FRM Part I Book 2: Quantitative Analysis

FRM 2017 PART I BOOK 2: QUANTITATIVE ANALYSIS

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Reading Assignments and Learning Objectives

The following material is a review of the Quantitative Analysis principles designed to address the learning objectives set forth by the Global Association of Risk Professionals.

Reading Assignments

Michael Miller, *Mathematics and Statistics for Financial Risk Management, 2nd Edition* (Hoboken, NJ: John Wiley & Sons, 2013).

James Stock and Mark Watson, *Introduction to Econometrics, Brief Edition* (Boston: Pearson, 2008).

Francis X. Diebold, *Elements of Forecasting, 4th Edition* (Mason, Ohio: Cengage Learning, 2006).

John Hull, *Risk Management and Financial Institutions, 4th Edition* (Hoboken, NJ: John Wiley & Sons, 2015).

Learning Objectives

15. Probabilities

After completing this reading, you should be able to:

1. Describe and distinguish between continuous and discrete random variables. (page 13)
2. Define and distinguish between the probability density function, the cumulative distribution function, and the inverse cumulative distribution function. (page 15)
3. Calculate the probability of an event given a discrete probability function. (page 16)
4. Distinguish between independent and mutually exclusive events. (page 19)
5. Define joint probability, describe a probability matrix, and calculate joint probabilities using probability matrices. (page 21)
6. Define and calculate a conditional probability, and distinguish between conditional and unconditional probabilities. (page 18)

16. Basic Statistics

After completing this reading, you should be able to:

1. Interpret and apply the mean, standard deviation, and variance of a random variable. (page 29)
2. Calculate the mean, standard deviation, and variance of a discrete random variable. (page 29)
3. Interpret and calculate the expected value of a discrete random variable. (page 34)
4. Calculate and interpret the covariance and correlation between two random variables. (page 38)
5. Calculate the mean and variance of sums of variables. (page 34)
6. Describe the four central moments of a statistical variable or distribution: mean, variance, skewness and kurtosis. (page 42)
7. Interpret the skewness and kurtosis of a statistical distribution, and interpret the concepts of coskewness and cokurtosis. (page 44)
8. Describe and interpret the best linear unbiased estimator. (page 48)

17. Distributions

After completing this reading, you should be able to:

1. Distinguish the key properties among the following distributions: uniform distribution, Bernoulli distribution, Binomial distribution, Poisson distribution, normal distribution, lognormal distribution, Chi-squared distribution, Student's t, and F-distributions, and identify common occurrences of each distribution. (page 53)
2. Describe the central limit theorem and the implications it has when combining independent and identically distributed (i.i.d.) random variables. (page 66)
3. Describe i.i.d. random variables and the implications of the i.i.d. assumption when combining random variables. (page 66)
4. Describe a mixture distribution and explain the creation and characteristics of mixture distributions. (page 70)

18. Bayesian Analysis

After completing this reading, you should be able to:

1. Describe Bayes' theorem and apply this theorem in the calculation of conditional probabilities. (page 75)

2. Compare the Bayesian approach to the frequentist approach. (page 80)
3. Apply Bayes' theorem to scenarios with more than two possible outcomes and calculate posterior probabilities. (page 81)

19. Hypothesis Testing and Confidence Intervals

After completing this reading, you should be able to:
1. Calculate and interpret the sample mean and sample variance. (page 90)
2. Construct and interpret a confidence interval. (page 96)
3. Construct an appropriate null and alternative hypothesis, and calculate an appropriate test statistic. (page 100)
4. Differentiate between a one-tailed and a two-tailed test and identify when to use each test. (page 102)
5. Interpret the results of hypothesis tests with a specific level of confidence. (page 113)
6. Demonstrate the process of backtesting VaR by calculating the number of exceedances. (page 121)

20. Linear Regression with One Regressor

After completing this reading, you should be able to:
1. Explain how regression analysis in econometrics measures the relationship between dependent and independent variables. (page 128)
2. Interpret a population regression function, regression coefficients, parameters, slope, intercept, and the error term. (page 129)
3. Interpret a sample regression function, regression coefficients, parameters, slope, intercept, and the error term. (page 130)
4. Describe the key properties of a linear regression. (page 131)
5. Define an ordinary least squares (OLS) regression and calculate the intercept and slope of the regression. (page 132)
6. Describe the method and three key assumptions of OLS for estimation of parameters. (page 133)
7. Summarize the benefits of using OLS estimators. (page 133)
8. Describe the properties of OLS estimators and their sampling distributions, and explain the properties of consistent estimators in general. (page 133)
9. Interpret the explained sum of squares, the total sum of squares, the residual sum of squares, the standard error of the regression, and the regression R^2.) (page 134)
10. Interpret the results of an OLS regression. (page 134)

21. Regression with a Single Regressor: Hypothesis Tests and Confidence Intervals

After completing this reading, you should be able to:
1. Calculate, and interpret confidence intervals for regression coefficients. (page 142)
2. Interpret the p-value. (page 144)
3. Interpret hypothesis tests about regression coefficients. (page 143)
4. Evaluate the implications of homoskedasticity and heteroskedasticity. (page 147)
5. Determine the conditions under which the OLS is the best linear conditionally unbiased estimator. (page 149)
6. Explain the Gauss-Markov Theorem and its limitations, and alternatives to the OLS. (page 149)
7. Apply and interpret the t-statistic when the sample size is small. (page 150)

22. Linear Regression with Multiple Regressors

After completing this reading, you should be able to:
1. Define and interpret omitted variable bias, and describe the methods for addressing this bias. (page 156)
2. Distinguish between single and multiple regression. (page 157)
3. Interpret the slope coefficient in a multiple regression. (page 158)
4. Describe homoskedasticity and heteroskedasticity in a multiple regression. (page 159)
5. Describe the OLS estimator in a multiple regression. (page 157)
6. Calculate and interpret measures of fit in multiple regression. (page 159)
7. Explain the assumptions of the multiple linear regression model. (page 162)
8. Explain the concept of imperfect and perfect multicollinearity and their implications. (page 162)

23. Hypothesis Tests and Confidence Intervals in Multiple Regression

After completing this reading, you should be able to:
1. Construct, apply, and interpret hypothesis tests and confidence intervals for a single coefficient in a multiple regression. (page 170)
2. Construct, apply, and interpret joint hypothesis tests and confidence intervals for multiple coefficients in a multiple regression. (page 176)
3. Interpret the F-statistic. (page 176)
4. Interpret tests of a single restriction involving multiple coefficients. (page 182)
5. Interpret confidence sets for multiple coefficients. (page 176)
6. Identify examples of omitted variable bias in multiple regressions. (page 183)
7. Interpret the R^2 and adjusted R^2 in a multiple regression. (page 181)

24. Modeling and Forecasting Trend

After completing this reading, you should be able to:
1. Describe linear and nonlinear trends. (page 189)
2. Describe trend models to estimate and forecast trends. (page 192)
3. Compare and evaluate model selection criteria, including mean squared error (MSE), s^2, the Akaike information criterion (AIC), and the Schwarz information criterion (SIC). (page 197)
4. Explain the necessary conditions for a model selection criterion to demonstrate consistency. (page 200)

25. Modeling and Forecasting Seasonality

After completing this reading, you should be able to:
1. Describe the sources of seasonality and how to deal with it in time series analysis. (page 206)
2. Explain how to use regression analysis to model seasonality. (page 208)
3. Explain how to construct an h-step-ahead point forecast. (page 210)

26. Characterizing Cycles

After completing this reading, you should be able to:
1. Define covariance stationary, autocovariance function, autocorrelation function, partial autocorrelation function, and autoregression. (page 214)
2. Describe the requirements for a series to be covariance stationary. (page 215)
3. Explain the implications of working with models that are not covariance stationary. (page 215)

4. Define white noise, and describe independent white noise and normal (Gaussian) white noise. (page 215)
5. Explain the characteristics of the dynamic structure of white noise. (page 215)
6. Explain how a lag operator works. (page 216)
7. Describe Wold's theorem.(page 216)
8. Define a general linear process.(page 216)
9. Relate rational distributed lags to Wold's theorem. (page 216)
10. Calculate the sample mean and sample autocorrelation, and describe the Box-Pierce Q-statistic and the Ljung-Box Q-statistic. (page 217)
11. Describe sample partial autocorrelation. (page 217)

27. Modeling Cycles: MA, AR, and ARMA Models

After completing this reading, you should be able to:
1. Describe the properties of the first-order moving average (MA(1)) process, and distinguish between autoregressive representation and moving average representation. (page 223)
2. Describe the properties of a general finite-order process of order q (MA(q)) process. (page 225)
3. Describe the properties of the first-order autoregressive (AR(1)) process, and define and explain the Yule-Walker equation. (page 225)
4. Describe the properties of a general p^{th} order autoregressive (AR(p)) process. (page 227)
5. Define and describe the properties of the autoregressive moving average (ARMA) process. (page 227)
6. Describe the application of AR and ARMA processes. (page 228)

28. Volatility

After completing this reading, you should be able to:
1. Define and distinguish between volatility, variance rate, and implied volatility. (page 233)
2. Describe the power law. (page 234)
3. Explain how various weighting schemes can be used in estimating volatility. (page 236)
4. Apply the exponentially weighted moving average (EWMA) model to estimate volatility. (page 237)
5. Describe the generalized autoregressive conditional heteroskedasticity (GARCH (p,q)) model for estimating volatility and its properties. (page 238)
6. Calculate volatility using the GARCH(1,1) model. (page 238)
7. Explain mean reversion and how it is captured in the GARCH(1,1) model. (page 239)
8. Explain the weights in the EWMA and GARCH(1,1) models. (page 237)
9. Explain how GARCH models perform in volatility forecasting. (page 240)
10. Describe the volatility term structure and the impact of volatility changes. (page 240)

29. Correlations and Copulas

After completing this reading, you should be able to:
1. Define correlation and covariance and differentiate between correlation and dependence. (page 245)
2. Calculate covariance using the EWMA and GARCH(1,1) models. (page 247)

3. Apply the consistency condition to covariance. (page 250)
4. Describe the procedure of generating samples from a bivariate normal distribution. (page 251)
5. Describe properties of correlations between normally distributed variables when using a one-factor model. (page 252)
6. Define copula and describe the key properties of copulas and copula correlation. (page 252)
7. Explain tail dependence. (page 256)
8. Describe the Gaussian copula, Student's t-copula, multivariate copula, and one factor copula. (page 255)

30. Simulation Methods

After completing this reading, you should be able to:

1. Describe the basic steps to conduct a Monte Carlo simulation. (page 263)
2. Describe ways to reduce Monte Carlo sampling error. (page 264)
3. Explain how to use antithetic variate technique to reduce Monte Carlo sampling error. (page 265)
4. Explain how to use control variates to reduce Monte Carlo sampling error and when it is effective. (page 266)
5. Describe the benefits of reusing sets of random number draws across Monte Carlo experiments and how to reuse them. (page 267)
6. Describe the bootstrapping method and its advantage over Monte Carlo simulation. (page 268)
7. Describe the pseudo-random number generation method and how a good simulation design alleviates the effects the choice of the seed has on the properties of the generated series. (page 269)
8. Describe situations where the bootstrapping method is ineffective. (page 269)
9. Describe disadvantages of the simulation approach to financial problem solving. (page 270)

The Time Value of Money

Exam Focus

This optional reading provides a tutorial for time value of money (TVM) calculations. Understanding how to use your financial calculator to make these calculations will be very beneficial as you proceed through the curriculum. In particular, for the fixed income material in Book 4, FRM candidates should be able to perform present value calculations using TVM functions. We have included Concept Checkers at the end of this reading for additional practice with these concepts.

Time Value of Money Concepts and Applications

The concept of **compound interest** or **interest on interest** is deeply embedded in time value of money (TVM) procedures. When an investment is subjected to compound interest, the growth in the value of the investment from period to period reflects not only the interest earned on the original principal amount but also on the interest earned on the previous period's interest earnings—the interest on interest.

TVM applications frequently call for determining the **future value** (FV) of an investment's cash flows as a result of the effects of compound interest. Computing FV involves projecting the cash flows forward, on the basis of an appropriate compound interest rate, to the end of the investment's life. The computation of the **present value** (PV) works in the opposite direction—it brings the cash flows from an investment back to the beginning of the investment's life based on an appropriate compound rate of return.

Being able to measure the PV and/or FV of an investment's cash flows becomes useful when comparing investment alternatives because the value of the investment's cash flows must be measured at some common point in time, typically at the end of the investment horizon (FV) or at the beginning of the investment horizon (PV).

Using a Financial Calculator

It is very important that you be able to use a financial calculator when working TVM problems because the FRM exam is constructed under the assumption that candidates have the ability to do so. There is simply no other way that you will have time to solve TVM problems. *GARP allows only four types of calculators to be used for the exam—the TI BAII Plus® (including the BAII Plus Professional), the HP 12C® (including the HP 12C Platinum), the HP 10bII®, and the HP 20b®. This reading is written primarily with the TI BAII Plus in mind.* If you don't already own a calculator, go out and buy a *TI BAII Plus!* However, if you already own one of the HP models listed and are comfortable with it, by all means continue to use it.

The TI BAII Plus comes preloaded from the factory with the periods per year function (P/Y) set to 12. This automatically converts the annual interest rate (I/Y) into monthly rates. While appropriate for many loan-type problems, this feature is not suitable for the vast majority of the TVM applications we will be studying. So prior to using our Study Notes, please set your P/Y key to "1" using the following sequence of keystrokes:

[2nd] [P/Y] "1" [ENTER] [2nd] [QUIT]

As long as you do not change the P/Y setting, it will remain set at one period per year until the battery from your calculator is removed (it does not change when you turn the calculator on and off). If you want to check this setting at any time, press [2nd] [P/Y]. The display should read P/Y = 1.0. If it does, press [2nd] [QUIT] to get out of the "programming" mode. If it doesn't, repeat the procedure previously described to set the P/Y key. With P/Y set to equal 1, it is now possible to think of I/Y as the interest rate per compounding period and N as the number of compounding periods under analysis. Thinking of these keys in this way should help you keep things straight as we work through TVM problems.

Before we begin working with financial calculators, you should familiarize yourself with your TI by locating the TVM keys noted below. These are the only keys you need to know to work virtually all TVM problems.

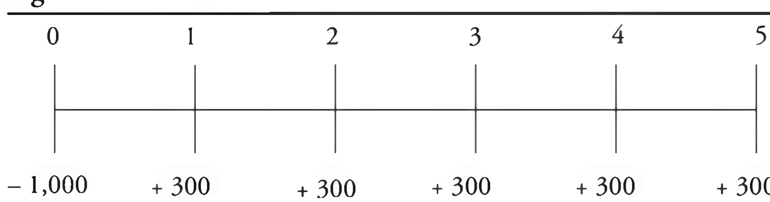
- N = Number of compounding periods.
- I/Y = Interest rate per compounding period.
- PV = Present value.
- FV = Future value.
- PMT = Annuity payments, or constant periodic cash flow.
- CPT = Compute.

Time Lines

It is often a good idea to draw a time line before you start to solve a TVM problem. A **time line** is simply a diagram of the cash flows associated with a TVM problem. A cash flow that occurs in the present (today) is put at time zero. Cash outflows (payments) are given a negative sign, and cash inflows (receipts) are given a positive sign. Once the cash flows are assigned to a time line, they may be moved to the beginning of the investment period to calculate the PV through a process called **discounting** or to the end of the period to calculate the FV using a process called **compounding**.

Figure 1 illustrates a time line for an investment that costs $1,000 today (outflow) and will return a stream of cash payments (inflows) of $300 per year at the end of each of the next five years.

Figure 1: Time Line



Please recognize that the cash flows occur at the end of the period depicted on the time line. Furthermore, note that the end of one period is the same as the beginning of the next period. For example, the end of the second year (t = 2) is the same as the beginning of the third year, so a cash flow at the beginning of year 3 appears at time t = 2 on the time line. Keeping this convention in mind will help you keep things straight when you are setting up TVM problems.

Professor's Note: Throughout the problems in this reading, rounding differences may occur between the use of different calculators or techniques presented in this document. So don't panic if you are a few cents off in your calculations.

Interest rates are our measure of the time value of money, although risk differences in financial securities lead to differences in their equilibrium interest rates. Equilibrium interest rates are the **required rate of return** for a particular investment, in the sense that the market rate of return is the return that investors and savers require to get them to willingly lend their funds. Interest rates are also referred to as **discount rates** and, in fact, the terms are often used interchangeably. If an individual can borrow funds at an interest rate of 10%, then that individual should *discount* payments to be made in the future at that rate in order to get their equivalent value in current dollars or other currency. Finally, we can also view interest rates as the **opportunity cost** of current consumption. If the market rate of interest on one-year securities is 5%, earning an additional 5% is the opportunity forgone when current consumption is chosen rather than saving (postponing consumption).

The **real risk-free rate** of interest is a theoretical rate on a single period loan that has no expectation of inflation in it. When we speak of a real rate of return, we are referring to an investor's increase in purchasing power (after adjusting for inflation). Since expected inflation in future periods is not zero, the rates we observe on U.S. Treasury bills (T-bills), for example, are risk-free rates but not *real* rates of return. T-bill rates are *nominal risk-free rates* because they contain an *inflation premium*. The approximate relation here is:

nominal risk-free rate = real risk-free rate + expected inflation rate

Securities may have one or more **types of risk**, and each added risk increases the required rate of return on the security. These types of risk are:

- **Default risk.** The risk that a borrower will not make the promised payments in a timely manner.
- **Liquidity risk.** The risk of receiving less than fair value for an investment if it must be sold for cash quickly.
- **Maturity risk.** As we will cover in detail in the readings on debt securities in Book 4, the prices of longer-term bonds are more volatile than those of shorter-term bonds. Longer maturity bonds have more maturity risk than shorter-term bonds and require a maturity risk premium.

Each of these risk factors is associated with a risk premium that we add to the nominal risk-free rate to adjust for greater default risk, less liquidity, and longer maturity relative to a very liquid, short-term, default risk-free rate such as that on T-bills. We can write:

required interest rate on a security	=	nominal risk-free rate
	+	default risk premium
	+	liquidity premium
	+	maturity risk premium

Present Value of a Single Sum

The PV of a single sum is today's value of a cash flow that is to be received at some point in the future. In other words, it is the amount of money that must be invested today, at a given rate of return over a given period of time, in order to end up with a specified FV. As previously mentioned, the process for finding the PV of a cash flow is known as *discounting* (i.e., future cash flows are "discounted" back to the present). The interest rate used in the discounting process is commonly referred to as the **discount rate** but may also be referred to as the **opportunity cost**, **required rate of return**, and the **cost of capital**. Whatever you want to call it, it represents the annual compound rate of return that can be earned on an investment.

The relationship between PV and FV is as follows:

$$PV = FV \times \left[\frac{1}{(1+I/Y)^N}\right] = \frac{FV}{(1+I/Y)^N}$$

Note that for a single future cash flow, PV is always less than the FV whenever the discount rate is positive.

The quantity $1/(1 + I/Y)^N$ in the PV equation is frequently referred to as the **present value factor, present value interest factor,** or **discount factor** for a single cash flow at I/Y over N compounding periods.

Example: PV of a single sum

Given a discount rate of 9%, **calculate** the PV of a $1,000 cash flow that will be received in five years.

Answer:

To solve this problem, input the relevant data and compute PV.

N = 5; I/Y = 9; FV = 1,000; CPT → PV = –$649.93 (ignore the sign)

Professor's Note: With single sum PV problems, you can either enter FV as a positive number and ignore the negative sign on PV or enter FV as a negative number.

This relatively simple problem could also be solved using the following PV equation.

$$PV = \frac{1{,}000}{(1+0.09)^5} = \$649.93$$

On the TI, enter 1.09 [y^x] 5 [=] [1/x] [×] 1,000 [=].

The PV computed here implies that at a rate of 9%, an investor will be indifferent between $1,000 in five years and $649.93 today. Put another way, $649.93 is the amount that must be invested today at a 9% rate of return in order to generate a cash flow of $1,000 at the end of five years.

Annuities

An **annuity** is a stream of *equal cash flows* that occurs at *equal intervals* over a given period. Receiving $1,000 per year at the end of each of the next eight years is an example of an annuity. The *ordinary annuity* is the most common type of annuity. It is characterized by cash flows that occur at the *end* of each compounding period. This is a typical cash flow pattern for many investment and business finance applications.

Computing the FV or PV of an annuity with your calculator is no more difficult than it is for a single cash flow. You will know four of the five relevant variables and solve for the fifth (either PV or FV). The difference between single sum and annuity TVM problems is that instead of solving for the PV or FV of a single cash flow, we solve for the PV or FV of a stream of equal periodic cash flows, where the size of the periodic cash flow is defined by the payment (PMT) variable on your calculator.

Example: FV of an ordinary annuity

What is the future value of an ordinary annuity that pays $150 per year at the end of each of the next 15 years, given the investment is expected to earn a 7% rate of return?

Answer:

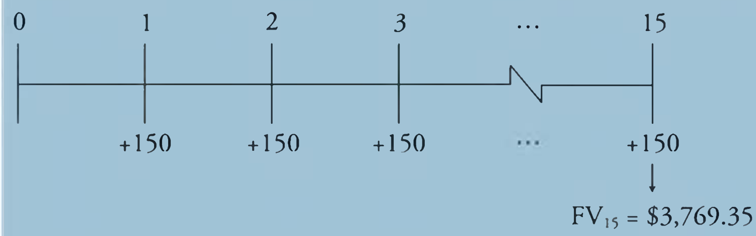
This problem can be solved by entering the relevant data and computing FV.

N = 15; I/Y = 7; PMT = –150; CPT → FV = $3,769.35

Implicit here is that PV = 0.

The time line for the cash flows in this problem is depicted in Figure 2.

Figure 2: FV of an Ordinary Annuity



As indicated here, the sum of the compounded values of the individual cash flows in this 15-year ordinary annuity is $3,769.35. Note that the annuity payments themselves amounted to $2,250 = 15 × $150, and the balance is the interest earned at the rate of 7% per year.

To find the PV of an ordinary annuity, we use the future cash flow stream, PMT, that we used with FV annuity problems, but we discount the cash flows back to the present (time = 0) rather than compounding them forward to the terminal date of the annuity.

Here again, the PMT variable is a *single* periodic payment, *not* the total of all the payments (or deposits) in the annuity. The PVA_O measures the collective PV of a stream of equal cash flows received at the end of each compounding period over a stated number of periods, N, given a specified rate of return, I/Y. The following example illustrates how to determine the PV of an ordinary annuity using a financial calculator.

Example: PV of an ordinary annuity

What is the PV of an annuity that pays $200 per year at the end of each of the next 13 years given a 6% discount rate?

Answer:

To solve this problem, enter the relevant information and compute PV.

N = 13; I/Y = 6; PMT = –200; CPT → PV = $1,770.54

The $1,770.54 computed here represents the amount of money that an investor would need to invest *today* at a 6% rate of return to generate 13 end-of-year cash flows of $200 each.

Present Value of a Perpetuity

A **perpetuity** is a financial instrument that pays a fixed amount of money at set intervals over an *infinite* period of time. In essence, a perpetuity is a perpetual annuity. British consul bonds and most preferred stocks are examples of perpetuities since they promise fixed interest or dividend payments forever. Without going into all the mathematical details, the discount factor for a perpetuity is just one divided by the appropriate rate of return (i.e., 1/r). Given this, we can compute the PV of a perpetuity.

$$PV_{perpetuity} = \frac{PMT}{I/Y}$$

The PV of a perpetuity is the fixed periodic cash flow divided by the appropriate periodic rate of return.

As with other TVM applications, it is possible to solve for unknown variables in the $PV_{perpetuity}$ equation. In fact, you can solve for any one of the three relevant variables, given the values for the other two.

Example: PV of a perpetuity

Assume the preferred stock of Kodon Corporation pays $4.50 per year in annual dividends and plans to follow this dividend policy forever. Given an 8% rate of return, what is the value of Kodon's preferred stock?

Answer:

Given that the value of the stock is the PV of all future dividends, we have:

$$PV_{perpetuity} = \frac{4.50}{0.08} = \$56.25$$

Thus, if an investor requires an 8% rate of return, the investor should be willing to pay $56.25 for each share of Kodon's preferred stock.

Example: Rate of return for a perpetuity

Using the Kodon preferred stock described in the preceding example, **determine** the rate of return that an investor would realize if she paid $75.00 per share for the stock.

Answer:

Rearranging the equation for $PV_{perpetuity}$, we get:

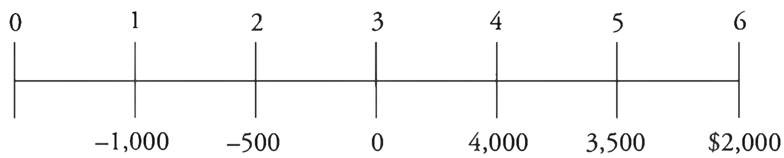
$$I/Y = \frac{PMT}{PV_{perpetuity}} = \frac{4.50}{75.00} = 0.06 = 6.0\%$$

This implies that the return (yield) on a $75 preferred stock that pays a $4.50 annual dividend is 6.0%.

PV and FV of Uneven Cash Flow Series

It is not uncommon to have applications in investments and corporate finance where it is necessary to evaluate a cash flow stream that is not equal from period to period. The time line in Figure 3 depicts such a cash flow stream.

Figure 3: Time Line for Uneven Cash Flows



This 6-year cash flow series is not an annuity since the cash flows are different every year. In fact, there is one year with zero cash flow and two others with negative cash flows. In essence, this series of uneven cash flows is nothing more than a stream of annual single sum cash flows. Thus, to find the PV or FV of this cash flow stream, all we need to do is sum the PVs or FVs of the individual cash flows.

Example: Computing the FV of an uneven cash flow series

Using a rate of return of 10%, **compute** the future value of the 6-year uneven cash flow stream described in Figure 3 at the end of the sixth year.

Answer:

The FV for the cash flow stream is determined by first computing the FV of each individual cash flow, then summing the FVs of the individual cash flows. Note that we need to preserve the signs of the cash flows.

FV_1: PV = –1,000; I/Y = 10; N = 5; CPT → FV = FV_1 = –1,610.51

FV_2: PV = –500; I/Y = 10; N = 4; CPT → FV = FV_2 = –732.05

FV_3: PV = 0; I/Y = 10; N = 3; CPT → FV = FV_3 = 0.00

FV_4: PV = 4,000; I/Y = 10; N = 2; CPT → FV = FV_4 = 4,840.00

FV_5: PV = 3,500; I/Y = 10; N = 1; CPT → FV = FV_5 = 3,850.00

FV_6: PV = 2,000; I/Y = 10; N = 0; CPT → FV = FV_6 = $\underline{2{,}000.00}$

FV of cash flow stream = $\sum FV_{\text{individual}}$ = 8,347.44

Example: Computing PV of an uneven cash flow series

Compute the present value of this 6-year uneven cash flow stream described in Figure 3 using a 10% rate of return.

Answer:

This problem is solved by first computing the PV of each individual cash flow, then summing the PVs of the individual cash flows, which yields the PV of the cash flow stream. Again the signs of the cash flows are preserved.

PV_1: FV = –1,000; I/Y = 10; N = 1; CPT → PV = PV_1 = –909.09

PV_2: FV = –500; I/Y = 10; N = 2; CPT → PV = PV_2 = –413.22

PV_3: FV = 0; I/Y = 10; N = 3; CPT → PV = PV_3 = 0

PV_4: FV = 4,000; I/Y = 10; N = 4; CPT → PV = PV_4 = 2,732.05

PV_5: FV = 3,500; I/Y = 10; N = 5; CPT → PV = PV_5 = 2,173.22

PV_6: FV = 2,000; I/Y = 10; N = 6; CPT → PV = PV_6 = <u>1,128.95</u>

PV of cash flow stream = $\sum PV_{individual}$ = $4,711.91

Solving TVM Problems When Compounding Periods are Other Than Annual

While the conceptual foundations of TVM calculations are not affected by the compounding period, more frequent compounding does have an impact on FV and PV computations. Specifically, since an increase in the frequency of compounding increases the effective rate of interest, it also *increases* the FV of a given cash flow and *decreases* the PV of a given cash flow.

Example: The effect of compounding frequency on FV and PV

Compute the FV and PV of a $1,000 single sum for an investment horizon of one year using a stated annual interest rate of 6.0% with a range of compounding periods.

Answer:

Figure 4: Compounding Frequency Effect

Compounding Frequency	*Interest Rate per Period*	*Effective Rate of Interest*	*Future Value*	*Present Value*
Annual (m = 1)	6.000%	6.000%	$1,060.00	$943.396
Semiannual (m = 2)	3.000	6.090	1,060.90	942.596
Quarterly (m = 4)	1.500	6.136	1,061.36	942.184
Monthly (m = 12)	0.500	6.168	1,061.68	941.905
Daily (m = 365)	0.016438	6.183	1,061.83	941.769

There are two ways to use your financial calculator to compute PVs and FVs under different compounding frequencies:

1. Adjust the number of periods per year (P/Y) mode on your calculator to correspond to the compounding frequency (e.g., for quarterly, P/Y = 4). **We do not recommend this approach!**
2. Keep the calculator in the annual compounding mode (P/Y = 1) and enter I/Y as the interest rate per compounding period, and *N* as the number of compounding periods in the investment horizon. Letting *m* equal the number of compounding periods per year, the basic formulas for the calculator input data are determined as follows:

$$\text{I/Y} = \text{the annual interest rate / m}$$

$$\text{N} = \text{the number of years} \times \text{m}$$

The computations for the FV and PV amounts in the previous example are:

PV_A: FV = –1,000; I/Y = 6/1 = 6; N = 1 × 1 = 1:
CPT → PV = PV_A = 943.396

PV_S: FV = –1,000; I/Y = 6/2 = 3; N = 1 × 2 = 2:
CPT → PV = PV_S = 942.596

PV_Q: FV = –1,000; I/Y = 6/4 = 1.5; N = 1 × 4 = 4:
CPT → PV = PV_Q = 942.184

PV_M: FV = –1,000; I/Y = 6/12 = 0.5; N = 1 × 12 = 12:
CPT → PV = PV_M = 941.905

PV_D: FV = –1,000; I/Y = 6/365 = 0.016438; N = 1 × 365 = 365:
CPT → PV = PV_D = 941.769

FV_A: PV = –1,000; I/Y = 6/1 = 6; N = 1 × 1 = 1:
CPT → FV = FV_A = 1,060.00

FV_S: PV = –1,000; I/Y = 6/2 = 3; N = 1 × 2 = 2:
CPT → FV = FV_S = 1,060.90

FV_Q: PV = –1,000; I/Y = 6/4 = 1.5; N = 1 × 4 = 4:
CPT → FV = FV_Q =1,061.36

FV_M: PV = –1,000; I/Y = 6/12 = 0.5; N = 1 × 12 = 12:
CPT → FV = FV_M = 1,061.68

FV_D: PV = –1,000; I/Y = 6/365 = 0.016438; N = 1 × 365 = 365:
CPT → FV = FV_D = 1,061.83

Example: FV of a single sum using quarterly compounding

Compute the FV of $2,000 today, five years from today using an interest rate of 12%, compounded quarterly.

Answer:

To solve this problem, enter the relevant data and compute FV:

N = 5 × 4 = 20; I/Y = 12 / 4 = 3; PV = –$2,000; CPT → FV = $3,612.22

Concept Checkers

1. The amount an investor will have in 15 years if $1,000 is invested today at an annual interest rate of 9% will be closest to:
   A. $1,350.
   B. $3,518.
   C. $3,642.
   D. $9,000.

2. How much must be invested today, at 8% interest, to accumulate enough to retire a $10,000 debt due seven years from today? The amount that must be invested today is closest to:
   A. $3,265.
   B. $5,835.
   C. $6,123.
   D. $8,794.

3. An analyst estimates that XYZ's earnings will grow from $3.00 a share to $4.50 per share over the next eight years. The rate of growth in XYZ's earnings is closest to:
   A. 4.9%.
   B. 5.2%.
   C. 6.7%.
   D. 7.0%.

4. If $5,000 is invested in a fund offering a rate of return of 12% per year, approximately how many years will it take for the investment to reach $10,000?
   A. 4 years.
   B. 5 years.
   C. 6 years.
   D. 7 years.

5. An investor is looking at a $150,000 home. If 20% must be put down and the balance is financed at 9% over the next 30 years, what is the monthly mortgage payment?
   A. $652.25.
   B. $799.33.
   C. $895.21.
   D. $965.55.

Concept Checker Answers

1. C N = 15; I/Y = 9; PV = –1,000; PMT = 0; CPT → FV = \$3,642.48

2. B N = 7; I/Y = 8; FV = –10,000; PMT = 0; CPT → PV = \$5,834.90

3. B N = 8; PV = –3; FV = 4.50; PMT = 0; CPT → I/Y = 5.1989

4. C PV = –5,000; I/Y = 12; FV = 10,000; PMT = 0; CPT → N = 6.12. Rule of 72 → 72/12 = six years.

   Note to HP12C users: One known problem with the HP12C is that it does not have the capability to round. In this particular question, you will come up with 7, although the correct answer is 6.1163.

5. D N = 30 × 12 = 360; I/Y = 9 / 12 = 0.75; PV = –150,000(1 – 0.2) = –120,000; FV = 0; CPT → PMT = \$965.55

The following is a review of the Quantitative Analysis principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Probabilities

Topic 15

Exam Focus

This topic covers important terms and concepts associated with probability theory. Random variables, events, outcomes, conditional probability, and joint probability are described. Specifically, we will examine the difference between discrete and continuous probability distributions, the difference between independent and mutually exclusive events, and the difference between unconditional and conditional probabilities. For the exam, be able to calculate probabilities based on the probability functions discussed.

Random Variables

LO 15.1: Describe and distinguish between continuous and discrete random variables.

- A **random variable** is an uncertain quantity/number.
- An **outcome** is an observed value of a random variable.
- An **event** is a single outcome or a set of outcomes.
- **Mutually exclusive events** are events that cannot happen at the same time.
- **Exhaustive events** are those that include all possible outcomes.

Consider rolling a 6-sided die. The number that comes up is a *random variable*. If you roll a 4, that is an *outcome*. Rolling a 4 is an event, and rolling an even number is an *event*. Rolling a 4 and rolling a 6 are *mutually exclusive events*. Rolling an even number and rolling an odd number is a set of mutually exclusive and *exhaustive events*.

A **probability distribution** describes the probabilities of all the possible outcomes for a random variable. The probabilities of all possible outcomes must sum to 1. A simple probability distribution is that for the roll of one fair die there are six possible outcomes and each one has a probability of 1/6, so they sum to 1. The probability distribution of all the possible returns on the S&P 500 Index for the next year is a more complex version of the same idea.

A **discrete random variable** is one for which the number of possible outcomes can be counted, and for each possible outcome, there is a measurable and positive probability. An example of a discrete random variable is the number of days it rains in a given month because there is a finite number of possible outcomes—the number of days it can rain in a month is defined by the number of days in the month.

A **probability function**, denoted p(x), specifies the probability that a random variable is equal to a specific value. More formally, p(x) is the probability that random variable X takes on the value x, or $p(x) = P(X = x)$.

The two key properties of a probability function are:

- $0 \le p(x) \le 1$.
- $\Sigma p(x) = 1$, the sum of the probabilities for *all* possible outcomes, *x*, for a random variable, *X*, equals 1.

Example: Evaluating a probability function

Consider the following function: $X = \{1, 2, 3, 4\}$, $p(x) = \frac{x}{10}$, else $p(x) = 0$

Determine whether this function satisfies the conditions for a probability function.

Answer:

Note that all of the probabilities are between 0 and 1, and the sum of all probabilities equals 1:

$$\sum p(x) = \frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \frac{4}{10} = 0.1 + 0.2 + 0.3 + 0.4 = 1$$

Both conditions for a probability function are satisfied.

A **continuous random variable** is one for which the number of possible outcomes is infinite, even if lower and upper bounds exist. The actual amount of daily rainfall between zero and 100 inches is an example of a continuous random variable because the actual amount of rainfall can take on an infinite number of values. Daily rainfall can be measured in inches, half inches, quarter inches, thousandths of inches, or even smaller increments. Thus, the number of possible daily rainfall amounts between zero and 100 inches is essentially infinite.

The assignment of probabilities to the possible outcomes for discrete and continuous random variables provides us with discrete probability distributions and continuous probability distributions. The difference between these types of distributions is most apparent for the following properties:

- For a *discrete distribution*, $p(x) = 0$ when *x* cannot occur, or $p(x) > 0$ if it can. Recall that $p(x)$ is read: "the probability that random variable $X = x$." For example, the probability of it raining 33 days in June is zero because this cannot occur, but the probability of it raining 25 days in June has some positive value.
- For a *continuous distribution*, $p(x) = 0$ even though *x* can occur. We can only consider $P(x_1 \le X \le x_2)$ where x_1 and x_2 are actual numbers. For example, the probability of receiving two inches of rain in June is zero because two inches is a single point in an infinite range of possible values. On the other hand, the probability of the amount of rain being between 1.99999999 and 2.00000001 inches has some positive value. In the case of continuous distributions, $P(x_1 \le X \le x_2) = P(x_1 < X < x_2)$ because $p(x_1) = p(x_2) = 0$.

In finance, some discrete distributions are treated as though they are continuous because the number of possible outcomes is very large. For example, the increase or decrease in the price of a stock traded on an American exchange is recorded in dollars and cents. Yet, the probability of a change of exactly \$1.33 or \$1.34 or any other specific change is almost zero. It is customary, therefore, to speak in terms of the probability of a range of possible price change, say between \$1.00 and \$2.00. In other words p(price change = 1.33) is essentially zero, but p(\$1 < price change < \$2) is greater than zero.

Distribution Functions

LO 15.2: Define and distinguish between the probability density function, the cumulative distribution function, and the inverse cumulative distribution function.

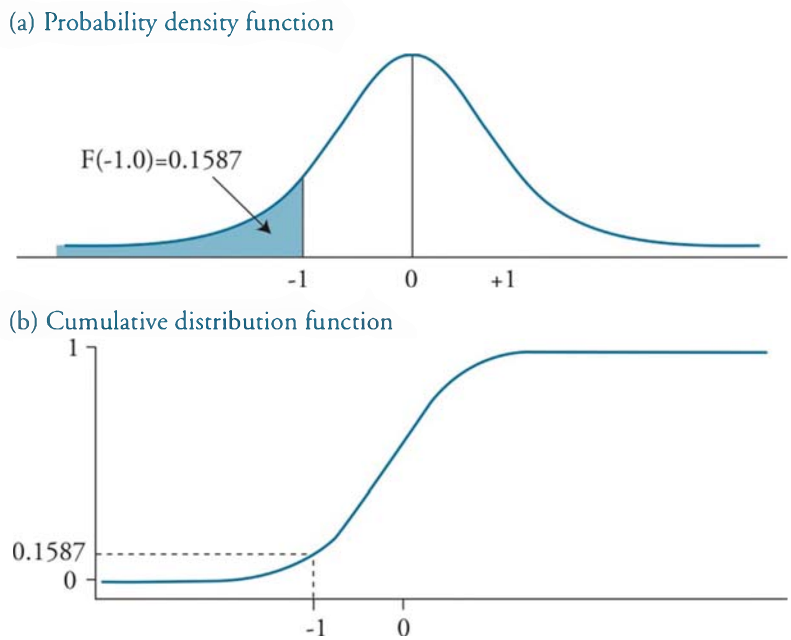
A **probability density function** (pdf) is a function, denoted f(x), that can be used to generate the probability that outcomes of a continuous distribution lie within a particular range of outcomes. For a continuous distribution, it is the equivalent of a *probability function* for a discrete distribution. Know that for a continuous distribution, the probability of any one particular outcome (of the infinite possible outcomes) is zero (e.g., the probability of receiving exactly two inches of rain in June is zero because two inches is a single point in an infinite range of possible values). A pdf is used to calculate the probability of an outcome between two values (i.e., the probability of the outcome falling within a specified range).

A **cumulative distribution function** (cdf), or simply *distribution function*, defines the probability that a random variable, *X*, takes on a value equal to or less than a specific value, *x*. It represents the sum, or *cumulative value*, of the probabilities for the outcomes up to and including a specified outcome. The cumulative distribution function for a random variable, *X*, may be expressed as $F(x) = P(X \leq x)$.

Consider the probability function defined earlier for X = {1, 2, 3, 4}, p(x) = x / 10. For this distribution, F(3) = 0.6 = 0.1 + 0.2 + 0.3, and F(4) = 1 = 0.1 + 0.2 + 0.3 + 0.4. This means that F(3) is the cumulative probability that outcomes 1, 2, or 3 occur, and F(4) is the cumulative probability that one of the possible outcomes occurs.

Figure 1 shows an example of a cumulative distribution function (for a standard normal distribution, described in Topic 17). There is a 15.87% probability of a value less than –1. This is the total area to the left of –1 in the pdf in Panel (a), and the y-axis value of the cdf for a value of –1 in Panel (b).

Figure 1: Standard Normal Probability Density and Cumulative Distribution Functions



Instead of finding the probability less than or equal to a specific value, x, the **inverse cumulative distribution function** can be used to find the value that corresponds to a specific probability. For example, it may be useful to know the value, x, where 15.87% of the distribution is less than or equal to x. From Figure 1, this value would be –1.

Consider a cumulative distribution function, $F(x) = p = x^2 / 25$, where $0 \leq x \leq 5$. F(3) finds the probability less than or equal to 3. In this case, $F(3) = 3^2 / 25 = 36\%$. The inverse function rearranges this cumulative function to instead input a probability and solve for x. Thus, the inverse cumulative distribution function in this example is: $F^{-1}(p) = x = 5\sqrt{p}$.

We can check the accuracy of this inverse function by testing the limits of the distribution ($0 \leq x \leq 5$). At p = 0, the minimum value is equal to 0, and at p = 1, the maximum value is equal to 5. By inputting a probability of 36% into the inverse function, we again see that 36% of the distribution is less than or equal to 3: $F^{-1}(0.36) = x = 5\sqrt{0.36} = 3$.

Discrete Probability Function

LO 15.3: Calculate the probability of an event given a discrete probability function.

A **discrete uniform random variable** is one for which the probabilities for all possible outcomes for a discrete random variable are equal. For example, consider the *discrete uniform probability distribution* defined as X = {1, 2, 3, 4, 5}, p(x) = 0.2. Here, the probability for each outcome is equal to 0.2 [i.e., p(1) = p(2) = p(3) = p(4) = p(5) = 0.2]. Also, the cumulative distribution function for the *n*th outcome, $F(x_n) = np(x)$, and the probability for a range of outcomes is p(x)k, where *k* is the number of possible outcomes in the range.

Example: Discrete uniform distribution

Determine p(6), F(6), and P($2 \le X \le 8$) for the discrete uniform distribution function defined as:

$$X = \{2, 4, 6, 8, 10\},\ p(x) = 0.2$$

Answer:

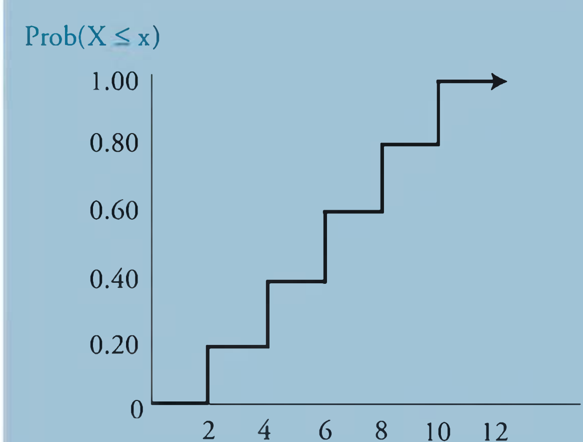
p(6) = 0.2, since p(x) = 0.2 for all *x*. F(6) = P($X \le 6$) = np(x) = 3(0.2) = 0.6. Note that n = 3 since 6 is the third outcome in the range of possible outcomes.

P($2 \le X \le 8$) = 4(0.2) = 0.8. Note that k = 4, since there are four outcomes in the range $2 \le X \le 8$. The following figures illustrate the concepts of a probability function and cumulative distribution function for this distribution.

Probability and Cumulative Distribution Functions

X = x	*Probability of x* *Prob (X = x)*	*Cumulative Distribution Function* *Prob (X < x)*
2	0.20	0.20
4	0.20	0.40
6	0.20	0.60
8	0.20	0.80

Cumulative Distribution Function for X ~ Uniform {2, 4, 6, 8, 10}



Conditional Probabilities

LO 15.6: Define and calculate a conditional probability, and distinguish between conditional and unconditional probabilities.

As noted earlier, there are two defining properties of probability:

- The probability of occurrence of any event (E_i) is between 0 and 1 (i.e., $0 \leq P(E_i) \leq 1$).
- If a set of events, E_1, E_2, … E_n, is mutually exclusive and exhaustive, the probabilities of those events sum to 1 (i.e., $\Sigma P(E_i) = 1$).

The first of the defining properties introduces the term $P(E_i)$, which is shorthand for the "probability of event *i*." If $P(E_i) = 0$, the event will never happen. If $P(E_i) = 1$, the event is certain to occur, and the outcome is not random.

The probability of rolling any one of the numbers 1–6 with a fair die is 1/6 = 0.1667 = 16.7%. The set of events—rolling a number equal to 1, 2, 3, 4, 5, or 6—is exhaustive, and the individual events are mutually exclusive, so the probability of this set of events is equal to 1. We are certain that one of the values in this set of events will occur.

Unconditional probability (i.e., *marginal probability*) refers to the probability of an event regardless of the past or future occurrence of other events. If we are concerned with the probability of an economic recession, regardless of the occurrence of changes in interest rates or inflation, we are concerned with the unconditional probability of a recession.

A **conditional probability** is one where the occurrence of one event affects the probability of the occurrence of another event. For example, we might be concerned with the probability of a recession *given* that the monetary authority increases interest rates. This is a conditional probability. The key word to watch for here is "given." Using probability notation, "the probability of A *given* the occurrence of B" is expressed as $P(A \mid B)$, where the vertical bar (|) indicates "given," or "conditional upon." For example, the probability of a recession *given* an increase in interest rates is expressed as P(*recession* | *increase in interest rates*). A conditional probability of an occurrence is also called its likelihood.

The **joint probability** of two events is the probability that they will both occur. We can calculate this from the conditional probability that A will occur given B occurs (a conditional probability) and the probability that B will occur (the unconditional probability of B). This calculation is sometimes referred to as the *multiplication rule of probability*. Using the notation for conditional and unconditional probabilities, we can express this rule as:

$$P(AB) = P(A \mid B) \times P(B)$$

This expression is read as follows: "The joint probability of A and B, P(AB), is equal to the conditional probability of A *given* B, $P(A \mid B)$, times the unconditional probability of B, P(B)."

This relationship can be rearranged to define the conditional probability of A given B as follows:

$$P(A \mid B) = \frac{P(AB)}{P(B)}$$

Example: Multiplication rule of probability

Consider the following information:

- P(I) = 0.4, the probability of the monetary authority increasing interest rates (I) is 40%.
- P(R | I) = 0.7, the probability of a recession (R) given an increase in interest rates is 70%.

What is P(RI), the joint probability of a recession *and* an increase in interest rates?

Answer:

Applying the multiplication rule, we get the following result:

$$P(RI) = P(R \mid I) \times P(I)$$

$$P(RI) = 0.7 \times 0.4$$

$$P(RI) = 0.28$$

Don't let the cumbersome notation obscure the simple logic of this result. If an interest rate increase will occur 40% of the time and lead to a recession 70% of the time when it occurs, the joint probability of an interest rate increase and a resulting recession is (0.4)(0.7) = (0.28) = 28%.

Independent and Mutually Exclusive Events

LO 15.4: Distinguish between independent and mutually exclusive events.

Independent events refer to events for which the occurrence of one has no influence on the occurrence of the others. The definition of independent events can be expressed in terms of conditional probabilities. Events A and B are independent if and only if:

$$P(A \mid B) = P(A), \text{ or equivalently, } P(B \mid A) = P(B)$$

If this condition is not satisfied, the events are **dependent events** (i.e., the occurrence of one is dependent on the occurrence of the other).

In our interest rate and recession example, recall that events I and R are not independent; the occurrence of I affects the probability of the occurrence of R. In this example, the independence conditions for I and R are violated because:

P(R) = 0.34, but P(R | I) = 0.7; the probability of a recession is greater when there is an increase in interest rates.

The best examples of independent events are found with the probabilities of dice tosses or coin flips. A die has "no memory." Therefore, the event of rolling a 4 on the second toss is independent of rolling a 4 on the first toss. This idea may be expressed as:

P(4 on second toss | 4 on first toss) = P(4 on second toss) = 1/6 or 0.167

The idea of independent events also applies to flips of a coin:

P(heads on first coin | heads on second coin) = P(heads on first coin) = 1/2 or 0.50

Calculating the Probability That at Least One of Two Events Will Occur

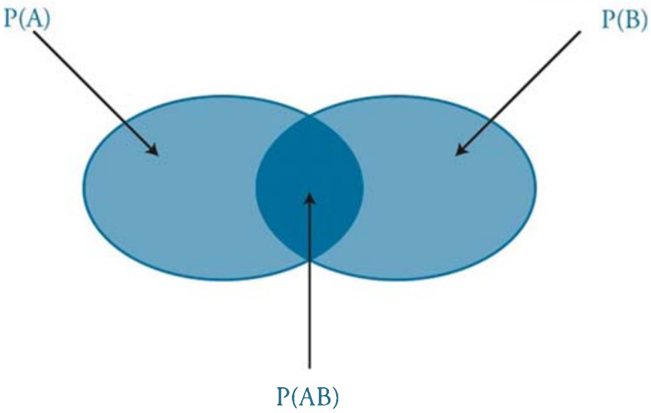
The *addition rule for probabilities* is used to determine the probability that at least one of two events will occur. For example, given two events, A and B, the addition rule can be used to determine the probability that either A or B will occur. If the events are *not* mutually exclusive, double counting must be avoided by subtracting the joint probability that *both* A and B will occur from the sum of the unconditional probabilities. This is reflected in the following general expression for the addition rule:

P(A or B) = P(A) + P(B) – P(AB)

For **mutually exclusive events** where the joint probability, P(AB), is zero, the probability that either A or B will occur is simply the sum of the unconditional probabilities for each event, P(A or B) = P(A) + P(B).

Figure 2 illustrates the addition rule with a Venn diagram and highlights why the joint probability must be subtracted from the sum of the unconditional probabilities. Note that if the events are mutually exclusive the sets do not intersect, P(AB) = 0, and the probability that one of the two events will occur is simply P(A) + P(B).

Figure 2: Venn Diagram for Events That Are Not Mutually Exclusive



Example: Addition rule of probability

Using the information in our previous interest rate and recession example and the fact that the unconditional probability of a recession, P(R), is 34%, **determine** the probability that either interest rates will increase *or* a recession will occur.

Answer:

Given that P(R) = 0.34, P(I) = 0.40, and P(RI) = 0.28, we can compute P(R or I) as follows:

P(R or I) = P(R) + P(I) – P(RI)

P(R or I) = 0.34 + 0.40 – 0.28

P(R or I) = 0.46

Calculating a Joint Probability of Any Number of Independent Events

LO 15.5: Define joint probability, describe a probability matrix, and calculate joint probabilities using probability matrices.

On the roll of two dice, the joint probability of getting two 4s is calculated as:

P(4 on first die and 4 on second die) = P(4 on first die) × P(4 on second die) = 1/6 × 1/6 = 1/36 = 0.0278

On the flip of two coins, the probability of getting two heads is:

P(heads on first coin and heads on second coin) = 1/2 × 1/2 = 1/4 = 0.25

Hint: When dealing with *independent events*, the word *and* indicates multiplication, and the word *or* indicates addition. In probability notation:

$$P(A \text{ or } B) = P(A) + P(B), \text{ and } P(A \text{ and } B) = P(A) \times P(B)$$

Professor's Note: On the exam, you may see A and B represented as A ∩ B. This notation means "the intersection of A and B" and refers to the event "both A and B." Similarly, you may see A or B represented as A ∪ B, which is "the union of A and B" and refers to the event "either A or B or both."

The multiplication rule we used to calculate the joint probability of two independent events may be applied to any number of independent events, as the following examples illustrate.

Example: Joint probability for more than two independent events (1)

What is the probability of rolling three 4s in one simultaneous toss of three dice?

Answer:

Since the probability of rolling a 4 for each die is 1/6, the probability of rolling three 4s is:

$$P(\text{three 4s on the roll of three dice}) = 1/6 \times 1/6 \times 1/6 = 1/216 = 0.00463$$

Similarly:

$$P(\text{four heads on the flip of four coins}) = 1/2 \times 1/2 \times 1/2 \times 1/2 = 1/16 = 0.0625$$

Example: Joint probability for more than two independent events (2)

Using empirical probabilities, suppose we observe that the DJIA has closed higher on two-thirds of all days in the past few decades. Furthermore, it has been determined that up and down days are independent. Based on this information, **compute** the probability of the DJIA closing higher for five consecutive days.

Answer:

$$P(\text{DJIA up five days in a row}) = 2/3 \times 2/3 \times 2/3 \times 2/3 \times 2/3 = (2/3)^5 = 0.132$$

Similarly:

$$P(\text{DJIA down five days in a row}) = 1/3 \times 1/3 \times 1/3 \times 1/3 \times 1/3 = (1/3)^5 = 0.004$$

Probability Matrix

Joint probabilities of independent events can be conveniently summarized using a probability matrix (sometimes known as a probability table). Suppose, for example, that we wanted to view how the state of the economy relates to the direction of interest rates. The probability matrix in Figure 3 shows the joint and unconditional probabilities of these two variables.

Figure 3: Joint and Unconditional Probabilities

		Interest Rates		
		Increase	No Increase	
Economy	Good	14%	6%	**20%**
	Normal	20%	30%	**50%**
	Poor	6%	24%	**30%**
		40%	**60%**	**100%**

From this probability matrix, we see that the joint probability of a poor economy and an increase in interest rates is 6%. Similarly, the joint probability of a normal economy and no increase in interest rates is 30%. Unconditional probabilities are shown as the sum of each column and each row. For example, the unconditional probability of a rate increase, irrespective of the state of the economy, is the sum of the joint probabilities, 14% + 20% + 6% = 40%. Also, the sum of all joint probabilities is equal to 100%, since one of these events must happen.

Example: Calculating joint probabilities using a probability matrix

Given the following incomplete probability matrix, **calculate** the joint probability of a normal economy and an increase in rates, and the unconditional probability of a good economy.

		Interest Rates		
		Increase	No Increase	
Economy	Good	15%	X2	**X3**
	Normal	X1	25%	**X4**
	Poor	10%	20%	**30%**
		50%	**50%**	**100%**

Answer:

Since the unconditional probability of an increase in rates, irrespective of the state of the economy, is 50%, we know the sum of each joint probability in the first column must equal 50%. By solving for X1, we find the joint probability of a normal economy and an increase in rates:

15% + X1 + 10% = 50%

X1 = 50% – 15% – 10% = 25%

The unconditional probability of a good economy, X3, can be computed by first solving for X2 (the joint probability of a good economy and no increase in interest rates) and then summing both joint probabilities in the first row.

X2 + 25% + 20% = 50%

X2 = 50% – 25% – 20% = 5%

X3 = 15% + X2 = 15% + 5% = 20%

Key Concepts

LO 15.1

A discrete random variable has positive probabilities associated with a finite number of outcomes.

A continuous random variable has positive probabilities associated with a range of outcome values—the probability of any single value is zero.

LO 15.2

A probability function specifies the probability that a random variable is equal to a specific value; P(X = x) = p(x).

A probability density function (pdf) is the expression for a probability function for a continuous random variable.

A cumulative distribution function (cdf) gives the probability of the random variable being equal to or less than each specific value. It is the area under the probability distribution to the left of a specified value.

LO 15.3

A discrete uniform distribution is one where there are *n* discrete, equally likely outcomes, so that for each outcome p(x) = 1/n.

LO 15.4

The probability of an independent event is unaffected by the occurrence of other events, but the probability of a dependent event is changed by the occurrence of another event.

Events A and B are independent if and only if:

P(A | B) = P(A), or equivalently, P(B | A) = P(B)

The probability that at least one of two events will occur is P(A or B) = P(A) + P(B) – P(AB). For mutually exclusive events, P(A or B) = P(A) + P(B), since P(AB) = 0.

LO 15.5

The joint probability of two events, P(AB), is the probability that they will both occur. P(AB) = P(A | B) × P(B). For independent events, P(A | B) = P(A), so that P(AB) = P(A) × P(B).

LO 15.6

Unconditional probability (marginal probability) is the probability of an event occurring.

Conditional probability, $P(A \mid B)$, is the probability of an event A occurring given that event B has occurred.

Concept Checkers

1. If events A and B are mutually exclusive, then:
   A. P(A | B) = P(A).
   B. P(A | B) = P(B).
   C. P(AB) = P(A) × P(B).
   D. P(A or B) = P(A) + P(B).

2. At a charity ball, 800 names were put into a hat. Four of the names are identical. On a random draw, what is the probability that one of these four names will be drawn?
   A. 0.004.
   B. 0.005.
   C. 0.010.
   D. 0.025.

3. Two events are said to be independent if the occurrence of one event:
   A. means the second event cannot occur.
   B. means the second event is certain to occur.
   C. affects the probability of the occurrence of the other event.
   D. does not affect the probability of the occurrence of the other event.

4. For a continuous random variable *X*, the probability of any single value of *X* is:
   A. one.
   B. zero.
   C. determined by the cdf.
   D. determined by the pdf.

5. Given the below incomplete probability matrix, what is the joint probability of a good economy and no increase in interest rates?

		Interest Rates		
		Increase	No Increase	
Economy	Good	20%	A	**B**
	Normal	C	20%	**D**
	Poor	10%	E	**20%**
		60%	**40%**	**100%**

   A. 0%.
   B. 10%.
   C. 20%.
   D. 30%.

Concept Checker Answers

1. D There is no intersection of events when events are mutually exclusive. P(AB) = P(A) × P(B) is only true for independent events. Note that since A and B are mutually exclusive (cannot both happen), P(A | B) and P(AB) must both be equal to zero, making answers A, B, and C incorrect.

2. B P(name 1 or name 2 or name 3 or name 4) = 1/800 + 1/800 + 1/800 + 1/800 = 4/800 = 0.005

3. D Two events are said to be independent if the occurrence of one event does not affect the probability of the occurrence of the other event.

4. B For a continuous distribution p(x) = 0 for all *X*; only ranges of value of *X* have positive probabilities.

5. B Because the unconditional probability of a poor economy, irrespective of interest rates, is 20%, we know that the sum of each joint probability in the poor economy row must equal 20%. By solving for *E*, we find the joint probability of a poor economy and no increase in rates:

   10% + E = 20%

   E = 20% – 10% = 10%

   The joint probability of a good economy and no increase in interest rates, *A*, can be computed by subtracting the joint probability of a normal economy and no increase in rates and the joint probability of a poor economy and no increase in rates from the unconditional probability of no increase in interest rates.

   A = 40% – 20% – E

   A = 40% – 20% – 10% = 10%