TMA4120 - Assignment 2

William Dugan

September 11, 2022

6.4.4

$$y'' + 16y = 4\delta(t - 3\pi),\ y(0) = 2,\ y'(0) = 0$$

$$\begin{aligned}
\mathcal{L}\text{ - transform:}\quad & \mathcal{L}\{y'' + 16y\} = \mathcal{L}\{4\delta(t - 3\pi)\} \\
& \to s^2Y - 2s + 16Y = 4e^{-3\pi s} \\
\text{Solve for } Y\text{:}\quad & (s^2 + 16)Y = 4e^{-3\pi s} + 2s \\
& \to Y = \tfrac{4e^{-3\pi s}}{s^2+16} + \tfrac{2s}{s^2+16} \\
\text{Inverse } \mathcal{L}\text{ -transform:}\quad & y = \mathcal{L}^{-1}\{Y\} = \sin(4(t - 3\pi))u(t - 3\pi) + 2\cos 4t
\end{aligned}$$

6.4.10

$$y'' + 5y' + 6y = \delta(t - \pi/2) + u(t - \pi)\cos t,\ y(0) = 0 = y'(0)$$

$$\begin{aligned}
\mathcal{L}\text{ - transform:}\quad & \mathcal{L}\{y'' + 5y' + 6y\} = \mathcal{L}\{\delta(t - \pi/2) + u(t - \pi)\cos t\} \\
& \to s^2Y + 5sY + 6Y = e^{-s\pi/2} + e^{-s\pi}\mathcal{L}\{\cos(t + \pi)\} \\
& \to (s^2 + 5s + 6)Y = e^{-s\pi/2} - \tfrac{e^{-s\pi}}{s^2+1} \\
\text{Solve for } Y\text{:}\quad & Y = \tfrac{e^{-s\pi/2}}{(s+2)(s+3)} - \tfrac{se^{-s\pi}}{(s+2)(s+3)(s^2+1)} \\
\text{Write as partial fraction:}\quad & Y = e^{-s\pi/2}\left[\tfrac{A}{s+2} + \tfrac{B}{s+3}\right] - e^{-s\pi}\left[\tfrac{Cs+D}{s^2+1} + \tfrac{E}{s+2} + \tfrac{F}{s+3}\right] \\
& \to A = 1,\ B = -1.\ C,\ D,\ E,\ F: \\
& \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 5 & 1 & 3 & 2 & 0 \\ 6 & 5 & 1 & 1 & 1 \\ 0 & 6 & 3 & 2 & 0 \end{bmatrix} \sim I_4,\ 1/10 \begin{bmatrix} 1 \\ 1 \\ -4 \\ 3 \end{bmatrix} \\
\text{Inverse } \mathcal{L}\text{ -transform:}\quad & y = \mathcal{L}^{-1}\{Y\} = u(t - \pi/2)[e^{-2(t-\pi/2)} - e^{-3(t-\pi/2)}] - \\
& \tfrac{u(t-\pi)}{10}[\cos(t-\pi) + \sin(t-\pi) - 4e^{-2(t-\pi)} + 3e{-}3(t - \pi)]
\end{aligned}$$

6.5.12

$$\begin{aligned} y(t) + \int_0^t y(\tau)\cosh(t-\tau)d\tau &= t + e^t \\ y + y * \cosh(t) &= t + e^t \end{aligned}$$

$$\begin{aligned} \mathcal{L}\{y\} + \mathcal{L}\{y\} \cdot \mathcal{L}\{\cosh(t)\} &= \mathcal{L}\{t + e^t\} \\ Y + Y\mathcal{L}\{\cosh(t)\} &= \frac{1}{s^2} + \frac{1}{s-1} \\ Y + Y\frac{s}{s^2-1} &= \frac{1}{s^2} + \frac{1}{s-1} \\ Y = \frac{s^2-1}{s^2(s-1)} &= \frac{1}{s^2} + \frac{1}{s} \\ y = \mathcal{L}^{-1}\{Y\} &= t + 1 \end{aligned}$$

6.5.19

$$\begin{aligned} F(s) &= \frac{2\pi s}{(s^2+\pi^2)^2} = 2 \cdot \frac{\pi}{s^2+\pi^2} \cdot \frac{s}{s^2+\pi^2} \\ \rightarrow f(t) &= 2\sin(\pi t) * \cos(\pi t) \\ &= 2\int_0^t \sin(\pi\tau)\cos(\pi(t-\tau))d\tau \\ &= 2\int_0^t \sin(\pi\tau)\cos(\pi t)\cos(\pi\tau)d\tau + 2\int_0^t \sin(\pi t)\sin^2(\pi\tau)d\tau \\ &= \dots \\ &= t\sin t \end{aligned}$$

6.5.22

$$\begin{aligned} F(s) &= \frac{e^{-as}}{s(s-2)} = \frac{e^{-as}}{s} \cdot \frac{1}{s-2} \\ \rightarrow f(t) &= u(t-a) * e^{2t} \\ &= \int_0^t u(\tau - a)e^{2(t-\tau)}d\tau \\ &= \int_a^t e^{2(t-\tau)}d\tau \\ &= \frac{1}{2}u(t-a)[e^{2(t-a)} - 1] \end{aligned}$$

6.6.7

$$\begin{aligned}
f(t) &= t^2 \sinh(2t),\ \ f_1(t) = \sinh(2t) \\
\mathcal{L}\{t^2 f_1\} &= \left(-\frac{d}{ds}\right)^2 \mathcal{L}\{f_1\} \\
&= \left(-\frac{d}{ds}\right)^2 \frac{2}{s^2-4} \\
&= \frac{16s^2}{(s^2-4)^3} - \frac{4}{(s^2-4)^2}
\end{aligned}$$

6.6.17

$$\begin{aligned}
F(s) &= \ln\left(\frac{s}{s-1}\right) = \ln s - \ln(s-1) \\
\frac{d}{ds}F &= \frac{1}{s} - \frac{1}{s-1} \\
\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s-1}\right\} &= 1 - e^t \\
\to f(t) &= \frac{e^t - 1}{t}
\end{aligned}$$

6.7.4

$$\begin{aligned}
y_1' &= 4y_2 - 8\cos(4t) & y_1(0) &= 0 \\
y_2' &= -3y_1 - 9\sin(4t) & y_2(0) &= 3
\end{aligned}$$

$\mathcal{L}$ - transform:

$$\begin{aligned}
sY_1 &= 4Y_2 - \frac{8s}{s^2+16} \\
sY_2 - 3 &= -3Y_1 - \frac{36}{s^2+16}
\end{aligned}$$

Solve for Y:

$$\begin{aligned}
sY_1 - 4Y_2 &= a \\
3Y_1 + sY_2 &= 3 - \frac{36}{s^2+16} = b
\end{aligned}$$

$$\begin{bmatrix} s & -4 & a \\ 3 & s & b \end{bmatrix} \sim \begin{bmatrix} 1 & -4/s & a/s \\ 0 & 1 & c \end{bmatrix} \text{ where } c = \tfrac{b-3a/s}{s+12/s}.$$
$$\sim \begin{bmatrix} 1 & 0 & a/s + 4c/s \\ 0 & 1 & c \end{bmatrix}$$

$$Y_2 = \left(\frac{-36}{s^2+16} + 3 + \frac{24}{s^2+16} \right) \cdot \frac{1}{s+12/s} = \frac{3s}{s^2+16}$$
$$Y_1 = \frac{a + 4Y_2}{s} = \frac{4}{s^2+16}$$

Inverse $\mathcal{L}$ -transform: $y_1 = \mathcal{L}^{-1}\{Y_1\} = \sin(4t)$
$y_2 = \mathcal{L}^{-1}\{Y_2\} = 3\cos(4t)$

Control: $y_1(0) = 0,\ y_2(0) = 3$ Ok.