MAT1110 - Mandatory assignment 2

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1

a

We have

$$\boldsymbol{F}(x,y) = -\frac{y}{2}\boldsymbol{i} + \frac{x}{2}\boldsymbol{j} \tag{1}$$

and the parameterization $\boldsymbol{r}$ of C oriented clockwise. If follows that $\boldsymbol{r}$ is piecewise smooth as C is piecewise smooth. Since C encloses an area including R, and the partial derivatives of $\boldsymbol{F}$ are continuous, we can use Greens' theorem two write the area enclosed by C as

$$\iint_R \Big| \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \Big| = \oint_C |\boldsymbol{F} \cdot d\boldsymbol{r}|$$

where we take the absolute value since Green's theorem uses a counter-clockwise parameterization.

b

Since C_k is the line piece connecting the points (a_k, b_k) and (a_{k+1}, b_{k+1}) can we write $\Delta x = a_{k+1} - a_k$ and $\Delta y = b_{k+1} - b_k$ from the point (a_k, b_k). The change will be linear as C_k are straight lines. If we put all this together, we get the parameterization

$$\boldsymbol{r}_k(t) = (a_k + t(a_{k+1} - a_k)), b_k + t(b_{k+1} - b_k)), \qquad t \in [0,1] \tag{2}$$

c

$$A_k = \int_{C_k} x dy \tag{3}$$

If we use $x = a_k + t(a_{k+1} - a_k)$ and $dy = (b_{k+1} - b_k)dt$ we get

$$\begin{aligned} A_k &= (b_{k+1} - b_k) \int_0^1 (a_k + t(a_{k+1} - a_k))dt \\ &= (b_{k+1} - b_k) \left[ta_k + \frac{1}{2}t^2(a_{k+1} - a_k) \right]_0^1 \\ &= \frac{1}{2}(a_{k+1} + a_k)(b_{k+1} - b_k) \end{aligned}$$

If we sum over all line-pieces we get

$$A = \frac{1}{2}\sum_{k=1}^{n}(a_{k+1} + a_k)(b_{k+1} - b_k) \tag{4}$$

d

We will now calculate the area of a triangle with corners $(0,0), (a,h)$ and $(g,0)$ using equation 4.

$$\begin{aligned}
A_{\text{triangle}} &= \frac{1}{2}[(g-0)(0-0) + (a+g)(h-0) + (0+a)(0-h)] \\
&= \frac{1}{2}[ah + gh - ah] \\
&= \frac{gh}{2}
\end{aligned}$$

For a rectangle with corners $(0,0), (g,0), (g,h)$ and $(0,h)$ we get

$$\begin{aligned}
A_{\text{rectangle}} &= \frac{1}{2}[(g+0)(0-0) + (g+g)(h-0) + (0+g)(h-h) + (0+0)(0-h)] \\
&= \frac{1}{2}[2gh] \\
&= gh.
\end{aligned}$$

2

a

We let

$$A = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix} \tag{5}$$

and (x, y, z) be a point on the unit shpere. In other words, $x^2 + y^2 + z^2 = 1$. We introduce the variable

$$\boldsymbol{u} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

To show that $\boldsymbol{u}$ is a point on the unit shpere, we need $|\boldsymbol{u}| = 1$.

$$\begin{aligned} |\boldsymbol{u}| &= \left(\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{2}} + \frac{z}{\sqrt{6}}\right)^2 + \left(-\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{2}} - \frac{z}{\sqrt{6}}\right)^2 + \left(\frac{x}{\sqrt{3}} + \frac{z}{\sqrt{6}}\right)^2 \\ &= \frac{x^2}{3} + \frac{y^2}{2} + \frac{z^2}{6} + \frac{x^2}{3} + \frac{y^2}{2} + \frac{z^2}{6} + \frac{x^2}{3} + \frac{4z^2}{6} \qquad \text{(Canceling like terms)} \\ &= x^2 + y^2 + z^2 \\ &= 1 \end{aligned}$$

b

We let

$$B = \begin{pmatrix} \frac{1}{6} & -\frac{1}{6} & 0 \\ -\frac{1}{6} & \frac{1}{2} & \frac{2}{3} \\ 0 & \frac{1}{6} & -\frac{1}{6} \end{pmatrix} \tag{6}$$

and

$$\boldsymbol{F} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = B \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \tag{7}$$

We define $\{\boldsymbol{v}_n\}$ such that $\boldsymbol{v}_{n+1} = \boldsymbol{F}(\boldsymbol{v}_n)$ and $\lim_{x\to\infty} \boldsymbol{v}_n = \boldsymbol{v}$.

$$\boldsymbol{F}(\boldsymbol{v}) = \lim_{x\to\infty} \boldsymbol{F}(\boldsymbol{v}_n) = \lim_{x\to\infty} \boldsymbol{v}_n = \boldsymbol{v}$$

Since $\{\boldsymbol{v}_n\}$ converges towards $\boldsymbol{v}$, $\boldsymbol{v}$ is a fixpoint for $\boldsymbol{F}$.

c

Since $\{\boldsymbol{F}(\boldsymbol{x}_{\mathrm{n}})\}$ converges for all $\boldsymbol{x_0}$, we start with the zero vector for convenience.

```python
import numpy as np

def F(v):
    B = np.asarray([
        [1/6, -1/6, 0],
        [-1/6, 1/2, 2/3],
        [0, 1/6, -1/6]
    ])
    return np.einsum('ij, j -> i', B, v) + (1, 0, 1)

def fixpoint(F):
    i = 0
    x = (0, 0, 0)
    while np.sum(F(x)-x) > 10e-8:
        x = F(x)
        i += 1

    print(f'Fixpoint: {x}, n = {i}')
    return x, i

fixpoint(F)

"""
Fixpoint: [1.0000001  0.9999997  0.99999994], n = 40
"""
```

The program shows that the vector $(1, 1, 1)$ is a fixpoint for $\boldsymbol{F}$.

3

We define $f : \mathbb{R}^2 \to \mathbb{R}^2$ where $f(\boldsymbol{v}) = A\boldsymbol{v} + \boldsymbol{b}$.

a

We let f be an isometry in the xy-plane. It is easy to show that f preserves norms:

$$\begin{aligned} |\boldsymbol{v} - \boldsymbol{w}| &= |f(\boldsymbol{v}) - f(\boldsymbol{w})| \\ &= |A\boldsymbol{v} + \boldsymbol{b} - (A\boldsymbol{w} + \boldsymbol{b})| \\ &= |A\boldsymbol{v} - A\boldsymbol{w}| \\ &\leq |A| \cdot |\boldsymbol{v} - \boldsymbol{w}| \\ &= |\boldsymbol{v} - \boldsymbol{w}| \end{aligned}$$

where we have used that $\det(A) = \pm 1$.

b

We let $f(\boldsymbol{v}) = A\boldsymbol{v} + \boldsymbol{b}$ be an isometry with $\det(A) = -1$. The vector $A\boldsymbol{b} + \boldsymbol{b}$ is an eigenvector if $(A - I)(A\boldsymbol{b} + \boldsymbol{b}) = 0$ for all choises of $\boldsymbol{b}$. Remembering that $A^2 = I$, we get

$$(A - I)(A\boldsymbol{b} + \boldsymbol{b}) = A^2\boldsymbol{b} + A\boldsymbol{b} - AI\boldsymbol{b} - I\boldsymbol{b} = 0$$

which shows that $A\boldsymbol{b} + \boldsymbol{b}$ is an eigenvector.

c

The transformation by f on the line perpendicular to $\boldsymbol{w}$ is

$$f(s\boldsymbol{w}) = sA\boldsymbol{w} + \boldsymbol{b}, \qquad s \in \mathbb{R}$$

Since $\boldsymbol{w}$ is an eigenvector $A\boldsymbol{w} = \lambda\boldsymbol{w}$. Remembering that $\lambda = 1$, we can then scale the transformation down by a half, and get

$$t\boldsymbol{w} + \frac{1}{2}\boldsymbol{b}, \qquad t \in \mathbb{R} \tag{8}$$

where we let $t = s/2$. This shows that f transforms the line in eq 8 on itself.