The following is a review of the Financial Markets and Products principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Interest Rates

Topic 37

Exam Focus

Spot, or zero, rates are computed from coupon bonds using a method known as bootstrapping. Forward rates can then be computed from the spot or zero curve. For the exam, understand how to use the bootstrapping method and how to compute forward rates from spot rates. Also, be familiar with the discrete and continuous compounding methods. Note that the fixed income readings in Book 4 will provide more information on the calculation of spot and forward rates as well as constructing the spot and forward rate curves. Duration and convexity are also mentioned in this topic but will be discussed in much more detail in Book 4.

Types of Rates

LO 37.1: Describe Treasury rates, LIBOR, and repo rates, and explain what is meant by the "risk-free" rate.

Three interest rates play a key role in interest rate derivatives: Treasury rates, LIBOR, and repo rates. Keep in mind that interest rates increase as the credit risk of the underlying instrument increases.

- **Treasury rates.** Treasury rates are the rates that correspond to government borrowing in its own currency. They are considered risk-free rates.
- **LIBOR.** The London Interbank Offered Rate (LIBOR) is the rate at which large international banks fund their activities. Some credit risk exists with LIBOR.
- **Repo rates.** The "repo" or repurchase agreement rate is the implied rate on a repurchase agreement. In a repo agreement, one party agrees to sell a security to another with the understanding that the selling party will buy it back later at a specified higher price. The interest rate implied by the price differential is the repo rate. The most common repo is the overnight repurchase agreement. Longer-term agreements are called term repos. Depending on the parties and structure involved, there is some credit risk with repurchase agreements.

Professor's Note: You may see reference to an inverse floater (a.k.a. reverse floater) on the exam. Just know that an inverse floater is a debt instrument whose coupon payments fluctuate inversely with the reference rate (e.g., LIBOR). For example, the inverse floater's coupon rate will increase when LIBOR decreases and vice versa.

As mentioned, Treasury rates (such as T-bill and T-bond rates) are often considered the benchmark for nominal risk-free rates. However, derivative traders view these rates as being too low to be considered risk free (since part of the demand for these bonds comes from fulfilling regulatory requirements, which drives prices up and rates down). As a result, traders instead use LIBOR rates for short-term risk-free rates, because LIBOR better reflects a trader's opportunity cost of capital.

Compounding

LO 37.2: Calculate the value of an investment using different compounding frequencies.

LO 37.3: Convert interest rates based on different compounding frequencies.

Derivative pricing often uses a framework called continuous time mathematics. In this framework, it is assumed that returns are continuously compounded. This is a theoretical construct only, as returns cannot literally be compounded continuously. Fortunately, converting discrete compounding to continuous compounding is straightforward.

If we have an initial investment of *A* that earns an annual rate *R*, compounded *m* times a year for *n* years, then it has a future value of:

$$FV_1 = A\left(1+\frac{R}{m}\right)^{m\times n}$$

If our same investment is continuously compounded over that period, it has a future value of:

$$FV_2 = Ae^{R\times n}$$

For any rate, *R*, FV_2 will always be greater than FV_1. The difference will decrease as *m* increases. In fact, as *m* becomes infinitely large, the difference goes to zero.

In most circumstances rates are discretely compounded, so we need to find the continuously compounded rate that gives the same future value. Using the previous two equations, the goal is to solve the following:

$$A\left(1+\frac{R}{m}\right)^{m\times n} = Ae^{R_c n}$$

where:
R_c = the continuous rate

We can solve for R_c as:

$$R_c = m\times \ln\left(1+\frac{R}{m}\right)$$

We can also solve for *R* as:

$$R = m\left(e^{R_c/m}-1\right)$$

Professor's Note: In order to algebraically solve for R or R_c, given one of the equations above, it is helpful to understand that e is the base of the natural log (ln). In other words, the natural log is the inverse function of the exponential function: $e^{\ln(x)} = \ln(e^x) = x$. So if you are given an equation such that $R = e^x$; x will be equal to: ln(R).

Example: Computing continuous rates

Suppose we have a 5% rate that is compounded semiannually. **Compute** the corresponding continuous rate. Repeat this for quarterly, monthly, weekly, and daily compounding.

Answer:

$$R_c = 2\ln\left(1+\frac{0.05}{2}\right) = 0.049385$$

The results for other compounding frequencies are shown in Figure 1.

Figure 1: Compounding Frequencies and Returns

m	R_c
4	0.049690
12	0.049896
52	0.049976
250	0.049995

Notice that as m increases, the difference between the rates decreases.

Example: Discrete compounding rate

A loan is quoted at 12% annually with continuous compounding. Interest is paid monthly. **Calculate** the equivalent rate with monthly compounding.

Answer:

$$R = 12(e^{0.12/12} - 1) = 12.06\%$$

Spot (Zero) Rates and Bond Pricing

LO 37.4: Calculate the theoretical price of a bond using spot rates.

Spot rates are the rates that correspond to zero-coupon bond yields. They are the appropriate discount rates for a single cash flow at a particular future time or maturity. Spot rates are also often called zero rates. Most interest rates that are observed in the market, such as coupon bond yields, are not spot rates.

Bond Pricing

A coupon bond makes a series of cash flows. Each cash flow considered in isolation is equivalent to a zero-coupon bond. Using this interpretation, a coupon bond is a series of zero-coupon bonds, and its value, assuming continuous compounding and semiannual coupons, is:

$$B = \left(\frac{c}{2} \times \sum_{j=1}^{N} e^{-\frac{z_j}{2} \times j}\right) + \left(FV \times e^{-\frac{z_N}{2} \times N}\right)$$

where:
c = the annual coupon
N = the number of semiannual payment periods
z_j = the bond equivalent spot rate that corresponds to j periods (j/2 years) on a continuously compounded basis
FV = the face value of the bond

Don't let this formula intimidate you. It simply says that the value of a bond is the present value of its cash flows, where each cash flow is discounted at the appropriate spot rate for its maturity. Notice that the negative sign on the rate just means that the coupon and principal payments are being discounted back to the present in a continuous fashion. The following example is a good illustration of the process.

Example: Calculating bond price

Compute the price of a $100 face value, 2-year, 4% semiannual coupon bond using the annualized spot rates in Figure 2.

Figure 2: Spot Rates

Maturity (Years)	*Spot Rate (%)*
0.5	2.5
1.0	2.6
1.5	2.7
2.0	2.9

Answer:

$$B = \left(\$2 \times e^{-\frac{0.025}{2} \times 1}\right) + \left(\$2 \times e^{-\frac{0.026}{2} \times 2}\right) + \left(\$2 \times e^{-\frac{0.027}{2} \times 3}\right) + \left(\$102 \times e^{-\frac{0.029}{2} \times 4}\right) = \$102.10$$

Bond Yield

The yield of a bond is the single discount rate that equates the present value of a bond to its market price. You can use a financial calculator to compute bond yield, as in the following example.

Example: Calculating bond yield

Compute the yield for the bond in the previous example.

Answer:

PMT = 2; N = 4; PV = –102.10; FV = 100; CPT → I/Y = 1.456;

Y = 1.456% × 2 ≈ 2.91%

The bond's **par yield** is the rate which makes the price of a bond equal to its par value. When the bond is trading at par, the coupon will be equal to the bond's yield.

Bootstrapping Spot Rates

The theoretical spot curve is derived by interpreting each Treasury bond (T-bond) as a package of zero-coupon bonds. Using the prices for each bond, the spot curve is computed using the bootstrapping methodology.

For example, suppose there is a T-bond maturing on a coupon date in exactly six months. Further assume that the bond is priced at 102.2969% of par and has a semiannual coupon of 6.125% How is the corresponding spot rate computed? In this case, this is truly a zero-coupon bond, since there is only one cash flow, which occurs in six months. Simply solve for z_1 in the bond valuation equation, given the price, as follows:

$$\$102.2969 = \left(\$100 + \frac{\$6.125}{2}\right) \times e^{-\frac{z_1}{2}}$$

Solving this for z_1:

$$z_1 = -2 \times \ln\left[\frac{\$102.2969}{\left(\$100 + \frac{\$6.125}{2}\right)}\right] = 1.491\%$$

The 6-month spot rate on a bond equivalent basis is 1.491%. Also note that the yield to maturity did not need to be computed in this case because the yield to maturity (YTM) and the spot rate are the same.

How is the spot rate that corresponds to one year found? Suppose a T-bond that matures in one year is priced at 104.0469% of par and has a semiannual coupon of 6.25%. From the previous computation, the 6-month spot rate is known, so the bond valuation equation can be written as:

$$\$104.0469 = \left(\frac{\$6.25}{2} \times e^{-\frac{0.01491}{2}}\right) + \left(\$100 + \frac{\$6.25}{2}\right) \times e^{-\frac{z_2}{2} \times 2}$$

$$\Rightarrow z_2 = 0.02136 = 2.136\%$$

The 1-year spot rate with continuous compounding is 2.136%.

Example: Bootstrapping spot rates

Compute the corresponding spot rate curve using the information in Figure 3. Note that we've already computed the first two spot rates.

Figure 3: Input Information to Bootstrap Spot Rates

Price as a Percentage of Par	*Coupon*	*Semiannual Period*	*Maturity (Years)*
102.2969	6.125	1	0.5
104.0469	6.250	2	1.0
104.0000	5.250	3	1.5
103.5469	4.750	4	2.0

Answer:

The spot rates derived by bootstrapping are shown in Figure 4.

Figure 4: Bootstrapped Spot Rate Curve

Price as a Percentage of Par	*Coupon*	*Semiannual Period*	*Maturity (Years)*	*Spot Rates*
102.2969	6.125	1	0.5	1.491%
104.0469	6.250	2	1.0	2.136%
104.0000	5.250	3	1.5	2.515%
103.5469	4.750	4	2.0	2.915%

An alternative verification is to use the spot rates to check if they result in the same prices using the bond valuation equation. For example, using the spot rates will ensure computation of the same price for the 2-year bond:

$$B = \left(\frac{\$4.75}{2} \times e^{-\frac{0.01491}{2} \times 1}\right) + \left(\frac{\$4.75}{2} \times e^{-\frac{0.02136}{2} \times 2}\right) + \left(\frac{\$4.75}{2} \times e^{-\frac{0.02515}{2} \times 3}\right) +$$

$$\left(\$100 + \frac{\$4.75}{2}\right) \times e^{-\frac{0.02915}{2} \times 4} = \$103.5469$$

This results in a bond price of $103.5469. Notice that this is exactly the price of the 2-year bond.

Forward Rates

LO 37.5: Derive forward interest rates from a set of spot rates.

Forward rates are interest rates implied by the spot curve for a specified future period. The spot rates in Figure 4 are the appropriate rates that an investor should expect to realize for various maturities. Suppose an investor is faced with the following two investments, which are based on the spot curve in Figure 4.

1. Invest for two years at 2.915%.
2. Invest for a year at 2.136% and then roll over that investment for another year at the forward rate.

It does not matter which investment is chosen if they both offer the same return at the end of two years. This is the same as stating that both strategies give the same future value at the end of two years. Equating the two future values:

$$e^{\frac{0.02915}{2}\times 4} = e^{\frac{0.02136}{2}\times 2} \times e^{\frac{R_{Forward}}{2}\times 2}$$

where:
$R_{Forward}$ = the 1-year forward rate one year from now

As we will show, for the two strategies to be equal, $R_{Forward}$ must be 3.693%.

We can simplify this calculation by using the following equation:

$$R_{Forward} = \frac{R_2T_2 - R_1T_1}{T_2 - T_1} = R_2 + (R_2 - R_1) \times \left(\frac{T_1}{T_2 - T_1}\right)$$

where:
R_i = the spot rate corresponding with T_i periods
$R_{Forward}$ = the forward rate between T_1 and T_2

For example, if the 1-year rate is 2.136% and the 2-year rate is 2.915%, the 1-year forward rate one year from now is:

$$R_{Forward} = 0.02915 + (0.02915 - 0.02136) \times \left(\frac{1}{2-1}\right) = 0.03694 = 3.694\%$$

This is the same forward rate (with slight rounding error) that was calculated before.

As a further example, consider the problem of finding the 1-year forward rate three years from now, given a 3-year spot rate of 7.424% and a 4-year spot rate of 8.216% (both continuously compounded annual rates). Based on the previous formula, the continuously compounded 1-year rate three years from now is:

$$0.08216 + (0.08216 - 0.07424) \times \frac{3}{4-3} = 0.10592$$

With this equation, generalizations can be made between the shape of the spot curve and the forward curve. The second term is always positive for an upward-sloping spot curve. Therefore, when there is an upward-sloping spot curve, the corresponding forward rate curve is upward-sloping and above the spot curve. Similarly, when there is a downward-sloping spot curve, the corresponding forward-rate curve is downward-sloping and below the spot curve.

Forward Rate Agreements

LO 37.6: Derive the value of the cash flows from a forward rate agreement (FRA).

A **forward rate agreement** (FRA) is a forward contract obligating two parties to agree that a certain interest rate will apply to a principal amount during a specified future time. Obviously, forward rates play a crucial role in the valuation of FRAs. The T_2 cash flow of an FRA that promises the receipt or payment of R_K is:

$$\text{cash flow (if receiving } R_K) = L \times (R_K - R) \times (T_2 - T_1)$$
$$\text{cash flow (if paying } R_K) = L \times (R - R_K) \times (T_2 - T_1)$$

where:
L = principal
R_K = annualized rate on L, expressed with compounding period $T_1 - T_2$
R = annualized actual rate, expressed with compounding period $T_1 - T_2$
T_i = time i, expressed in years

The value of an FRA if we're receiving or paying is:

$$\text{value (if receiving } R_K) = L \times (R_K - R_{Forward}) \times (T_2 - T_1) \times e^{-R_2 \times T_2}$$
$$\text{value (if paying } R_K) = L \times (R_{Forward} - R_K) \times (T_2 - T_1) \times e^{-R_2 \times T_2}$$

where:
$R_{Forward}$ = forward rate between T_1 and T_2

Note that R_2 is expressed as a continuously compounded rate.

Example: Computing the payoff from an FRA

Suppose an investor has entered into an FRA where he has contracted to pay a fixed rate of 3% on $1 million based on the quarterly rate in three months. Assume that rates are compounded quarterly. **Compute** the payoff of the FRA if the quarterly rate is 1% in three months.

Answer:

For this FRA, the payoff will take place in six months. The net payoff will be the difference between the fixed-rate payment and the floating rate receipt. If the floating rate is 1% in three months, the payoff at the end of the sixth month will be:

$$\$1{,}000{,}000\ (0.01 - 0.03)(0.25) = -\$5{,}000$$

Example: Computing the value of an FRA

Suppose the 3-month and 6-month LIBOR spot rates are 4% and 5%, respectively (continuously compounded rates). An investor enters into an FRA in which she will receive 8% (assuming quarterly compounding) on a principal of $5,000,000 between months 3 and 6. **Calculate** the value of the FRA.

Answer:

$$R_{\text{Forward}} = 0.05 + (0.05 - 0.04) \times \left(\frac{1}{2-1}\right) = 0.06 = 6\%$$

$$R_{\text{Forward}}\,(\text{with quarterly compounding}) = 4 \times \left(e^{\frac{0.06}{4}} - 1\right) = 0.060452 = 6.05\%$$

$$\text{value} = \$5{,}000{,}000 \times (0.0800 - 0.0605) \times (0.50 - 0.25) \times e^{-(0.05)(0.5)} = \$23{,}773$$

Duration

LO 37.7: Calculate the duration, modified duration and dollar duration of a bond.

The duration of a bond is the average time until the cash flows on the bond are received. For a zero-coupon bond, this is simply the time to maturity. For a coupon bond, its duration will be necessarily shorter than its maturity. The weights on the time in years until each cash flow is to be received are the proportion of the bond's value represented by each of the coupon payments and the maturity payment. The formula for duration using continuously compounded discounting of the cash flows is:

$$\text{duration} = \sum_{i=1}^{n} t_i \left[\frac{c_i e^{-y t_i}}{B}\right]$$

where:
t_i = the time (in years) until cash flow c_i is to be received
y = the continuously compounded yield (discount rate) based on a bond price of B

The usefulness of the duration measure lies in the fact that the approximate change in a bond's price, B, for a parallel shift in the yield curve of Δy is:

$$\frac{\Delta B}{B} = -\text{duration} \times \Delta y$$

The change in yield is often expressed as a **basis point** change. One basis point is equivalent to 0.01%. So a 100 basis point change is a change of 1% in the yield.

Modified duration is used when the yield given is something other than a continuously compounded rate. When the yield is expressed as a semiannually compounded rate, for example, modified duration = duration/(1 + y/2). In general we can express this relation as: $\text{modified duration} = \frac{\text{duration}}{1+y/m}$, where m is the number of compounding periods per year.

Note that as m goes to infinity (continuous compounding), the two measures are equal and there is no difference between the two.

On the exam, you may also see a reference to **dollar duration**. Dollar duration is simply modified duration multiplied by the price of the bond.

Convexity

LO 37.8: Evaluate the limitations of duration and explain how convexity addresses some of them.

Duration is a good approximation of price changes for an option-free bond, but it's only good for relatively small changes in interest rates. As rate changes grow larger, the curvature of the bond price/yield relationship becomes more important, meaning that a linear estimate of price changes, such as duration, will contain errors.

In fact, the relationship between bond price and yield is not linear (as assumed by duration) but convex. This convexity shows that the difference between actual and estimated prices widens as the yield swings grow. That is, the widening error in the estimated price is due to the curvature of the actual price path. This is known as the **degree of convexity**.

Fortunately, the amount of convexity in a bond can be measured and used to supplement duration in order to achieve a more accurate estimate of the change in price. It's important to note that all convexity does is account for the amount of error in the estimated price change based on duration. In other words, it picks up where duration leaves off and converts the straight (estimated price) line into a curved line that more closely resembles the convex (actual price) line.

Using Convexity to Improve Price Change Estimates

In order to obtain an estimate of the percentage change in price due to convexity, or the amount of price change that is not explained by duration, the following calculation will need to be made:

$$\text{convexity effect} = 1/2 \times \text{convexity} \times \Delta y^2$$

The convexity effect is typically quite small. However, remember that convexity is simply correcting for the error embedded in the duration, so you would expect convexity to have a much smaller effect than duration. Also note that for an option-free bond, the convexity effect is always positive, no matter which direction interest rates move. Thus, for option-free bonds, convexity is always added to duration to modify the price volatility errors embedded in duration. This decreases the drop in price (due to an increase in yields) and adds to the rise in price (due to a fall in yields).

LO 37.9: Calculate the change in a bond's price given its duration, its convexity, and a change in interest rates.

By combining duration and convexity, we can obtain a far more accurate estimate of the percentage change in the price of a bond, especially for large swings in yield. That is, you can account for the amount of convexity embedded in a bond by adding the convexity effect to the duration effect.

Example: Estimating price changes with the duration/convexity approach

Estimate the effect of a 100 basis point increase and decrease on a 10-year, 5%, option-free bond currently trading at par, using the duration/convexity approach. The bond has a duration of 7 and a convexity of 90.

Answer:

Using the duration/convexity approach:

percentage bond price change ≈ duration effect + convexity effect

$$\Delta B_{+\Delta y} \approx [-7 \times 0.01] + [(1/2) \times 90 \times (0.01^2)]$$

$$\approx -0.07 + 0.0045 = -0.0655 = -6.55\%$$

$$\Delta B_{-\Delta y} \approx [-7 \times -0.01] + [(1/2) \times 90 \times (-0.01^2)]$$

$$\approx 0.07 + 0.0045 = 0.0745 = 7.45\%$$

Theories of the Term Structure

LO 37.10: Compare and contrast the major theories of the term structure of interest rates.

The **expectations theory** suggests that forward rates correspond to expected future spot rates. That is, forward rates are good predictors of expected future spot rates. In reality, the expectations theory fails to explain all future spot rate expectations. The **market segmentation theory** states that the bond market is segmented into different maturity sectors and that supply and demand for bonds in each maturity range dictate rates in that maturity range. The **liquidity preference theory** suggests that most depositors prefer short-term liquid deposits. In order to coax them to lend longer term, the intermediary will raise longer-term rates by adding a liquidity premium.

Key Concepts

LO 37.1

Three types of interest rates are particularly relevant in the interest rate derivative markets: Treasury rates, London Interbank Offered Rate (LIBOR), and repo rates. Treasury rates (such as T-bill and T-bond rates) are often considered the benchmark for nominal risk-free rates.

LO 37.2

If we have an initial investment of *A* that earns an annual rate *R*, compounded *m* times a year for *n* years, then it has a future value of:

$$FV = A\left(1+\frac{R}{m}\right)^{m \times n}$$

LO 37.3

In most circumstances, rates are discretely compounded so we need to find the continuously compounded rate that gives the same future value. The continuous rate can be solved as follows:

$$R_c = m \times \ln\left(1+\frac{R}{m}\right)$$

LO 37.4

Zero (spot) rates correspond to the interest earned on a single cash flow at a single point in time. Bond prices are computed using the spot curve by discounting each cash flow at the appropriate spot rate.

The yield of a bond is the single discount rate that equates the present value of a bond to its market price.

Zero rates are computed using the bootstrapping methodology.

LO 37.5

Forward rates are computed from spot rates. When the spot curve is upward-sloping, the corresponding forward rate curve is upward-sloping and above the spot curve. When the spot curve is downward-sloping, the corresponding forward rate curve is downward-sloping and below the spot curve.

LO 37.6

A forward-rate agreement is a contract between two parties that an interest rate will apply to a specific principal during some future time period.

LO 37.7

Duration and modified duration are the same when continuously compounded yields are used, and they both estimate the percentage price change of a bond from an absolute change in yield. Dollar duration is modified duration multiplied by the price of the bond.

LO 37.8

Duration is only good for relatively small changes in interest rates. As rate changes grow larger, the curvature of the bond price/yield relationship becomes more important, meaning that a linear estimate of price changes, such as duration, will contain errors. The amount of convexity in a bond can be measured and used to supplement duration in order to achieve a more accurate estimate of the change in price.

LO 37.9

The approximate change in a bond's price, B, for a parallel shift in the yield curve of Δy is:

$$\frac{\Delta B}{B} = -\text{duration} \times \Delta y$$

In order to obtain an estimate of the percentage change in price due to convexity, the following calculation will need to be made:

$$\text{convexity effect} = \frac{1}{2} \times \text{convexity} \times \Delta y^2$$

Combining duration and convexity creates a more accurate estimate of the percentage change in the price of a bond:

$$\text{percentage bond price change} \approx \text{duration effect} + \text{convexity effect}$$

LO 37.10

The expectations theory suggests that forward rates correspond to expected future spot rates. The market segmentation theory states that bonds are segmented into different maturity sectors and that supply and demand dictate rates in the segmented maturity sectors. The liquidity preference theory suggests that longer-term rates incorporate a liquidity premium.

Concept Checkers

1. What is the continuously compounded rate of return for an investment that has a value today of \$86.50 and will have a future value of \$100 in one year?
   A. 13.62%.
   B. 14.50%.
   C. 15.61%.
   D. 16.38%.

2. Assume that the continuously compounded 10-year spot rate is 5% and the 9-year spot rate is 4.9%. Which of the following is closest to the 1-year forward rate nine years from now?
   A. 4.1%.
   B. 5.1%.
   C. 5.9%.
   D. 6.0%.

3. An investor enters into a 1-year forward rate agreement (FRA) where she will receive the contracted rate on a principal of \$1 million. The contracted rate is a 1-year rate at 5%. Which of the following is closest to the cash flow if the actual rate is 6% at maturity of the underlying asset (loan)?
   A. –\$10,000.
   B. –\$1,000.
   C. +\$1,000.
   D. +\$10,000.

4. What is the bond price of a \$100 face value, 2.5-year, 3% semiannual coupon bond using the following annual continuously compounded spot rates: $z_1 = 3\%$, $z_2 = 3.1\%$, $z_3 = 3.2\%$, $z_4 = 3.3\%$, and $z_5 = 3.4\%$?
   A. \$97.27.
   B. \$97.83.
   C. \$98.15.
   D. \$98.99.

5. A \$100 face value, 1-year, 4% semiannual bond is priced at 99.806128. If the annualized 6-month spot rate (z_1) is 4.1%, what is the 1-year spot rate (z_2)? (Both spots are continuously compounded rates.)
   A. 4.07%.
   B. 4.16%.
   C. 4.20%.
   D. 4.26%.

Concept Checker Answers

1. B The formula to solve this problem is:

$$R_c = m \times \ln\left(1+\frac{R}{m}\right)$$

First, we need to compute *R* as the rate earned on the $86.50 investment:

$$R = \frac{\$100 - \$86.50}{\$86.50} = 0.15607$$

This is essentially the effective rate earned over one year with annual compounding. So, m = 1, and $R_c = 1 \times \ln(1.15607) = 0.1450$. Alternatively, since m = 1,

$$\ln\left(\frac{100}{86.50}\right) = 0.1450 = 14.50\%$$

2. C $R_{Forward} = R_2 + (R_2 - R_1) \times [T_1 / (T_2 - T_1)] = 0.05 + (0.05 - 0.049) \times [9 / (10 - 9)] = 5.9\%$

3. A $1,000,000 (0.05 – 0.06)(1) = –$10,000

4. D $B = 1.5 \times e^{[(-0.03/2)\times 1]} + 1.5 \times e^{[(-0.031/2)\times 2]} + 1.5 \times e^{[(-0.032/2)\times 3]} + 1.5 \times e^{[(-0.033/2)\times 4]} + 101.5 \times e^{[(-0.034/2)\times 5]} = 1.48 + 1.45 + 1.43 + 1.40 + 93.23 = \$98.99$

5. B $B = 2 \times e^{[(-z_1/2)\times 1]} + 102 \times e^{[(-z_2/2)\times 2]}$; $\$99.806128 = 2 \times e^{[(-0.041/2)\times 1]} + 102 \times e^{[(-z_2/2)\times 2]}$; $\$97.846711 = 102 \times e^{[(-z_2/2)\times 2]}$; $z_2 = 0.0415707 = 4.16\%$

The following is a review of the Financial Markets and Products principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Determination of Forward and Futures Prices

Topic 38

Exam Focus

Both forward and futures contracts are obligations regarding a future transaction. Because the difference in pricing between these contract types is small, forward contract pricing and futures contract pricing are often presented interchangeably. The basic model for forward prices is the cost-of-carry model, which essentially connects the forward price to the cost incurred from purchasing and storing the underlying asset until the contract maturity date. Cash flows over the life of the contract are easily incorporated into the pricing model. Futures contracts contain delivery options that benefit the short seller of the contract. These delivery options must be incorporated into the futures pricing model.

Investment and Consumption Assets

LO 38.1: Differentiate between investment and consumption assets.

An **investment asset** is an asset that is held for the purpose of investing. This type of asset is held by many different investors for the sake of investment. Examples of investment assets include stocks and bonds. A **consumption asset** is an asset that is held for the purpose of consumption. Examples of consumption assets include commodities such as oil and natural gas.

Short-Selling and Short Squeeze

LO 38.2: Define short-selling and calculate the net profit of a short sale of a dividend-paying stock.

Short sales are orders to sell securities that the seller does not own. Short selling is also known as "shorting" and is possible with some investment assets. For a short sale, the short seller (1) simultaneously borrows and sells securities through a broker, (2) must return the securities at the request of the lender or when the short sale is closed out, and (3) must keep a portion of the proceeds of the short sale on deposit with the broker.

The short seller may be forced to close his position if the broker runs out of securities to borrow. This is known as a **short squeeze,** and the seller will need to close his short position immediately.

Why would anyone ever want to sell securities short? The seller thinks the current price is too high and that it will fall in the future, so the short seller hopes to sell high and then buy low. If a short sale is made at \$30 per share and the price falls to \$20 per share, the short seller can buy shares at \$20 to replace the shares borrowed and keep \$10 per share as profit.

Two rules currently apply to short selling:

1. The short seller must pay all dividends due to the lender of the security.
2. The short seller must deposit collateral to guarantee the eventual repurchase of the security.

Example: Net profit of a short sale of a dividend-paying stock

Assume that trader Alex Rodgers sold short XYZ stock in March by borrowing 200 shares and selling them for $50/share. In April, XYZ stock paid a dividend of $2/share. **Calculate** the net profit from the short sale assuming Rodgers bought back the shares in June for $40/share in order to replace the borrowed shares and close out his short position.

Answer:

The cash flows from the short sale on XYZ stock are as follows:

March: borrow 200 shares and sell them for $50/share	+$10,000
April: short seller dividend payment to lender of $2/share	–$400
June: buyback shares for $40/share to close short position	–$8,000
Total net profit =	+$1,600

Forward and Futures Contracts

LO 38.3: Describe the differences between forward and futures contracts and explain the relationship between forward and spot prices.

LO 38.4: Calculate the forward price given the underlying asset's spot price, and describe an arbitrage argument between spot and forward prices.

LO 38.9: Calculate, using the cost-of-carry model, forward prices where the underlying asset either does or does not have interim cash flows.

Futures contracts and **forward contracts** are *similar* in that both:

- Can be either deliverable or cash settlement contracts.
- Are priced to have zero value at the time an investor enters into the contract.

Futures contracts *differ* from forward contracts in the following ways:

- Futures contracts trade on organized exchanges. Forwards are private contracts and do not trade on an exchange.
- Futures contracts are highly standardized. Forwards are customized contracts satisfying the needs of the parties involved.

- A single clearinghouse is the counterparty to all futures contracts. Forwards are contracts with the originating counterparty.
- The government regulates futures markets. Forward contracts are usually not regulated.

Forward Prices

The pricing model used to compute forward prices makes the following assumptions:

- No transaction costs or short-sale restrictions.
- Same tax rates on all net profits.
- Borrowing and lending at the risk-free rate.
- Arbitrage opportunities are exploited as they arise.

For the development of a forward pricing model, we will use the following notation:

- T = time to maturity (in years) of the forward contract.
- S_0 = underlying asset price today ($t = 0$).
- F_0 = forward price today.
- r = continuously compounded risk-free annual rate.

The forward price may be written as:

Equation 1

$$F_0 = S_0 e^{rT}$$

The right-hand side of Equation 1 is the cost of borrowing funds to buy the underlying asset and carrying it forward to time *T*. Equation 1 states that this cost must equal the forward price. If $F_0 > S_0e^{rT}$, then arbitrageurs will profit by selling the forward and buying the asset with borrowed funds. If $F_0 < S_0e^{rT}$, arbitrageurs will profit by selling the asset, lending out the proceeds, and buying the forward. Hence, the equality in Equation 1 must hold. Note that this model assumes perfect markets.

As it turns out, actual short sales are not necessary for Equation 1 to hold. All that is necessary is a sufficient number of investors who are not only holding the investment asset but also are willing to sell the asset if the forward price becomes too low. In the event that the forward price is too low, the investor will sell the asset and take a long position in the forward contract. This is important since the arbitrage relationship in Equation 1 must hold for all investment assets even though short selling is not available for every asset.

Example: Computing a forward price with no interim cash flows

Suppose we have an asset currently worth $1,000. The current continuously compounded rate is 4% for all maturities. **Compute** the price of a 6-month forward contract on this asset.

Answer:

$$F_0 = \$1{,}000e^{0.04(0.5)} = \$1{,}020.20$$

Forward Price With Carrying Costs

If the underlying pays a known amount of cash over the life of the forward contract, a simple adjustment is made to Equation 1. Since the owner of the forward contract does not receive any of the cash flows from the underlying asset between contract origination and delivery, the present value of these cash flows must be deducted from the spot price when calculating the forward price. This is most easily seen when the underlying asset makes a periodic payment. With this in mind, we let *I* represent the *present value* of the cash flows over *T* years. Equation 1 then becomes:

Equation 2

$$F_0 = (S_0 - I)\, e^{rT}$$

The same arbitrage arguments used for Equation 1 are used here. The only modification is that the arbitrageur must account for the known cash flows.

Example: Forward price when underlying asset has a cash flow

Compute the price of a 6-month forward on a coupon bond worth $1,000 that pays a 5% coupon semiannually. A coupon is to be paid in three months. Assume the risk-free rate is 4%.

Answer:

The cost of carry (income) in this case is computed as:

$$I = 25e^{-0.04(0.25)} = \$24.75125$$

Using Equation 2:

$$F_0 = (\$1{,}000 - \$24.75125)e^{0.04(0.5)} = \$994.95$$

The Effect of a Known Dividend

When the underlying asset for a forward contract pays a dividend, we assume that the dividend is paid continuously. Letting *q* represent the continuously compounded dividend yield paid by the underlying asset expressed on a per annum basis, Equation 1 becomes:

Equation 3

$$F_0 = S_0 e^{(r-q)T}$$

Once again, the same arbitrage arguments are used to prove that Equation 3 must be true.

Example: Forward price when the underlying asset pays a dividend

Compute the price of a 6-month forward contract for which the underlying asset is a stock index with a value of 1,000 and a continuous dividend yield of 1%. Assume the risk-free rate is 4%.

Answer:

Using Equation 3:

$$F_0 = 1{,}000e^{(0.04-0.01)0.5} = 1{,}015.11$$

Value of a Forward Contract

The initial value of a forward contract is zero. After its inception, the contract can have a positive value to one counterparty (and a negative value to the other). Since the forward price at every moment in time is computed to prevent arbitrage, the value at inception of the contract must be zero. The forward contract can take on a non-zero value only after the contract is entered into and the obligation to buy or sell has been made. If we denote the obligated delivery price after inception as *K*, then the value of the long contract on an asset with no cash flows is computed as $S_0 - Ke^{-rT}$; with cash flows (with present value *I*) it is $S_0 - I - Ke^{-rT}$; and with a continuous dividend yield of *q*, it is $S_0e^{-qT} - Ke^{-rT}$.

Example: Value of a stock index forward contract

Using the stock index forward in the previous example, **compute** the value of a long position if the index increases to 1,050 immediately after the contract is purchased.

Answer:

In this case, $K = 1{,}015.11$ and $S_0 = 1{,}050$, so the value is:

$$1{,}050e^{-0.01(0.5)} - 1{,}015.11e^{-0.04(0.5)} = 49.75$$

Currency Futures

LO 38.6: Calculate a forward foreign exchange rate using the interest rate parity relationship.

Interest rate parity (IRP) states that the forward exchange rate, *F* (measured in domestic per unit of foreign currency), must be related to the spot exchange rate, *S*, and to the interest rate differential between the domestic and the foreign country, $r - r_f$.

The general form of the interest rate parity condition is expressed as:

$$F = Se^{(r-r_f)T}$$

This equation is a no-arbitrage relationship. Using our notation from earlier, we can state the interest rate parity relationship as:

Equation 4

$$F_0 = S_0 e^{(r-r_f)T}$$

Note that this is equivalent to Equation 3 with r_f replacing q. Just as the continuous dividend yield q was used to adjust the cost of carry, we use the continuous yield on a foreign currency deposit here.

Example: Currency futures pricing

Suppose we wish to **compute** the futures price of a 10-month futures contract on the Mexican peso. Each contract controls 500,000 pesos and is quoted in terms of dollar/peso. Assume that the continuously compounded risk-free rate in Mexico (r_f) is 14%, the continuously compounded risk-free rate in the United States is 2%, and the current exchange rate is 0.12.

Answer:

Applying Equation 4:

$$F_0 = \$0.12e^{(0.02-0.14)\frac{10}{12}} = \$0.10858\,/\,\text{peso}$$

Professor's Note: The concept of interest rate parity will show up again in the foreign exchange risk topic (Topic 49).

Forward Prices vs. Futures Prices

LO 38.5: Explain the relationship between forward and futures prices.

The most significant difference between forward contracts and futures contracts is the daily marking to market requirement on futures contracts. When interest rates are known over the life of a contract, *T*, forward and futures prices can be shown to be the same. Various relationships can be derived, depending on the assumptions made between the value of the underlying and the level of change in interest rates. In general, when *T* is small, the price differences are usually very small and can be ignored. Empirical research comparisons of forwards and futures prices are mixed. Some studies conclude a significant difference and others do not. The important concept to understand here is that assuming the two are the same is an approximation, and under certain circumstances the approximation can be inaccurate.

Commodity Futures

LO 38.7: Define income, storage costs, and convenience yield.

LO 38.8: Calculate the futures price on commodities incorporating income/storage costs and/or convenience yields.

Professor's Note: Topic 45 later in this book is devoted to commodity forwards and futures. In that topic, you will learn more about storage costs and convenience yield as well as the arbitrage relationships that must hold with commodity futures.

Income and Storage Costs

When the underlying is considered a consumption asset, the pricing relationships developed above do not adequately capture all the necessary characteristics of the asset. *Consumption assets have actual storage costs associated with them.* These costs increase the carrying costs. The costs can be expressed either as a known cash flow or as a yield. Let U denote the present value of known storage cost over the life of the forward contract. Equation 1 then becomes:

Equation 5

$$F_0 = (S_0 + U)e^{rT}$$

If we express the storage costs in terms of a continuous yield, u:

Equation 6

$$F_0 = S_0e^{(r+u)T}$$

The arbitrage relationships are the same except we need to account for the additional carrying costs over T years. However, when the owner of these assets is reluctant to sell the asset, Equations 5 and 6 are replaced by:

Equation 7

$$F_0 \leq (S_0 + U)e^{rT}$$

And:

Equation 8

$$F_0 \leq S_0e^{(r+u)T}$$

Convenience Yield

Equations 7 and 8 suggest there is a *benefit to owning the underlying consumable asset compared to owning the futures contract*. If we introduce a **convenience yield**, *y*, to balance Equations 7 and 8, we have:

$$F_0e^{yT} = (S_0 + U)e^{rT} = S_0e^{(r+u)T}$$

This formula can be reduced to:

Equation 9

$$F_0 = S_0e^{(r+u-y)T}$$

In other words, the convenience yield is simply the yield required to produce an equality and is thus a measure of the benefit of owning spot, or physical, consumption commodities.

Delivery Options in the Futures Market

LO 38.10: Describe the various delivery options available in the futures markets and how they can influence futures prices.

Some futures contracts grant **delivery options** to the short—options on what, where, and when to deliver. Some Treasury bond contracts give the short a choice of several bonds that are acceptable to deliver and options as to when to deliver during the expiration month. Physical assets, such as gold or corn, may offer a choice of delivery locations to the short. These options can be of significant value to the holder of the short position in a futures contract.

As shown in the previous discussion on commodity futures, if the cost of carrying the asset is greater than the convenience yield (benefit from holding the physical asset), it is ideal for the short position to deliver the contract early. This scenario suggests that the futures price will increase over time; hence, the short has an incentive to deliver early. The opposite relationship holds true when the cost of carry is less than the convenience yield. In this case, the short position will delay delivery since the futures price is expected to fall over time.

Futures and Expected Future Spot Prices

LO 38.11: Explain the relationship between current futures prices and expected future spot prices, including the impact of systematic and nonsystematic risk.

The cost of carry model is a widely used method for estimating the appropriate price of a futures contract, but other theories exist for explaining the futures price. One intuitively appealing model expresses the futures price as a function of the expected spot price (S_T).

$$F_0 = E(S_T)$$

For obvious reasons, this is called the **expectations model** and states that the current futures price for delivery at time *T* is equal to the expected spot price at time *T*. Similar to the no-arbitrage rule, this model acts to keep the current futures price in line with the expected spot rate at that time. If the futures price is less than the expected price, aggressive buying of the futures would push up the futures price. If the futures price is greater than the expected spot rate, aggressive selling of the futures would lead to lower the futures price. Although intuitively appealing, other factors probably play a role in the pricing mechanism. Indeed, if the expectations model limited traders to a risk-free rate of return, there would be no incentive to buy or sell contracts.

Cost of Carry vs. Expectations

Economist John Maynard Keynes found the expectations model to be flawed precisely because it provided no justification for speculators to enter the market. Futures contracts provide a mechanism to transfer risk from those who need to hedge their positions (e.g., farmers who are long the commodities) to speculators. In order to entice speculators to bear the risk of these contracts, there has to exist an expectation of profit greater than the risk-free rate. For this to occur, the futures contract price must be less than the expected spot rate at maturity [$F_0 < E(S_T)$] and must continually increase during the term of the contract. Keynes referred to this as **normal backwardation**. This relationship suggests that the asset underlying the futures contract exhibits positive systematic risk, since this is the risk that remains after diversifying away all nonsystematic risk.

On the other side of the contracts are those who are users of the commodity who want to shift some of the risk of rising market prices to speculators. They wish to purchase futures contracts from speculators. The speculators have to be enticed into assuming this risk by the expectation of profits that would exceed the risk-free rate. From this perspective, the futures price must be higher than the expected spot price at maturity [$F_0 > E(S_T)$] and must continually decrease during the term of the contract. Keynes referred to this expectation as **contango** (a.k.a. normal contango). This relationship suggests that the asset underlying the futures contract exhibits negative systematic risk.

Contango and Backwardation

LO 38.12: Define and interpret contango and backwardation, and explain how they relate to the cost-of-carry model.

Backwardation refers to a situation where the futures price is below the spot price. For this to occur, there must be a significant benefit to holding the asset. Backwardation might occur if there are benefits to holding the asset that offset the opportunity cost of holding the asset (the risk-free rate) and additional net holding costs.

Contango refers to a situation where the futures price is above the spot price. If there are no benefits to holding the asset (e.g., dividends, coupons, or convenience yield), contango will occur because the futures price will be greater than the spot price.

Professor's Note: In this case, the reference to backwardation and contango refers to the relationship between the futures price and the current spot price, not the expected spot price.

Key Concepts

LO 38.1

An investment asset is an asset that is held for the purpose of investing. A consumption asset is an asset that is held for the purpose of consumption.

LO 38.2

Short sales are orders to sell securities that the seller does not own. A short squeeze results if the broker runs out of securities to borrow.

LO 38.3

Forward and futures contracts are similar because they are both future obligations to transact an asset on some future date. Forward contracts do not trade on an exchange, are not standardized, and do not normally close out prior to expiration.

The relationship between forward and spot prices is as follows:

$$F = S_0 e^{rT}$$

LO 38.4

The cost-of-carry model is used to price forward and futures contracts. It states that the total cost of carrying the underlying asset to expiration must be the futures price. Any other price results in arbitrage.

LO 38.5

When interest rates are known over the life of a contract, forward and futures prices can be shown to be the same. Various relationships can be derived, depending on the assumptions made between the value of the underlying and the level of change in interest rates.

LO 38.6

Interest rate parity states that the forward exchange rate, F (measured in domestic per unit of foreign currency), must be related to the spot exchange rate, S, and to the interest rate differential between the domestic and the foreign country:

$$F = S_0 e^{(r_{DC} - r_{FC})T}$$

LO 38.7

Consumption assets have actual storage costs (known as carrying costs) associated with them.

If there is a benefit to owning the underlying consumable asset compared to owning the futures, the futures price will incorporate a convenience yield.

LO 38.8

Futures price with storage costs, *u*: $F = S_0e^{(r+u)T}$

Futures price with convenience yield, *y*: $F = S_0e^{(r+u-y)T}$

LO 38.9

The futures price or cost-of-carry model is easily accommodated for interim cash flows from the underlying asset. If the underlying asset pays a known amount of cash, *I*, over the life of the forward contract, a simple adjustment is made to the cost-of-carry model:

$$F = (S_0 - I)e^{rT}$$

When the underlying asset pays a dividend, *q*, we assume that the dividend is paid continuously:

$$F = S_0e^{(r-q)T}$$

LO 38.10

Physical assets, such as gold or corn, may offer a choice of delivery locations to the short. These options can be of significant value to the holder of the short position in a futures contract. Futures contracts are typically "offset" by buying or selling a contract before the delivery date. Only a small percentage of contracts result in physical delivery.

LO 38.11

The expectations model states that the current futures price for delivery at time *T* is equal to the expected spot price at time *T*. This model acts to keep the current futures price in line with the expected spot rate at that time.

LO 38.12

Contango is the situation in which the futures price is above the current spot price. Backwardation is the opposite relationship.

Concept Checkers

Use the following data to answer Questions 1 and 2.

An investor has an asset that is currently worth $500, and the continuously compounded risk-free rate at all maturities is 3%.

1. Which of the following is the closest to the no-arbitrage price of a 3-month forward contract?
   A. $496.26.
   B. $500.00.
   C. $502.00.
   D. $503.76.

2. If the asset pays a continuous dividend of 2%, which of the following is the closest to the no-arbitrage price of a 3-month forward contract?
   A. $494.24.
   B. $498.75.
   C. $501.25.
   D. $506.29.

3. A bond pays a semiannual coupon of $40 and has a current value of $1,109. The next payment on the bond is in four months and the interest rate is 6.50%. Using the continuous time model, the price of a 6-month forward contract on this bond is closest to:
   A. $995.62.
   B. $1,011.14.
   C. $1,035.65.
   D. $1,105.20.

4. The owner of 300,000 bushels of corn wishes to hedge his position for a sale in 150 days. The current price of corn is $1.50/bushel and the contract size is 5,000 bushels. The interest rate is 7%, compounded daily. The storage cost for the corn is $18/day. Assume the cost of storage as a percentage of the contract per year is 1.46%. The price for the appropriate futures contract used to hedge the position is closest to:
   A. $6,635.
   B. $7,248.
   C. $7,656.
   D. $7,765.

5. Backwardation refers to a situation where:
   A. spot prices are above futures prices.
   B. spot prices are below futures prices.
   C. expected future spot prices are above futures prices.
   D. expected future spot prices are below futures prices.

Concept Checker Answers

1. **D** Using Equation 1:

   $500e^{(0.03)(0.25)} = \$503.76$

   where S = 500, T = 0.25, and r = 0.03

2. **C** Using Equation 3:

   $500e^{(0.03-0.02)0.25} = \$501.25$

3. **D** Use the formula $F_0 = (S_0 - I)e^{rT}$, where *I* is the present value of \$40 to be received in 4 months, or 0.333 years. At a discount rate of 6.50%:

   $I = \$40 \times e^{-0.065 \times 0.333} = \$39.14$

   $F_0 = (\$1{,}109 - 39.14) \times e^{(0.065 \times 0.5)} = \$1{,}105.20$

4. **D** Since both the interest and the storage costs compound on a daily basis, a continuous time model is appropriate to approximate the price of the contract.

   The cost of storage as a percentage of the contract per year is:

   $$u = 365 \times \frac{18}{1.50 \times 300{,}000} = 0.0146$$

   Using Equation 6, the futures price per bushel is:

   $F = \$1.50 \times e^{(0.07 + 0.0146)(150/365)} = \$1.553 \times 5{,}000 \text{ bushels per contract} = \$7{,}765.34$

5. **A** Backwardation refers to a situation where spot prices are higher than futures prices. Significant monetary benefits of the asset or a relatively high convenience yield can lead to this result.