MAT1110 - Mandatory assignment 2

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1

a

We have

$$\boldsymbol{F}(x, y) = -\frac{y}{2}\boldsymbol{i} + \frac{x}{2}\boldsymbol{j} \tag{1}$$

and the parameterization $\boldsymbol{r}$ of C oriented counter-clockwise. If follows that $\boldsymbol{r}$ is piecewise smooth as C is piecewise smooth. Since C encloses an area including R, and the partial derivatives of $\boldsymbol{F}$ are continuous, we can use Greens' theorem two write the area enclosed by C as

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

b

Since C_k is the line piece connecting the points (a_k, b_k) and (a_{k+1}, b_{k+1}) can we write $\Delta x = a_{k+1} - a_k$ and $\Delta y = b_{k+1} - b_k$ from the point (a_k, b_k). The change will be linear as C_k are straight lines. If we put all this together, we get the parameterization

$$\boldsymbol{r}_k(t) = (a_k + t(a_{k+1} - a_k)), b_k + t(b_{k+1} - b_k)), \qquad t \in [0, 1] \tag{2}$$

c

$$A_k = \int_{C_k} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{C_k} x dy$$

If we use $x = a_k + t(a_{k+1} - a_k)$ and $dx = (b_{k+1} - b_k)dt$ we get

$$\begin{aligned} A_k &= (b_{k+1} - b_k) \int_0^1 (a_k + t(a_{k+1} - a_k))dt \\ &= (b_{k+1} - b_k) \left[ta_k + \frac{1}{2}t^2(a_{k+1} - a_k)\right]_0^1 \\ &= \frac{1}{2}(a_{k+1} + a_k)(b_{k+1} - b_k) \end{aligned}$$

If we sum over all line-pieces we get

$$A = \frac{1}{2}\sum_{k=1}^{n-1}(a_{k+1} + a_k)(b_{k+1} - b_k) \tag{3}$$

d

We will now calculate the area of a triangle with corners $(0,0), (a,h)$ and $(g,0)$ using equation 3.

$$\begin{aligned} A_{\text{triangle}} &= \frac{1}{2}[(g-0)(0-0) + (a+g)(h-0) + (0+a)(0-h)] \\ &= \frac{1}{2}[ah + gh - ah] \\ &= \frac{gh}{2} \end{aligned}$$

For a rectangle with corners $(0,0), (g,0), (g,h)$ and $(0,h)$ we get

$$\begin{aligned} A_{\text{rectangle}} &= \frac{1}{2}[(g+0)(0-0) + (g+g)(h-0) + (0+g)(h-h) + (0+0)(0-h)] \\ &= \frac{1}{2}[2gh] \\ &= gh. \end{aligned}$$

2

3

We define $f : \mathbb{R}^2 \to \mathbb{R}^2$ where $f(\boldsymbol{v}) = A\boldsymbol{v} + \boldsymbol{b}$.

a

We let f be an isometry in the xy-plane. It is easy to show that f preserves norms:

$$\begin{aligned}
||\boldsymbol{v} - \boldsymbol{w}|| &= ||f(\boldsymbol{v}) - f(\boldsymbol{w}))|| \\
&= ||A\boldsymbol{v} + \boldsymbol{b} - (A\boldsymbol{w} + \boldsymbol{b})|| \\
&= ||A\boldsymbol{v} - A\boldsymbol{w}|| \\
&= ||A|| \cdot ||\boldsymbol{v} - \boldsymbol{w}|| \\
&= ||\boldsymbol{v} - \boldsymbol{w}||
\end{aligned}$$

where we have used that $\det(A) = \pm 1$.

b

We let $f(\boldsymbol{v}) = A\boldsymbol{v} + \boldsymbol{b}$ be an isometry with $\det(A) = -1$. The vector $A\boldsymbol{b} + \boldsymbol{b}$ is an eigenvector if $(A - I)(A\boldsymbol{b} + \boldsymbol{b}) = 0$ for all choises of $\boldsymbol{b}$. Remembering that $A^2 = I$, we get

$$(A - I)(A\boldsymbol{b} + \boldsymbol{b}) = A^2\boldsymbol{b} + A\boldsymbol{b} - AI\boldsymbol{b} - I\boldsymbol{b} = 0$$

which shows that $A\boldsymbol{b} + \boldsymbol{b}$ is an eigenvector.

c

The transformation by f on the line perpendicular to $\boldsymbol{w}$ is

$$f(s\boldsymbol{w}) = sA\boldsymbol{w} + \boldsymbol{b}, \qquad s \in \mathbb{R}$$

Since $\boldsymbol{w}$ is an eigenvector $A\boldsymbol{w} = \lambda\boldsymbol{w}$. Remembering that $\lambda = 1$, we can then scale the transformation down by a half, and get

$$t\boldsymbol{w} + \frac{1}{2}\boldsymbol{b}, \qquad t \in \mathbb{R} \tag{4}$$

where we let $t = s/2$. This shows that f transforms the line in eq 4 on itself.