

MAT1110 - Mandatory assignment 2

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1

a

We have

$$\mathbf{F}(x, y) = -\frac{y}{2}\mathbf{i} + \frac{x}{2}\mathbf{j} \quad (1)$$

and the parameterization \mathbf{r} of C oriented clockwise. It follows that \mathbf{r} is piecewise smooth as C is piecewise smooth. Since C encloses an area including R , and the partial derivatives of \mathbf{F} are continuous, we can use Green's theorem to write the area enclosed by C as

$$\left| \oint_C \mathbf{F} \cdot d\mathbf{r} \right| = \left| \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \right| = \left| \iint_R dA \right|$$

where we take the absolute value since Green's theorem uses a counter-clockwise parameterization.

b

Since C_k is the line piece connecting the points (a_k, b_k) and (a_{k+1}, b_{k+1}) we can write $\Delta x = a_{k+1} - a_k$ and $\Delta y = b_{k+1} - b_k$ from the point (a_k, b_k) . The change will be linear as C_k are straight lines. If we put all this together, we get the parameterization

$$\mathbf{r}_k(t) = (a_k + t(a_{k+1} - a_k), b_k + t(b_{k+1} - b_k)), \quad t \in [0, 1] \quad (2)$$

c

$$A_k = \int_{C_k} x dy \quad (3)$$

If we use $x = a_k + t(a_{k+1} - a_k)$ and $dy = (b_{k+1} - b_k)dt$ we get

$$\begin{aligned} A_k &= (b_{k+1} - b_k) \int_0^1 (a_k + t(a_{k+1} - a_k)) dt \\ &= (b_{k+1} - b_k) \left[ta_k + \frac{1}{2} t^2 (a_{k+1} - a_k) \right]_0^1 \\ &= \frac{1}{2} (a_{k+1} + a_k) (b_{k+1} - b_k) \end{aligned}$$

If we sum over all line-pieces we get

$$A = \frac{1}{2} \sum_{k=1}^n (a_{k+1} + a_k) (b_{k+1} - b_k) \quad (4)$$

d

We will now calculate the area of a triangle with corners $(0, 0)$, (a, h) and $(g, 0)$ using equation 4.

$$\begin{aligned} A_{\text{triangle}} &= \frac{1}{2} [(g+0)(0-0) + (a+g)(h-0) + (0+a)(0-h)] \\ &= \frac{1}{2} [ah + gh - ah] \\ &= \frac{gh}{2} \end{aligned}$$

For a rectangle with corners $(0, 0)$, $(g, 0)$, (g, h) and $(0, h)$ we get

$$\begin{aligned} A_{\text{rectangle}} &= \frac{1}{2} [(g+0)(0-0) + (g+g)(h-0) + (0+g)(h-h) + (0+0)(0-h)] \\ &= \frac{1}{2} [2gh] \\ &= gh. \end{aligned}$$

2

a

We let

$$A = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix} \quad (5)$$

and (x, y, z) be a point on the unit sphere. In other words, $x^2 + y^2 + z^2 = 1$. We introduce the variable

$$\mathbf{u} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

To show that \mathbf{u} is a point on the unit sphere, we need $|\mathbf{u}| = 1$.

$$\begin{aligned} |\mathbf{u}| &= \left[\left(\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{2}} + \frac{z}{\sqrt{6}} \right)^2 + \left(-\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{2}} - \frac{z}{\sqrt{6}} \right)^2 + \left(\frac{x}{\sqrt{3}} - \frac{2z}{\sqrt{6}} \right)^2 \right]^{\frac{1}{2}} \\ &= \left[\frac{x^2}{3} + \frac{y^2}{2} + \frac{z^2}{6} + \frac{x^2}{3} + \frac{y^2}{2} + \frac{z^2}{6} + \frac{x^2}{3} + \frac{4z^2}{6} \right]^{\frac{1}{2}} \quad (\text{Canceling like terms}) \\ &= [x^2 + y^2 + z^2]^{\frac{1}{2}} \\ &= x^2 + y^2 + z^2 \\ &= 1 \end{aligned}$$

b

We let

$$B = \begin{pmatrix} \frac{1}{6} & -\frac{1}{6} & 0 \\ -\frac{1}{6} & \frac{1}{2} & \frac{2}{3} \\ 0 & \frac{1}{6} & -\frac{1}{6} \end{pmatrix} \quad (6)$$

and

$$\mathbf{F} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = B \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \quad (7)$$

We define $\{\mathbf{v}_n\}$ such that $\mathbf{v}_{n+1} = \mathbf{F}(\mathbf{v}_n)$ and $\lim_{n \rightarrow \infty} \mathbf{v}_n = \mathbf{v}$.

$$\mathbf{F}(\mathbf{v}) = \lim_{n \rightarrow \infty} \mathbf{F}(\mathbf{v}_n) = \lim_{n \rightarrow \infty} \mathbf{v}_n = \mathbf{v}$$

Since $\{\mathbf{v}_n\}$ converges towards \mathbf{v} , \mathbf{v} is a fixpoint for \mathbf{F} .

C

Since $\{F(x_n)\}$ converges for all x_0 , we start with the zero vector for convenience.

```
1 import numpy as np
2
3 def F(v):
4     B = np.asarray([
5         [1/6, -1/6, 0],
6         [-1/6, 1/2, 2/3],
7         [0, 1/6, -1/6]
8     ])
9     return np.einsum('ij, j -> i', B, v) + (1, 0, 1)
10
11 def fixpoint(F):
12     i = 0
13     x = (0, 0, 0)
14     while np.sum(F(x)-x) > 10e-8:
15         x = F(x)
16         i += 1
17
18     print(f'Fixpoint: {x}, n = {i}')
19     return x, i
20
21 fixpoint(F)
22
23 """
24 Fixpoint: [1.0000001  0.9999997  0.99999994], n = 40
25 """
```

The program shows that the vector $(1, 1, 1)$ is a fixpoint for F .

3

We define $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ where $f(\mathbf{v}) = A\mathbf{v} + \mathbf{b}$.

a

We let f be an isometry in the xy-plane. It is easy to show that f preserves norms:

$$\begin{aligned} |\mathbf{v} - \mathbf{w}| &= |f(\mathbf{v}) - f(\mathbf{w})| \\ &= |A\mathbf{v} + \mathbf{b} - (A\mathbf{w} + \mathbf{b})| \\ &= |A\mathbf{v} - A\mathbf{w}| \\ &\leq |A| \cdot |\mathbf{v} - \mathbf{w}| \\ &\leq \pm |\mathbf{v} - \mathbf{w}| \\ &= |\mathbf{v} - \mathbf{w}| \end{aligned}$$

where we have used that $\det(A) = \pm 1$.

b

We let $f(\mathbf{v}) = A\mathbf{v} + \mathbf{b}$ be an isometry with $\det(A) = -1$. The vector $A\mathbf{b} + \mathbf{b}$ is an eigenvector if $(A - I)(A\mathbf{b} + \mathbf{b}) = 0$ for all choices of \mathbf{b} . Remembering that $A^2 = I$, we get

$$(A - I)(A\mathbf{b} + \mathbf{b}) = A^2\mathbf{b} + A\mathbf{b} - A\mathbf{b} - \mathbf{b} = 0$$

which shows that $A\mathbf{b} + \mathbf{b}$ is an eigenvector.

c

The transformation by f on the line perpendicular to \mathbf{w} is

$$f(s\mathbf{w}) = sA\mathbf{w} + \mathbf{b}, \quad s \in \mathbb{R}$$

Since \mathbf{w} is an eigenvector $A\mathbf{w} = \lambda\mathbf{w}$. Remembering that $\lambda = 1$, we can then scale the transformation down by a half, and get

$$t\mathbf{w} + \frac{1}{2}\mathbf{b}, \quad t \in \mathbb{R} \tag{8}$$

where we let $t = s/2$. This shows that f transforms the line in eq 8 on itself.