Heat Transfer:

$$u_t - u_{xx} = f(x,t) \qquad (x,t) \in (0,1)\times(0,1)$$

$u(x,0) = \sin(\pi x)$ - initial condition

$u(0,t) = u(1,t) = 0$ - boundary conditions

Weak form derivation:

multiply by a virtual function $v(x)$ and integrate over spatial domain.

$$\int_0^1 \frac{du}{dt} v(x)\, dx - \int_0^1 \frac{\partial^2 u}{dx^2} v(x)\, dx = \int_0^1 f(x,t) v(x)\, dx \qquad (1)$$

Rearrange

$$\int_0^1 \frac{du}{dt} v(x)\, dx - \int_0^1 \frac{\partial^2 u}{dx^2} v(x)\, dx - \int_0^1 f(x,t) v(x)\, dx = 0 \qquad (2)$$

Apply integration by parts to the middle term

$$\int_0^1 \frac{\partial^2 u}{\partial x^2} v(x)\, dx = v(x) \frac{\partial u}{\partial x}\Big|_0^1 - \int_0^1 \frac{\partial u}{\partial x} \frac{\partial v}{\partial x}\, dx \qquad (3)$$

Because $v(x)$ should match the boundary conditions we know

$v(0) = v(1) = 0$. Hence

$$\int_0^1 \frac{\partial^2 u}{\partial x^2} v(x)\, dx = -\int_0^1 \frac{\partial u}{\partial x} \frac{\partial v}{\partial x}\, dx \qquad (4)$$

Substitute (4) into (2)

$$\boxed{\int_0^1 \frac{du}{dt} v(x)\, dx + \int_0^1 \frac{\partial u}{dx} \frac{\partial v}{\partial x}\, dx - \int_0^1 f(x,t) v(x)\, dx = 0}$$

This must be satisfied for all suitable virtual function $v(x)$