

Prime Numbers and the Riemann Hypothesis

(Writing a book with Barry Mazur)

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Abstract

Abstract

In 2004, Barry Mazur and I started writing the book “Prime Numbers and the Riemann Hypothesis”, which we recently published with Cambridge University Press. I’ll talk about what’s in the book and why, describe some of the technical aspects of production of the book, and tradeoffs for us between self publishing online versus with a traditional publisher.

This is talk about talking about the Riemann Hypothesis.

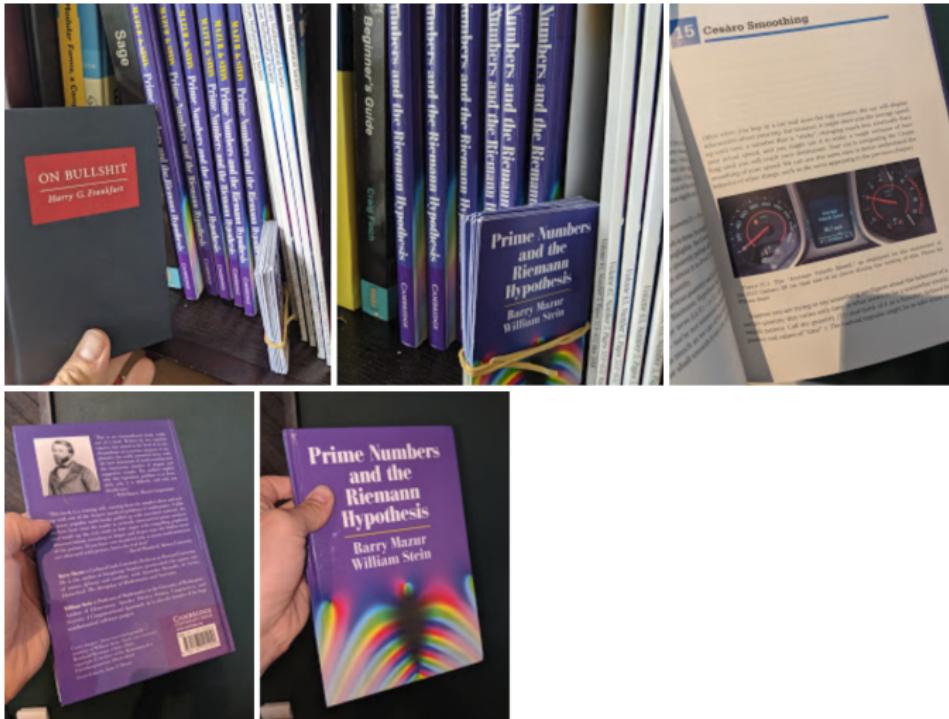
Overview

1 A Public Lecture

2 Write a Book

3 Publish a Book

The book...



Clay Math Institute public lecture (MIT, May 3, 2005)

Are there still unsolved problems about the numbers 1, 2, 3, 4, ... ?

“Sell” number theory to the general public

- Immediately accessible
- Immediately interesting
- Primes and how erratic they are
- Cicada's every 13, 17 years...
- Lots of examples that are “opening, interesting questions/issues.”
- People can immediately make computations of their own
- Barry Mazur got his father who had done NO math hooked on Goldbach Conjecture, so thought primes would work.

SageMath



I started SageMath

I launched Sage a few months before this 2005 CMI public lecture.

- Sage is a free open source alternative to Mathematica, Maple, Magma, and Matlab.
- Early Sage development was motivated by this talk
 - Linking Sage to Mathematica to compute the incomplete Gamma function.
 - Early plotting functionality.

More about what was in the public lecture...

Let's write a book...

Turn this lecture into a book.

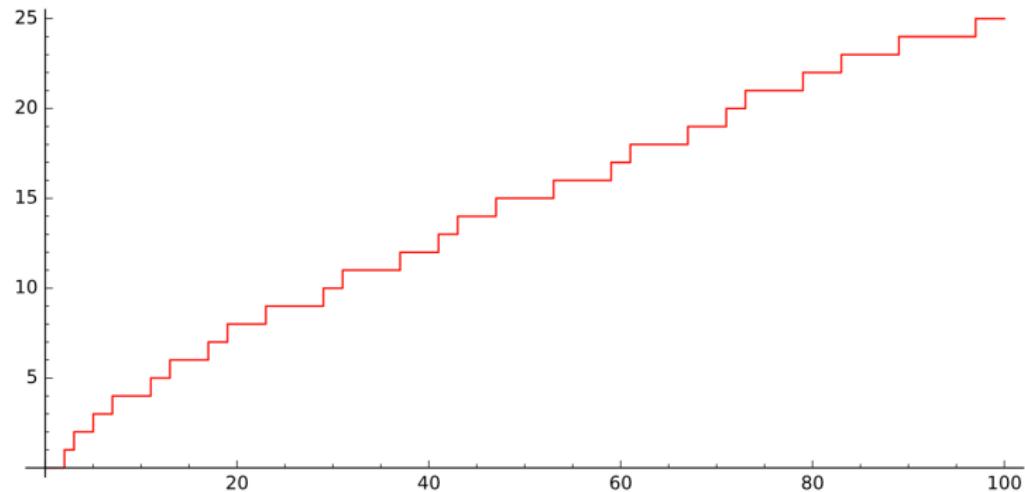
- Write something for a popular audience.
- Small and readable.
- Full of *mathematics*, not stories of people.
- Profusely illustrated.
- Meet for a few weeks each year and focus on this.

The Prime Counting Problem

Let $\pi(x)$ be the number of primes $\leq x$.

Problem: give a “good approximation” for $\pi(x)$.

```
plot(prime_pi, 0, 100, color='red', figsize=[8,4])
```

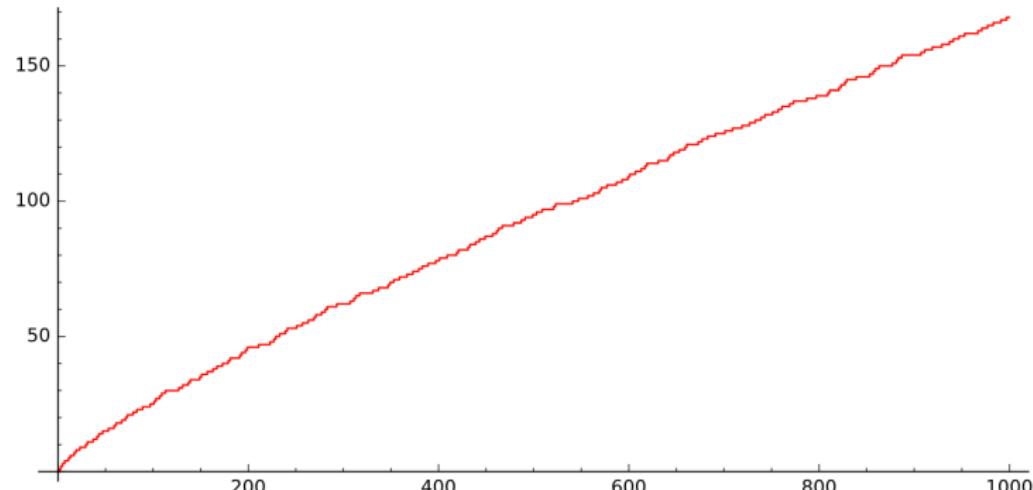


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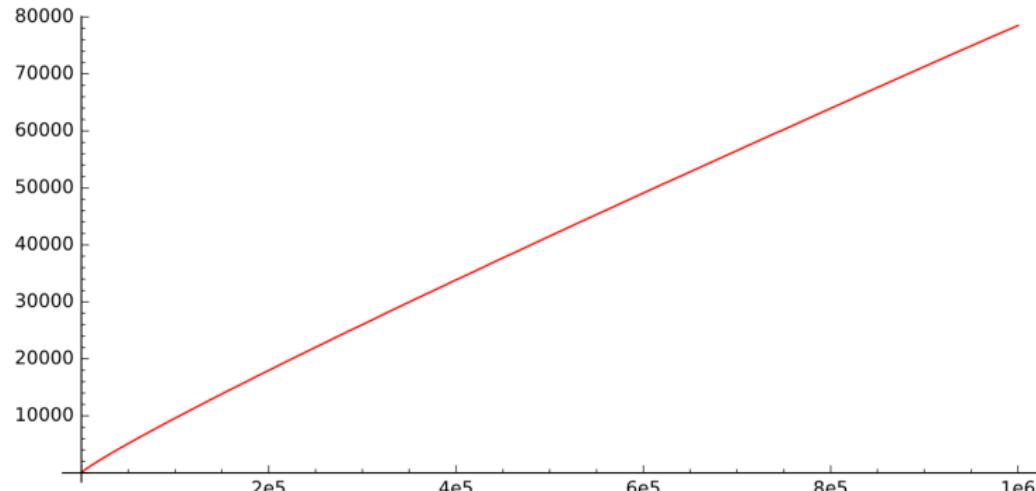


The Prime Counting Problem

Let $\pi(x)$ be the number of primes $\leq x$.

Problem: give a “good approximation” for $\pi(x)$.

```
plot(prime_pi, 0, 10^6, color='red', figsize=[8,4])
```



Answer: The Riemann Hypothesis (first formulation)

The number of prime numbers less than X is approximately $\text{Li}(X)$ and this approximation is essentially square root accurate.

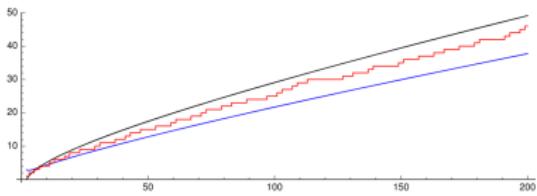


Figure 13.1: Plots of $\text{Li}(X)$ (top), $\pi(X)$ (in the middle), and $X/\log(X)$

$$\pi(X) = 18,435,599,767,349,200,867,866$$

$$\text{Li}(X) = 18,435,599,767,366,347,775,143.10580\dots$$

$$X/(\log(X) - 1) = 18,429,088,896,563,917,716,962.93869\dots$$

$$\text{Li}(X) - \pi(X) = 17,146,907,277.105803\dots$$

$$\sqrt{X} \cdot \log(X) = 55,262,042,231,857.096416\dots$$

Answer: The Riemann Hypothesis (second formulation)

The prime power staircase $\psi(X)$ is essentially square root close to the 45 degree straight line.

$\psi(x)$: Our staircase starts on the ground at $x = 0$ and the height of the riser of the step at $x = 1$ will be $\log(2\pi)$. The height of the riser of the step at $x = p^n$ will not be 1 but rather: the step at $x = p^n$ will have the height of its riser equal to $\log p$.

Answer: The Riemann Hypothesis (third formulation)

Basically, *the explicit formula*: the Fourier transform of the derivative of $\psi(X)$ “is” a discrete distribution (supported at the real parts of the zeros of the Riemann Zeta function). We deleted this from book, since it was too technical to precisely state. .)

(illustrative plot.)

Answer: The Riemann Hypothesis (fourth formulation)

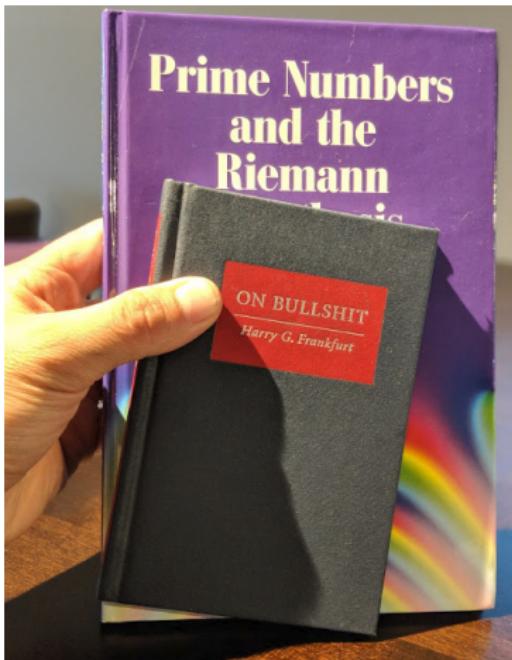
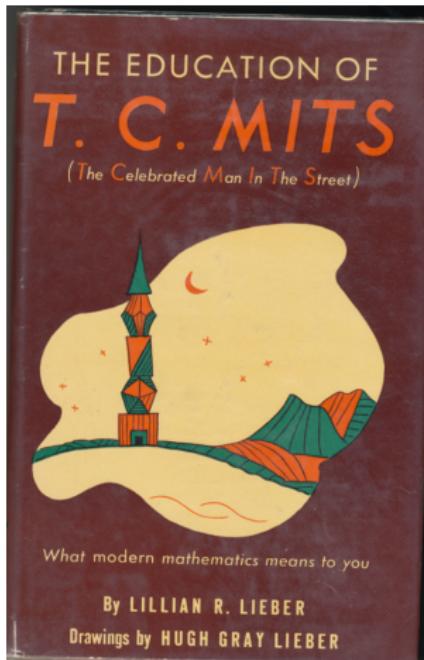
All the nontrivial zeroes of $\zeta(s)$ lie on the vertical line in the complex plane consisting of the complex numbers with real part equal to $1/2$.

What kind of book?

- Book motivated by the prime counting problem (connect with other project – explicit formula projects...).
- Mostly math and no “stories of people”, since many other books on RH do that already.
- Similar problem to guessing rank of an elliptic curve from a_p , which we had been thinking about.

What Sort of Book? Big? Small?

Like T. C. MITS or like ON BULLSHIT?



Our Approach

- Do not emphasize people/history/stories, since that's done already in many other books on RH.
- Go back 150+ years and explain what RH is more from the point of view of real classical Fourier analysis.
 - We embraced this mid-19th century real perspective.
 - We left complex numbers to the very, very end.



Target Audience?

Who are we writing this book for? Those who want to read about mathematics (not people and history).

- **High School students?** Tested at SIMUW 2007.
- **Retired electrical engineers?** Tested with original MIT talk, and online materials we shared.

SageMath again

Used Sage to compute with prime numbers, zeros, etc., and generally to plot everything in the book.

- We drew tons of plots to illustrate everything.
- The plots are absolutely essential to the exposition, and in fact really drove it!
- It is surprising that you see the spikes with such little computation.
- Central observations:
 - Fourier transform of discrete distribution at primes, gives discrete distribution of zeros of ζ .
 - The reverse gets the discrete distribution of prime powers.
- This what got us thinking about “how explicit is the explicit formula if you compute?”

Collaborative L^AT_EX via CoCalc

- Using CoCalc's L^AT_EX editor.
 - Web browser...
 - Gives a sense of the collaborative spirit
 - Plug: a brand new version of this was just released.
- We put a rough PDF of book on the web at every stage.
- GitHub: <https://github.com/williamstein/rh>

How to Publish the book: Self publish!?

- Self publishing: just put it on my website and see what happens.
 - Enormous number of people that proof read it.
 - Will Hearst pushed us for proper publication.

Finding a commercial publisher

- Finding a publisher
- What expected to give is \LaTeX ...

Typos and Mistakes

- Dozens of people carefully read drafts of the book and provided incredibly useful feedback. **THANK YOU!**
- The publisher also had a copy editor read the book, and provided complementary feedback.
- Don't expect your publisher to catch the sort of mistakes a mathematician catches:

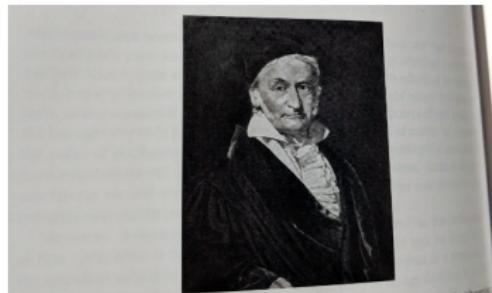


Figure 10.1. Carl Friedrich Gauss (1777–1855). Courtesy of the Smithsonian Libraries, Washington, D.C.

The search for such approximating curves began, in fact, two centuries ago. In 1796, Carl Gauss defined a certain beautiful curve that, expressing it in terms of the distribution of primes,

Creating a Cover

• ...

Endorsements for the back cover

• ...

Production

- Cambridge Univ Press might have made some positive steps toward \LaTeX (mention Oguis).
- Working with CUP has been a very positive experience overall.

Published!

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Prime Numbers and the Riemann Hypothesis 1st Edition, Kindle Edition

by Barry Mazur (Author), William Stein (Author)

★★★★★ 37 customer reviews

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Barry Mazur
William Stein

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Oliver

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June 27, 2016

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A couple of books on the Riemann hypothesis have appeared for the general public: Derbyshire 2003, Du Sautoy 2003, Sabbagh 2003, Rockmore 2005, Watkins 2015, van der Veen and van der Craats 2015 and now Mazur-Stein 2016. More for mathematicians are Koblitz 1977, Edwards 2001, and Stopple 2003. From general expositions, one should also mention the paper of Conrey of 2003 which won the Conant prize for expository writing as well as a nice paper of Bombieri of 1992. Is this too much for the subject? No. A problem like the Riemann hypothesis can never be written too much about, especially if texts are written by experts. It is the open problems which drive mathematics. The Riemann hypothesis is the most urgent of all the open problems in math and like a good wine, the problem has become more valuable over time. What helped also is that since the time of Riemann, more and more connections with other fields of mathematics have emerged. The book of Veen-Craats and Mazur-Stein have emerged about at the same time. They are both small and well structured. Veen-Craats has been field tested with high school students and has focus mostly on the gorgeous Mangoldt explicit formula for the Chebychev prime distribution function, sometimes called the "music of the primes". Mazur-Stein do it similarly, however stress more on the Riemann spectrum and go didactically rather gently into the mathematics of Fourier theory as well as the theory of distributions. The book is carefully typeset, has color prints and some computer code for Sage. While Veen-Craats has many nice exercises, an exercise of Mazur-Stein led me to abandon other things for a couple of weeks, since it was so captivating. So be careful! A student who has taken basic calculus courses, should be able to read it. By the way, except Sabagh's book "Dr Riemann's zeros", which was written by a writer and journalist, the other books were created by professional mathematicians. The Mazur-Stein book has probably the best "street cred" among the RH books for the general audience: both have done important work in number theory, also related to zeta functions: Mazur's name is on one of the grand generalizations of the Riemann zeta function, the Artin-Mazur zeta function which has exploded into a major tool under the lead of Duall who made it

discussion of the Reimann spec:

Some good graphics and a good Reimann spectrum. I found Derb Obsession" overall more interest for the non-expert.

Published 1 year ago



Ricardo J. Menéndez

★★★★★ Five Stars

Excellent presentation ! Ricardo
Published 1 year ago



Saul

★★★★★ Very Good for Ge

Some Math Background

Very informative. Not for the prc
about it already, of course.

Published 1 year ago



david bailey

★★★★★ Four Stars

satisfactory

Published 1 year ago

Negative reviews due to *production issues*, both with the physical book and the Kindle edition....

Reception by Readers

- Feedback
- Sarnak's review in Bulletins
- Other reviews
- Prizes

Royalties

We sold some copies, so Cambridge University Press sent us some money. I bought a puppy named Bella!



Translation: what to expect?

Dear Professor Stein,

Prime Numbers and the Riemann Hypothesis

I am delighted to inform you that we are currently concluding an agreement with Nippon Hyoron Sha for a Japanese language edition of your book. They plan to print an edition of 2,500 copies initially, which will be sold at approximately 2,200 JPY per copy.

What to expect?

- Also Korean?
- Will they bother with French, etc.?

Future Plans

- Online fully interactive version.
- Related research on L -series of elliptic curves, etc.