The following is a review of the Quantitative Analysis principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Simulation Methods

Topic 30

Exam Focus

Simulation methods model uncertainty by generating random inputs that are assumed to follow an appropriate probability distribution. This topic discusses the basic steps for conducting a Monte Carlo simulation and compares this simulation method to the bootstrapping technique. For the exam, be able to explain ways to reduce Monte Carlo sampling error, including the use of antithetic and control variates. Also, understand the pseudo-random number generation method and the benefits of reusing sets of random number draws in Monte Carlo experiments. Finally, be able to describe the advantages and disadvantages of the bootstrapping technique in comparison to the traditional Monte Carlo approach.

Monte Carlo Simulation

LO 30.1: Describe the basic steps to conduct a Monte Carlo simulation.

Monte Carlo simulations are often used to model complex problems or to estimate variables when there are small sample sizes. A few practical finance applications of Monte Carlo simulations are: pricing exotic options, estimating the impact to financial markets of changes in macroeconomic variables, and examining capital requirements under stress-test scenarios.

There are four basic steps required to conduct a Monte Carlo simulation.

Step 1: Specify the data generating process (DGP)
Step 2: Estimate an unknown variable or parameter
Step 3: Save the estimate from step 2
Step 4: Go back to step 1 and repeat this process *N* times

The first step of conducting a simulation requires generating random inputs that are assumed to follow a specific probability distribution. The DGP could be a simple time series model or a more complex full structural model that requires multiple DGPs.

The second step of the simulation generates scenarios or trials based on randomly generated inputs drawn from a pre-specified probability distribution. The most common probability distribution used is the standard normal distribution. However, Student's *t* distribution is often used if the user believes it is a better fit for the data. A well-defined simulation model requires the generation of variables that follow appropriate probability distributions.

The last two steps in the simulation process allow for data analysis related to the properties of the probability distributions of the output variables. In other words, rather than making just one output estimate for a problem, the model generates a probability distribution of estimates. This provides the user with a better understanding of the range of possible outcomes. The quantity N in step four is the number of times the simulation is repeated. This is referred to as the number of replications or iterations and is typically 1,000 to 10,000 times depending on how costly it is to generate the sample size.

For example, suppose we are managing an investment portfolio and desire to estimate the ending capital in the portfolio in one year, C_1. The initial capital investment, C_0, is \$100 invested in the Standard & Poor's 500 index (S&P 500). The return is a random variable that depends on how the market performs over the next year.

If we assume the return over the next year is equal to a historical mean return, we can calculate one point estimate of the ending capital based on the equation: $C_1 = C_0(1 + r)$. The return over the next period is a random variable, and a simulation model estimates multiple scenarios to represent future returns based on a probability distribution of possible outcomes. The output variable is an estimate of an ending amount of capital that is also a random variable. The simulation model allows us to visualize the output and analyze the probability distribution of the ending capital amounts generated by the model.

Reducing Monte Carlo Sampling Error

LO 30.2: Describe ways to reduce Monte Carlo sampling error.

The sampling variation for a Monte Carlo simulation is quantified as the standard error estimate. The standard error of the true expected value is computed as $s/\sqrt{N}$, where s is the standard deviation of the output variables and N is the number of scenarios or replications in the simulation. Based on this equation, it intuitively follows that in order to reduce the standard error estimate by a factor of 10, the analyst must increase N by a factor of 100. (Because the square root of 100 is 10, if we increase the sample size 100 times it will reduce the standard error estimate by dividing by 10.)

Suppose we continue the illustration from the previous example and run a simulation to estimate the ending capital amount for an initial investment portfolio of \$100. The number of replications is initially 100 (i.e., N = 100), resulting in a mean ending capital of \$110 and a standard deviation of \$14.798. For this example, the standard error estimate is computed as \$1.4798 (i.e., \$14.798 / 10). Now, suppose we want to increase the accuracy by reducing the standard error estimate. How can we increase the accuracy of the simulation?

The accuracy of simulations depends on the standard deviation and the number of scenarios run. We cannot control the standard deviation, but we can control the number of replications. Assume we rerun the previous simulation with 400 replications that result in the same mean ending capital of \$110, and the standard deviation remains at \$14.798. The standard error estimate for the simulation with 400 replications is then \$0.7399 (i.e., 14.798 / 20). With four times the number of scenarios (4 × N, or 400, in this example) the standard error estimate is cut in half to \$0.7399. In other words, quadrupling the number of scenarios will improve the accuracy twofold.

However, increasing the number of generated scenarios can become costly for more complex multi-period simulations. Variance reduction techniques offer an alternative way to reduce the sampling error of a Monte Carlo simulation. The two most commonly used techniques for reducing the standard error estimate are antithetic variates and control variates.

Antithetic Variates

LO 30.3: Explain how to use antithetic variate technique to reduce Monte Carlo sampling error.

One reason sampling error occurs is because there are often a wide range of possible outcomes for a particular experiment or problem. Thus, in order to replicate the entire range of possible outcomes the sampling sets must be recreated numerous times. However, increasing the number of samples drawn may be too costly and time consuming. As an alternative approach, the **antithetic variate technique** can reduce Monte Carlo sampling error by rerunning the simulation using a *complement* set of the original set of random variables.

If the original set of random draws is denoted u_t for each replication, then the simulation is rerun with the complement set of random numbers denoted $-u_t$. By definition, the use of antithetic variates results in a lower covariance and variance, because the two sets are perfectly negatively correlated [i.e., $\text{corr}(u_t, -u_t) = -1$]. The following example illustrates how the standard error for a Monte Carlo simulation is reduced by using the antithetic variate technique.

First, consider a simulation of two sets that does not use the antithetic variate technique. Suppose the average parameter estimate is determined by two Monte Carlo simulations using different random sample sets. The average output parameter value, $\bar{x}$, for the two simulations using different random sample replications is simply calculated as:

$$\bar{x} = (x_1 + x_2)/2$$

Where x_1 and x_2 are the average output parameter values for simulation sets 1 and 2, respectively.

Next, we can calculate the variance of the average of the two sets as follows:

$$\text{var}(\bar{x}) = \frac{\text{var}(x_1) + \text{var}(x_2) + 2\,\text{cov}(x_1, x_2)}{4}$$

Without using antithetic variates, the two sets of Monte Carlo replications are independent. Thus, the covariance will be zero and the variance of $\bar{x}$ is simply reduced to the following:

$$\text{var}(\bar{x}) = \frac{\text{var}(x_1) + \text{var}(x_2)}{4}$$

The use of antithetic variates results in a negative covariance between the original random draws and their complements (i.e., antithetic variates). Thus, the use of antithetic variates causes the error terms to be independent for the two sets, which results in a negative covariance term in the variance equation. This negative relationship means that the Monte Carlo sampling error must always be smaller using this approach.

Control Variates

LO 30.4: Explain how to use control variates to reduce Monte Carlo sampling error and when it is effective.

The **control variate technique** is a widely used method to reduce the sampling error in Monte Carlo simulations. A control variate involves replacing a variable *x* (under simulation) that has unknown properties with a similar variable y that has known properties.

Suppose two separate simulations are conducted on variable *x* with unknown properties and control variable y with known properties using the same set of random numbers. Also assume that the Monte Carlo simulation estimated variables for *x* and *y* are denoted as $\hat{x}$ and $\hat{y}$, respectively. The original estimate *x* can be redefined as x* as follows:

$$x^* = y + (\hat{x} - \hat{y})$$

The new x* variable estimate will have a smaller sampling error than the original *x* variable if the control statistic and statistic of interest are highly correlated. The Monte Carlo results for the new x* variable are assumed to have similar properties to the known *y* control variable.

The following mathematical equations help illustrate the condition that is necessary to reduce the sampling error using control variates. Consider taking the variance of both sides of the equation that defines the new variable such that:

$$\text{var}(x^*) = \text{var}[y + (\hat{x} - \hat{y})]$$

The control variable y does not have a sampling error because it has known properties. Thus, the var(y) equals zero. Now, the variance of the remaining two variables can be rewritten as follows:

$$\text{var}(x^*) = \text{var}(\hat{x}) + \text{var}(\hat{y}) - 2\,\text{cov}(\hat{x}, \hat{y})$$

The control variate method will only reduce the sampling error in Monte Carlo simulations if var(x*) is less than var($\hat{x}$). Another way of expressing this condition is as follows:

$$\text{var}(\hat{y}) - 2\,\text{cov}(\hat{x}, \hat{y}) < 0$$

This relationship can be simplified as follows:

$$\text{cov}(\hat{x}, \hat{y}) > \frac{\text{var}(\hat{y})}{2}$$

The covariance can be converted to correlation by dividing both sides of the previous inequality by the product of the standard deviations as follows:

$$\operatorname{corr}(\hat{x},\hat{y}) > \frac{1}{2}\sqrt{\frac{\operatorname{var}(\hat{y})}{\operatorname{var}(\hat{x})}}$$

A practical financial example of applying control variates is the use of Monte Carlo simulations in pricing Asian options (which will be discussed in Book 4). An Asian option is priced based on the average value of the underlying asset over the lifespan of the option. The use of a similar derivative, such as a European option, with known statistical properties can be used as a control variate. The price of the European option, P_{BS}, is determined by the Black-Scholes-Merton option pricing model. Next, simulated prices are determined for the Asian option and the European option and denoted P_A and P_{BS}^*, respectively. The new estimate of the Asian option price, P_A^*, could then be determined based on the following equation:

$$P_A^* = (P_A - P_{BS}) + P_{BS}^*$$

Reusing Sets of Random Numbers

LO 30.5: Describe the benefits of reusing sets of random number draws across Monte Carlo experiments and how to reuse them.

Reusing sets of random number draws across Monte Carlo experiments reduces the estimate variability across experiments by using the same set of random numbers for each simulation. Normally, a user would not desire to reuse the same random draws. However, in certain situations this technique is useful. Two examples of reusing sets of random numbers are for testing the power of the Dickey-Fuller test (used to determine whether a time series is covariance stationary) or for different experiments with options using time series data.

Dickey-Fuller (DF) test. Suppose an analyst wants to examine the DF test for sample sizes of 1,000 to test whether or not a particular market follows a random walk or contains a drift element. The analyst could reuse the same set of standard normal random variables for each simulation run while testing with different DF parameters. Using the same set of random numbers for each Monte Carlo experiment reduces the sampling variation across experiments. In this case, the sampling variability is reduced, but the accuracy of the actual estimates is not increased.

Different experiments. Another example where reusing sample data is useful is in testing differences among options. For example, suppose an analyst is examining option prices that are similar in all aspects except for time to maturity. The analyst could simulate a long time series of random draws and then split this longer time series into shorter time frames. A six-month time series of data could be subdivided into three sets of two-month maturity options or six sets of one-month maturity options. Using the same random number data set reduces the variability of simulated option prices across maturities.

Bootstrapping Method

LO 30.6: Describe the bootstrapping method and its advantage over Monte Carlo simulation.

Another way to generate random numbers is the bootstrapping method. The bootstrapping approach draws random return data from a sample of historical data. Under traditional Monte Carlo simulation, data sets are created by selecting random variables drawn from a pre-determined probability distribution. The bootstrapping method uses actual historical data instead of random data from a probability distribution. In addition, bootstrapping repeatedly draws data from a historical data set and replaces the data so it can be drawn again.

For example, suppose an analyst uses the bootstrapping method to estimate parameter θ. The analyst begins by obtaining sample historical data over a specific time period. This historical data is denoted:

$$y = y_1, y_2, \ldots, y_T$$

The statistical properties of parameter $\hat{\theta}_T$ are then estimated based on the bootstrapping sample data. The analyst creates N samples of T variables with replacement from the original y data sample. The parameter estimate $\hat{\theta}$ is calculated for every sample to create N estimates. In other words, the samples that are drawn are not totally random, but are drawn from a pre-determined historical sample set y. The statistical properties of this sample of $\hat{\theta}$ estimates are then analyzed.

An obvious advantage of the bootstrapping approach is that no assumptions are made regarding the true distribution of the parameter estimate that is being examined. This implies that it can include extreme events that have occurred in the past (e.g., during a financial crisis). Inclusion of outliers will produce a distribution that has fatter tails than the normal distribution, which allows for a more realistic view of actual return data. Thus, the bootstrapping methodology generates a collection of data sets with approximately the same distribution properties as the original data. However, any dependency of variables or autocorrelations in the original data set will no longer be present, because variables are not drawn in the same sequence as the original data set.

The following example describes how bootstrapping is used with a regression model. Assume that the bootstrapping approach is used to re-sample data with respect to the following standard regression model:

$$y = u + X\beta$$

The first step of the bootstrapping approach is to generate a sample size T of the historical data by drawing samples with replacement that take all related data corresponding to each observation y_i. In other words, for the 21st data observation, y_{21}, the approach takes this estimate along with all values of the explanatory variables for the 21st observation.

Next the coefficient matrix, $\hat{\beta}^*$, is estimated for this bootstrap sample. This process is then repeated a total of *N* times. Every time data is resampled, a sample size of *T* is generated from the original sample data with replacement and a coefficient matrix is estimated. This results in a set of *N* coefficient vectors that will all be unique, and a distribution of estimates is created for each coefficient.

This bootstrapping approach has a methodological problem resulting from sampling from regressors rather than using a fixed estimate in repeated samples. To correct for this problem, the approach can be slightly modified where re-sampling occurs with the residuals. Thus, the first step would be to sample actual data, estimate the value $\hat{y}$ and calculate the residuals, $\hat{u}$. The coefficient vector is then created using a modified dependent variable that is the sum of the fitted values and the bootstrap residuals $\hat{u}^*$ as follows:

$$y^* = \hat{y} + \hat{u}^*$$

LO 30.8: Describe situations where the bootstrapping method is ineffective.

Two situations that cause the bootstrapping method to be ineffective are *outliers* in the data and *non-independent data*.

If outliers exist in the data, the inferences drawn from parameter estimates may not be accurate depending on how many times the outliers are included in the bootstrapped sample. Because replacement is used in the bootstrap method, outliers could be drawn more often, causing the bootstrap distribution to have fatter tails. Alternatively, not drawing the outlier in the bootstrapped sample may lead to the opposite conclusions regarding the parameter estimate statistical properties. Recall that a major advantage of the bootstrapping approach over traditional approaches is that it does not require any assumptions of the probability distribution of the sampled data. Thus, the best way to mitigate this issue is to have a large number of replications.

If autocorrelation exists in the original sample data, then the original historical data are not independent of one another. A technique known as a *moving block bootstrap* is used to overcome the problem of autocorrelation. Blocks of data are examined at one time in order to preserve the original data dependency.

Random Number Generation

LO 30.7: Describe the pseudo-random number generation method and how a good simulation design alleviates the effects the choice of the seed has on the properties of the generated series.

A good random number generator has the ability to reproduce a random sequence and analyze characteristics of random numbers. Simulation software programs are able to reproduce the same sequence of iterations by starting sequences with a seed random number. The algorithms used to generate these random sequences are referred to as **pseudo-random number generators.** These number generators are advantageous because risk managers can improve models by reducing the estimate variance or debugging computer

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codes if the same sequence of random numbers is reproduced when programming the model.

A very common pseudo-random number generator is one that generates random number sequences uniformly distributed between 0 and 1. Each number has an equal probability of being drawn from this uniform (0,1) distribution. Numbers can be drawn from a discrete or continuous distribution. The term *pseudo* implies that these computer-generated numbers are *not truly random*, because they are actually generated from a formula. For example, suppose random numbers are generated from a continuous uniform (0,1) distribution based on the following formula:

$$y_{i+1} = (ay_i + c) \text{ modulo } m, \; i = 0,1,2,\ldots,T$$

In the above formula, T is the total number of random numbers drawn, y_0 is the initial value of y, which is referred to as the **seed**, a is a constant multiplier, and c is an incremental value. The statement "modulo m" in the above formula refers to modulo operator, which is a clocklike process where the generator returns to 1 when the value m is reached.

In order to run a simulation, the user must first define the initial seed value, y_0. The choice of seed value will influence the properties of the random number distribution that is generated. The effect is strongest for the early draws in a series, but eventually the impact fades away. Therefore, the best way to control for this problem is to generate a very large number of observations and then discard the earliest observations.

For example, if a user requires 800 observations, then 1,000 random numbers are generated and the first 200 are eliminated from the sample. This ensures that the statistical properties of the sample reflect those of true random numbers that are not based on a pre-specified formula. Eventually random number sequences will repeat. Therefore, a good random number generator uses sequences with long cycles that require numerous iterations before a sequence is repeated.

Disadvantages of Simulation Approaches

LO 30.9: Describe disadvantages of the simulation approach to financial problem solving.

Disadvantages of the simulation approach to financial problem solving include:

- High computation costs
- Results are imprecise
- Results are difficult to replicate
- Results are experiment-specific

Some problems may require a large number of replications to obtain more accurate results. If estimated parameters are complex, the computations may take an extremely long time to run. Computer processor times have improved exponentially. However, the complexity of markets and issues that are examined have also become increasingly complex, leading to *high computation costs*.

Imprecise results may be present even with a very large number of simulation iterations when the assumptions of model inputs or the data generating process are unrealistic. A common mis-specified model assumption is related to the underlying probability distribution of inputs. For example, option prices are typically fat-tailed, but a model could erroneously draw option prices from a normal distribution. This would lead to inaccurate results regardless of the number of replications.

In practice, users seldom use a defined seed for the start of random draws in simulations. Without the use of an initial seed, it is *not possible to replicate results* from previous experiments. The best way to overcome this problem and reduce the variation of results is to use a very large number of replications. Thus, it is common to use at least 10,000 replications in Monte Carlo simulations if it is computationally cost-effective.

Simulation results are experiment-specific because financial problems are analyzed based on a specific data generating process and set of equations. If alternate assumptions are made in the equations or data generating process, the results may differ substantially.

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Key Concepts

LO 30.1

The basic steps of a Monte Carlo simulation are: (1) specify the data generating process (DGP), (2) estimate an unknown variable, (3) save the estimate from step 2, and (4) go back to step 1 and repeat this process *N* times.

LO 30.2

The standard error estimate of a Monte Carlo simulation, $s/\sqrt{N}$, can be reduced by a factor of 10 by increasing *N* by a factor of 100.

LO 30.3

The antithetic variate technique reduces Monte Carlo sampling error by rerunning the simulation using a complement set of the original set of random variables.

LO 30.4

The control variate technique replaces a variable *x* that has unknown properties in a Monte Carlo simulation with a similar variable *y* that has known properties. The new *x** variable estimate will have a smaller sampling error than the original *x* variable if the control statistic and statistic of interest are highly correlated.

LO 30.5

Reusing sets of random number draws across Monte Carlo experiments reduces the estimate variability across experiments.

LO 30.6

Bootstrapping simulations repeatedly draw data from historical data sets and replace the data so it can be re-drawn. The bootstrapping technique requires no assumptions with respect to the true distribution of the parameter estimates.

LO 30.7

Pseudo-random numbers are not truly random, because they are actually generated from a formula. The choice of the initial seed value influences the properties of the random number distribution that is generated. Thus, when using a seed value, increasing the number of replications and eliminating early estimates from the sample can mitigate any biases.

LO 30.8

The bootstrapping method is ineffective when there are outliers in the data or when the data is non-independent.

LO 30.9

Disadvantages of the simulation approach to financial problem solving include: high computation costs, imprecise results, difficulty with replicating results, and experiment-specific results.

Concept Checkers

1. Suppose an analyst is concerned about Monte Carlo sampling error. Based on an initial Monte Carlo simulation with 100 replications, the results indicated a standard deviation of 12.64. The simulation was rerun with 900 replications and the standard deviation remained at 12.64. What are the standard error estimates for the simulations with 100 replications and 900 replications, respectively?

	N = 100	N = 900
A.	0.126	0.014
B.	0.126	0.140
C.	1.264	0.421
D.	1.264	0.214

2. A concern for Monte Carlo simulations is the size of the sampling error. One way to reduce the sampling error is to use the antithetic variate technique. Which of the following statements best describe this technique?
   A. The simulation is rerun using a complement set of the original set of random variables.
   B. The number of replications is increased significantly to reduce sampling error.
   C. Sample data is replaced after every replication to ensure it has an equal probability of being redrawn.
   D. The data generating process is approximated by redefining the unknown variable with a variable that has known properties.

3. Suppose an analyst is testing the robustness of the Dickey-Fuller test by changing the drift parameter for several different experiments. Reusing sets of random number draws across Monte Carlo experiments will most likely result in:
   A. increasing the accuracy of the drift estimates for each experiment.
   B. increasing the sampling variance across experiments.
   C. reducing the accuracy of the drift estimates for each experiment.
   D. reducing the sampling variance across experiments.

4. Suppose a pseudo-random number generator is used that generates random number sequences uniformly and continuously distributed between 0 and 1. An analyst begins by defining the initial seed value for the number generator process. The analyst knows that the choice of seed value will influence the properties of the generated random number distribution. The best way to reduce this problem is by using a:
   A. large number of replications and discarding the outliers.
   B. large number of replications and discarding the earliest draws.
   C. small seed or initial value.
   D. large seed or initial value.

5. Monte Carlo simulation is a widely used technique in solving economic and financial problems. Which of the following statements is not a limitation of the Monte Carlo technique when solving problems of this nature?
   A. High computational costs arise with complex problems.
   B. Simulation results are experiment-specific because financial problems are analyzed based on a specific data generating process and set of equations.
   C. Results of most Monte Carlo experiments are difficult to replicate.
   D. If the input variables have fat tails, Monte Carlo simulations are not relevant because it always draws random variables from a normally distributed population.

Concept Checker Answers

1. C The standard error is determined by dividing the standard deviation by the square root of the number of replications $s / \sqrt{N}$. The standard error estimate for the first simulation of 100 replications is 1.264 (i.e., 12.64 / 10). With 900 replications, the standard error estimate is reduced to 0.4213 (i.e., 12.64 / 30).

2. A The antithetic variate technique reduces Monte Carlo sampling error by rerunning the simulation using a complement set of the original set of random variables.

3. D Using the same set of random numbers for each Monte Carlo experiment reduces the sampling variation across experiments. Although the sampling variability is reduced, the accuracy of the actual estimates in each case is not influenced.

4. B The best way to control for this problem is to generate a very large number of observations and then discard the earliest observations. This ensures that the statistical properties of the sample reflect those of true random numbers that are not based on a pre-specified formula.

5. D A disadvantage of Monte Carlo simulation is that imprecise results may be present when the assumptions of model inputs or data generating process are unrealistic. The distribution of input variables does not need to be the normal distribution. The problem arises when a variable in the real world is fat-tailed, but a model could erroneously draw option prices from a normal distribution.

Self-Test: Quantitative Analysis

10 Questions: 24 Minutes

1. Given the following probability data for the return on the market and the return on Best Oil, calculate the covariance of returns between Best Oil and the market.

Probability Matrix

	R_{Best} = 20%	R_{Best} = 10%	R_{Best} = 5%
R_{Mkt} = 15%	40%	0	0
R_{Mkt} = 10%	0	20%	0
R_{Mkt} = 0%	0	0	40%

A. 44.0.
B. 12.0.
C. 2.8.
D. 22.5.

2. Rob Conniff has encountered a difficult section on a multiple-choice exam. There are five questions in this section and each question has three equally likely answer choices. Which of the following amounts is closest to the probability that he will get three or more questions correct by randomly guessing?
A. 4.5%.
B. 16.5%.
C. 21.0%.
D. 79.0%.

3. You are forecasting the sales of a building materials supplier by assessing the expansion plans of its largest customer, a homebuilder. You estimate the probability that the customer will increase its orders for building materials to 25%. If the customer does increase its orders, you estimate the probability that the homebuilder will start a new development at 70%. If the customer does not increase its orders from this supplier, you estimate only a 20% chance that it will start the new development. Later, you find out that the homebuilder will start the new development. In light of this new information, what is your new (updated) probability that the builder will increase its orders from this supplier?
A. 17.50%.
B. 32.55%.
C. 53.85%.
D. 60.00%.

4. In performing hypothesis testing as a quantitative analyst, you have recently encountered some unsatisfactory results. You consult your boss and he suggests that you consider increasing the significance level in your testing activities. Which of the following outcomes would most likely occur with such an increase?
A. Increased probability of making a Type I error.
B. Increased probability of making a Type I or II error.
C. Decreased probability of making a Type I error.
D. Decreased probability of making a Type I or II error.

Use the following information to answer Question 5.

An analyst is given the data in the following table for a regression of the annual sales for Company XYZ, a maker of paper products, on paper product industry sales.

Parameters	*Coefficient*	*Standard Error of the Coefficient*
Intercept	–94.88	32.97
Slope (industry sales)	0.2796	0.0363

The correlation between company and industry sales is 0.9757. The regression was based on five observations.

5. Which of the following is closest to the value and reports the most likely interpretation of the R^2 for this regression? The R^2 is:
A. 0.048, indicating that the variability of industry sales explains about 4.8% of the variability of company sales.
B. 0.048, indicating that the variability of company sales explains about 4.8% of the variability of industry sales.
C. 0.952, indicating that the variability of industry sales explains about 95.2% of the variability of company sales.
D. 0.952, indicating that the variability of company sales explains about 95.2% of the variability of industry sales.

Use the following information to answer Questions 6 through 8.

Theresa Miller is attempting to forecast sales for Alton Industries based on a multiple regression model. The model Miller estimates is:

$$\text{sales} = b_0 + (b_1 \times \text{DOL}) + (b_2 \times \text{IP}) + (b_3 \times \text{GDP}) + \varepsilon_t$$

where:
sales = change in sales adjusted for inflation
DOL = change in the real value of the \$ (rates measured in €/\$)
IP = change in industrial production adjusted for inflation (millions of \$)
GDP = change in inflation-adjusted GDP (millions of \$)

All changes in variables are in percentage terms.

Miller runs the regression using monthly data for the prior 180 months. The model estimates (with coefficient standard errors in parentheses) are:

$$\underset{(5.4)}{\text{sales} = 10.2} + (\underset{(3.5)}{5.6} \times \text{DOL}) + (\underset{(4.2)}{6.3} \times \text{IP}) + (\underset{(5.3)}{9.2} \times \text{GDP})$$

The sum of squared residuals (SSR) is 145.6 and the total sum of squares (TSS) is 357.2.

Figure 1: Partial Student's *t*-distribution (one-tailed probabilities)

df	p = 0.10	p =0.05	p = 0.025	p = 0.01	p = 0.005
170	1.287	1.654	1.974	2.348	2.605
176	1.286	1.654	1.974	2.348	2.604
180	1.286	1.653	1.973	2.347	2.603

Figure 2: Partial *F*-Table critical values for right-hand tail area equal to 0.05

	df 1 = 1	df 1 = 3	df 1 = 5
df 2 = 170	3.90	2.66	2.27
df 2 = 176	3.89	2.66	2.27
df 2 = 180	3.89	2.65	2.26

Figure 3: Partial *F*-Table critical values for right-hand tail area equal to 0.025

	df 1 = 1	df 1 = 3	df 1 = 5
df 2 = 170	5.11	3.19	2.64
df 2 = 176	5.11	3.19	2.64
df 2 = 180	5.11	3.19	2.64

6. The unadjusted R^2 and the standard error of the regression (SER) are closest to:

	R^2	SER
A.	59.2%	1.425
B.	59.2%	0.910
C.	40.8%	0.910
D.	40.8%	1.425

7. The appropriate decision with regard to the *F*-statistic for testing the null hypothesis that all of the independent variables are simultaneously equal to zero at the 5% significance level is to:
   A. reject the null hypothesis because the *F*-statistic is larger than the critical *F*-value of 3.19.
   B. fail to reject the null hypothesis because the *F*-statistic is smaller than the critical *F*-value of 3.19.
   C. reject the null hypothesis because the *F*-statistic is larger than the critical *F*-value of 2.66.
   D. fail to reject the null hypothesis because the *F*-statistic is smaller than the critical *F*-value of 2.66.

8. What is the width of the 99% confidence interval for GDP, and is zero in that 99% confidence interval?

	Width of 99% CI	Zero in interval
A.	13.8	Yes
B.	3.8	No
C.	27.6	Yes
D.	27.6	No

9. The GTEC Corporation uses an exponentially weighted moving average (EWMA) model with a decay factor of 0.75 to model the daily volatility of a stock. The current estimate of daily volatility 1.8%. The closing price of the stock was $38 yesterday and $35 today. Using continuously compounded returns, what is the updated estimate of volatility?
   A. 5.39%.
   B. 4.39%.
   C. 3.39%.
   D. 2.39%.

10. A risk manager estimates the daily variance using a GARCH(1,1) model on daily returns (r_t):

$$h_t = \alpha_0 + \alpha_1 r^2_{t-1} + \beta h_{t-1}$$

The model parameter values are:

$\alpha_0 = 0.0000008$
$\alpha_1 = 0.050$
$\beta = 0.93$

Using the model, what is the long-run annualized volatility estimate (assuming 252 trading days in a year and that volatility increases by the square root of time)?
   A. 0.52%.
   B. 0.63%.
   C. 9.89%.
   D. 10.04%.

Self-Test Answers: Quantitative Analysis

1. A $E(R_{Best}) = 0.4(20\%) + 0.2(10\%) + 0.4(5\%) = 12\%$

$E(R_{Mkt}) = 0.4(15\%) + 0.2(10\%) + 0.4(0\%) = 8\%$

$Cov(R_{Best}, R_{Mkt}) = 0.4(20\% - 12\%)(15\% - 8\%)$

$+ \quad 0.2(10\% - 12\%)(10\% - 8\%)$

$+ \quad 0.4(5\% - 12\%)(0\% - 8\%)$

$= \quad 0.4(8)(7) + 0.2(-2)(2) + 0.4(-7)(-8) = 44$

The units of covariance (like variance) are percent squared here. We used whole number percents in the calculations and got 44; if we had used decimals, we would have gotten 0.0044.

(See Topic 16)

2 C The number of questions correct would follow a binomial distribution. Probability of success is 1/3 and the number of trials is 5. The probability of getting three or more questions correct is the sum of the following:

$P(3) = 10 \times (1/3)^3 \times (2/3)^2 = 0.1646$

$P(4) = 5 \times (1/3)^4 \times (2/3)^1 = 0.0412$

$P(5) = 1 \times (1/3)^5 \times (2/3)^0 = 0.0041$

$0.1646 + 0.0412 + 0.0041 = 21.0\%$

(See Topic 17)

3. C The prior probability that the builder will increase its orders is 25%.

P(increase) = 0.25

P(no increase) = 0.75

There are four possible outcomes:
- Builder increases its orders and starts new development.
- Builder increases its orders and does not start new development.
- Builder does not increase its orders and starts new development.
- Builder does not increase its orders and does not start new development.

The probabilities of each outcome are as follows:
- P(increase and development) = (0.25)(0.70) = 0.175.
- P(increase and no development) = (0.25)(0.30) = 0.075.
- P(no increase and development) = (0.75)(0.20) = 0.15.
- P(no increase and no development) = (0.75)(0.80) = 0.60.

We want to update the probability of an increase in orders, given the new information that the builder is starting the development. We can apply Bayes' formula:

$$\text{P(increase | development)} = \frac{\text{P(development | increase)} \times \text{P(increase)}}{\text{P(development)}}$$

From our assumptions, P(development | increase) = 0.70, and P(increase) = 0.25, so the numerator is (0.70)(0.25) = 0.175.

P(development) is the sum of P(development and increase) and P(development and no increase).

P(development) = 0.175 + 0.15 = 0.325

Thus, $\text{P(increase | development)} = \frac{(0.7)\times(0.25)}{0.175+0.15} = \frac{0.175}{0.325} = 0.5385$, or 53.85%

(See Topic 18)

4. A An increase in the significance level (from 1% to 5%, for example) means that a researcher is more likely to reject the null hypothesis since the critical value will be lower. Therefore, there is a greater probability of making a Type I error (rejecting the null hypothesis when it is actually true).

(See Topic 19)

5. C The R^2 is computed as the correlation squared: $(0.9757)^2 = 0.952$.

The interpretation of this R^2 is that 95.2% of the variation in Company XYZ's sales is explained by the variation in industry sales. Answer D is incorrect because it is the independent variable (industry sales) that explains the variation in the dependent variable (company sales). This interpretation is based on the economic reasoning used in constructing the regression model.

(See Topic 20)

6. B $\text{SER} = \sqrt{\frac{145.6}{180-3-1}} = 0.910$

unadjusted $R^2 = \frac{357.2-145.6}{357.2} = 0.592$

(See Topic 22)

7. C ESS = 357.2 – 145.6 = 211.6, *F*-statistic = (211.6 / 3) / (145.6 / 176) = 85.3. The critical value for a one-tailed 5% *F*-test with 3 and 176 degrees of freedom is 2.66. Because the *F*-statistic is greater than the critical *F*-value, the null hypothesis that all of the independent variables are simultaneously equal to zero should be rejected.

(See Topic 23)

8. C The confidence interval is 9.2 +/– (5.3 × 2.604), where 2.604 is the two-tailed 1% *t*-statistic with 176 degrees of freedom (which is the same as a one-tailed 0.5% *t*-statistic with 176 degrees of freedom). The interval is –4.6 to 23.0, which has a width of 27.6 and zero is in that interval.

(See Topic 23)

9. **B** Updated volatility estimate = $[\lambda \times (\text{volatility}_{t-1})^2 + (1 - \lambda) \times (\text{current return})^2]^{0.5}$

Current return = ln(price today / price yesterday)

$\ln(35/38) = -8.223\%$

$$\begin{aligned}\text{Updated volatility estimate} &= [0.75 \times (0.018)^2 + 0.25 \times (-0.08223)^2]^{0.5} \\ &= [0.000243 + 0.001690443]^{0.5} \\ &= 4.39\%\end{aligned}$$

(See Topic 28)

10. **D** Remember that when questions ask for volatility, they are referring to the standard deviation.

We first calculate the daily variance, which then needs to be adjusted to an annualized variance and finally we can take the square root to find the annualized volatility (standard deviation).

$$\begin{aligned}\text{Long-run daily variance} &= \alpha_0 / (1 - \alpha_1 - \beta) \\ &= 0.0000008 / (1 - 0.05 - 0.93) = 0.00004\end{aligned}$$

$$\text{Long-run daily standard deviation} = \sqrt{\text{variance}} = \sqrt{0.00004} = 0.6325\%$$

$$\begin{aligned}\text{Annualized standard deviation} &= \text{daily standard deviation} \times \sqrt{\text{time}} \\ &= 0.6325\% \times \sqrt{252} = 10.04\%\end{aligned}$$

(See Topic 28)

FORMULAS

Topic 15

joint probability: $P(AB) = P(A \mid B) \times P(B)$

conditional probability: $P(A \mid B) = \dfrac{P(AB)}{P(B)}$

independent events: $P(A \mid B) = P(A)$

Topic 16

expected value: $E(X) = \Sigma P(x_i)x_i$

variance: $\text{Var}(X) = E[(X - \mu)^2]$

covariance: $\text{Cov}(R_i, R_j) = E\{[R_i - E(R_i)][R_j - E(R_j)]\}$

correlation: $\text{Corr}(R_i, R_j) = \dfrac{\text{Cov}(R_i, R_j)}{\sigma(R_i)\sigma(R_j)}$

portfolio variance: $\text{Var}(R_P) = w_A^2\sigma^2(R_A) + w_B^2\sigma^2(R_B) + 2w_Aw_B\sigma(R_A)\sigma(R_B)\rho(R_A,R_B)$

$$\text{skewness} = \frac{E\left[(R-\mu)^3\right]}{\sigma^3}$$

$$\text{kurtosis} = \frac{E\left[(R-\mu)^4\right]}{\sigma^4}$$

Topic 17

Poisson distribution: $P(X = x) = \dfrac{\lambda^x e^{-\lambda}}{x!}$

binomial probability function: (number of ways to choose x from n) $p^x(1 - p)^{n-x}$

expected value of a binomial random variable: E(X) = np

variance of a binomial random variable: np(1 – p) = npq

uniform distribution range: $P(x_1 \le X \le x_2) = (x_2 - x_1)/(b - a)$

mean of uniform distribution: $E(x) = \dfrac{a+b}{2}$

variance of uniform distribution: $Var(x) = \dfrac{(b-a)^2}{12}$

Topic 18

Bayes' theorem: $P(A \mid B) = \dfrac{P(B \mid A) \times P(A)}{P(B)}$

Topic 19

population mean: $\mu = \dfrac{\sum_{i=1}^{N} X_i}{N}$

sample mean: $\overline{X} = \dfrac{\sum_{i=1}^{n} X_i}{n}$

population variance: $\sigma^2 = \dfrac{\sum_{i=1}^{N} (X_i - \mu)^2}{N}$

population standard deviation: $\sigma = \sqrt{\dfrac{\sum_{i=1}^{N} (X - \mu)^2}{N}}$

sample variance: $s^2 = \dfrac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n-1}$

sample standard deviation: $s = \sqrt{\dfrac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n-1}}$

sample covariance: $\text{covariance} = \sum_{i=1}^{n} \frac{\left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right)}{n-1}$

sample correlation coefficient: $r_{XY} = \frac{\text{Cov}(X,Y)}{(s_X)(s_Y)}$

$$z = \frac{\text{observation} - \text{population mean}}{\text{standard deviation}} = \frac{x - \mu}{\sigma}$$

sampling error of the mean = sample mean – population mean = $\bar{x} - \mu$

standard error of the sample mean: $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$

chi-squared test statistic: $\chi^2_{n-1} = \frac{(n-1)s^2}{\sigma_0^2}$

$$F\text{-test} = \frac{s_1^2}{s_2^2}$$

$$\text{test statistic} = \frac{\text{sample statistic} - \text{hypothesized value}}{\text{standard error of the sample statistic}}$$

confidence interval:

$$\left\{\left[\begin{matrix}\text{sample}\\\text{statistic}\end{matrix}\right] - \left(\begin{matrix}\text{critical}\\\text{value}\end{matrix}\right)\left(\begin{matrix}\text{standard}\\\text{error}\end{matrix}\right)\right] < \begin{matrix}\text{population}\\\text{parameter}\end{matrix} < \left[\begin{matrix}\text{sample}\\\text{statistic}\end{matrix} + \left(\begin{matrix}\text{critical}\\\text{value}\end{matrix}\right)\left(\begin{matrix}\text{standard}\\\text{error}\end{matrix}\right)\right]\right\}$$

t-statistic: $t_{n-1} = \frac{\bar{x} - \mu_0}{s / \sqrt{n}}$

z-statistic $= \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$

Topic 20

sample regression function: $Y_i = b_0 + b_1 \times X_i + e_i$

residual: $e_i = Y_i - (b_0 + b_1 \times X_i)$

regression slope coefficient: $b_1 = \dfrac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(X_i - \overline{X})^2} = \dfrac{\text{Cov}(X,Y)}{\text{Var}(X)}$

regression intercept: $b_0 = \overline{Y} - b_1\overline{X}$

where:
$\overline{Y}$ = mean of Y
$\overline{X}$ = mean of X

sum of squared residuals (SSR) = $\sum e_i^2 = \sum (Y_i - \hat{Y}_i)^2$

total sum of squares = explained sum of squares + sum of squared residuals

$$\sum(Y_i - \overline{Y})^2 = \sum(\hat{Y} - \overline{Y})^2 + \sum(Y_i - \hat{Y})^2$$

TSS = ESS + SSR

coefficient of determination:

$$R^2 = \frac{ESS}{TSS} = \frac{\sum(\hat{Y}_i - \overline{Y})^2}{\sum(Y_i - \overline{Y})^2}$$

$$R^2 = 1 - \frac{SSR}{TSS} = 1 - \frac{\sum(Y_i - \hat{Y}_i)^2}{\sum(Y_i - \overline{Y})^2}$$

Topic 22

standard error of the regression: $SER = \sqrt{s_e^2} = \sqrt{\dfrac{SSR}{n-k-1}}$

F-statistic = (ESS / df) / (SSR / df)

adjusted $R^2 = 1 - (1 - R^2) \times \dfrac{n-1}{n-k-1}$

Topic 23

homoskedasticity-only *F*-statistic: $F = \dfrac{(R_{ur}^2 - R_r^2)/m}{(1 - R_{ur}^2)/(n - k_{ur} - 1)}$

Topic 24

linear trend model: $y_t = \beta_0 + \beta_1(t)$

quadratic trend model: $y_t = \beta_0 + \beta_1(t) + \beta_2(t)^2$

exponential trend model: $y_t = \beta_0 e^{\beta_1(t)}$

mean squared error (MSE): $\text{MSE} = \dfrac{\sum_{t=1}^{T} e_t^2}{T}$

unbiased mean squared error (s^2): $s^2 = \left(\dfrac{T}{T-k}\right)\dfrac{\sum_{t=1}^{T} e_t^2}{T}$

Akaike information criterion (AIC): $\text{AIC} = e^{\left(\frac{2k}{T}\right)}\dfrac{\sum_{t=1}^{T} e_t^2}{T}$

Schwarz information criterion (SIC): $\text{SIC} = T^{\left(\frac{k}{T}\right)}\dfrac{\sum_{t=1}^{T} e_t^2}{T}$

Topic 25

pure seasonal dummy model: $y_t = \sum_{i=1}^{s} \gamma_i \left(D_{i,t}\right) + \varepsilon_t$

trend model with seasonality: $y_t = \beta_1(t) + \sum_{i=1}^{s} \gamma_i \left(D_{i,t}\right) + \varepsilon_t$

Topic 26

first-difference operator: $\Delta y_t = (1 - L)y_t = y_t - y_{t-1}$

Topic 27

first-order moving average [MA(1)] process:

$$y_t = \varepsilon_t + \theta\varepsilon_{t-1}$$

where:
y_t = the time series variable being estimated
ε_t = current random white noise shock
ε_{t-1} = one-period lagged random white noise shock
θ = coefficient for the lagged random shock

MA(q) process:

$$y_t = \varepsilon_t + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q}$$

where:
y_t = the time series variable being estimated
ε_t = current random white noise shock
ε_{t-1} = one-period lagged random white noise shock
ε_{t-q} = q^{th}-period lagged random white noise shock
θ = coefficients for the lagged random shocks

first-order autoregressive [AR(1)] process:

$$y_t = \phi y_{t-1} + \varepsilon_t$$

where:
y_t = the time series variable being estimated
y_{t-1} = one-period lagged observation of the variable being estimated
ε_t = current random white noise shock
ϕ = coefficient for the lagged observation of the variable being estimated

Yule-Walker equation: $\rho_t = \phi^t$ for $t = 0, 1, 2, \ldots$

AR(p) process:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + \varepsilon_t$$

where:
y_t = the time series variable being estimated
y_{t-1} = one-period lagged observation of the variable being estimated
y_{t-p} = p^{th}-period lagged observation of the variable being estimated
ε_t = current random white noise shock
ϕ = coefficients for the lagged observations of the variable being estimated

autoregressive moving average (ARMA) process:

$$y_t = \phi y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}$$

where:
y_t = the time series variable being estimated
ϕ = coefficient for the lagged observations of the variable being estimated
y_{t-1} = one-period lagged observation of the variable being estimated
ε_t = current random white noise shock
θ = coefficient for the lagged random shocks
ε_{t-1} = one-period lagged random white noise shock

Topic 28

the power law: $P(V > X) = K \times X^{-\alpha}$

continuously compounded return: $u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$

exponentially weighted moving average (EWMA) model (volatility):

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_{n-1}^2$$

where:
λ = weight on previous volatility estimate (λ between zero and one)

GARCH(1,1) model (volatility):

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta\sigma_{n-1}^2$$

where:
α = weighting on the previous period's return
β = weighting on the previous volatility estimate
ω = weighted long-run variance $= \gamma V_L$
V_L = long-run average variance $= \frac{\omega}{1-\alpha-\beta}$
$\alpha + \beta + \gamma = 1$
$\alpha + \beta < 1$ for stability so that γ is not negative

Topic 29

exponentially weighted moving average (EWMA) model (covariance):

$$\text{cov}_n = \lambda \text{cov}_{n-1} + (1-\lambda)X_{n-1}Y_{n-1}$$

where:
λ = the weight for the most recent covariance on day $n - 1$
X_{n-1} = the percentage change for variable X on day $n - 1$
Y_{n-1} = the percentage change for variable Y on day $n - 1$

GARCH(1,1) model (covariance): $\text{cov}_n = \omega + \alpha X_{n-1}Y_{n-1} + \beta \text{cov}_{n-1}$

covariance consistency condition: $\rho_{12}^2 + \rho_{13}^2 + \rho_{23}^2 - 2\rho_{12}\rho_{13}\rho_{23} \leq 1$

factor model: $U_i = \alpha_i F + \sqrt{1-\alpha_i^2}Z_i$

Using the Cumulative Z-Table

Probability Example

Assume that the annual earnings per share (EPS) for a large sample of firms is normally distributed with a mean of \$5.00 and a standard deviation of \$1.50. What is the approximate probability of an observed EPS value falling between \$3.00 and \$7.25?

If EPS = x = \$7.25, then $z = (x - \mu)/\sigma = (\$7.25 - \$5.00)/\$1.50 = +1.50$

If EPS = x = \$3.00, then $z = (x - \mu)/\sigma = (\$3.00 - \$5.00)/\$1.50 = -1.33$

For z-value of 1.50: Use the row headed 1.5 and the column headed 0 to find the value 0.9332. This represents the area under the curve to the left of the critical value 1.50.

For z-value of –1.33: Use the row headed 1.3 and the column headed 3 to find the value 0.9082. This represents the area under the curve to the left of the critical value +1.33. The area to the left of –1.33 is 1 – 0.9082 = 0.0918.

The area between these critical values is 0.9332 – 0.0918 = 0.8414, or 84.14%.

Hypothesis Testing—One-Tailed Test Example

A sample of a stock's returns on 36 non-consecutive days results in a mean return of 2.0%. Assume the population standard deviation is 20.0%. Can we say with 95% confidence that the mean return is greater than 0%?

H_0: $\mu \leq 0.0\%$, H_A: $\mu > 0.0\%$. The test statistic = *z*-statistic = $\frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$ = (2.0 – 0.0) / (20.0 / 6) = 0.60.

The significance level = 1.0 – 0.95 = 0.05, or 5%.

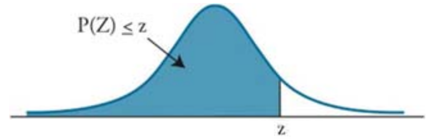
Since this is a one-tailed test with an alpha of 0.05, we need to find the value 0.95 in the cumulative *z*-table. The closest value is 0.9505, with a corresponding critical *z*-value of 1.65. Since the test statistic is less than the critical value, we fail to reject H_0.

Hypothesis Testing—Two-Tailed Test Example

Using the same assumptions as before, suppose that the analyst now wants to determine if he can say with 99% confidence that the stock's return is not equal to 0.0%.

H_0: $\mu = 0.0\%$, H_A: $\mu \neq 0.0\%$. The test statistic (*z*-value) = (2.0 – 0.0) / (20.0 / 6) = 0.60. The significance level = 1.0 – 0.99 = 0.01, or 1%.

Since this is a two-tailed test with an alpha of 0.01, there is a 0.005 rejection region in both tails. Thus, we need to find the value 0.995 (1.0 – 0.005) in the table. The closest value is 0.9951, which corresponds to a critical *z*-value of 2.58. Since the test statistic is less than the critical value, we fail to reject H_0 and conclude that the stock's return equals 0.0%.



Cumulative Z-Table

$P(Z \leq z) = N(z)$ for $z \geq 0$
$P(Z \leq -z) = 1 - N(z)$

z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.937	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.983	0.9834	0.9838	0.9842	0.9846	0.985	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.989
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.994	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

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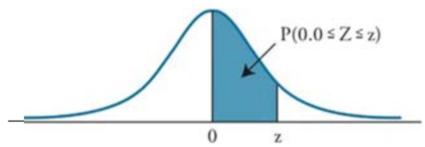

Alternative Z-Table

$P(Z \leq z) = N(z)$ for $z \geq 0$

$P(Z \leq -z) = 1 - N(z)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3356	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4939	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990

Student's T-Distribution

Level of Significance for One-Tailed Test						
df	0.100	0.050	0.025	0.01	0.005	0.0005
Level of Significance for Two-Tailed Test						
df	0.20	0.10	0.05	0.02	0.01	0.001
1	3.078	6.314	12.706	31.821	63.657	636.619
2	1.886	2.920	4.303	6.965	9.925	31.599
3	1.638	2.353	3.182	4.541	5.841	12.294
4	1.533	2.132	2.776	3.747	4.604	8.610
5	1.476	2.015	2.571	3.365	4.032	6.869
6	1.440	1.943	2.447	3.143	3.707	5.959
7	1.415	1.895	2.365	2.998	3.499	5.408
8	1.397	1.860	2.306	2.896	3.355	5.041
9	1.383	1.833	2.262	2.821	3.250	4.781
10	1.372	1.812	2.228	2.764	3.169	4.587
11	1.363	1.796	2.201	2.718	3.106	4.437
12	1.356	1.782	2.179	2.681	3.055	4.318
13	1.350	1.771	2.160	2.650	3.012	4.221
14	1.345	1.761	2.145	2.624	2.977	4.140
15	1.341	1.753	2.131	2.602	2.947	4.073
16	1.337	1.746	2.120	2.583	2.921	4.015
17	1.333	1.740	2.110	2.567	2.898	3.965
18	1.330	1.734	2.101	2.552	2.878	3.922
19	1.328	1.729	2.093	2.539	2.861	3.883
20	1.325	1.725	2.086	2.528	2.845	3.850
21	1.323	1.721	2.080	2.518	2.831	3.819
22	1.321	1.717	2.074	2.508	2.819	3.792
23	1.319	1.714	2.069	2.500	2.807	3.768
24	1.318	1.711	2.064	2.492	2.797	3.745
25	1.316	1.708	2.060	2.485	2.787	3.725
26	1.315	1.706	2.056	2.479	2.779	3.707
27	1.314	1.703	2.052	2.473	2.771	3.690
28	1.313	1.701	2.048	2.467	2.763	3.674
29	1.311	1.699	2.045	2.462	2.756	3.659
30	1.310	1.697	2.042	2.457	2.750	3.646
40	1.303	1.684	2.021	2.423	2.704	3.551
60	1.296	1.671	2.000	2.390	2.660	3.460
120	1.289	1.658	1.980	2.358	2.617	3.373
∞	1.282	1.645	1.960	2.326	2.576	3.291

F-Table at 5%

Critical values of the *F*-distribution at a 5% level of significance

Degrees of freedom for the numerator along top row
Degrees of freedom for the denominator along side row

	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40
1	161	200	216	225	230	234	237	239	241	242	244	246	248	249	250	251
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.5	19.5	19.5
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	6.01	2.94	2.90	2.86	2.83
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39

F-Table at 2.5%

Critical values of the *F*-distribution at a 2.5% level of significance

Degrees of freedom for the numerator along top row
Degrees of freedom for the denominator along side row

	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40
1	648	799	864	900	922	937	948	957	963	969	977	985	993	997	1001	1006
2	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.43	39.45	39.46	39.46	39.47
3	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.34	14.25	14.17	14.12	14.08	14.04
4	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.75	8.66	8.56	8.51	8.46	8.41
5	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.52	6.43	6.33	6.28	6.23	6.18
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.37	5.27	5.17	5.12	5.07	5.01
7	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.67	4.57	4.47	4.41	4.36	4.31
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.20	4.10	4.00	3.95	3.89	3.84
9	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.87	3.77	3.67	3.61	3.56	3.51
10	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.62	3.52	3.42	3.37	3.31	3.26
11	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.43	3.33	3.23	3.17	3.12	3.06
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.28	3.18	3.07	3.02	2.96	2.91
13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.15	3.05	2.95	2.89	2.84	2.78
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.05	2.95	2.84	2.79	2.73	2.67
15	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.96	2.86	2.76	2.70	2.64	2.59
16	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.89	2.79	2.68	2.63	2.57	2.51
17	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98	2.92	2.82	2.72	2.62	2.56	2.50	2.44
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.77	2.67	2.56	2.50	2.44	2.38
19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.72	2.62	2.51	2.45	2.39	2.33
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.68	2.57	2.46	2.41	2.35	2.29
21	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80	2.73	2.64	2.53	2.42	2.37	2.31	2.25
22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.60	2.50	2.39	2.33	2.27	2.21
23	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73	2.67	2.57	2.47	2.36	2.30	2.24	2.18
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.54	2.44	2.33	2.27	2.21	2.15
25	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.51	2.41	2.30	2.24	2.18	2.12
30	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.41	2.31	2.20	2.14	2.07	2.01
40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.29	2.18	2.07	2.01	1.94	1.88
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.17	2.06	1.94	1.88	1.82	1.74
120	5.15	3.80	3.23	2.89	2.67	2.52	2.39	2.30	2.22	2.16	2.05	1.94	1.82	1.76	1.69	1.61
∞	5.02	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.94	1.83	1.71	1.64	1.57	1.48

Chi-Squared Table

Values of χ^2 (Degrees of Freedom, Level of Significance)
Probability in Right Tail

Degrees of Freedom	0.99	0.975	0.95	0.9	0.1	0.05	0.025	0.01	0.005
1	0.000157	0.000982	0.003932	0.0158	2.706	3.841	5.024	6.635	7.879
2	0.020100	0.050636	0.102586	0.2107	4.605	5.991	7.378	9.210	10.597
3	0.1148	0.2158	0.3518	0.5844	6.251	7.815	9.348	11.345	12.838
4	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.554	0.831	1.145	1.610	9.236	11.070	12.832	15.086	16.750
6	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.647	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	4.107	5.009	5.892	7.041	19.812	22.362	24.736	27.688	29.819
14	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	42.796
23	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.558
25	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	12.878	14.573	16.151	18.114	36.741	40.113	43.195	46.963	49.645
28	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.994
29	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.335
30	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
50	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
80	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
100	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.170