



Computational Science:
Computational Methods in Engineering

Finite-Difference Analysis of Transmission Lines

Outline

- Introduction to Transmission Lines
- High Level Solution Approach
- Formulation
 - Derivation of governing equations
 - Finite-difference approximation of governing equations
- Implementation
 - Building the device on a grid
 - 2x grid technique
 - Solving the matrix equations
 - Calculating the transmission line parameters
- Example – Weird Coax

Introduction to Transmission Lines

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What Are Transmission Lines?

	Homogeneous	Inhomogeneous
Single-Ended	<ul style="list-style-type: none"> • Has TEM mode. • Has TE and TM modes.  <p>coaxial</p>  <p>stripline</p>	<ul style="list-style-type: none"> • Supports only quasi-(TEM, TE, & TM) modes.  <p>microstrip</p>  <p>coplanar</p>
Differential	 <p>buried parallel plate</p>  <p>shielded pair</p>	 <p>coplanar strips</p>  <p>slotline</p>

Transmission lines are essentially high-frequency electrical cables composed of two or more metal wires.

Transmission Line Parameters RLGC

Think transmission lines as being composed of millions of tiny little circuit elements that are distributed along the length of the line.

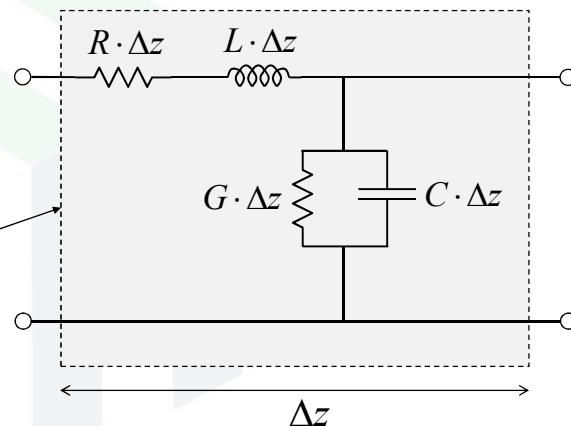
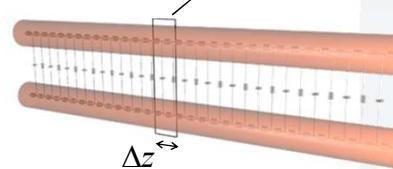


In fact, these circuit element are not discrete, but continuous along the length of the transmission line.

RLGC Circuit Model

This model is accurate when the size Δz of the equivalent circuit is very small compared to the wavelength of the signal on the transmission line.

$$\lim_{\Delta z \rightarrow 0}$$



Distributed TL Parameters

Distributed Circuit Parameters

R (Ω/m)

Resistance per unit length. Arises due to resistivity in the conductors.

L (H/m)

Inductance per unit length. Arises due to stored magnetic energy around the line.

G ($1/\Omega \cdot m$)

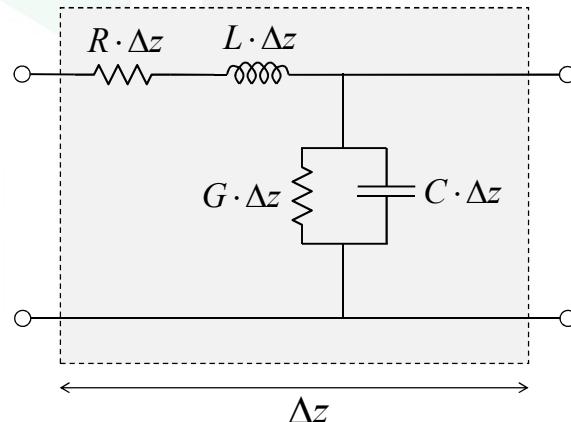
Conductance per unit length. Arises due to conductivity in the dielectric separating the conductors.

$$G = \frac{1}{R}$$

C (F/m)

Capacitance per unit length. Arises due to stored electric energy between the conductors.

There are many possible circuit models for transmission lines, but most produce the same equations after analysis.



Characteristic Impedance, Z_0

The characteristic impedance is the voltage divided the current at any point along the transmission line.

$$Z_0 = \frac{V}{I}$$

If we perform a circuit analysis of the equivalent circuit, we get

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

When we ignore loss (usually accurate to do this), we get

$$Z_0 = \sqrt{\frac{L}{C}}$$

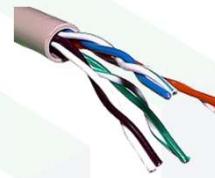
Example RLGC Parameters

RG-59 Coax



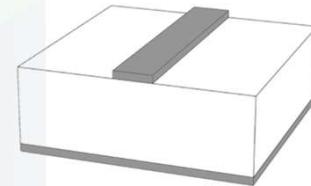
$$\begin{aligned} R &= 36 \text{ m}\Omega/\text{m} \\ L &= 430 \text{ nH/m} \\ G &= 10 \mu\text{S}/\text{m} \\ C &= 69 \text{ pF/m} \\ Z_0 &= 75 \Omega \end{aligned}$$

CAT5 Twisted Pair



$$\begin{aligned} R &= 176 \text{ m}\Omega/\text{m} \\ L &= 490 \text{ nH/m} \\ G &= 2 \mu\text{S}/\text{m} \\ C &= 49 \text{ pF/m} \\ Z_0 &= 100 \Omega \end{aligned}$$

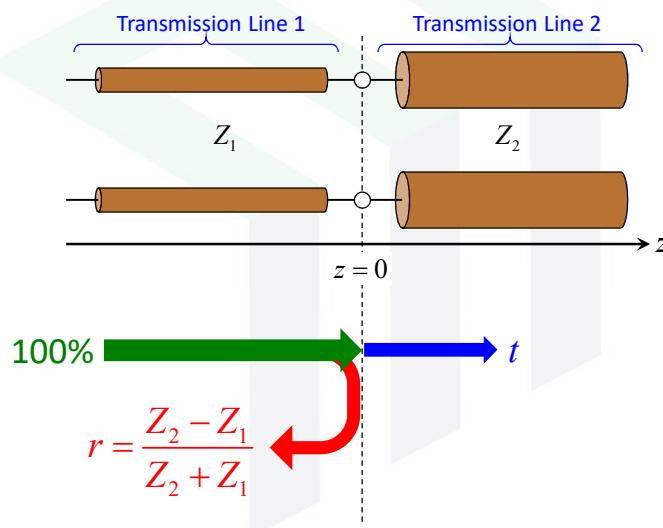
Microstrip



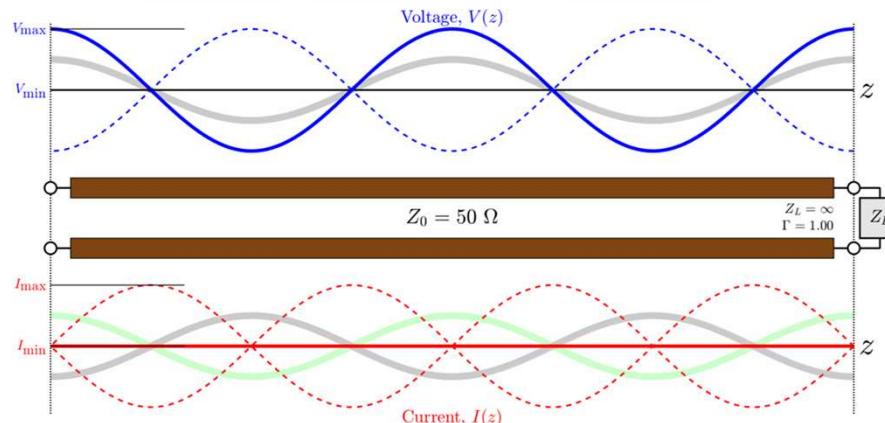
$$\begin{aligned} R &= 150 \text{ m}\Omega/\text{m} \\ L &= 364 \text{ nH/m} \\ G &= 3 \mu\text{S}/\text{m} \\ C &= 107 \text{ pF/m} \\ Z_0 &= 50 \Omega \end{aligned}$$

Surprisingly, almost all transmission lines have parameters very close to these same values.

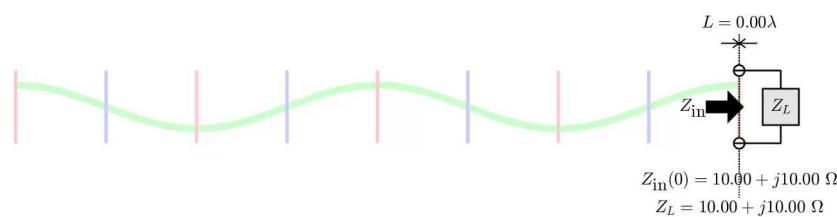
Reflections From Impedance Discontinuities



Voltage Standing Wave Ratio



Impedance Transformation



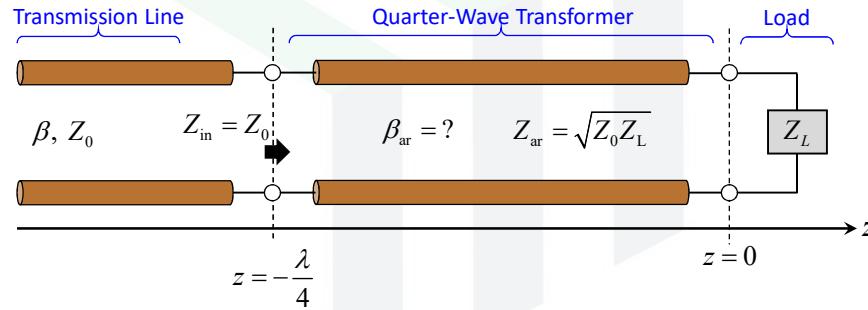
As we back away from the load, the impedance experienced by the source changes!

Short circuits can look like open circuits, open circuits can look like short circuits, inductors can look like capacitors, capacitors can look like inductors,...its crazy!

$$Z_{in}(-\ell) = Z_0 \frac{Z_L + jZ_0 \tan(\beta\ell)}{Z_0 + jZ_L \tan(\beta\ell)}$$

Impedance Matching

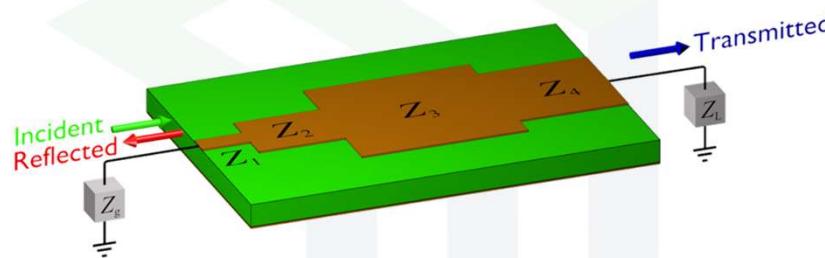
Similar to the anti-reflection layer for waves, we can match a transmission line to a load impedance by inserting a quarter-wave section of a second transmission line.



We must perform an electromagnetic analysis of the transmission line to determine β_{ar} .

$$\ell = \frac{\lambda}{4} = \frac{\pi}{2\beta_{ar}}$$

Multi-Segment Circuits



We can design filters, pulse shaper, and much more.

High Level Solution Approach

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Maxwell's Equations

Maxwell's equations describe classical electromagnetics.

$$\nabla \bullet \vec{D} = \rho_v$$

Gauss' law

$$\nabla \bullet \vec{B} = 0$$

Gauss' law for magnetic fields

$$\nabla \times \vec{E} = -\partial \vec{B} / \partial t$$

Faraday's law

$$\nabla \times \vec{H} = \vec{J} + \partial \vec{D} / \partial t$$

Ampere's circuit law

\vec{E} ≡ electric field intensity (V/m)

\vec{D} ≡ electric flux density (C/m²)

\vec{H} ≡ magnetic field intensity (A/m)

\vec{B} ≡ magnetic flux density (Wb/m²)

ρ_v ≡ volume charge density (C/m³)

\vec{J} ≡ electric current density (A/m²)

The constitutive relations describe how the electromagnetic fields interact with matter.

$$\vec{D} = \epsilon \vec{E}$$

Electric response

$$\vec{B} = \mu \vec{H}$$

Magnetic response

ϵ ≡ permittivity (F/m)

μ ≡ permeability (H/m)

Physical Constants

The permittivity and permeability are usually expressed in terms of the relative parameters.

$$\epsilon = \epsilon_0 \epsilon_r \quad \text{Permittivity}$$

$$\mu = \mu_0 \mu_r \quad \text{Permeability}$$

$$\epsilon_0 = 8.8541878176 \times 10^{-12} \text{ F/m}$$

$$1 \leq \epsilon_r < \infty$$

$$\epsilon \equiv \text{permittivity (F/m)}$$

$$\epsilon_0 \equiv \text{free space permittivity (F/m)}$$

$$\epsilon_r \equiv \text{relative permittivity (no units)}$$

$$\mu_0 = 1.2566370614 \times 10^{-6} \text{ H/m}$$

$$1 \leq \mu_r < \infty$$

$$\mu \equiv \text{permeability (H/m)}$$

$$\mu_0 \equiv \text{free space permeability (H/m)}$$

$$\mu_r \equiv \text{relative permeability (no units)}$$

ϵ_r is commonly called the dielectric constant.

The speed of light in free space can be calculated from the permittivity and permeability as follows.

$$c_0 = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 299,792,458 \text{ m/s}$$

$$c_0 \equiv \text{speed of light in vacuum (m/s)}$$

Electrostatic Approximation

Transmission lines are waveguides. To be rigorous, they should be modeled as such. This can be rather computationally intensive. An alternative is to analyze transmission lines in the electrostatic approximation using Laplace's equation.

The dimensions of a transmission line are typically much smaller than the operating wavelength so the wave nature is less important to consider. Therefore, we can solve Maxwell's equations assuming static fields.

$$\begin{aligned} \frac{\partial}{\partial t} \rightarrow 0 \quad \rightarrow & \nabla \bullet \vec{B} = 0 \\ & \nabla \bullet \vec{D} = \rho_v \\ & \nabla \times \vec{H} = \vec{J} + \cancel{\frac{\partial \vec{D}}{\partial t}} \\ & \nabla \times \vec{E} = \cancel{\frac{\partial \vec{B}}{\partial t}} \end{aligned} \quad \begin{aligned} & \nabla \bullet \vec{B} = 0 \\ & \nabla \bullet \vec{D} = \rho_v \\ & \nabla \times \vec{H} = \vec{J} \\ & \nabla \times \vec{E} = 0 \end{aligned}$$

Electric Potential V

The vector electric field \vec{E} and the scalar electric potential V describe the same physical phenomenon. This allows equations to be solved in terms of a scalar quantity instead of a vector quantity.

In electrostatics, we have $\nabla \times \vec{E} = 0$.

In vector calculus, the curl of a gradient is always zero, $\nabla \times (\nabla V) = 0$.

Based on this, if a vector field does not have any curl, it must be possible to express it as the gradient of a scalar field.

$$\vec{E} = -\nabla V$$

V \equiv electric potential (volts)

The negative sign is incorporated to be consistent with the sign conventions used with charges and fields.

Inhomogeneous Laplace's Equation

Away from charges ρ_v , the divergence condition for the electric field is

$$\nabla \bullet \vec{D} = 0 \quad \text{Eq. (1)}$$

But, we know that $\vec{D} = \epsilon_r \vec{E}$, so Eq. (1) can be written in terms of \vec{E} .

$$\nabla \bullet (\epsilon_r \vec{E}) = 0 \quad \text{Eq. (2)}$$

The electric field \vec{E} is related to the scalar potential V as follows.

$$\vec{E} = -\nabla V \quad \text{Eq. (3)}$$

The inhomogeneous Laplace's equation is derived by substituting Eq. (3) into Eq. (2).

$$\nabla \bullet [\epsilon_r (\nabla V)] = 0$$

Homogeneous Laplace's Equation

When the transmission line is embedded in a homogeneous dielectric, the permittivity is not a function of position and drops out of Laplace's equation.

$$\nabla \bullet [\epsilon_r (\nabla V)] = 0$$

$$\epsilon_r \nabla \bullet [(\nabla V)] = 0$$

$$\nabla \bullet [(\nabla V)] = 0$$

$$\nabla^2 V = 0$$

$$\boxed{\nabla^2 V = 0}$$

Distributed Capacitance C

In the electrostatic approximation, the transmission line is a capacitor. The total energy U stored in a capacitor is the total energy in the fields \vec{D} and \vec{E} :

$$U = \frac{1}{2} \iint_A (\vec{D} \bullet \vec{E}) dA$$

The integral is taken over the entire cross section A of the device. For open devices like microstrips, this is an infinite area. In practice, integration happens over a large enough area to incorporate as much of the electric field as possible.

From circuit theory, the capacitance C is related to the total stored energy U through

$$U = \frac{CV_0^2}{2} \quad V_0 \text{ is the voltage across the capacitor.}$$

If the above equations are set equal, an equation is derived that calculates the distributed capacitance C from the electric field functions \vec{D} and \vec{E} .

$$\boxed{C = \frac{1}{V_0^2} \iint_A (\vec{D} \bullet \vec{E}) dA}$$

Distributed Inductance L

The voltage along the transmission line embedded in a homogeneous material travels at the same velocity as the electric field so we can write

$$v_V = v_E \rightarrow \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu_h \epsilon_h}} \rightarrow LC = \frac{\mu_{r,h} \epsilon_{r,h}}{c_0^2}$$

Solving this equation for L , we get

$$L = \frac{\mu_{r,h} \epsilon_{r,h}}{c_0^2 C}$$

This means that for transmission lines embedded in homogeneous materials, we can calculate the distributed inductance L directly from the distributed capacitance C .

Dielectric materials should not alter the inductance. However if we use the value of C calculated previously, it will. This is incorrect. The solution is to calculate distributed capacitance with air dielectric C_h and then calculate the distributed inductance L from this.

$$L = \frac{\mu_{r,h}}{c_0^2 C_h}$$

For most materials $\mu_{rh} = 1$

Calculating Transmission Line Parameters

The characteristic impedance Z_c is calculated from the distributed inductance L and distributed capacitance C through

$$Z_c = \sqrt{\frac{L}{C}}$$

The velocity v of a signal travelling along the transmission line is

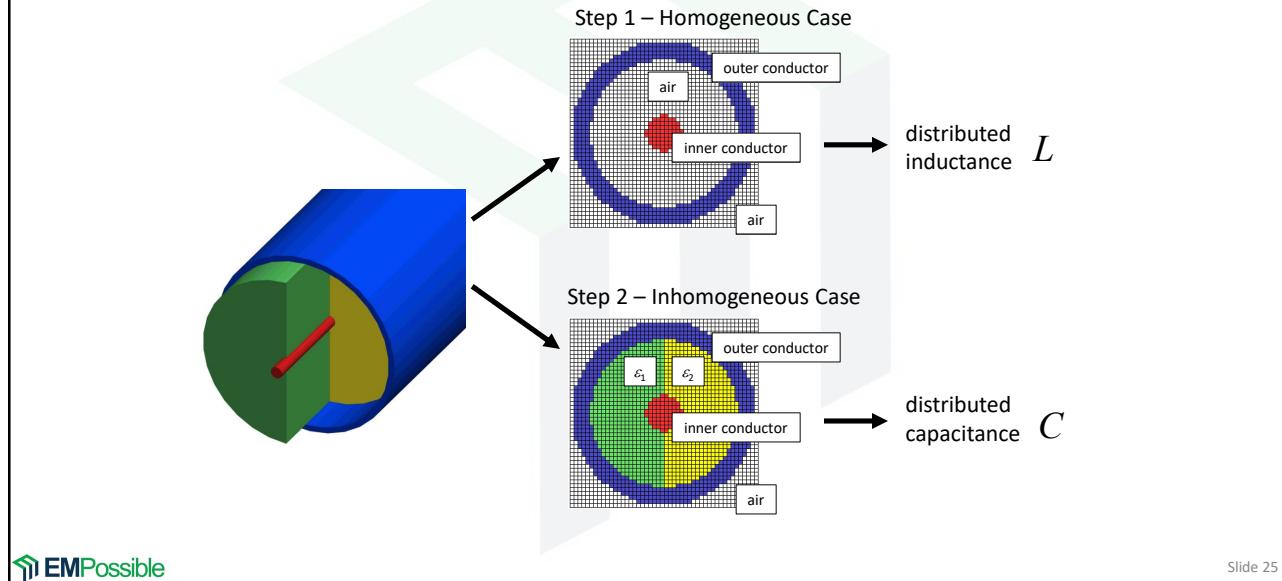
$$v = \frac{1}{\sqrt{LC}}$$

The effective refractive index n_{eff} is therefore

$$n_{eff} = \frac{c_0}{v} = c_0 \sqrt{LC}$$

Both Z_c and n_{eff} are needed to analyze transmission line circuits.

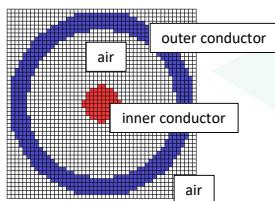
Two Step Modeling Approach



How We Will Analyze Transmission Lines

Step 1

Construct homogeneous TL



Step 2

Construct and solve matrix equation

$$\nabla^2 V_h = 0$$

$$\Downarrow$$

$$\mathbf{L}_h \mathbf{v}_h = \mathbf{v}_{src}$$

$$\Downarrow$$

$$\mathbf{v}_h = \mathbf{L}_h^{-1} \mathbf{v}_{src}$$

Step 3

Calculate distributed inductance

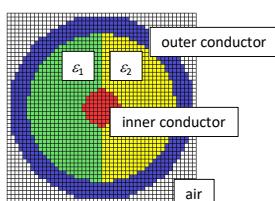
$$\vec{E}_h = -\nabla V_h$$

$$C_h = \frac{\epsilon_0}{V_0^2} \iint_A (\vec{D}_h \bullet \vec{E}_h) dA$$

$$L = \frac{\mu_{rh}}{c_0^2 C_h}$$

Step 4

Construct inhomogeneous TL



Step 5

Construct and solve matrix equation

$$\nabla \bullet [\epsilon_r (\nabla V)] = 0$$

$$\Downarrow$$

$$\mathbf{L} \mathbf{v} = \mathbf{v}_{src}$$

$$\Downarrow$$

$$\mathbf{v} = \mathbf{L}^{-1} \mathbf{v}_{src}$$

Step 6

Calculate distributed capacitance

$$\vec{E} = -\nabla V$$

$$C = \frac{\epsilon_0}{V_0^2} \iint_A (\vec{D} \bullet \vec{E}) dA$$

Step 7

Calculate TL parameters

$$Z_c = \sqrt{\frac{L}{C}} \quad n = c_0 \sqrt{LC}$$

Formulation: Derivation of Governing Equations

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Equations of Electrostatics

Any electromagnetic analysis begins with Maxwell's equations.

For electrostatic problems, start with the following.

$$\nabla \bullet \vec{D} = 0 \quad \text{Eq. (1)}$$

$$\vec{D} = [\epsilon] \vec{E} \quad \text{Eq. (2)}$$

$$\vec{E} = -\nabla V \quad \text{Eq. (3)}$$

Expand Equations

$$\nabla \bullet \vec{D} = 0 \rightarrow \left[\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z} \right] \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = 0$$

$$\vec{D} = [\epsilon] \vec{E} \rightarrow \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$$\vec{E} = -\nabla V \rightarrow \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = - \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} V$$

2D Analysis of Cross Section

It is desired to analyze a transmission line that is uniform in the direction that the wave propagates.

Let this direction be z .

Since the device is uniform in z , nothing changes in the z direction.

$$\frac{\partial}{\partial z} = 0$$

Reduced Equations for 2D

$$\nabla \bullet \vec{D} = 0 \rightarrow \left[\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \cancel{\frac{\partial}{\partial z}} \right] \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = 0 \rightarrow \left[\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \right] \begin{bmatrix} D_x \\ D_y \end{bmatrix} = 0 \\ D_z = 0$$

$$\vec{D} = \epsilon \vec{E} \rightarrow \begin{bmatrix} D_x \\ D_y \\ \cancel{D_z} \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ \cancel{E_z} \end{bmatrix} \rightarrow \begin{bmatrix} D_x \\ D_y \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & 0 \\ 0 & \epsilon_{yy} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} \\ E_z = 0$$

$$\vec{E} = -\nabla V \rightarrow \begin{bmatrix} E_x \\ E_y \\ \cancel{E_z} \end{bmatrix} = - \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \cancel{\partial/\partial z} \end{bmatrix} V \rightarrow \begin{bmatrix} E_x \\ E_y \end{bmatrix} = - \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} V$$

$$D_z = E_z = 0$$

This implies TEM.

Final Governing Equations

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = 0$$

$$\left[\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \right] \begin{bmatrix} D_x \\ D_y \end{bmatrix} = 0$$

$$\begin{bmatrix} D_x \\ D_y \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & 0 \\ 0 & \epsilon_{yy} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = - \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} V$$

$$D_x = \epsilon_{xx} E_x$$

$$D_y = \epsilon_{yy} E_y$$

$$E_x = - \frac{\partial V}{\partial x}$$

$$E_y = - \frac{\partial V}{\partial y}$$

Formulation: Finite-Difference Approximation of Governing Equations

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Grid Strategy (1 of 2)

The manner in which we stagger the functions across the grid comes from the governing equations themselves.

From the constitutive equations, we see that D_x and ε_{xx} will lie on the same points as E_x .

$$D_x = \varepsilon_{xx} E_x \quad \Rightarrow \quad D_x^{i,j} = \varepsilon_0 \varepsilon_{xx}^{i,j} E_x^{i,j}$$

Similarly, D_y and ε_{yy} will lie on the same points as E_y .

$$D_y = \varepsilon_{yy} E_y \quad \Rightarrow \quad D_y^{i,j} = \varepsilon_0 \varepsilon_{yy}^{i,j} E_y^{i,j}$$

Thus, the constitutive relations do not require any staggering.

Grid Strategy (2 of 2)

Next, inspect the equations relating \vec{E} and V .

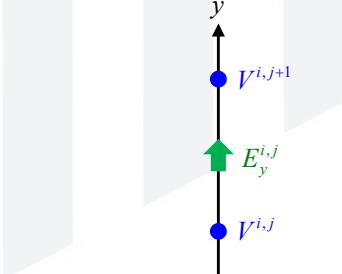
Observe that V will need to be staggered around E_x in the x direction.

$$E_x = -\frac{\partial V}{\partial x} \rightarrow E_x^{i,j} = -\frac{V^{i+1,j} - V^{i,j}}{\Delta x}$$



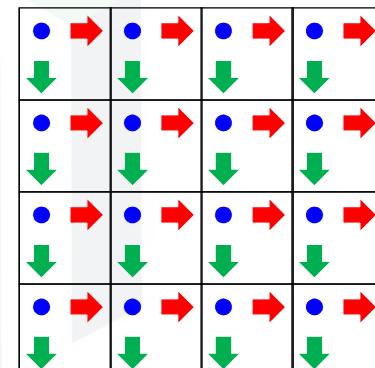
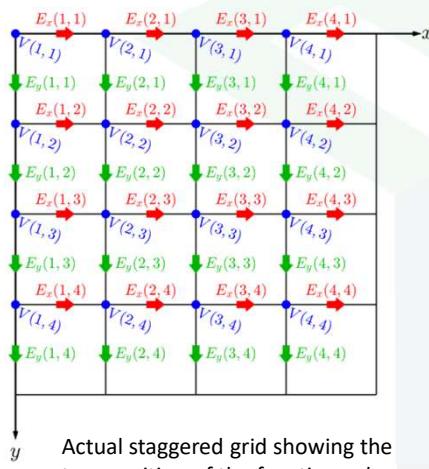
Observe that V will need to be staggered around E_y in the y direction.

$$E_y = -\frac{\partial V}{\partial y} \rightarrow E_y^{i,j} = -\frac{V^{i,j+1} - V^{i,j}}{\Delta y}$$



Grid Strategy (3 of 3)

Putting all of this together, the staggered grid is...



An alternative view of the staggered grid that more clearly conveys which cell each function value resides in.

Matrix Form of Governing Equations

There are five coupled equations. These can immediately be written in matrix form as:

$$\begin{aligned} \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} &= 0 & \frac{D_x^{i,j} - D_x^{i-1,j}}{\Delta x} + \frac{D_y^{i,j} - D_y^{i,j-1}}{\Delta y} &= 0 & \mathbf{D}_x^e \mathbf{d}_x + \mathbf{D}_y^e \mathbf{d}_y &= \mathbf{0} \\ D_x &= \varepsilon_{xx} E_x & D_x^{i,j} &= \varepsilon_0 \varepsilon_{xx}^{i,j} E_x^{i,j} & \mathbf{d}_x &= \varepsilon_0 \boldsymbol{\varepsilon}_{xx} \mathbf{e}_x \\ D_y &= \varepsilon_{yy} E_y & D_y^{i,j} &= \varepsilon_0 \varepsilon_{yy}^{i,j} E_y^{i,j} & \mathbf{d}_y &= \varepsilon_0 \boldsymbol{\varepsilon}_{yy} \mathbf{e}_y \\ E_x &= -\frac{\partial V}{\partial x} & E_x^{i,j} &= -\frac{V^{i+1,j} - V^{i,j}}{\Delta x} & \mathbf{e}_x &= -\mathbf{D}_x^v \mathbf{v} \\ E_y &= -\frac{\partial V}{\partial y} & E_y^{i,j} &= -\frac{V^{i,j+1} - V^{i,j}}{\Delta y} & \mathbf{e}_y &= -\mathbf{D}_y^v \mathbf{v} \end{aligned}$$

Block Matrix Form

Write the matrix equations in block matrix form.

$$\begin{aligned} \left[\begin{array}{cc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{array} \right] \begin{bmatrix} D_x \\ D_y \end{bmatrix} &= 0 & \left[\begin{array}{cc} \mathbf{D}_x^e & \mathbf{D}_y^e \end{array} \right] \begin{bmatrix} \mathbf{d}_x \\ \mathbf{d}_y \end{bmatrix} &= \mathbf{0} \\ \begin{bmatrix} D_x \\ D_y \end{bmatrix} &= \begin{bmatrix} \varepsilon_{xx} & 0 \\ 0 & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} & \begin{bmatrix} \mathbf{d}_x \\ \mathbf{d}_y \end{bmatrix} &= \varepsilon_0 \begin{bmatrix} \boldsymbol{\varepsilon}_{xx} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\varepsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} \\ \begin{bmatrix} E_x \\ E_y \end{bmatrix} &= -\begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \end{bmatrix} V & \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} &= -\begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} \mathbf{v} \end{aligned}$$

Eliminate \vec{D} Field

Eliminate the \vec{D} field by substituting the second equation into the first.

$$\begin{aligned} \begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \mathbf{d}_x \\ \mathbf{d}_y \end{bmatrix} = \mathbf{0} \\ \begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \left(\epsilon_0 \begin{bmatrix} \boldsymbol{\epsilon}_{xx} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\epsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} \right) = \mathbf{0} \\ \begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \boldsymbol{\epsilon}_{xx} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\epsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} = \mathbf{0} \end{aligned}$$

Eliminate \vec{E} Field

The \vec{E} field can be eliminated in a similar way.

$$\begin{aligned} \begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \boldsymbol{\epsilon}_{xx} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\epsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} = \mathbf{0} \\ \begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \boldsymbol{\epsilon}_{xx} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\epsilon}_{yy} \end{bmatrix} \left(-\begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} \mathbf{v} \right) = \mathbf{0} \\ \begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \boldsymbol{\epsilon}_{xx} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\epsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} \mathbf{v} = \mathbf{0} \end{aligned}$$

There is now a single scalar differential equation in matrix form.

Final Matrix Equations

The final matrix equation is the matrix form of the inhomogeneous Laplace's equation.

Inhomogeneous Laplace's Equation

$$\nabla \cdot [\epsilon_r (\nabla V)] = 0 \quad \rightarrow \quad \begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \boldsymbol{\epsilon}_{xx} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\epsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} = \mathbf{0}$$

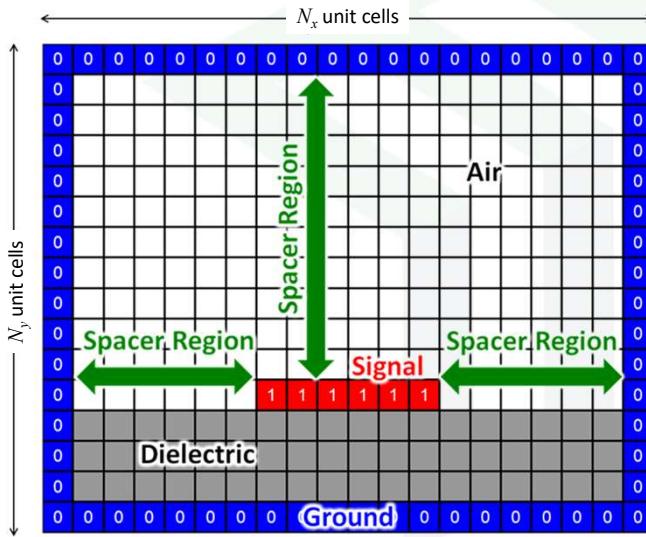
Homogeneous Laplace's Equation

We can easily write the block matrix form of the homogeneous Laplace's equation from this result because $\epsilon_r = 1$ everywhere.

$$\nabla^2 V_h = 0 \quad \rightarrow \quad \begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} = \mathbf{0}$$

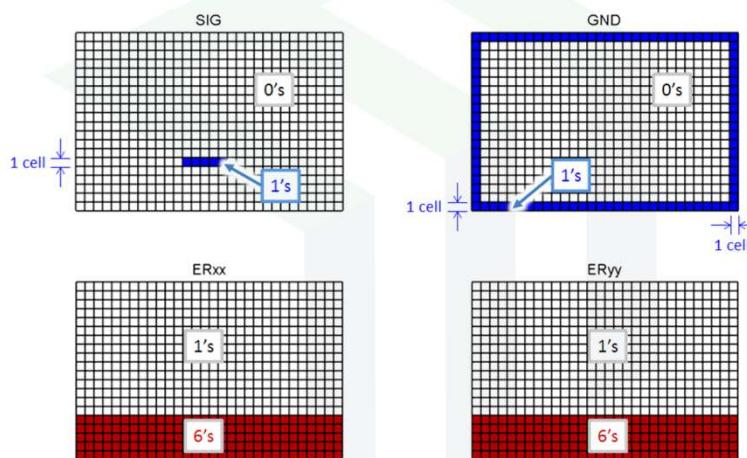
Implementation: Build the Device on a Grid

Grid Representation of a Microstrip Transmission Line



The spacer regions are needed so that the transmission line is not affected by the edges of the grid.

Four Arrays to Describe a Transmission Line ($\epsilon_r = 6$)



Note: If you wish to simulate a transmission line with more than two conductors, you will need one array for each conductor in addition to the two permittivity arrays.

Building the Diagonal Matrices ϵ_{xx} and ϵ_{yy}

1. Build ERxx and ERyy using the 2x grid technique.
2. Reshape these 2D arrays into 1D arrays (column vectors).


```
ERxx = ERxx(:);  
ERyy = ERyy(:);
```
3. Declare the 1D arrays as sparse.


```
ERxx = sparse(ERxx);  
ERyy = sparse(ERyy);
```
4. Convert the 1D sparse arrays into sparse diagonal matrices.


```
ERxx = diag(sparse(ERxx));  
ERyy = diag(sparse(ERyy));
```

Actually, all of this can be accomplished in a single line of MATLAB code:

```
ERxx = diag(sparse(ERxx(:)));  
ERyy = diag(sparse(ERyy(:)));
```

The Forced Potentials Grid

$v_f(x, y)$															
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$$\begin{bmatrix} v_f \end{bmatrix} =$$


Reshaped to a
column vector.
 $v_f = v_f(:);$

Implementation: 2× Grid Technique

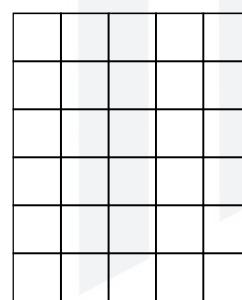
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2× Grid Technique (1 of 9)

Define the ordinary “1×” grid as usual.

The output of this step is the number of cells in the grid,
 N_x and N_y , and the size of the cells in the grid, d_x and d_y .

```
% DEFINE GRID  
Nx = 5;  
Ny = 6;  
dx = 1;  
dy = 1;
```

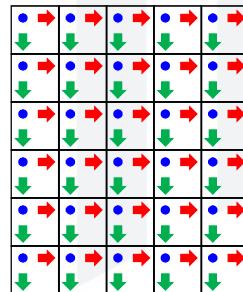


2x Grid Technique (2 of 9)

Recall how the various functions overlay onto the grid.

Functions assigned to the same grid cell are in physically different positions and may reside in different materials as a result.

```
% DEFINE GRID
Nx = 5;
Ny = 6;
dx = 1;
dy = 1;
```



2x Grid Technique (3 of 9)

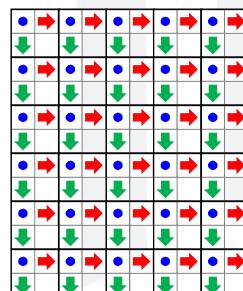
It is like we are getting twice the resolution due to the staggering of the functions.

In order to sort out what values go where, we construct a “2x” grid at twice the resolution of the original grid.

The 2x grid occupies the same physical amount of space as the original grid.

```
% DEFINE GRID
Nx = 5;
Ny = 6;
dx = 1;
dy = 1;
```

```
% 2X GRID
Nx2 = 2*Nx;
Ny2 = 2*Ny;
dx2 = dx/2;
dy2 = dy/2;
```



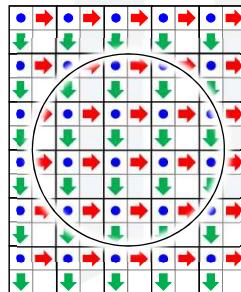
2x Grid Technique (4 of 9)

Let's say we wish to construct a cylinder of radius 2 on our grid.

```
% DEFINE GRID
Nx = 5;
Ny = 6;
dx = 1;
dy = 1;

% 2X GRID
Nx2 = 2*Nx;
Ny2 = 2*Ny;
dx2 = dx/2;
dy2 = dy/2;

% CREATE CYLINDER
r = 2;
```



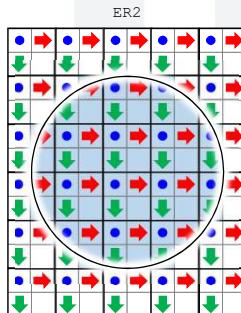
2x Grid Technique (5 of 9)

We start by building our object on the 2x grid, ignoring anything about our original grid for now.

```
% DEFINE GRID
Nx = 5;
Ny = 6;
dx = 1;
dy = 1;

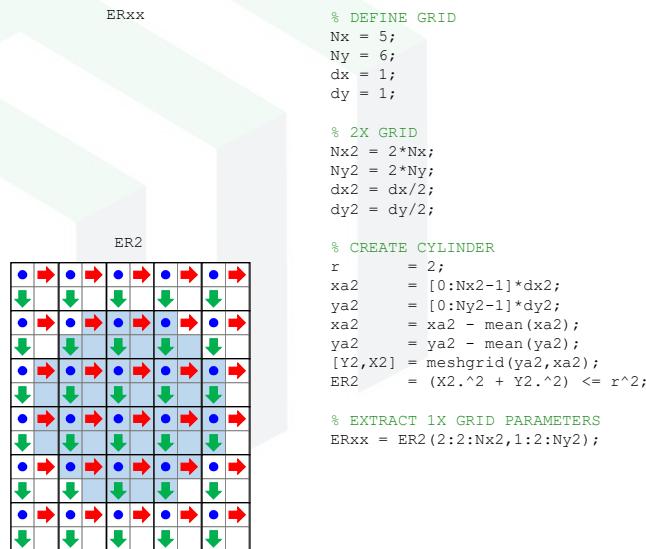
% 2X GRID
Nx2 = 2*Nx;
Ny2 = 2*Ny;
dx2 = dx/2;
dy2 = dy/2;

% CREATE CIRCLE
r = 2;
xa2 = [0:Nx2-1]*dx2;
ya2 = [0:Ny2-1]*dy2;
xa2 = xa2 - mean(xa2);
ya2 = ya2 - mean(ya2);
[Y2,X2] = meshgrid(ya2,xa2);
ER2 = (X2.^2 + Y2.^2) <= r^2;
```



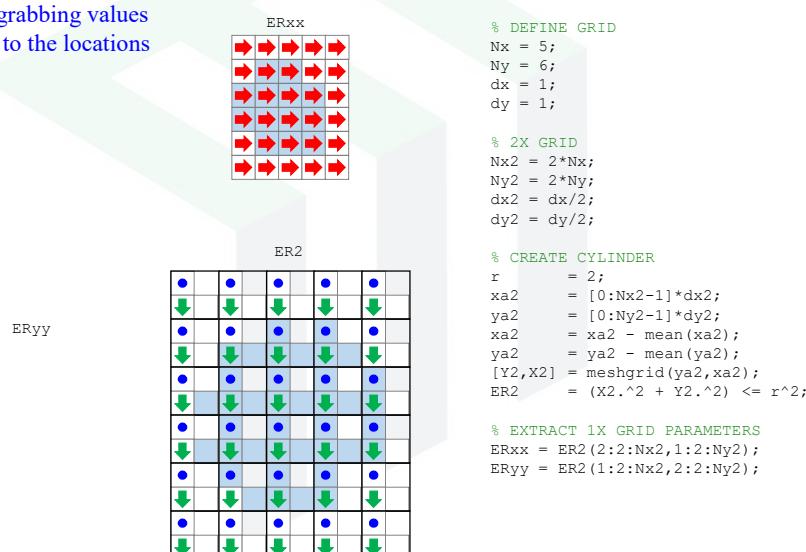
2x Grid Technique (6 of 9)

Given the object on the 2x grid, we extract ERxx by grabbing values from ER2 that correspond to the locations of ERxx.

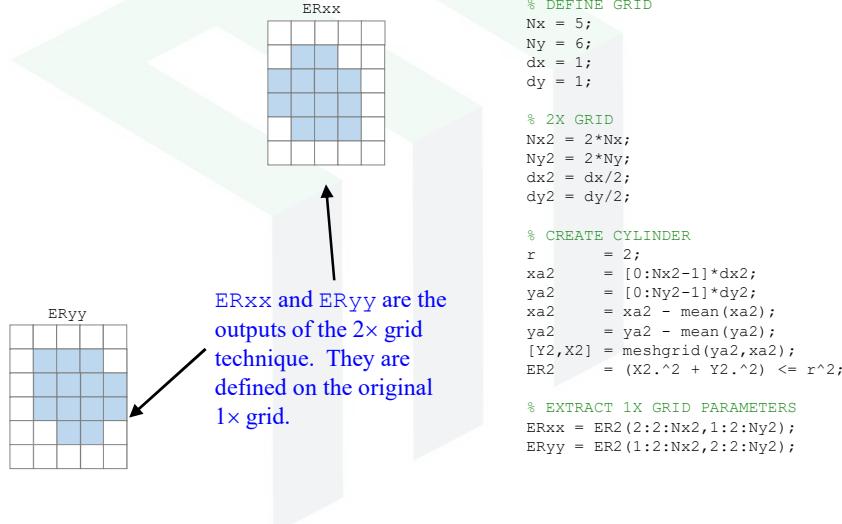


2x Grid Technique (7 of 9)

We then extract ERyy by grabbing values from ER2 that correspond to the locations of ERyy.



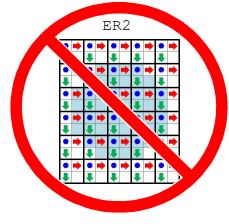
2x Grid Technique (8 of 9)



2x Grid Technique (9 of 9)

After building ERxx and ERyy, the 2x grid is no longer used anywhere.

All of the 2x grid parameters may be deleted at this point because they are no longer needed.



```
% DEFINE GRID
Nx = 5;
Ny = 6;
dx = 1;
dy = 1;

% 2X GRID
Nx2 = 2*Nx;
Ny2 = 2*Ny;
dx2 = dx/2;
dy2 = dy/2;

% CREATE CYLINDER
r = 2;
xa2 = [0:Nx2-1]*dx2;
ya2 = [0:Ny2-1]*dy2;
xa2 = xa2 - mean(xa2);
ya2 = ya2 - mean(ya2);
[Y2,X2] = meshgrid(ya2,xa2);
ER2 = (X2.^2 + Y2.^2) <= r^2;

% EXTRACT 1X GRID PARAMETERS
ERxx = ER2(2:2:Nx2,1:2:Ny2);
ERyy = ER2(1:2:Nx2,2:2:Ny2);
```

No more 2x grid!!!

Implementation: Solving the Matrix Equation

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Are We Ready to Solve?

So far we have discussed:

- (1) Construction of the permittivity matrices, $\boldsymbol{\epsilon}_{xx}$ and $\boldsymbol{\epsilon}_{yy}$.
- (2) Construction of the derivative operators, \mathbf{D}_x^e and \mathbf{D}_y^e .
- (3) Construction of the matrix problem.

That puts us here:

$$\begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \boldsymbol{\epsilon}_{xx} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\epsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} = \mathbf{0}$$

How do we solve this for \mathbf{v} ?

$$\mathbf{L} = \begin{bmatrix} \mathbf{D}_x^d & \mathbf{D}_y^d \end{bmatrix} \begin{bmatrix} \boldsymbol{\epsilon}_{xx} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\epsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix}$$

$$\mathbf{L}\mathbf{v} = \mathbf{0}$$

$$\mathbf{v} = \mathbf{L}^{-1} \mathbf{0} = \mathbf{0} \longrightarrow \text{We can only find a trivial solution!}$$

What is Missing?

$$\mathbf{Lv} = \mathbf{0} \rightarrow \mathbf{v} = \mathbf{0}$$

We are missing the excitation $\mathbf{b} \rightarrow$ The forced potentials.

The metals of the transmission line must be set to some known voltage.

$$\mathbf{L}'\mathbf{v} = \mathbf{b} \rightarrow \boxed{\mathbf{v} = \mathbf{L}'^{-1}\mathbf{b}}$$

How do we construct \mathbf{b} ?

Constructing \mathbf{b} (1 of 2)

It turns out we must also modify \mathbf{L} in addition to constructing \mathbf{b} .

For each point where there is a forced potential, we do the following to the corresponding row in the matrix equation:

1. Replace entire row in \mathbf{L} with all zeros.
2. Place a 1 along the diagonal element.
3. Place the forced potential in \mathbf{b} .

$$\begin{matrix}
 \left[\begin{array}{ccccccc} (\#) & (\#) & (\#) & \cdots & (\#) & (\#) & (\#) \\ (\#) & (\#) & (\#) & \cdots & (\#) & (\#) & (\#) \\ \vdots & & \ddots & & \vdots & & \end{array} \right] & \left[\begin{array}{c} V_1 \\ V_2 \\ \vdots \\ V_m \\ \vdots \\ V_{N_x N_y - 1} \\ V_{N_x N_y} \end{array} \right] & = & \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ V_{\text{applied}} \\ \vdots \\ 0 \\ 0 \end{array} \right] \\
 \mathbf{L}' & \mathbf{v} & & \mathbf{b}
 \end{matrix}$$

The applied voltages are usually just 0 and V_0 , but you may use other values if you are doing something special.

Constructing \mathbf{b} (2 of 2)

Define two functions:

F ≡ Force matrix

Diagonal matrix containing 1's in the diagonal positions corresponding to values we wish to force (i.e. metals). 0's otherwise.

\mathbf{v}_f ≡ forced potentials

Column vector containing the forced potentials.
Numbers in positions not being forced are ignored.

1. Replace rows in \mathbf{L} with all zeros that correspond to metals on the grid.

$$\mathbf{L}'' = (\mathbf{I} - \mathbf{F})\mathbf{L}$$

2. Then place a 1 in the diagonal element by adding \mathbf{F} to \mathbf{L}' .

$$\boxed{\mathbf{L}' = \mathbf{F} + (\mathbf{I} - \mathbf{F})\mathbf{L}} = \mathbf{F} + \mathbf{L}''$$

3. Place the forced potentials in **b**. The remaining elements in **b** must be 0 in order to be consistent with Laplace's equation..

$$\mathbf{b} = \mathbf{F}\mathbf{v}_f$$

The Force Matrix

The force matrix \mathbf{F} starts off as an array $F(x,y)$ containing 1's in the positions we wish to force to some potential and 0's everywhere else.

We already have two arrays defined this same way that describe the two conductors of the transmission line.

We simply combine all of the conductor arrays to construct the force array.

Last, we construct the force matrix by diagonalizing the force array.

```
F = diag(sparse(F(:, :)));
```

Complete Matrix Solution

We are now ready to solve the problem. The scalar potential V is calculated as

$$\mathbf{v} = (\mathbf{L}')^{-1} \mathbf{b}$$

We can then calculate the \vec{E} field from V .

$$\begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} = - \begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} \mathbf{v}$$

We can then calculate the \vec{D} field from the \vec{E} field and the permittivity tensor.

$$\begin{bmatrix} \mathbf{d}_x \\ \mathbf{d}_y \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \mathbf{0} \\ \mathbf{0} & \epsilon_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix}$$

There should be an ϵ_0 here. We have removed it from this equation and will incorporate it in our final calculation of distributed capacitance in order to keep our functions normalized.

Extract the Vector Components

We must extract the terms \mathbf{e}_x and \mathbf{e}_y from column vectors.

```
v = L\b;
e = -[DVX; DVY]*v;
M = Nx*Ny;
ex = e(1:M);
ey = e(M+1:2*M);
```

$$\mathbf{e} = - \begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} \mathbf{v}$$

$$\begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} = \mathbf{e}$$

Reshape Back to 2D Arrays

The terms \mathbf{v} , \mathbf{e}_x , \mathbf{e}_y , \mathbf{d}_x , and \mathbf{d}_y are column vectors, or 1D arrays.

These need to be reshaped back to 2D arrays before they can be visualized.

```
v = reshape(v, Nx, Ny);  
ex = reshape(ex, Nx, Ny);  
ey = reshape(ey, Nx, Ny);
```

Implementation: Calculating the Transmission Line Parameters

Distributed Capacitance, C

The distributed capacitance C is calculated by numerical integration of

$$C = \frac{\epsilon_0}{V_0^2} \iint_A (\vec{D} \bullet \vec{E}) dA$$

This is where we incorporate the constant ϵ_0 .

Assuming $V_0 = 1$, this integral is easily evaluated as

$$C = \mathbf{d}^T \mathbf{e} (\epsilon_0 \Delta x \Delta y)$$

```
C = d.' * e * (e0*dx*dy);
```

Distributed Inductance, L

First, the distributed capacitance C_h for the homogeneous case is calculated.

$$C_h = \mathbf{d}_h^T \mathbf{e}_h (\epsilon_0 \Delta x \Delta y)$$

```
Ch = dh.' * eh * (e0*dx*dy);
```

Second, the distributed inductance L is calculated from C_h .

$$L = \frac{\mu_{r,h}}{c_0^2 C_h}$$

```
L = urh / (c0^2 * Ch);
```

Characteristic Impedance, Z_0

Given the distributed inductance L and distributed capacitance C , the characteristic impedance of the transmission line is

$$Z_0 = \sqrt{\frac{L}{C}}$$

$$z_0 = \text{sqrt}(L/C);$$

Effective Refractive Index, n_{eff}

Given the distributed inductance L and distributed capacitance C , the effective refractive index n_{eff} of the transmission line is

$$n = c_0 \sqrt{LC}$$

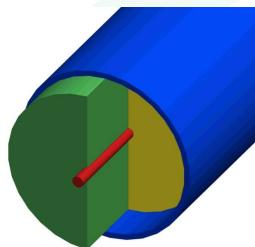
$$n_{\text{eff}} = c_0 * \text{sqrt}(L*C);$$

Example – Weird Coax

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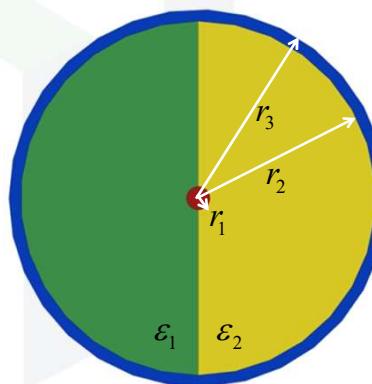
Step 1: Choose a Coaxial Transmission Line

Choose a transmission line.



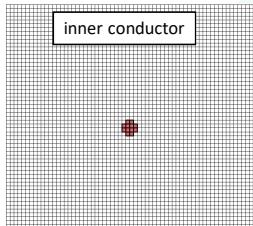
$$\begin{aligned}r_1 &= 0.15 \text{ mm} \\r_2 &= 2.5 \text{ mm} \\r_3 &= 2.7 \text{ mm} \\\varepsilon_{r1} &= 2.3 \\\varepsilon_{r2} &= 2.3\end{aligned}$$

Look only at the cross section.



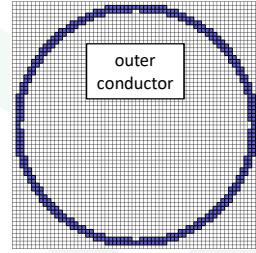
Step 2: Build Materials Arrays

```
% BUILD INNER CONDUCTOR
CIN = (RSQ<=r1^2);
```



1x Grid

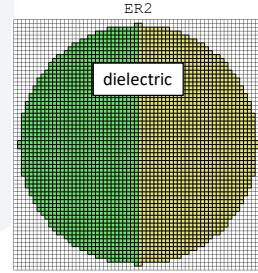
```
% BUILD OUTER CONDUCTOR
COUT = (RSQ<r3^2 & RSQ>=r2^2);
```



1x Grid

ER2

dielectric



2x Grid

```
% BUILD DIELECTRIC
r23 = (r2 + r3)/2;
ER2 = ones(Nx2,Ny2);
ER2 = ER2 + (er1 - 1)*(RSQ2<r23^2 & X2<0);
ER2 = ER2 + (er2 - 1)*(RSQ2<r23^2 & X2>=0);
```

```
%middle of outer conductor
%air
%dielectric 1
%dielectric 2
```



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Step 3: Form Diagonal Materials Matrices

This task reshapes the dielectric functions into 1D arrays and then places them along the diagonal of two matrices.

$$\begin{bmatrix} \epsilon_{xx} & \mathbf{0} \\ \mathbf{0} & \epsilon_{yy} \end{bmatrix}$$

```
% EXTRACT ERxx AND ERyy FROM ER2
ERxx = ER2(2:2:Nx2,1:2:Ny2);
ERyy = ER2(1:2:Nx2,2:2:Ny2);

% FORM DIAGONAL PERMITTIVITY MATRICES
ERxx = diag(sparse(ERxx(:)));
ERyy = diag(sparse(ERyy(:)));

% FORM PERMITTIVITY TENSOR
Z = sparse(Nx*Ny,Nx*Ny);
ER = [ ERxx , Z ; Z , ERyy ];
```



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Step 4: Construct Derivative Operators

This task will be performed by the function `tlder()`.

```
% CALL FUNCTION TO CONSTRUCT DERIVATIVE OPERATORS
NS = [Nx Ny]; %grid size
RES = [dx dy]; %grid resolution
[DVX,DVY,DDX,DDY] = tlder(NS,RES); %build matrices
```

$$\mathbf{D}_x^e = \frac{1}{\Delta_x} \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & 0 & 1 \\ & & -1 & 1 \\ & & & -1 & 1 \\ & & & & 0 & 1 \\ & & & & -1 & 1 \\ & & & & & -1 & 1 \\ & & & & & & 1 \end{bmatrix}$$

$$\mathbf{D}_y^e = \frac{1}{\Delta_y} \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ -1 & 0 & 0 & 1 & \\ -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & \\ -1 & 0 & 0 & 1 & \\ -1 & 0 & 0 & 1 & \\ -1 & 0 & 0 & 1 & \\ -1 & 0 & 0 & 1 & \end{bmatrix}$$

$$\mathbf{D}_x^v = \frac{1}{\Delta_x} \begin{bmatrix} -1 & 1 & & & & \\ -1 & -1 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & -1 & 1 & 1 & \\ & & & -1 & 1 & \\ & & & & -1 & 0 & 1 \\ & & & & & -1 & 1 & \\ & & & & & & -1 & 1 & \\ & & & & & & & -1 & 1 \\ & & & & & & & & 1 \end{bmatrix}$$

$$\mathbf{D}_y^v = \frac{1}{\Delta_y} \begin{bmatrix} -1 & 0 & 1 & & & & \\ -1 & 0 & 0 & 1 & & & \\ -1 & 0 & 0 & 0 & 1 & & \\ -1 & 0 & 0 & 1 & & & \\ -1 & 0 & 0 & 0 & 1 & & \\ -1 & 0 & 0 & 1 & & & \\ -1 & 0 & 0 & 0 & 1 & & \\ -1 & 0 & 0 & 1 & & & \\ -1 & 0 & 0 & 0 & 1 & & \\ -1 & 0 & 0 & 1 & & & \end{bmatrix}$$

Step 5: Build Matrix Equations

Inhomogeneous Laplace's Equation

$$\mathbf{L} = \begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \boldsymbol{\epsilon}_{xx} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\epsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix}$$

```
L = [DEX DEY] * ER * [DVX ; DVY];
```

Homogeneous Laplace's Equation

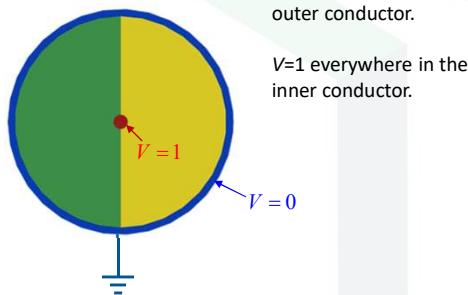
$$\mathbf{L}_h = \begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix}$$

```
Lh = [DEX DEY] * [DVX ; DVY];
```

$$\mathbf{L} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

Step 6: Force Known Potentials

This same procedure should be done for both the inhomogeneous and homogeneous matrix equations.



```
% FORCE MATRIX
F = CIN | COUT;
F = diag(sparse(F(:))) ;

% FORCED POTENTIALS
vf = 1*CIN + 0*COUT;

% FORCE KNOWN POTENTIALS
I = speye(M,M);
L = (I - F)*L + F;
Lh = (I - F)*Lh + F;
b = F*vf(:,');
```

Step 7: Compute the Fields

Compute the Potentials

$$\mathbf{v} = (\mathbf{L}')^{-1} \mathbf{b}$$

$$\mathbf{v}_h = (\mathbf{L}'_h)^{-1} \mathbf{b}$$

WARNING: DO NOT compute the inverse of \mathbf{L} .

```
% COMPUTE POTENTIALS
v = L\b;
vh = Lh\b;
```

Compute the E Fields

$$\begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} = - \begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} \mathbf{v}$$

$$\begin{bmatrix} \mathbf{e}_{x,h} \\ \mathbf{e}_{y,h} \end{bmatrix} = - \begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} \mathbf{v}_h$$

```
% COMPUTE E FIELDS
e = - [ DVX ; DVY ] * v;
eh = - [ DVX ; DVY ] * vh;
```

Compute the D Fields

$$\begin{bmatrix} \mathbf{d}_x \\ \mathbf{d}_y \end{bmatrix} = \begin{bmatrix} \mathbf{\epsilon}_{xx} & \mathbf{0} \\ \mathbf{0} & \mathbf{\epsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{d}_{x,h} \\ \mathbf{d}_{y,h} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_{x,h} \\ \mathbf{e}_{y,h} \end{bmatrix}$$

Note: We dropped the ϵ_0 in these equations and will incorporate that when we calculate C .

```
% COMPUTE D FIELDS
d = ER*e;
dh = eh;
```

Step 8: Calculate TL Parameters

Distributed Capacitance

$$C = \frac{\epsilon_0}{V_0^2} \iint_A (\vec{D} \bullet \vec{E}) dA$$

Note: We added the ϵ_0 constant here.

% DISTRIBUTED CAPACITANCE

C = d.' * e0 * (e0 * dx * dy);

We have assumed $V_0 = 1$.

Characteristic Impedance

$$Z_0 = \sqrt{\frac{L}{C}}$$

% CHARACTERISTIC IMPEDANCE

Z0 = sqrt(L/C);

Distributed Inductance

$$C_h = \frac{\epsilon_0}{V_0^2} \iint_A (\vec{D}_h \bullet \vec{E}_h) dA$$

$$L = \frac{1}{c_0^2 C_h}$$

Recall that we have
 $\mu_h = \epsilon_{rh} = 1$

% DISTRIBUTED INDUCTANCE

Ch = dh.' * eh * (e0 * dx * dy);

L = 1 / (c0^2 * Ch);

We have assumed $V_0 = 1$.

Effective Refractive Index

$$n_{\text{eff}} = c_0 \sqrt{LC}$$

% EFFECTIVE REFRACTIVE INDEX

neff = c0 * sqrt(L * C);

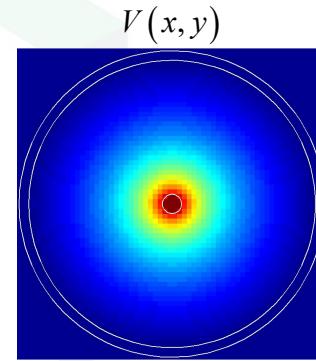
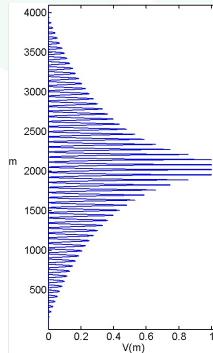


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Step 9: Reshape the Data

We need to reshape the column vectors back to a 2D grid.

$$\mathbf{v} = \begin{bmatrix} (\#) \\ (\#) \\ \vdots \\ (\#) \\ (\#) \end{bmatrix}$$



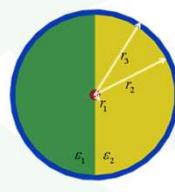
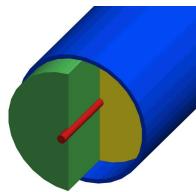
% RESHAPE THE FUNCTIONS BACK TO A 2D GRID
Vh = reshape(Vh, Nx, Ny);
V = reshape(V, Nx, Ny);



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Step 10: Show the results!

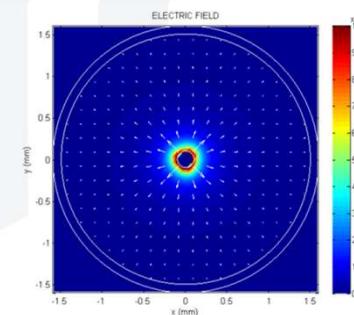
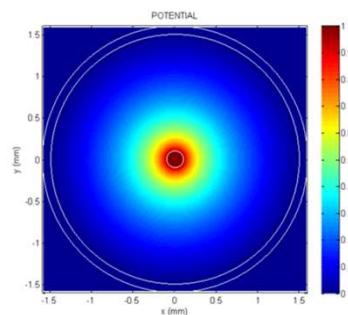
Problem definition:



Standard RG-59

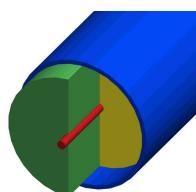
$$\begin{aligned}r_1 &= 0.3 \text{ mm} & C &\cong 70.73 \text{ pF/m} \\r_2 &= 2.5 \text{ mm} & L &\cong 361.8 \text{ nH/m} \\r_3 &= 2.7 \text{ mm} & Z_0 &\cong 71.5 \Omega \\\epsilon_1 &= 2.3 & n &\cong 1.52 \\\epsilon_2 &= 2.3\end{aligned}$$

Problem results:



Step 10: Another Case

Problem definition:



$$\begin{aligned}r_1 &= 0.1 \text{ mm} & C &\cong 121.1 \text{ pF/m} \\r_2 &= 1.5 \text{ mm} & L &\cong 551.2 \text{ nH/m} \\r_3 &= 1.6 \text{ mm} & Z_0 &\cong 67.5 \Omega \\\epsilon_1 &= 3.0 & n &\cong 2.45 \\\epsilon_2 &= 9.0\end{aligned}$$

Problem results:

