Accelerating Monte Carlo Simulation Path Generation using Antithetic Variates

Antithetic Variates is a very powerful and classic Monte Carlo optimization technique.

Its core idea is: **Rather than trying to *eliminate* randomness, it *utilizes* the symmetry of randomness to "cancel" it out.**

1. Concept

Antithetic Variates is a Monte Carlo variance reduction technique. Its core idea is: **Instead of generating N completely independent simulation paths, $N/2$ pairs of paths are generated. The two paths in each pair are "antithetic," meaning they are driven by mutually opposite random numbers, thereby creating a statistical negative correlation.**

By averaging the (payoff) results of these two negatively correlated paths, a paired estimator with a smaller variance is obtained. Ultimately, the variance of the entire simulation will be significantly lower than that of a standard Monte Carlo simulation using N independent paths.

2. Principle: Why Can Negative Correlation Optimize the Simulation? (The Principle)

Let us elaborate on the principle mathematically.

Assume our goal is to estimate the expected value $\theta = E[X]$ of some random variable X. (In finance, X is typically the discounted payoff of a derivative, and θ is the derivative's price.)

A. Standard Monte Carlo (Standard MC)

We generate N independent samples $X_1, X_2, \dots, X_N$, all identically distributed as X. The estimator is: $$ \hat{\theta}{MC} = \frac{1}{N} \sum{i=1}^N X_i $$ Its variance is: $$ Var(\hat{\theta}{MC}) = Var\left(\frac{1}{N} \sum{i=1}^N X_i \right) = \frac{1}{N^2} \sum_{i=1}^N Var(X_i) = \frac{1}{N^2} \cdot N \cdot Var(X) = \frac{Var(X)}{N} $$ (This utilizes the independence between X_i, where the covariance is 0).

B. Antithetic Variates (AV)

Our goal is to construct a new estimator with a variance less than $\frac{Var(X)}{N}$.

We do not generate N independent samples, but rather $N/2$ pairs of samples (Y_1, Y_2). These two are **generated antithetically** ✅ = \theta$.

We construct a new paired variable: $$ Y^* = \frac{Y_1 + Y_2}{2} $$ The expected value of this new variable is clearly correct: $$ E[Y^] = E\left[\frac{Y_1 + Y_2}{2} \right] = \frac{E[Y_1] + E[Y_2]}{2} = \frac{\theta + \theta}{2} = \theta $$ Therefore, $Y^$ is an unbiased estimator of θ.

Our total estimator is the average of $N/2$ such $Y^$ variables (using a total of N paths): $$ \hat{\theta}{AV} = \frac{1}{N/2} \sum{i=1}^{N/2} Y_i^ $$ Its variance is: $$ Var(\hat{\theta}_{AV}) = \frac{Var(Y^*)}{N/2} $$

C. Proof of Variance Reduction

Now, let us analyze $Var(Y^*)$: $$ Var(Y^*) = Var\left(\frac{Y_1 + Y_2}{2} \right) = \frac{1}{4} Var(Y_1 + Y_2) $$ Using the variance and covariance property $Var(A+B) = Var(A) + Var(B) + 2Cov(A, B)$: $$ Var(Y^*) = \frac{1}{4} [Var(Y_1) + Var(Y_2) + 2Cov(Y_1, Y_2)] $$ Because Y_1 and Y_2 have the same distribution as X, $Var(Y_1) = Var(Y_2) = Var(X)$. $$ Var(Y^*) = \frac{1}{4} [2Var(X) + 2Cov(Y_1, Y_2)] = \frac{1}{2} [Var(X) + Cov(Y_1, Y_2)] $$ Substituting this back into $Var(\hat{\theta}_{AV})$: $$ Var(\hat{\theta}_{AV}) = \frac{Var(Y^*)}{N/2} = \frac{\frac{1}{2} [Var(X) + Cov(Y_1, Y_2)]}{N/2} = \frac{Var(X) + Cov(Y_1, Y_2)}{N} $$

D. Comparing the Variance

Now we compare the variance of Standard MC and AV:

- **Standard MC:** $Var(\hat{\theta}_{MC}) = \frac{Var(X)}{N}$
- **Antithetic Variates:** $Var(\hat{\theta}_{AV}) = \frac{Var(X)}{N} + \frac{Cov(Y_1, Y_2)}{N}$

The key to the optimization is: If $Cov(Y_1, Y_2) < 0$ (i.e., Y_1 and Y_2 are negatively correlated), then $Var(\hat{\theta}_{AV})$ will be less than $Var(\hat{\theta}_{MC})$. **The more negative the covariance, the greater the variance reduction effect.**

3. How it Optimizes: Application in Path Generation

The goal of the antithetic variates method is to design a generation method for (Y_1, Y_2) such that they are naturally negatively correlated.

In Monte Carlo path generation (e.g., simulating stock prices), the Payoff Y is typically a function of standard normal random numbers Z, i.e., $Y = f(Z)$. (If the path has multiple steps, $Y = f(Z_1, Z_2, \dots, Z_M)$).

We utilize the **symmetry of the standard normal distribution** to construct the antithetic variables.

A. Basic Method: Inverting Random Numbers

1. Instead of generating $Z \sim N(0, 1)$, we first generate $U \sim U[0, 1]$ (a uniformly distributed random number).
2. Using the **Inverse Transform Method**, we get the normal random number: $Z = \Phi^{-1}(U)$, where Φ^{-1} is the inverse of the standard normal cumulative distribution function (CDF).
3. **Path 1 (Y1):** Use U to generate $Z_1 = \Phi^{-1}(U)$, and use Z_1 to drive the simulation, yielding payoff $Y_1 = f(Z_1)$.
4. **Path 2 (Y2):** Use $1-U$ to generate Z_2.
   - $Z_2 = \Phi^{-1}(1 - U)$
   - **Key Insight:** Due to the symmetry of the standard normal distribution about 0, we have $\Phi^{-1}(1-U) = -\Phi^{-1}(U)$.
   - Therefore, $Z_2 = -Z_1$.
   - We use Z_2 (i.e., $-Z_1$) to drive the simulation, yielding payoff $Y_2 = f(Z_2) = f(-Z_1)$.

B. Condition for Optimization: Monotonic Function

How do we guarantee that $Cov(Y_1, Y_2) < 0$? That is, $Cov(f(Z_1), f(-Z_1)) < 0$?

The answer is: **If $f(z)$ is a Monotonic Function, this condition will be met.**

- **Intuitive Understanding:**

- Assume $f(z)$ is **monotonically increasing** (e.g., a European call option).
- If Z_1 is a large positive number (e.g., 2.0), then $Y_1 = f(2.0)$ will be large.
- Its antithetic variable $Z_2 = -2.0$ is a small negative number, so $Y_2 = f(-2.0)$ will be small.
- Conversely, if Z_1 is small (-2.0), then Y_1 is small; Z_2 is large (2.0), then Y_2 is large.
- Y_1 and Y_2 always tend to move in opposite directions, which creates **negative correlation**.

C. Application in Multi-Step Path Generation

When simulating a complete asset path, such as Geometric Brownian Motion (GBM), we may need M random numbers to simulate M time steps. $$ S_{t_i} = S_{t_{i-1}} \exp\left((r - \frac{1}{2}\sigma^2)\Delta t + \sigma \sqrt{\Delta t} Z_i \right), \quad i=1, \dots, M $$ where $Z_i \sim N(0, 1)$.

The optimization steps are as follows:

1. **Generate $N/2$ random vectors:** Generate $N/2$ M-dimensional standard normal random vectors $\mathbf{Z}^{(j)} = (Z_1^{(j)}, Z_2^{(j)}, \dots, Z_M^{(j)})$, where $j = 1, \dots, N/2$.

2. **Generate two paths for each j:**

   - **Path A:** Use the random vector $\mathbf{Z}^{(j)}$ to drive the GBM, generating path $\mathbf{S}_A^{(j)}$, and calculate its payoff $Y_A^{(j)}$.
   - **Path B (Antithetic Path):** Use the **inverted** random vector $-\mathbf{Z}^{(j)} = (-Z_1^{(j)}, -Z_2^{(j)}, \dots, -Z_M^{(j)})$ to drive the GBM, generating path $\mathbf{S}_B^{(j)}$, and calculate its payoff $Y_B^{(j)}$.

3. **Calculate the paired estimator:** For each pair j, calculate $Y^{*(j)} = \frac{Y_A^{(j)} + Y_B^{(j)}}{2}$.

4. **Calculate the final price:** The final estimated price is $\hat{\theta}_{AV} = \frac{1}{N/2} \sum_{j=1}^{N/2} Y^{*(j)}$.

For many common derivatives (such as European options, Asian options), their final payoff, as a function of the underlying random drivers $(Z_1, \dots, Z_M)$, is largely monotonic. Therefore, the payoffs Y_A and Y_B from the two paths generated using $\mathbf{Z}$ and $-\mathbf{Z}$ are almost always negatively correlated, thereby achieving significant variance reduction.

Summary

Antithetic Variates systematically **introduces negative correlation** between pairs of simulated paths by **exploiting the symmetry of the random number generator (e.g., U and $1-U$, or Z and $-Z$)**.

The mathematical principle relies on $Var(\frac{Y_1+Y_2}{2}) = \frac{Var(Y) + Cov(Y_1, Y_2)}{2}$. When $Cov(Y_1, Y_2) < 0$, the variance of this paired estimator, $Var(Y^*)$, will be less than the variance of a single path, $Var(Y)$.

This allows the Monte Carlo simulation to converge faster, yielding a more precise price estimate with a smaller variance for the same total number of N paths.