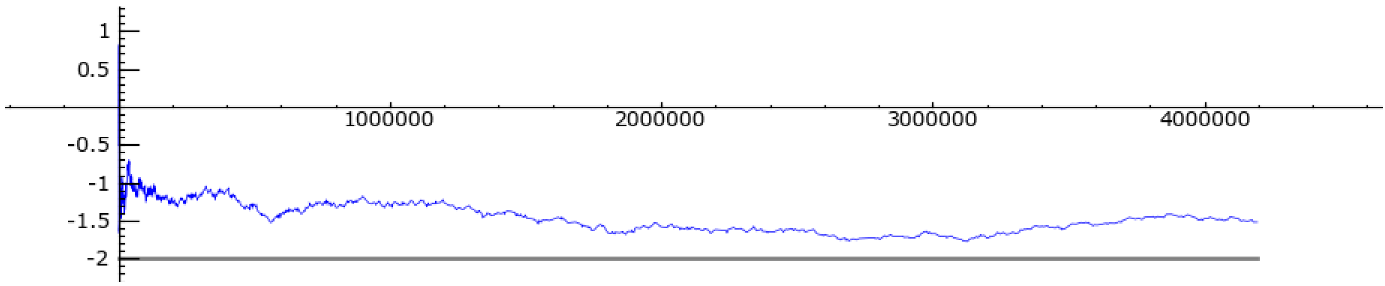

Figure 4.3:

Rank $r = 3$: $\mathcal{E}$ **=5077A.**

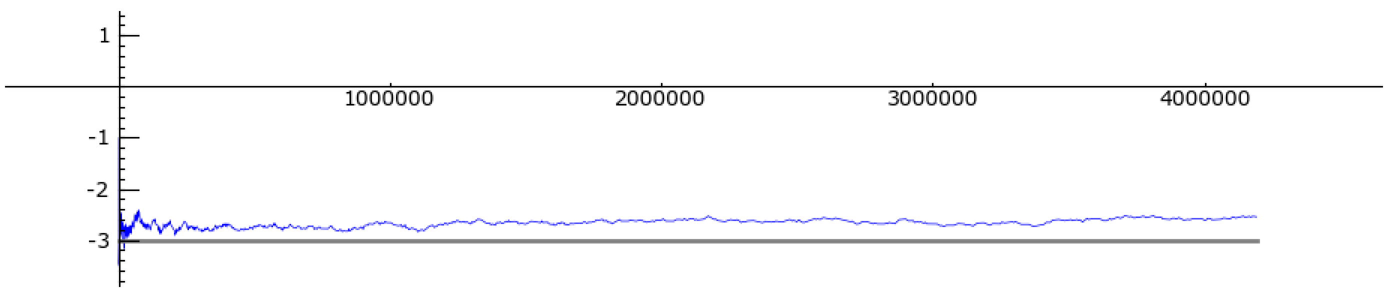

Figure 4.4:

Rank $r = 4$**.**

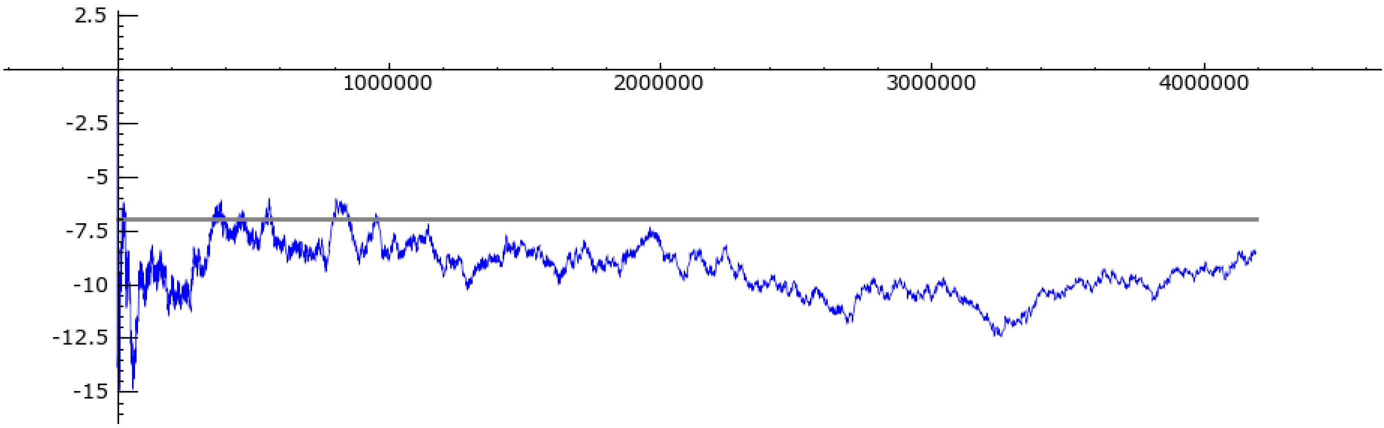

Figure 4.5:

Rank $r = 6$**.**

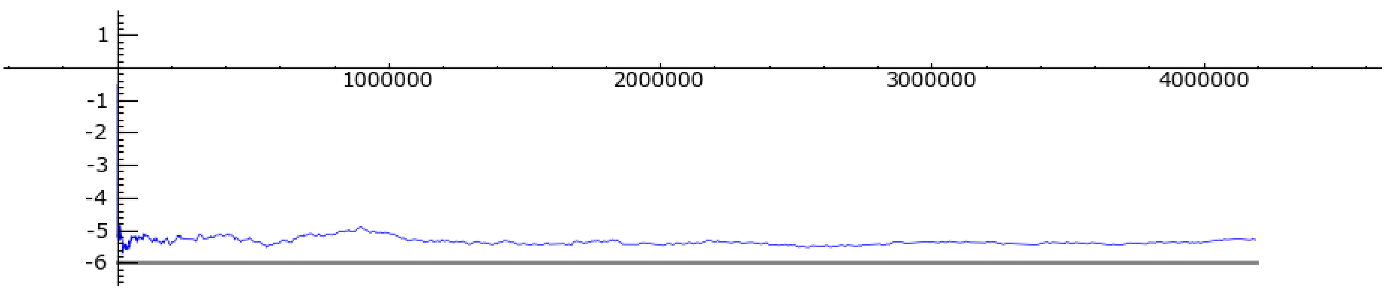

Figure 4.6:

5 The medium-rare data

(**Graphs of** $X \mapsto \mathcal{D}_E(X) = \frac{\log X}{\sqrt{X}} \sum_{p \le X} \frac{a_{\mathcal{E}}(p)}{\sqrt{p}}$)

Rank $r = 0$**:** $\mathcal{E} =$**11A.**

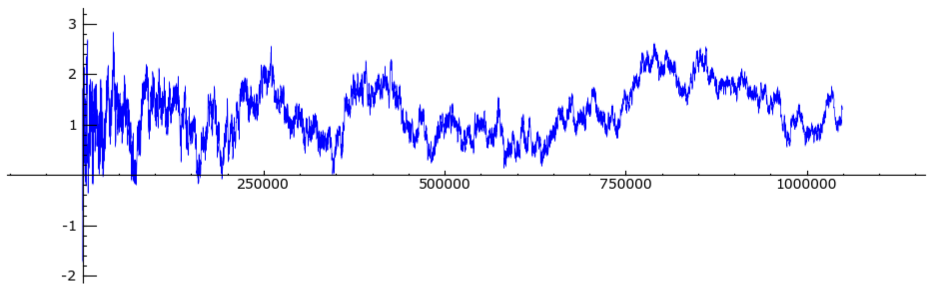

Figure 5.1:

Rank $r = 1$**:** $\mathcal{E} =$**37A.**

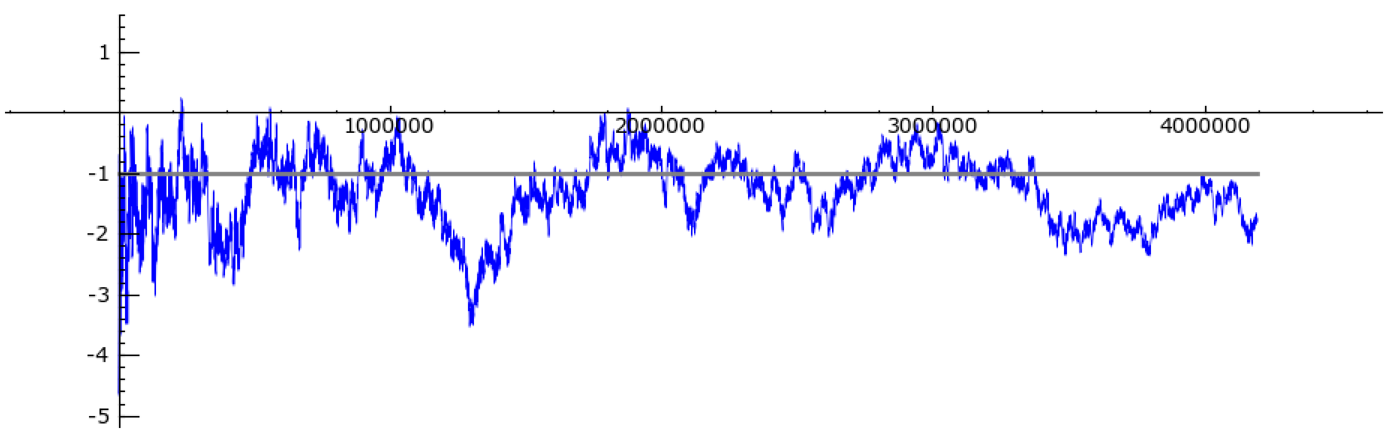

Figure 5.2:

Rank $r = 2$: $\mathcal{E}$ =**389A.**

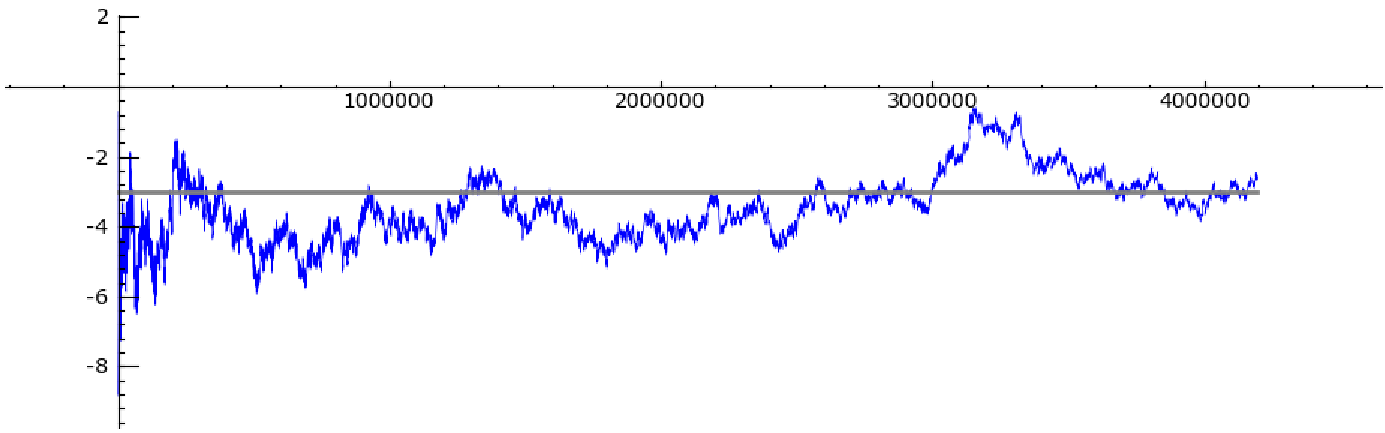

Figure 5.3:

Rank $r = 3$: $\mathcal{E}$ =**5077A.**

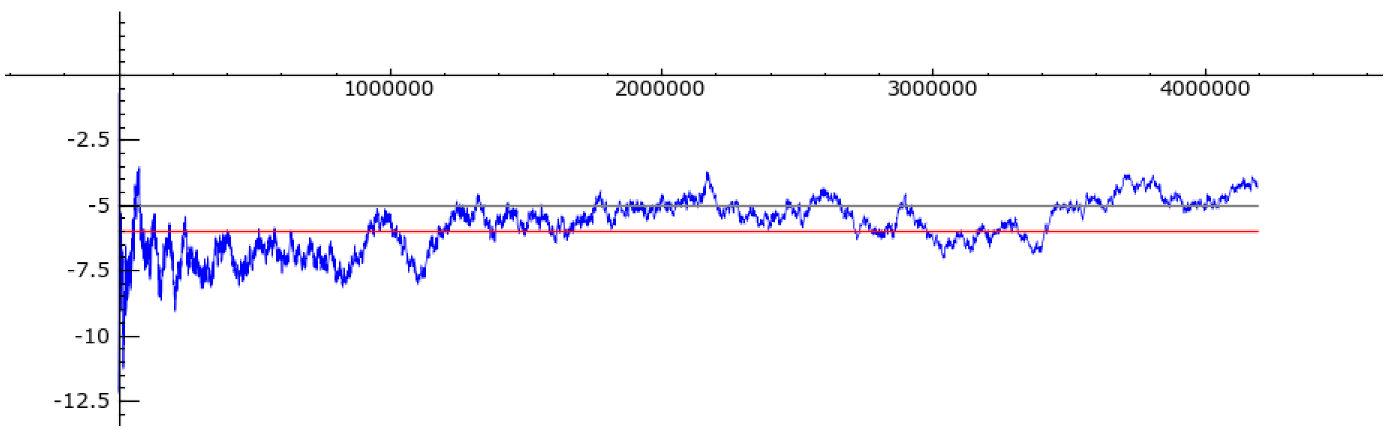

Figure 5.4:

Rank $r = 4$**.**

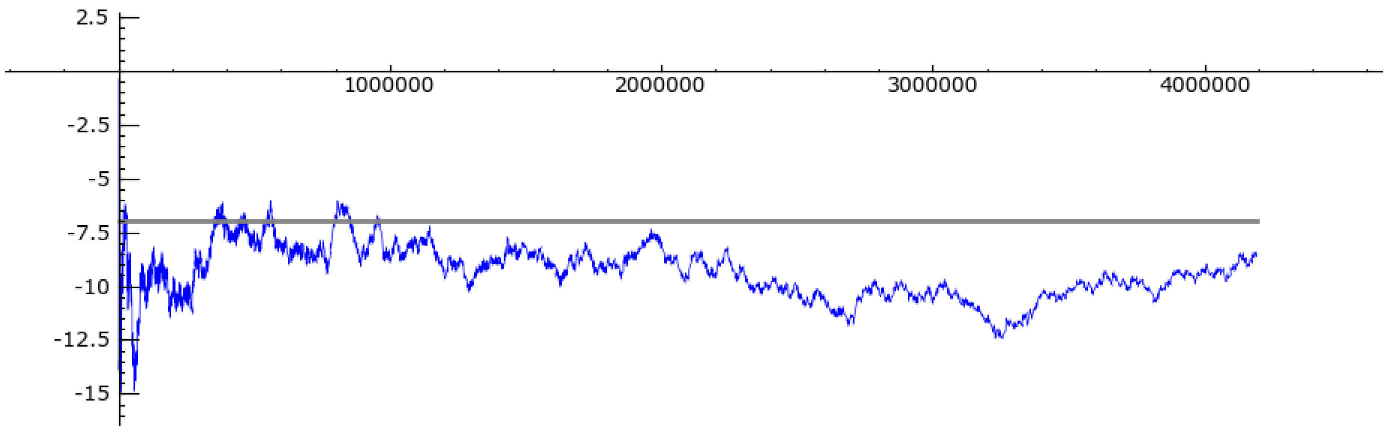

Figure 5.5:

Rank $r = 5$**.**

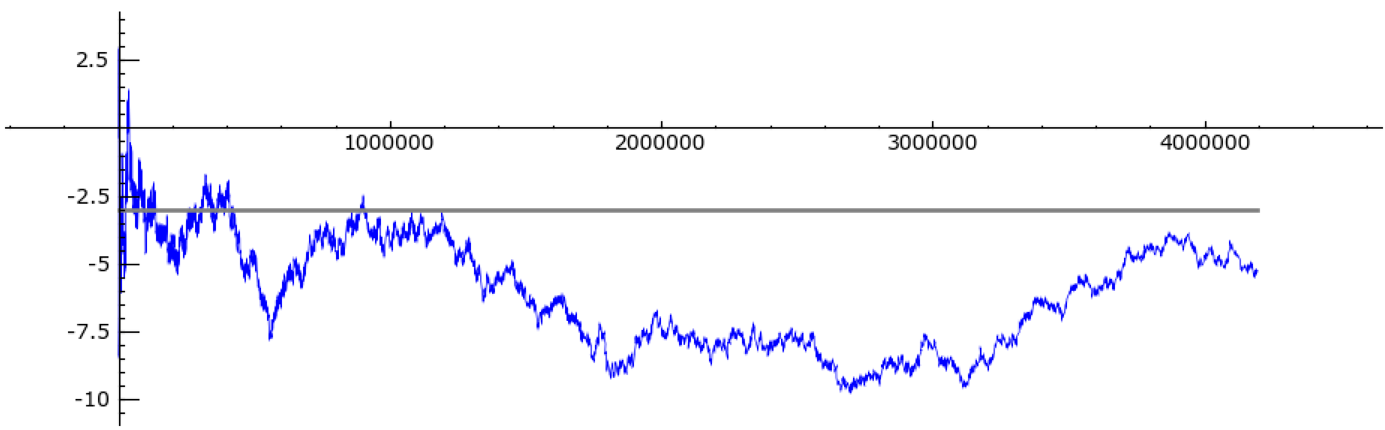

Figure 5.6:

Rank $r = 6$.

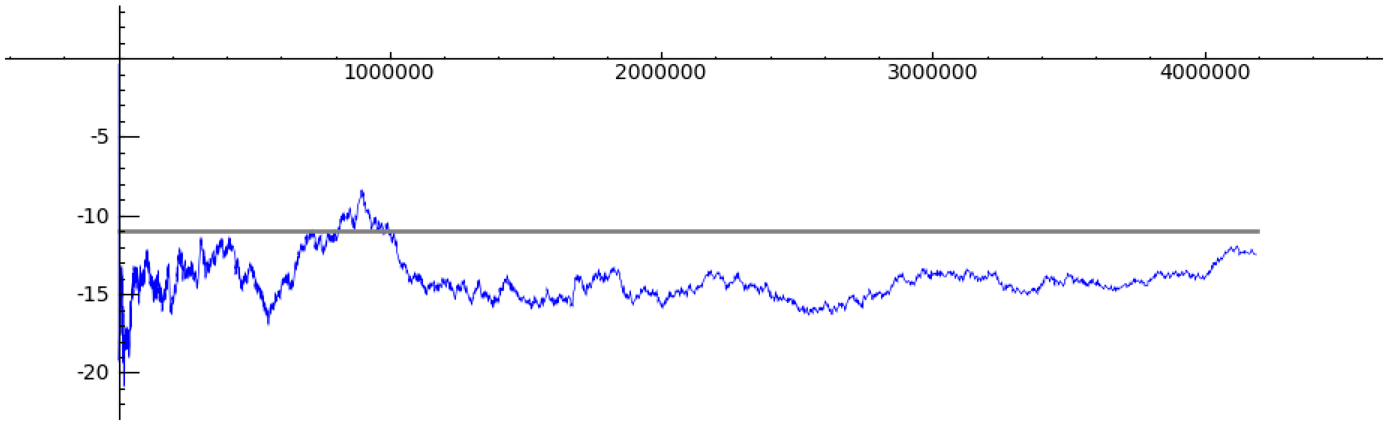

Figure 5.7:

6 The raw data

(Graphs of $X \mapsto D_E(X) = \frac{\log X}{\sqrt{X}} \#\{p < X \mid a_E(p) > 0\} \; - \; \#\{p < X \mid a_E(p) < 0\})$

Rank $r = 0$**:** $\mathcal{E}$ =**11A.**

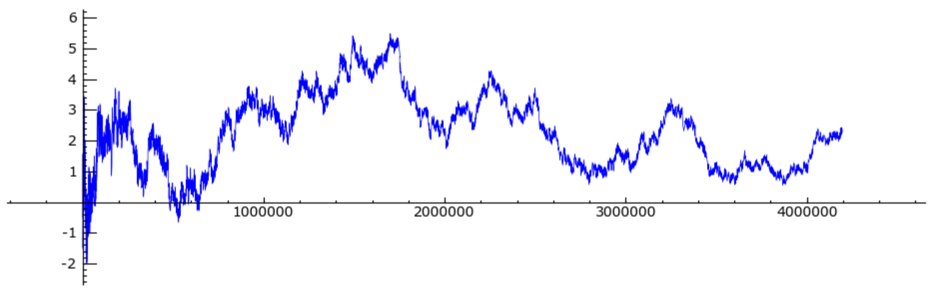

Figure 6.1:

Rank $r = 1$**:** $\mathcal{E}$ =**37A.**

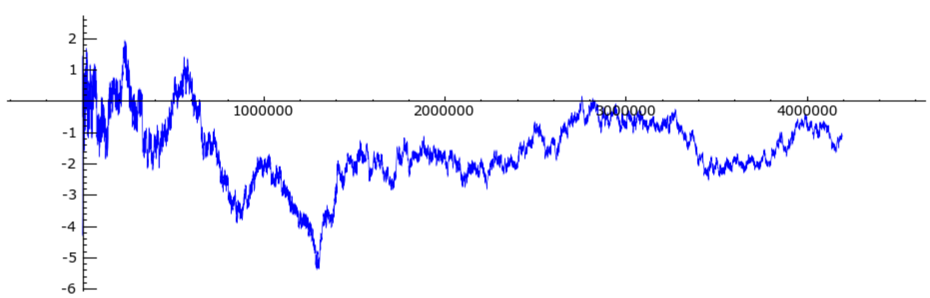

Figure 6.2:

Rank $r = 2$**:** $\mathcal{E} =$**389A.**

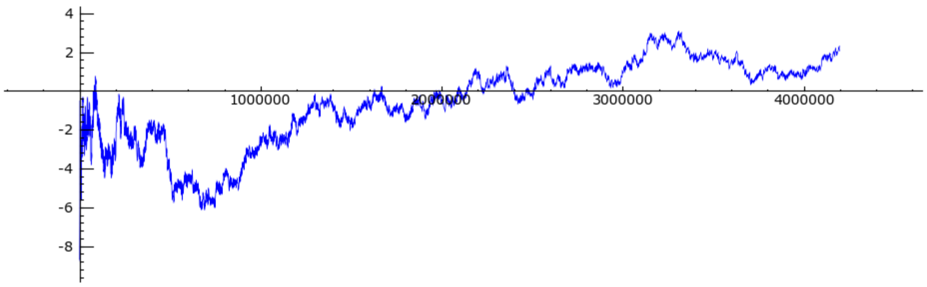

Figure 6.3:

Rank $r = 3$**:** $\mathcal{E} =$**5077A.**

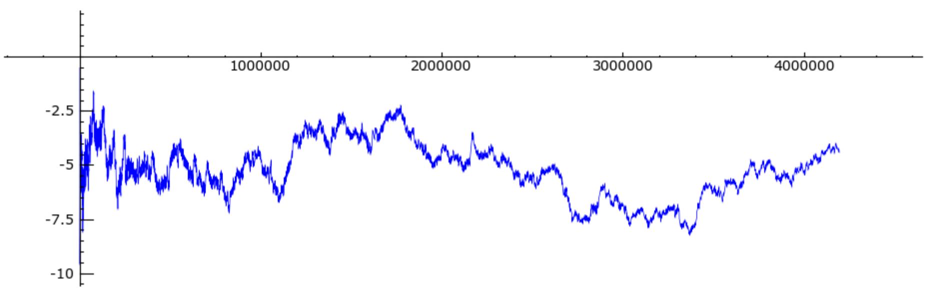

Figure 6.4: