VaR Methods

Exam Focus

Value at risk (VaR) was developed as an efficient, inexpensive method to determine economic risk exposure of banks with complex diversified asset holdings. In this reading, we define VaR, demonstrate its calculation, discuss how VaR can be converted to longer time periods, and examine the advantages and disadvantages of the three main VaR estimation methods. For the exam, be sure you know when to apply each VaR method and how to calculate VaR using each method. VaR is one of GARP's favorite testing topics and it appears in many assigned readings throughout the FRM Part I and Part II curricula.

Defining VaR

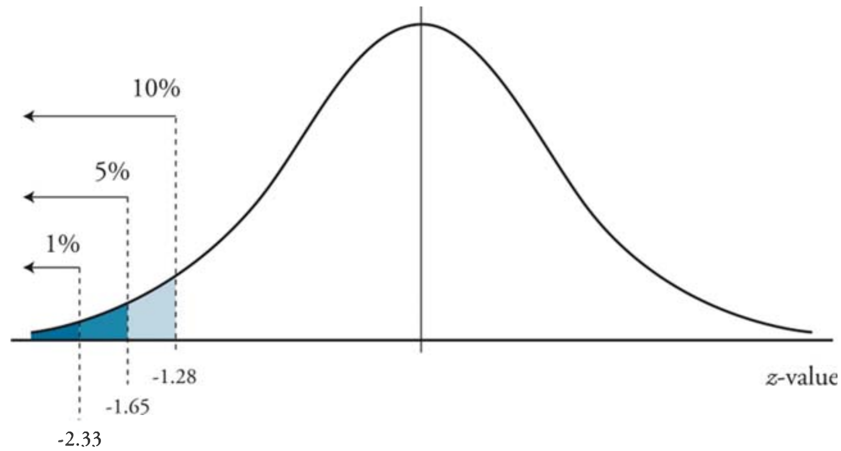
Value at risk (VaR) is a probabilistic method of measuring the potential loss in portfolio value over a given time period and for a given distribution of historical returns. VaR is the dollar or percentage loss in portfolio (asset) value that will be equaled or exceeded only *X* percent of the time. In other words, there is an *X* percent probability that the loss in portfolio value will be equal to or greater than the VaR measure. VaR can be calculated for any percentage probability of loss and over any time period. A 1%, 5%, and 10% VaR would be denoted as VaR(1%), VaR(5%), and VaR(10%), respectively. The risk manager selects the *X* percent probability of interest and the time period over which VaR will be measured. Generally, the time period selected (and the one we will use) is one day.

A brief example will help solidify the VaR concept. Assume a risk manager calculates the daily 5% VaR as $10,000. The VaR(5%) of $10,000 indicates that there is a 5% chance that on any given day, the portfolio will experience a loss of $10,000 or more. We could also say that there is a 95% chance that on any given day the portfolio will experience either a loss less than $10,000 or a gain. If we further assume that the $10,000 loss represents 8% of the portfolio value, then on any given day there is a 5% chance that the portfolio will experience a loss of 8% or greater, but there is a 95% chance that the loss will be less than 8% or a percentage gain greater than zero.

Calculating VaR

Calculating delta-normal VaR is a simple matter but requires assuming that asset returns conform to a standard normal distribution. Recall that a standard normal distribution is defined by two parameters, its mean ($\mu = 0$) and standard deviation ($\sigma = 1$), and is perfectly symmetric with 50% of the distribution lying to the right of the mean and 50% lying to the left of the mean. Figure 1 illustrates the standard normal distribution and the cumulative probabilities under the curve.

Figure 1: Standard Normal Distribution and Cumulative Probabilities



From Figure 1, we observe the following: the probability of observing a value more than 1.28 standard deviations below the mean is 10%; the probability of observing a value more than 1.65 standard deviations below the mean is 5%; and the probability of observing a value more than 2.33 standard deviations below the mean is 1%. Thus, we have critical *z*-values of –1.28, –1.65, and –2.33 for 10%, 5%, and 1% lower tail probabilities, respectively. We can now define percent VaR mathematically as:

$$\text{VaR}(X\%) = z_{X\%}\sigma$$

where:

VaR (X%) = the X% probability value at risk

$z_{X\%}$ = the critical *z*-value based on the normal distribution and the selected X% probability

σ = the standard deviation of daily returns on a percentage basis

Professor's Note: VaR is a one-tailed test, so the level of significance is entirely in one tail of the distribution. As a result, the critical values will be different than a two-tailed test that uses the same significance level.

In order to calculate VaR(5%) using this formula, we would use a critical *z*-value of –1.65 and multiply by the standard deviation of percent returns. The resulting VaR estimate would be the percentage loss in asset value that would only be exceeded 5% of the time. VaR can also be estimated on a dollar rather than a percentage basis. To calculate VaR on a dollar basis, we simply multiply the percent VaR by the asset value as follows:

$$\text{VaR}(X\%)_{\text{dollar basis}} = \text{VaR}(X\%)_{\text{decimal basis}} \times \text{asset value}$$

$$= (z_{X\%}\sigma) \times \text{asset value}$$

To calculate VaR(5%) using this formula, we multiply VaR(5%) on a percentage basis by the current value of the asset in question. This is equivalent to taking the product of the critical *z*-value, the standard deviation of percent returns, and the current asset value. An estimate of VaR(5%) on a dollar basis is interpreted as the dollar loss in asset value that will only be exceeded 5% of the time.

Example: Calculating percentage and dollar VaR

A risk management officer at a bank is interested in calculating the VaR of an asset that he is considering adding to the bank's portfolio. If the asset has a daily standard deviation of returns equal to 1.4% and the asset has a current value of $5.3 million, **calculate** the VaR (5%) on both a percentage and dollar basis.

Answer:

The appropriate critical *z*-value for a VaR (5%) is –1.65. Using this critical value and the asset's standard deviation of returns, the VaR (5%) on a percentage basis is calculated as follows:

$$\text{VaR }(5\%) = z_{5\%}\sigma = -1.65(0.014) = -0.0231 = -2.31\%$$

The VaR(5%) on a dollar basis is calculated as follows:

$$\begin{aligned}\text{VaR }(5\%)_{\text{dollar basis}} &= \text{VaR }(5\%)_{\text{decimal basis}} \times \text{asset value} = -0.0231 \times \$5{,}300{,}000 \\ &= -\$122{,}430\end{aligned}$$

Thus, there is a 5% probability that, on any given day, the loss in value on this particular asset will equal or exceed 2.31%, or $122,430.

If an expected return other than zero is given, VaR becomes the expected return minus the quantity of the critical value multiplied by the standard deviation.

$$\text{VaR} = [E(R) - z\sigma]$$

In the example above, the expected return value is zero and thus ignored. The following example demonstrates how to apply an expected return to a VaR calculation.

Example: Calculating VaR given an expected return

For a $100,000,000 portfolio, the expected 1-week portfolio return and standard deviation are 0.00188 and 0.0125, respectively. **Calculate** the 1-week VaR at 5% significance.

Answer:

$$
\begin{aligned}
\text{VaR} &= [E(R) - z\sigma] \times \text{portfolio value} \\
&= [0.00188 - 1.65(0.0125)] \times \$100{,}000{,}000 \\
&= -0.018745 \times \$100{,}000{,}000 \\
&= -\$1{,}874{,}500
\end{aligned}
$$

The manager can be 95% confident that the maximum 1-week loss will not exceed $1,874,500.

VaR Conversions

VaR, as calculated previously, measured the risk of a loss in asset value over a short time period. Risk managers may, however, be interested in measuring risk over longer time periods, such as a month, quarter, or year. VaR can be converted from a 1-day basis to a longer basis by multiplying the daily VaR by the square root of the number of days (J) in the longer time period (called the **square root rule**). For example, to convert to a weekly VaR, multiply the daily VaR by the square root of 5 (i.e., five business days in a week). We can generalize the conversion method as follows:

$$\text{VaR}(X\%)_{\text{J-days}} = \text{VaR}(X\%)_{\text{1-day}}\sqrt{J}$$

Example: Converting daily VaR to other time bases

Assume that a risk manager has calculated the daily VaR $(10\%)_{\text{dollar basis}}$ of a particular asset to be $12,500. **Calculate** the weekly, monthly, semiannual, and annual VaR for this asset. Assume 250 days per year and 50 weeks per year.

Answer:

The daily dollar VaR is converted to a weekly, monthly, semiannual, and annual dollar VaR measure by multiplying by the square root of 5, 20, 125, and 250, respectively.

$$\text{VaR}(10\%)_{\text{5-days (weekly)}} = \text{VaR}(10\%)_{\text{1-day}}\sqrt{5} = \$12{,}500\sqrt{5} = \$27{,}951$$

$$\text{VaR}(10\%)_{\text{20-days (monthly)}} = \text{VaR}(10\%)_{\text{1-day}}\sqrt{20} = \$12{,}500\sqrt{20} = \$55{,}902$$

$$\text{VaR}(10\%)_{\text{125-days}} = \text{VaR}(10\%)_{\text{1-day}}\sqrt{125} = \$12{,}500\sqrt{125} = \$139{,}754$$

$$\text{VaR}(10\%)_{\text{250-days}} = \text{VaR}(10\%)_{\text{1-day}}\sqrt{250} = \$12{,}500\sqrt{250} = \$197{,}642$$

VaR can also be converted to different confidence levels. For example, a risk manager may want to convert VaR with a 95% confidence level to VaR with a 99% confidence level. This conversion is done by adjusting the current VaR measure by the ratio of the updated confidence level to the current confidence level.

Example: Converting VaR to different confidence levels

Assume that a risk manager has calculated VaR at a 95% confidence level to be $16,500. Now assume the risk manager wants to adjust the confidence level to 99%. **Calculate** the VaR at a 99% confidence level.

Answer:

$$\text{VaR}(1\%) = \text{VaR}(5\%) \times \frac{z_{1\%}}{z_{5\%}}$$

$$\text{VaR}(1\%) = \$16{,}500 \times \frac{2.33}{1.65} = \$23{,}300$$

The VaR Methods

The three main VaR methods can be divided into two groups: linear methods and full valuation methods.

1. **Linear methods** replace portfolio positions with linear exposures on the appropriate risk factor. For example, the linear exposure used for option positions would be delta while the linear exposure for bond positions would be duration. This method is used when calculating VaR with the delta-normal method.
2. **Full valuation methods** fully reprice the portfolio for each scenario encountered over a historical period, or over a great number of hypothetical scenarios developed through historical simulation or Monte Carlo simulation. Computing VaR using full revaluation is more complex than linear methods. However, this approach will generally lead to more accurate estimates of risk in the long run.

Linear Valuation: The Delta-Normal Valuation Method

The **delta-normal approach** begins by valuing the portfolio at an initial point as a relationship to a specific risk factor, *S* (consider only one risk factor exists):

$$V_0 = V(S_0)$$

With this expression, we can describe the relationship between the change in portfolio value and the change in the risk factor as:

$$dV = \Delta_0 \times dS$$

Here, Δ_0 is the sensitivity of the portfolio to changes in the risk factor, *S*. As with any linear relationship, the biggest change in the value of the portfolio will accompany the biggest change in the risk factor. The VaR at a given level of significance, *z*, can be written as:

$$VaR = |\Delta_0| \times (z\sigma S_0)$$

where:
$z\sigma S_0 = VaR_S$

Generally speaking, VaR developed by a delta-normal method is more accurate over shorter horizons than longer horizons.

Consider, for example, a fixed income portfolio. The risk factor impacting the value of this portfolio is the change in yield. The VaR of this portfolio would then be calculated as follows:

$$VaR = \text{modified duration} \times z \times \text{annualized yield volatility} \times \text{portfolio value}$$

Notice here that the volatility measure applied is the volatility of changes in the yield. In future examples, the volatility measured used will be the standard deviation of returns.

Since the delta-normal method is only accurate for linear exposures, non-linear exposures, such as convexity, are not adequately captured with this VaR method. By using a Taylor series expansion, convexity can be accounted for in a fixed income portfolio by using what is known as the **delta-gamma method**. You will see this method in Topic 53. For now, just take note that complexity can be added to the delta-normal method to increase its reliability when measuring non-linear exposures.

Full Valuation: Monte Carlo and Historic Simulation Methods

The **Monte Carlo simulation** approach revalues a portfolio for a large number of risk factor values, randomly selected from a normal distribution. **Historical simulation** revalues a portfolio using actual values for risk factors taken from historical data. These full valuation approaches provide the most accurate measurements because they include all nonlinear relationships and other potential correlations that may not be included in the linear valuation models.

Comparing the Methods

The delta-normal method is appropriate for large portfolios without significant option-like exposures. This method is fast and efficient.

Full-valuation methods, either based on historical data or on Monte Carlo simulations, are more time consuming and costly. However, they may be the only appropriate methods for large portfolios with substantial option-like exposures, a wider range of risk factors, or a longer-term horizon.

Delta-Normal Method

The delta-normal method (a.k.a. the variance-covariance method or the analytical method) for estimating VaR requires the assumption of a **normal distribution**. This is because the method utilizes the expected return and standard deviation of returns. For example, in calculating a daily VaR, we calculate the standard deviation of daily returns in the past and assume it will be applicable to the future. Then, using the asset's expected 1-day return and standard deviation, we estimate the 1-day VaR at the desired level of significance.

The assumption of normality is troublesome because many assets exhibit skewed return distributions (e.g., options), and equity returns frequently exhibit leptokurtosis (fat tails). When a distribution has "fat tails," VaR will tend to underestimate the loss and its associated probability. Also know that delta-normal VaR is calculated using the historical standard deviation, which may not be appropriate if the composition of the portfolio changes, if the estimation period contained unusual events, or if economic conditions have changed.

Example: Delta-normal VaR

The expected 1-day return for a $100,000,000 portfolio is 0.00085 and the historical standard deviation of daily returns is 0.0011. **Calculate** daily value at risk (VaR) at 5% significance.

Answer:

To locate the value for a 5% VaR, we use the Alternative *z*-Table in the appendix to this book. We look through the body of the table until we find the value that we are looking for. In this case, we want 5% in the lower tail, which would leave 45% below the mean that is not in the tail. Searching for 0.45, we find the value 0.4505 (the closest value we will find). Adding the *z*-value in the left hand margin and the *z*-value at the top of the column in which 0.4505 lies, we get 1.6 + 0.05 = 1.65, so the *z*-value coinciding with a 95% VaR is 1.65. (Notice that we ignore the negative sign, which would indicate the value lies below the mean.)

You will also find a Cumulative *z*-Table in the appendix. When using this table, you can look directly for the significance level of the VaR. For example, if you desire a 5% VaR, look for the value in the table which is closest to (1 – significance level) or 1 – 0.05 = 0.9500. You will find 0.9505, which lies at the intersection of 1.6 in the left margin and 0.05 in the column heading.

$$\text{VaR} = \left[\hat{R}_P - (z)(\sigma)\right]V_P$$

$$= [0.00085 - 1.65(0.0011)](\$100{,}000{,}000)$$

$$= -0.000965(\$100{,}000{,}000)$$

$$= -\$96{,}500$$

where:
$\hat{R}_P$ = expected 1-day return on the portfolio
V_P = value of the portfolio
z = z-value corresponding with the desired level of significance
σ = standard deviation of 1-day returns

The interpretation of this VaR is that there is a 5% chance the *minimum* 1-day loss is 0.0965%, or $96,500. (There is 5% probability that the 1-day loss will exceed $96,500.) Alternatively, we could say we are 95% confident the 1-day loss will not exceed $96,500.

If you are given the standard deviation of annual returns and need to calculate a daily VaR, the daily standard deviation can be estimated as the annual standard deviation divided by the square root of the number of (trading) days in a year, and so forth:

$$\sigma_{\text{daily}} \cong \frac{\sigma_{\text{annual}}}{\sqrt{250}};\sigma_{\text{monthly}} \cong \frac{\sigma_{\text{annual}}}{\sqrt{12}}$$

Delta-normal VaR is often calculated assuming an expected return of zero rather than the portfolio's actual expected return. When this is done, VaR can be adjusted to longer or shorter periods of time quite easily. For example, daily VaR is estimated as annual VaR divided by the square root of 250 (as when adjusting the standard deviation).

Likewise, the annual VaR is estimated as the daily VaR multiplied by the square root of 250. If the true expected return is used, VaR for different length periods must be calculated independently.

Professor's Note: Assuming a zero expected return when estimating VaR is a conservative approach because the calculated VaR will be greater (i.e., farther out in the tail of the distribution) than if the expected return is used.

Since portfolio values are likely to change over long time periods, it is often the case that VaR over a short time period is calculated and then converted to a longer period. The Basel Accord (discussed in the FRM Part II curriculum) recommends the use of a two-week period (10 days).

Professor's Note: For the exam, you will likely be required to make these time conversation calculations since VaR is often calculated over a short time frame.

Advantages of the delta-normal VaR method include the following:

- Easy to implement.
- Calculations can be performed quickly.
- Conducive to analysis because risk factors, correlations, and volatilities are identified.

Disadvantages of the delta-normal method include the following:

- The need to assume a normal distribution.
- The method is unable to properly account for distributions with fat tails, either because of unidentified time variation in risk or unidentified risk factors and/or correlations.
- Nonlinear relationships of option-like positions are not adequately described by the delta-normal method. VaR is misstated because the instability of the option deltas is not captured.

Historical Simulation Method

The historical method for estimating VaR is often referred to as the **historical simulation** method. The easiest way to calculate the 5% daily VaR using the historical method is to accumulate a number of past daily returns, rank the returns from highest to lowest, and identify the lowest 5% of returns. The highest of these lowest 5% of returns is the 1-day, 5% VaR.

Example: Historical VaR

You have accumulated 100 daily returns for your $100,000,000 portfolio. After ranking the returns from highest to lowest, you identify the lowest six returns:

–0.0011, –0.0019, –0.0025, –0.0034, –0.0096, –0.0101

Calculate daily value at risk (VaR) at 5% significance using the historical method.

Answer:

The lowest five returns represent the 5% lower tail of the "distribution" of 100 historical returns. The fifth lowest return (–0.0019) is the 5% daily VaR. We would say there is a 5% chance of a daily loss exceeding 0.19%, or $190,000.

As you will see in Topic 52, the historical simulation method may weight observations and take an average of two returns to obtain the historical VaR. Each observation can be viewed as having a probability distribution with 50% to the left and 50% to the right of a given observation. When considering the previous example, 5% VaR with 100 observations would take the average of the fifth and sixth observations [i.e., (–0.0011 + –0.0019) / 2 = –0.0015]. Therefore, the 5% historical VaR in this case would be $150,000. Either approach (using a given percentile or an average of two) is acceptable for calculating historical VaR, however, using a given percentile, as provided in the previous example, will yield a more conservative estimate since the calculated VaR estimate will be lower.

Professor's Note: On past FRM exams, GARP has calculated historical VaR in a similar fashion to the previous example.

Advantages of the historical simulation method include the following:

- The model is easy to implement when historical data is readily available.
- Calculations are simple and can be performed quickly.
- Horizon is a positive choice based on the intervals of historical data used.
- Full valuation of portfolio is based on actual prices.
- It is not exposed to model risk.
- It includes all correlations as embedded in market price changes.

Disadvantages of the historical simulation method include the following:

- It may not be enough historical data for all assets.
- Only one path of events is used (the actual history), which includes changes in correlations and volatilities that may have occurred only in that historical period.
- Time variation of risk in the past may not represent variation in the future.
- The model may not recognize changes in volatility and correlations from structural changes.
- It is slow to adapt to new volatilities and correlations as old data carries the same weight as more recent data. However, exponentially weighted average (EWMA) models can be used to weigh recent observations more heavily.
- A small number of actual observations may lead to insufficiently defined distribution tails.

Monte Carlo Simulation Method

The Monte Carlo method refers to computer software that generates hundreds, thousands, or even millions of possible outcomes from the distributions of inputs *specified by the user*. For example, a portfolio manager could enter a distribution of possible 1-week returns for each of the hundreds of stocks in a portfolio. On each "run" (the number of runs is specified by the user), the computer selects one weekly return from each stock's distribution of possible returns and calculates a weighted average portfolio return.

The several thousand weighted average portfolio returns will naturally form a distribution, which will approximate the normal distribution. Using the portfolio expected return and the standard deviation, which are part of the Monte Carlo output, VaR is calculated in the same way as with the delta-normal method.

Example: Monte Carlo VaR

A Monte Carlo output specifies the expected 1-week portfolio return and standard deviation as 0.00188 and 0.0125, respectively. **Calculate** the 1-week VaR at 1% significance.

Answer:

$$\text{VAR} = \left[\widehat{R}_P - (z)(\sigma)\right] V_P$$

$$= [0.00188 - 2.33(0.0125)](\$100{,}000{,}000)$$

$$= -0.027245(\$100{,}000{,}000)$$

$$= -\$2{,}724{,}500$$

The manager can be 99% confident that the maximum 1-week loss will not exceed $2,724,500. Alternatively, the manager could say there is a 1% probability that the minimum loss will be $2,724,500 or greater (the portfolio will lose at least $2,724,500).

Advantages of the Monte Carlo method include the following:

- It is the most powerful model.
- It can account for both linear and nonlinear risks.
- It can include time variation in risk and correlations by aging positions over chosen horizons.
- It is extremely flexible and can incorporate additional risk factors easily.
- Nearly unlimited numbers of scenarios can produce well-described distributions.

Disadvantages of the Monte Carlo method include the following:

- There is a lengthy computation time as the number of valuations escalates quickly.
- It is expensive because of the intellectual and technological skills required.
- It is subject to model risk of the stochastic processes chosen.
- It is subject to sampling variation at lower numbers of simulations.

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Quantifying Volatility in VaR Models

Topic 52

Exam Focus

Obtaining an accurate estimate of an asset's value that is at risk of loss hinges greatly on the measurement of the asset's volatility (or possible deviation in value over a certain time period). Asset value can be evaluating using a normal distribution; however, deviations from normality will create challenges for the risk manager in measuring both volatility and value at risk (VaR). In this topic, we will discuss issues with volatility estimation and different weighting methods that can be used to determine VaR. The advantages, disadvantages, and underlying assumptions of the various methodologies will also be discussed. For the exam, understand why deviations from normality occur and have a general understanding of the approaches to measuring VaR (parametric and nonparametric).

LO 52.1: Explain how asset return distributions tend to deviate from the normal distribution.

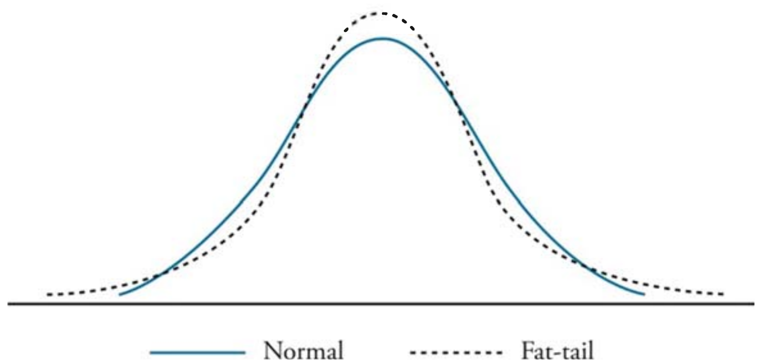
LO 52.2: Explain reasons for fat tails in a return distribution and describe their implications.

LO 52.3: Distinguish between conditional and unconditional distributions.

Three common deviations from normality that are problematic in modeling risk result from asset returns that are fat-tailed, skewed, or unstable.

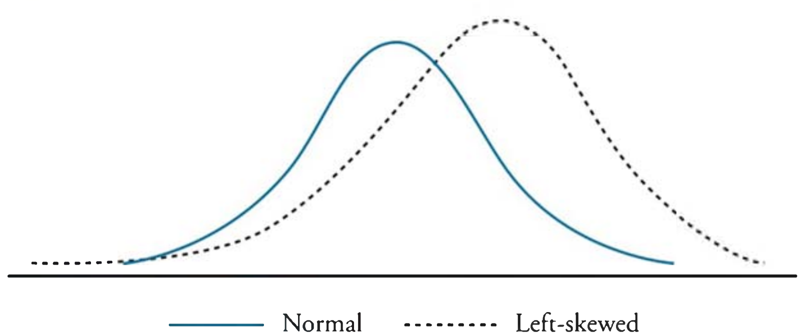
Fat-tailed refers to a distribution with a higher probability of observations occurring in the tails relative to the normal distribution. As illustrated in Figure 1, there is a larger probability of an observation occurring further away from the mean of the distribution. The first two moments (mean and variance) of the distributions are similar for the fat-tailed and normal distribution. However, in addition to the greater mass in the tails, there is also a greater probability mass around the mean for the fat-tailed distribution. Furthermore, there is less probability mass in the intermediate range (around +/– one standard deviation) for the fat-tailed distribution compared to the normal distribution.

Figure 1: Illustration of Fat-Tailed and Normal Distributions



A distribution is **skewed** when the distribution is not symmetrical. A risk manager is more concerned when there is a higher probability of a large negative return than a large positive return. This is referred to as left-skewed and is illustrated in Figure 2.

Figure 2: Left-Skewed and Normal Distributions



In modeling risk, a number of assumptions are necessary. If the parameters of the model are **unstable**, they are not constant but vary over time. For example, if interest rates, inflation, and market premiums are changing over time, this will affect the volatility of the returns going forward.

Deviations From the Normal Distribution

The phenomenon of "fat tails" is most likely the result of the volatility and/or the mean of the distribution changing over time. If the mean and standard deviation are the same for asset returns for any given day, the distribution of returns is referred to as an **unconditional distribution** of asset returns. However, different market or economic conditions may cause the mean and variance of the return distribution to change over time. In such cases, the return distribution is referred to as a **conditional distribution**.

Assume we separate the full data sample into two normally distributed subsets based on market environment with **conditional** means and variances. Pulling a data sample at different points of time from the full sample could generate fat tails in the unconditional distribution even if the conditional distributions are normally distributed with similar means but different volatilities. If markets are efficient and all available information is reflected in stock prices, it is not likely that the first moments or conditional means of the distribution vary enough to make a difference over time.

The second possible explanation for "fat tails" is that the second moment or volatility is time-varying. This explanation is much more likely given observed changes in interest rate volatility (e.g., prior to a much-anticipated Federal Reserve announcement). Increased market uncertainty following significant political or economic events results in increased volatility of return distributions.

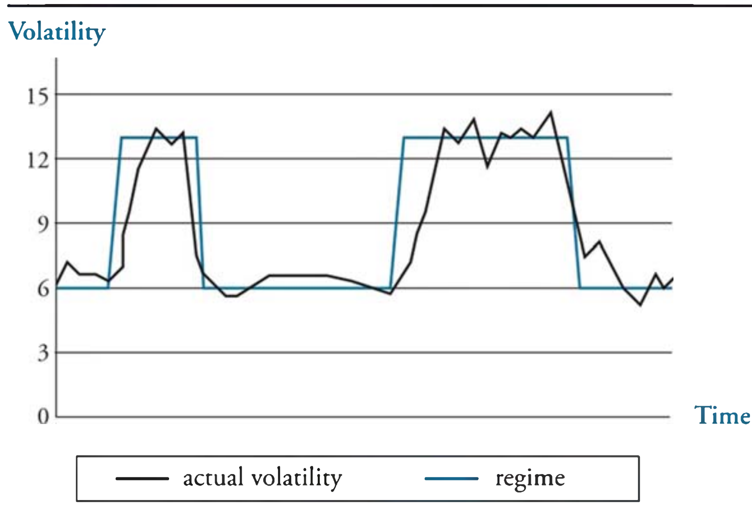
Market Regimes and Conditional Distributions

LO 52.4: Describe the implications of regime switching on quantifying volatility.

A **regime-switching volatility model** assumes different market regimes exist with high or low volatility. The conditional distributions of returns are always normal with a constant mean but either have a high or low volatility. Figure 3 illustrates a hypothetical regime-switching model for interest rate volatility. Note that the true interest rate volatility depicted by the solid line is either 13 basis points per day (bp/day) or 6bp/day. The actual observed returns deviate around the high volatility 13bp/day level and the low volatility 6bp/day. In this example, the unconditional distribution is not normally distributed. However, assuming time-varying volatility, the interest rate distributions are *conditionally normally distributed*.

The probability of large deviations from normality occurring are much less likely under the regime-switching model. For example, the interest rate volatility in Figure 3 ranges from 5.7bp/day to 13.6bp/day with an overall mean of 8.52bp/day. However, the 13.6bp/day has a difference of only 0.6bp/day from the conditional high volatility level compared to a 5.08bp/day difference from the unconditional distribution. This would result in a fat-tailed unconditional distribution. The regime-switching model captures the conditional normality and may resolve the fat-tail problem and other deviations from normality.

Figure 3: Actual Conditional Return Volatility Under Market Regimes



If we assume that volatility varies with time and that asset returns are conditionally normally distributed, then we may be able to tolerate the fat-tail issue. In the next section we demonstrate how to estimate conditional means and variances. However, despite efforts to more accurately model financial data, extreme events do still occur. The model (or distribution) used may not capture these extreme movements. For example, value at risk (VaR) models are typically utilized to model the risk level apparent in asset prices. VaR assumes asset returns follow a normal distribution, but as we have just discussed, asset return distributions tend to exhibit fat tails. As a result, VaR may underestimate the actual loss amount.

However, some tools exist that serve to complement VaR by examining the data in the tail of the distribution. For example, stress testing and scenario analysis can examine extreme events by testing how hypothetical and/or past financial shocks will impact VaR. Also, extreme value theory (EVT) can be applied to examine just the tail of the distribution and some classes of EVT apply a separate distribution to the tail. Despite not being able to accurately capture events in the tail, VaR is still useful for approximating the risk level inherent in financial assets.

Value at Risk

LO 52.5: Explain the various approaches for estimating VaR.

LO 52.6: Compare and contrast different parametric and non-parametric approaches for estimating conditional volatility.

LO 52.7: Calculate conditional volatility using parametric and non-parametric approaches.

A value at risk (VaR) method for estimating risk is typically either a historical-based approach or an implied-volatility-based approach. Under the historical-based approach, the shape of the conditional distribution is estimated based on historical time series data.

Historical-based approaches typically fall into three sub-categories: parametric, nonparametric, and hybrid.

1. The **parametric approach** requires specific assumptions regarding the asset returns distribution. A parametric model typically assumes asset returns are normally or lognormally distributed with time-varying volatility. The most common example of the parametric method in estimating future volatility is based on calculating historical variance or standard deviation using "mean squared deviation." For example, the following equation is used to estimate future variance based on a window of the *K* most recent returns data.[1]

$$\sigma_t^2 = \left(r^2_{t-K,t-K+1} + \cdots + r^2_{t-3,t-2} + r^2_{t-2,t-1} + r^2_{t-1,t}\right)\Big/K$$

1. In order to adjust for one degree of freedom related to the conditional mean, the denominator in the formula is K – 1. In practice, adjusting for the degrees of freedom makes little difference when large sample sizes are used.

If we assume asset returns follow a random walk, the mean return is zero. Alternatively, an analyst may assume a conditional mean different from zero and a volatility for a specific period of time.

Professor's Note: The delta-normal method is an example of a parametric approach.

Example: Estimating a conditional mean

Assuming K = 100 (an estimation window using the most recent 100 asset returns), **estimate** a conditional mean assuming the market is known to decline 15%.

Answer:

The daily conditional mean asset return, μ_t, is estimated to be –15bp/day.

$\mu_t = -1500\text{bp}/100\text{days} = -15\text{bp/day}$

2. The **nonparametric approach** is less restrictive in that there are no underlying assumptions of the asset returns distribution. The most common nonparametric approach models volatility using the historical simulation method.

3. As the name suggests, the **hybrid approach** combines techniques of both parametric and nonparametric methods to estimate volatility using historical data.

The **implied-volatility-based approach** uses derivative pricing models such as the Black-Scholes-Merton option pricing model to estimate an implied volatility based on current market data rather than historical data.

Parametric Approaches for VaR

The RiskMetrics® [i.e., exponentially weighted moving average (EWMA) model] and GARCH approaches are both exponential smoothing weighting methods. RiskMetrics® is actually a special case of the GARCH approach. Both exponential smoothing methods are similar to the historical standard deviation approach because all three methods:

- Are parametric.
- Attempt to estimate conditional volatility.
- Use recent historical data.
- Apply a set of weights to past squared returns.

Professor's Note: The RiskMetrics® approach is just an EWMA model that uses a pre-specified decay factor for daily data (0.94) and monthly data (0.97).

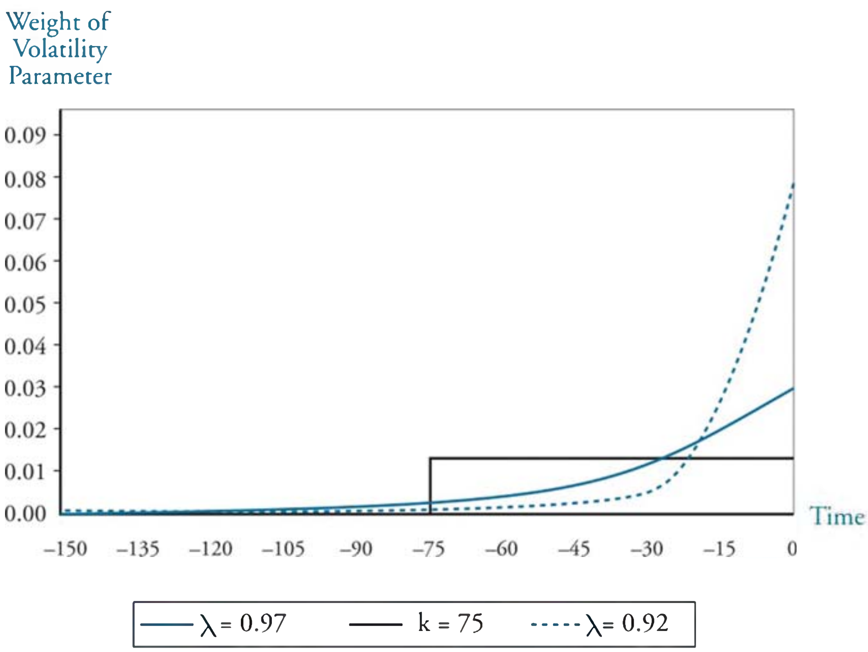
The only major difference between the historical standard deviation approach and the two exponential smoothing approaches is with respect to the weights placed on historical returns that are used to estimate future volatility. The historical standard deviation approach assumes all K returns in the window are equally weighted. Conversely, the exponential smoothing methods place a higher weight on more recent data, and the weights decline exponentially to zero as returns become older. The rate at which the weights change, or smoothness, is determined by a parameter λ (known as the decay factor) raised to a power. The parameter λ must fall between 0 and 1 (i.e., $0 < \lambda < 1$); however, most models use parameter estimates between 0.9 and 1 (i.e., $0.9 < \lambda < 1$).

Figure 4 illustrates the weights of the historical volatility for the historical standard deviation approach and RiskMetrics® approach. Using the RiskMetrics® approach, conditional variance is estimated using the following formula:

$$\sigma_t^2 = (1-\lambda)\left(\lambda^0 r^2_{t-1,t} + \lambda^1 r^2_{t-2,t-1} + \lambda^2 r^2_{t-3,t-2} + \cdots + \lambda^N r^2_{t-N-1,t-N}\right)$$

where:
N = the number of observations used to estimate volatility

Figure 4: Comparison of Exponential Smoothing and Historical Standard Deviation



Professor's Note: You may have noticed in Figure 4 that K *(the number of observations used to calculate the historical standard deviation) is 75, but* N *(the number of terms in the RiskMetrics® formula) is more than 75. There is no inconsistency here because the series* $[(1 - \lambda)\lambda^0 + (1 - \lambda)\lambda^1 + \ldots]$ *only sums to one if* N *is infinite. In practice,* N *is chosen so that the first* K *terms (in this example) sum to a number close to one.*

Example: Calculating weights using the RiskMetrics® approach

Using the RiskMetrics® approach, **calculate** the weight for the most current historical return, t = 0, when $\lambda = 0.97$.

Answer:

The weight for the most current historical return, t = 0, when $\lambda = 0.97$ is calculated as follows:

$$(1-\lambda)\,\lambda^t = (1-0.97)0.97^0 = 0.03$$

Example: Calculating weights using the historical standard deviation approach

Calculate the weight for the most recent return using historical standard deviation approach with K = 75.

Answer:

All historical returns are equally weighted. Therefore, the weights will all be equal to 0.0133 (i.e., 1 / K = 1 / 75 = 0.0133).

Figure 5 summarizes the most recent weights for the volatility parameters using the three approaches used in Figure 4. Parameter λ values of 0.92 and 0.97 are used for the example of the RiskMetrics® approaches in Figure 4.

Figure 5: Summary of RiskMetrics® and Historical Standard Deviation Calculations

	Weight of Volatility Parameter		
	$(1-\lambda)\lambda^t$	1/k	$(1-\lambda)\lambda^t$
t	$\lambda = 0.97$	k = 75	$\lambda = 0.92$
0	0.0300	0.0133	0.0800
–1	0.0291	0.0133	0.0736
–2	0.0282	0.0133	0.0677
–3	0.0274	0.0133	0.0623
–4	0.0266	0.0133	0.0573

Example: Applying a shorter estimation window

How would a shorter estimation window of K = 40 impact forecasts using the historical standard deviation method?

Answer:

Using a shorter estimation window (K = 40) for the historical standard deviation method results in forecasts that are more volatile. This is in part due to the fact that each observation has more weight, and extreme observations therefore have a greater impact on the forecast. However, an advantage of using a smaller *K* for the estimation window is the model adapts more quickly to changes.

Example: Applying a smaller λ parameter

How would a smaller λ parameter in the RiskMetrics® approach impact forecasts?

Answer:

Using a smaller *K* in the historical simulation model is similar to using a smaller λ parameter in the RiskMetrics® approach. It results in a higher weight to recent observations and a smaller sample window. As illustrated by Figure 4, a λ parameter closer to one results in less weight on recent observations and a larger sample window with a slower exponential smoothing decay in information.

GARCH

A more general exponential smoothing model is the GARCH model. This is a time-series model used by analysts to predict time-varying volatility. Volatility is measured with a general GARCH(p,q) model using the following formula:

$$\sigma_t^2 = a + b_1 r_{t-1,t}^2 + b_2 r_{t-2,t-1}^2 + \cdots + b_p r_{t-p,t-p+1}^2 + c_1\sigma_{t-1}^2 + c_2\sigma_{t-2}^2 + \cdots + c_q\sigma_{t-q}^2$$

where:

parameters a, b_1 through b_p, and c_1 through c_q = parameters estimated using historical data with *p* lagged terms on historical returns squared and *q* lagged terms on historical volatility

A GARCH(1,1) model would look like this:

$$\sigma_t^2 = a + b r_{t-1,t}^2 + c\sigma_{t-1}^2$$

Example: GARCH vs. RiskMetrics®

Show how the GARCH(1,1) time-varying process with a = 0 and b + c = 1 is identical to the RiskMetrics® model.

Answer:

Using these assumptions and substituting 1 – c for *b* results in the following special case of the GARCH(1,1) model as follows:

$$\sigma_t^2 = (1-c) r_{t-1,t}^2 + c\sigma_{t-1}^2$$

Substituting λ for *c* in this equation results in the common notation for the RiskMetrics® approach. Therefore, the GARCH model is less restrictive and more general than the RiskMetrics® model. The GARCH model using a larger number of parameters can more accurately model historical data. However, a model with more parameters to estimate also incurs more estimation risk, or noise, that can cause the GARCH model to have less ability to forecast future returns.

Nonparametric vs. Parametric VaR Methods

Three common types of nonparametric methods used to estimate VaR are: (1) historical simulation, (2) multivariate density estimation, and (3) hybrid. These nonparametric methods exhibit the following advantages and disadvantages over parametric approaches.

Advantages of nonparametric methods compared to parametric methods:

- Nonparametric models do not require assumptions regarding the entire distribution of returns to estimate VaR.
- Fat tails, skewness, and other deviations from some assumed distribution are no longer a concern in the estimation process for nonparametric methods.
- Multivariate density estimation (MDE) allows for weights to vary based on how relevant the data is to the current market environment, regardless of the timing of the most relevant data.
- MDE is very flexible in introducing dependence on economic variables (called *state variables* or *conditioning variables*).
- Hybrid approach does not require distribution assumptions because it uses a historical simulation approach with an exponential weighting scheme.

Disadvantages of nonparametric methods compared to parametric methods:

- Data is used more efficiently with parametric methods than nonparametric methods. Therefore, large sample sizes are required to precisely estimate volatility using historical simulation.
- Separating the full sample of data into different market regimes reduces the amount of usable data for historical simulations.
- MDE may lead to data snooping or over-fitting in identifying required assumptions regarding the weighting scheme identification of relevant conditioning variables and the number of observations used to estimate volatility.
- MDE requires a large amount of data that is directly related to the number of conditioning variables used in the model.

Nonparametric Approaches for VaR

Historical Simulation Method

The six lowest returns for an estimation window of 100 days (K = 100) are listed in Figure 6. Under the historical simulation, all returns are weighted equally based on the number of observations in the estimation window (1/K). Thus, in this example, each return has a weight of 1/100, or 0.01.

Example: Calculating VaR using historical simulation

Calculate VaR of the 5th percentile using historical simulation and the data provided in Figure 6.

Figure 6: Historical Simulation Example

Six Lowest Returns	*Historical Simulation Weight*	*HS Cumulative Weight*
–4.70%	0.01	0.0100
–4.10%	0.01	0.0200
–3.70%	0.01	0.0300
–3.60%	0.01	0.0400
–3.40%	0.01	0.0500
–3.20%	0.01	0.0600

Answer:

Calculating VaR of 5% requires identifying the 5th percentile. With 100 observations, the 5th percentile would be the 5th lowest return. However, observations must be thought of as a random event with a probability mass centered where the observation occurs, with 50% of its weight to the left and 50% of its weight to the right. Thus, the 5th percentile is somewhere between the 5th and 6th observation. In our example, the 5th lowest return, –3.40%, represents the 4.5th percentile, and we must interpolate to obtain the 5th percentile at –3.30% [calculated as (–3.4% + –3.20%) / 2].

Professor's Note: As was mentioned in the VaR Methods reading, the calculation of historical VaR may differ depending on the method used. You may use a given percentile return or interpolate to obtain the percentile return as was done in the previous example. On past FRM exams, GARP has just used the percentile in question, so in the previous example, the historical VaR of 5% would be based on –3.4%.

Notice that regardless of how far away in the 100-day estimation window the lowest observations occurred, they will still carry a weight of 0.01. The hybrid approach described next uses exponential weighting similar to the RiskMetrics® approach to adjust the weighting more heavily toward recent returns.

Hybrid Approach

The hybrid approach uses historical simulation to estimate the percentiles of the return and weights that decline exponentially (similar to GARCH or RiskMetrics®). The following three steps are required to implement the hybrid approach.

Step 1: Assign weights for historical realized returns to the most recent *K* returns using an exponential smoothing process as follows:

$$[(1 - \lambda) / (1 - \lambda^K)], [(1 - \lambda) / (1 - \lambda^K)]\lambda^1, \ldots, [(1 - \lambda) / (1 - \lambda^K)]\lambda^{K-1}$$

Step 2: Order the returns.

Step 3: Determine the VaR for the portfolio by starting with the lowest return and accumulating the weights until *x* percentage is reached. Linear interpolation may be necessary to achieve an exact *x* percentage.

In Step 1, there are several equations in between the second and third terms. These equations change the exponent on the last decay factor term to reflect observations that have occurred *t* days ago. For example, assume 100 observations and a decay factor of 0.96. For the hybrid weight for an observation that occurred one period ago, you would use the following equation: $[(1 - 0.96) / (1 - 0.96^{100})] = 0.0407$. For the hybrid weight of an observation two periods ago, you use this equation: $[(1 - 0.96) / (1 - 0.96^{100})] \times 0.96^{(100-99)} = 0.0391$. The hybrid weight five periods ago would equal: $[(1 - 0.96) / (1 - 0.96^{100})] \times 0.96^{(100-96)} = 0.0346$.

Example: Calculating weight using the hybrid approach

Suppose an analyst is using a hybrid approach to determine a 5% VaR with the most recent 100 observations (K = 100) and λ = 0.96 using the data in Figure 7. Note that the data in Figure 7 are already ranked as described in Step 2 of the hybrid approach. Therefore, the six lowest returns out of the most recent 100 observations are listed in Figure 7. The weights for each observation are based on the number of observations (K = 100) and the exponential weighting parameter (λ = 0.96) using the formula provided in Step 1.

Figure 7: Hybrid Example Illustrating Six Lowest Returns (where K = 100 and λ = 0.96)

Rank	*Six Lowest Returns*	*Number of Past Periods*	*Hybrid Weight*	*Hybrid Cumulative Weight**
1	–4.70%	2	0.0391	0.0391
2	–4.10%	5	0.0346	0.0736
3	–3.70%	55	0.0045	0.0781
4	–3.60%	25	0.0153	0.0934
5	–3.40%	14	0.0239	0.1173
6	–3.20%	7	0.0318	0.1492

*Cumulative weights are slightly affected by rounding error.

Calculate the hybrid weight assigned to the lowest return, –4.70%.

Answer:

The hybrid weight is calculated as follows:

$$[(1 - \lambda) / (1 - \lambda^K)]\lambda^1 = [(1 - 0.96) / (1 - 0.96^{100})]0.96 = 0.0391$$

Note: Since this observation is only two days old, it has the second highest weight assigned out of the 100 total observations in the estimation window.

Example: Calculating VaR using the hybrid approach

Using the information in Figure 7, **calculate** the initial VaR at the 5th percentile using the hybrid approach.

Answer:

The lowest and second lowest returns have cumulative weights of 3.91% and 7.36%, respectively. Therefore, we must interpolate to obtain the 5% VaR percentile. The point halfway between the two lowest returns is interpolated as –4.40% [(–4.70% + –4.10%) / 2] with a cumulative weight of 5.635% calculated as follows: (7.36% + 3.91%) / 2. Further interpolation is required to find the 5th percentile VaR level somewhere between –4.70% and –4.40%.

For the initial period represented in Figure 7, the 5% VaR using the hybrid approach is calculated as:

$$4.7\% - (4.70\% - 4.40\%)[(0.05 - 0.03910) / (0.05635 - 0.03910)]$$
$$= 4.70\% - 0.3\%(0.63188) = 4.510\%$$

Example: Calculating revised VaR

Assume that over the next 20 days there are no extreme losses. Therefore, the six lowest returns will be the same returns in 20 days, as illustrated in Figure 8. Notice that the weights are less for these observations because they are now further away. **Calculate** the revised VaR at the 5th percentile using the information in Figure 8.

Figure 8: Hybrid Example Illustrating Six Lowest Return After 20 Days (where K = 100 and λ = 0.96)

Rank	*Six Lowest Returns*	*Number of Past Periods*	*Hybrid Weight*	*Hybrid Cumulative Weight**
1	–4.70%	22	0.0173	0.0173
2	–4.10%	25	0.0153	0.0325
3	–3.70%	75	0.0020	0.0345
4	–3.60%	45	0.0068	0.0413
5	–3.40%	34	0.0106	0.0519
6	–3.20%	27	0.0141	0.0659

*Cumulative weights are slightly affected by rounding error.

Answer:

The 5th percentile for calculating VaR is somewhere between –3.6% and –3.4%. The point halfway between these points is interpolated as –3.5% with a cumulative weight of 4.66% [(4.13% + 5.19%) / 2]. The 5% VaR using the hybrid approach is calculated as:

$$3.5\% - (3.5\% - 3.4\%)[(0.05 - 0.0466) / (0.0519 - 0.0466)]$$
$$= 3.5\% - 0.1\%(0.6415) = 3.436\%$$

Multivariate Density Estimation (MDE)

Under the MDE model, conditional volatility for each market state or regime is calculated as follows:

$$\sigma_t^2 = \sum_{i=1}^{K} \omega(x_{t-i}) r_{t-i}^2$$

where:

x_{t-i} = the vector of relevant variables describing the market state or regime at time $t - i$

$\omega(x_{t-i})$ = the weight used on observation $t - i$, as a function of the "distance" of the state x_{t-i} from the current state x_t

The kernel function, $\omega(x_{t-i})$, is used to measure the relative weight in terms of "near" or "distant" from the current state. The MDE model is very flexible in identifying dependence on state variables. Some examples of relevant state variables in an MDE model are interest rate volatility dependent on the level of interest rates or the term structure of interest rates, equity volatility dependent on implied volatility, and exchange rate volatility dependent on interest rate spreads.

Return Aggregation

LO 52.8: Explain the process of return aggregation in the context of volatility forecasting methods.

When a portfolio is comprised of more than one position using the RiskMetrics® or historical standard deviation approaches, a single VaR measurement can be estimated by assuming asset returns are all normally distributed. The covariance matrix of asset returns is used to calculate portfolio volatility and VaR. The delta-normal method requires the calculation of N variances and $[N \times (N - 1)] / 2$ covariances for a portfolio of N positions. The model is subject to estimation error due to the large number of calculations. In addition, some markets are more highly correlated in a downward market, and in such cases, VaR is underestimated.

The historical simulation approach requires an additional step that aggregates each period's historical returns weighted according to the relative size of each position. The weights are based on the market value of the portfolio positions today, regardless of the actual allocation of positions K days ago in the estimation window. A major advantage of this approach compared to the delta-normal approach is that no parameter estimates are required. Therefore, the model is not subject to estimation error related to correlations and the problem of higher correlations in downward markets.

A third approach to calculating VaR estimates the volatility of the vector of aggregated returns and assumes normality based on the strong law of large numbers. The strong law of large numbers states that an average of a very large number of random variables will end up converging to a normal random variable. However, this approach can only be used in a well-diversified portfolio.

Implied Volatility

LO 52.9: Evaluate implied volatility as a predictor of future volatility and its shortcomings.

Estimating future volatility using historical data requires time to adjust to current changes in the market. An alternative method for estimating future volatility is implied volatility. The Black-Scholes-Merton model is used to infer an implied volatility from equity option prices. Using the most liquid at-the-money put and call options, an average implied volatility is extrapolated using the Black-Scholes-Merton model.

A big *advantage* of implied volatility is the forward-looking predictive nature of the model. Forecast models based on historical data require time to adjust to market events. The implied volatility model reacts immediately to changing market conditions.

The implied volatility model does, however, exhibit some *disadvantages*. The biggest disadvantage is that implied volatility is model dependent. A major assumption of the model is that asset returns follow a continuous time lognormal diffusion process. The volatility parameter is assumed to be constant from the present time to the contract maturity date. However, implied volatility varies through time; therefore, the Black-Scholes-Merton model is misspecified. Options are traded on the volatility of the underlying asset with what is known as "vol" terms. In addition, at a given point in time, options with the same underlying assets may be trading at different vols. Empirical results suggest implied volatility is on average greater than realized volatility. In addition to this upward bias in implied volatility, there is the problem that available data is limited to only a few assets and market factors.

Mean Reversion and Long Time Horizons

LO 52.10: Explain long horizon volatility/VaR and the process of mean reversion according to an AR(1) model.

LO 52.11: Calculate conditional volatility with and without mean reversion.

LO 52.12: Describe the impact of mean reversion on long horizon conditional volatility estimation.

To demonstrate mean reversion, consider a time series model with one lagged variable:

$$X_i = a + b \times X_{i-1}$$

This type of regression, with a lag of its own variable, is known as an autoregressive (AR) model. In this case, since there is only one lag, it is referred to as an AR(1) model. The long-run mean of this model is evaluated as [a / (1 – b)]. The key parameter in this long-run mean equation is *b*. Notice that if b = 1, the long-run mean is infinite (i.e., the process is a random walk). If *b*, however, is less than 1, then the process is mean reverting (i.e., the time series will trend toward its long-run mean). In the context of risk management, it is helpful to evaluate the impact of mean revision on variance.

Note that the single-period conditional variance of the rate of change is σ^2 and that the two-period variance is $(1 + b^2)\sigma^2$. If b = 1, the typical variance (i.e., square root volatility) would occur as this represents a random walk. If b < 1, the process is mean reverting. For example, the two-period volatility *without* mean reversion would be equal to:

$$\sqrt{2\sigma^2} = 1.41\sigma$$

With mean reversion (e.g., b = 0.8), the two-period volatility would be less:

$$\sqrt{(1+0.8^2)\sigma^2} = 1.28\sigma$$

Understanding the impact of mean reversion is especially important in the context of arbitrage and other trading strategies. For example, a convergence trade assumes explicitly that the spread between a long and short position is mean reverting. If mean reversion exists, the long horizon risk (and the resulting VaR calculation) is smaller than square root volatility.

Professor's Note: Remember that, like volatility, VaR can be extended to a longer-term basis by multiplying VaR by the square root of the number of days (i.e., the square root rule). For example, to convert daily VaR to weekly VaR, multiply the daily VaR by the square root of 5.

Backtesting VaR Methodologies

Backtesting is the process of comparing losses predicted by the value at risk (VaR) model to those actually experienced over the sample testing period. If a model were completely accurate, we would expect VaR to be exceeded (this is called an *exception*) with the same frequency predicted by the confidence level used in the VaR model. In other words, the probability of observing a loss amount greater than VaR is equal to the significance level (x%). This value is also obtained by calculating one minus the confidence level.

For example, if a VaR of $10 million is calculated at a 95% confidence level, we expect to have exceptions (losses exceeding $10 million) 5% of the time. If exceptions are occurring with greater frequency, we may be underestimating the actual risk. If exceptions are occurring less frequently, we may be overestimating risk.

There are three desirable attributes of VaR estimates that can be evaluated when using a backtesting approach. The first desirable attribute is that the VaR estimate should be ***unbiased***. To test this property, we use an indicator variable to record the number of times an exception occurs during a sample period. For each sample return, this indicator variable is recorded as 1 for an exception or 0 for a non-exception. The average of all indicator variables over the sample period should equal x% (i.e., the significance level) for the VaR estimate to be unbiased.

A second desirable attribute is that the VaR estimate is ***adaptable***. For example, if a large return increases the size of the tail of the return distribution, the VaR amount should also be increased. Given a large loss amount, VaR must be adjusted so that the probability of the next large loss amount again equals x%. This suggests that the indicator variables, discussed previously, should be independent of each other. It is necessary that the VaR estimate account for new information in the face of increasing volatility.

A third desirable attribute, which is closely related to the first two attributes, is for the VaR estimate to be ***robust***. A strong VaR estimate produces only a small deviation between the number of expected exceptions during the sample period and the actual number of exceptions. This attribute is measured by examining the statistical significance of the autocorrelation of extreme events over the backtesting period. A statistically significant autocorrelation would indicate a less reliable VaR measure.

By examining historical return data, we can gain some clarity regarding which VaR method actually produces a more reliable estimate in practice. In general, VaR approaches that are nonparametric (e.g., historical simulation and the hybrid approach) do a better job at producing VaR amounts that mimic actual observations when compared to parametric methods such as an exponential smoothing approach (e.g., GARCH). The likely reason for this performance difference is that nonparametric approaches can more easily account for the presence of fat tails in a return distribution. Note that higher levels of λ (the exponential weighing parameter) in the hybrid approach will perform better than lower levels of λ. Finally, when testing the autocorrelation of tail events, we find that the hybrid approach performs better than exponential smoothing approaches. In other words, the hybrid approach tends to reject the null hypothesis that autocorrelation is equal to zero *fewer times* than exponential smoothing approaches.

Key Concepts

LO 52.1

Three common deviations from normality that are problematic in modeling risk result from asset returns that are fat-tailed, skewed, or unstable. Fat-tailed refers to a distribution with a higher probability of observations occurring in the tails relative to the normal distribution. A distribution is skewed when the distribution is not symmetrical and there is a higher probability of outliers. Parameters of the model that vary over time are said to be unstable.

LO 52.2

The phenomenon of "fat tails" is most likely the result of the volatility and/or the mean of the distribution changing over time.

LO 52.3

If the mean and standard deviation are the same for asset returns for any given day, the distribution of returns is referred to as an unconditional distribution of asset returns. However, different market or economic conditions may cause the mean and variance of the return distribution to change over time. In such cases, the return distribution is referred to as a conditional distribution.

LO 52.4

A regime-switching volatility model assumes different market regimes exist with high or low volatility. The probability of large deviations from normality (such as fat tails) occurring are much less likely under the regime-switching model because it captures the conditional normality.

LO 52.5

Historical-based approaches of measuring VaR typically fall into three sub-categories: parametric, nonparametric, and hybrid.

- The parametric approach typically assumes asset returns are normally or lognormally distributed with time-varying volatility (i.e., historical standard deviation or exponential smoothing).
- The nonparametric approach is less restrictive in that there are no underlying assumptions of the asset returns distribution (i.e., historical simulation).
- The hybrid approach combines techniques of both parametric and nonparametric methods to estimate volatility using historical data.

LO 52.6

A major difference between the historical standard deviation approach and the two exponential smoothing approaches is with respect to the weights placed on historical returns. Exponential smoothing approaches give more weight to recent returns, and the historical standard deviation approach weights all returns equally.

LO 52.7

The RiskMetrics® and GARCH approaches are both exponential smoothing weighting methods. RiskMetrics® is actually a special case of the GARCH approach. Exponential smoothing methods are similar to the historical standard deviation approach because they are parametric, attempt to estimate conditional volatility, use recent historical data, and apply a set of weights to past squared returns.

LO 52.8

When a portfolio is comprised of more than one position using the RiskMetrics® or historical standard deviation approaches, a single VaR measurement can be estimated by assuming asset returns are all normally distributed. The historical simulation approach for calculating VaR for multiple portfolios aggregates each period's historical returns weighted according to the relative size of each position. The weights are based on the market value of the portfolio positions today, regardless of the actual allocation of positions *K* days ago in the estimation window. A third approach to calculating VaR for portfolios with multiple positions estimates the volatility of the vector of aggregated returns and assumes normality based on the strong law of large numbers.

LO 52.9

The implied-volatility-based approach for measuring VaR uses derivative pricing models such as the Black-Scholes-Merton option pricing model to estimate an implied volatility based on current market data rather than historical data.

LO 52.10

With an AR(1) model, long-run mean is computed as: [a / (1 – b)]. If b equals 1, the long-run mean is infinite (i.e., the process is a random walk). If b is less than 1, then the process is mean reverting.

LO 52.11

Under the context of mean reversion, the single-period conditional variance of the rate of change is σ^2, and the two-period variance is $(1 + b^2)\sigma^2$. Without mean reversion (i.e., b = 1), the two-period volatility is equal to the square root of $2\sigma^2$. With mean reversion (i.e., b < 1), the two-period volatility will be less than the volatility from no mean reversion.

LO 52.12

If mean reversion exists, the long horizon risk (and resulting VaR calculation) will be smaller than square root volatility.