A Very Short Introduction to OTFS

(Orthogonal Time Frequency Spacing)

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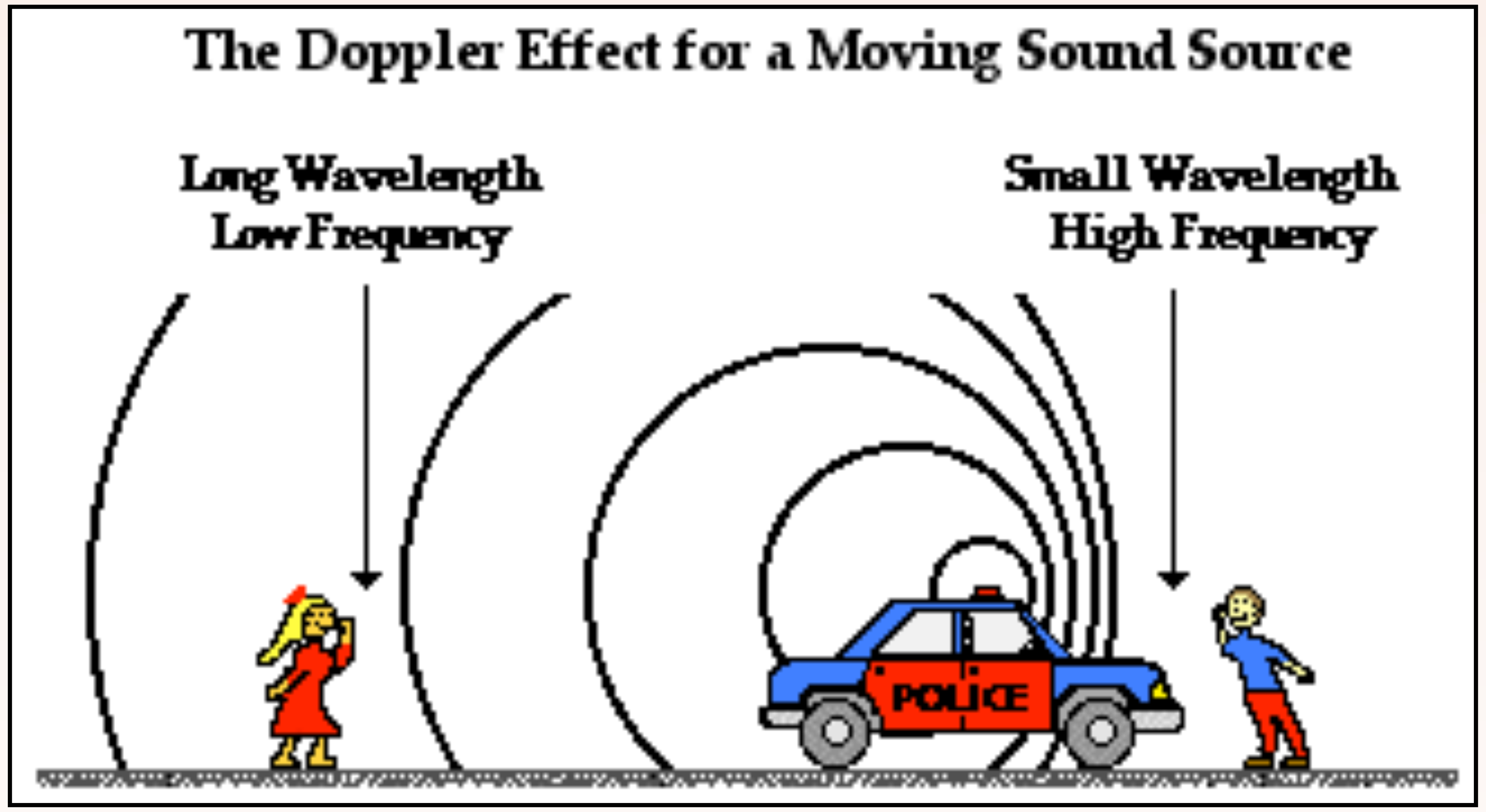
Content

- This < 1 hour talk will mainly focus on the basics of **OTFS (orthogoal time frequency spacing)**, and an inevitable wider concept: “**delay-Dopper communication/(de)modulation**”
  - Including: motivations, some derivations, the mathematical expressions, possibly with some examples.
- I tried to make this talk (and the slides as well) as simple & self-contained as they should be. So we can all learn something. Hopefully to develop some understanding.
- Also, as a humble learner on all the following topics, who knows little on this field. I have limited knowledge of the drawbacks and challenges associated with OTFS modulation.

- Specifically, we want to cover (not in the sequel):
  - **Motivation of OTFS (focus)**
  - **Must-know equations and expressions of OTFS (focus)**
  - **How to do channel estimation and message detection in DD domain (focus).**
  - Comparison between OTFS and OFDM
  - Some wonderful resources to learn OTFS
  - Code implementation for OTFS Tx-Rx and Mod/Demod (Toy example demo)

Delay and "Doppler" (OTFS modulation happens in the "delay-Doppler domain"/DD domain)

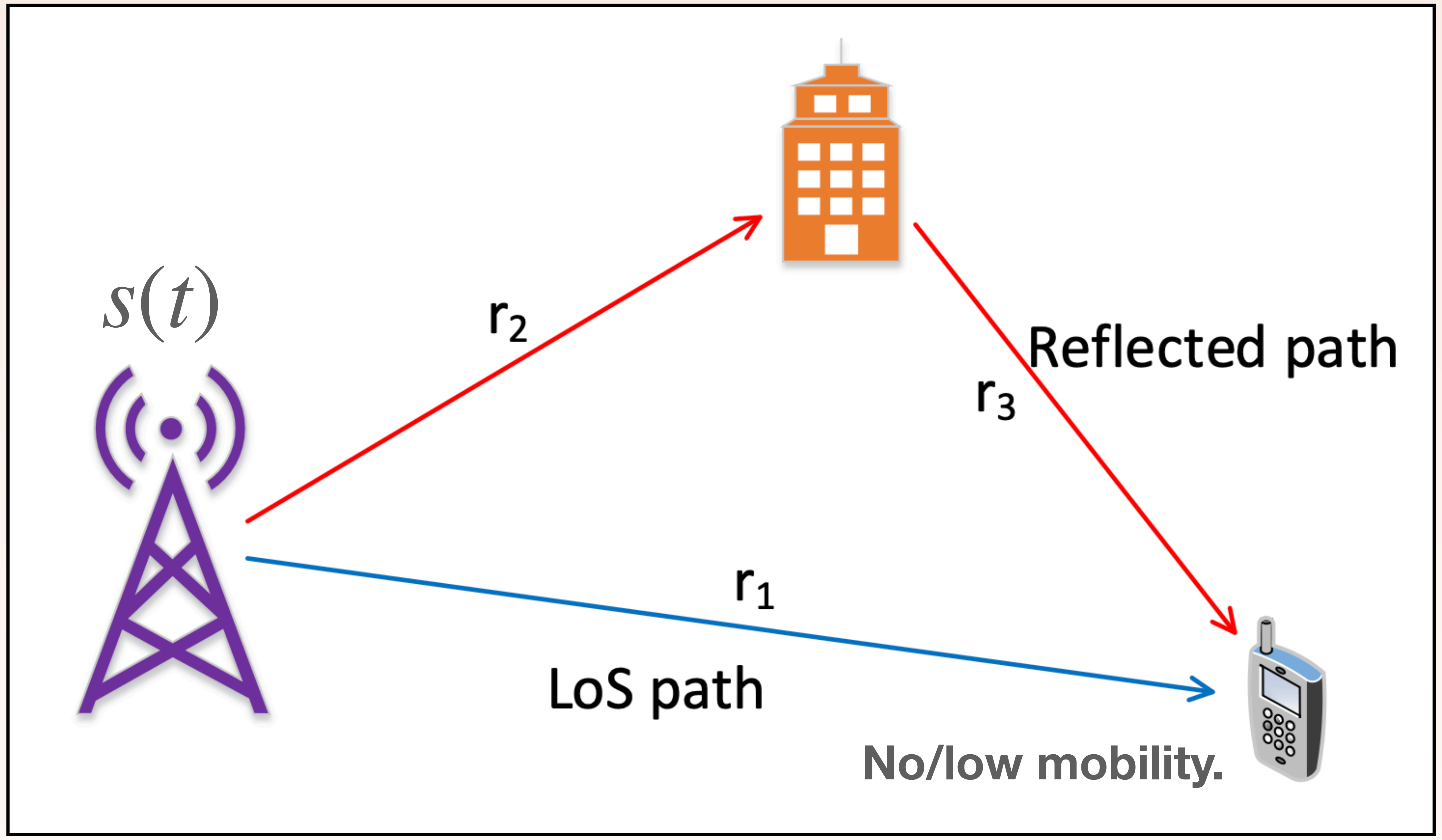
- What is delay? —A meeting is planned to start at 4pm, but eventually started at 5pm. This is called **delay**.
- In some sense, "delay" means the amount of inconsistency / shift in the time domain.



C. Doppler (1803 - 1853)

- What is "Doppler" ("Doppler shift")?
  - — Doppler is a person. :)
  - Something should be f_c Hz, but received as $f_c + f_d$ Hz. This additional amount f_d is termed as "**Doppler**".
  - *" A Doppler shift by f_d Hz is equivalent to modulating/upconverting the transmitted signal with $e^{j2\pi f_d t}$ "*.
  - "Doppler" can be interpreted as: the amount of inconsistency/ shift / difference in the frequency domain.
  - **What brings Doppler? —— Mobility!**

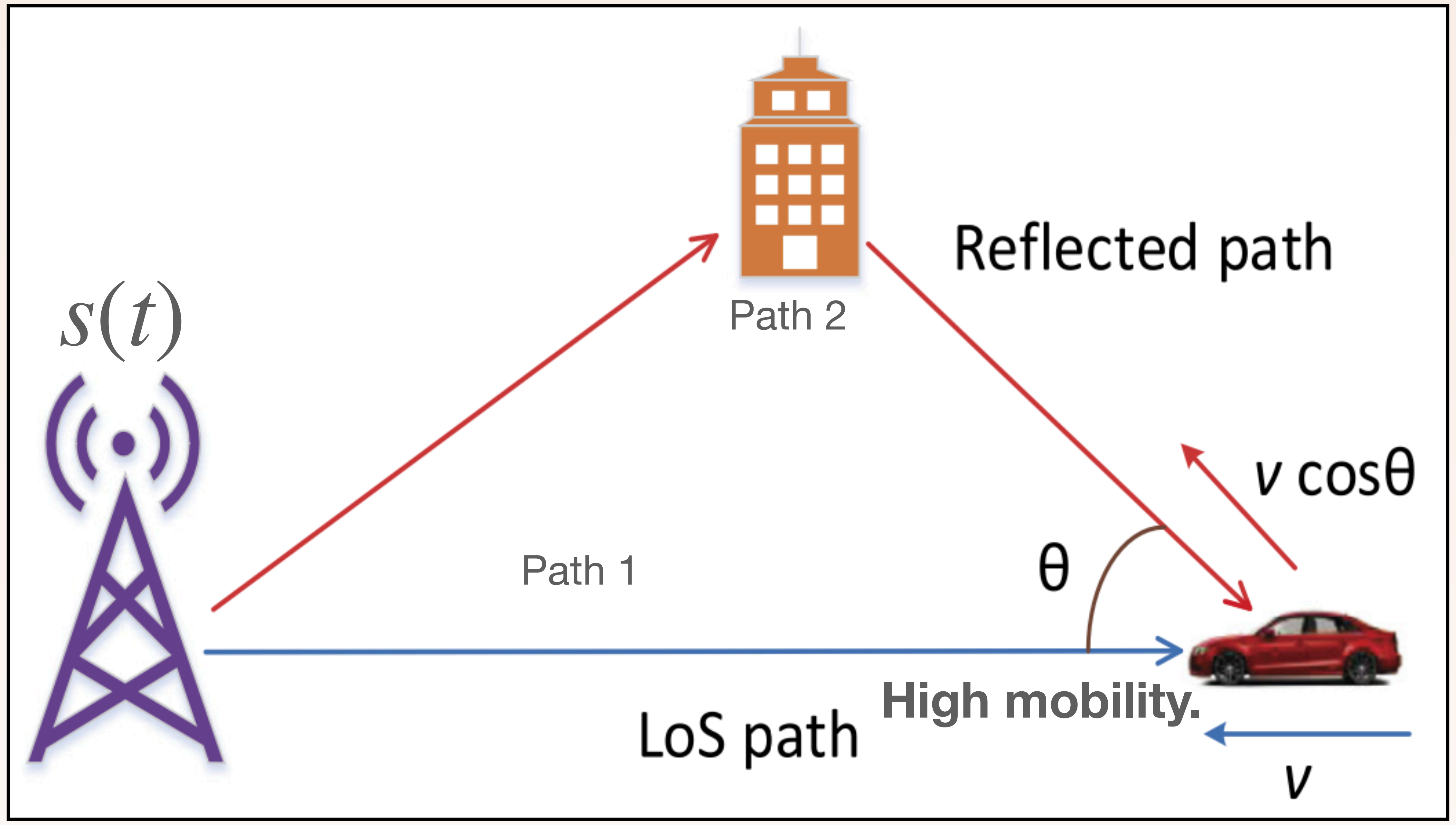
Low-mobility wireless channels



- Delay of LoS path: $\tau_1 = r_1/c$
- Delay of reflected path: $\tau_2 = (r_2 + r_3)/c$
- Delay spread: $\tau_2 - \tau_1$

- Suppose the signal sent by the BS is $s(t)$. Then what is received at the user (the phone) can be expressed as:
- $$r(t) = g_1 s(t - \tau_1) + g_2 s(t - \tau_2),$$
- where g_1, g_2 are (baseband equivalent) complex channel gains. And $c = 3 \cdot 10^8 m/s$ is the speed of light.

High-mobility wireless channels



- Original signal: $s(t)$
- On path $i, i = 1{,}2$ considering the Doppler shift (an additional subcarrier): $s(t)e^{j2\pi\nu_i t}$
- In this example: $\nu_1 = \dfrac{v}{c} f_c$. And $\nu_2 = \dfrac{v\cos\theta}{c} f_c$.
- Consider delay: path i contribute $s(t-\tau_i)e^{j2\pi\nu_i(t-\tau_i)}$

- Hence, finally at the receiver, it receives (no noise yet)
- $$r(t) = g_1 e^{j2\pi\nu_1(t-\tau_1)} s(t-\tau_1) + g_2 e^{j2\pi\nu_2(t-\tau_2)} s(t-\tau_2)$$
- Note that, if there is no mobility $\Longrightarrow$ No Doppler $\Longleftrightarrow$ $\nu_i = 0 \Longrightarrow$
- $g_1 s(t-\tau_1) + g_2 s(t-\tau_2)$ (What we wrote on the last slide)

	$f_c = 2$ GHz	$f_c = 60$ GHz
$\upsilon = 1.5$ m/s $= 5.5$ km/h	$\nu_{\max} = 10$ Hz	$\nu_{\max} = 300$ Hz
$\upsilon = 3$ m/s $= 11$ km/h	$\nu_{\max} = 20$ Hz	$\nu_{\max} = 600$ Hz
$\upsilon = 30$ m/s $= 110$ km/h	$\nu_{\max} = 200$ Hz	$\nu_{\max} = 6$ kHz
$\upsilon = 150$ m/s $= 550$ km/h	$\nu_{\max} = 1$ kHz	$\nu_{\max} = 30$ kHz

Doppler spread for some typical wireless channels.

Let take the last row as an example:

$$\nu_{\max} = \frac{v}{c} f_c = \frac{150}{3\times10^8} 2\times10^9\text{Hz} = 10^3\text{Hz} = 1\text{kHz}$$

Why OFDM is not perfect in high-mobility scenarios

- OFDM stacks its info in frequency domain: $x \in \mathbb{C}^{M\times 1}$. After a $M-$point FFT, it becomes the time domian OFDM (symbol) vector $\tilde{x} = \mathbf{F}_M^{\dagger} x \in \mathbb{C}^{M\times 1}$. (For rectangular pulse-shaping waveforms), we equip the CP part as:

- $$s_{CP} = [\tilde{x}[M - l_{\max}], \ldots, \tilde{x}[M-1]]^T, \text{ and obtain } \underbrace{s' = [s_{CP}^T, \tilde{x}^T]}_{\text{To transmit, length is } M + l_{\max}}.$$

- Suppose the channel is **static**, $(\nu_i = 0)$, then: $h(t) = \sum_{i=1}^{P} g_i \delta(t - \tau_i)$. Discrete-time equivalent channel vector: $h' \triangleq [h_0, \ldots, h_{l_{\max}}]^T$. $h_l = g_i$ if $l = l_i, \forall i = 1,\ldots,P$

- The equivalent baseband signal (including CP) is given by: (In time domain, convolution happens)

- $$\underbrace{r' = h' * s' + w'}_{\text{To receive}} \in \mathbb{C}^{M + l_{\max}}.$$

- Take away the CP, we have $r = \underbrace{[r'[l_{\max}], \ldots, r'[M + l_{\max} - 1]]^T = h \circledast \tilde{x} + w}_{\text{Removed CP, length is } M}$. $h = [h_0, \ldots, h_{\max}, 0, \ldots, 0]^T \in \mathbb{C}^M$.

- Equivalently, we can express $r = \mathbf{H}\tilde{x} + w$ for some appropriate matrix $\mathbf{H}$.

$$\mathbf{H} = \underbrace{\begin{bmatrix} h_0 & 0 & \cdots & 0 & h_{l_{\max}} & h_{l_{\max}-1} & \cdots & h_1 \\ h_1 & h_0 & \cdots & 0 & 0 & h_{l_{\max}} & \cdots & h_2 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ h_{l_{\max}-1} & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & h_{l_{\max}} \\ h_{l_{\max}} & h_{l_{\max}-1} & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & h_{l_{\max}} & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & h_0 & 0 \\ 0 & 0 & \cdots & h_{l_{\max}} & h_{l_{\max}-1} & \cdots & h_1 & h_0 \end{bmatrix}}_{M\times M \text{ Circulant matrix}},$$

$\mathbf{H}$ is a circulant matrix.

Why OFDM is not perfect in high-mobility scenarios

- Equivalently, we can express $r = \mathbf{H}\tilde{x} + w$

$$\mathbf{H} = \sum_{i=1}^{P} g_i \Pi^{l_i}\, \mathbf{D}^{k_i}, \qquad \mathbf{D} \triangleq \mathrm{diag}(1, \omega, \ldots, \omega^{M-1})$$

$$\mathbf{H} =$$

$$\mathbf{H} = \sum_{i=1}^{P} g_i \Pi^{l_i} \quad \underbrace{\begin{bmatrix} h_0 & 0 & \cdots & 0 & h_{l_{\max}} & h_{l_{\max}-1} & \cdots & h_1 \\ h_1 & h_0 & \cdots & 0 & 0 & h_{l_{\max}} & \cdots & h_2 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ h_{l_{\max}-1} & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & h_{l_{\max}} \\ h_{l_{\max}} & h_{l_{\max}-1} & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & h_{l_{\max}} & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & h_0 & 0 \\ 0 & 0 & \cdots & h_{l_{\max}} & h_{l_{\max}-1} & \cdots & h_1 & h_0 \end{bmatrix}}_{M \times M \text{ Circulant matrix}},$$

Static channel (we saw on the last page)

- Note that on the LHS, **H** *is a circulant matrix: which can be diagonalized via Fourier transform matrices pairs* (DFT matrices form a orthonormal basis of circulant matrices)

- $\mathbf{H} = \mathbf{F}_M^{\dagger} \mathrm{diag}(\mathbf{F}_M h) \mathbf{F}_M.$

$$\Pi = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}$$

- $y = \mathbf{F}_M r = \hat{h} \odot \tilde{x} + \hat{w}$

- Where $\odot$ denotes element-wise product. Such orthogonality enables us to use **single-tap equalizer** to estimate $\tilde{x}$ **entry-by-entry.**

- How to understand this? —— Each path $i \in [P]$. The signal is first "modulated" by a carrier k_i. Then shifted by Π^{l_i}. Then they composite together to make the received r, with noise.

$$= \begin{bmatrix} h_0 & 0 & \cdots & h_{l_{\max}}\omega^{k_P(M-l_{\max}))} & \cdots & h_1\omega^{k_1(M-1)} \\ \vdots & h_0\omega^{k_0} & \cdots & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & h_{l_{\max}}\omega^{k_P(M-1)} \\ h_{l_{\max}} & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & h_{l_{\max}}\omega^{k_P} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & h_{l_{\max}}\omega^{k_P(M-l_{\max}-1)} & \cdots & \cdots & h_0\omega^{k_0(M-1)} \end{bmatrix}$$

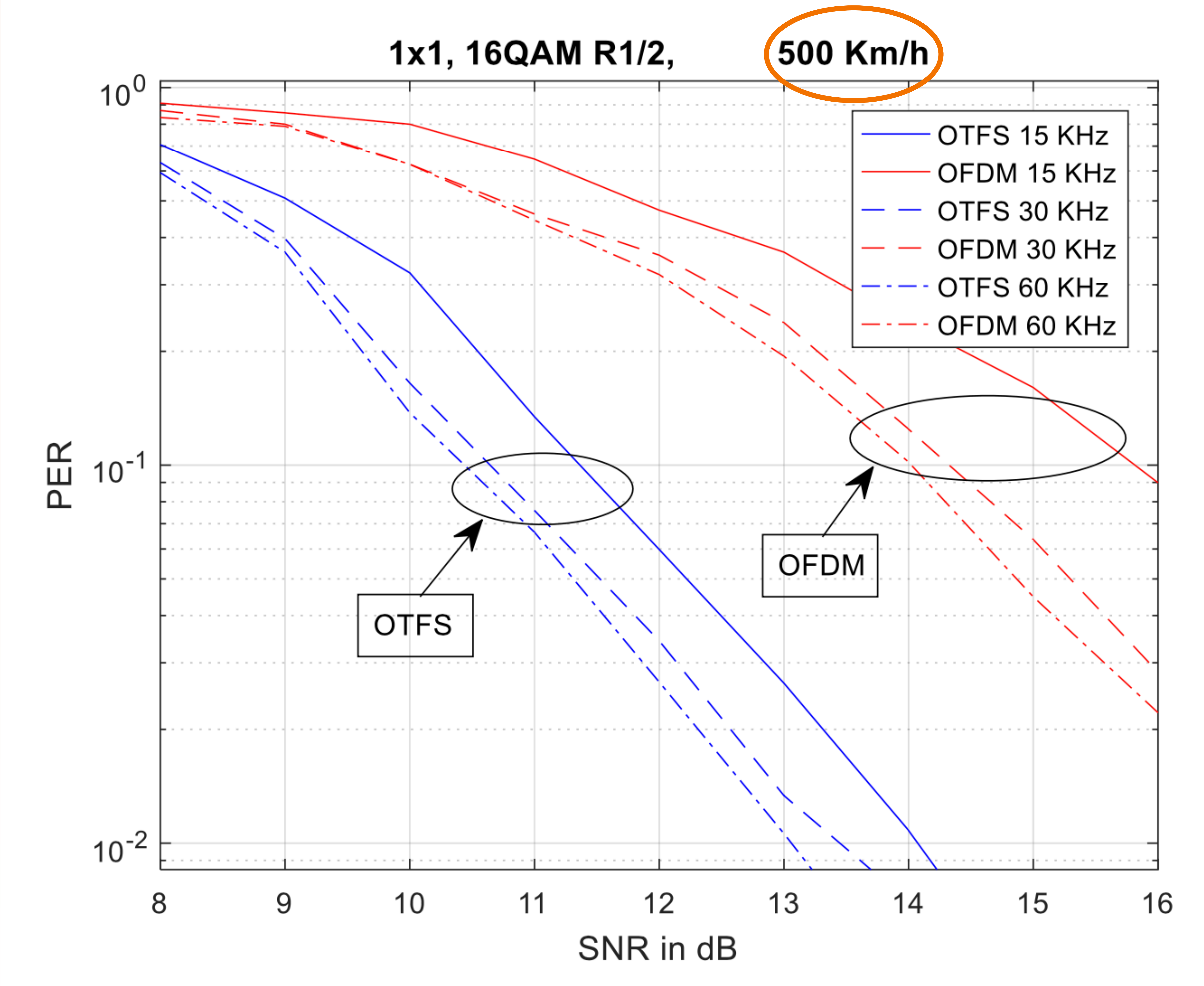
High-mobility channel

- However, on the RHS, there are some additional $\omega^{\cdots}$ factors, caused by Doppler shifts, makes this matrix no longer circulant. $\Longrightarrow$ cannot be eigen-decomposited via FFT matrices!!! (Where $\omega = e^{\frac{j2\pi}{M}}$)

  - Continuous case: $s(t) \to s(t)e^{j2\pi f_m t}$. Discrete case: $s[m] \to s[m]e^{j2\pi k_i m}$

- $\Longrightarrow$ Gives us performance degradation. If we do LHS decoding anyway $\Longrightarrow$ need to endure some ICI. If those $k_m = 0$, then RHS $\rightarrow$LHS.

With Dopper, OTFS outperforms OFDM



This plot should be self-explanatory.

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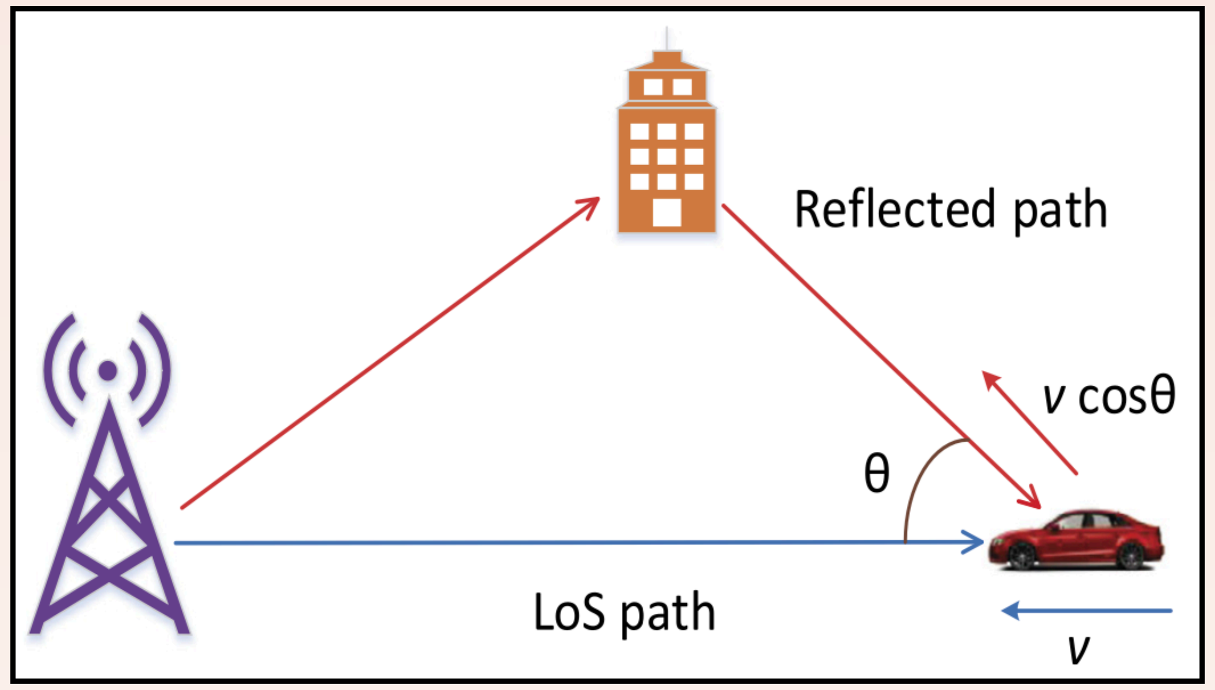
High-mobility wireless channels



$g_1 e^{j2\pi\nu_1(t-\tau_1)} s(t-\tau_1) + g_2 e^{j2\pi\nu_2(t-\tau_2)} s(t-\tau_2)$

- The 2-path result can be generalized for P paths as the following:
- $$r(t) = \sum_{i=1}^{P} g_i e^{j2\pi\nu_i(t-\tau_i)} s(t-\tau_i) = \sum_{i=1}^{P} g_i e^{j2\pi\nu_i t} s(t-\tau_i) e^{-j2\pi\nu_i\tau_i}$$
- We define the delay-Doppler channel representation as:
- $$h(\tau,\nu) \triangleq \sum_{i=1}^{P} g_i e^{-j2\pi\nu_i\tau_i} \delta(\tau-\tau_i)\delta(\nu-\nu_i)$$ [this function only has value on (τ_i, ν_i)]
- One can see the received signal $r(t)$ as:
- $$r(t) = \int\int h(\tau,\nu) e^{j2\pi\nu t} s(t-\tau) d\nu d\tau$$
- **The cool thing here is:** the channel $h(\tau,\nu)$ (in DD domain) is completely represented by these quantities: $(g_i, \tau_i, \nu_i), i = 1,2,\ldots,P$. The # of quantities you need is small $\Longrightarrow$ sparsity.
- We will relate the above new channel representation with something we knew.

Different (Linear time variant) channel representations

- $$h(\tau,\nu) \triangleq \sum_{i=1}^{P} g_i e^{-2\pi\nu_i\tau_i}\delta(\tau-\tau_i)\delta(\nu-\nu_i) \quad \text{(delay-Doppler)}$$
  This is the delay-time domain representation of the channel.

- If we go back one step: $r(t) = \sum_{i=1}^{P} g_i e^{j2\pi\nu_i(t-\tau_i)} s(t-\tau_i)$

- $$r(t) = \sum_{i=1}^{P} \underbrace{g_i e^{j2\pi\nu_i(t-\tau_i)}}_{g(\tau_i,t)} s(t-\tau_i) \implies \text{generalize} \int_0^\infty \underbrace{g(\tau,t)}_{\text{Delay-time domain}} s(t-\tau)d\tau.$$

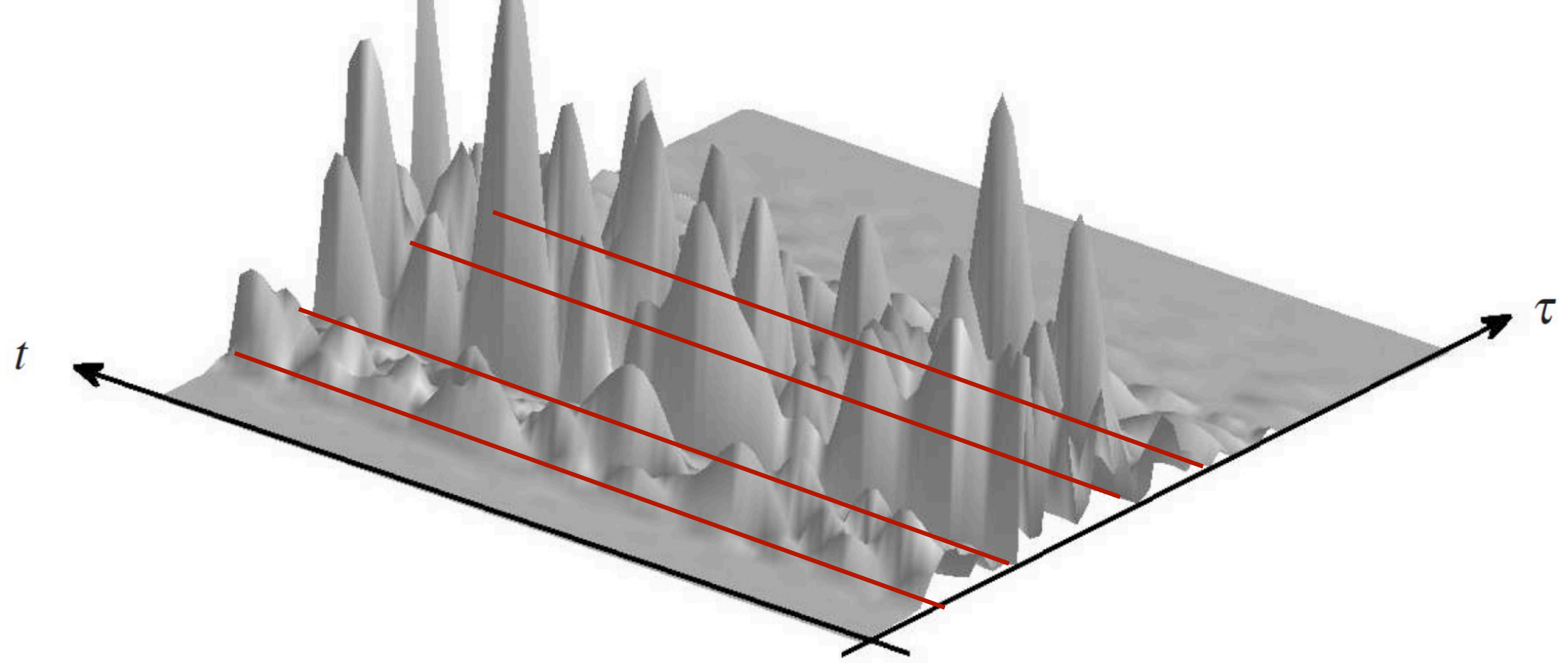
- Another immediate equivalent representation is the time-frequency representation:

- $$r(t) = \sum_{i=1}^{P} \underbrace{g_i e^{j2\pi\nu_i(t-\tau_i)}}_{g(\tau_i,t)} s(t-\tau_i) \implies \int_0^\infty g(\tau,t)s(t-\tau)d\tau = \int \underbrace{H(f,t)}_{\text{Time-frequency domain}} S(f)e^{j2\pi ft}df$$

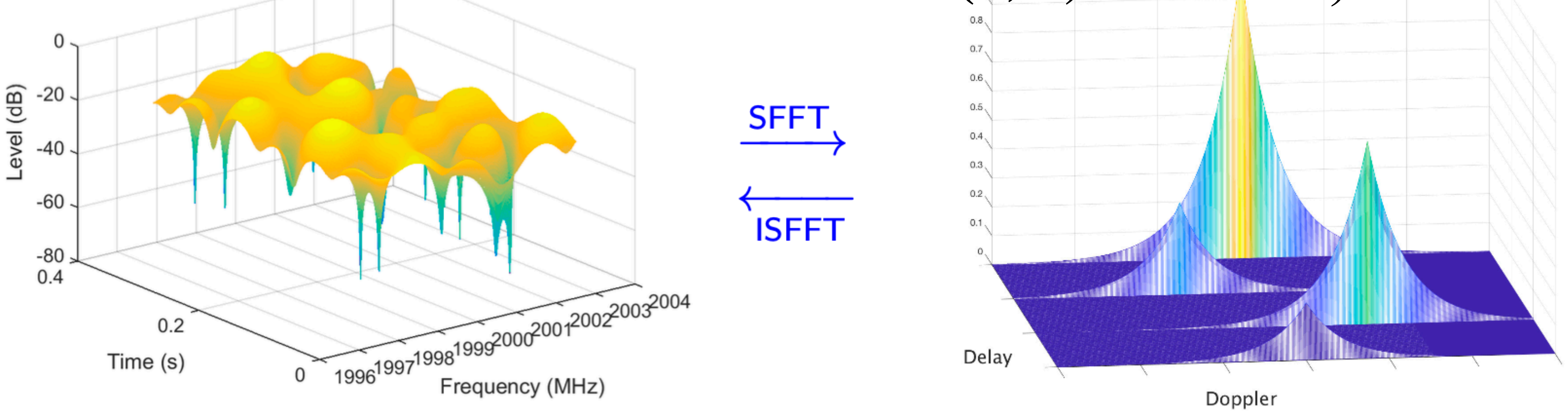
- The last step can be understood as consider product in frequency domain, then use IFT back to time domain.

Time-variant impulse response $g(t,\tau)$

- $g(\tau, t)$ be like :(

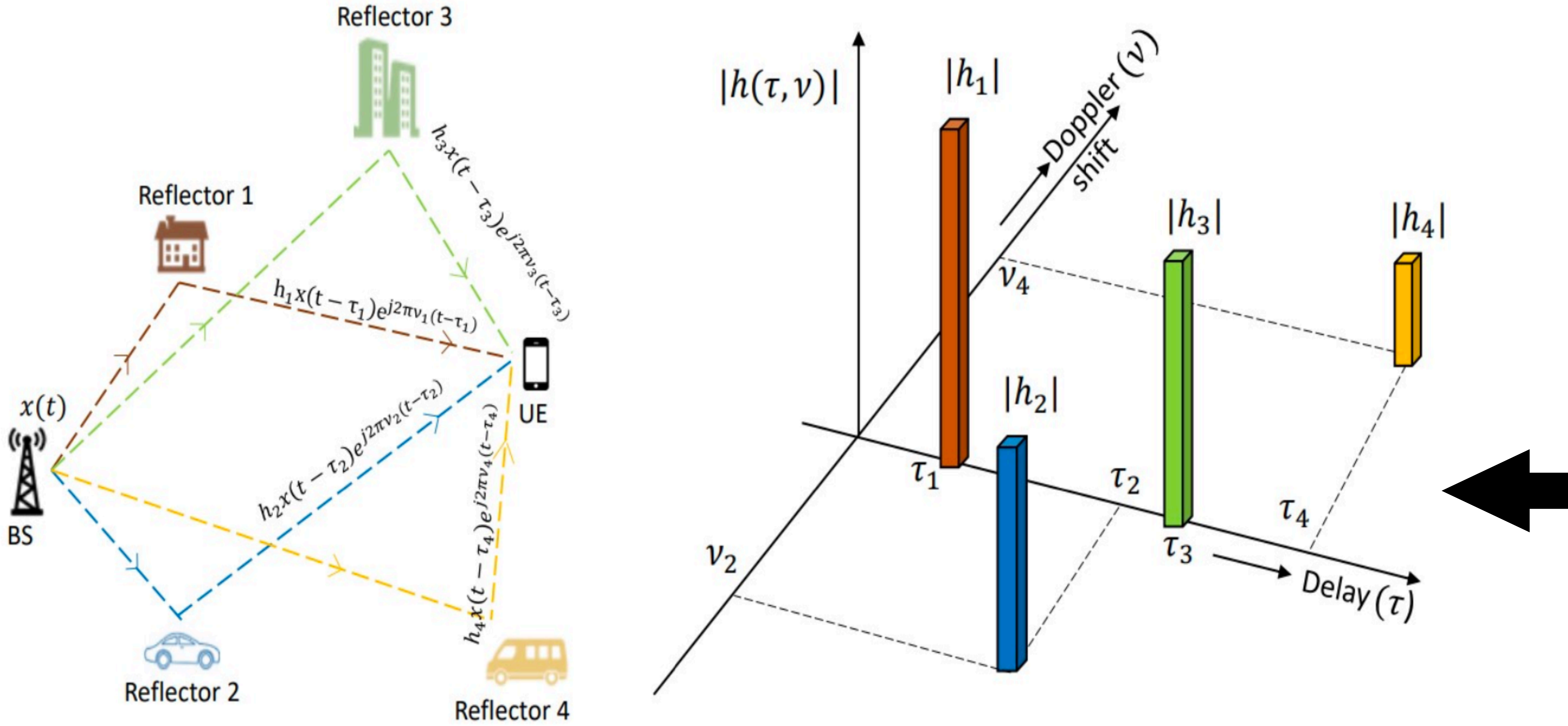


- $h(\tau, \nu)$ be like :)



- $H(t, f)$ be like :(

Channel in Time–frequency $H(t,f)$ and delay–Doppler $h(\tau,\nu)$

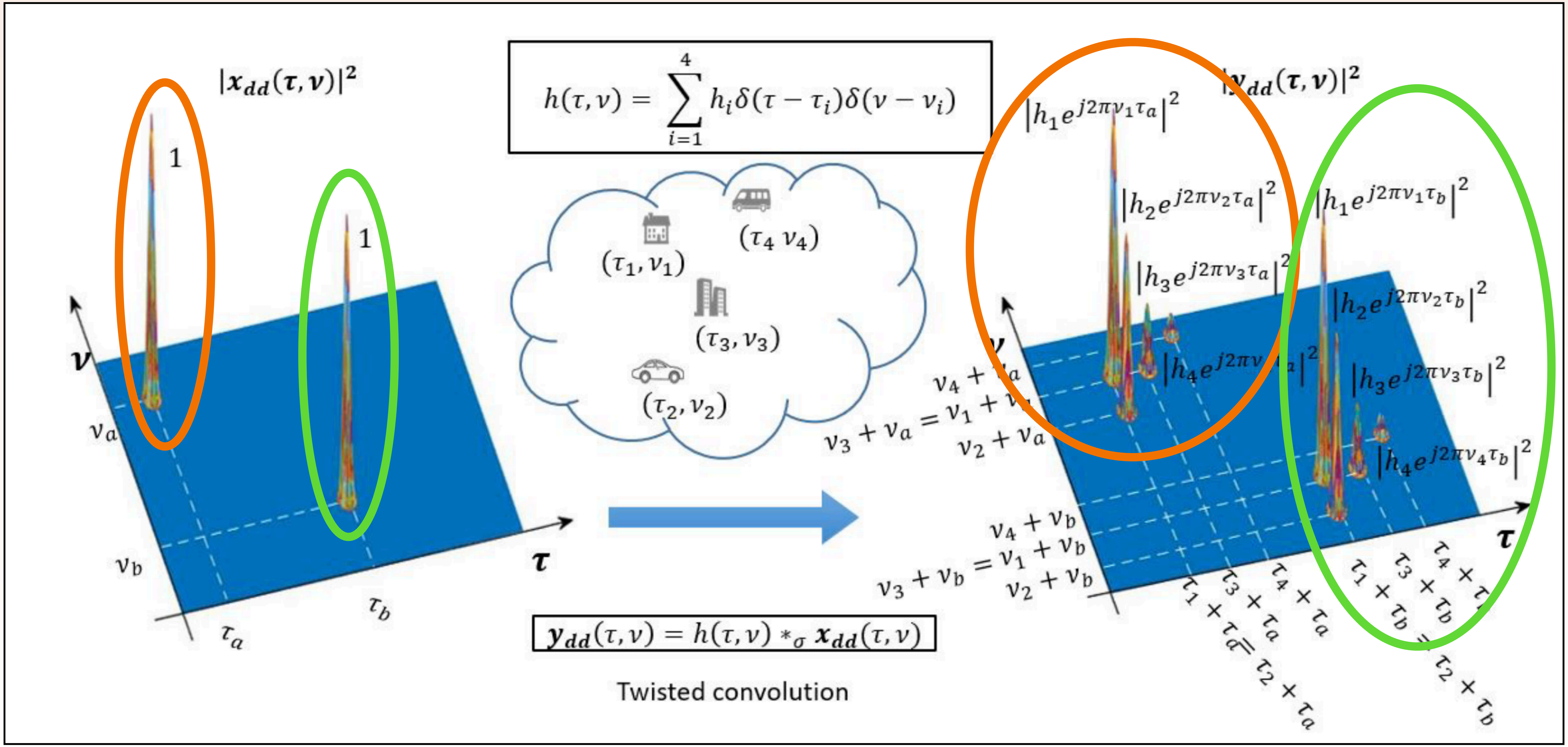


- DD domain notation sparsify the channel representation!!

- The channel representation should be simple as this!!!

TX and RX signal in *delay-Dopper domain*

- OK. But does the DD domain stuff ever make our lives easier?? —— Probabily, yes.



- We will come back to this later. Now just get some intuitive feeling of how signals will interact with the channel in DD domain.

Parameters and their choice in OTFS

- In an OTFS system, we need to specify a bunch of parameters.
- We assume the OTFS system operates on a P-path high mobility channel, with a bandwidth B.
- Maximum delay spread $\tau_{\max}$, maximum Doppler shift $\nu_{\max}$.
- Sampling frequency $f_s = B = \dfrac{1}{T_s}$, where T_s is referred as "sampling interval".
- Each OTFS frame contains NM samples divided into N time slots. M samples per block.
- OTFS frame duration $T_f = NMT_s = NT$. $T = MT_S$ denotes the duration of each block.
- $\Delta f = \dfrac{1}{T}$. This is the "space of spectrum samples".

OTFS modulation

- In OFDM, we store our information in frequency domain vectors. In OTFS, we place them in **"delay-Doppler"** domain matrix $\mathbf{X} \in \mathbb{C}^{M\times N}$, with entries $\mathbf{X}[m,n], m \in [M], n \in [N]$.

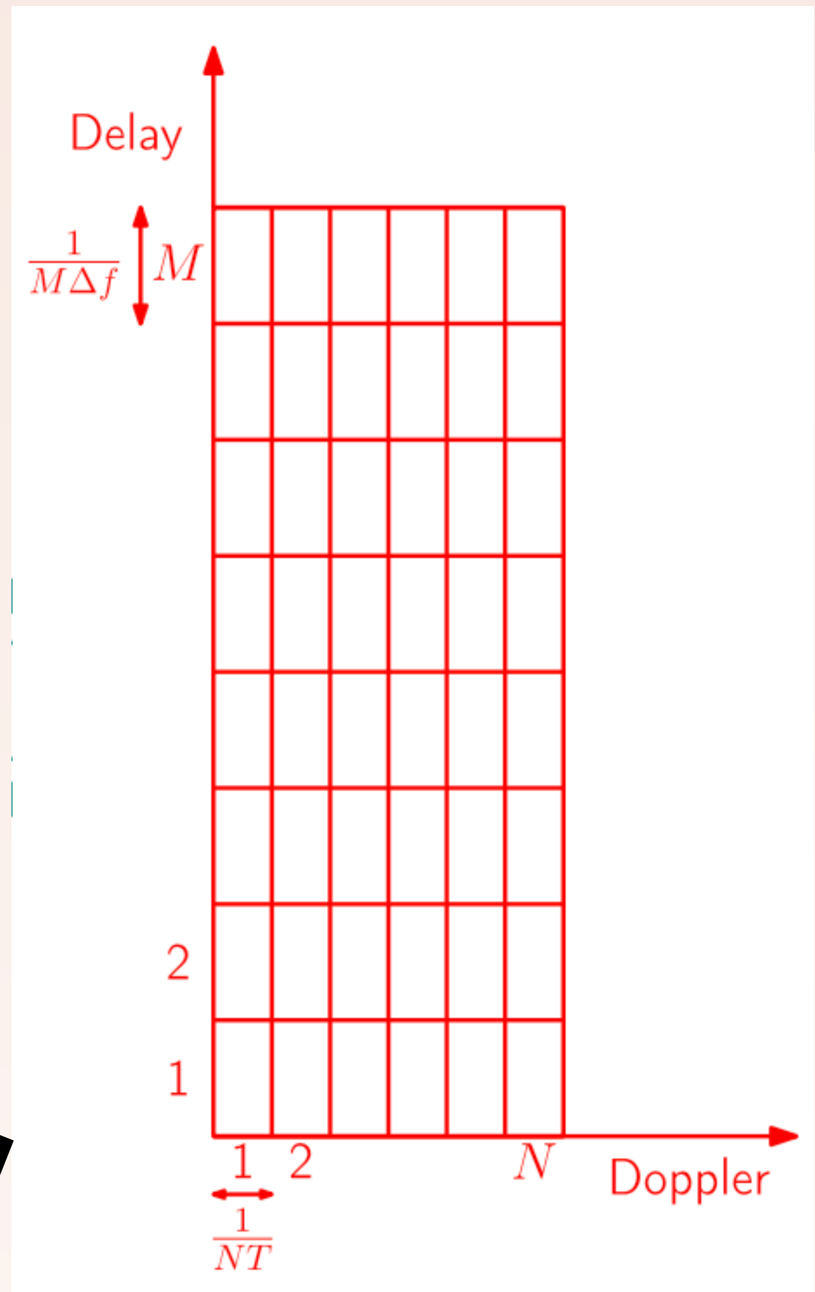
- We first convert $\mathbf{X}$ into frequency-time domain $\mathbf{X}_{ft} \in \mathbb{C}^{M\times N}$ by applying (N-pt) IFFT along the rows of $\mathbf{X}$, then apply (M-pt) FFT on columns of it. (Can we swap the order? —— yes) Put it in math, we have:

- $$\mathbf{X}_{ft} = \frac{1}{\sqrt{MN}} \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} \mathbf{X}[m,n] e^{j2\pi(\frac{nk}{N}-\frac{ml}{M})} = \mathbf{F}_M \underbrace{\mathbf{X}}_{M\times N} \mathbf{F}_N^H$$ (this is called "ISFFT" inverse symplectic FFT)

- Then to convert the 2D $\mathbf{X}_{ft}$ to a time domain signal, we do the following:

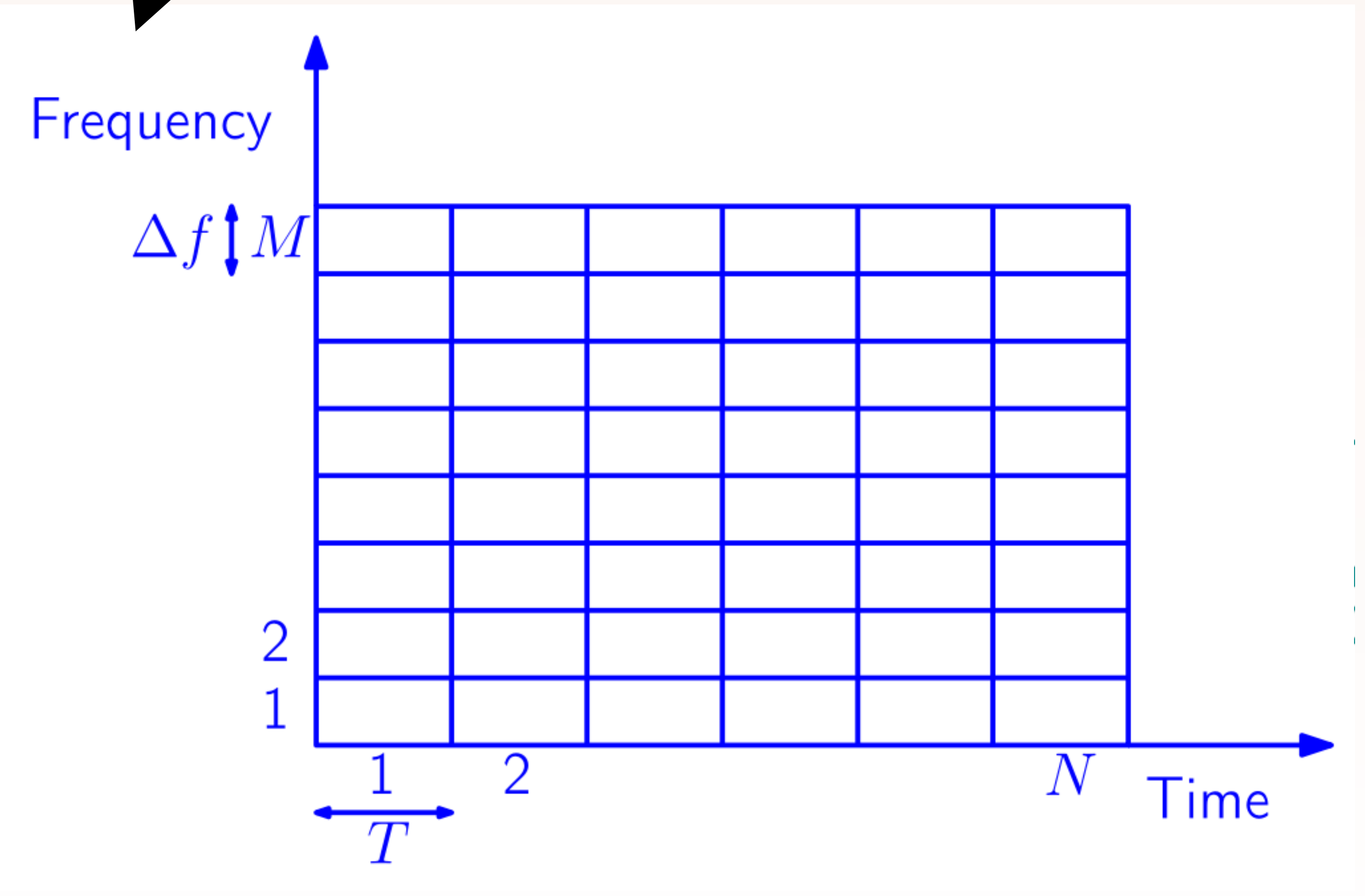
- $$s(t) = \sum_{k=0}^{N-1}\sum_{l=0}^{M-1} \mathbf{X}_{ft}[l,k] \underbrace{g_{tx}(t-kT)}_{\text{Some transmit waveform}} e^{j2\pi l\Delta f(t-kT)}$$ (this is called Heisenberg transform)

- The above $s(t)$ is ready to transmit.



delay-Doppler grid, where $\mathbf{X}$ lives.

ISFFT



Frequency-time grid, where $\mathbf{X}_{ft}$ lives in.

OTFS: channel (revisited)

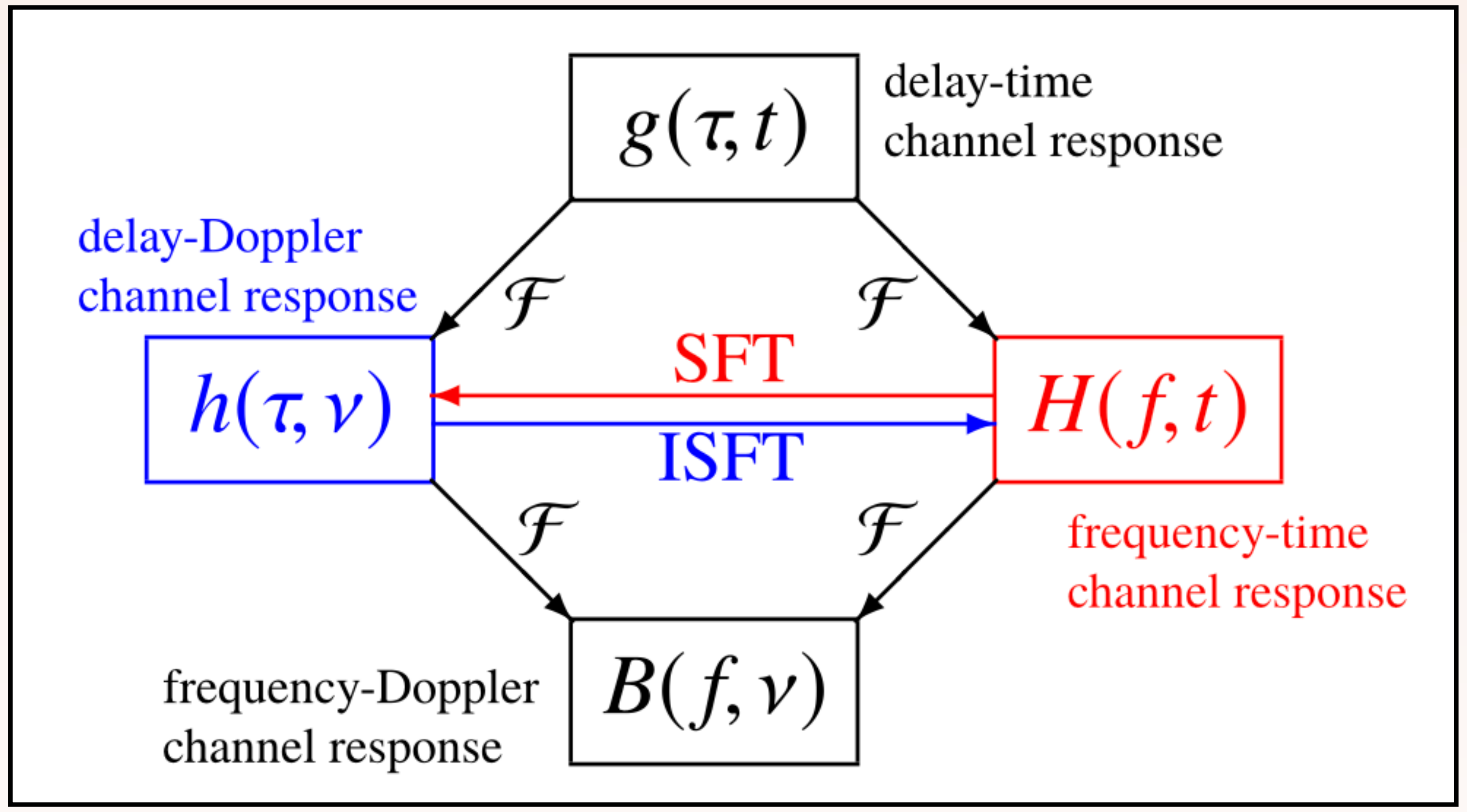
- $$s(t) = \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} \mathbf{X}_{ft}[l, k] g_{tx}(t - kT) e^{j2\pi l \Delta f (t-kT)}$$ (what to transmit)

- Imparied by the channel, we can expressed the received signal as the following:

- $$r(t) = \int\int h(\tau, \nu) e^{j2\pi\nu t} s(t - \tau) d\nu d\tau$$ *(delay-Doppler channel).*

- $$= \int \underbrace{H(t, f)}_{\text{time-frequency response}} S(f) e^{j2\pi f t} df$$ *(Time-frequency channel).*

- $$= \int g(t, \tau) s(t - \tau) d\tau$$ *[delay - time expression, should be the most intuitive]*



Relations betwee different representations of LTV channels

OTFS: demodulation

- $s(t) = \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} \mathbf{X}_{ft}[l,k] g_{tx}(t-kT) e^{j2\pi l \Delta f (t-kT)}, \; r(t) = \int\int h(\tau,\nu) e^{j2\pi\nu t} s(t-\tau) d\nu d\tau$ (other expressions available)
- With a matched filter, we process like the following:

Explanation needed

- $$Y(f,t) = A_{g_{rx},r}(f,t) \triangleq \int g_{rx}^*(t'-t) r(t') e^{-j2\pi f(t'-t)} dt'$$ (This is called "Wigner transform")
- "Cross-ambiguity function" between g_1, g_2 is defined as $A_{g_1,g_2}(f,t) \triangleq \int g_1(t') g_2^*(t'-t) e^{-j2\pi f(t'-t)} dt'$.
- Intuition? $A_{g_1,g_2}(f,t)$ defines the correlation between $g_1(t)$ and a delayed-by-t & frequency shifted by f version of $g_2(t)$.
- $$\mathbf{Y}_{ft}[l,k] = Y(f,t)\,|_{f=l\Delta f, t=kT} \in \mathbb{C}^{M\times N}$$ This lives in frequency-time domain
- Finally, we use the SFFT to process $\mathbf{Y}_{ft}[l,k]$, leading to a matrix $\mathbf{Y}$:
- $$\mathbf{Y}[m,n] = \frac{1}{\sqrt{NM}} \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} \mathbf{Y}_{ft}[l,k] e^{-j2\pi(\frac{nk}{N} - \frac{ml}{M})}$$ This lives in DD domain
- *This finishes the "standard procedure" of OTFS modulation & demodulation.* (But understand little ☹️)

Delay-Doppler domain analysis

- The delay-Doppler transmitted and received samples matrices are related to those in the frequency-time domian by SFFT: $\mathbf{Y} = \mathrm{SFFT}(\mathbf{Y}_{ft}), \quad \mathbf{X} = \mathrm{SFFT}(\mathbf{X}_{ft}).$

- Also, the delay-Doppler domain channel matrix (of size $M \times N$) is related to the frequency-time channel matrix as: $\mathbf{H}_{dd} = \mathrm{SFFT}(\mathbf{H}_{ft}).$

- *(In frequency-time domain, ideally)* $\mathbf{Y}_{ft}[l,k] = \mathbf{H}_{ft}[l,k]\mathbf{X}_{ft}[l,k]\,.$ ⬅️ Omit/fix the k, nothing but $Y(f) = H(f)S(f)$

- Then we can actually derive:

$$\begin{aligned}\mathbf{Y}[m,n] &= \mathbf{H}_{dd}[m,n] \circledast \mathbf{X}[m,n] \\ &= \sum_{m'} \sum_{n'} \mathbf{H}_{dd}[m',n']\mathbf{X}[[m-m']_M, [n-n']_N] \\ &= \sum_{i=1}^{P} g_i \mathbf{X}[[m-l_i]_M, [n-k_i]_N]\end{aligned}$$

- Lemma: $\mathbf{Z} = \mathbf{U} \circledast \mathbf{V}$ can be equivalently written as $\mathrm{vec}(\mathbf{Z}) = \mathbf{B}_{\mathrm{U}} \cdot \mathrm{vec}(\mathbf{V})$, where $\mathbf{B}_{\mathrm{U}} = \mathrm{circ}[\mathrm{circ}(U_1), \ldots, \mathrm{circ}(U_N)]$.

$$\mathrm{circ}(a_0, \ldots, a_{N-1}) = \begin{pmatrix} a_0 & a_{N-1} & \cdots & & a_1 \\ a_1 & a_0 & & & a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{N-2} & & & a_0 & a_{N-1} \\ a_{N-1} & a_{N-2} & \cdots & a_1 & a_0 \end{pmatrix}$$

Delay-Doppler domain analysis

- (In frequency-time domain, ideally) $\mathbf{Y}_{ft}[l,k] = \mathbf{H}_{ft}[l,k]\mathbf{X}_{ft}[l,k]$.

⬇️ want to derive the bottom

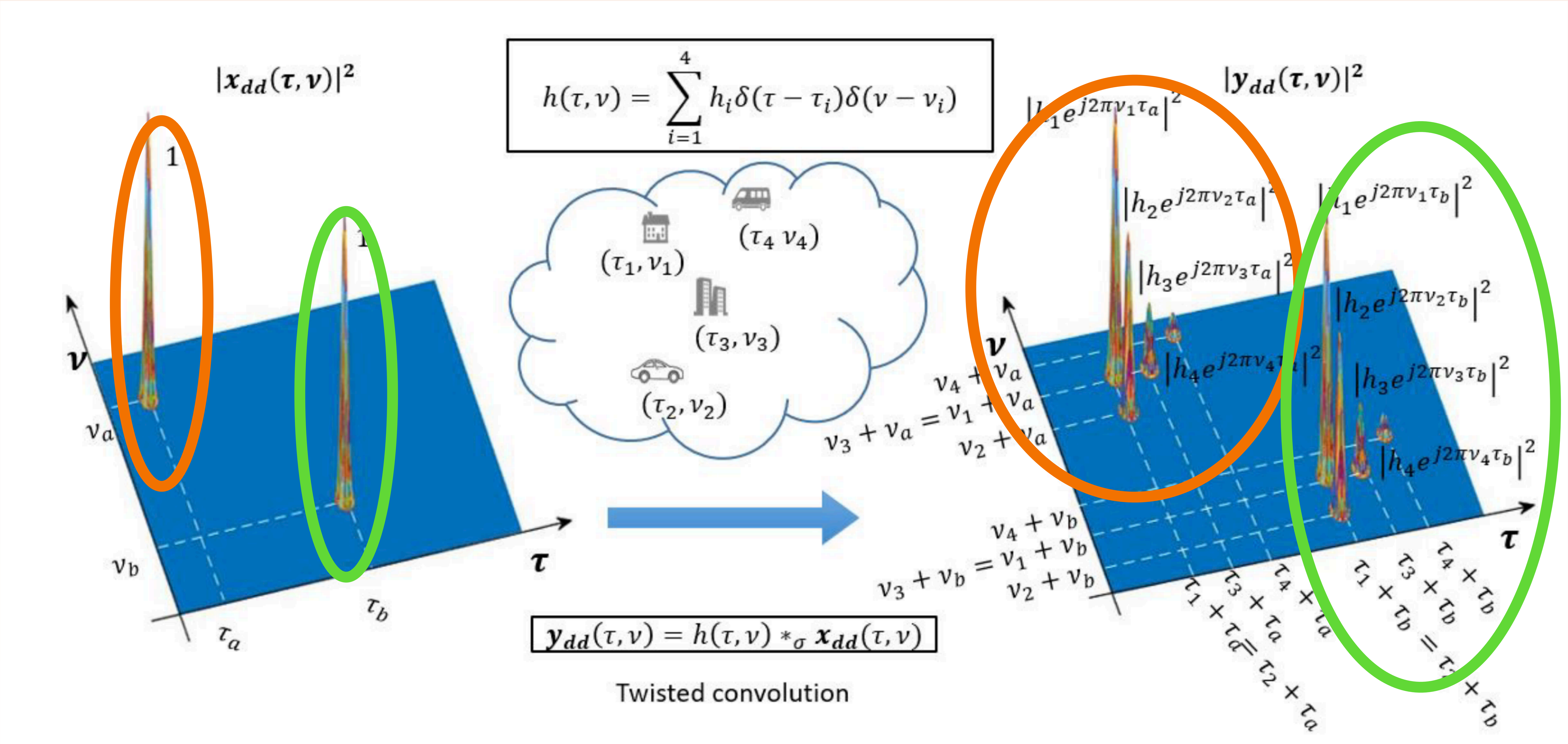
- $$\mathbf{Y}[m,n] = \mathbf{H}_{dd}[m,n] \circledast \mathbf{X}[m,n] \triangleq \sum_{m'} \sum_{n'} \mathbf{H}_{dd}[m',n']\mathbf{X}[[m-m']_M,[n-n']_N] .$$

- Proof sketch (details is quite tedious, if we do not use higher level theorems)
  - Suffice to prove $\mathrm{vec}(\mathbf{Y}) = \mathbf{B}_{\mathbf{H}_{dd}} \cdot \mathrm{vec}(\mathbf{X})$
  - $\mathbf{Y}_{ft}[l,k] = \mathbf{H}_{ft}[l,k]\mathbf{X}_{ft}[l,k] \iff \mathrm{vec}(\mathbf{Y}_{ft}) = \mathrm{vec}(\mathbf{H}_{ft}) \odot \mathrm{vec}(\mathbf{X}_{ft})$ $\quad\odot$ *element-wise / Hadarmard product*
  - $$\iff (\mathbf{F}_N \otimes \mathbf{F}_M^\dagger)^{-1}\underbrace{(\mathbf{F}_N \otimes \mathbf{F}_M^\dagger)\mathrm{vec}(\mathbf{Y}_{ft})}_{=\mathrm{vec}(\mathbf{Y})} = \mathrm{diag}(\mathrm{vec}(\mathbf{H}_{ft}))(\mathbf{F}_N \otimes \mathbf{F}_M^\dagger)^{-1}\underbrace{(\mathbf{F}_N \otimes \mathbf{F}_M^\dagger)\mathrm{vec}(\mathbf{X}_{ft})}_{=\mathrm{vec}(\mathbf{X})}$$

    $\otimes$ Kronecker product
  - $$\iff \mathrm{vec}(\mathbf{Y}) = \underbrace{(\mathbf{F}_N \otimes \mathbf{F}_M^\dagger)\ \mathrm{diag}(\mathrm{vec}(\mathbf{H}_{ft}))(\mathbf{F}_N \otimes \mathbf{F}_M^\dagger)^{-1}}_{\text{This, surprisingly, equals } \mathbf{B}_{\mathbf{H}_{dd}}}\ \mathrm{vec}(\mathbf{X})$$

TX and RX signal in DD domain (revisited)

$\mathbf{X}[m,n]$



$\mathbf{Y}[m,n]$

$$\mathbf{Y}[m,n] = \mathbf{H}_{dd}[m,n] \circledast \mathbf{X}[m,n]$$

OTFS: modulation & demodulation summary

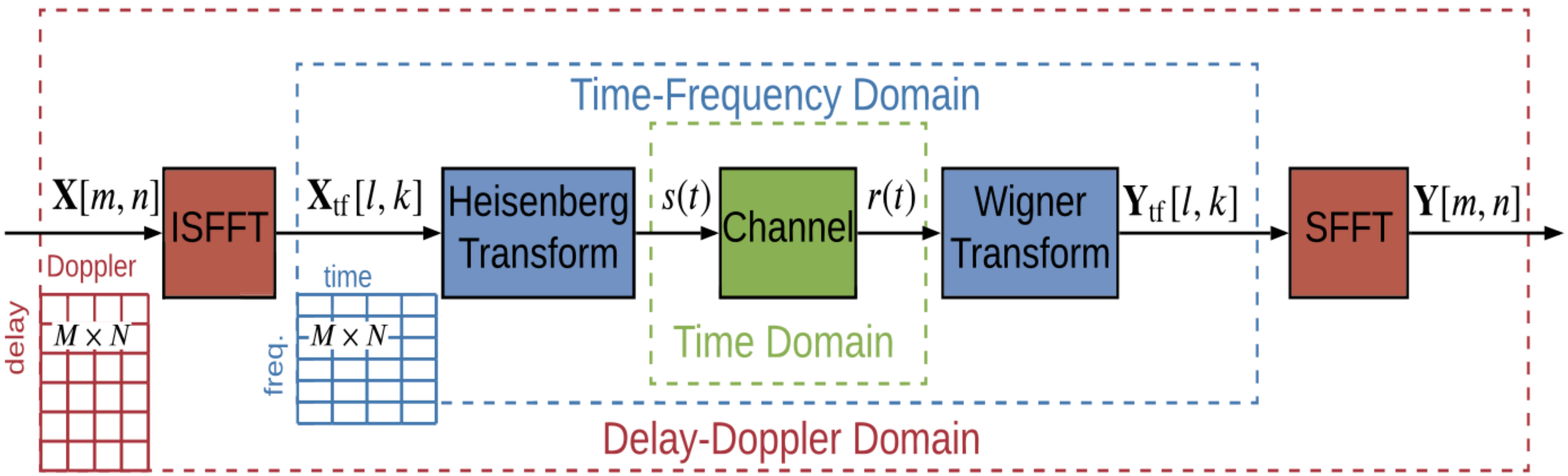


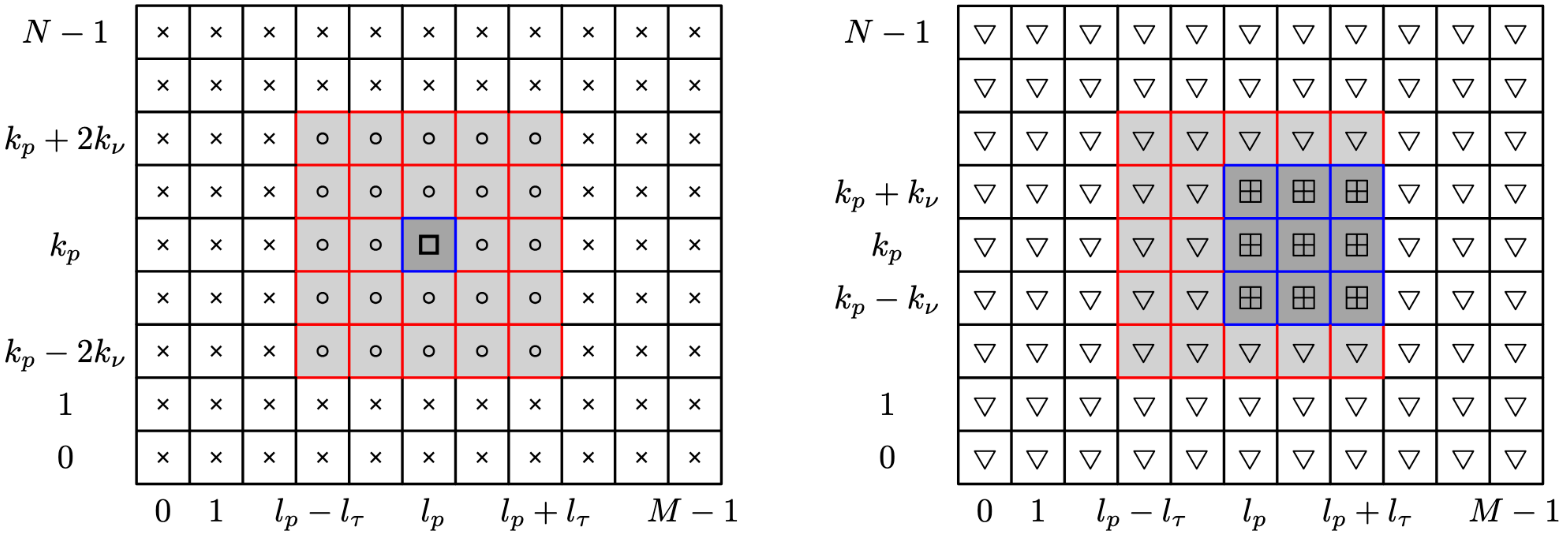
FIGURE 4.1 OTFS system diagram in its original form.

- As shown above, the OTFS modulator maps $\mathbf{X}[m,n]$ in the delay-Doppler domain to $\mathbf{X}_{ft}[l,k]$ in the time-frequency domain (using ISFFT). Then the Heisenberg transform is applied to $\mathbf{X}_{ft}[l,k]$ to generate time domain signal $s(t)$.

- At the receiver, $r(t)$ is transformed to the time-frequency domain by Wigner transform. Then to DD domian via SFFT.

Channel estimation (in delay-Doppler way)

- Esstimation in DD domain. (There are other domian-based channel estimation methods.)



(a) Tx symbol arrangement (□: pilot; ○: guard symbols; ×: data symbols)

Zero symbols

(b) Rx symbol pattern (▽: data detection, ⊞: channel estimation)

With channel estimation, how to decode?

- Suppose one have estimated the (delay-Doppler domain) channel matrix $\mathbf{H} \in \mathbb{C}^{NM \times NM}$, then we are facing (equivalently) a linear system with NM equations:
- $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w}$, (Comes from $\mathbf{Y} = \mathbf{H}_{dd} \circledast \mathbf{X}$, all $M \times N$ matices, using the previous lemma)
- Where where $\mathbf{y}, \mathbf{x}, \mathbf{w} \in \mathbb{C}^{NM \times 1}$ are $\mathbf{x}[k,l], \mathbf{y}[k,l], \mathbf{w}[k,l]$ in vectorized form, $k \in [N], l \in [M]$. $\mathbf{H}$ is treated as given.
- Given the sparse nature of $\mathbf{H}$ (each row sum=each column sum = P) we can solve it by message passing algorithm.

$$\widehat{\mathbf{x}} = \underset{\mathbf{x} \in \mathbb{A}^{NM \times 1}}{\arg\max} \Pr\left(\mathbf{x} \middle| \mathbf{y}, \mathbf{H}\right),$$

which has a complexity exponential in NM. Since the joint MAP detection can be intractable for practical values of N and M, we consider the symbol-by-symbol MAP detection rule for $c = 1, \ldots, NM$

$$\widehat{x}[c] = \underset{a_j \in \mathbb{A}}{\arg\max} \Pr\left(x[c] = a_j \middle| \mathbf{y}, \mathbf{H}\right) \tag{31}$$

$$= \underset{a_j \in \mathbb{A}}{\arg\max} \frac{1}{Q} \Pr\left(\mathbf{y} \middle| x[c] = a_j, \mathbf{H}\right) \tag{32}$$

$$\approx \underset{a_j \in \mathbb{A}}{\arg\max} \prod_{d \in \mathcal{J}_c} \Pr\left(y[d] \middle| x[c] = a_j, \mathbf{H}\right). \tag{33}$$

With channel estimation, how to decode?

- $$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w},$$

Message passings from observation nodes $y[d]$ to variable nodes $x[c], c \in \mathcal{I}(d)$: The mean $\mu_{d,c}^{(i)}$ and variance $(\sigma_{d,c}^{(i)})^2$ of the interference, approximately modeled as a Gaussian random variable $\zeta_{d,c}^{(i)}$ defined as:

$$y[d] = x[c]H[d,c] + \underbrace{\sum_{e \in \mathcal{I}(d), e \neq c} x[e]H[d,e] + z[d]}_{\zeta_{d,c}^{(i)}}, \tag{34}$$

can be computed as:

$$\mu_{d,c}^{(i)} = \sum_{e \in \mathcal{I}(d), e \neq c} \sum_{j=1}^{Q} p_{e,d}^{(i-1)}(a_j) a_j H[d,e], \tag{35}$$

and

$$(\sigma_{d,c}^{(i)})^2 = \sum_{e \in \mathcal{I}(d), e \neq c} \left(\sum_{j=1}^{Q} p_{e,d}^{(i-1)}(a_j) |a_j|^2 |H[d,e]|^2 - \left| \sum_{j=1}^{Q} p_{e,d}^{(i-1)}(a_j) a_j H[d,e] \right|^2 \right) + \sigma^2. \tag{36}$$

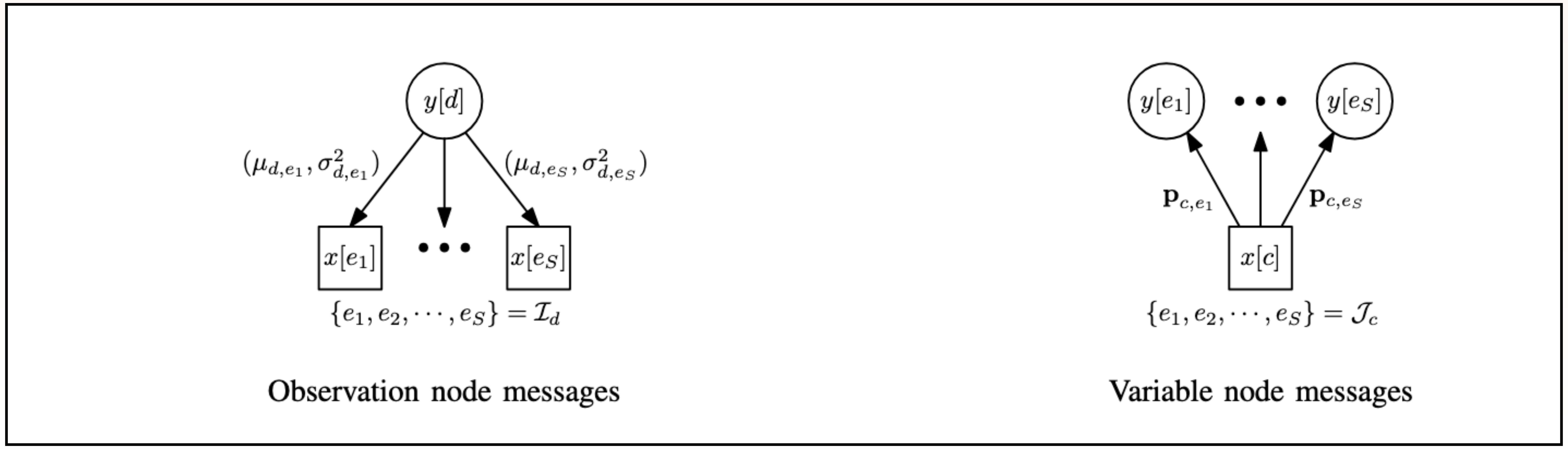
Message passings from variable nodes $x[c]$ to observation nodes $y[d], d \in \mathcal{J}(c)$: The pmf vector $\mathbf{p}_{c,d}^{(i)}$ can be updated as:

$$p_{c,d}^{(i)}(a_j) = \Delta \cdot \tilde{p}_{c,d}^{(i)}(a_j) + (1-\Delta) \cdot p_{c,d}^{(i-1)}(a_j), a_j \in \mathbb{A} \tag{37}$$

where $\Delta \in (0,1]$ is the *damping factor* used to improve the performance by controlling the convergence speed [18], and

$$\begin{aligned} \tilde{p}_{c,d}^{(i)}(a_j) &\propto \prod_{e \in \mathcal{J}(c), e \neq d} \Pr\left(y[e] \middle| x[c] = a_j, \mathbf{H} \right) \\ &= \prod_{e \in \mathcal{J}(c), e \neq d} \frac{\xi^{(i)}(e,c,j)}{\sum_{k=1}^{Q} \xi^{(i)}(e,c,k)}, \end{aligned} \tag{38}$$

where $\xi^{(i)}(e,c,k) = \exp\left(\frac{-\left| y[e] - \mu_{e,c}^{(i)} - H_{e,c} a_k \right|^2}{(\sigma_{e,c}^{(i)})^2} \right)$.



The message passing algorithm (copied from paper)

Algorithm 1 MP algorithm for OTFS symbol detection

Input: Received signal $\mathbf{y}$, channel matrix $\mathbf{H}$.

Initialization: pmf $\mathbf{p}_{c,d}^{(0)} = 1/Q, c = 1, \cdots, NM, d \in \mathcal{J}(c)$, iteration count $i = 1$.

repeat

- Observation nodes $y[d]$ compute the means $\mu_{d,c}^{(i)}$ and variances $(\sigma_{d,c}^{(i)})^2$ of Gaussian random variables $\zeta_{d,c}^{(i)}$ using $\mathbf{p}_{c,d}^{(i-1)}$ and pass them to variables nodes $x[c], c \in \mathcal{I}(d)$.
- Variable nodes $x[c]$ update $\mathbf{p}_{c,d}^{(i)}$ using $\mu_{d,c}^{(i)}$, $(\sigma_{d,c}^{(i)})^2$, and $\mathbf{p}_{c,d}^{(i-1)}$ and pass them to observation nodes $y[d], d \in \mathcal{J}(c)$.
- Compute convergence indicator $\eta^{(i)}$.
- Update the decision on the transmitted symbols $\widehat{x}[c], c = 1, \ldots, NM$ if needed.
- $i = i + 1$

until *Stopping criteria*;

Output: The decision on transmitted symbols $\widehat{x}[c]$.

With channel estimation, how to decode?

- Suppose one have estimated the (delay-Doppler domain) channel matrix $\mathbf{H} \in \mathbb{C}^{NM \times NM}$, then we are facing (equivalently) a linear system with NM equations:
- $$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w},$$
- Where where $\mathbf{y}, \mathbf{x}, \mathbf{w}$ are $\mathbf{x}[k, l], \mathbf{y}[k, l], \mathbf{w}[k, l]$ in vectorized form, $k \in [N], l \in [M]$. $\mathbf{H}$ is treated given.

- Easier ways to do decoding?
- —MMSE! *(We use this in our MatLab code demo later)*
- $$\hat{\mathbf{x}} \triangleq (\mathbf{H}^{\dagger}\mathbf{H} + \lambda \mathbf{I})^{-1}\mathbf{H}^{\dagger}\mathbf{y}$$
- Note that: this involves a matrix inversion of a $NM \times NM$ matrix. Whose (in general) complexity is $\mathcal{O}(M^3N^3)$. Usually, when N, M are not too small. This will be huge…
- Side note: In fact, maybe we can leverage the property of $\mathbf{H}$ being sparse to accelarate it. But the result is shown not good as MP.

What if there is no Doppler?

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OTFS Performance on Static Multipath Channels

P. Raviteja, Emanuele Viterbo, and Yi Hong

- "…… In this letter, we **show that, in static multipath channels, the system structure of OTFS is equivalent to the asymmetric orthogonal frequency division multiplexing (A-OFDM, an OFDM-like scheme with unequal subcarrier spacing),** a scheme … bridging between cyclic prefix single carrier (CPSC) and traditional OFDM.

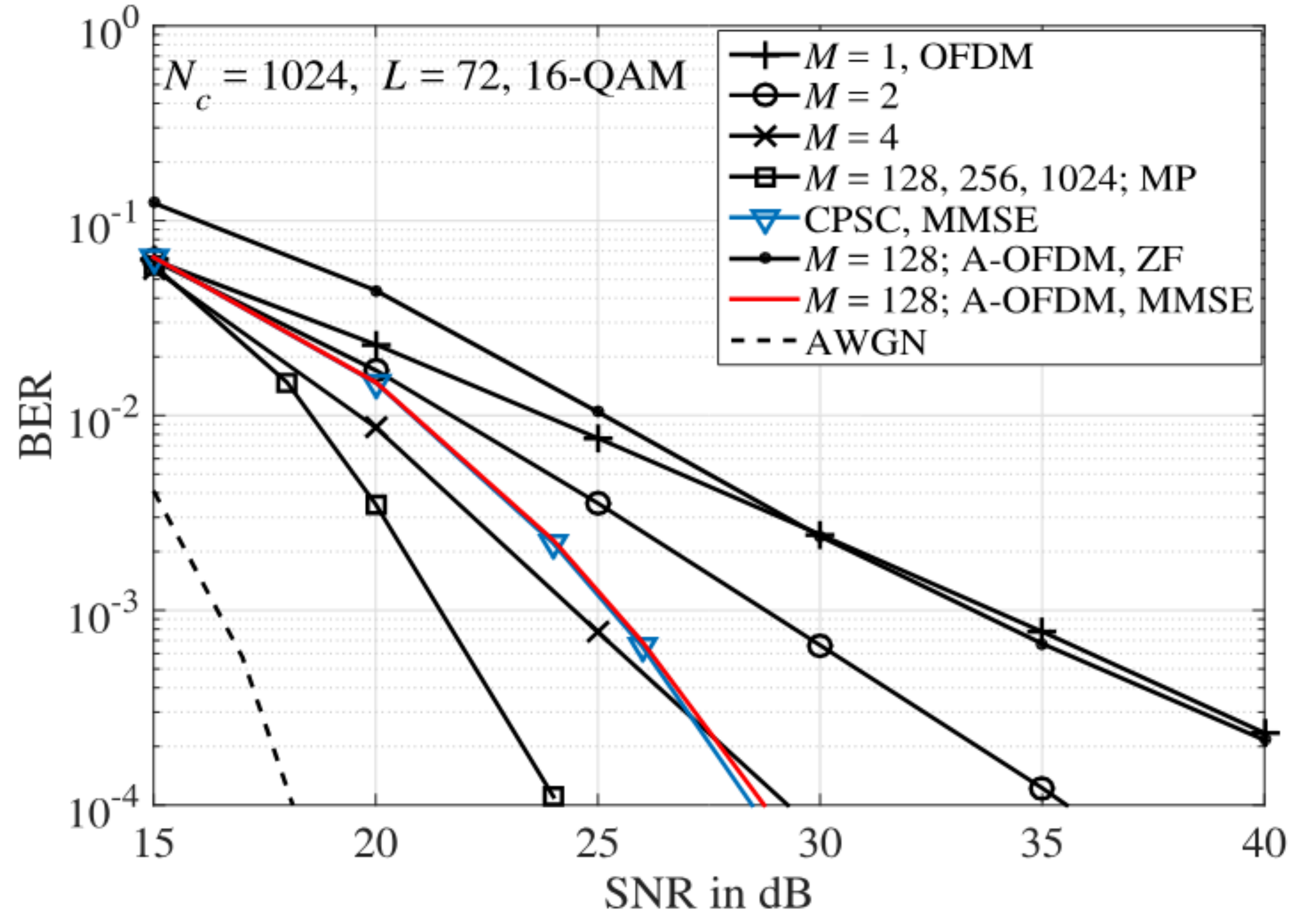


Fig. 3. BER of OTFS for different M with $N_c = 1024$, $L = 72$, and 16-QAM.

Matlab Code Demo

- Try some different SNRs and see the results.

- N = 16, M = 64. (The DD domain matrix is 16 x 64)
- Will output a number, indicating out of the 16x64 = 1024 data, how many are incorrect.

- 4QAM

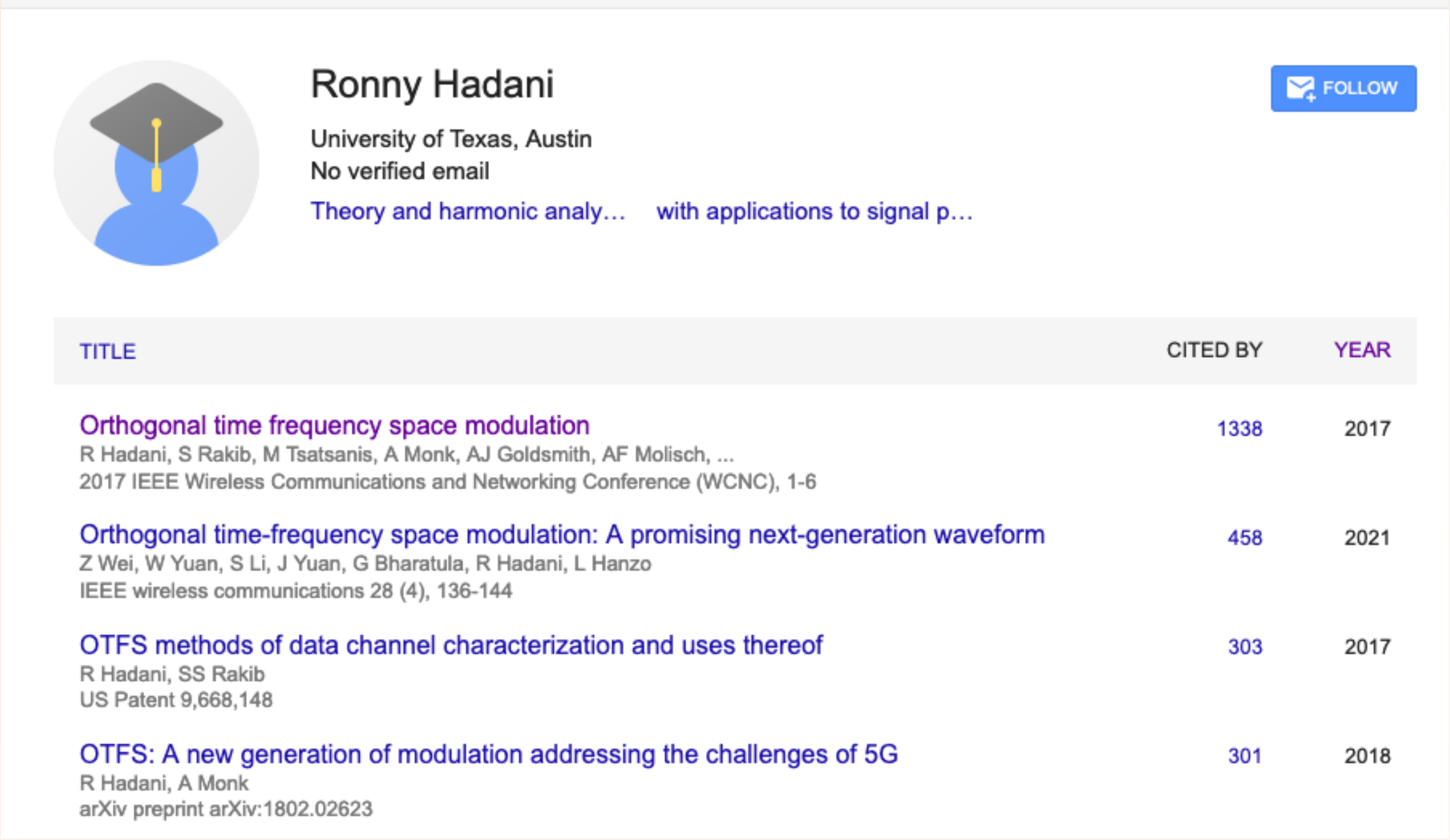
- About $\approx$ 9 taps.
- Rayleigh fading + AWGN
- **Assuming channel matrix is given**

Other concepts related to OTFS, but not mentioned

- Quasi-periodicity
- Pulsone: (the waveform) of OTFS
- Zak transform: a useful tool in simplifying the modulation / demodulation of OTFS
- …

OTFS community

- It was first presented in WCNC 2017, by Ronny Hadani.
- Since then, many (≥ 1000) works have been published.



- There is an official small community called “OTFS special interest group”. Members are experts and pioneers in OTFS. They host online webinar on a monthly basis.
- 

References

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- Saif Khan Mohammed, , “OTFS - A Mathematical Foundation for Communication and Radar Sensing in the Delay-Doppler Domain”, IEEE, 2022.
- Emanuele Viterbo, “An introduction to Orthogonal Time Frequency Space (OTFS) modulation for high mobility communications”, 2019. SIG on OTFS. [slides]
- Emanuele Viterbo, “OTFS Detection and Channel Estimation: a Low-Complexity Practical Solution”, 2021. SIG on OTFS. [slides]
- P. Raviteja, Khoa T. Phan, Yi Hong, and Emanuele Viterbo, “Interference Cancellation and Iterative Detection for Orthogonal Time Frequency Space Modulation” [The paper talked about message passing decoding of OTFS]
- Brian Olson, Steven Shaw, Chengzhi Shi, Christophe Pierre, Robert Parker, “Circulant Matrices and Their Application to Vibration Analysis”

OTFS: channel matrix representation

- Received signal in vector form in time domain (assuming noiseless)

$$\mathbf{r} = \mathbf{G}\mathbf{s}$$

- $\mathbf{G}$ is an $MN \times MN$ matrix of the following form

$$\mathbf{G} = \sum_{i=1}^{P} h_i' \mathbf{\Pi}^{l_i} \mathbf{\Delta}^{(k_i)},$$

where, $\mathbf{\Pi}$ is the permutation matrix (forward cyclic shift), and $\mathbf{\Delta}^{(k_i)}$ is the diagonal matrix

$$\underbrace{\mathbf{\Pi} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}_{MN \times MN}}_{\text{Delay (similar to OFDM)}}, \underbrace{\mathbf{\Delta}^{(k_i)} = \begin{bmatrix} e^{\frac{j2\pi k_i(0)}{MN}} & 0 & \cdots & 0 \\ 0 & e^{\frac{j2\pi k_i(1)}{MN}} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\frac{j2\pi k_i(MN-1)}{MN}} \end{bmatrix}}_{\text{Doppler}}$$