Writing a book with Barry Mazur

"Prime Numbers and the Riemann Hypothesis"

William Stein

SageMath, Inc. and University of Washington

June 4, 2018 at Harvard University

Abstract

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In 2004, Barry Mazur and I started a decade project to write the book "Prime Numbers and the Riemann Hypothesis". This talk is about what's in the book and why, and some aspects of production of the book.

Prelude: collaborate with great co-authors!

Writing John Tate's *Galois Cohomology* notes for PCMI 1999...

> *"Everybody is so jealous of you getting to talk with John Tate!"*
> – David Savitt

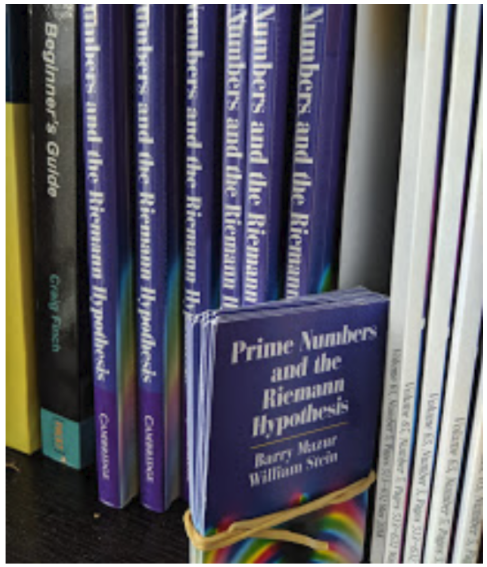
If you ever get the chance to write something with someone incredible, **take it!!**

(I next wrote a long paper with Ken Ribet from that same PCMI.)

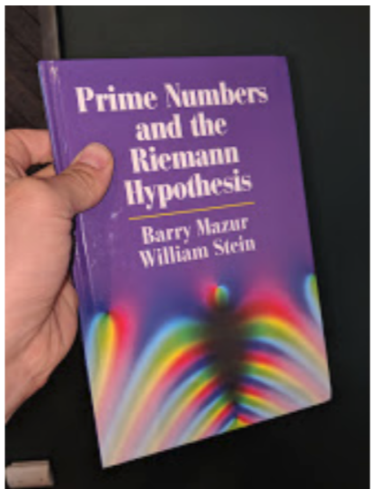
Overview

The book


Sage
ON BULLSHIT
Harry G. Frankfurt


Beginner's Guide
Numbers and the Riemann Hypothesis
Prime Numbers and the Riemann Hypothesis
Barry Mazur
William Stein


15 Cesàro Smoothing


Prime Numbers and the Riemann Hypothesis
Barry Mazur
William Stein

§1. Barry's Public Lecture

Clay Math Institute public lecture (MIT, May 3, 2005)

"Are there still unsolved problems about the numbers 1, 2, 3, 4, ... ?"

Use primes to "sell" number theory to the general public

- Immediately accessible
- Immediately interesting
- Primes and how eratic they are
- Cicada's every 13, 17 years...
- Many examples of "open, interesting questions"
- People can immediately make computations of their own
- Barry got his father, who had done NO math, hooked on the Goldbach Conjecture, so thought primes would work.

SageMath

2005: I started SageMath

I launched Sage a few months before this 2005 CMI public lecture.

- Sage is a **free open source** alternative to Mathematica, Maple, Magma, and Matlab.
- Early Sage development motivated by this talk
  - Linking Sage to Mathematica to compute Γ
  - Early visualization functionality
  - Prime enumeration

More about what was in Barry's public lecture...

Barry's Public Lecture: Topics

- Primes as atoms (factorization)
- Record large primes
- Enumerating primes (Sieve of Eratosthenes)
- Twin primes
- Counting primes
- Gauss's Conjecture – Prime Number Theorem
- Riemann: Fourier Adjustments to get smooth approx to $\pi(x)$
- Riemann Harmonics (zeros of $\zeta(s)$)

It worked!

§2. Writing a Book

"Let's write a book..." – Barry

Could we turn this public lecture into a popular book?

- Write something for a popular audience
- Small and readable
- Full of *mathematics*, not stories of people
- Extremely profusely illustrated
- Meet for a few weeks and focus on this

What kind of book?

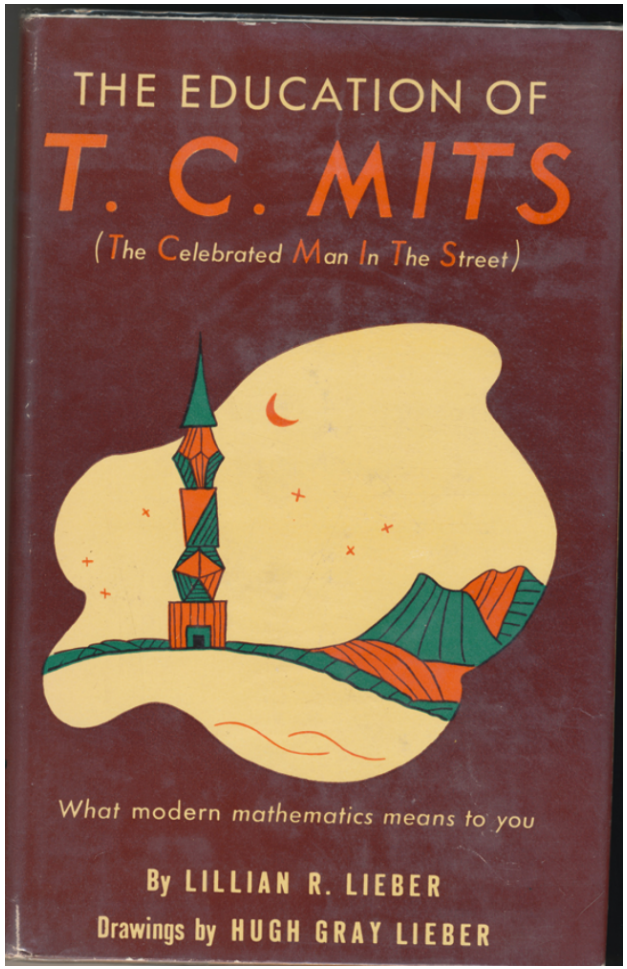
There are already a bunch of books on the Riemann Hypothesis. Why write another?

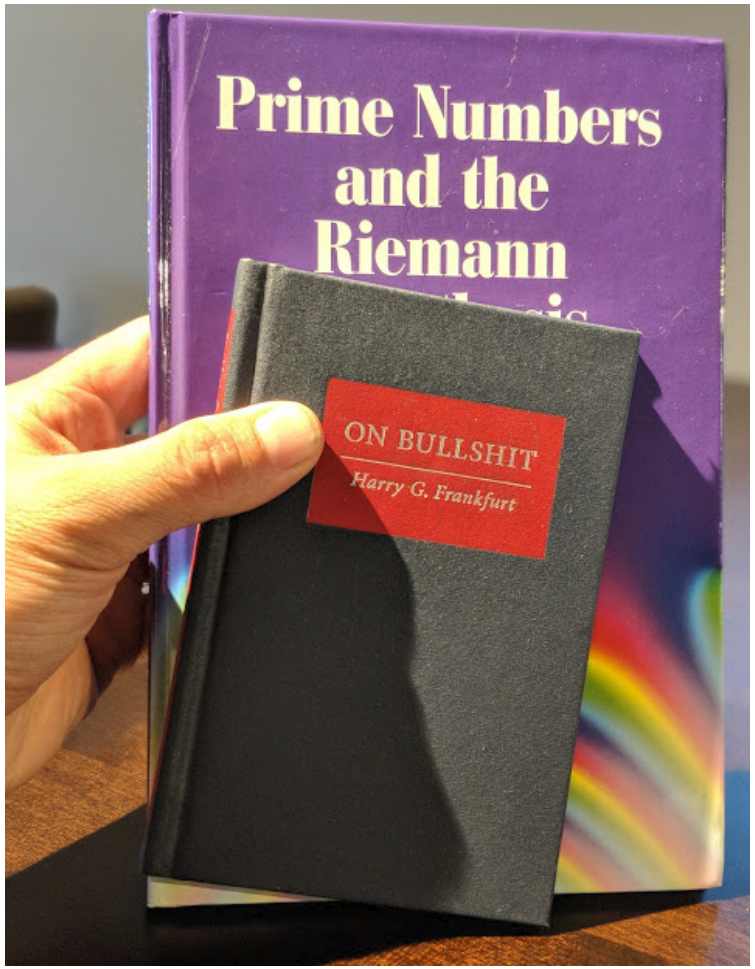
Our general audience book will be unique

- Motivated by deeply discussing the prime counting problem, and connecting it with our other research project on *the explicit formula*.
- Mostly math and not "stories of people", since many other books on RH do that already really well.

What Sort of Book? Big? Small?

Like T. C. MITS or like ON BULLSHIT?





Our Approach

Go back 150+ years and explain what RH is more from the point of view of real classical Fourier analysis.

- We embraced this mid-19th century very $\mathbb{R}$eal perspective.
- We left $\mathbb{C}$omplex numbers to the very, very end.

Target Audience?

Who are we writing this book for?

Lovers of number theory who want to read about mathematics (not history).

- **High school students?** Tested at SIMUW 2007.
- **Retired electrical engineers?** Tested with original MIT talk, and online materials we shared.

SageMath again

Computation with Sage drove the exposition

We used Sage to compute with prime numbers, zeros, etc., and generally to plot everything in the book.

- The numerous plots are absolutely essential to the exposition, and in fact really drove it!
- Surprising to see so much with such little computation.
- Central observation of the book:
  - Fourier transform of discrete distribution at primes, gives discrete distribution of zeros of ζ.
  - The reverse gets the discrete distribution of prime powers.
- This is also what got us thinking about "how explicit is the explicit formula?" (another research project...)

Collaborative LaTeX via CoCalc

How we wrote the book

- Using CoCalc's LaTeX editor.
  - In a web browser...
  - Multiple simultaneous editors
  - Precise history of all changes
  - Gives a sense of the collaborative spirit
  - Plug: I just released a new version!
- Rough PDF's of book on the web at every stage.
- GitHub tracking of changes
- Sage graphics computations run in same place as editing book.

CoCalc's Collaborative LaTeX Editor

Build | Source | PDF - Preview

```
346   50 Million Prime Numbers}:
347
348 % I took this myself when we were hiking in Oberwolfach once...
349 \ill{zagier}{.35}{Don Zagier}
350
351
352 \begin{quote}
353   ``There are two facts about the distribution of prime numbers of
354   which I hope to convince you so overwhelmingly that they will be
355   permanently engraved in your hearts. The first is that, [they are]
356   the most arbitrary and ornery objects studied by mathematicians:
357   they grow like weeds among the natural numbers, seeming to obey no
358   other law than that of chance, and nobody can predict where the next
359   one will sprout. The second fact is even more astonishing, for it
360   states just the opposite: that the prime numbers exhibit stunning
361   regularity, that there are laws governing their behavior, and that
362   they obey these laws with almost military precision.''
363 \end{quote}
364
365
366  Mathematics is flourishing.  Each year sees new exciting initiatives
that extend and sharpen the applications of our subject,   new
directions for  deep  exploration---and finer understanding---of
classical as well as very contemporary mathematical domains. We are
aided in such explorations by the development of more and more powerful
tools. We  see resolutions of centrally important questions.  And
through all of this, we are treated to surprises and dramatic changes
of viewpoint; in short: marvels.
367
368  And what an array of wonderful techniques  allow mathematicians to do
their work:  framing {\it definitions;} producing {\it constructions;}
formulating {\it analogies relating disparate concepts, and disparate
```

Figure 3: Don Zagier

"There are two facts about the distribution of prime numbers of which I hope to convince you so overwhelmingly that they will be permanently engraved in your hearts. The first is that, [they are] the most arbitrary and ornery objects studied by mathematicians: they grow like weeds among the natural numbers, seeming to obey no other law than that of chance, and nobody can predict where the

Errors and Warnings

Errors

None

Typesetting

Build Control

```
latexmk -pdf -f -g -bibtex -synctex=1 -interaction=nonstopmode rh.tex
```

Engine

LaTeX Output (3.9 seconds)

```
Latexmk: applying rule 'makeindex rh.idx'...
Latexmk: applying rule 'bibtex rh'...
For rule 'bibtex rh', running '&run_bibtex(  )' ...
This is BibTeX, Version 0.99d (TeX Live 2017/Debian)
```

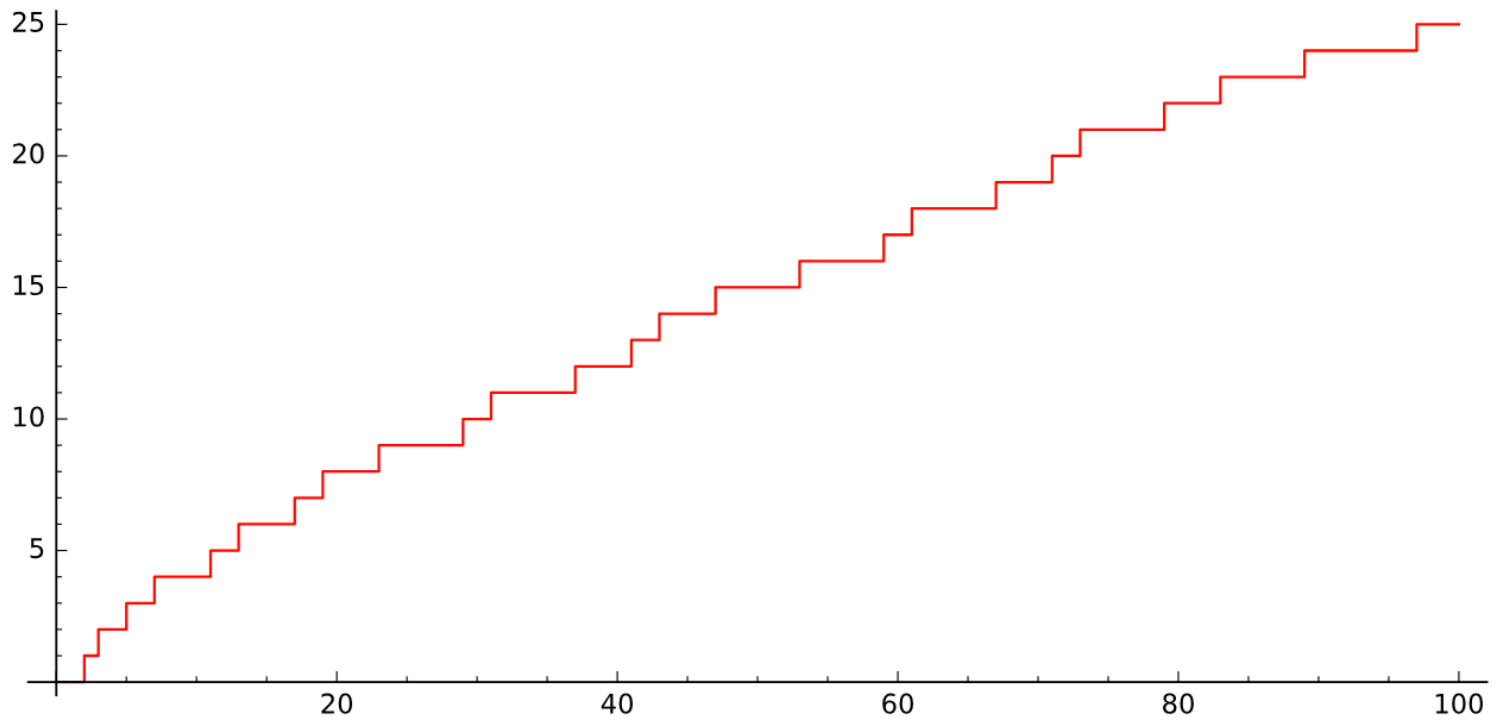
The Actual Book

(in a few slides)

Focus on The Prime Counting Problem

Let $\pi(x)$ be the number of primes $\leq x$.
Problem: give a "good approximation" for $\pi(x)$.

```
plot(prime_pi, 0, 100, color='red', figsize=[8,4])
```



Focus on The Prime Counting Problem

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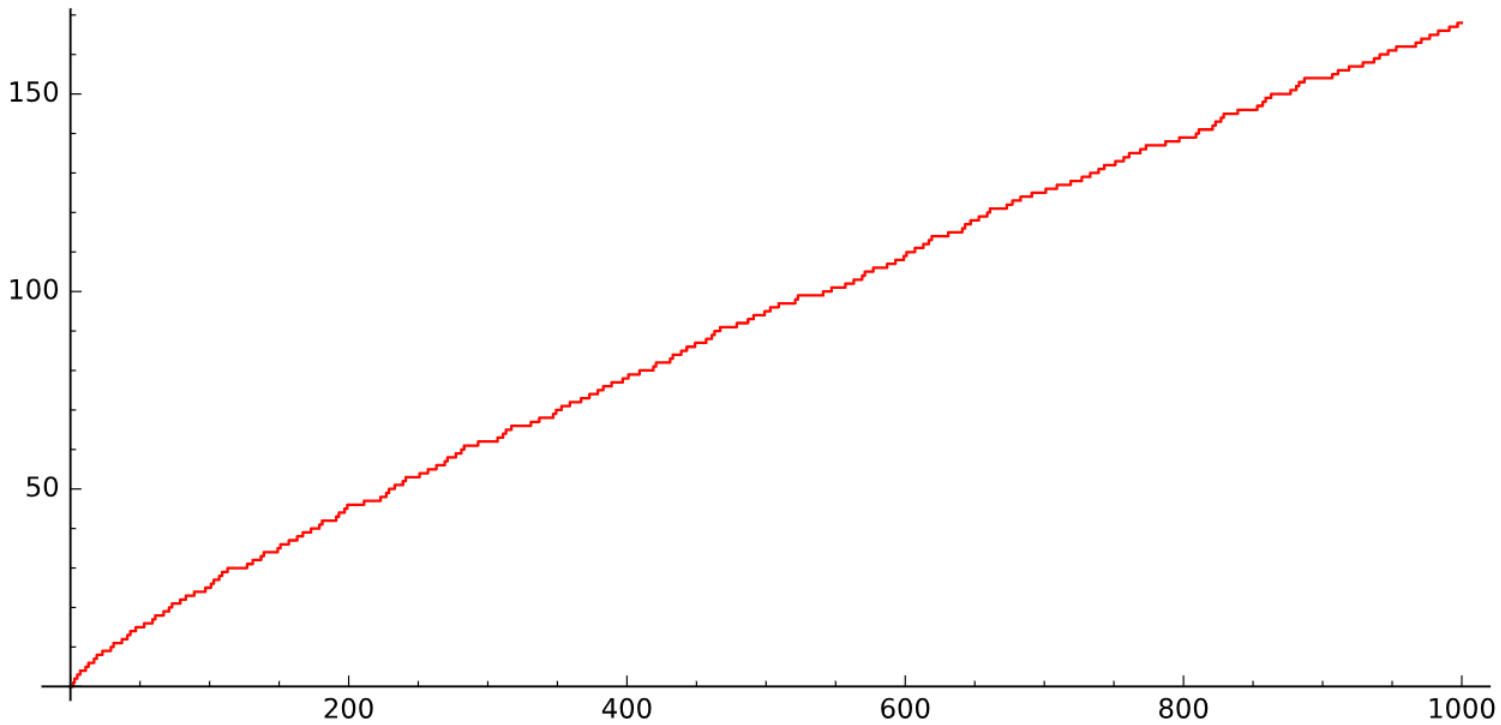
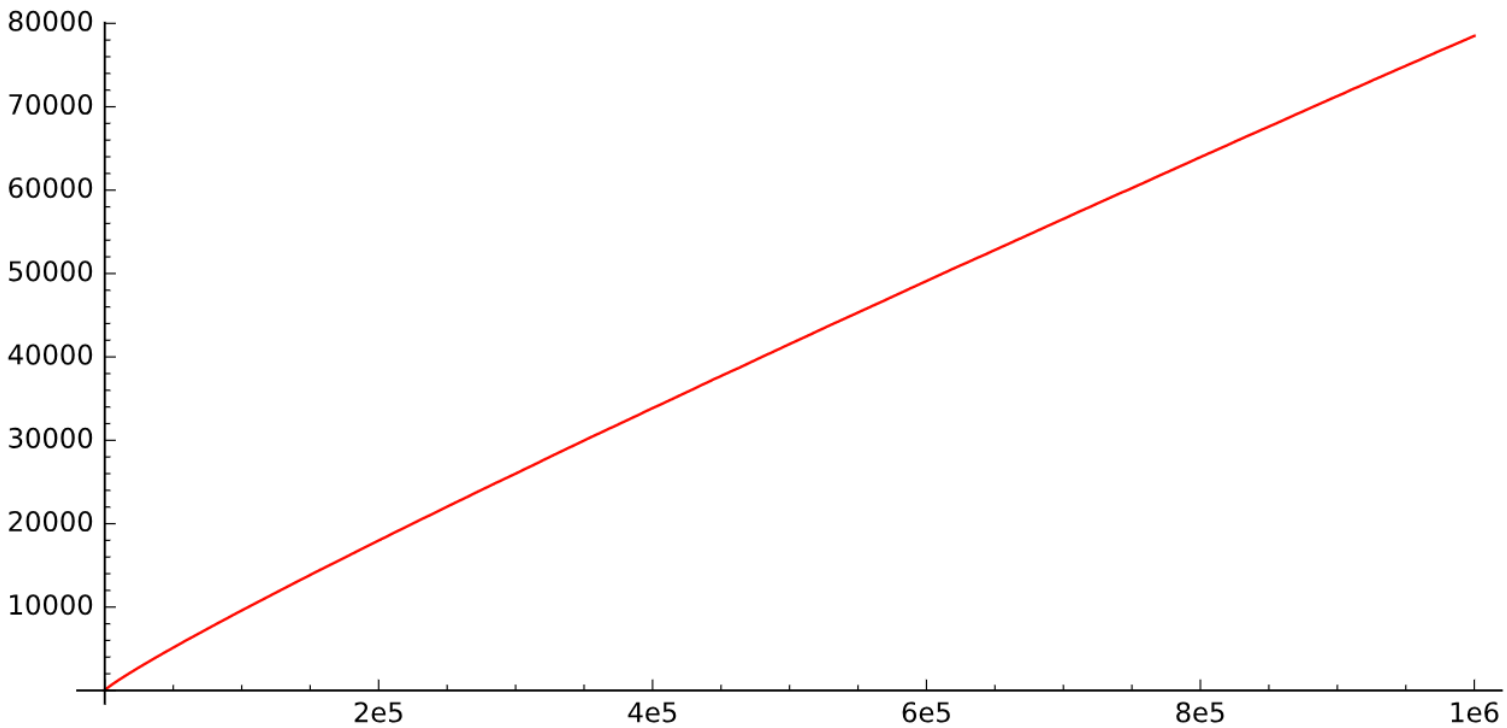
```
plot(prime_pi, 0, 1000, color='red', figsize=[8,4])
```



Focus on The Prime Counting Problem

Let $\pi(x)$ be the number of primes $\leq x$.
Problem: give a "good approximation" for $\pi(x)$.

```
plot(prime_pi, 0, 10^6, color='red', figsize=[8,4])
```



Answer: The Riemann Hypothesis (first formulation)

The number of prime numbers less than X is approximately $\mathrm{Li}(X)$ and this approximation is essentially square root accurate.

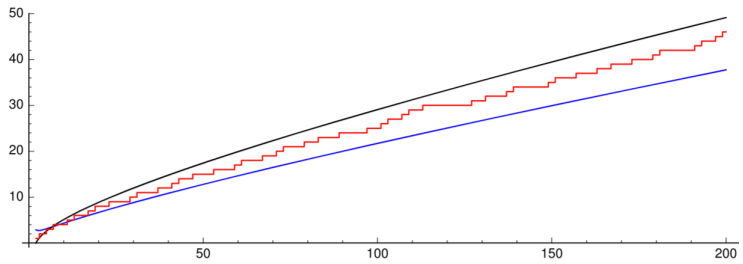


Figure 13.1: Plots of $\mathrm{Li}(X)$ (top), $\pi(X)$ (in the middle), and $X/\log(X)$

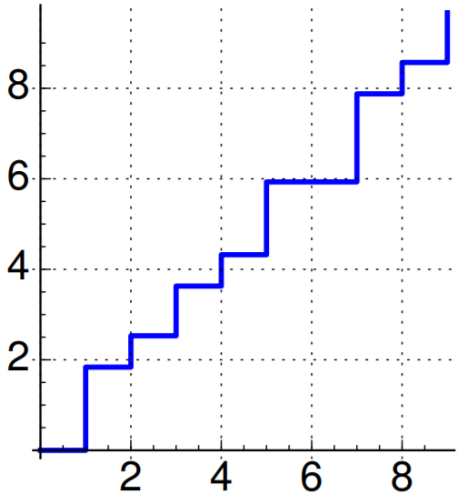
$$\begin{aligned}
\pi(X) &= 18{,}435{,}599{,}767{,}349{,}200{,}867{,}866 \\
\mathrm{Li}(X) &= 18{,}435{,}599{,}767{,}366{,}347{,}775{,}143.10580\ldots \\
X/(\log(X)-1) &= 18{,}429{,}088{,}896{,}563{,}917{,}716{,}962.93869\ldots \\
\mathrm{Li}(X)-\pi(X) &= 17{,}146{,}907{,}277.105803\ldots \\
\sqrt{X}\cdot\log(X) &= 55{,}262{,}042{,}231{,}857.096416\ldots
\end{aligned}$$

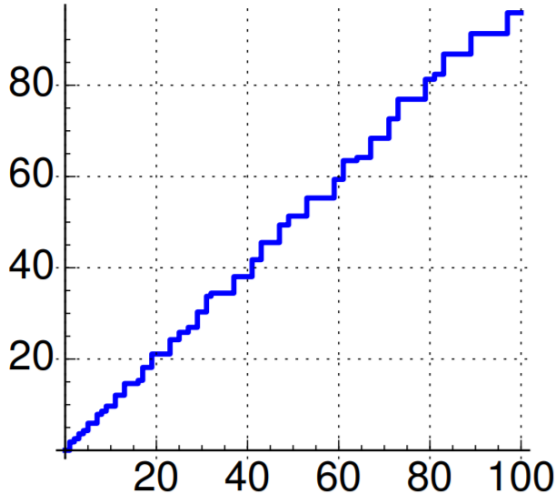
$$\pi(X) = \#\{\text{primes } \leq X\} \sim \mathrm{Li}(X) = \int_2^X \frac{1}{\log(t)}\,dt$$

Answer: The Riemann Hypothesis (second formulation)

The prime power staircase $\psi(X)$ is essentially square root close to the 45 degree straight line.

$\psi(x)$: "Our staircase starts on the ground at $x = 0$ and the height of the riser of the step at $x = 1$ will be $\log(2\pi)$. The height of the riser of the step at $x = p^n$ will not be 1 but rather: the step at $x = p^n$ will have the height of its riser equal to $\log p$."





Answer: The Riemann Hypothesis (third formulation)

The Fourier transform of the derivative of $\psi(X)$ *"is basically"* a discrete distribution supported at the imaginary parts of the zeros of the Riemann Zeta function.

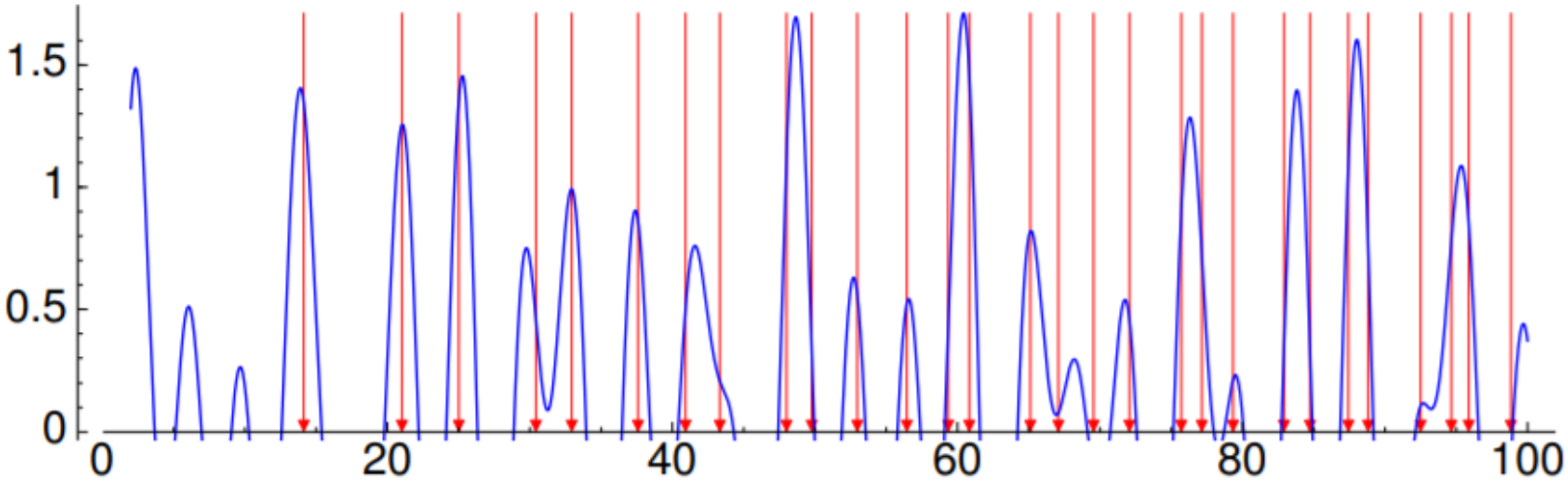
We deleted this formulation from the book, since it was too technical to state properly (it's the *explicit formula*).[1]

Instead, we illustrate the heck out of this, as follows...

[1] We deleted this formulation from the book, but accidentally didn't relabel the "fourth formulation", which confused readers.

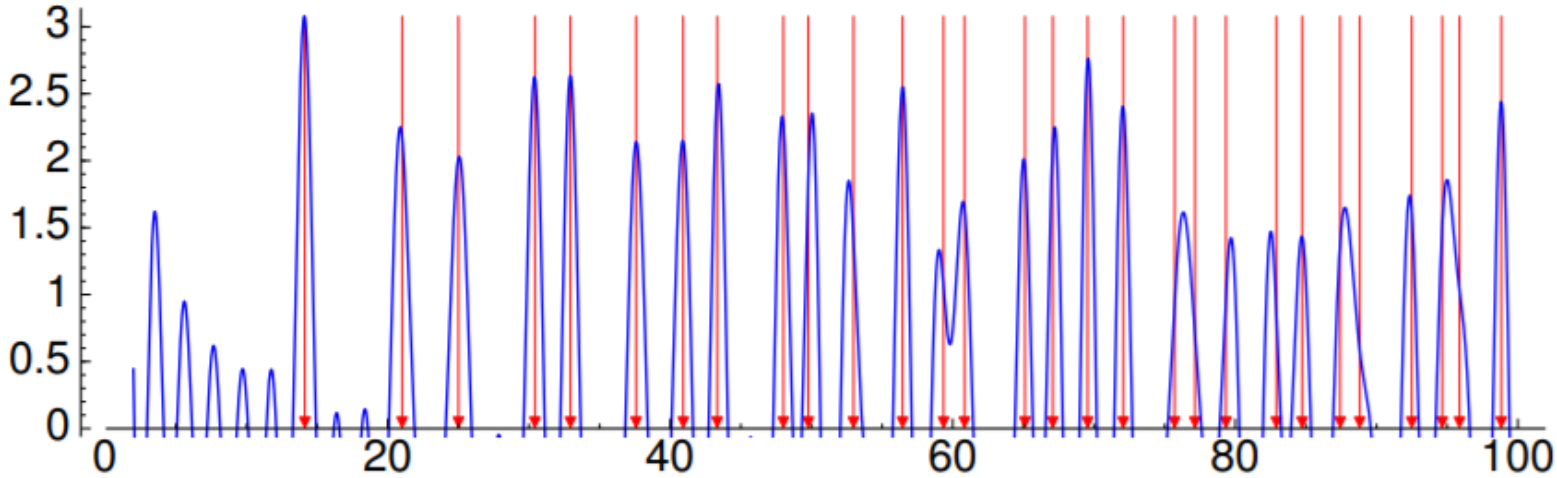
Fourier transform of $\Psi'(x)$ (first four terms)

$$f(t) = -\frac{\log(2)}{2^{1/2}}\cos(t\log(2)) - \frac{\log(3)}{3^{1/2}}\cos(t\log(3)) - \frac{\log(2)}{4^{1/2}}\cos(t\log(4)) - \frac{\log(5)}{5^{1/2}}\cos(t\log(5))$$



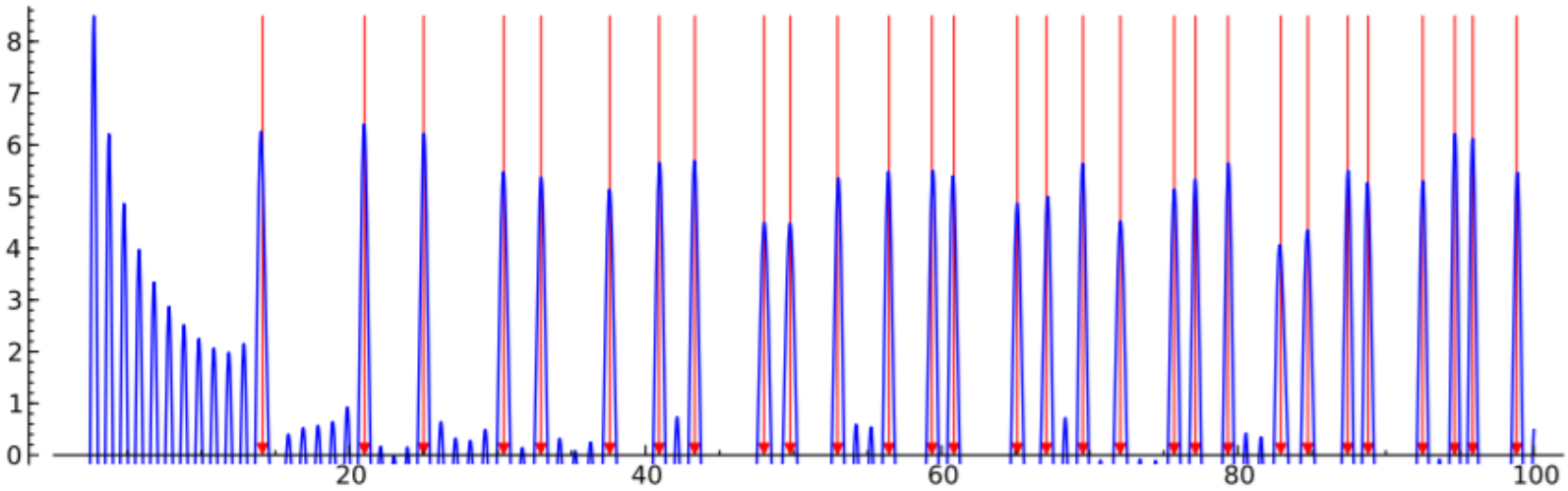
- Anybody can easily plot this.
- Arrows point to imaginary parts of zeros of $\zeta(s)$!

Fourier transform of $\Psi'(x)$ (first 20 terms)



$$-\sum_{p^n \leq 20} \frac{\log(p)}{p^{n/2}} \cos(t \log(p^n))$$

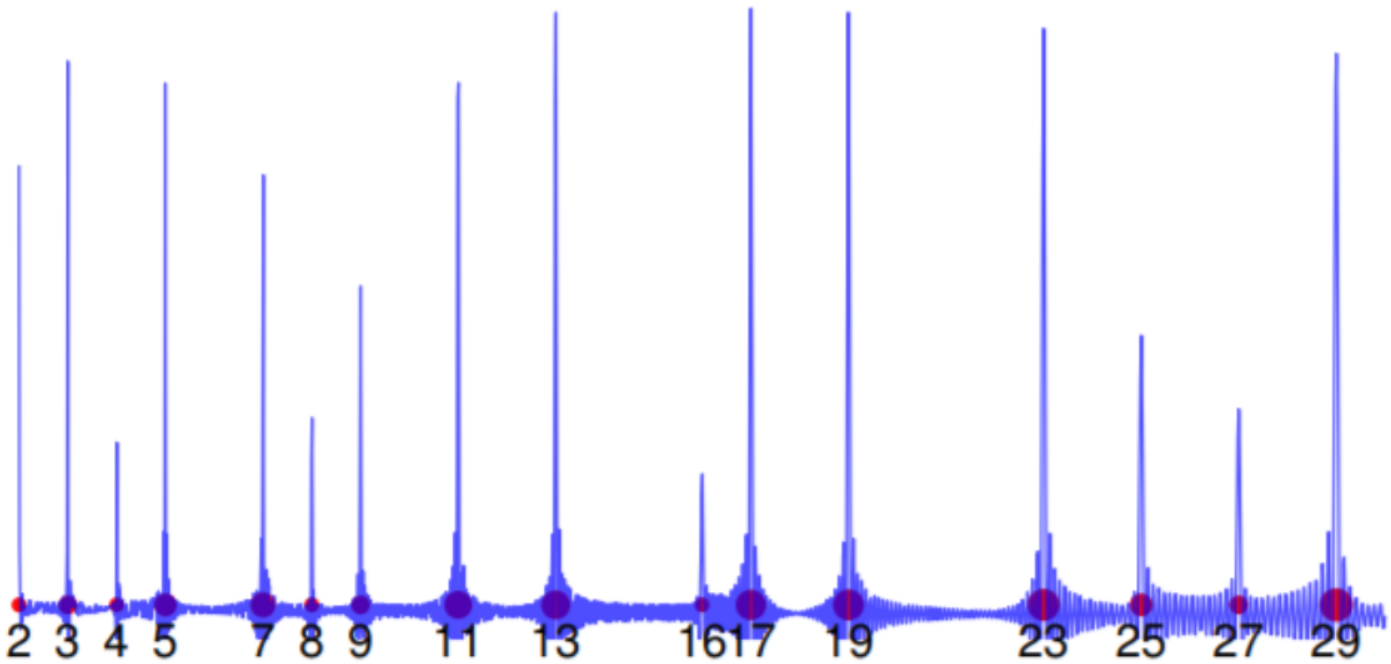
Fourier transform of $\Psi'(x)$ (first 500 terms)



$$-\sum_{p^n \leq 500} \frac{\log(p)}{p^{n/2}} \cos(t \log(p^n))$$

Take this home: *The Fourier transform of the derivative of the prime power staircase "is" the zeros of the Riemann zeta function.*

And Fourier transform of the zeros "is" prime powers



$$-\sum_{i=1}^{1000} \cos(\log(s)\theta_i)$$

Untangle to get $\pi(x)$...

Finish book with manipulation to approximate $\pi(x)$ by a sum of smooth functions involving the θ_i.

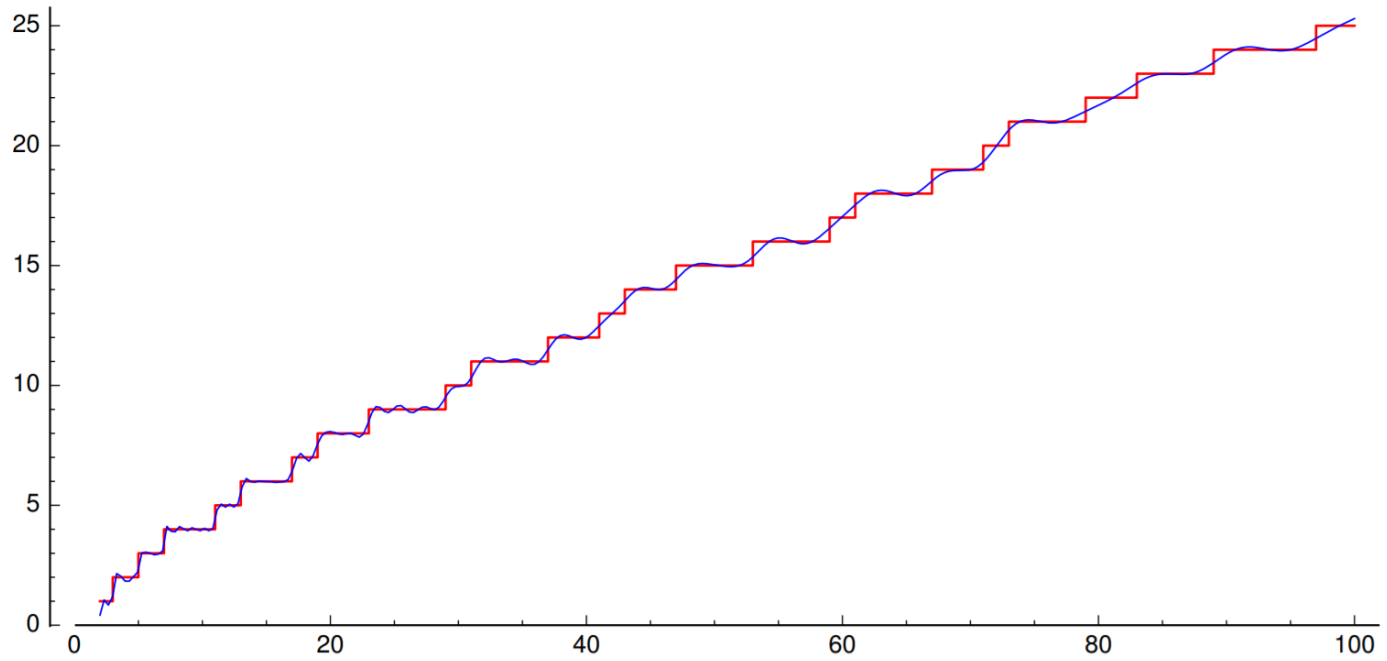


Figure 36.8: The function R_{25} approximating the staircase of primes up to 100

Inspired by Zagier's lecture "The First 50 million prime numbers".

R_{50} approximates $\pi(x)$ very well!

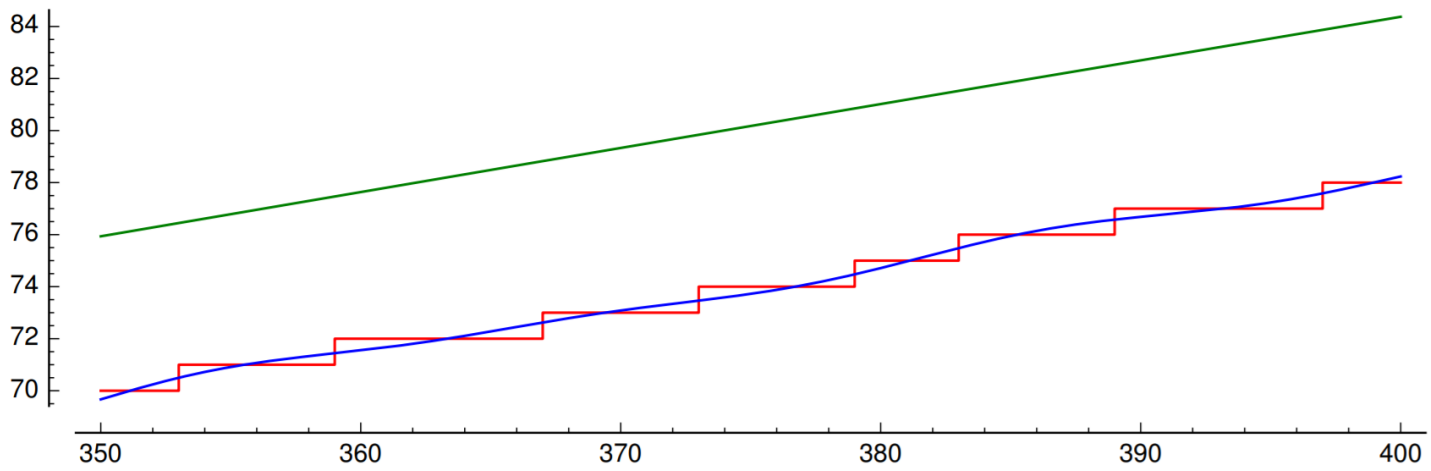


Figure 36.11: The function $\mathrm{Li}(X)$ (top, green), the function $R_{50}(X)$ (in blue), and the staircase of primes on the interval from 350 to 400.

Answer: The Riemann Hypothesis (fourth formulation)

All the nontrivial zeroes of $\zeta(s)$ lie on the vertical line in the complex plane consisting of the complex numbers with real part equal to $1/2$.

§3. Publishing a Book

How to Publish the book: Self publish!?

Self publishing?

Just put it on my website and see what happens.

- May people looked at it...
- But it didn't really get **significant traction**.

Will Hearst convinced us to publish with a commercial publisher. Maybe he was tired of printing out copies to give to people. :-)

Finding a commercial publisher

Finding our publisher...

- Barry and I have both published a few books with a couple of publishers, over the years.
- Talked to many editors (JMM was very helpful!)
- Looked at reputation, similar books, and who followed up
- Balanced competing goals (e.g., price, quality, rights)
- Kaitlin from Cambridge University Press won for this.

Typos and Mistakes

Or, making the book easier for people to read

- Dozens of people carefully read drafts of the book and provided incredibly useful feedback. **THANK YOU!!**
- The publisher also had a copy editor read the book, and provided complementary feedback.
- Don't expect your publisher to catch the sort of mistakes a mathematician catches:

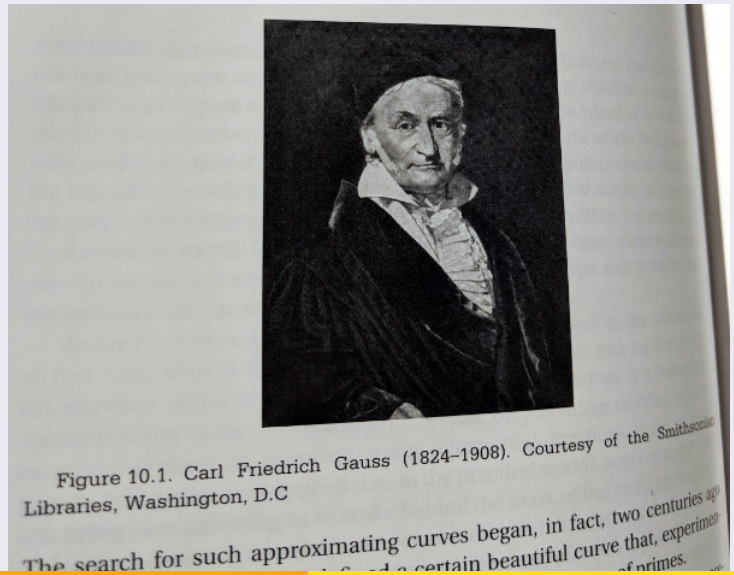


Figure 10.1. Carl Friedrich Gauss (1824–1908). Courtesy of the Smithsonian Libraries, Washington, D.C

The search for such approximating curves began, in fact, two centuries ago

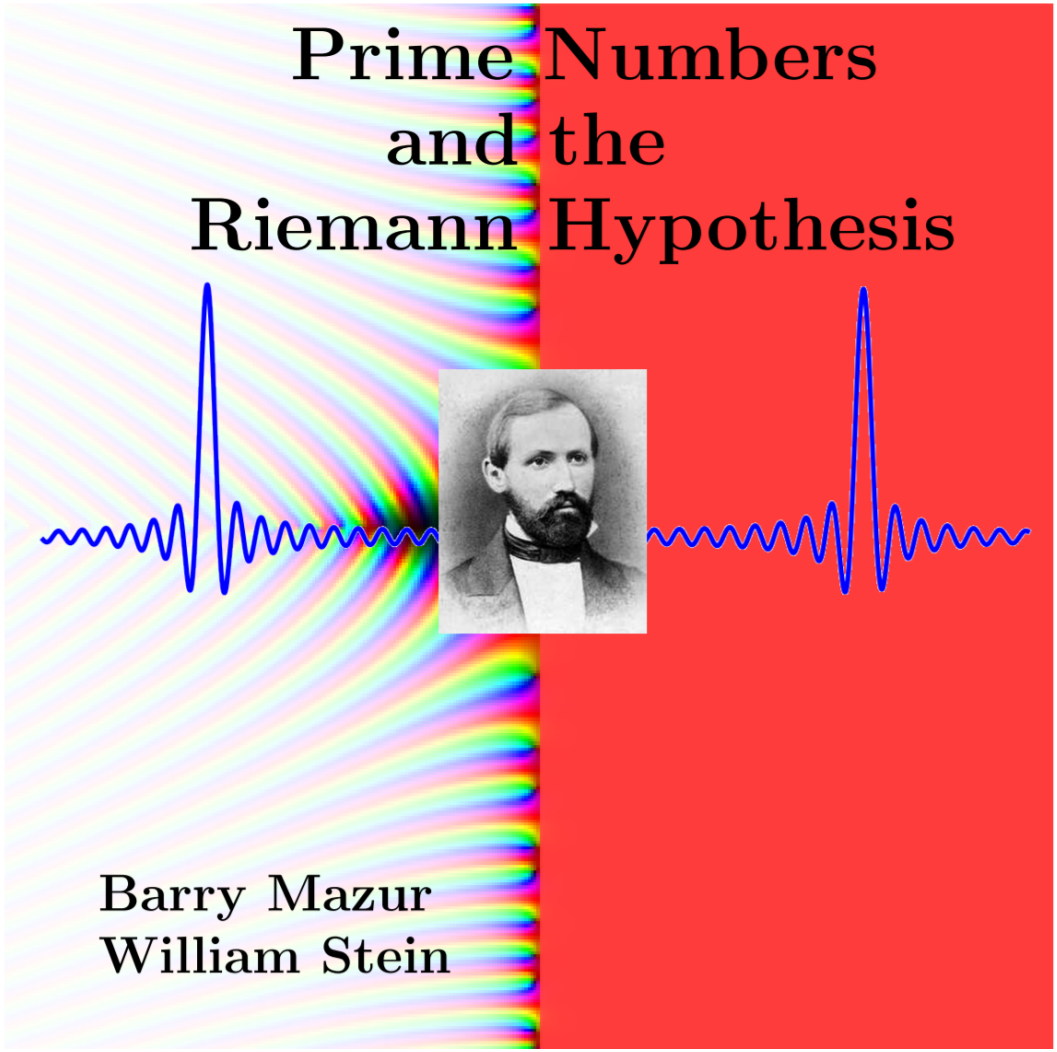
Creating a Cover

Ideas for Components Included...

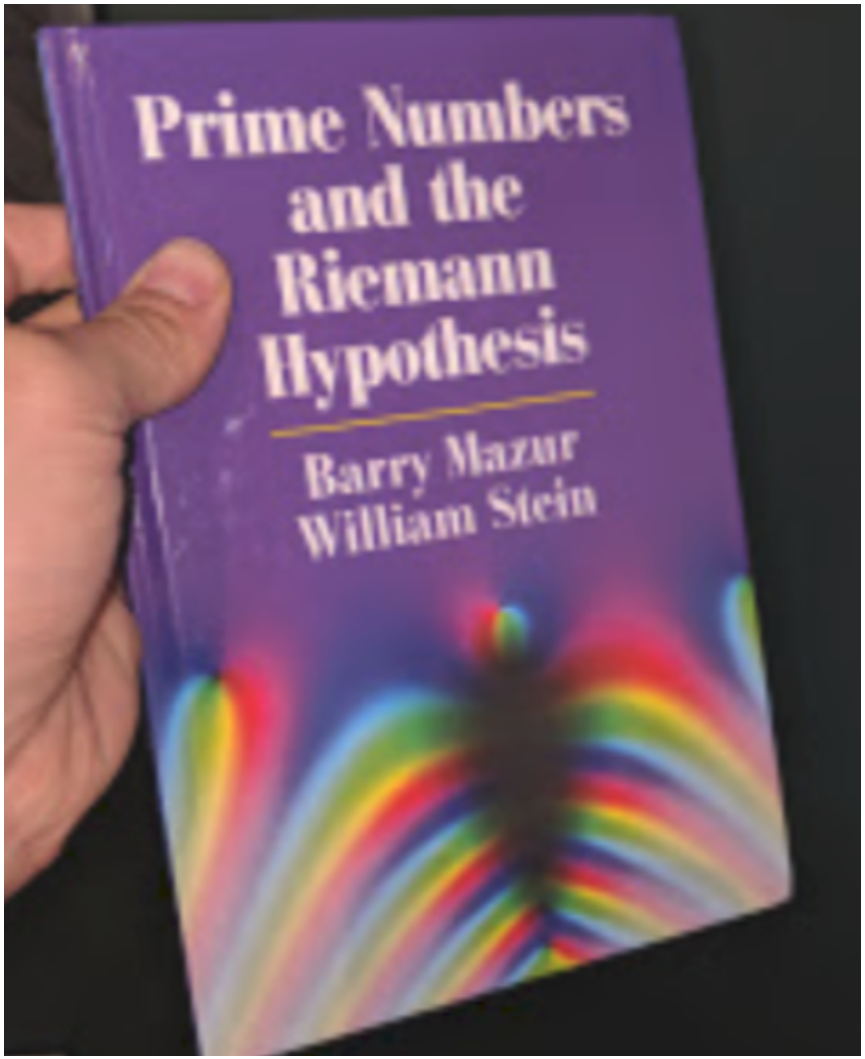
- Plot of Zeta Function (using Sage's `complex_plot`)
- Our names
- Title of book
- Portrait of Riemann, the star of the book!
- Plots illustrating the main ideas of the book
- A "classic" look

Natural design tension: publisher vs author

What we proposed



The Actual Cover



Endorsements for the back cover

Will Hearst and John Cremona kindly wrote about our book...

Production

Producing the book

- Initial friction with production, e.g., "Please provide your Microsoft Word document."
- Evidently, Cambridge Univ Press might have made some new positive steps toward better LaTeX support.
- Some unfortunate physical issues with the very first printing.
- CUP strongly supported and marketed the book.
- Working with CUP has been a *very positive experience* overall.

Published!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Prime Numbers and the Riemann Hypothesis 1st Edition, Kindle Edition
by Barry Mazur (Author), William Stein (Author)
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Oliver

★★★★★ **short, gorgeous, inspring and affordable.**

June 27, 2016

Format: Paperback | Verified Purchase

A couple of books on the Riemann hypothesis have appeared for the general public: Derbeshire 2003, Du Sautoiy 2003, Sabbagh 2003, Rockmore 2005, Watkins 2015, van der Veen and van der Craats 2015 and now Mazur-Stein 2016. More for mathematicians are Koblitz 1977, Edwards 2001, and Stopple 2003. From general expositions, one should also mention the paper of Conrey of 2003 which won the Conant prize for expository writing as well as a nice paper of Bombieri of 1992. Is this too much for the subject? No. A problem like the Riemann hypothesis can never be written too much about, especially if texts are written by experts. It is the open problems which drive mathematics. The Riemann hypothesis is the most urgent of all the open problems in math and like a good wine, the problem has become more valuable over time. What helped also is that since the time of Riemann, more and more connections with other fields of mathematics have emerged. The book of Veen-Craats and Mazur-Stein have emerged about at the same time. They are both small and well structured. Veen-Craats has been field tested with high school students and has focus mostly on the gorgeous Mangoldt explicit formula for the Chebychev prime distribution function, sometimes called the "music of the primes". Mazur-Stein do it similarly, however stress more on the Riemann spectrum and go didactically rather gently into the mathematics of Fourier theory as well as the theory of distributions. The book is carefully typeset, has color prints and some computer code for Sage. While Veen-Craats has many nice exercises, an exercise of Mazur-Stein led me to abandon other things for a couple of weeks, since it was so captivating. So be careful! A student who has taken basic calculus courses, should be able to read it. By the way, except Sabagh's book "Dr Riemann's zeros", which was written by a writer and journalist, the other books were created by professional mathematicians. The Mazur-Stein book has probably the best "street cred" among the RH books for the general audience: both have done important work in number theory, also related to zeta functions: Mazur's name is on one of the grand generalizations of the Riemann zeta functions, the Artin-Mazur zeta function which has exploded into a major tool under the lead of Ruelle who made it

discussion of the Reimann spec
Some good graphics and a good Reimann spectrum. I found Derb Obsession" overall more interest for the non-expert.
Published 1 year ago

Ricardo J. Menéndez

★★★★★ **Five Stars**
Excellent presentation ! Ricardo
Published 1 year ago

Saul

★★★★★ **Very Good for Ge**
Some Math Background
Very informative. Not for the prc about it already, of course.
Published 1 year ago

david bailey

★★★★☆ **Four Stars**
satisfactory
Published 1 year ago

Negative reviews due to *production issues*, both with the physical book and the Kindle edition....

Reception by Readers

- Sarnak's review in Bulletins
- Other reviews
- Prizes

Royalties

We sold some copies, so Cambridge University Press sent us some money. I'm spending the money on expenses for my dream dog:

Translation: what to expect?

Dear Professor Stein,

Prime Numbers and the Riemann Hypothesis

I am delighted to inform you that we are currently concluding an agreement with Nippon Hyoron Sha for a Japanese language edition of your book. They plan to print an edition of 2,500 copies initially, which will be sold at approximately 2,200 JPY per copy.

What to expect?

- Also Korean?
- Will they bother with French, etc.?

Future plans

- Online fully interactive version.
- Related research on L-series of elliptic curves, etc. There could be a similar book or article about elliptic curves.