besvarelse

May 1, 2022

1 STK1100 - Mandatory assignment 2

1.1 William Dugan

1.1.1 Task 1a

$$f(x,y) = \begin{cases} k(x+2y) & 0 \leq x, y \leq 1, x+y \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$

To find k, we need to have $\iint_{\mathbb{R}} f(x,y)dxdy = 1$.

$$\begin{aligned} \iint_{\mathbb{R}} f(x,y)dxdy &= k \int_0^1 \int_0^{1-x} (x+2y)dydx \\ &= k \int_0^1 (1-x)dx \\ &= \frac{k}{2} \\ \implies k &= 2 \end{aligned}$$

1.1.2 Task 1b

$$\begin{aligned} f_Y(y) &= \int_{\mathbb{R}} f(x,y)dx \\ &= \int_0^{1-y} 2(x+2y)dx \\ &= [x^2 + 4xy]_{x=0}^{x=1-y} \\ &= 1 - 2y + y^2 + 4y - 4y^2 \\ &= 1 + 2y - 3y^2, \qquad 0 \leq y \leq 1 \end{aligned}$$

1.1.3 Task 1c

$$\begin{aligned} f_{X|Y}(y) &= \frac{f(x,y)}{f_Y(y)} \\ &= \frac{2(x+2y)}{1+2y-3y^2} \\ &= \frac{2x+4y}{1+2y-3y^2} \end{aligned}$$

1.1.4 Task 1d

X and Y are independent if $ f_X(x) f_Y(y) = f(x, y)$. First we need to find $f_X(x)$:

$$f_X(x) = \int_{\mathbb{R}} f(x,y)dy = \int_0^{1-x} (2x+4y)dy = 2-2x$$

We can now calculate $f_X(x)f_Y(y)$:

$$\begin{aligned} f_X(x)f_Y(y) &= (2-2x)(1+2y-3y^2) \\ &= 2+4y-6y^2-2x-4xy+6xy^2 \\ &\neq f(x,y) \end{aligned}$$

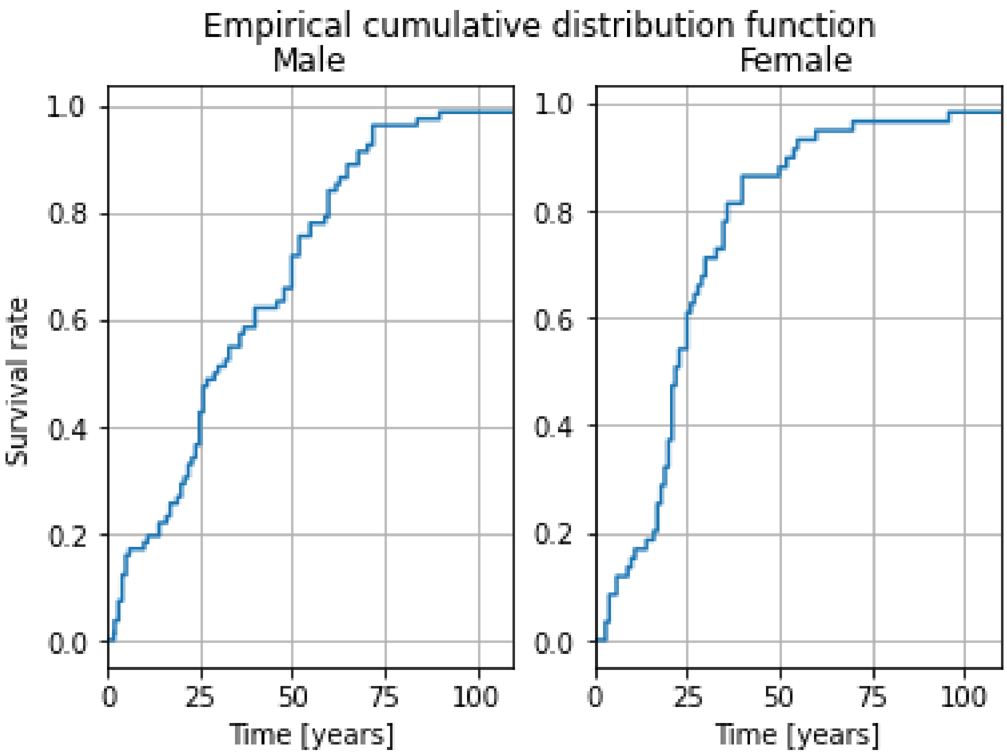
which means that X and Y are dependent.

1.1.5 Task 2a

[ ]:
```python
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
import scipy.stats, scipy.interpolate
```

[ ]:
```python
time = pd.read_csv(
    'https://www.uio.no/studier/emner/matnat/math/STK1100/data/egypt_data.txt',
    header=None
)

t_m = np.asarray(time[:82]).flatten()
t_f = np.asarray(time[82:]).flatten()

def empirical_cdf(x):
    return scipy.interpolate.interp1d(
                np.sort(x),
                np.arange(len(x))/float(len(x)),
                kind = 'zero',
                fill_value = 'extrapolate'
            )

fig, ax = plt.subplots(ncols=2, sharex=True)
fig.suptitle('Empirical cumulative distribution function')
z = np.linspace(0, 110, 1000)

F_m = empirical_cdf(t_m)
ax[0].step(z, F_m(z))
ax[0].set(
    xlabel = 'Time [years]',
    ylabel = 'Survival rate',
    title = 'Male',
    xlim = (0, 110)
```

```
)
ax[0].grid()

F_f = empirical_cdf(t_f)
ax[1].step(z, F_f(z))
ax[1].set(
    xlabel = 'Time [years]',
    title = 'Female',
    xlim = (0, 110)
)
ax[1].grid()

plt.show()
```



1.1.6 Task 2b

We can find α and β by using $\mathbb{E}(T) = \alpha\beta$ and $\mathrm{Var}(T) = \alpha\beta^2$. This is done in `estimate_moments_gamma(t)`.

[ ]:
```
def estimate_moments_gamma(t):
    t_mean = np.mean(t)
    s_sq = np.var(t)
```

```
    beta = s_sq / t_mean
    alpha = t_mean / beta
    Ex = alpha * beta

    return alpha, beta, Ex

alpha_m, beta_m, Ex_m = estimate_moments_gamma(t_m)
alpha_f, beta_f, Ex_f = estimate_moments_gamma(t_f)

print(f'Gender: Male')
print(f'Alpha = {alpha_m:.3f}')
print(f'Beta = {beta_m:.3f}')
print(f'E(X) = {Ex_m:.2f} years \n')

print(f'Gender: Male')
print(f'Alpha = {alpha_f:.3f}')
print(f'Beta = {beta_f:.3f}')
print(f'E(X) = {Ex_f:.2f} years')
```

```
Gender: Male
Alpha = 2.211
Beta = 15.430
E(X) = 34.12 years

Gender: Male
Alpha = 2.240
Beta = 11.572
E(X) = 25.92 years
```

We observe that the life expectancy is higher for males compared to females.

1.1.7 Task 2c

The transformation is given by $g(y) = f(v(y))|v'(y)|$. We set $Y = e^X$ where $X \sim N(\mu, \sigma)$, which yields $x = \ln y$. This gives $|v'(y)| = v'(y) = \frac{1}{y}$. Using the PDF of a normally distributed random variable, we find

$$g(y) = \frac{1}{y\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ln y - \mu)^2}{2\sigma^2}\right]$$

for $y > 0$.

1.1.8 Task 2d

Rearranging the given relationships we get

$$\mu = \ln \bar{t} - \frac{\sigma^2}{2}, \quad \sigma^2 = \ln\left[1 + \frac{S^2}{\bar{t}^2}\right]$$

[ ]:
```python
def estimate_moments_lognormal(t):
    t_mean = np.mean(t)
    s_sq = np.var(t)

    sigma_sq = np.log(s_sq/(t_mean**2) + 1)
    mu = np.log(t_mean) - sigma_sq/2
    Ex = np.exp(mu + sigma_sq/2)

    return mu, sigma_sq, Ex

mu_m, sigma_sq_m, Ex_m = estimate_moments_lognormal(t_m)
mu_f, sigma_sq_f, Ex_f = estimate_moments_lognormal(t_f)

print(f'Gender: Male')
print(f'Mu = {mu_m:.3f}')
print(f'Sigma^2 = {sigma_sq_m:.3f}')
print(f'E(T) = {Ex_m:.2f} years \n')

print(f'Gender: Female')
print(f'Mu = {mu_f:.3f}')
print(f'Sigma^2 = {sigma_sq_f:.3f}')
print(f'E(T) = {Ex_f:.2f} years')
```

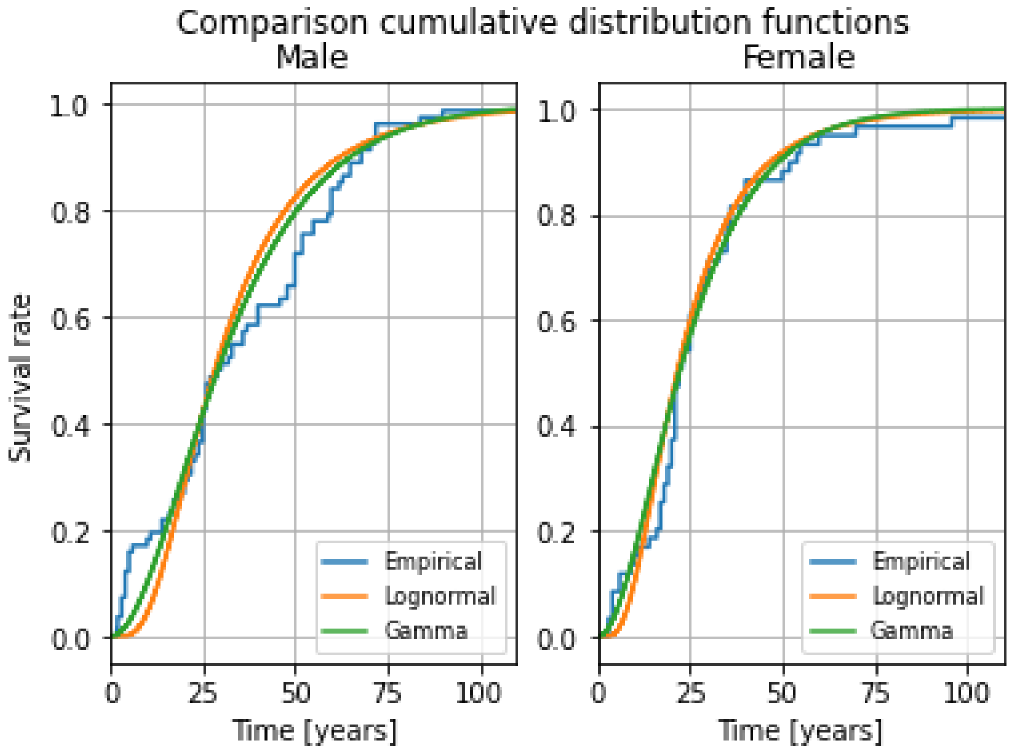
```
Gender: Male
Mu = 3.343
Sigma^2 = 0.373
E(T) = 34.12 years

Gender: Female
Mu = 3.071
Sigma^2 = 0.369
E(T) = 25.92 years
```

We observe that the life expectancy is longer for males compared to females, and the estimation yields the same result as in (b).

1.1.9 Task 2e

[ ]:
```python
fig, ax = plt.subplots(ncols=2, sharex=True)
fig.suptitle('Comparison cumulative distribution functions')
z = np.linspace(0, 110, 1000)

for i, t in enumerate((t_m, t_f)):
    F = empirical_cdf(t)
    ax[i].step(z, F(z), label='Empirical')

    mu, sigma_sq, Ex = estimate_moments_lognormal(t)
    ax[i].step(
```

```python
        z,
        scipy.stats.lognorm.cdf(z, s=np.sqrt(sigma_sq),
        scale=np.exp(mu)),
        label='Lognormal'
    )

    alpha, beta, Ex = estimate_moments_gamma(t)
    ax[i].step(
        z,
        scipy.stats.gamma.cdf(z, a=alpha, scale=beta),
        label='Gamma'
    )

    ax[i].grid('True')
    ax[i].legend(loc='lower right', prop={'size': 8})
    ax[i].set(
        xlabel='Time [years]',
        xlim=(0, 110)
    )

ax[0].set(
    ylabel='Survival rate',
    title='Male',
)

ax[1].set(title='Female')

plt.show()
```



Both the log-normal and the gamma distribution seem to fit the data. There are some minor differences, and it looks as the gamma distribution is most appropriate for males. For females, the differences between log-normal and gamma are even smaller.

1.1.10 Task 3a

Let $U \sim \text{uniform}(0, 1)$. The inverse of the cumulative distribution function for a uniformly distributed variable is $F^{-1}(p) = a + p(b - a). Since $a = 0,\ b = 1,\ F^{-1}(p) = p$. It follows that the cdf of $X = F^{-1}(U)$ is $F(x)$.

1.1.11 Task 3b

$$f_X(x) = \begin{cases} \frac{\alpha}{\lambda}(1 + \frac{x}{\lambda})^{-(\alpha+1)}) & x > 0 \\ 0 & \text{Otherwise} \end{cases}$$

where $\alpha,\ \lambda > 0$. The CDF of X is

$$\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} f_X(x)dx = \int_{0}^{x} f_X(x)dx \\
&= \alpha \int_0^x u^{-\alpha-1}du, \qquad u = 1 + \frac{x}{\lambda} \\
&= \left[\left(1 + \frac{x^*}{\lambda} \right)^{-\alpha} \right]_{x^*=x}^{x^*=0} \\
&= 1 - \left(1 + \frac{x}{\lambda} \right)^{-\alpha}, \qquad x > 0
\end{aligned}$$

To find the median we set $F_X(x) = 1/2$. Solving for $\bar{X}$:

$$\begin{aligned}
1 - \left(1 + \frac{\bar{X}}{\lambda} \right)^{-\alpha} &= \frac{1}{2} \\
\frac{\bar{X}}{\lambda} &= \sqrt[\alpha]{2} - 1 \\
\bar{X} &= \lambda(\sqrt[\alpha]{2} - 1)
\end{aligned}$$

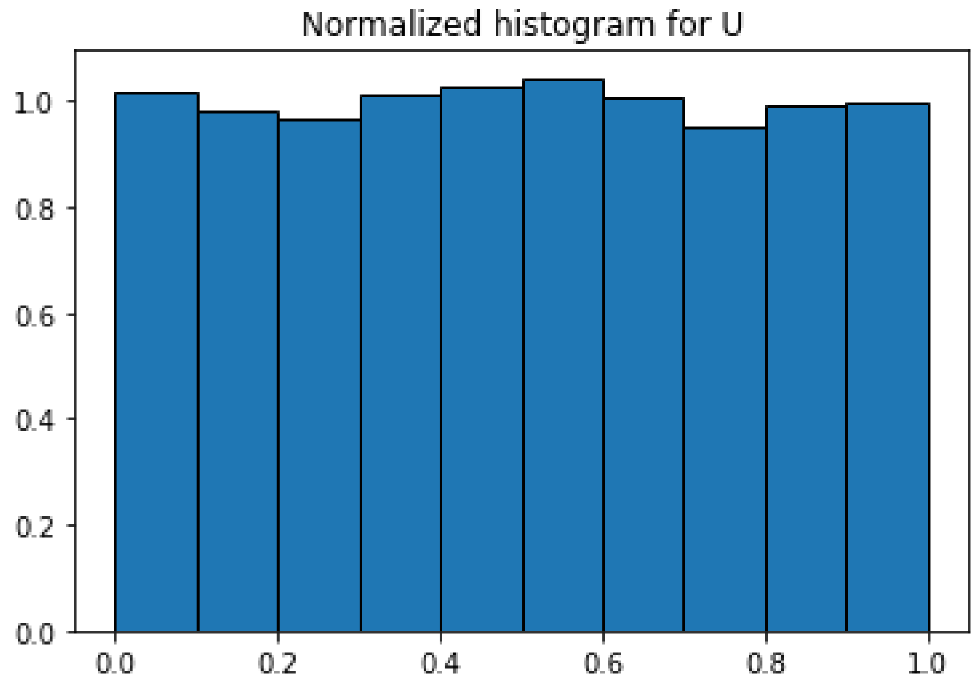
1.1.12 Task 3c

We can generate n Lomax-distributed observations with `scipy.stats.lomax` using a shape parameter `c` $= \alpha$ and scale parameter `scale` $= \lambda$.

1.1.13 Task 3d

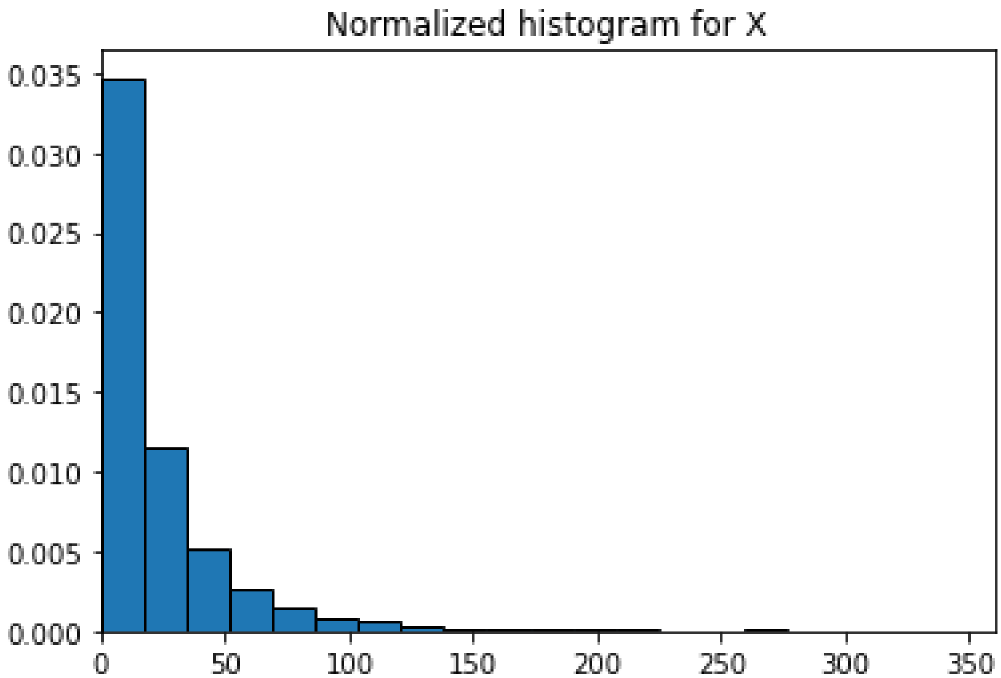
[ ]:
```
n = 10_000
lmbda = 48
alpha = 3

u = np.random.uniform(0, 1, n)
plt.hist(u, density=True, edgecolor='black')
plt.title('Normalized histogram for U')

x = scipy.stats.lomax(c=alpha, scale=lmbda)
X = x.rvs(size=n)
print(f'Analytical median = {x.median():.3f}')
print(f'Calculated median = {np.median(X):.3f}')

plt.show()
```

```
Analytical median = 12.476
Calculated median = 12.653
```



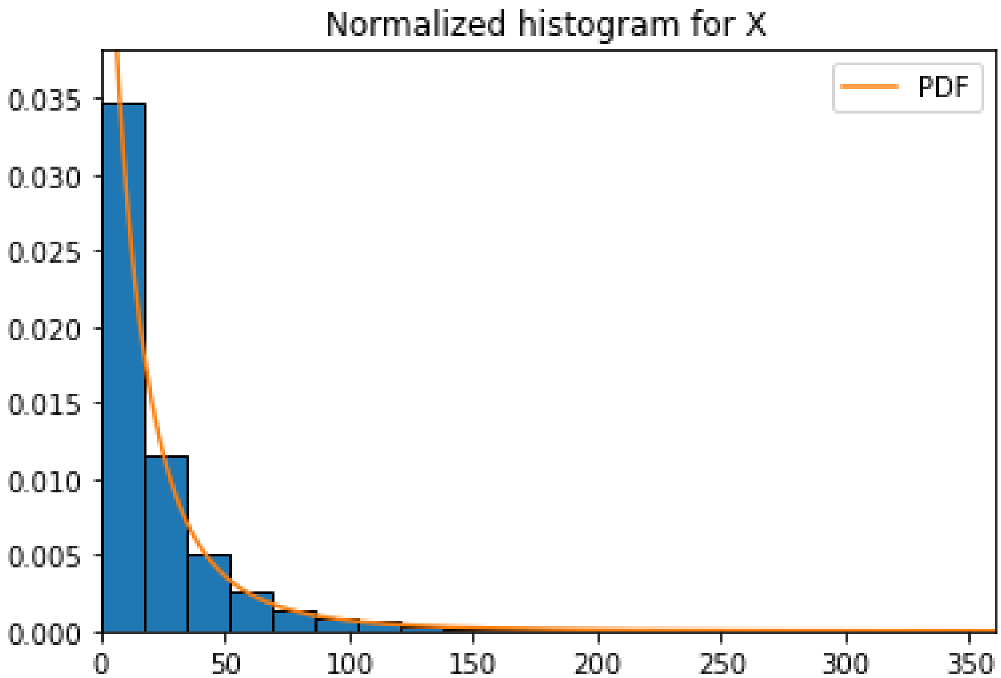
1.1.14 Task 3e

[ ]:
```
plt.hist(X, bins=50, density=True, edgecolor='black')
plt.title('Normalized histogram for X')
plt.xlim(0, 360)
plt.show()
```



1.1.15 Task 3f

[ ]:
```
y, *_ = plt.hist(X, bins=50, density=True, edgecolor='black')

t = np.linspace(0, 360, n)
plt.plot(t, x.pdf(t), label='PDF')

plt.title('Normalized histogram for X')
plt.xlim(0, 360)
plt.ylim(None, 1.1*y.max())
plt.legend()
plt.show()
```

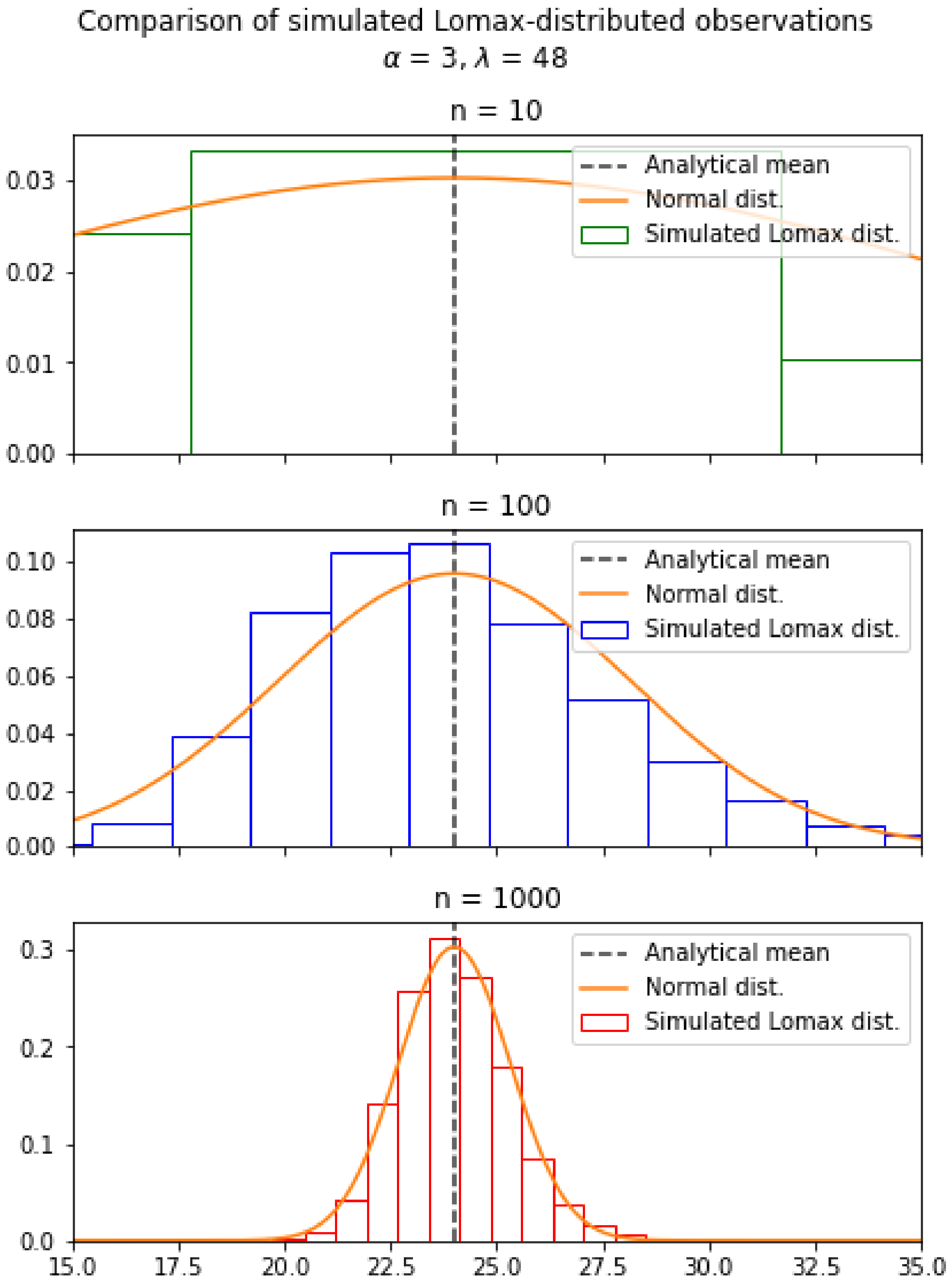


We observe that our method of generating observations from the Lomax-distribution fits with the probability density.

1.1.16 Task 3g

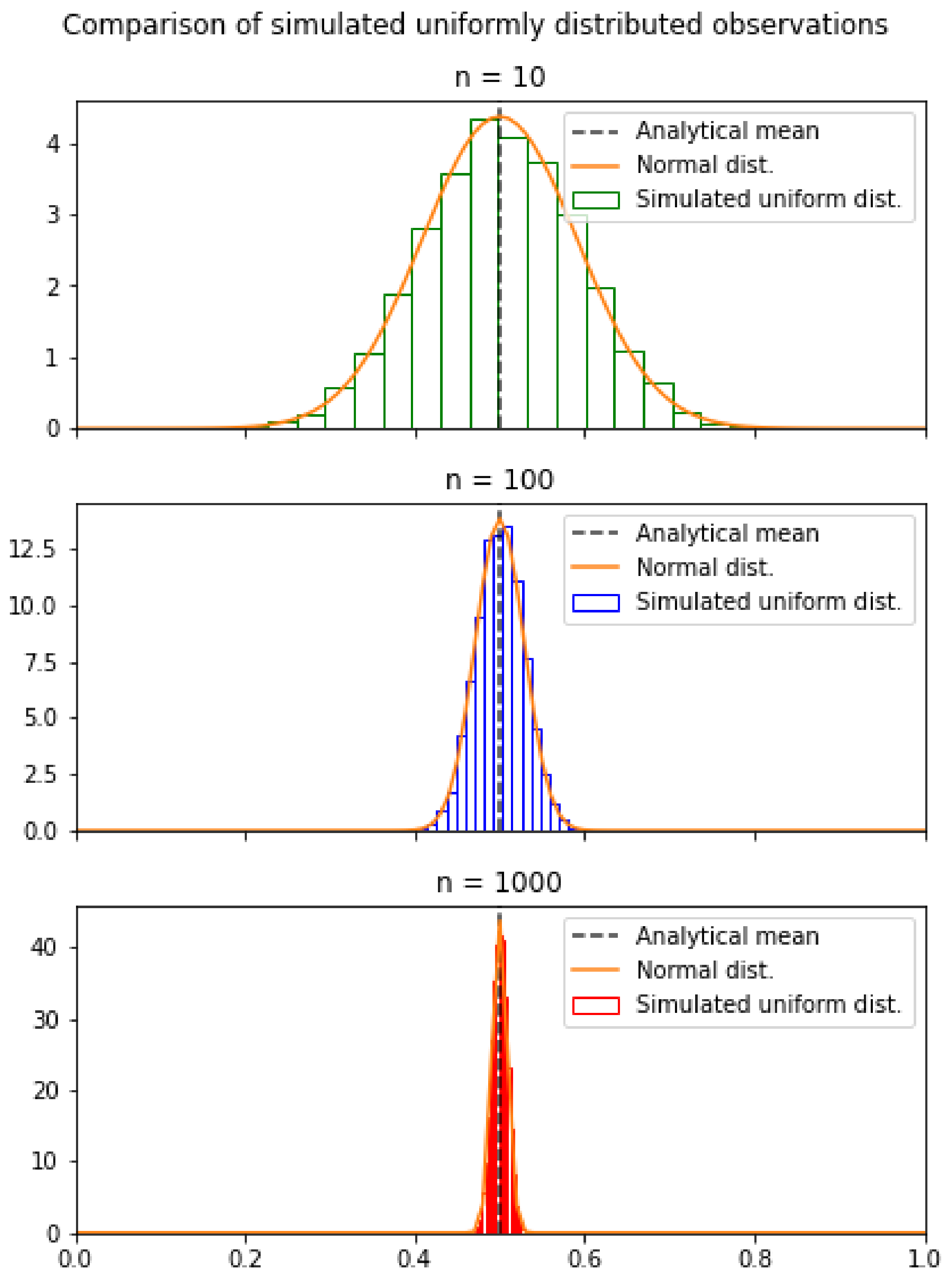
[ ]:
```
runs = 10_000
averages = np.zeros((3, runs))
x = np.linspace(0, 100, 10_000)
n_list = (10, 100, 1000)
colors = ('green', 'blue', 'red')

ex, var = scipy.stats.lomax.stats(c=alpha, scale=lmbda, moments='mv')

for i, n in enumerate(n_list):
    for j in range(runs):
        X = scipy.stats.lomax.rvs(c=alpha, scale=lmbda, size=n)
        averages[i, j] = np.mean(X)

fig, ax = plt.subplots(nrows=3, figsize=(6, 8), sharex=True, tight_layout=True)

fig.suptitle(
    ''.join(
        str(_) for _ in (
            r'Comparison of simulated Lomax-distributed observations',
            '\n',
```

```python
            r'$\alpha$ = ',
            alpha,
            r', $\lambda$ = ',
            lmbda
        )
    )
)

for i, (n, color) in enumerate(zip(n_list, colors)):
    ax[i].hist(
        averages[i],
        bins=25,
        density=True,
        fc='none',
        ec=color,
        label='Simulated Lomax dist.'
    )
    ax[i].set(
        xlim = (15, 35),
        title = f'n = {n}'
    )
    ax[i].axvline(
        ex,
        ls='--',
        color='k',
        label='Analytical mean',
        alpha=0.8
    )
    ax[i].plot(
        x,
        scipy.stats.norm.pdf(x, loc=ex, scale=np.sqrt(var/n)),
        label='Normal dist.'
    )
    ax[i].legend(loc='upper right', prop={'size': 10})

plt.show()
```




```
[ ]: averages = np.zeros((3, runs))
```

```python
ex, var = scipy.stats.uniform.stats(moments='mv')

for i, n in enumerate(n_list):
    for j in range(runs):
        X = scipy.stats.uniform.rvs(size=n)
        averages[i, j] = np.mean(X)

fig, ax = plt.subplots(nrows=3, figsize=(6, 8), sharex=True, tight_layout=True)

fig.suptitle('Comparison of simulated uniformly distributed observations')

for i, (n, color) in enumerate(zip(n_list, colors)):
    ax[i].hist(
        averages[i],
        bins=20,
        density=True,
        fc='none',
        ec=color,
        label='Simulated uniform dist.'
    )
    ax[i].set(
        xlim = (0., 1),
        title = f'n = {n}'
    )
    ax[i].axvline(
        ex,
        ls='--',
        color='k',
        label='Analytical mean',
        alpha=0.8
    )
    ax[i].plot(
        x,
        scipy.stats.norm.pdf(x, loc=ex, scale=np.sqrt(var/n)),
        label='Normal dist.'
    )
    ax[i].legend(loc='upper right', prop={'size': 10})

plt.show()
```



We observe that both simulations tends toward a normal distribution, even though the original variables are Lomax and Uniformly distributed. This is due to the central limit theorem, which says that when independent variables are summed up, their normalized sum tends toward a normal distribution.