

Writing a book with Barry Mazur

"Prime Numbers and the Riemann Hypothesis"

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SageMath, Inc. and University of Washington

June 4, 2018 at Harvard University - Mazur Celebration

Abstract

Abstract

In 2005, Barry Mazur and I started a decade+ project to write the book “Prime Numbers and the Riemann Hypothesis”. This talk is about what’s in the book and why, and how the book was produced.

Thank you to Barry and the organizers!

Encourage people to rudely interrupt me during my talk and ask questions!

Prelude: collaborate with great co-authors!

Writing John Tate's *Galois Cohomology* notes for PCMI 1999...

"Everybody is so jealous of you getting to talk with John Tate!"
– David Savitt

If you ever get the chance to write something with someone incredible, **take it!!**

(I next wrote a long paper with Ken Ribet from that same PCMI.)

Overview

1 §1. Barry's Public Lecture

2 §2. Writing a Book

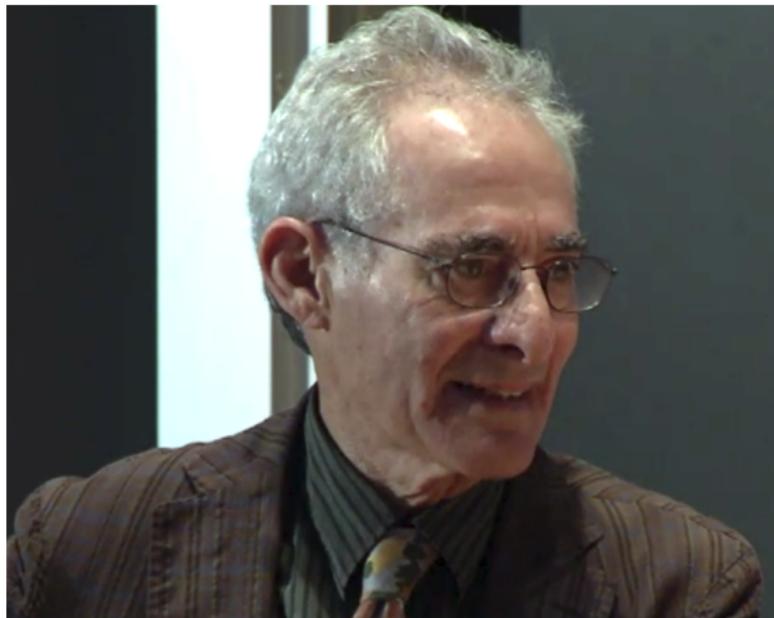
3 §3. Publishing a Book

§1. BARRY'S PUBLIC LECTURE

§1. Barry's Public Lecture

§2. Writing a Book

§3. Publishing a Book



Clay Math Institute public lecture (MIT, May 3, 2005)

"Are there still unsolved problems about the numbers 1, 2, 3, 4, ... ?"

Use primes to “sell” number theory to the general public

- Immediately accessible
- Immediately interesting
- How erratic primes are
- Cicada's every 13, 17 years...
- Many examples of “open, interesting questions”
- People can immediately make computations of their own
- Barry got his father, who had done NO math, hooked on the Goldbach Conjecture, so thought primes would work.

SageMath



I also launched SageMath in 2005

I launched Sage a few months before this 2005 CMI public lecture.

- Sage is a **free open source** alternative to Mathematica, Maple, Magma, and Matlab.
- Early Sage development motivated by this talk
 - Linking Sage to Mathematica to compute Γ
 - Early visualization functionality
 - Prime enumeration (via PARI)

More about what was in Barry's public lecture...

Topics

- Primes as atoms: integer factorization
- The largest known prime
- Enumerating primes: Sieve of Eratosthenes
- Twin primes
- Counting primes
- Gauss's Conjecture: The Prime Number Theorem
- Riemann: Fourier style smooth approximations $R_k(x)$ to $\pi(x)$
- Riemann's Harmonics: zeros of $\zeta(s)$

It worked!

§2. WRITING A BOOK

“Let’s write a book...” – Barry

Could we turn this public lecture into a “popular book”?

- Write something for a general audience
- Small and readable
- Full of *mathematics*, not stories of people
- Profusely illustrated
- Meet for a few weeks in his country house and focus on this

What kind of book?

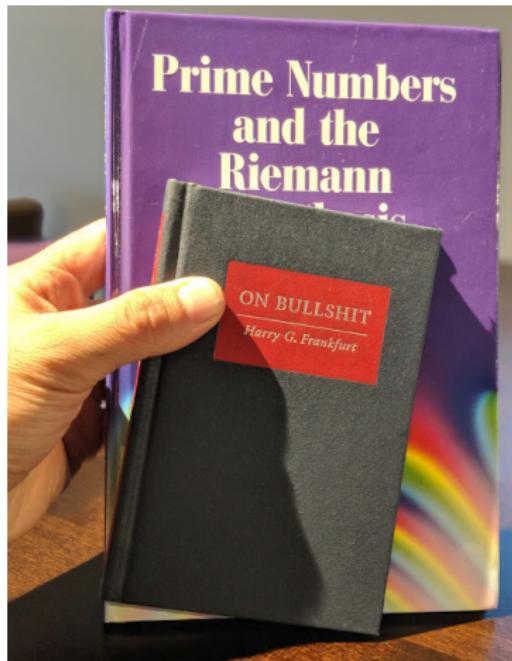
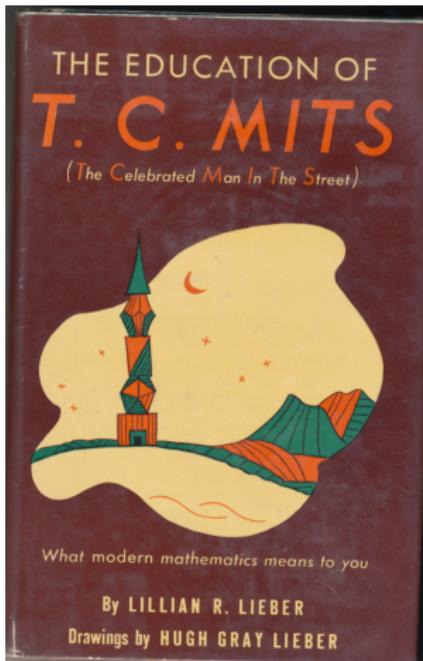
There are already 4 recent popular books on the Riemann Hypothesis. Why write another?

Our book could be unique

- Motivate by connecting the prime counting problem with our other research on *the explicit formula*
- Mostly math and not “stories of people” (other books on RH already do the stories well)

What Sort of Book: Small, Medium or Large?

Like T. C. MITS or like ON BULLSHIT?



Our Approach

Go back 150+ years and explain what RH is more from the point of view of classical Fourier analysis.

- Embrace a mid-19th century very Real perspective
- Leave Complex numbers to the very, very end



Target Audience?

Who are we writing this book for?

Lovers of number theory who want to read about mathematics.

- Bright high school students?^a
- Retired electrical engineers?

^aNeither Barry nor I graduated from high school.

SageMath again

Computations with Sage drove the exposition

We used Sage to compute with prime numbers, zeros, etc., and generally to plot everything in the book.

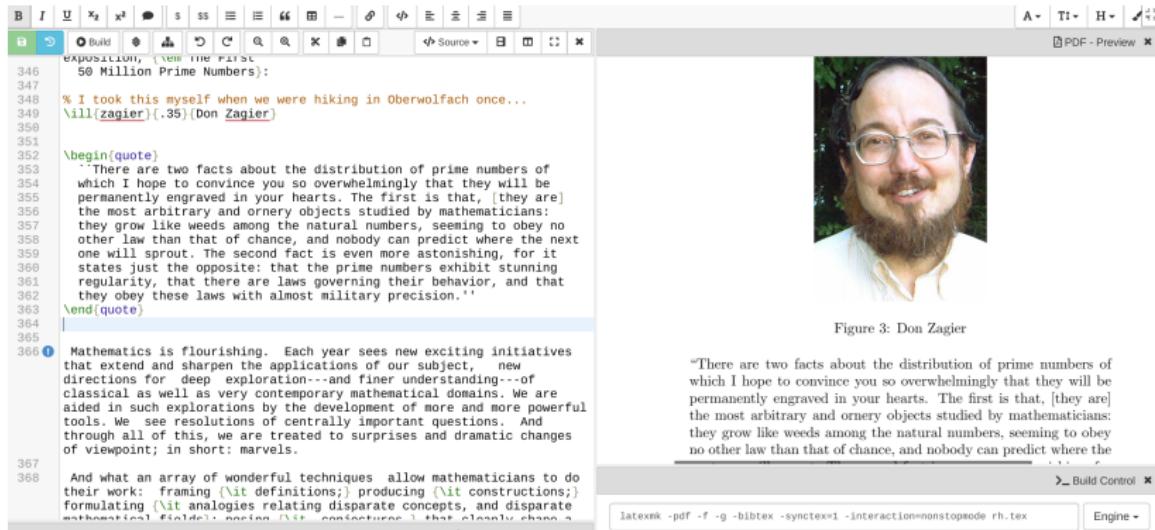
- Numerous plots that are absolutely essential to the exposition, and in fact really drove it!
- Surprising to see so much with such little computation.
- Central hook of the book appears from computation:
“Fourier transform links the discrete distribution at prime powers and the discrete distribution of zeros of $\zeta(s)$.”
- This is also what got us thinking about “how explicit is the explicit formula?” (another research project...)

Collaborative L^AT_EX via CoCalc

How we wrote the book

- I wrote CoCalc's L^AT_EX editor **for this book project:**
 - In a web browser
 - Both of us simultaneously editing the same file
 - Precise history of all changes
 - Gives a sense of the collaborative spirit
- Rough PDF of book on the web at every stage
- GitHub tracking of changes
- Sage computations run in the same place as editing book
- Barry very closely read and understood the Sage code

CoCalc's Collaborative L^AT_EX Editor



The screenshot shows the CoCalc LaTeX editor interface. At the top, there's a toolbar with various icons for text styling (B, I, U), alignment (X₁, X₂, X₃), tables (T₁, T₂, T₃), and other document functions. Below the toolbar is a menu bar with "Build", "File", "Edit", "View", "Insert", "Search", "Help", and "Source". A preview window on the right shows a portrait of a man with glasses and a beard, identified as Don Zagier.

The main text area contains LaTeX code:

```
346 \exposition{ (view The First  
347 50 Million Prime Numbers):  
348 % I took this myself when we were hiking in Oberwolfach once...  
349 \vfill{zagier}{.35}{(Don Zagier)}  
350  
351  
352 \begin{quote}  
353 ``There are two facts about the distribution of prime numbers of  
354 which I hope to convince you so overwhelmingly that they will be  
355 permanently engraved in your hearts. The first is that, [they are]  
356 the most arbitrary and ornery objects studied by mathematicians:  
357 they grow like weeds among the natural numbers, seeming to obey no  
358 other law than that of chance, and nobody can predict where the next  
359 one will sprout. The second fact is even more astonishing, for it  
360 states just the opposite: that the prime numbers exhibit stunning  
361 regularity, that there are laws governing their behavior, and that  
362 they obey these laws with almost military precision.''  
363 \end{quote}  
364  
365  
366 ① Mathematics is flourishing. Each year sees new exciting initiatives  
367 that extend and sharpen the applications of our subject, new  
368 directions for deep exploration--and finer understanding--of  
classical as well as very contemporary mathematical domains. We are  
aided in such explorations by the development of more and more powerful  
tools. We see resolutions of centrally important questions. And  
through all of this, we are treated to surprises and dramatic changes  
of viewpoint; in short: marvels.  
369  
370 And what an array of wonderful techniques allow mathematicians to do  
371 their work: framing (\texttt{At} definitions); producing (\texttt{At} constructions);  
372 formulating (\texttt{At} analogies relating disparate concepts, and disparate  
373 mathematical fields); posing (\texttt{At} conjectures) that cleanly change a  
374
```

Below the text area, there are sections for "Errors" (None) and "Typesetting". At the bottom, there's a "Build Control" panel with "latexmk -pdf -f -g -bibtex -synctex=1 -interaction=nonstopmode rh.tex" and an "Engine" dropdown. The "Latex Output" section shows the command being run and its results.

Figure 3: Don Zagier

“There are two facts about the distribution of prime numbers of which I hope to convince you so overwhelmingly that they will be permanently engraved in your hearts. The first is that, [they are] the most arbitrary and ornery objects studied by mathematicians: they grow like weeds among the natural numbers, seeming to obey no other law than that of chance, and nobody can predict where the next one will sprout. The second fact is even more astonishing, for it states just the opposite: that the prime numbers exhibit stunning regularity, that there are laws governing their behavior, and that they obey these laws with almost military precision.”

CoCalc's Collaborative L^AT_EX Editor and Sage Worksheet

The screenshot shows a dual-pane interface. The left pane is a L^AT_EX editor for a file named `talk.tex`, displaying a document page titled "Page 18" with the header "§1. Barry's Public Lecture", "§2. Writing a Book", and "§3. Publishing a Book". The main content of the page discusses the distribution of prime numbers and includes a portrait of Barry Mazur. The right pane is a Sage worksheet with a code cell showing a plot of the prime-counting function $\pi(x)$ from 0 to 100, colored red, with a figure showing a step-like curve. Below the code cell, the output shows the generated L^AT_EX code for the plot. A status bar at the bottom indicates "Errors and Warnings" with none listed.

Figure 3: Screenshot of CoCalc's Collaborative L^AT_EX Editor

We sold some copies, so Cambridge University Press sent us some money. I'm spending my share on expenses for my dream dog:



HERE IS THE BOOK!

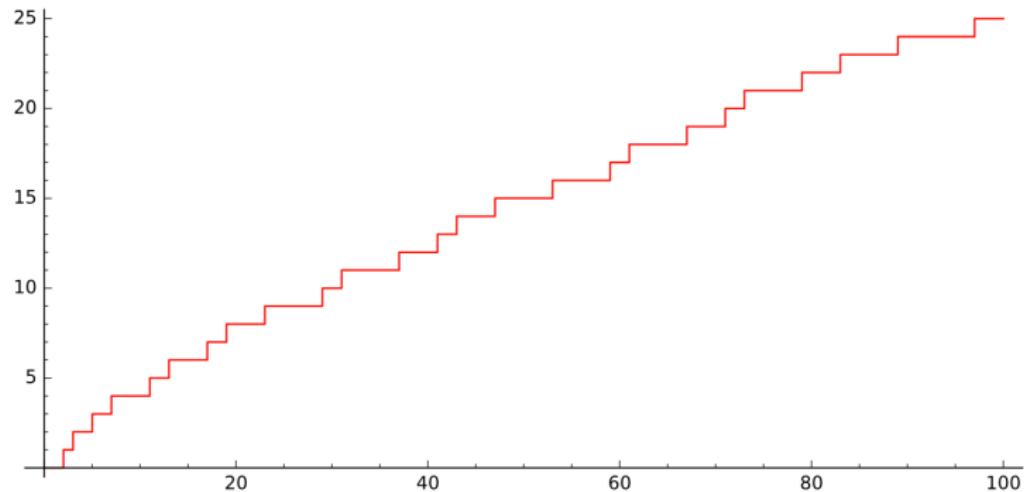
(in just a few slides)

The Prime Counting Problem

Let $\pi(x)$ be the number of primes $\leq x$.

Problem: Give a “good approximation” for $\pi(x)$.

```
plot(prime_pi, 0, 100, color='red', figsize=[8,4])
```

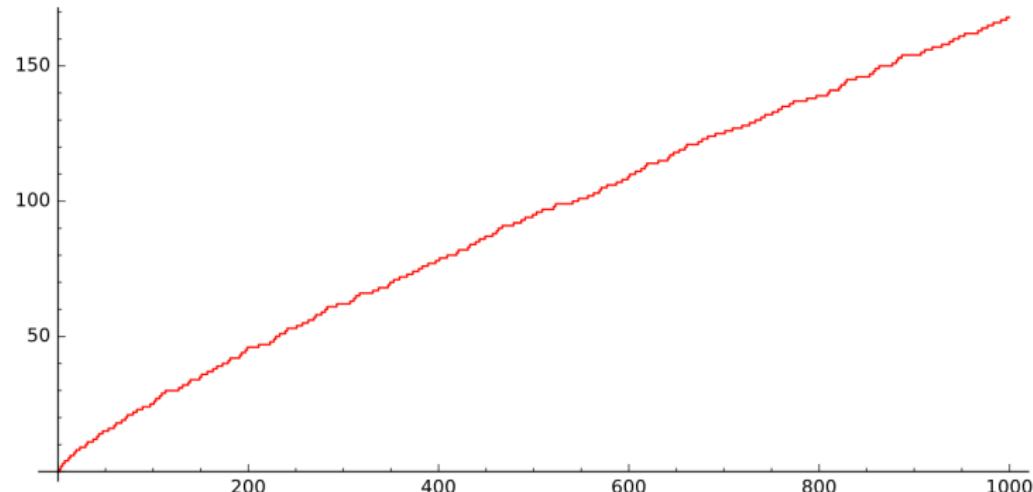


The Prime Counting Problem

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plot(prime_pi, 0, 1000, color='red', figsize=[8,4])
```

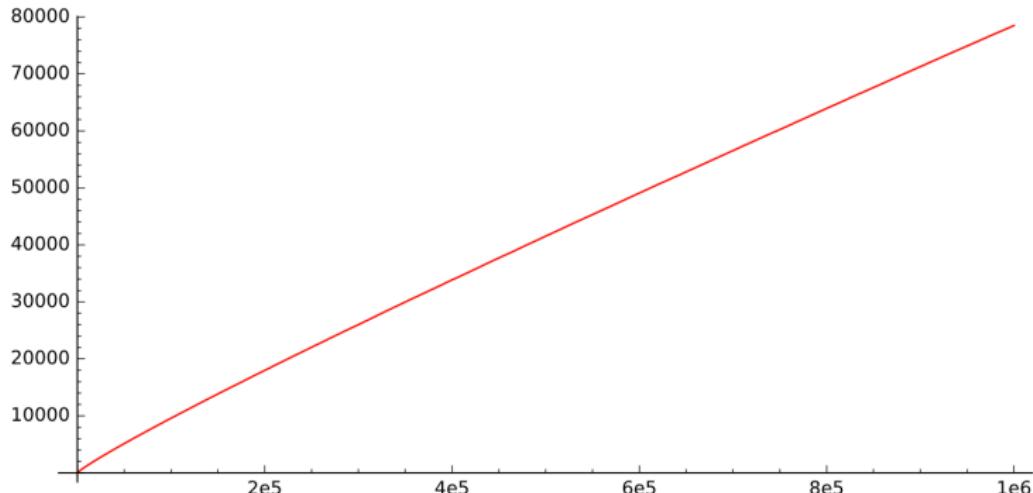


Focus on The Prime Counting Problem

Let $\pi(x)$ be the number of primes $\leq x$.

Problem: Give a “good approximation” for $\pi(x)$.

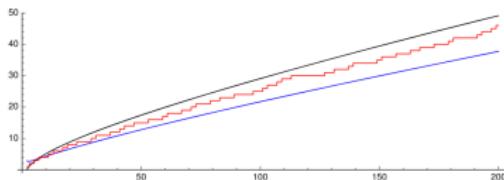
```
plot(prime_pi, 0, 10^6, color='red', figsize=[8,4])
```



Answer: The Riemann Hypothesis (first formulation)

$\text{Li}(X) = \int_2^X \frac{1}{\log(t)} dt$ is good approx to $\pi(X) = \#\{\text{primes} \leq X\}$

For $X = 10^{24}$:



$$\begin{aligned}\pi(X) &= 18,435,599,767,349,200,867,866 \\ \text{Li}(X) &= 18,435,599,767,366,347,775,143.10580\dots \\ X/(\log(X) - 1) &= 18,429,088,896,563,917,716,962.93869\dots \\ \text{Li}(X) - \pi(X) &= 17,146,907,277.10580\dots \\ \sqrt{X} \cdot \log(X) &= 55,262,042,231,857.096416\dots\end{aligned}$$

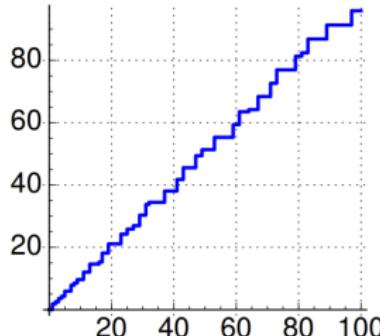
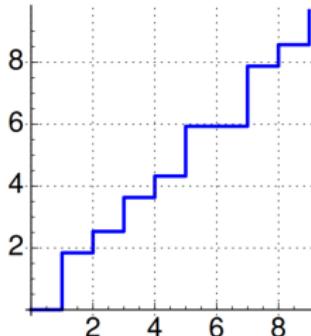
Figure 13.1: Plots of $\text{Li}(X)$ (top), $\pi(X)$ (in the middle), and $X/\log(X)$

RH1: The number of prime numbers less than X is approximately $\text{Li}(X)$ and this approximation is essentially square root accurate.

Gauss wrote in his 1849 letter that there are 216,745 prime numbers less than three million.
This is wrong: the actual number of these primes is 216,816.

Answer: The Riemann Hypothesis (second formulation)

$\psi(x)$: "A new staircase that starts on the ground at $x = 0$ and the height of the riser of the step at $x = 1$ will be $\log(2\pi)$. The height of the riser of the step at $x = p^n$ will not be 1 but rather: the step at $x = p^n$ will have the height of its riser equal to $\log p$."



RH2: The prime power staircase $\psi(X)$ is essentially square root close to the 45 degree straight line.

Answer: The Riemann Hypothesis (third formulation)

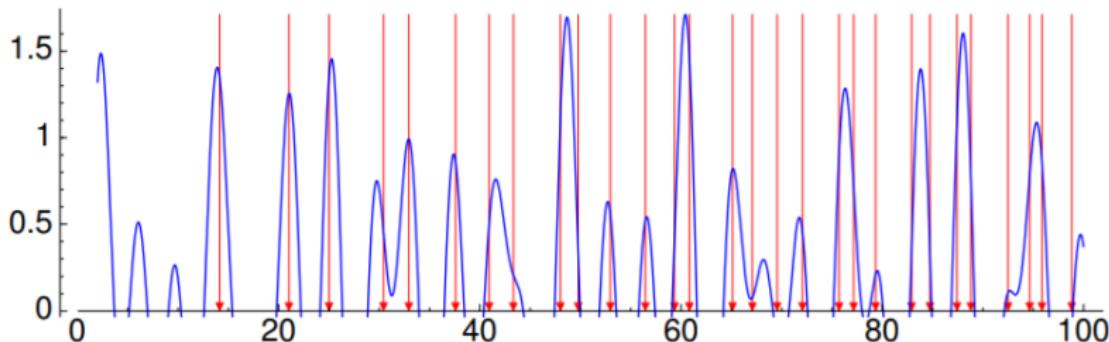
We deleted this formulation from the book, since it was too technical to state properly (it's the *explicit formula*). After deleting this, we accidentally didn't relabel the “fourth formulation”, which confused readers.

Instead, we illustrate the heck out of it!

RH3: The Fourier transform of $\psi'(X)$ “*is basically*” a discrete distribution supported at the imaginary parts of the (nontrivial) zeros of $\zeta(s)$.

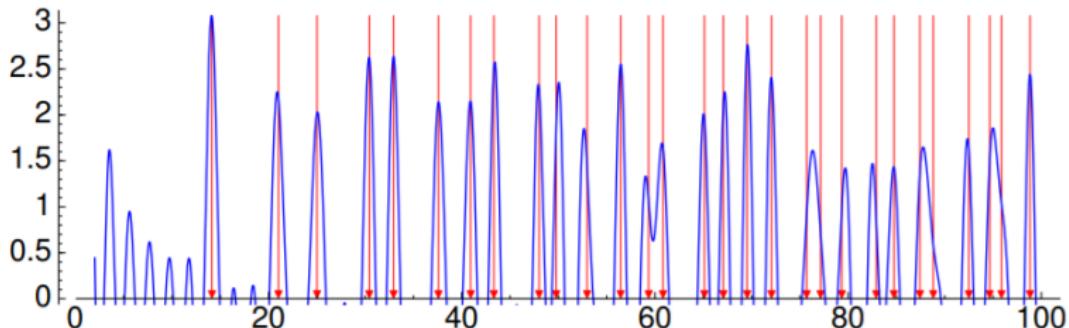
Fourier transform of $\Psi'(x)$ (just four terms!)

$$f(t) = -\frac{\log(2)}{2^{1/2}} \cos(t \log(2)) - \frac{\log(3)}{3^{1/2}} \cos(t \log(3)) \\ - \frac{\log(2)}{4^{1/2}} \cos(t \log(4)) - \frac{\log(5)}{5^{1/2}} \cos(t \log(5))$$



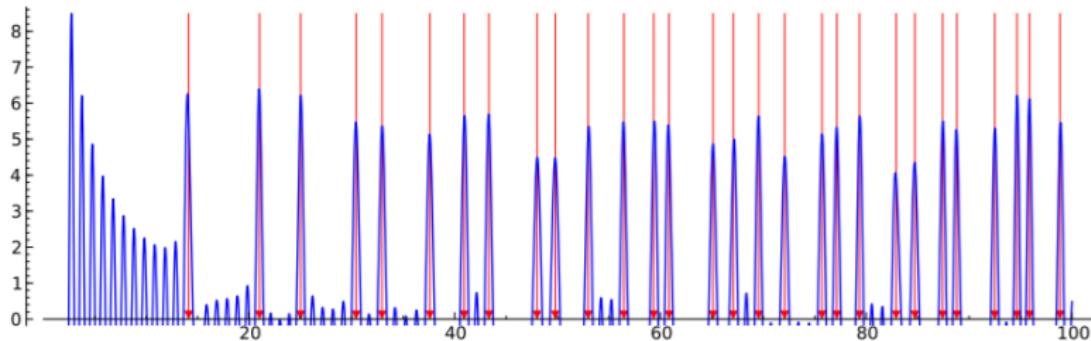
- Anybody can easily plot this.
- Arrows point to imaginary parts of zeros of $\zeta(s)$!

Fourier transform of $\Psi'(x)$ (first 20 terms)



$$-\sum_{p^n \leq 20} \frac{\log(p)}{p^{n/2}} \cos(t \log(p^n))$$

Fourier transform of $\Psi'(x)$ (first 500 terms)

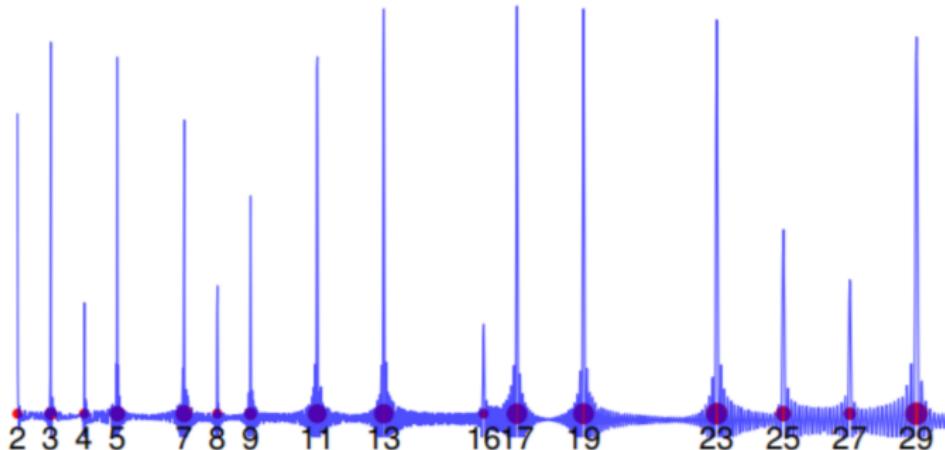


$$-\sum_{p^n \leq 500} \frac{\log(p)}{p^{n/2}} \cos(t \log(p^n))$$

Take this home: *The Fourier transform of the derivative of the prime power staircase “is” the zeros of the Riemann zeta function.*

It goes both ways!

The Fourier transform of the zeros “is” prime powers:



$$-\sum_{i=1}^{1000} \cos(\log(s)\theta_i)$$

Riemann untangled this to get $\pi(x) \dots$

We finish book with manipulation to approximate $\pi(x)$ by a sum of smooth functions $R_k(x)$ involving the θ_i .

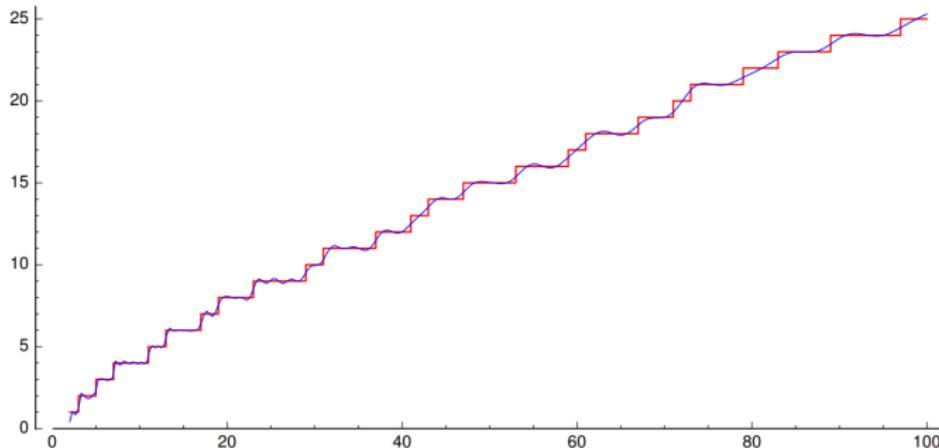


Figure 36.8: The function R_{25} approximating the staircase of primes up to 100

Inspiration: Zagier's lecture "The First 50 million prime numbers".

R_{50} approximates $\pi(x)$ very well!

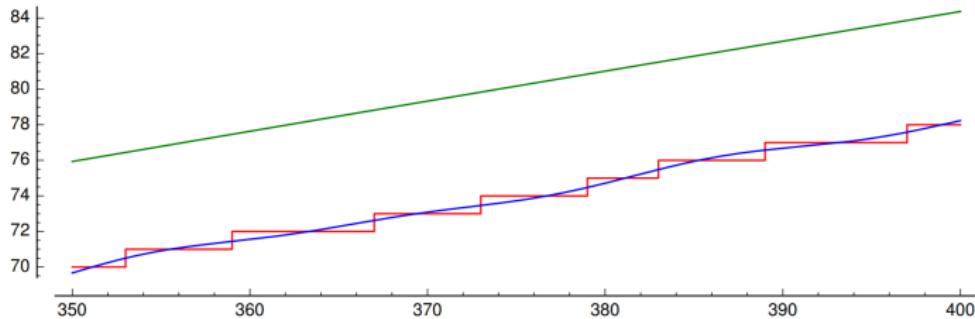


Figure 36.11: The function $\text{Li}(X)$ (top, green), the function $R_{50}(X)$ (in blue), and the staircase of primes on the interval from 350 to 400.

Answer: The Riemann Hypothesis (fourth formulation)

RH4: All the nontrivial zeroes of $\zeta(s)$ lie on the vertical line in the complex plane consisting of the complex numbers with real part equal to $1/2$.

§3. PUBLISHING A BOOK

How to Publish the book: Self publish!?

Self publishing?

Just put it on my website and see what happens.

- Some people read it...
- It didn't really get **significant traction**.
- There was still that key **missing quality** step.

Will Hearst convinced us to publish with a commercial publisher.
Maybe he was tired of printing out copies to give to people?



Finding a publisher

Finding the right publisher for *this book*...

- Barry and I have both published a few books with a couple of publishers, over the years.
- Talked to many editors (the JMM was **very** helpful!)
- Looked at reputation, similar books, and who followed up
- Balanced competing goals (e.g., price, quality, rights)
- Kaitlin Leach from Cambridge University Press won.

Typos and Mistakes

Or, making the book easier for people to read

- Dozens of people carefully read drafts of the book and provided incredibly useful feedback. **THANK YOU!!**
- The publisher had a copy editor read the book, and provided complementary feedback.
- Don't expect your publisher to catch the sort of mistakes a mathematician would catch (should be 1777–1855):



Figure 10.1. Carl Friedrich Gauss (1824–1908). Courtesy of the Smithsonian Libraries, Washington, D.C.

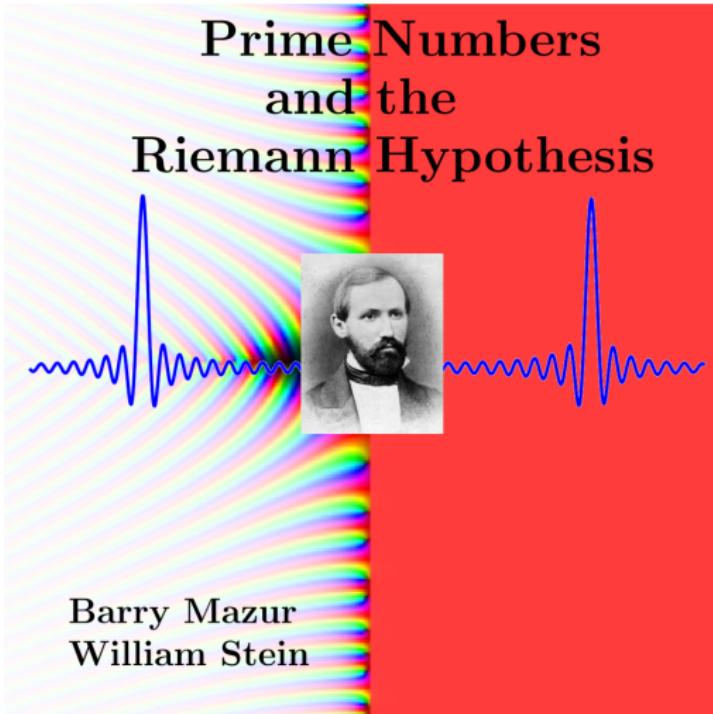
Creating a Cover

Ideas for Components Included...

- Title of book
- Our names
- Plot of $\zeta(s)$, using Sage's complex_plot
- Portrait of Riemann, the star of the book!
- Plots illustrating the main ideas of the book
- A “classical” look

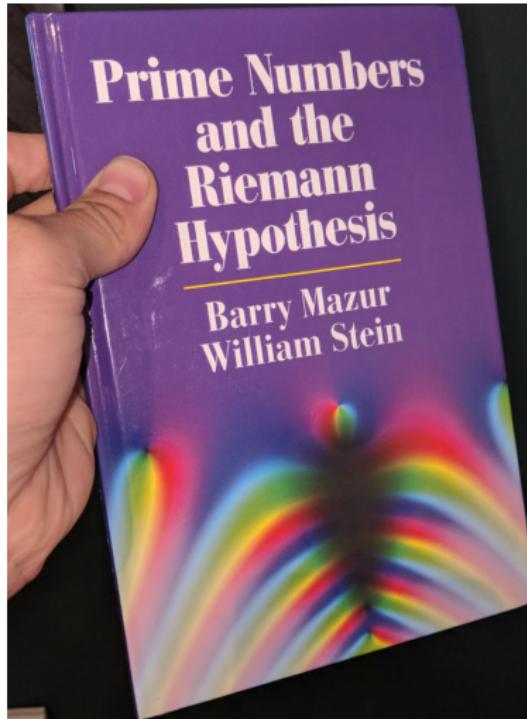
There is a natural tension here: publisher vs author vs marketer

What We Created



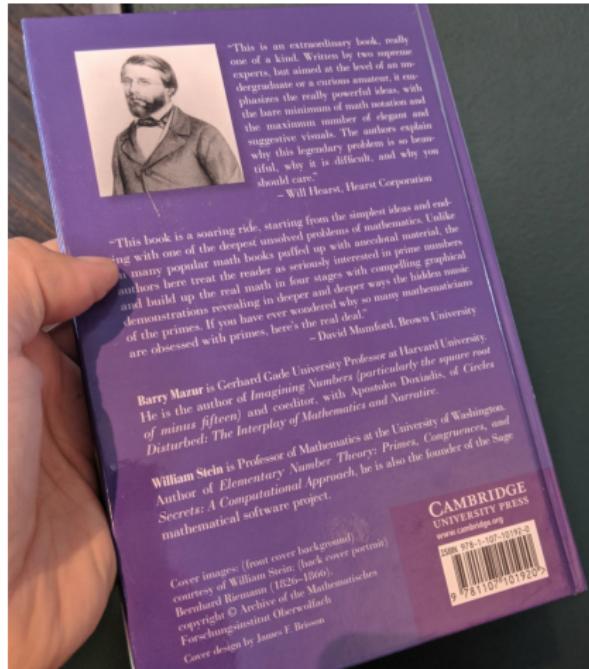
- §1. Barry's Public Lecture
- §2. Writing a Book
- §3. Publishing a Book

The Actual Cover



Endorsements for the back cover

Will Hearst and David Mumford kindly wrote about our book...



Production

Producing the book

- Initial friction with production, e.g., “Please provide Microsoft Word document.” (Cambridge Univ Press has made many positive steps toward better \LaTeX support.)
- An unfortunate physical issue with some of the first printing.
- CUP strongly supported and marketed the book.
- Working with CUP has been a *very positive experience* overall.

Published!

amazon prime

Deliver to William Seattle 98122

Departments ▾ Browsing History ▾ William's Amazon.com Today's Deals Gift Cards Registry Sell

Buy a Kindle Kindle eBooks Kindle Unlimited Prime Reading Best Sellers & More Kindle Book Deals Free Reading Apps Kindle Singles Newsstand

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Prime Numbers and the Riemann Hypothesis 1st Edition, Kindle Edition

by Barry Mazur (Author), William Stein (Author)

★★★★★ 37 customer reviews

The book cover is dark purple with white text. It features the title "Prime Numbers and the Riemann Hypothesis" in large, bold, serif font, and the authors' names "Barry Mazur" and "William Stein" in a smaller, sans-serif font below it.

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User Reviews

Top customer reviews



Oliver

★★★★★ short, gorgeous, inspiring and affordable.

June 27, 2016

Format: Paperback | **Verified Purchase**

A couple of books on the Riemann hypothesis have appeared for the general public: Derbyshire 2003, Du Sautoy 2003, Sabbagh 2003, Rockmore 2005, Watkins 2015, van der Veen and van der Craats 2015 and now Mazur-Stein 2016. More for mathematicians are Koblitz 1977, Edwards 2001, and Stopple 2003. From general expositions, one should also mention the paper of Conrey of 2003 which won the Conant prize for expository writing as well as a nice paper of Bombieri of 1992. Is this too much for the subject? No. A problem like the Riemann hypothesis can never be written too much about, especially if texts are written by experts. It is the open problems which drive mathematics. The Riemann hypothesis is the most urgent of all the open problems in math and like a good wine, the problem has become more valuable over time. What helped also is that since the time of Riemann, more and more connections with other fields of mathematics have emerged. The book of Veen-Craats and Mazur-Stein have emerged about at the same time. They are both small and well structured. Veen-Craats has been field tested with high school students and has focus mostly on the gorgeous Mangoldt explicit formula for the Chebychev prime distribution function, sometimes called the "music of the primes". Mazur-Stein do it similarly, however stress more on the Riemann spectrum and go didactically rather gently into the mathematics of Fourier theory as well as the theory of distributions. The book is carefully typeset, has color prints and some computer code for Sage. While Veen-Craats has many nice exercises, an exercise of Mazur-Stein led me to abandon other things for a couple of weeks, since it was so captivating. So be careful! A student who has taken basic calculus courses, should be able to read it. By the way, except Sabagh's book "Dr Riemann's zeros", which was written by a writer and journalist, the other books were created by professional mathematicians. The Mazur-Stein book has probably the best "street cred" among the RH books for the general audience: both have done important work in number theory, also related to zeta functions: Mazur's name is on one of the grand generalizations of the Riemann zeta function, the Artin-Mazur zeta function which has exploded into a major tool under the lead of Buell who made it

discussion of the Riemann spec:
Some good graphics and a good Riemann spectrum. I found Derby Obsession" overall more interesting for the non-expert.

Published 1 year ago



Ricardo J. Menéndez

★★★★★ Five Stars

Excellent presentation ! Ricardo
Published 1 year ago



Saul

★★★★★ Very Good for General Math Background
Very informative. Not for the person about it already, of course.
Published 1 year ago



david bailey

★★★★☆ Four Stars

satisfactory
Published 1 year ago

Magazine & Blog Reviews

- Sarnak review in Bulletins: “make effective use of such technology and do a marvelous job of integrating all this information into an exposition of the underlying mathematics.”
- Avner Ash in American Math Monthly
- MAA review: “This book is a splendid piece of work, informative and valuable.”
- John Baez: “It’s the best elementary introduction to the connection between prime numbers and zeros of the Riemann zeta function.”
- Cathy O’Neil: “If I have one complaint it’s all the pictures of white male mathematicians. [...] it would have been even better if it focused on the ideas more and the people less.”

\$ Royalties \$

We sold some copies, so Cambridge University Press sent us some money. I'm spending my share on expenses for my dream dog:



Translations

25 Apr 2018

Dear Professor Stein,

Prime Numbers and the Riemann Hypothesis

I am delighted to inform you that we are currently concluding an agreement with Nippon Hyoron Sha for a Japanese language edition of your book. They plan to print an edition of 2,500 copies initially, which will be sold at approximately 2,200 JPY per copy.

Future plans

Someday we hope to...

- Create online fully interactive version of all the plots, which don't require knowing Sage to use.
- Finish related research on L -series of elliptic curves, connecting the rank to statistical behavior of the a_p .

Thank You!