MEK1100 - Mandatory assignment 1

William Dugan

March 5, 2022

1 Scaling

A ball thrown with initial velocity v_0 and at an angle θ to the horizontal at time $t = 0$ will follow the curve defined by

$$x(t) = v_0 t \cos\theta \tag{1}$$

$$y(t) = v_0 t \sin\theta - \frac{1}{2} g t^2. \tag{2}$$

a.

The time taken for the ball to reach $y = 0$ is

$$\begin{aligned} y(t) &= 0 \\ v_o t \sin\theta &= \frac{1}{2} g t^2 \\ t &= \frac{2 v_0 \sin\theta}{g} = t_m \end{aligned}$$

The x-component at this time is

$$\begin{aligned} x(t_m) &= \frac{2 v_0^2 \sin\theta \cos\theta}{g} \\ &= \frac{v_0^2 \sin(2\theta)}{g} = x_m \end{aligned}$$

b.

We introduce the following dimensionless variables:

$$\begin{aligned} x^* &= \frac{x}{x_m} \implies x = x^* x_m \\ y^* &= \frac{y}{x_m} \implies y = y^* x_m \end{aligned}$$

We insert these into (1) and get

$$
\begin{aligned}
v_0 t\cos\theta &= x^* \frac{v_0^2\sin(2\theta)}{g} \\
x^* &= \frac{gt}{2v_0\sin\theta} \\
v_0 t\sin\theta - \frac{1}{2}gt^2 &= y^* \frac{v_0^2\sin(2\theta)}{g} \\
y^* &= \frac{g(v_0 t\sin\theta - \frac{1}{2}gt^2)}{v_0^2\sin\theta\cos\theta} \\
y^* &= \frac{gt}{2v_0\cos\theta} - \frac{g^2t^2}{4v_0^2\sin\theta\cos\theta} \\
y^* &= \frac{gt}{2v_0\sin\theta}\cdot\frac{sin\theta}{\cos\theta}\cdot\left(1 - \frac{gt}{2v_0\sin\theta}\right) \\
y^* &= x^*\tan\theta(1 - x^*)
\end{aligned}
$$

Scaling time is done by $t^* = t/t_m = x^*$. θ is dimensionless, hence we do not need to scale it.

c.

```python
import numpy as np
import matplotlib.pyplot as plt

v0 = 5
g = 9.81

for theta in (np.pi/6, np.pi/4, np.pi/3):
    t = np.linspace(0, 2*v0*np.sin(theta)/g, 100)
    x = g*t/(2*v0*np.sin(theta))
    y = x*np.tan(theta)*(1-x)
    plt.plot(x, y)

plt.legend(['pi/6', 'pi/4', 'pi/3'])
plt.xlabel('x*')
plt.ylabel('y*')
plt.show()
```

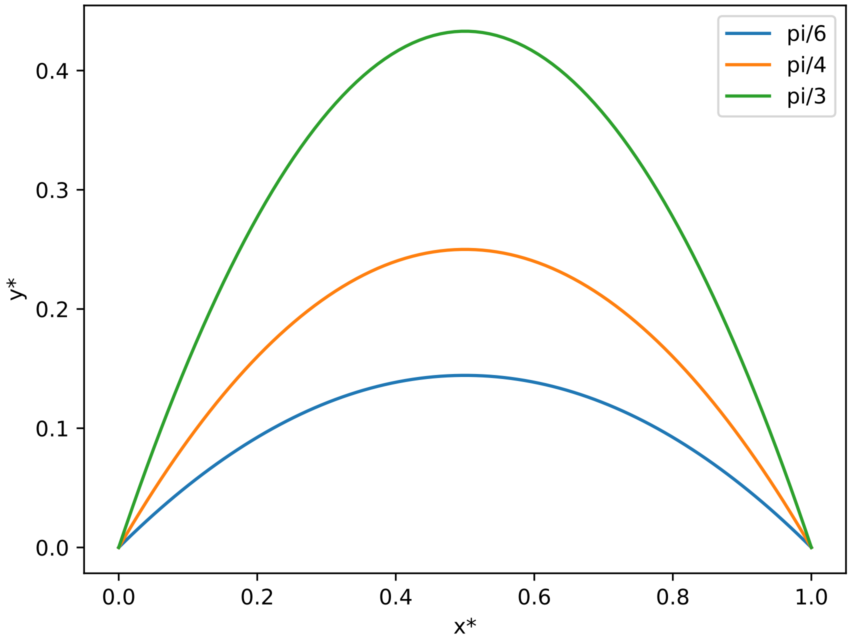

Figure 1: Plot of x^*, y^* for $\theta = \pi/6, \pi/4, \pi/3$.

Since we have used dimensionless variables, the trajectory of the ball will only depend on the angle θ.

2 Stream lines for a two-dimensional field

We are given the velocity field

$$\boldsymbol{v} = v_x \boldsymbol{i} + v_y \boldsymbol{j} = xy\boldsymbol{i} + y\boldsymbol{j}. \tag{3}$$

a.

To find the stream lines we integrate both sides as described in chapter 2.4 in Gjevik & Fagerland (2021).

$$\int xy dy = \int y dx$$
$$\int dy = \int \frac{1}{x} dx$$
$$y = \ln|x| + C$$

It is clear that if $y = 0$ the whole x-axis is a solution to the initial differential equation.

b.

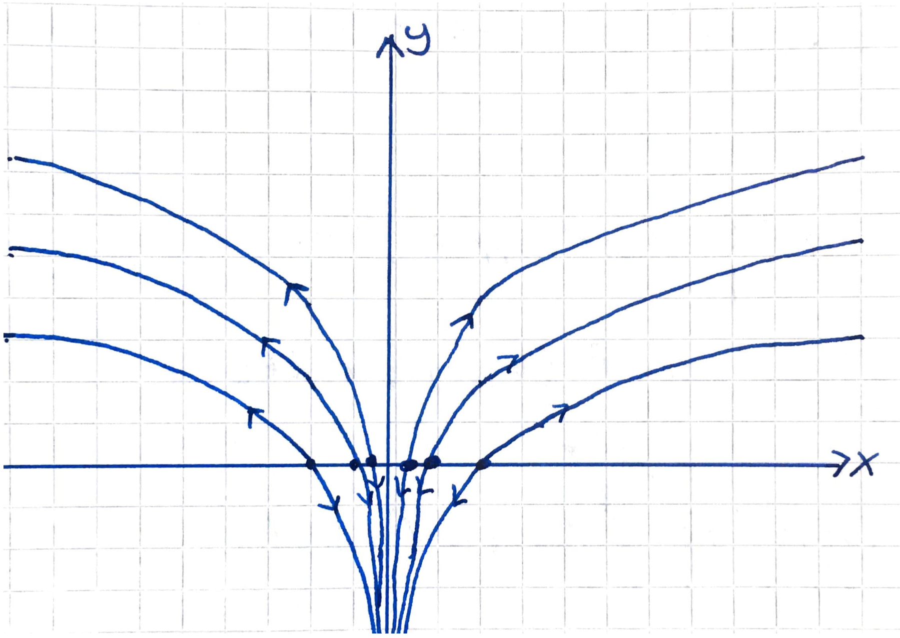

Figure 2: Hand-drawn stream lines for velocity field in (3)

The contour lines are defined when $y = \ln|x| + C$ is constant, meaning $z = \ln|x| - y$. All stagnation points will lie along the x-axis.

```
import numpy as np
import matplotlib.pyplot as plt

t = np.linspace(-2, 2, 1001)
x, y = np.meshgrid(t, t)
z = np.log(np.abs(x)) - y
plt.contour(x, y, z)
plt.show()
```

See also Figure 3 on next page.

c.

According to 4.6 in Gjevik & Fagerland (2021), there is a scalar potential $\psi = \psi(x, y)$ if

$$\nabla \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{4}$$

We can calculate the divergence of $\boldsymbol{v}$ in (3).

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = y + 1 \neq 0$$

Hence, there is no ψ for the given velocity field.

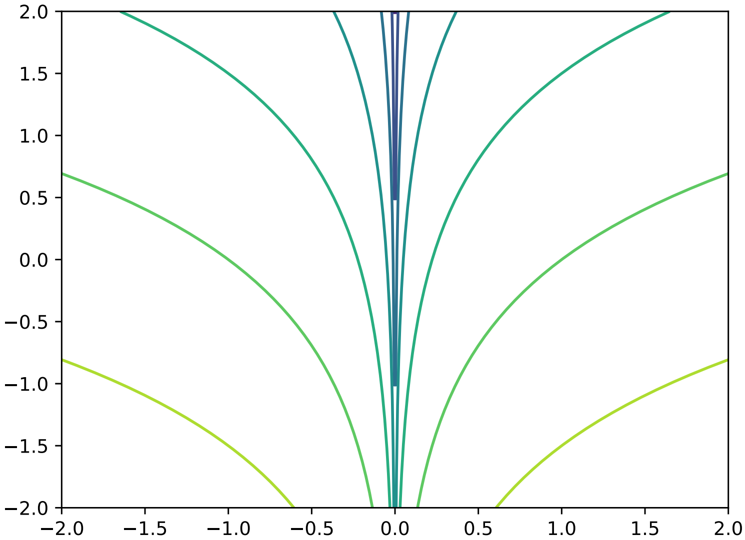

Figure 3: Stream lines drawn in python for velocity field in (3)

3 Another two dimensional stream field

A velocity field in the xy-plane is given by $\boldsymbol{v} = v_x \boldsymbol{i} + v_y \boldsymbol{j}$ where

$$v_x = \cos(x)\sin(y), \quad v_y = -\sin(x)\cos(y). \tag{5}$$

a.

Divergence:

$$\nabla \cdot \boldsymbol{v} = -\sin(s)\sin(y) + \sin(x)\sin(y) = 0$$

Curl:

$$\nabla \times \boldsymbol{v} = \left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y}\right)\boldsymbol{k} = (-2\cos(x)\cos(y))\boldsymbol{k}$$

b.

The given task was quite vague, so I simply plotted the field in python.

```
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(-2, 2, 21)
y = x.copy()
vx, vy = np.meshgrid(np.cos(x)*np.sin(y), np.sin(x)*np.cos(y))

plt.quiver(x, y, vx, -vy)
plt.axis('equal')
plt.show()
```

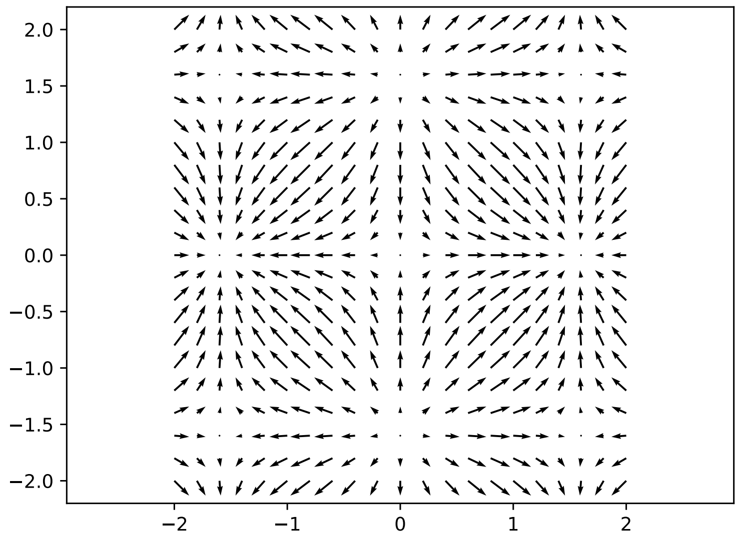

Figure 4: Plot of $\boldsymbol{v}$ for $x, y \in [-2, 2]$

c.

In this section, we will use the formula for a line integral of a vector field over a curve

$$\int_C \boldsymbol{F} \cdot \boldsymbol{dr} \tag{6}$$

We separate the curve into four separate pieces. Consider the paramererisation $\boldsymbol{r}(t) = (t, -\pi/2)$

for $t \in [-\pi/2, \pi/2]$. Differentiating this we get $\boldsymbol{dr} = (1, 0)$. Hence, our integral is

$$\begin{aligned}
&\int_{-\pi/2}^{\pi/2} (v_x, v_y) \cdot (1, 0) dt \\
=&\int_{-\pi/2}^{\pi/2} \cos(t) \sin(-\pi/2) dt \\
=&\,[-sin(t)]_{-\pi/2}^{\pi/2} \\
=&-2
\end{aligned}$$

Since the divergence of $\boldsymbol{v}$ is zero, it is a conservative field. This results in the circulation around the square defined as $-\pi/2 \leq x, y \leq \pi/2$ to be $-2 * 4 = -8$.

d.

We have already concluded that there is a scalar potential ψ since $\boldsymbol{v}$ is conservative. To check if

$$\psi = \cos(x) \cos(y) \tag{7}$$

is a valid scalar potential for $\boldsymbol{v}$, we turn to 4.6 in Gjevik & Fagerland (2021) once again. We have that

$$v_x = -\frac{\partial \psi}{\partial y}, \quad v_y = \frac{\partial \psi}{\partial x} \tag{8}$$

if $\psi(x, y)$ is a scalar potential for $\boldsymbol{v} = (v_x, v_y)$. We test for our given ψ and $\boldsymbol{v}$:

$$\begin{aligned}
\frac{\partial \psi}{\partial y} &= -\cos(x) \sin(y) = -v_x \\
\frac{\partial \psi}{\partial x} &= \sin(x) \cos(y) = v_y
\end{aligned}$$

which shows that (7) is a scalar potential for (5).

e.

To find the Taylor approximation of $\psi(0, 0)$ we use the formula stated in chapter 2.2 in Gjevik & Fagerland (2021).

$$\begin{aligned}
f(x, y) \cong f(x_0, y_0) &+ \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) \\
&+ \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x_0, y_0)(x - x_0)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial y^2}(x_0, y_0)(y - y_0)^2 \\
&+ \frac{\partial^2 f}{\partial x \partial y}(x - x_0)(y - y_0)
\end{aligned}$$

To save on some typing, I will only include the parts that do not become zero. We use $(x_0, y_0) = (0, 0)$.

$$
\begin{aligned}
& & \psi(0,0) &= 1 \\
\frac{\partial^2 \psi}{\partial x^2} &= -\cos(x)\cos(y) \implies & \frac{\partial^2 \psi}{\partial x^2}(0,0) &= -1 \\
\frac{\partial^2 \psi}{\partial y^2} &= -\cos(x)\cos(y) \implies & \frac{\partial^2 \psi}{\partial y^2}(0,0) &= -1
\end{aligned}
$$

If we insert these values into our formula we get

$$
\psi(x, y) \cong 1 - \frac{x^2}{2} - \frac{y^2}{2} \tag{9}
$$

4 Stream lines and velocity field in Python

a.

`strlin.py`:

```
import numpy as np
import matplotlib.pyplot as plt
from streamfun import streamfun

for n in (5, 30):
    x, y, psi = streamfun(n)
    plt.contour(x, y, psi)
    plt.axis('equal')
    plt.show()
```

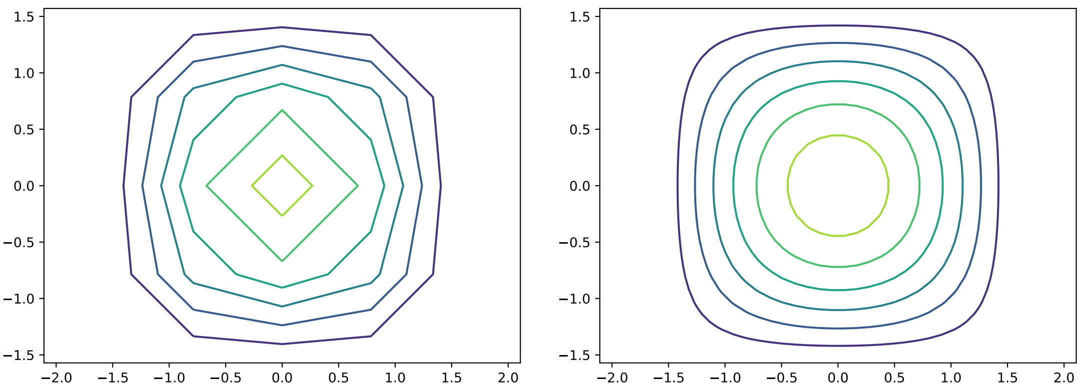

Figure 5: Showing streamlines for $n = 5, 30$.

We observe that if we use a higher n, such as 30, the shape is more close to the circles described by (9).

b.

`velfield.py`:

```
import numpy as np

def velfield(n):
    t = np.linspace(-np.pi/2, np.pi/2, n)
    x, y = np.meshgrid(t, t)
    u, v = np.cos(x)*np.sin(y), np.sin(x)*np.cos(y)

    return x, y, u, -v
```

`vec.py`:

```
import numpy as np
import matplotlib.pyplot as plt
from velfield import velfield

x, y, u, v = velfield(21)
plt.quiver(x, y, u, v)
plt.axis('equal')
plt.show()
```

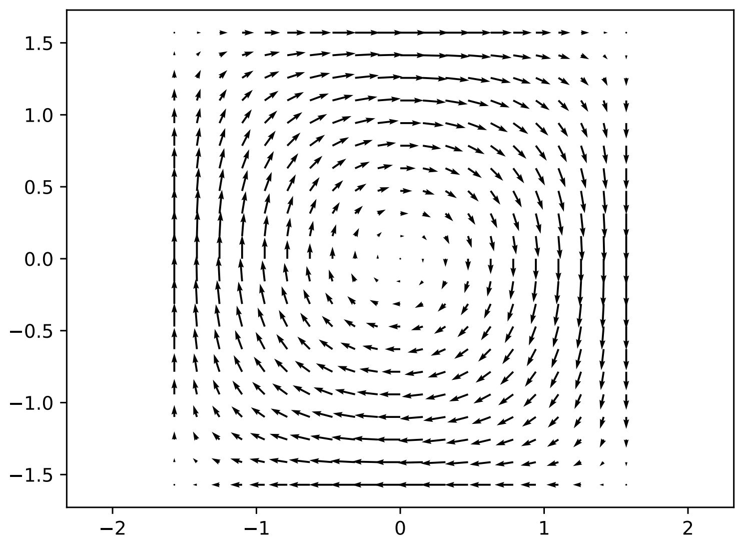

Figure 6: Plot of velocity field with $n = 21$.