Problem 1 Let λ_k be $n+1$ distinct real numbers. Let t_j be $n+1$ distinct real numbers.

(a) Show that

$$a(t) = \sum_{k=0}^{n} a_k e^{\lambda_k t}$$

can vanish for all real t only if $a_0 = a_1 = \cdots = a_n = 0$.

(b) Show that for the exponential interpolation problem

$$a(t_j) = \sum_{k=0}^{n} a_k e^{\lambda_k t_j} = f_j \qquad 0 \leq j \leq n$$

there exists a unique solution $a(t)$ for any data values f_j.

(c) Interpolate the function

$$f(t) = \frac{1}{1+t^6}$$

by $n+1$ exponentials with $\lambda_k = -k/n$, $k = 0$ through n, at $n+1$ equidistant points $t_j = 5j/n$ for $j = 0$ through n on the interval $[0, 5]$ and tabulate the error for $n = 3, 5, 9, 17, 33$.

Problem 2 For equidistant points $x_j = j$, $0 \le j \le n$, n even, let

$$\omega(x) = (x - x_0)(x - x_1) \dots (x - x_n)$$

Use Stirling's formula to estimate the ratio $\omega(1/2)/\omega(n/2 + 1/2)$ for large n. Define and explain the Runge phenomenon.

Problem 3 Interpolate the function

$$f(x) = \frac{1}{1 + x^6}$$

on the interval $[0, 5]$ at

(a) $n + 1$ equidistant points $x_k = 5k/n$, and

(b) $n + 1$ Chebyshev points $x_k = (5 + 5\cos((2k + 1)\pi/(2n + 2)))/2$.

Use $n = 3, 5, 9, 17, 33$ and for each case

(1) tabulate the maximum error over 1000 random points $y_k \in [0, 5]$, and

(2) plot $\ln(1 + |\omega(x)|) = \ln(1 + |(x - x_0)(x - x_1) \ldots (x - x_n)|)$.

Problem 4 (See BBF 3.4.11) (a) Show that $H_{2n+1}(x)$ is the unique polynomial p agreeing with f and f' at $x_0, \ldots, x_n$. (Hint: Find a square system of linear equations that determine the coefficients of p in some basis for degree-$(2n+1)$ polynomials. Show that a (possibly non-unique) solution always exists. Use linear algebra.)

(b) Derive the error term in Theorem 3.9. (Hint: Use the same method as in the Lagrange error derivation, Theorem 3.3, defining

$$g(t) = f(t) - H_{2n+1}(t) - \frac{(t-x_0)^2 \cdots (t-x_n)^2}{(x-x_0)^2 \cdots (x-x_n)^2}(f(x) - H_{2n+1}(x))$$

and using the fact that $g'(t)$ has $2n+2$ distinct zeroes in $[a, b]$.)

(c) Separate the error into three factors and explain why each factor is inevitable.

Problem 5 Let p be a positive integer and

$$f(x) = 2^x$$

for $0 \le x \le 2$.

(a) Find a formula for the pth derivative $f^{(p)}(x)$.

(b) For $p = 0, 1, 2$ find a formula for the polynomial H_p of degree $2p + 1$ such that

$$H_p^{(k)}(x_j) = f^{(k)}(x_j)$$

for $0 \le k \le p$, $0 \le j \le 1$, $x_0 = 0$, $x_1 = 2$.

(c) For general p prove that

$$|f(x) - H_p(x)| \le \left(\frac{1}{p+1}\right)^{2p+2}$$

for $0 \le x \le 2$.

(d) Show that one step of Newton's method for solving

$$g(y) = x \ln 2 - \ln y = 0$$

starting from $y_0 = H_4(x)$ gives $y_1 = f(x) = 2^x$ to almost double precision accuracy for $0 \le x \le 2$.

Problem 6 Let $n \geq m \geq 0$, $a \in R$, and $n+1$ distinct interpolation points x_0, $x_1, \ldots, x_n$. Let $\delta^m_{nk}(a)$ be the differentiation coefficients

$$\delta^m_{nk}(a) = \left(\frac{d}{dx}\right)^m L^n_k(x)|_{x=a}$$

such that the degree-n polynomial $p(x)$ which interpolates $n+1$ values f_j at $n+1$ points x_j satisfies

$$p^{(m)}(a) = \sum_{k=0}^{n} \delta^m_{nk}(a) f_k.$$

(a) Derive the recurrence relation

$$\delta^m_{nk}(a) = \frac{m}{x_k - x_n}\delta^{m-1}_{n-1,k}(a) + \frac{a - x_n}{x_k - x_n}\delta^m_{n-1,k}(a)$$

for $0 \leq k \leq n-1$.

(b) Write a Matlab code which evaluates $\delta^m_{nk}(a)$ for $0 \leq m \leq M$, given n and the points a and x_j.

(c) Validate your coefficients $\delta^m_{nk}(a)$ by verifying $O(h^{n-m})$ accuracy for the mth derivative of $f(x) = \mathrm{e}^x$ evaluated at $n+1$ equidistant points $x_j = jh$.

(d) Fix interpolation points x_j and form an $(n+1) \times (n+1)$ matrix A_m of differentiation coefficients with

$$(A_m)_{ij} = \delta^m_{nj}(x_i).$$

Is $A_m = A_1^m$? Why or why not?