

Problem 1 The Fibonacci numbers f_n are defined by

$$f_{n+2} = f_{n+1} + f_n$$

with $f_0 = 0$ and $f_1 = 1$.

(a) Show that

$$\frac{f_{n+1}}{f_n} \rightarrow \varphi = \frac{1 + \sqrt{5}}{2}$$

as $n \rightarrow \infty$.

(b) Determine the rate of convergence of f_{n+1}/f_n to φ .

Problem 2 Consider the fixed point iteration

$$x_{n+1} = \frac{-x_n^2 - c}{2b}, \quad (*)$$

where b and c are fixed real parameters.

(a) If $x_n \rightarrow x$, what does x solve?

(b) Analyze and sketch the region of (b, c) values where $(*)$ converges at a rate $\mathcal{O}(2^{-n})$ or better from an interval of starting values x_0 near x .

Problem 3 Consider the fixed point iteration

$$x_{n+1} = -b - \frac{c}{x_n} = g(x_n) \quad (**)$$

- (a) Show that $|g'(x)| \leq \frac{1}{2}$ whenever $x^2 \geq 2|c|$.
- (b) Show that $g(x)^2 \geq 2|c|$ whenever $x^2 \geq 2|c|$ and $b^2 \geq \frac{9}{2}|c|$.
- (c) Draw the region of (b, c) -space where $(**)$ converges at a rate at least $\mathcal{O}(2^{-n})$ from any starting point x_0 with $x_0^2 \geq 2|c|$.

Problem 4 Fix $a > 0$ and consider the fixed point iteration

$$x_{n+1} = x_n(2 - ax_n). \quad (***)$$

- (a) Show that if $x_n \rightarrow x$ then $x = \frac{1}{a}$ or $x = 0$.
- (b) Find an interval (α, β) containing $\frac{1}{a}$ such that $(***)$ converges to $1/a$ whenever $x_0 \in (\alpha, \beta)$.
- (c) Find the rate of convergence in (b).

Problem 5 (a) Write down Newton's method in the form

$$x_{k+1} = g(x_k)$$

for solving

$$f(x) = x^2 - 2bx + b^2 - d^2 = 0$$

where $b > 0$ and $d > 0$ are parameters.

(b) Show that $|g'(x)| \leq 1/2$ whenever $|x - b| \geq d/\sqrt{2}$.

(c) Show that $|g(x) - b| \geq d/\sqrt{2}$ whenever $|x - b| \geq d/\sqrt{2}$.

(d) Sketch the graph of $f(x)$ with the roots of $f(x) = 0$ and the intervals of x where Newton's method is guaranteed to converge.

Problem 6 Implement Newton's method in a Matlab or Octave program `newton.m` of the form

```
function r = newton(x0, f, p, n)
% x0: initial estimate of the root
% f: function and derivative handle [ y, yp ] = f(x, p)
% p: parameters to pass through to f
% n: number of steps
```

(a) Use `newton.m` to find an approximation to within ϵ to the first positive value of x with $x = 2 \sin x$. Report the number of steps, the final result, and the absolute and relative errors. Characterize the convergence as linear or quadratic by tabulating the number of correct bits at each step of the iteration.

(b) Use `newton.m` as many times as necessary to find all solutions $x > 0$ of the equation

$$f(x) = \frac{1}{x} + \ln x - 2 = 0.$$

Report the number of steps, the final result, and the absolute and relative errors. Characterize the convergence as linear or quadratic by tabulating the number of correct bits at each step of the iteration.

(c) Use `newton.m` to solve the equation

$$f(x) = (x - \epsilon^3)^3 = 0$$

Report the number of steps, the final result, and the absolute and relative errors. Characterize the convergence as linear or quadratic by tabulating the number of correct bits at each step of the iteration. Explain your results.

(d) Use `newton.m` to solve the equation

$$f(x) = \arctan(x - \epsilon^2) = 0$$

for a diverse selection of starting values. Find starting values which lead to convergence, divergence and oscillation. Report the number of steps, the final result, and the absolute and relative errors.