Fra Kreyszig (10th), avsnitt 11.4

$\boxed{2}$ Vi vet at koeffisientene som minimerer $\int_{-\pi}^{\pi}(f(x)-F(x))^2\ \mathrm{d}x$ er Fourier-koeffisientene til f. Ettersom f er odde, finner vi at $A_n = 0$ og

$$\begin{aligned} B_n &= \frac{2}{\pi}\int_0^\pi f(x)\sin nx\ \mathrm{d}x \\ &= \frac{2}{\pi}\int_0^\pi x\sin nx\ \mathrm{d}x \\ &= \frac{2}{\pi}\left(-\left.\right|_0^\pi x\frac{\cos nx}{n} + \frac{1}{n}\int_0^\pi \cos nx\ \mathrm{d}x\right) \\ &= \frac{2}{\pi}\left(-\pi\frac{(-1)^n}{n} + 0\right) \\ &= (-1)^{n+1}\frac{2}{n} \end{aligned}$$

Dvs.

$$x = f(x) \approx F_N(x) = 2\sum_{n=1}^{N}\frac{(-1)^{n+1}}{n}\sin nx$$

på $(-\pi,\pi)$. Feilen er gitt ved

$$\begin{aligned} E_N &= \int_{-\pi}^{\pi}(f(x))^2\ \mathrm{d}x - \pi\left(2a_0^2 + \sum_{n=1}^{N}\left(a_n^2+b_n^2\right)\right) \\ &= 2\int_0^\pi x^2\ \mathrm{d}x - \pi\sum_{n=1}^{N}b_n^2 \\ &= \frac{2}{3}\pi^3 - 4\pi\sum_{n=1}^{N}\frac{1}{n^2} = 4\pi\left(\frac{\pi^2}{6} - \sum_{n=1}^{N}\frac{1}{n^2}\right). \end{aligned}$$

Dette gir

$$\begin{aligned} E_1 &= 4\pi\left(\frac{\pi^2}{6} - 1\right) \\ &\approx 8.10, \end{aligned}$$

$$\begin{aligned} &\approx 8.10, \\ E_2 &= 4\pi\left(\frac{\pi^2}{6} - 1 - 1/4\right) \\ &\approx 4.96, \\ E_3 &= 4\pi\left(\frac{\pi^2}{6} - 1 - 1/4 - 1/9\right) \\ &\approx 3.57, \end{aligned}$$

$$
\begin{aligned}
E_4 &= 4\pi\left(\frac{\pi^2}{6} - 1 - 1/4 - 1/9 - 1/16\right) \\
&\approx 2.78, \\
E_5 &= 4\pi\left(\frac{\pi^2}{6} - 1 - 1/4 - 1/9 - 1/16 - 1/25\right) \\
&\approx 2.28.
\end{aligned}
$$

$\boxed{3}$ Vi vet at koeffisientene som minimerer $\int_{-\pi}^{\pi}(f(x) - F(x))^2 \,\mathrm{d}x$ er Fourier-koeffisientene til f. Ettersom f er jevn, finner vi at $B_n = 0$,

$$
A_0 = \frac{1}{\pi}\int_0^{\pi} |x| \mathrm{d}x = \frac{1}{2\pi}\pi^2 = \frac{\pi}{2}
$$

og

$$
\begin{aligned}
A_n &= \frac{2}{\pi}\int_0^{\pi} x\cos nx \,\mathrm{d}x \\
&= \frac{2}{\pi}\left(|_0^{\pi}\, x\frac{\sin nx}{n} - \frac{1}{n}\int_0^{\pi} \sin nx \,\mathrm{d}x\right) \\
&= \frac{2}{\pi}\left(0 + \frac{1}{n}\bigg|_0^{\pi} \frac{\cos nx}{n}\right) \\
&= \frac{2}{n^2\pi}\left((-1)^n - 1\right) \\
&= \begin{cases} -\frac{4}{n^2\pi}, & n \text{ odde} \\ 0, & n \text{ jevn.} \end{cases}
\end{aligned}
$$

Dvs.

$$
|x| = f(x) \approx F_N(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{(N+1)/2} \frac{\cos(2n-1)x}{(2n-1)^2}
$$

for N odde og $F_N = F_{N-1}$ for N jevn. Feilen er gitt ved

$$
\begin{aligned}
E_N &= \int_{-\pi}^{\pi} (f(x))^2 \,\mathrm{d}x - \pi\left(2a_0^2 + \sum_{n=1}^{N}\left(a_n^2 + b_n^2\right)\right) \\
&= 2\int_0^{\pi} x^2 \,\mathrm{d}x - \pi\left(\frac{\pi^2}{2} + \sum_{n=1}^{(N+1)/2} a_{2n-1}^2\right), \quad N \text{ odd} \\
&= \frac{\pi^3}{6} - \frac{16}{\pi}\sum_{n=1}^{(N+1)/2}\frac{1}{(2n-1)^4} \\
&= \frac{16}{\pi}\left(\frac{\pi^4}{96} - \sum_{n=1}^{(N+1)/2}\frac{1}{(2n-1)^4}\right)
\end{aligned}
$$

og $E_N = E_{N-1}$ for N jevn. Dette gir

$$\begin{aligned}
E_1 &= \frac{16}{\pi}\left(\frac{\pi^4}{96} - 1\right) \\
&\approx 0.0748 \\
E_2 &= E_1 \\
E_3 &= \frac{16}{\pi}\left(\frac{\pi^4}{96} - 1 - 1/3^4\right) \\
&\approx 0.01187 \\
E_4 &= E_3 \\
E_5 &= \frac{16}{\pi}\left(\frac{\pi^4}{96} - 1 - 1/3^4 - 1/5^4\right) \\
&\approx 0.00373.
\end{aligned}$$

$\boxed{13}$ Skal ved hjelp av oppgave 11.1.17 vise at

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + ... = \frac{\pi^4}{96}$$

Oppgave 11.1.17 sier at Fourier-rekka til

$$f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$$

er

$$F(x) = \frac{\pi}{2} + \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos(3x) + \frac{1}{25}\cos(5x) + ...\right)$$

Ser at koeffisientene må være

$$\begin{aligned}
a_0 &= \frac{\pi}{2} \\
a_n &= \frac{2}{\pi n^2}(1 - (-1)^n) \\
b_n &= 0
\end{aligned}$$

Trikset her er å sette disse uttrykkene inn i Parsevals identitet:

$$\begin{aligned}
2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,\mathrm{d}x \\
\frac{\pi^2}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{\pi n^2}(1 - (-1)^n)\right)^2 &= \frac{2}{\pi}\int_{0}^{\pi}(\pi - x)^2\,\mathrm{d}x \\
\frac{\pi^2}{2} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1 - (-1)^n}{n^4} &= \frac{2}{\pi}\left(\pi^3 - \pi^3 + \frac{\pi^3}{3}\right) \\
\sum_{n=1}^{\infty}\frac{1 - (-1)^n}{n^4} &= \frac{\pi^4}{48} \\
2\left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + ...\right) &= \frac{\pi^4}{48}
\end{aligned}$$

Som gir det endelige svaret:

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + ... = \frac{\pi^4}{96}$$

9-utg9 Fra oppgave 11.4.2 er

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

der $b_n = (-1)^{n+1}\frac{2}{n}$. Vi ønsker å finne de komplekse koeffisientene c_n slik at

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Skriv Vi har at

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} = c_0 + \sum_{n=1}^{\infty} \left(c_n e^{inx} + c_{-n} e^{-inx}\right)$$

$$e^{inx} = \cos nx + i \sin nx$$
$$e^{-inx} = \cos nx - i \sin nx$$

så dermed må $c_0 = 0$ og

$$b_n \sin nx = (c_n + c_{-n}) \cos nx + i\,(c_n - c_{-n}) \sin nx.$$

Dvs. $c_n = -c_{-n}$ og

$$b_n = i\,(c_n - c_{-n}) = 2ic_n.$$

Dette gir $c_n = \frac{b_n}{2i} = -i\frac{b_n}{2} = i\frac{(-1)^n}{n}$ og Fourier-rekken

$$f(x) = i \sum_{n=-\infty, n\neq 0}^{\infty} \frac{(-1)^n}{n} e^{inx}.$$

Fra Kreyszig (10th), avsnitt 11.7

1 Definér funksjonen f på $\mathbb{R}$ ved $f(x) = \pi e^{-x}$ for $x \geq 0$ og $f(x) = 0$ for $x < 0$. Da er f lik sitt Fourier-integral for alle x bortsett fra diskontinuiteten i $x = 0$. Dvs

$$f(x) = \int_0^\infty (A(w) \cos wx + B(w) \sin wx) \mathrm{d}w, \quad x \neq 0.$$

Nå er

$$\begin{aligned}
A(w) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos wt \ \mathrm{d}t \\
&= \int_0^\infty e^{-t} \cos wt \ \mathrm{d}t \\
&= |_{s=1} \int_0^\infty e^{-st} \cos wt \ \mathrm{d}t \\
&= \mathcal{L}\{\cos wt\}(1) \\
&= |_{s=1} \frac{s}{s^2 + w^2} \\
&= \frac{1}{1 + w^2}
\end{aligned}$$

og

$$\begin{aligned} B(w) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin wt \ \mathrm{d}t \\ &= \int_0^{\infty} e^{-t}\sin wt \ \mathrm{d}t \\ &= |_{s=1} \int_0^{\infty} e^{-st}\sin wt \ \mathrm{d}t \\ &= \mathcal{L}\{\sin wt\}(1) \\ &= |_{s=1} \frac{w}{s^2+w^2} \\ &= \frac{w}{1+w^2} \end{aligned}$$

I $x = 0$ vil verdien av integralet være gjennomsnittet av grenseverdiene fra høyre og venstre. Dvs.

$$\int_0^{\infty} A(w)\mathrm{d}w = \frac{\lim_{x\to 0-f} f(x) + \lim_{x\to 0+} f(x)}{2} = \frac{0+\pi}{2}.$$

Dermed er

$$\begin{aligned} \int_0^{\infty} \frac{\cos xw + w\sin xw}{1+w^2} \ \mathrm{d}w &= \int_0^{\infty} (A(w)\cos wx + B(w)\sin wx)\mathrm{d}w \\ &= \begin{cases} f(x), & x \neq 0 \\ \pi/2, & x = 0 \end{cases} \\ &= \begin{cases} 0, & x < 0 \\ \pi/2, & x = 0 \\ \pi e^{-x}, & x > 0. \end{cases} \end{aligned}$$

Fra Kreyszig (10th), avsnitt 11.9

5

$$f(x) = \begin{cases} e^x & \text{for } -a < x < a \\ 0 & \text{ellers} \end{cases}$$

Fouriertransformasjon:

$$\begin{aligned} \hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-iwx} dx \\ \implies \hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^x e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{(1-iw)x} dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{1-iw} e^{(1-iw)x}\Big|_{-a}^{a} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{1-iw} (e^{(1-iw)a} - e^{-(1-iw)a}) \end{aligned}$$

7

$$
\begin{aligned}
\hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\ \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{a} x e^{-iwx}\ \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \left(- \left.\right|_0^a x \frac{e^{-iwx}}{iw} + \frac{1}{iw} \int_0^a e^{-iwx}\ \mathrm{d}x \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(-\frac{1}{iw} a e^{-iwa} - \frac{1}{(iw)^2} \bigg|_0^a e^{-iwx} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{i}{w} a e^{-iwa} + \frac{1}{w^2} \left(e^{-iwa} - 1 \right) \right) \\
&= \frac{(iaw+1)e^{-iwa} - 1}{\sqrt{2\pi} w^2}.
\end{aligned}
$$

9

$$
f(x) = \begin{cases} |x| & \text{for } -1 < x < 1 \\ 0 & \text{ellers} \end{cases}
$$

$$
\begin{aligned}
\hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \left(\int_{-1}^{0} -x e^{-iwx} dx + \int_0^1 x e^{-iwx} dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{iw} x e^{-iwx} \bigg|_{-1}^{0} - \int_{-1}^{0} \frac{1}{iw} e^{-iwx} dx - \frac{1}{iw} x e^{-iwx} \bigg|_0^1 + \int_0^1 \frac{1}{iw} e^{-iwx} dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{iw} e^{iw} + \frac{1}{(iw)^2} e^{-iwx} \bigg|_{-1}^{0} - \frac{1}{iw} e^{-iw} - \frac{1}{(iw)^2} e^{-iwx} \bigg|_0^1 \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{iw} e^{iw} - \frac{1}{w^2} + \frac{1}{w^2} e^{iw} - \frac{1}{iw} e^{-iw} + \frac{1}{w^2} e^{-iw} - \frac{1}{w^2} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(-\frac{2}{w^2} + \frac{2}{w^2} w \sin w + \frac{2}{w^2} \cos w \right) \\
&= \frac{\sqrt{2}}{\sqrt{\pi} w^2} \left(\cos w + w \sin w - 1 \right)
\end{aligned}
$$

Brukte at $e^{iw} = \cos w + i \sin w$