Writing a book with Barry Mazur

"Prime Numbers and the Riemann Hypothesis"

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SageMath, Inc. and University of Washington

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Abstract

Abstract

In 2004, Barry Mazur and I started a decade project to write the book "Prime Numbers and the Riemann Hypothesis". This talk is about what's in the book and why, and some aspects of production of the book.

Prelude: collaborate with great co-authors!

Writing John Tate's *Galois Cohomology* notes for PCMI 1999...

> *"Everybody is so jealous of you getting to talk with John Tate!"*
> – David Savitt

If you ever get the chance to write something with someone incredible, **take it!!**

(I next wrote a long paper with Ken Ribet from that same PCMI.)

Overview

§1. Barry's Public Lecture

Clay Math Institute public lecture (MIT, May 3, 2005)

"Are there still unsolved problems about the numbers 1, 2, 3, 4, ... ?"

Use primes to "sell" number theory to the general public

- Immediately accessible
- Immediately interesting
- Primes and how eratic they are
- Cicada's every 13, 17 years...
- Many examples of "open, interesting questions"
- People can immediately make computations of their own
- Barry got his father, who had done NO math, hooked on the Goldbach Conjecture, so thought primes would work.

SageMath

2005: I started SageMath

I launched Sage a few months before this 2005 CMI public lecture.

- Sage is a **free open source** alternative to Mathematica, Maple, Magma, and Matlab.
- Early Sage development motivated by this talk
  - Linking Sage to Mathematica to compute Γ
  - Early visualization functionality
  - Prime enumeration

More about what was in Barry's public lecture...

Barry's Public Lecture: Topics

- Primes as atoms: factorization
- The largest known prime
- Enumerating primes: Sieve of Eratosthenes
- Twin primes
- Counting primes
- Gauss's Conjecture: The Prime Number Theorem
- Riemann: Fourier style smooth approximations $R_k(x)$ to $\pi(x)$
- Riemann's Harmonics: zeros of $\zeta(s)$

It worked!

§2. Writing a Book

"Let's write a book..." – Barry

Could we turn this public lecture into a popular book?

- Write something for a popular audience
- Small and readable
- Full of *mathematics*, not stories of people
- Profusely illustrated
- Meet for a few weeks in his country house and focus on this

What kind of book?

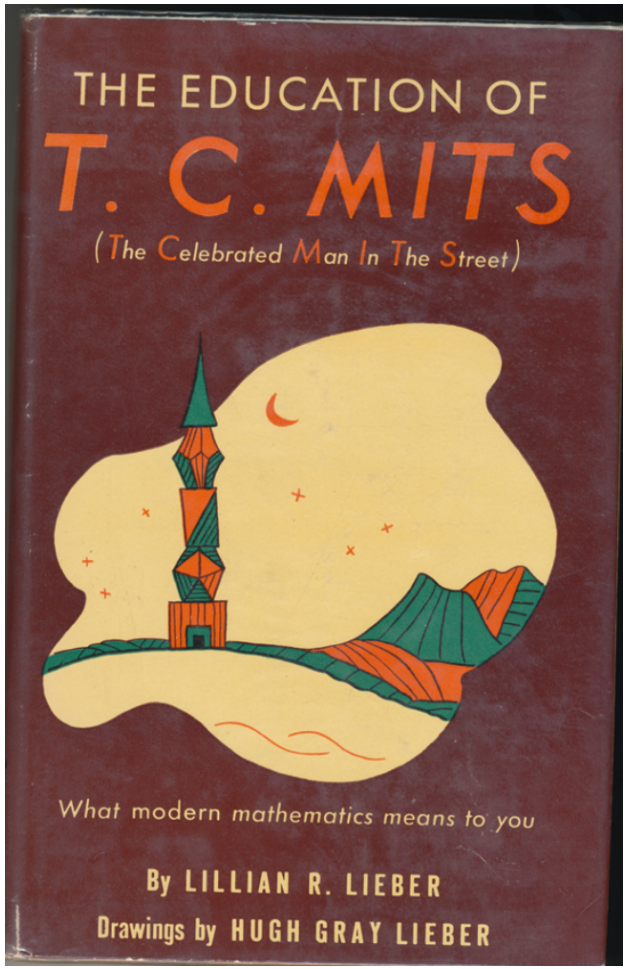
There are already 4 recent popular books on the Riemann Hypothesis. Why write another?

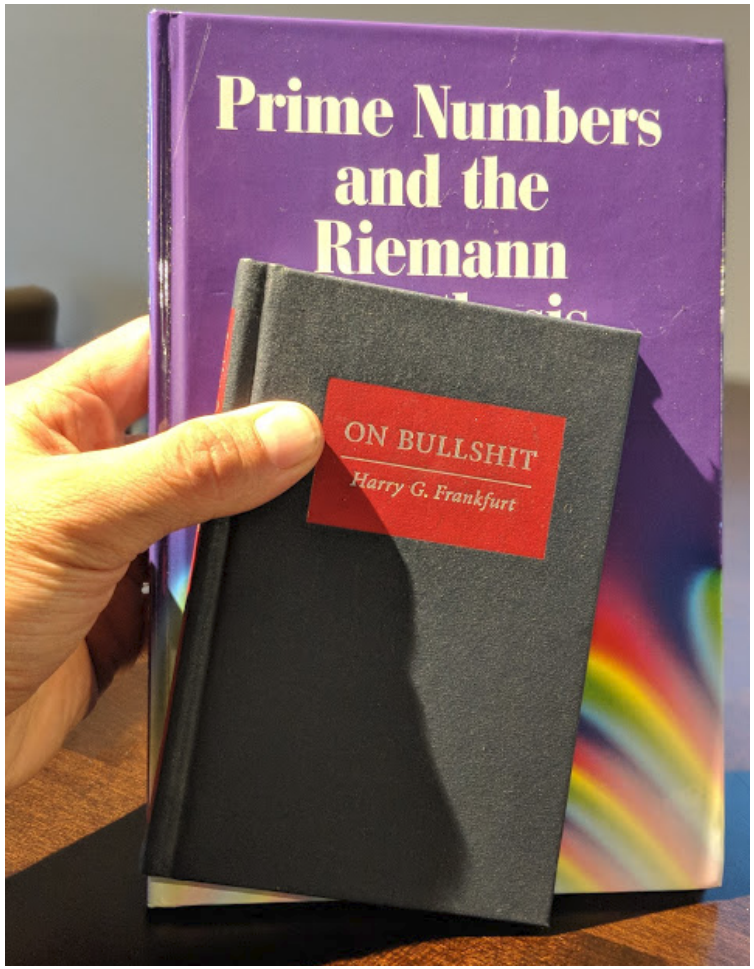
Our general audience book could be unique

- Motivated by deeply discussing the prime counting problem, and connecting it with our other research project on *the explicit formula*.
- Mostly math and not "stories of people" (other books on RH already do the stories well).

What Sort of Book: Small, Medium or Large?

Like T. C. MITS or like ON BULLSHIT?





Our Approach

Go back 150+ years and explain what RH is more from the point of view of real classical Fourier analysis.

- Embrace a mid-19th century very $\mathbb{R}$eal perspective
- Leave $\mathbb{C}$omplex numbers to the very, very end

Target Audience?

Who are we writing this book for?

Lovers of number theory, who want to read about mathematics.

- **High school students?**[a] Tested at SIMUW 2007.
- **Retired electrical engineers?** Tested with original MIT talk, and online materials we shared.

[a] Neither Barry nor I graduated from high school...

SageMath again

Computations with Sage drove the exposition

We used Sage to compute with prime numbers, zeros, etc., and generally to plot everything in the book.

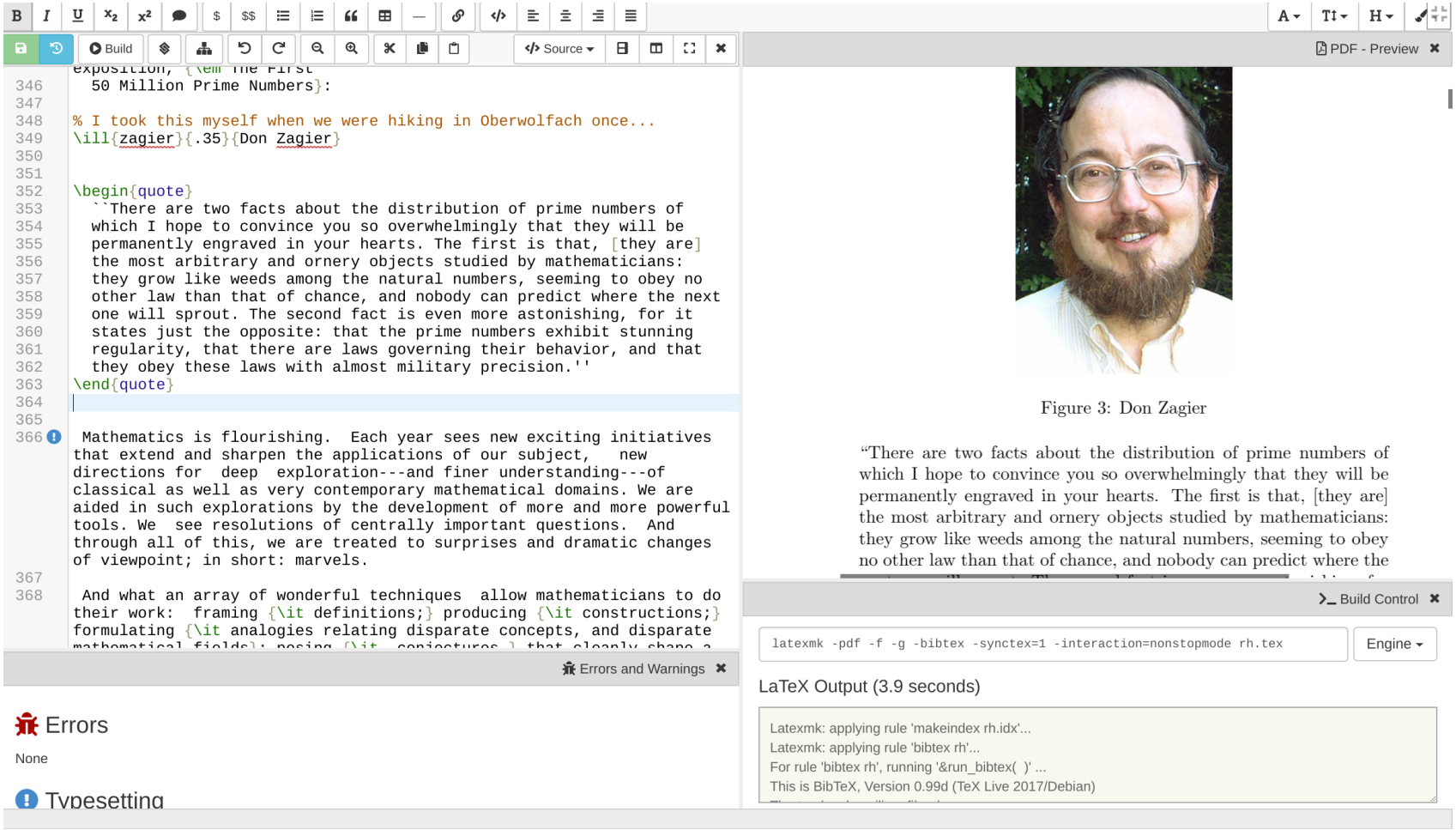
- Numerous plots that are absolutely essential to the exposition, and in fact really drove it!
- Surprising to see so much with such little computation.
- Central hook of the book appears from computation: *"Fourier transform links the discrete distribution at prime powers and the discrete distribution of zeros of ζ."*
- This is also what got us thinking about "how explicit is the explicit formula?" (another research project...)

Collaborative LaTeX via CoCalc

How we wrote the book

- Using CoCalc's LaTeX editor.
  - In a web browser...
  - Both of us simultaneously editing the same file.
  - Precise history of all changes
  - Gives a sense of the collaborative spirit
  - Plug: I just released a new version of the editor!
- Rough PDF's of book on the web at every stage.
- GitHub tracking of changes
- Sage computations run in the same place as editing book.
- Barry did very closely read and understand all the Sage code!

Screenshot of CoCalc's Collaborative LaTeX Editor



Here it is!

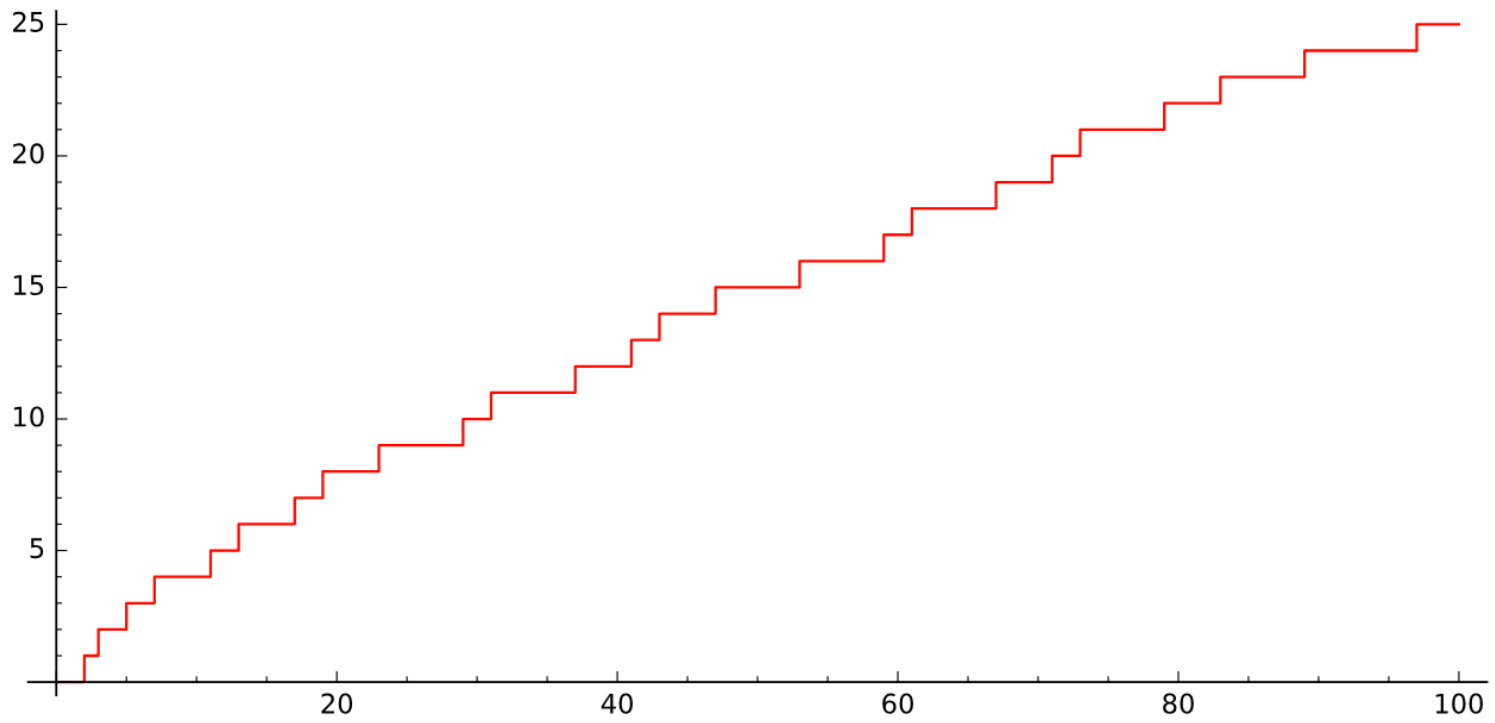
(in just a few slides)

The Prime Counting Problem

Let $\pi(x)$ be the number of primes $\leq x$.

Problem: *Give a "good approximation" for* $\pi(x)$.

```
plot(prime_pi, 0, 100, color='red', figsize=[8,4])
```



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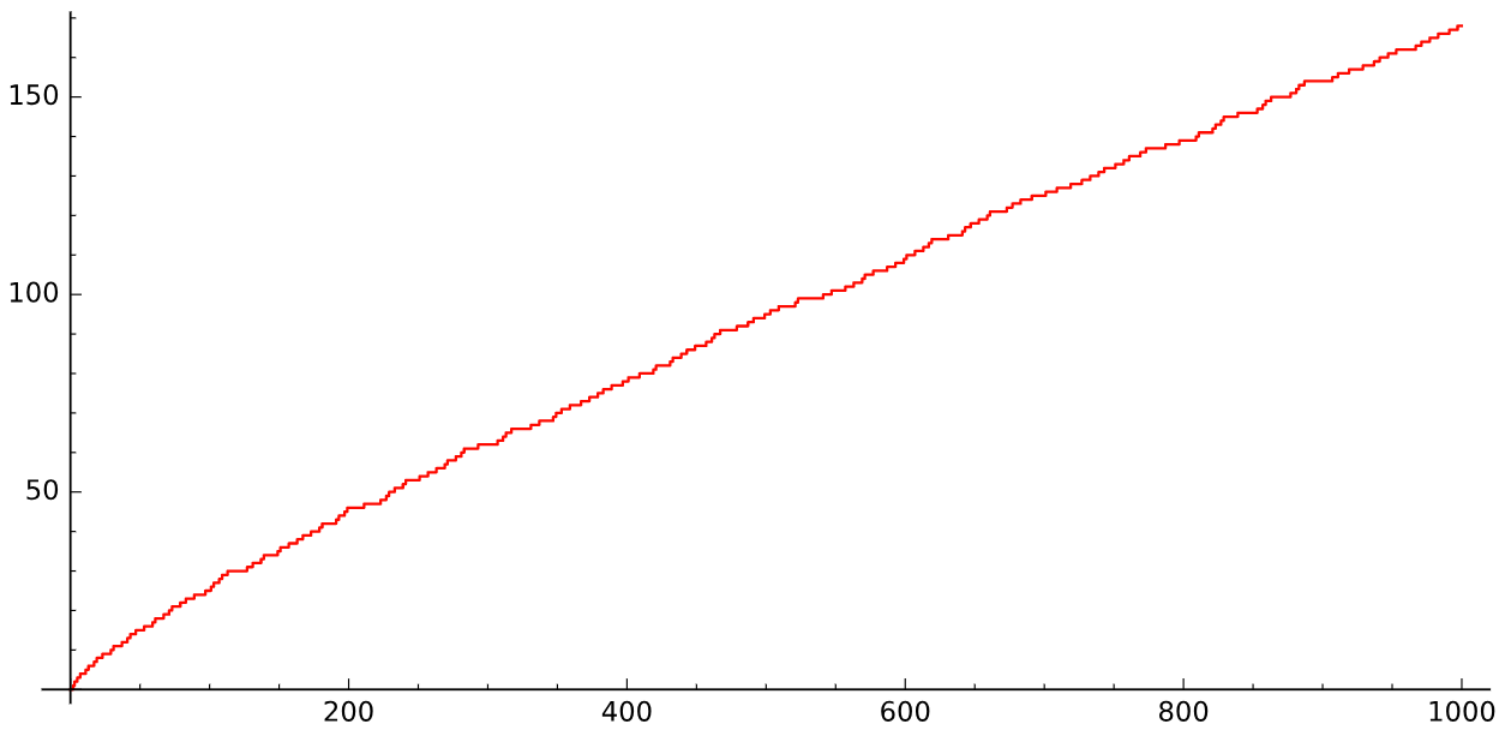
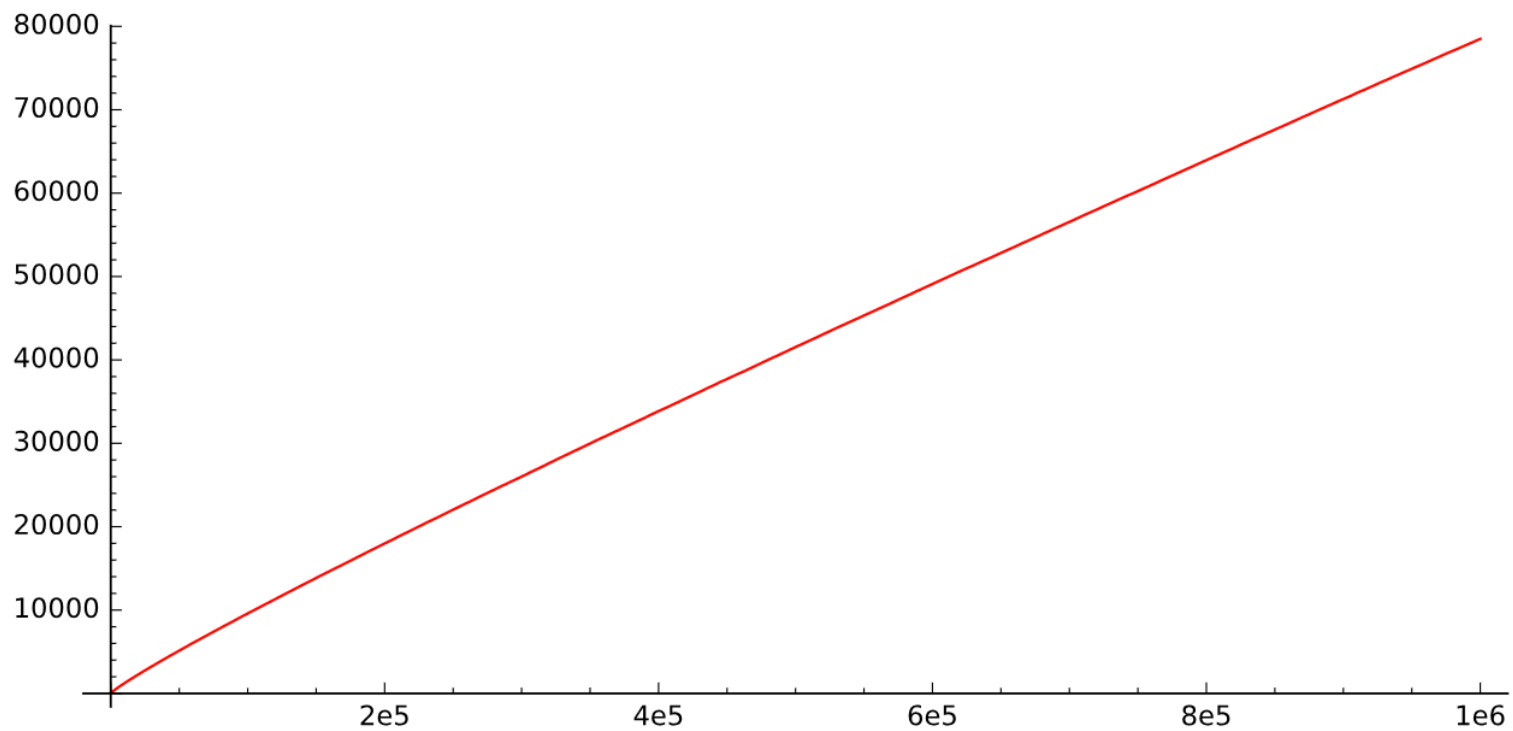


Focus on The Prime Counting Problem

Let $\pi(x)$ be the number of primes $\leq x$.

Problem: *Give a "good approximation" for* $\pi(x)$.

```
plot(prime_pi, 0, 10^6, color='red', figsize=[8,4])
```



Answer: The Riemann Hypothesis (first formulation)

The number of prime numbers less than X is approximately $\mathrm{Li}(X)$ and this approximation is essentially square root accurate.

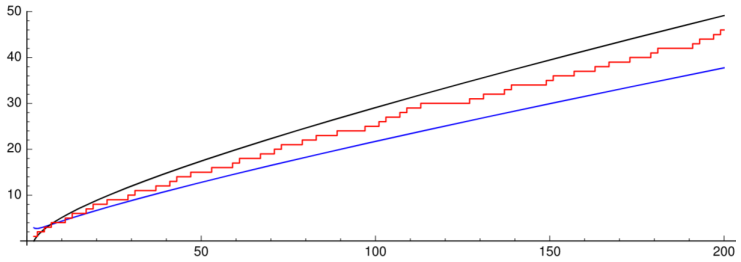


Figure 13.1: Plots of $\mathrm{Li}(X)$ (top), $\pi(X)$ (in the middle), and $X/\log(X)$

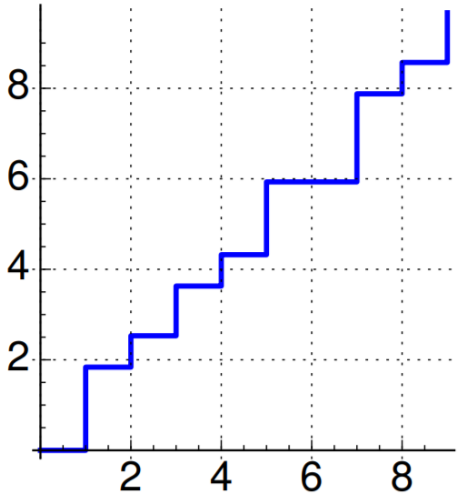
$$\begin{aligned}
\pi(X) &= 18{,}435{,}599{,}767{,}349{,}200{,}867{,}866 \\
\mathrm{Li}(X) &= 18{,}435{,}599{,}767{,}366{,}347{,}775{,}143.10580\ldots \\
X/(\log(X)-1) &= 18{,}429{,}088{,}896{,}563{,}917{,}716{,}962.93869\ldots \\
\mathrm{Li}(X)-\pi(X) &= 17{,}146{,}907{,}277.105803\ldots \\
\sqrt{X}\cdot\log(X) &= 55{,}262{,}042{,}231{,}857.096416\ldots
\end{aligned}$$

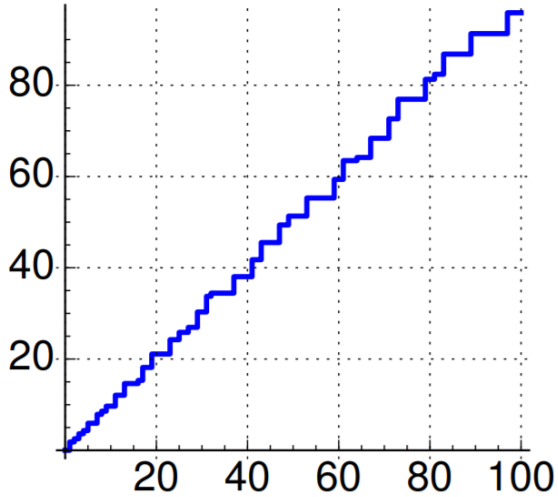
$$\pi(X) = \#\{\text{primes } \leq X\} \sim \mathrm{Li}(X) = \int_2^X \frac{1}{\log(t)}\,dt$$

Answer: The Riemann Hypothesis (second formulation)

The prime power staircase $\psi(X)$ is essentially square root close to the 45 degree straight line.

$\psi(x)$: "Our staircase starts on the ground at $x = 0$ and the height of the riser of the step at $x = 1$ will be $\log(2\pi)$. The height of the riser of the step at $x = p^n$ will not be 1 but rather: the step at $x = p^n$ will have the height of its riser equal to $\log p$."





Answer: The Riemann Hypothesis (third formulation)

The Fourier transform of the derivative of $\psi(X)$ *"is basically"* a discrete distribution supported at the imaginary parts of the zeros of the Riemann Zeta function.

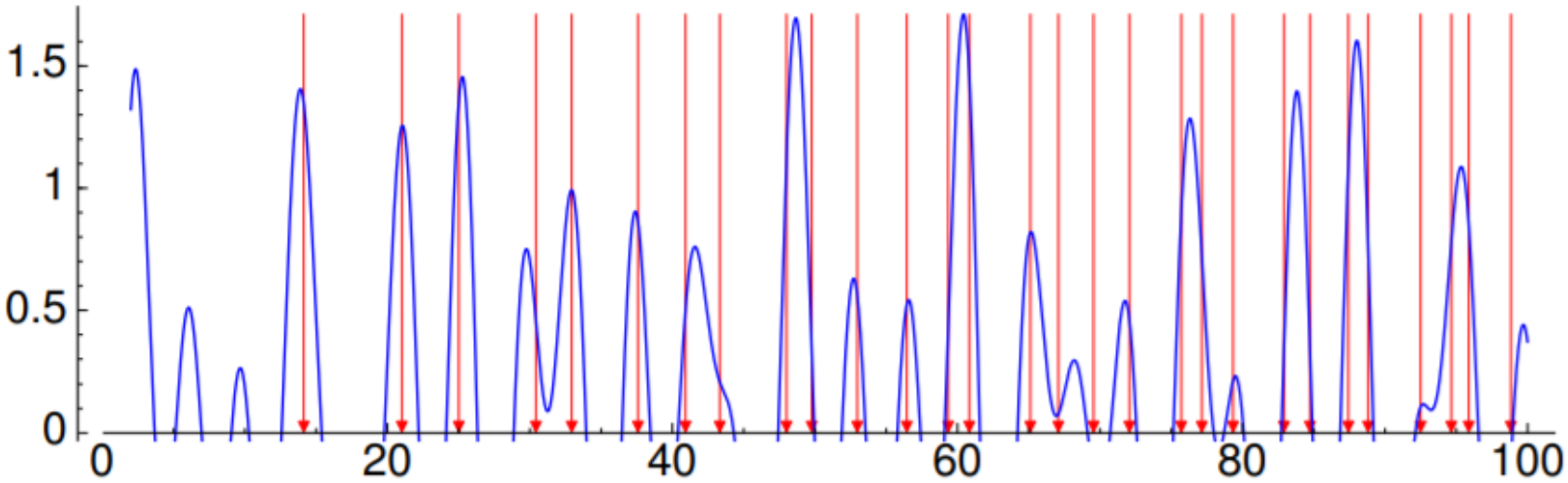
We deleted this formulation from the book, since it was too technical to state properly (it's the *explicit formula*).[1]

Instead, we illustrate the heck out of it!

[1] After deleting this, we accidentally didn't relabel the "fourth formulation", which confused readers.

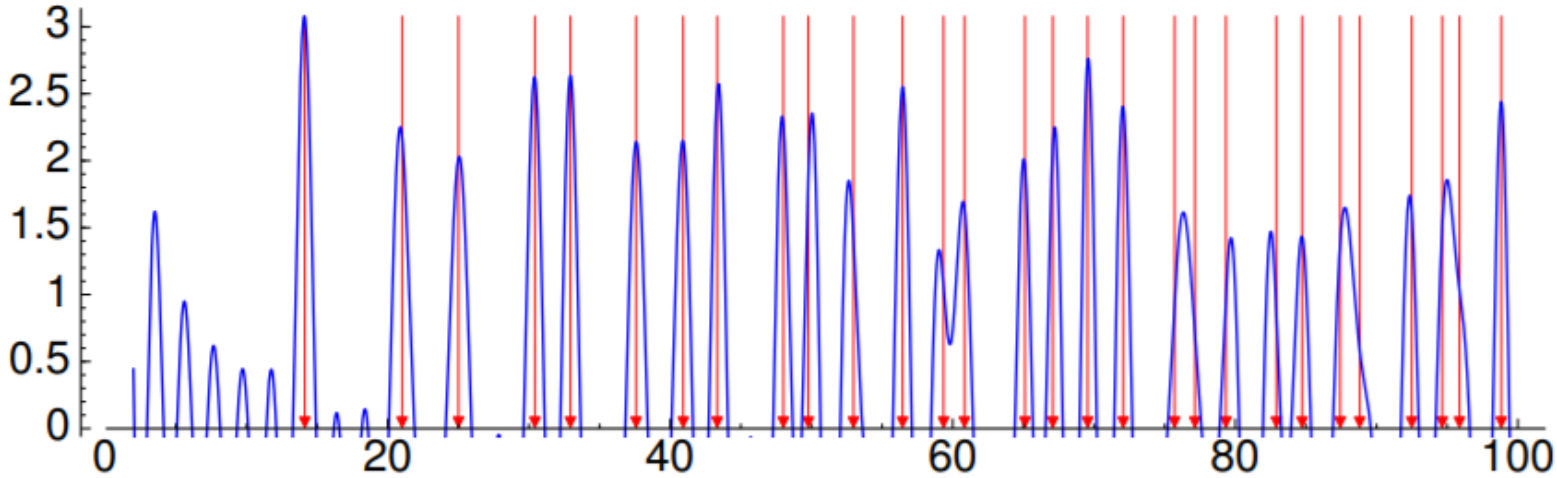
Fourier transform of $\Psi'(x)$ (just four terms!)

$$f(t) = -\frac{\log(2)}{2^{1/2}}\cos(t\log(2)) - \frac{\log(3)}{3^{1/2}}\cos(t\log(3)) - \frac{\log(2)}{4^{1/2}}\cos(t\log(4)) - \frac{\log(5)}{5^{1/2}}\cos(t\log(5))$$



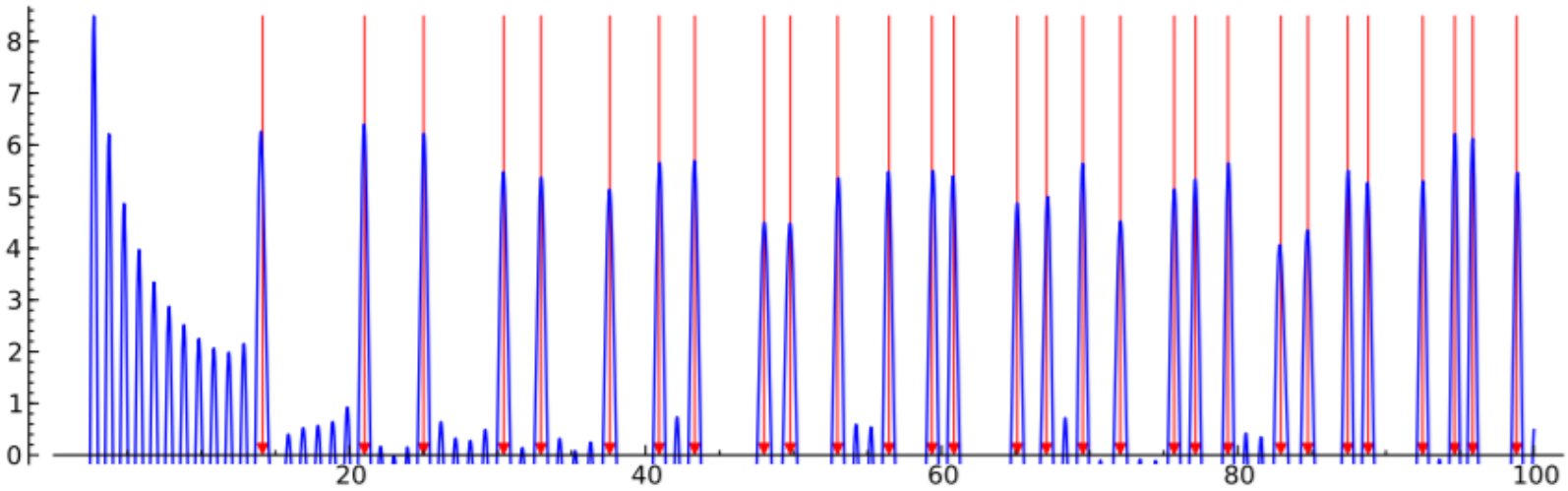
- Anybody can easily plot this.
- Arrows point to imaginary parts of zeros of $\zeta(s)$!

Fourier transform of $\Psi'(x)$ (first 20 terms)



$$-\sum_{p^n \leq 20} \frac{\log(p)}{p^{n/2}} \cos(t \log(p^n))$$

Fourier transform of $\Psi'(x)$ (first 500 terms)

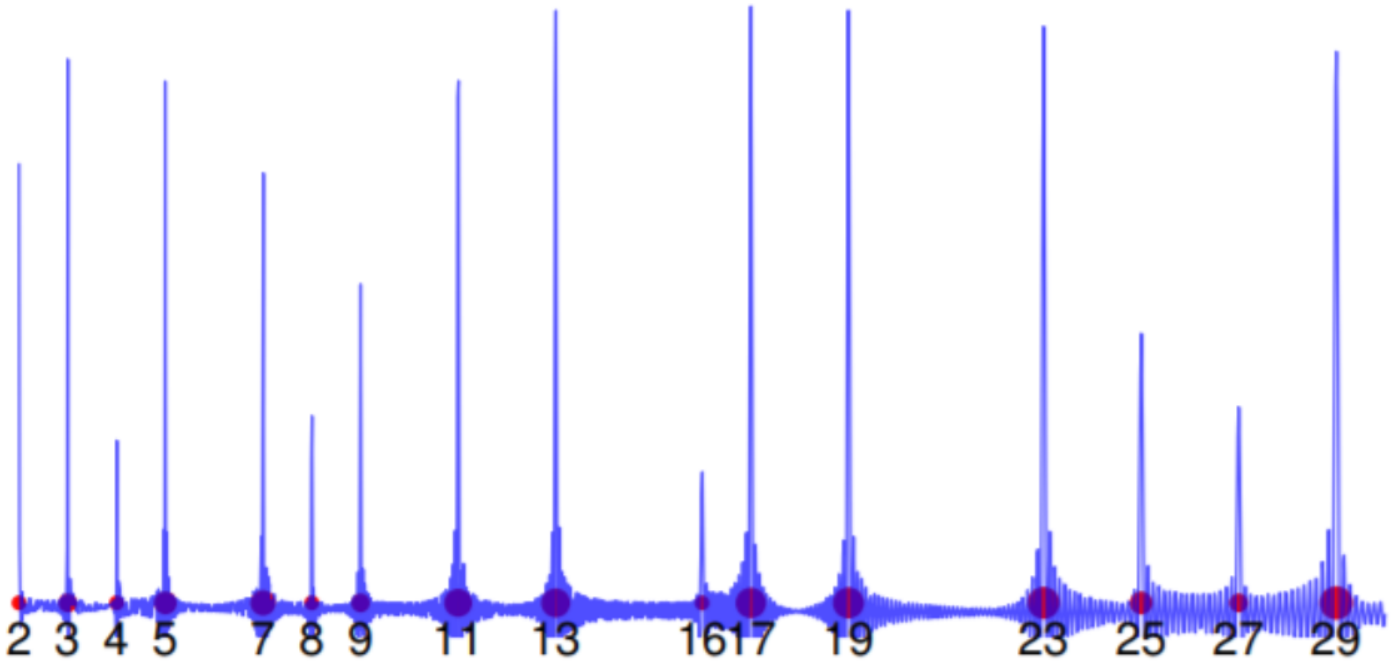


$$-\sum_{p^n \leq 500} \frac{\log(p)}{p^{n/2}} \cos(t \log(p^n))$$

Take this home: *The Fourier transform of the derivative of the prime power staircase "is" the zeros of the Riemann zeta function.*

It goes both ways!

The Fourier transform of the zeros "is" prime powers:



$$-\sum_{i=1}^{1000} \cos(\log(s)\theta_i)$$

Riemann untangled this to get $\pi(x)$...

We finish book with manipulation to approximate $\pi(x)$ by a sum of smooth functions $R_k(x)$ involving the θ_i.

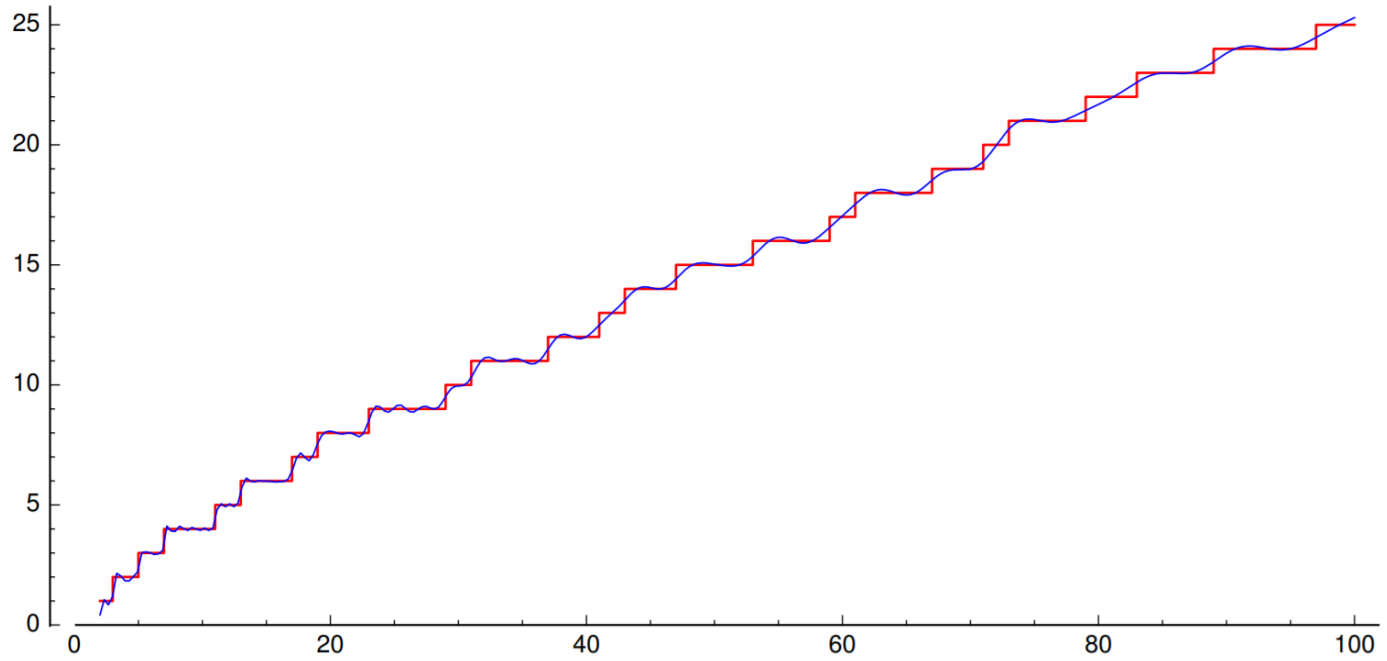


Figure 36.8: The function R_{25} approximating the staircase of primes up to 100

Inspiration: Zagier's lecture "The First 50 million prime numbers".

R_{50} approximates $\pi(x)$ very well!

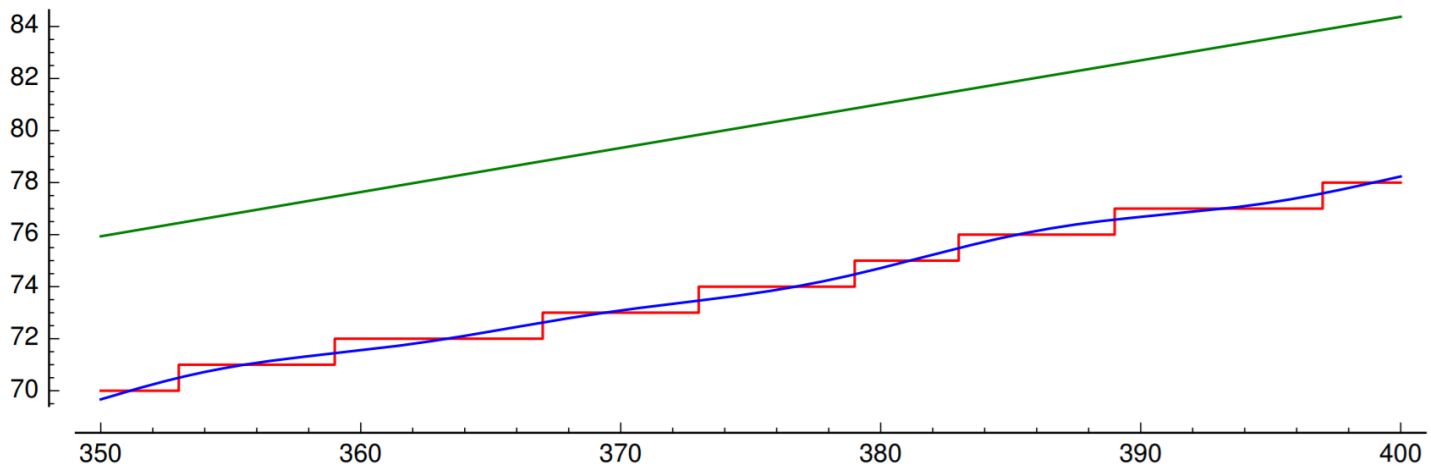


Figure 36.11: The function $\mathrm{Li}(X)$ (top, green), the function $R_{50}(X)$ (in blue), and the staircase of primes on the interval from 350 to 400.

Answer: The Riemann Hypothesis (fourth formulation)

All the nontrivial zeroes of $\zeta(s)$ lie on the vertical line in the complex plane consisting of the complex numbers with real part equal to $1/2$.

§3. Publishing a Book

How to Publish the book: Self publish!?

Self publishing?

Just put it on my website and see what happens.

- Some people read it...
- It didn't really get **significant traction**.
- There was still that key **missing quality** step.

Will Hearst convinced us to publish with a commercial publisher.
Maybe he was tired of printing out copies to give to people?

Finding a publisher

Finding the right publisher for *this book*...

- Barry and I have both published a few books with a couple of publishers, over the years.
- Talked to many editors (the JMM was **very** helpful!)
- Looked at reputation, similar books, and who followed up
- Balanced competing goals (e.g., price, quality, rights)
- Kaitlin from Cambridge University Press won for this.

Typos and Mistakes

Or, making the book easier for people to read

- Dozens of people carefully read drafts of the book and provided incredibly useful feedback. **THANK YOU!!**
- The publisher had a copy editor read the book, and provided complementary feedback.
- Don't expect your publisher to catch the sort of mistakes a mathematician catches:

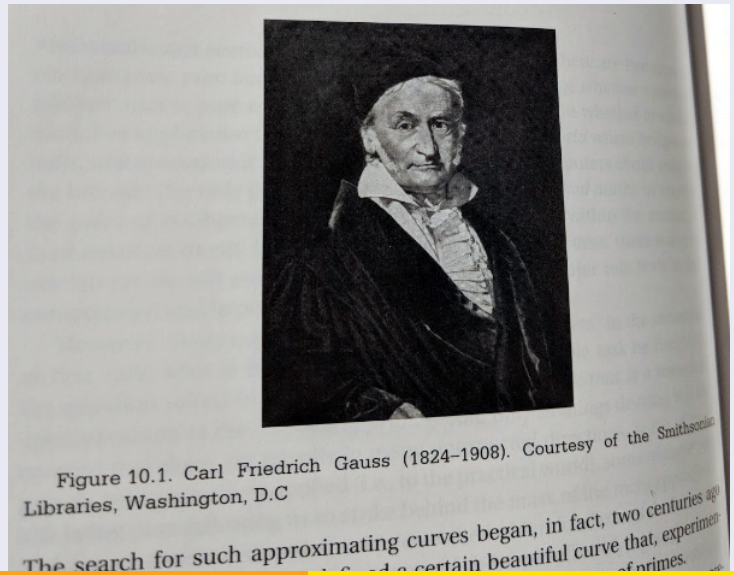


Figure 10.1. Carl Friedrich Gauss (1824–1908). Courtesy of the Smithsonian Libraries, Washington, D.C

The search for such approximating curves began, in fact, two centuries ago

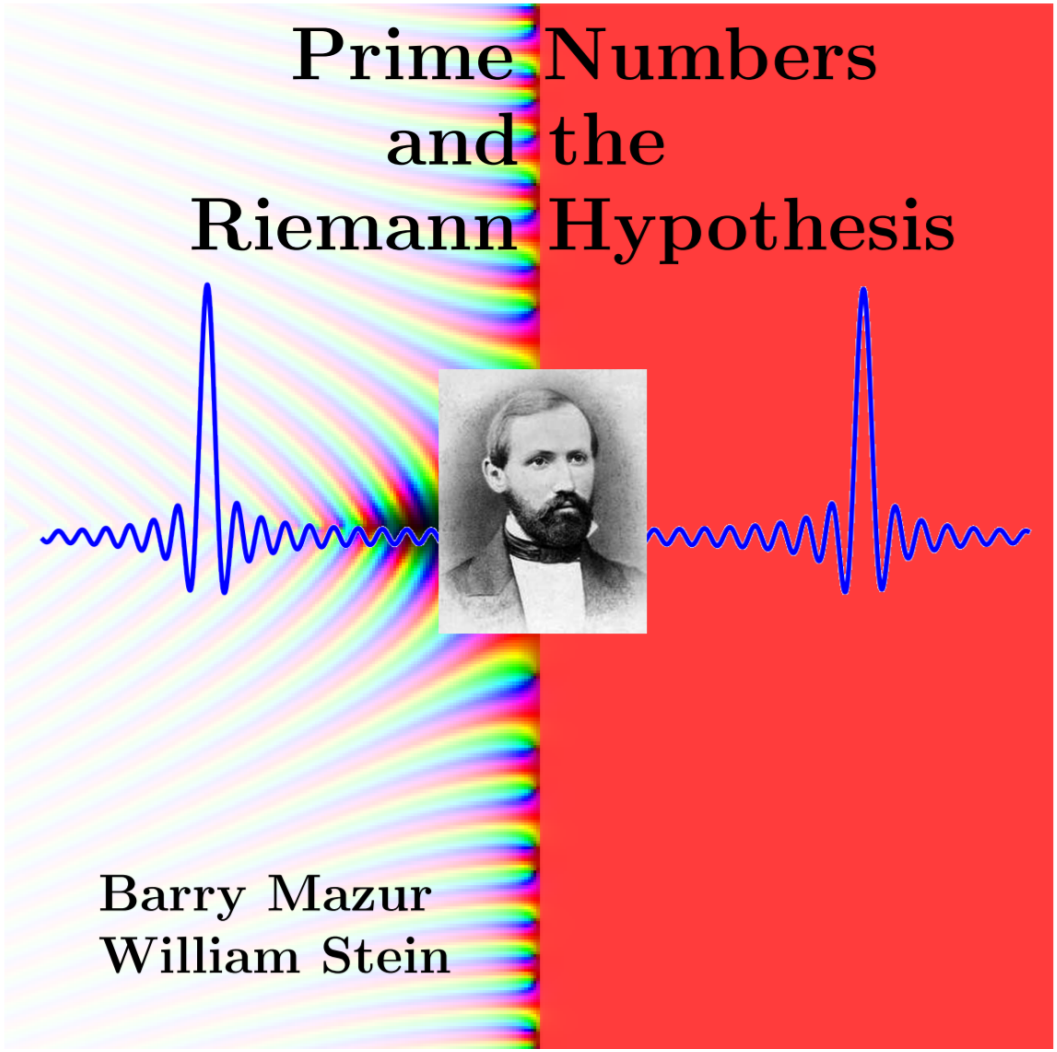
Creating a Cover

Ideas for Components Included...

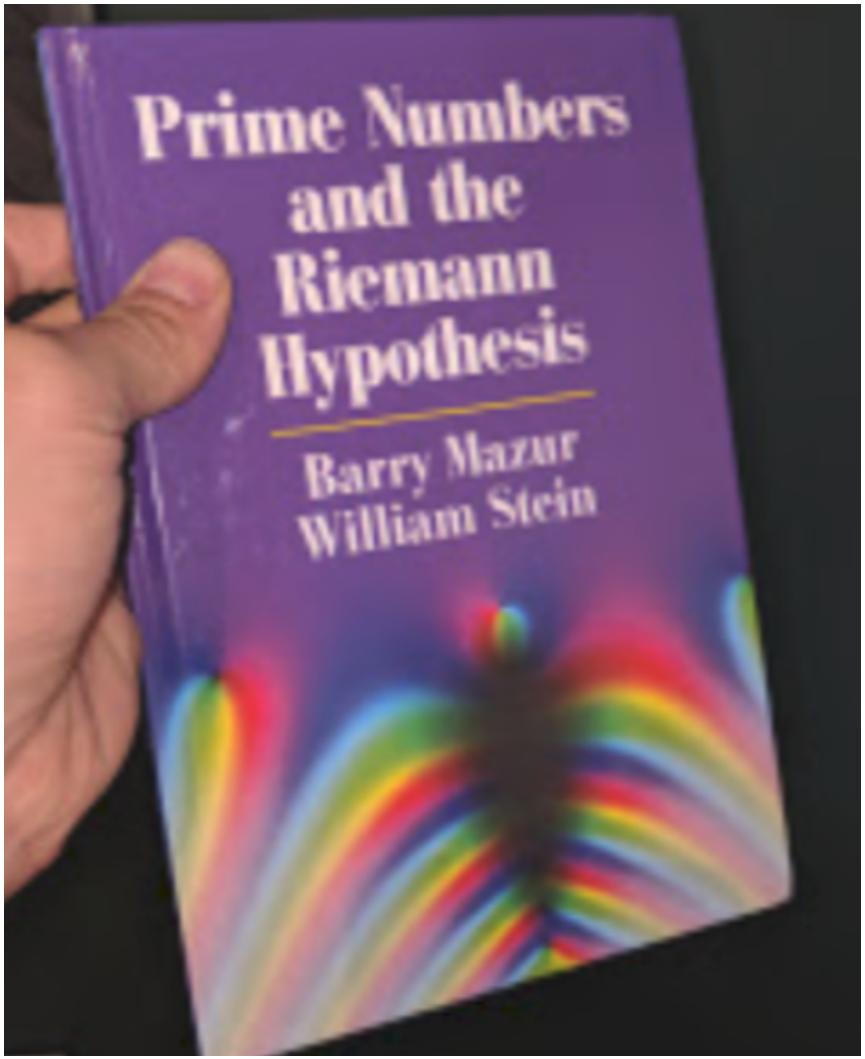
- Title of book
- Our names
- Plot of $\zeta(s)$, using Sage's `complex_plot`
- Portrait of Riemann, the star of the book!
- Plots illustrating the main ideas of the book
- A "classical" look

There is a natural tension here: publisher vs author vs marketer

What We Created



The Actual Cover



Endorsements for the back cover

Will Hearst and John Cremona kindly wrote about our book...

Production

Producing the book

- Initial friction with production, e.g., "Please provide Microsoft Word document."
- Evidently, Cambridge Univ Press might have made some new positive steps toward better LaTeX support.
- Some unfortunate physical issues with the very first printing.
- CUP strongly supported and marketed the book.
- Working with CUP has been a *very positive experience* overall.

Published!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by Barry Mazur (Author), William Stein (Author)
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★★★★★ **short, gorgeous, inspring and affordable.**

June 27, 2016

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A couple of books on the Riemann hypothesis have appeared for the general public: Derbeshire 2003, Du Sautoiy 2003, Sabbagh 2003, Rockmore 2005, Watkins 2015, van der Veen and van der Craats 2015 and now Mazur-Stein 2016. More for mathematicians are Koblitz 1977, Edwards 2001, and Stopple 2003. From general expositions, one should also mention the paper of Conrey of 2003 which won the Conant prize for expository writing as well as a nice paper of Bombieri of 1992. Is this too much for the subject? No. A problem like the Riemann hypothesis can never be written too much about, especially if texts are written by experts. It is the open problems which drive mathematics. The Riemann hypothesis is the most urgent of all the open problems in math and like a good wine, the problem has become more valuable over time. What helped also is that since the time of Riemann, more and more connections with other fields of mathematics have emerged. The book of Veen-Craats and Mazur-Stein have emerged about at the same time. They are both small and well structured. Veen-Craats has been field tested with high school students and has focus mostly on the gorgeous Mangoldt explicit formula for the Chebychev prime distribution function, sometimes called the "music of the primes". Mazur-Stein do it similarly, however stress more on the Riemann spectrum and go didactically rather gently into the mathematics of Fourier theory as well as the theory of distributions. The book is carefully typeset, has color prints and some computer code for Sage. While Veen-Craats has many nice exercises, an exercise of Mazur-Stein led me to abandon other things for a couple of weeks, since it was so captivating. So be careful! A student who has taken basic calculus courses, should be able to read it. By the way, except Sabagh's book "Dr Riemann's zeros", which was written by a writer and journalist, the other books were created by professional mathematicians. The Mazur-Stein book has probably the best "street cred" among the RH books for the general audience: both have done important work in number theory, also related to zeta functions: Mazur's name is on one of the grand generalizations of the Riemann zeta functions, the Artin-Mazur zeta function which has exploded into a major tool under the lead of Ruelle who made it

discussion of the Reimann spec
Some good graphics and a good Reimann spectrum. I found Derb Obsession" overall more interest for the non-expert.
Published 1 year ago

Ricardo J. Menéndez

★★★★★ **Five Stars**
Excellent presentation ! Ricardo
Published 1 year ago

Saul

★★★★★ **Very Good for Ge**
Some Math Background
Very informative. Not for the prc about it already, of course.
Published 1 year ago

david bailey

★★★★☆ **Four Stars**
satisfactory
Published 1 year ago

Negative reviews mainly due to *production issues*, both with the physical book and the Kindle edition, which CUP fully addressed.

Reception by Readers

- Sarnak's review in Bulletins
- Ash's review in ???
- Other reviews? Granville?
- Prizes

Royalties

We sold some copies, so Cambridge University Press sent us some money. I'm spending my share on expenses for my dream dog:

Translations

Dear Professor Stein,

Prime Numbers and the Riemann Hypothesis

I am delighted to inform you that we are currently concluding an agreement with Nippon Hyoron Sha for a Japanese language edition of your book. They plan to print an edition of 2,500 copies initially, which will be sold at approximately 2,200 JPY per copy.

What to expect?

- Also Korean?
- Will they bother with French, etc.?

Future plans

Someday we hope to...

- Create online fully interactive version of all the plots, which don't require knowing Sage to use.
- Finish related research on L-series of elliptic curves, connecting the rank to statistical behavior of the a_p.