Problem 1 Let λ_k be $n+1$ distinct real numbers. Let t_j be $n+1$ distinct real numbers.

(a) Show that

$$a(t) = \sum_{k=0}^{n} a_k \mathrm{e}^{\lambda_k t}$$

can vanish for all real t only if $a_0 = a_1 = \cdots = a_n = 0$.

(b) Show that for the exponential interpolation problem

$$a(t_j) = \sum_{k=0}^{n} a_k \mathrm{e}^{\lambda_k t_j} = f_j \qquad 0 \le j \le n$$

there exists a unique solution $a(t)$ for any data values f_j.

(c) Interpolate the function

$$f(t) = \frac{1}{1+t^6}$$

by $n+1$ exponentials with $\lambda_k = -k/n$, $k = 0$ through n, at $n+1$ equidistant points $t_j = 5j/n$ for $j = 0$ through n on the interval $[0, 5]$ by and tabulate the error for $n = 3, 5, 9, 17, 33$.

Solution 1 (3 parts x 5 pts = 15 pts)

Let $\lambda_0, \ldots, \lambda_n$ denote $n+1$ unique real numbers. Given numbers $a_0, \ldots, a_n$, define the function

$$a(t) = \sum_{k=0}^{n} a_k\, e^{\lambda_k t}.$$

1. The statement that $a(t) = 0$ for all $t \in \mathbb{R}$ implies $a_0 = \cdots = a_n = 0$ is the same thing as saying that the exponential functions $\{e^{\lambda_0 t}, \ldots, e^{\lambda_k t}\}$ are linearly independent as functions on $\mathbb{R}$. Four different solutions are given.

   (a) If $a(t)$ vanishes identically on $\mathbb{R}$, then so do all the derivatives of $a(t)$. The jth derivative of $a(t)$ satisfies

   $$a^{(j)}(t) = \sum_{k=0}^{n} \lambda_k^j\, a_k\, e^{\lambda_k t} = 0 \text{ for all } t.$$

   Plugging in $t = 0$ and keeping the first $n+1$ derivatives, this gives a system of $n+1$ equations for $a_0, \ldots, a_n$,

   $$\begin{aligned} a_0 + \ldots + a_n &= 0 \\ \lambda_0 a_0 + \ldots + \lambda_n a_n &= 0 \\ &\vdots \\ \lambda_0^n a_0 + \ldots + \lambda_n^n a_n &= 0. \end{aligned} \tag{1}$$

The matrix

$$V^T = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ \lambda_0 & \lambda_1 & \cdots & \lambda_{n-1} & \lambda_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_0^n & \lambda_1^n & \cdots & \lambda_{n-1}^n & \lambda_n^n \end{pmatrix}.$$

of this linear system is the *transpose* of the matrix V which determines the coefficients of a polynomial interpolant $p(\lambda) = p_0 + p_1\lambda + \cdots + p_n\lambda^n$ at distinct points $\lambda_0, \ldots, \lambda_n$. By the uniqueness of polynomial interpolation, V is therefore nonsingular. Since $\det(V) = \det(V^T)$, V^T is also nonsingular. Hence the system (1) only has the trivial solution $a_0 = \ldots = a_n = 0$.

(b) If $a(t) = 0$ for all t, then $a(0) = a(1) = \ldots = a(n) = 0$. The matrix associated with these $n+1$ equations is

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ e^{\lambda_0} & e^{\lambda_1} & \cdots & e^{\lambda_{n-1}} & e^{\lambda_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (e^{\lambda_0})^n & (e^{\lambda_1})^n & \cdots & (e^{\lambda_{n-1}})^n & (e^{\lambda_n})^n \end{pmatrix},$$

This is also a Vandermonde matrix, which is nonsingular since $e^{\lambda_0}, \ldots, e^{\lambda_n}$ are distinct provided $\lambda_0, \ldots, \lambda_n$ are distinct.

(c) Suppose that $a(t) = 0$. If a_j were nonzero for some $j \in \{0, \ldots, n\}$, then after dividing by a_j,

$$e^{\lambda_j t} = \sum_{k \neq j} b_k e^{\lambda_k t}, \tag{2}$$

where $b_k = -a_k/a_j$. The right hand side of (2) is annihilated by the differential operator

$$P_j = (\partial_t - \lambda_0) \cdots \widehat{(\partial_t - \lambda_j)} \cdots (\partial_t - \lambda_n).$$

On the other hand P_j applied to the left hand side of (2) is

$$P_j(e^{\lambda_k t}) = (\lambda_j - \lambda_0) \cdots \widehat{(\lambda_j - \lambda_j)} \cdots (\lambda_j - \lambda_n),$$

which does not vanish for distinct $\lambda_0, \ldots, \lambda_n$. This contradicts the assumption that $a_j \neq 0$.

(d) After relabeling, it may be assumed that

$$\lambda_0 < \lambda_1 < \ldots < \lambda_{n-1} < \lambda_n.$$

Dividing $a(t)$ by the fastest-growing term $e^{\lambda_n t}$ gives

$$a_n = -\sum_{k=0}^{n-1} a_k e^{(\lambda_k - \lambda_n)t}. \tag{3}$$

Letting $t \to \infty$ in (3) shows that the right hand side tends to zero, hence $a_n = 0$. Now apply the same argument to a_{n-1} and so on to show that each a_k vanishes.

2. According to the previous part, the functions $\{e^{\lambda_0 t}, \dots, e^{\lambda_n t}\}$ span an $(n+1)$-dimensional subspace of functions on the real line. Given $n+1$ distinct points $t_0, \dots, t_n$ on the line, define the evaluation map

$$E : \operatorname{span}\{e^{\lambda_0 t}, \dots, e^{\lambda_n t}\} \to \mathbb{R}^{n+1}$$

by

$$E(a(t)) = (a(t_0), \dots, a(t_n)).$$

Solving the exponential interpolation problem uniquely is equivalent to showing that the linear map E is an isomorphism.

Since the domain and target space of E have the same dimension, it suffices to show that E is one-to-one. Thus we want to show that if

$$a(t_0) = a(t_1) = \ldots = a(t_n) = 0 \tag{4}$$

for distinct points $t_0, \dots, t_n$, then $a(t) = 0$ for all t.
Lemma 1 Let $a(t) = \sum_{k=0}^{n} a_k e^{\lambda_k t}$, where $\lambda_0, \dots, \lambda_n$ are $n+1$ distinct real numbers. If $a(t)$ is not identically zero, then $a(t)$ has at most n real roots.

Proof. The proof is by induction. The base case $n = 0$ is clear since $a(t) = e^{\lambda_0 t}$ has no roots. Now suppose the result is true for $n-1$. To prove it for n, assume that $a(t)$ has at least $n+1$ roots; we want to show that $a(t) = 0$ for all t in that case. Differentiate

$$\left(e^{-\lambda_n t} \cdot a(t)\right)' = \sum_{k=0}^{n-1} a_k(\lambda_k - \lambda_n) e^{(\lambda_k - \lambda_n)t}.$$

If $b(t)$ denotes this derivative, then $b(t)$ is an exponential sum with n unique exponents $\lambda_0 - \lambda_n, \dots, \lambda_{n-1} - \lambda_n$. Furthermore, $b(t)$ has at least n roots: the function $e^{-\lambda_n t} a(t)$ has the same roots as $a(t)$ (so at least $n+1$ roots), and by the mean value theorem $b(t)$ has a root between any two roots of $a(t)$ (so at least n roots).

By the inductive hypothesis, $b(t)$ is identically zero. By part (a), all the coefficients of $b(t)$ must vanish, which implies $a_0 = \ldots = a_{n-1} = 0$. Therefore $a(t) = a_n e^{\lambda_n t}$, and since $a(t)$ has at least $n+1$ roots, this also implies $a_n = 0$. Therefore $a_0 = \ldots = a_n = 0$ as desired. ■

The proof that E is one-to-one is now finished, since if (4) is true, then $a(t)$ has $n+1$ roots, hence must be identically zero by the lemma.

3. Let $\lambda_k = -k/n$, where $k = 0, 1, \dots, n$. If $t_0, \dots, t_n$ are the nodes, define $\mu_k = e^{-t_k/n}$. Then the exponential interpolation problem is equivalent to

the system of equations

$$a_0 + a_1\,\mu_k + \ldots + a_n\,\mu_k^n = f_k$$

for $k = 0, 1, \ldots, n$. In terms of the variable $\mu = e^{-t/n}$ this is just polynomial interpolation. Using the barycentric formula for polynomial interpolation, define

$$w_k = \frac{1}{\prod_{k \neq j}(e^{-t_k/n} - e^{-t_j/n})}.$$

Then set

$$a(t) = \left(\sum_{k=0}^{n} \frac{f_k w_k}{e^{-t/n} - e^{-t_k/n}}\right) \Big/ \left(\sum_{k=0}^{n} \frac{w_k}{e^{-t/n} - e^{-t_k/n}}\right).$$

This is the desired interpolation formula. Choosing 1000 random points in the interval $[0, 5]$ and computing the maximum errors gave the following:

$n = 3$	$n = 5$	$n = 9$	$n = 17$	$n = 33$
0.1049442668	0.0220255979	0.0009452483022	$1.28252704 \times 10^{-05}$	$8.33593909 \times 10^{-10}$

Problem 2 For equidistant points $x_j = j$, $0 \le j \le n$, n even, let

$$\omega(x) = (x - x_0)(x - x_1) \ldots (x - x_n)$$

Use Stirling's formula to estimate the ratio $\omega(1/2)/\omega(n/2 + 1/2)$ for large n. Define and explain the Runge phenomenon.

Solution 2 (10 pts)

Let $\omega_n = t(t-1)\cdots(t-n)$, where n is an even integer. We are interested in the ratio

$$\frac{|\omega_n\left((n+1)/2\right)|}{|\omega_n\left(1/2\right)|}. \tag{5}$$

First compute

$$|\omega_n(1/2)| = \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\cdots\left(\frac{2n-3}{2}\right)\left(\frac{2n-1}{2}\right).$$

Multiply and divide this quantity by $(n-1)!$, written as

$$(n-1)! = \left(\frac{2}{2}\right)\left(\frac{4}{2}\right)\cdots\left(\frac{2n-4}{2}\right)\left(\frac{2n-2}{2}\right);$$

After some algebra this gives

$$|\omega_n(1/2)| = \frac{1}{2^{2n+1}}\frac{(2n)!}{n!}.$$

Next, calculate $|\omega_n\left((n+1)/2\right)|$ by noting that the product of terms for $j \in \{1, \ldots, n/2\}$ is the same in absolute value as the product of the terms for $j \in \{n/2+1, \ldots, n\}$ in absolute value. Therefore

$$|\omega_n((n+1)/2)| = \frac{n+1}{2}\left(\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\cdots\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\right)^2.$$

As before, multiply and divide this by $((n/2)!)^2$ to see that

$$|\omega_n((n+1)/2)| = \frac{1}{2^{2n+1}}\frac{(n+1)!\,n!}{((n/2)!)^2}.$$

Therefore the ratio is given by

$$\frac{|\omega_n(1/2)|}{|\omega_n\left((n+1)/2\right)|} = \frac{(2n)!}{(n+1)\,n!}\left(\frac{(n/2)!}{n!}\right)^2.$$

It remains to plug in Stirling's approximation $k! \sim \sqrt{2\pi k}\,(k/e)^k$, valid for large k. This gives

$$\frac{|\omega_n(1/2)|}{|\omega_n\left((n+1)/2\right)|} \sim \frac{2^n}{n\sqrt{2}} = \frac{\sqrt{2}\cdot 2^{n-1}}{n}$$

for n large.

Problem 3 Interpolate the function

$$f(x) = \frac{1}{1+x^6}$$

on the interval $[0, 5]$ at

(a) $n+1$ equidistant points $x_k = 5k/n$, and

(b) $n+1$ Chebyshev points $x_k = (5 + 5\cos((2k+1)\pi/(2n+2)))/2$.

Use $n = 3, 5, 9, 17, 33$ and for each case

(1) tabulate the maximum error over 1000 random points $y_k \in [0, 5]$, and

(2) plot $\ln(1 + |\omega(x)|) = \ln(1 + |(x - x_0)(x - x_1)\ldots(x - x_n)|)$.

Solution 3 (2 parts x 10 pts = 20 pts)

Say you want to approximate a function f on an interval $[a, b]$ using n equispaced nodes

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b,$$

where n is even. Then the difference between $f(x)$ and the interpolating polynomial $P_n(x)$ satisfies

$$f(x) - P_n(x) = (x - x_0)\cdots(x - x_n) \cdot \frac{f^{(n+1)}(\xi)}{(n+1)!},$$

where ξ is a point between x and the nearest node. If $h > 0$ is the spacing between consecutive nodes, then this can also be written as

$$h^{n+1}\,\omega_n(t)\,\frac{f^{(n+1)}(\xi)}{(n+1)!},$$

where t is related to x by $x = x_0 + th$. If $f^{(n+1)}$ is slowly varying, then the ratio (5) indicates that the error at $X = x_0 + h/2$ is roughly $(\sqrt{2}\cdot 2^{n-1})/n$ times larger than the corresponding error at $X' = x_0 + h(n+1)/2$. Note that X is near the edge of the interval, while X' is near the center of the interval.

This is at least an indication that the error near the boundary of the interval grows exponentially compared to the error near the center of the interval as more nodes are added.

Below is a table of the maximum errors in each case.

n	even	cheb	geom
3	0.4665	0.3799	0.3934
5	0.1796	0.2093	0.5639
9	0.1534	0.0470	0.5783
17	0.9200	0.0112	$1.8299 \times 10^{+13}$
33	8.1001	1.8713×10^{-04}	$2.4484 \times 10^{+44}$

In the case of even spacing, $\omega(x)$ was largest near the endpoints and smallest in the middle. For Chebyshev points, it was constrained everywhere. For geometric spacing, it was smallest near the point 10/3 (3.3333) and exceptionally large near zero.

Problem 4 (See BBF 3.4.11) (a) Show that $H_{2n+1}(x)$ is the unique polynomial p agreeing with f and f' at $x_0, \ldots, x_n$. (Hint: Find a square system of linear equations that determine the coefficients of p in some basis for degree-$(2n+1)$ polynomials. Show that a (possibly non-unique) solution always exists. Use linear algebra.)

(b) Derive the error term in Theorem 3.9. (Hint: Use the same method as in the Lagrange error derivation, Theorem 3.3, defining

$$g(t) = f(t) - H_{2n+1}(t) - \frac{(t-x_0)^2 \cdots (t-x_n)^2}{(x-x_0)^2 \cdots (x-x_n)^2}(f(x) - H_{2n+1}(x))$$

and using the fact that $g'(t)$ has $2n+2$ distinct zeroes in $[a, b]$.)

(c) Separate the error into three factors and explain why each factor is inevitable.

Solution 4 (3 parts x 5 pts = 15 pts)

1. If $P(x) = a_{2n+1}x^{2n+1} + \cdots + a_1 x + a_0$, setting $P(x_j) = f(x_j)$ and $P'(x_j) = f'(x_j)$ leads to the $(2n+2) \times (2n+2)$ square linear system

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 & \ldots & x_0^{2n+1} \\ 0 & 1 & 2x_0 & 3x_0^2 & \ldots & (2n+1)x_0^{2n} \\ 1 & x_1 & x_1^2 & x_1^3 & \ldots & x_1^{2n+1} \\ & & & & & \\ & & & & & \\ 0 & 1 & 2x_n & 3x_n^2 & \ldots & (2n+1)x_n^{2n} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ a_{2n+1} \end{bmatrix} = \begin{pmatrix} f(x_0) \\ f'(x_0) \\ f(x_1) \\ \vdots \\ f'(x_n) \end{pmatrix}.$$

   We know by using the Newton basis for Hermite interpolation that a solution exists for every right-hand side. Then since the matrix is square it must be invertible, and therefore the solution is unique.

   Alternate solution: Assume that $P(x)$ and $H_{2n+1}(x)$ both agree with f and f' at $x_0, \ldots, x_n$, and consider the polynomial $D = H_{2n+1} - P$. Since D is a degree $2n+1$ polynomial such that $D(x_i) = D'(x_i) = 0$ for $0 \leq i \leq n$, the polynomial D has $2n+2$ zeros and must therefore be the zero polynomial. Since $D = 0$, we conclude that $H_{2n+1}(x) = P(x)$ and therefore that the interpolating polynomial is unique. Since the system of linear equations is square, the matrix is invertible and so a solution always exists.

2. As for the error term, define the polynomial $g(t)$ by

$$g(t) = f(t) - H_{2n+1}(t) - (f(x) - H_{2n+1}(x)) \prod_{i=0}^{n} \frac{(t-x_i)^2}{(x-x_i)^2}.$$

   Then $g(x_i) = g'(x_i) = 0$ for $0 \leq i \leq n$, and we can additionally check that $g(x) = f(x) - H_{2n+1}(x) - [f(x) - H_{2n+1}(x)] = 0$. Thus g has at least

$2n + 3$ zeros in the interval $[a, b]$, so by the Generalized Rolle's Theorem there exists some $\xi \in [a, b]$ such that

$$0 = g^{(2n+2)}(\xi) = f^{(2n+2)}(\xi) - \frac{(2n + 2)!}{\prod_{i=0}^{n}(x - x_i)^2}(f(x) - H_{2n+1}(x)).$$

Rearranging gives

$$f(x) = H_{2n+1}(x) + \frac{\prod_{i=0}^{n}(x - x_i)^2}{(2n + 2)!} f^{(2n+2)}(\xi),$$

as desired.

Problem 5 Let p be a positive integer and

$$f(x) = 2^x$$

for $0 \le x \le 2$.

(a) Find a formula for the pth derivative $f^{(p)}(x)$.

(b) For $p = 0, 1, 2$ find a formula for the polynomial H_p of degree $2p + 1$ such that

$$H_p^{(k)}(x_j) = f^{(k)}(x_j)$$

for $0 \le k \le p$, $0 \le j \le 1$, $x_0 = 0$, $x_1 = 2$.

(c) For general p prove that

$$|f(x) - H_p(x)| \le \left(\frac{1}{p+1}\right)^{2p+2}$$

for $0 \le x \le 2$.

(d) Show that one step of Newton's method for solving

$$g(y) = x \ln 2 - \ln y = 0$$

starting from $y_0 = H_4(x)$ gives $y_1 = f(x) = 2^x$ to almost double precision accuracy for $0 \le x \le 2$.

Solution 5 (4 parts x 5 pts = 20 pts)

(a) $f(x) = 2^x = e^{x \ln 2}$, so $f^{(p)}(x) = (\ln 2)^p e^{x \ln 2} = (\ln 2)^p 2^x$.

(b) Define $k = \ln 2$. Then we get the (vertically-oriented) divided-difference table

0	0	0	2	2	2
1	1	1	4	4	4
k	k	$\frac{3}{2}$	$4k$	$4k$	
k^2	$\frac{3}{4} - \frac{k}{2}$	$2k - \frac{3}{4}$	$4k^2$		
$-\frac{k^2}{2} - \frac{k}{4} + \frac{3}{8}$	$\frac{5}{4}k - \frac{3}{4}$	$2k^2 - k + \frac{3}{8}$			
$\frac{k^2}{4} + \frac{3}{4}k - \frac{9}{16}$	$k^2 - \frac{9}{8}k + \frac{9}{16}$				
$\frac{3}{8}k^2 - \frac{15}{16}k + \frac{9}{16}$					

Note that this table for $p = 2$ also contains inside it the tables for $p = 0$ and

$p = 1$. We then get the polynomials

$$\begin{aligned}
H_0(x) &= 1 + \frac{3}{2}x \\
H_1(x) &= 1 + kx + \left(\frac{3}{4} - \frac{k}{2}\right)x^2 + \left(\frac{5}{4}k - \frac{3}{4}\right)x^2(x-2) \\
H_2(x) &= 1 + kx + k^2x^2 + \left(-\frac{k^2}{2} - \frac{k}{4} + \frac{3}{8}\right)x^3 \\
&\quad + \left(\frac{k^2}{4} + \frac{3}{4}k - \frac{9}{16}\right)x^3(x-2) \\
&\quad + \left(\frac{3}{8}k^2 - \frac{15}{16}k + \frac{9}{16}\right)x^3(x-2)^2.
\end{aligned}$$

(c) For general $p \geq 1$ and for $0 \leq x \leq 2$ we get the error bound

$$\begin{aligned}
|f(x) - H_p(x)| &\leq \frac{|f^{(2p+2)}(\xi)|}{(2p+2)!} \prod_{j=0}^{1} |(x - x_j)|^{p+1} \\
&\leq \frac{4\ln(2)^{2p+2}}{(2p+2)!} \\
&\leq \frac{4\ln(2)^{2p+2}}{\sqrt{2\pi(2p+2)}(2(p+1)/e)^{2p+2}} \\
&= \frac{4}{\sqrt{2\pi(2p+2)}} \left(\frac{1}{p+1}\right)^{2p+2} \left(\frac{e\ln(2)}{2}\right)^{2p+2} \\
&\leq \frac{4}{\sqrt{8\pi}} \left(\frac{e\ln(2)}{2}\right)^2 \left(\frac{1}{p+1}\right)^{2p+2} \\
&\leq \left(\frac{1}{p+1}\right)^{2p+2}.
\end{aligned}$$

The assumption $p \geq 1$ was used in the second-to-last inequality, substituting $(2p+2) \geq 4$. Checking the case $p = 0$ separately, we find that

$$\frac{4\ln(2)^{2p+2}}{(2p+2)!} = 2\ln(2)^2 < 1 = \left(\frac{1}{p+1}\right)^{2p+2},$$

which completes the proof.

(d) If we want to use Newton's method to solve the equation

$$g(y) = x\ln 2 - \ln y = 0,$$

then we get the fixed-function (and corresponding derivatives)

$$\begin{aligned}
h(y) &= y(1 + x\ln 2 - \ln y) \\
h'(y) &= x\ln 2 - \ln y \\
h''(y) &= -1/y.
\end{aligned}$$

Expanding h as a Taylor series around $y_* = 2^x$, we get that

$$h(y_* + \delta) = h(y_*) + \delta h'(y_*) + \delta^2 h''(\xi)/2$$

for some $\xi \in [y_*, y_* + \delta]$ and therefore (since $h'(y_*) = 0$) that

$$\begin{aligned}
|y_1 - y_*| &= \delta^2 |h''(\xi)|/2 \\
&= |y_0 - y_*|^2/2\xi \\
&\le |y_0 - y_*|^2/2
\end{aligned}$$

From the previous part we know that

$$\begin{aligned}
|2^x - y_0| &\le \frac{4}{\sqrt{2\pi(2p+2)}} \left(\frac{1}{p+1}\right)^{2p+2} \left(\frac{e\ln(2)}{2}\right)^{2p+2} && (6) \\
&= \frac{2}{\sqrt{5\pi}} \left(\frac{e\ln(2)}{10}\right)^{10} && (7) \\
&\le 2^{-25} && (8)
\end{aligned}$$

so combining this bound with the result above gives that

$$|2^x - y_1| \le \frac{1}{2}(2^{-25})^2 = 2^{-51}.$$

Problem 6 Let $n \geq m \geq 0$, $a \in R$, and $n+1$ distinct interpolation points x_0, $x_1, \ldots, x_n$. Let $\delta^m_{nk}(a)$ be the differentiation coefficients

$$\delta^m_{nk}(a) = \left(\frac{d}{dx}\right)^m L^n_k(x)|_{x=a}$$

such that the degree-n polynomial $p(x)$ which interpolates $n+1$ values f_j at $n+1$ points x_j satisfies

$$p^{(m)}(a) = \sum_{k=0}^{n} \delta^m_{nk}(a) f_k.$$

(a) Derive the recurrence relation

$$\delta^m_{nk}(a) = \frac{m}{x_k - x_n}\delta^{m-1}_{n-1,k}(a) + \frac{a - x_n}{x_k - x_n}\delta^m_{n-1,k}(a)$$

for $0 \leq k \leq n-1$.

(b) Write a Matlab code which evaluates $\delta^m_{nk}(a)$ for $0 \leq m \leq M$, given n and the points a and x_j.

(c) Validate your coefficients $\delta^m_{nk}(a)$ by verifying $O(h^{n-m})$ accuracy for the mth derivative of $f(x) = \mathrm{e}^x$ evaluated at $n+1$ equidistant points $x_j = jh$.

(d) Fix interpolation points x_j and form an $(n+1) \times (n+1)$ matrix A_m of differentiation coefficients with

$$(A_m)_{ij} = \delta^m_{nj}(x_i).$$

Is $A_m = A_1^m$? Why or why not?

Solution 6 (4 parts x 5 pts = 20 pts)

1. First we prove the recurrence relation efficiently. Observe that the Lagrange basis polynomials satisfy a recursion

$$L^n_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i} = \frac{x - x_n}{x_k - x_n} \prod_{\substack{i=0 \\ i \neq k}}^{n-1} \frac{x - x_i}{x_k - x_i} = \frac{x - x_n}{x_k - x_n} L^{n-1}_k(x)$$

when $0 \leq k < n$. Evaluating at $x = a$ gives the recurrence relation for $m = 0$:

$$\delta^0_{nk} = \frac{a - x_n}{x_k - x_n}\delta^0_{n-1,k}.$$

Differentiating the recursion for Lagrange basis polynomials gives

$$\frac{d}{dx}L^n_k(x) = \frac{1}{x_k - x_n}L^{n-1}_k(x) + \frac{x - x_n}{x_k - x_n}\frac{d}{dx}L^{n-1}_k(x),$$

and evaluating at $x = a$ gives the recurrence relation for $m = 1$:

$$\delta^1_{nk} = \frac{1}{x_k - x_n}\delta^0_{n-1,k} + \frac{a - x_n}{x_k - x_n}\delta^1_{n-1,k}.$$

In order to calculate higher derivatives, we observe that

$$\begin{aligned}\left(\frac{d}{dx}\right)^m (f(x)g(x)) &= \left(\frac{d}{dx}\right)^{m-1}(f'(x)g(x) + f(x)g'(x)) \\ &= \left(\frac{d}{dx}\right)^{m-2}(2f'(x)g'(x) + f(x)g''(x)) \\ &= mf'(x)g^{(m-1)}(x) + f(x)g^{(m)}(x)\end{aligned}$$

whenever f is a linear function so that $f'' = 0$. Thus

$$\left(\frac{d}{dx}\right)^m L^n_k(x) = \frac{m}{x_k - x_n}\left(\frac{d}{dx}\right)^{m-1} L^{n-1}_k(x) + \frac{x - x_n}{x_k - x_n}\left(\frac{d}{dx}\right)^m L^{n-1}_k(x).$$

Evaluating at $x = a$ gives the desired recurrence relation.

We also prove the recurrence relation as in lecture, because it gives a useful consequence. The key observation is that because the coefficients δ^m_{nj} are mth derivatives of $L^n_j(t)$ evaluated at $t = a$, there is a Taylor expansion of the Lagrange basis polynomial:

$$L^n_j(t) = \sum_{m=0}^{n} \frac{\delta^m_{nj}}{m!}(t - a)^m.$$

Since we also know from the explicit formula that

$$L^n_j(t) = \frac{t - t_n}{t_j - t_n} L^{n-1}_j(t)$$

for $0 \le j < n$, we can equate two power series:

$$\sum_{m=0}^{n} \frac{\delta^m_{nj}}{m!}(t - a)^m = \frac{t - a + a - t_n}{t_j - t_n} \sum_{m=0}^{n-1} \frac{\delta^m_{n-1,j}}{m!}(t - a)^m.$$

Shifting the index of summation gives

$$\sum_{m=0}^{n} \frac{\delta^m_{nj}}{m!}(t-a)^m = \frac{1}{t_j - t_n}\sum_{m=1}^{n} \frac{\delta^{m-1}_{n-1,j}}{(m-1)!}(t-a)^m + \frac{a - t_n}{t_j - t_n}\sum_{m=0}^{n-1} \frac{\delta^m_{n-1,j}}{m!}(t-a)^m.$$

When two power series are equal, their coefficients must be equal. Thus equating coefficients gives the result.

A useful consequence of the key observation is the evaluation of integration weights. In approximating the integral

$$\int_a^b f(x)dx$$

a standard approach is to evaluate $f_j = f(x_j)$ at $n+1$ interpolation points x_j and then approximate the integral by integrating the degree-n polynomial interpolant

$$p(x) = \sum_{j=0}^{n} f_j L_j(x).$$

The resulting formula is

$$\int_a^b f(x)dx = \sum_{j=0}^{n} w_j f_j$$

where the weights are given by

$$w_j = \int_a^b L_j(x)dx.$$

These integrals can be complicated to evaluate for high-degree polynomials, but the key observation above yields the simple formula

$$w_j = \int_a^b \sum_{m=0}^{n} \frac{\delta_{nj}^m}{m!}(x-a)^m dx = \sum_{m=0}^{n} \frac{\delta_{nj}^m}{(m+1)!}(b-a)^{m+1} dx$$

once the differentiation weights δ_{nj}^m are evaluated at $x = a$.

2. The above recurrence relation relies on the inequality $k < n$ holding. This can be trivially enforced by re-indexing, so that $k = 0$. It is clear from the recurrence relation that $\delta_{n,k}^m$ depends on $\delta_{n-1,k}^m$ and $\delta_{n-1,k}^{m-1}$, which depend on $\delta_{n-2,k}^m$, $\delta_{n-2,k}^{m-1}$, and $\delta_{n-2,k}^{m-2}$. More generally, each set of the form $\delta_{n,k}^*$ depends on $\delta_{n-1,k}^*$. Thus, the recurrence relation can build all differentiation coefficients from those of the form

$$\delta_{1,k=0}^m(a) = \begin{cases} \frac{a-x_1}{x_0-x_1} & m=0 \\ \frac{1}{x_0-x_1} & m=1 \\ 0 & m>1 \end{cases}$$

and

$$\delta_{n,k}^0(a) = \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{a-x_i}{x_k-x_i}$$

These form the first row and zeroth column of a matrix containing all the differentiation coefficients, which can be constructed moving down the entries, using the recurrence relation. Alternatively, the zeroth column and zeroth row can be used to start, with

$$\delta_{0,0}^m = \begin{cases} 1 & m = 0 \\ 0 & m > 0 \end{cases}$$

With the first row and column predetermined this way, the coefficients can be created by applying the recurrence relation and extracting the last row of the matrix. Below is a MATLAB code that uses this implementation of the recurrence relation:

```
function d = fornberg(M,x0,alphas)
N = length(alphas) - 1;
deltas(1,1,1) = 1; %Delta(M,N,K) for the order of indexing.
c1 = 1;
for n = 1:N
c2 = 1;
for v = 0:(n-1)
c3 = alphas(n+1)-alphas(v+1);
c2 = c2*c3;
if n <= M, deltas(n+1,n,v+1) = 0; end
for m = 0:min(n,M)
if m == 0, D = 0; else , D = m*deltas(m,n,v+1); end
D2 = (alphas(n+1)-x0)*deltas(m+1,n,v+1);
deltas(m+1,n+1,v+1) = (D2-D)/c3;
end
end
for m = 0:min(n,M)
if m == 0, D = 0; else , D = m*deltas(m,n,n); end
D2 = (alphas(n)-x0)*deltas(m+1,n,n);
deltas(m+1,n+1,n+1) = (c1/c2)*(D-D2);
end
c1 = c2;
end
d = deltas(end,end,:);
d = reshape(d,1,[]);
end
```

3. The formula given,

$$p^{(m)}(a) = \sum_{k=0}^{n} \delta_{nk}^m(a) f_k,$$

can be described as a vector-matrix multiplication of the row vector $\vec{v}$ of f evaluated at all the points in the given x vector, which has entries $v_j = f(jh)$, multiplied by the matrix D whose entries are given by

$$D_{k,m} = \delta_{nk}^m(a)$$

Then the vector $\vec{v}D$ gives the approximations of the derivatives of the function f. Setting $h = 0.1$, $n = 4$, $a = 0.5$, and $M = 5$ we get the following errors:

$$\begin{bmatrix} 1.28670303 \times 10^{-05} & 2.96019564 \times 10^{-04} & 4.92727672 \times 10^{-03} & 5.72299634 \times 10^{-02} & 4.25281314 \times 10 \end{bmatrix}$$

Setting $h = 0.0666666667$, i.e. dividing h by 1.5, we get the errors

$$\begin{bmatrix} 5.63954233 \times 10^{-05} & 8.36240212 \times 10^{-04} & 9.79325784 \times 10^{-03} & 8.55452261 \times 10^{-02} & 5.05243784 \times 10 \end{bmatrix}$$

Thus the ratio of each pair of errors is

$$\begin{bmatrix} 4.3829401 & 2.82494914 & 1.98755994 & 1.4947629 & 1.18802253 \end{bmatrix}$$

As the errors should be on the order h^{n-m}, this ratio should ideally be 1.5^{n-m}, or

$$\begin{bmatrix} 5.0625 & 3.375 & 2.25 & 1.5 & 1 \end{bmatrix}$$

The results are close enough to claim order h^{n-m} convergence.

4. In approaching this problem, it helps to see the matrix A_m as an operator that takes the mth derivative of p when it multiplies f. Thus, it is reasonable to see that m applications of A_1 should be equivalent to taking the first derivative m times, giving, finally, the mth derivative. Thus, intuition would suggest that $A_m = A_1^m$. To prove this:

It is clearly true that $A_m = A_1^m$ when $m = 1$, as this is simply $A_1 = A_1^1$, and any matrix is equal to itself to the first power. Assume that $A_b = A_1^b$ for some integer b. Now, setting $m = 1$ in the given interpolation formula,

$$p'(a) = \sum_{k=0}^{n} \delta_{nk}^1(a) f_k$$

This can be applied to the case where $f = \left(\frac{d}{dx}\right)^b L_j^n$, and $a = x_i$. Since f is a polynomial of degree less than n, interpolation is exact, and $p = f$. Then $f_k = p(x_k) = \left(\frac{d}{dx}\right)^b L_j^n(x)|_{x=x_k} = \delta_{nj}^b(x_k)$. Therefore,

$$\left(\frac{d}{dx}\right)^{b+1} L_j^n(x)|_{x=x_i} = p'(x_i) = p'(a) = \sum_{k=0}^{n} \delta_{nk}^1(a) f_k = \sum_{k=0}^{n} \delta_{nk}^1(x_i)\delta_{nj}^b(x_k)$$

Observe that matrix multiplication of A_1 and A_b gives

$$(A_1 A_b)_{ij} = \sum_{k=0}^{n} (A_1)_{ik}(A_b)_{kj} = \sum_{k=0}^{n} \delta_{nk}^1(x_i)\delta_{nj}^b(x_k)$$

Furthermore,

$$\left(\frac{d}{dx}\right)^{b+1} L_j^n(x)|_{x=x_i} = \delta_{nj}^{b+1}(x_i) = (A_{b+1})_{ij}$$

Therefore

$$(A_{b+1})_{ij} = (A_1 A_b)_{ij}$$

Since it is assumed that $A_b = A_1^b$, then

$$(A_{b+1})_{ij} = (A_1 A_b)_{ij} = (A_1 A_1^b)_{ij} = (A_1^{b+1})_{ij}$$

Since every element agrees, $A_{b+1} = A_1^{b+1}$, and, by induction, $A_m = A_1^m$ for every integer m.