MAT1110 - Mandatory assignment 1

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1

a.

The curve C is given the following parametrisation:

$$\boldsymbol{r}(t) = \left(a \cdot \operatorname{arcsinh}\left(\frac{t}{a}\right), \sqrt{t^2 + a^2}\right) \tag{1}$$

for $-b \leq t \leq b$. $\boldsymbol{r}'(t)$ is

$$\begin{aligned}
\boldsymbol{r}'(t) &= \left(\frac{a}{\sqrt{1 + \left(\frac{t}{a}\right)^2}} \cdot \frac{1}{a}, \frac{1}{2\sqrt{t^2 + a^2}} \cdot 2t\right) \\
&= \left(\frac{a}{\sqrt{t^2 + a^2}}, \frac{t}{\sqrt{t^2 + a^2}}\right).
\end{aligned}$$

Its length is

$$\begin{aligned}
||\boldsymbol{r}'(t)|| =& \sqrt{\left(\frac{a}{\sqrt{t^2 + a^2}}\right)^2 + \left(\frac{t}{\sqrt{t^2 + a^2}}\right)^2} \\
&= \sqrt{\frac{a^2 + t^2}{a^2 + t^2}} \\
&= 1
\end{aligned}$$

b.

The length of a curve is given by

$$s = \int_a^b v(t) dt = \int_a^b ||\boldsymbol{r}'(t)|| dt \tag{2}$$

Hence, the length of C is

$$s = \int_{-b}^{b} dt = 2b$$

c.

See python code at the end of the paper.

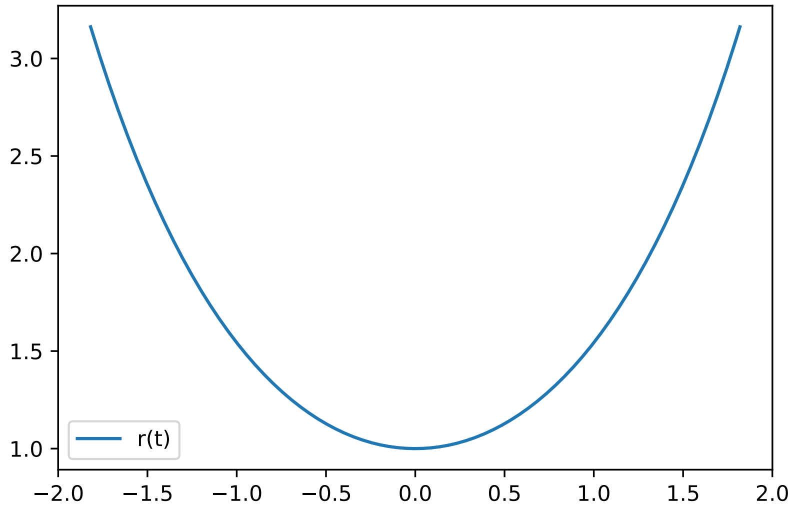

Figure 1: Plot of the catenary curve using $a = 1, b = 3$

d.

$$\rho_t = \frac{\partial \rho}{\partial t} = \left(\frac{a}{\sqrt{t^2 + a^2}}, \frac{t}{\sqrt{t^2 + a^2}} \cos\theta, \frac{t}{\sqrt{t^2 + a^2}} \sin\theta \right) \tag{3}$$

$$\rho_\theta = \frac{\partial \rho}{\partial \theta} = \left(0, -\sqrt{t^2 + a^2} \sin\theta, \sqrt{t^2 + a^2} \cos\theta \right) \tag{4}$$

We define the surface unit normal as

$$\boldsymbol{n} = \frac{\boldsymbol{f}}{f}$$

where $\boldsymbol{f} = \rho_t \times \rho_\theta$ and $f = ||\boldsymbol{f}||$. Taking the cross product of ρ_t and $\rho\theta$ gives

$$\boldsymbol{f} = (t, -a \cdot \cos\theta, -a \cdot \sin\theta)$$

which has a length

$$\begin{aligned} f &= ||\boldsymbol{f}|| \\ &= \sqrt{t^2 + (-a)^2 \cdot (\cos\theta + \sin\theta)} \\ &= \sqrt{t^2 + a^2} \end{aligned}$$

Hence the surface unit normal $\boldsymbol{n}$ is

$$\boldsymbol{n} = \frac{\boldsymbol{f}}{f} = \left(\frac{t}{\sqrt{t^2 + a^2}}, -\frac{a \cot \cos\theta}{\sqrt{t^2 + a^2}}, -\frac{a \cdot \sin\theta}{\sqrt{t^2 + a^2}} \right) \tag{5}$$

e.

We define the following:

$$
\begin{aligned}
E &= ||\rho_t||^2, & F &= \rho_t \cdot \rho_\theta, & G &= ||\rho_\theta||^2 \\
L &= \rho_{tt} \cdot \boldsymbol{n}, & M &= \rho_{t\theta} \cdot \boldsymbol{n}, & N &= \rho_{\theta\theta} \cdot \boldsymbol{n}
\end{aligned}
$$

The mean curvature of a surface S is given by

$$
H = \frac{1}{2}\frac{EN - 2FM + GL}{EG - F^2} \tag{6}
$$

and S is a minimal surface if $H = 0$ at all points. For this to be true, the numerator has to be zero for all t. Since ρ_t and ρ_θ is normal to each other, $F = 0$, hence $-2FM = 0$. We need $EN - GL = 0$.

$$
\begin{aligned}
E &= \sqrt{\frac{a^2}{t^2 + a^2} + \frac{t^2 \cdot \cos^2\theta}{t^2 + a^2} + \frac{t^2 \cdot \sin^2\theta}{t^2 + a^2}} = 1 \\
G &= \sqrt{(t^2 + a^2) \cdot \sin^2\theta + (t^2 + a^2) \cdot \cos^2\theta}^2 = t^2 + a^2 \\
L &= -\frac{a \cdot t^2}{(t^2 + a^2)^2} - \frac{a^3 \cdot \cos^2\theta}{(t^2 + a^2)^2} - \frac{a^3 \cdot \sin^2\theta}{(t^2 + a^2)^2} = -\frac{a}{t^2 + a^2} \\
N &= a \cdot \cos^2\theta + a \cdot \sin^2\theta = a
\end{aligned}
$$

If we put this together, we get

$$
1 \cdot a - (t^2 + a^2) \cdot \frac{a}{(t^2 + a^2)} = 0
$$

and S is a minimal surface.

2

$$\begin{aligned}
\boldsymbol{F}(x, y) &= (ax + by, cx + dy) \\
\boldsymbol{F}^{\perp}(x, y) &= (-cx - dy, ax + by) \\
\phi(x, y) &= -\frac{c}{2}x^2 + axy + \frac{b}{2}y^2
\end{aligned}$$

a.

$\boldsymbol{F}$ and $\boldsymbol{F}^{\perp}$ are orthogonal if the dot product between them is zero.

$$(ax + by)(-cx - dy) + (xc + dy)(ax + by) = 0.$$

b.

For any field $\boldsymbol{F}$ to be conservative, the following condition must be true for all $\vec{x}, i, j$:

$$\frac{\partial \boldsymbol{F}_i}{\partial x_j}(\vec{x}) = \frac{\partial \boldsymbol{F}_j}{\partial x_i}(\vec{x}) \tag{7}$$

In our case we have

$$\frac{\partial \boldsymbol{F}_{\mathbf{1}}^{\perp}}{\partial y} = -d, \qquad \frac{\partial \boldsymbol{F}_{\mathbf{2}}^{\perp}}{\partial x} = a$$

Hence, $\boldsymbol{F}^{\perp}$ is conservative when $d = -a$. Furthermore,

$$\begin{aligned}
\nabla\phi(x, y) &= (-cx + ay, ax + by) \\
&= (-cx - dy, ax + by) \\
&= \boldsymbol{F}^{\perp}
\end{aligned}$$

c.

We use the parametrisation $\boldsymbol{r}(t) = (x(t), y(t))$. This gives $\phi(x, y) = \phi(\boldsymbol{r}(t))$. The contours of $\boldsymbol{F}$ is given when $\phi(\boldsymbol{r}(t))$ is constant. This gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\phi(\boldsymbol{r}(t)) &= 0 \\
\nabla\phi(\boldsymbol{r}(t)) \cdot (\boldsymbol{r}'(t)) &= 0
\end{aligned}$$

which shows that $\nabla\phi$ is perpendicular to the contour lines.

d.

Since the contour lines to ϕ is perpendicular to $\boldsymbol{F}^{\perp}$ (as we found in the previous task), and $\boldsymbol{F}^{\perp}$ is perpendicular to $\boldsymbol{F}$, the contour lines are parallel with $\boldsymbol{F}$. This means that the contour lines are tangential to $\boldsymbol{F}$ and gives the field lines for the vector field$\boldsymbol{F}$.

e.

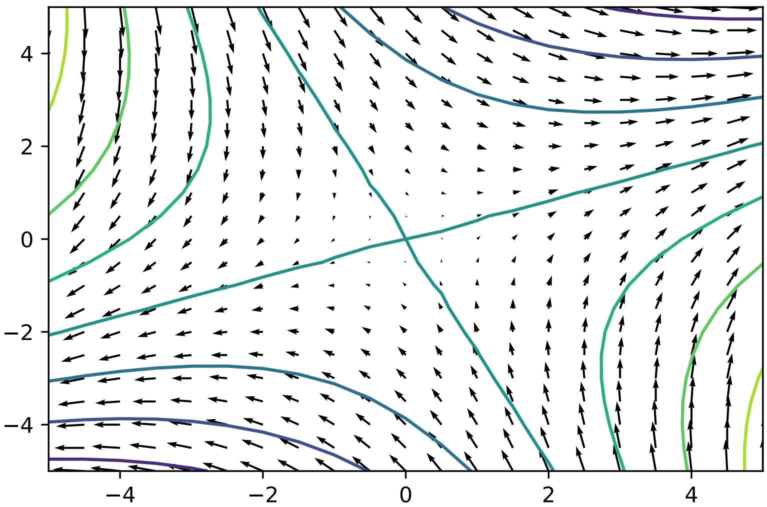

Figure 2: Plot of vector field $\boldsymbol{F}$ and its contour lines. $a = b = c = 1$.

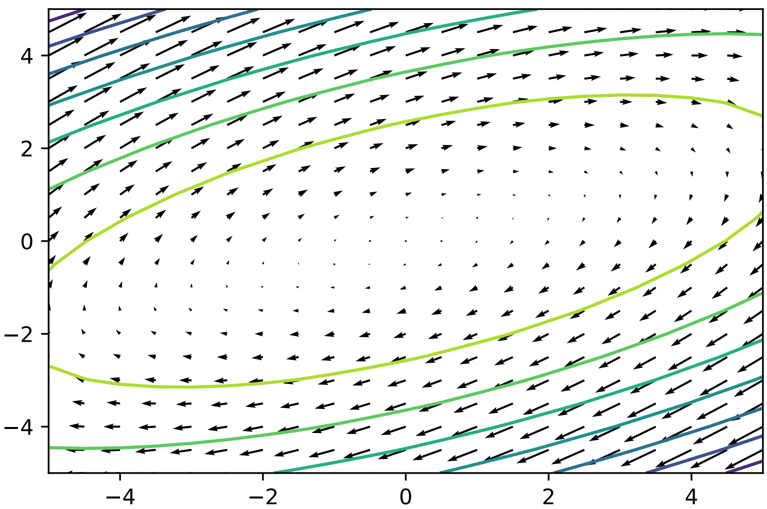

Figure 3: Plot of vector field $\boldsymbol{F}$ and its contour lines. $a = -1, b = 3, c = -1$.

Python code

```python
import numpy as np
import matplotlib.pyplot as plt

a = 1
b = 3

t = np.linspace(-b, b, 1001)
r = np.asarray([a*np.arcsinh(t/a), np.sqrt(t**2+a**2)])

plt.plot(r[0], r[1], label='r(t)')
plt.legend()
plt.show()

for a, b, c in ((1, 1, 1), (-1, 3, -1)):
    plt.clf()
    d = -a

    t = np.linspace(-5, 5, 21)
    x, y = np.meshgrid(t, t)

    u, v = a*x + b*y, c*x + d*y
    phi = c*x**2 - 2*a*x*y - b*y**2

    plt.quiver(x, y, u, v)
    plt.contour(x, y, phi)
    plt.show()
```