TMA4120 - Assignment 1

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6.1.1

$$
\begin{aligned}
f(t) &= 2t + 8 \\
\mathscr{L}\{f\}(s) &= 2 \cdot \mathscr{L}\{t\}(s) + 8 \cdot \mathscr{L}\{1\}(s) && \text{(Linearity)} \\
&= \frac{2}{s^2} + \frac{8}{s} \\
&= \frac{8s + 2}{s^2}
\end{aligned}
$$

6.1.12

From the graph we can obtain the following equation for f;

$$
f(t) = \begin{cases} t & 0 \leq t < 1 \\ 1 & 1 < t < 2 \\ 0 & t \geq 2. \end{cases}
$$

$$
\begin{aligned}
\mathscr{L}\{f\}(s) &= \mathscr{L}\{t[H(t) - H(t-1)]\} + \mathscr{L}\{H(t-1) - H(t-2)\} \\
&= \mathscr{L}\{tH(t)\} - \mathscr{L}\{tH(t-1)\} + \mathscr{L}\{H(t-1)\} - \mathscr{L}\{H(t-2)\} \\
&= \frac{1}{s^2} - \quad + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}
\end{aligned}
$$

6.1.23

Given $\mathscr{L}\{f(t)\}(s) = F(s)$ and a positive constant $c > 0$, we have

$$
\begin{aligned}
\mathscr{L}\{f(ct)\}(s) &= \int_0^\infty e^{-st} f(ct) dt \\
&= \frac{1}{c} \int_0^\infty e^{-sx/c} f(x) dx, \qquad x = ct,\ dt = dx/c \\
&= \frac{1}{c} F(s/c)
\end{aligned}
$$

Using $\mathscr{L}\{\cos bt\}(s) = \frac{s}{s^2+b}$ we get

$$\begin{aligned}
\mathscr{L}\{\cos \omega t\}(s) &= \frac{1}{\omega}\frac{s/\omega}{(s/\omega)^2+1} \\
&= \frac{s}{s^2+\omega^2}
\end{aligned}$$

6.1.26

$$\begin{aligned}
F(s) &= \frac{5s+1}{s^2-25} = \frac{5s+1}{(s+5)(s-5)} = \frac{A}{s+5} + \frac{B}{s-5} \to \frac{12}{5(s+5)} + \frac{13}{5(s-5)} \\
\mathscr{L}^{-1}\{F(s)\}(t) &= \mathscr{L}^{-1}\left\{\frac{12}{5(s+5)}\right\} + \mathscr{L}^{-1}\left\{\frac{13}{5(s+5)}\right\} \\
&= \frac{12}{5}e^{-5t} + \frac{13}{5}e^{5t} \\
&= \frac{1}{5}(12e^{-5t} + 12e^{5t})
\end{aligned}$$

6.1.36

$$\begin{aligned}
f(t) &= \sinh t \cos t \\
&= \frac{1}{2}(e^t - e^{-t})\cos t \\
&= \frac{1}{2}e^t \cos t - \frac{1}{2}e^{-t}\cos t \\
\mathscr{L}\{f\} &= \frac{1}{2}\mathscr{L}\{e^t \cos t\} - \frac{1}{2}\mathscr{L}\{e^{-t}\cos t\} \\
&= \frac{1}{2}[F(s-1) - F(s+1)] \\
&= \frac{1}{2}\left[\frac{s-1}{(s-1)^2+1} - \frac{s+1}{(s+1)^2+1}\right]
\end{aligned}$$

6.1.40

$$\begin{aligned}
F(s) &= \frac{4}{s^2-2s-3} = \frac{4}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1} \to \frac{1}{s-3} - \frac{1}{s+1} \\
\mathscr{L}^{-1}\{F\} &= \mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{3t} - e^{-t}
\end{aligned}$$

6.2.4

I wrote down the wrong values from the book so I will be solving the IVP $y'' + y' = 10e^{-t}$ with $y(0) = 0 = y'(0)$.

$$
\begin{aligned}
\mathscr{L}\text{ - transform:}\quad & \mathscr{L}\{y'' + y'\} = \mathscr{L}\{10e^{-t}\} \to s^2Y + sY = \tfrac{10}{s+1} \\
\text{Solve for } Y\text{:}\quad & Y = \tfrac{10}{s(s+1)^2} \\
\text{Write as partial fraction:}\quad & (A+C)s^2 + (A+B+2C)s + C = 10 \\
& \to A = -10,\ B = -10,\ C = 10 \\
& \to Y = \tfrac{10}{s} - \tfrac{10}{s+1} - \tfrac{10}{(s+1)^2} \\
\text{Inverse } \mathscr{L}\text{ -transform:}\quad & y = \mathscr{L}^{-1}\{Y\} = -10e^{-t} - 10te^{-t} + 10 \text{ (from table.)} \\
\text{Control:}\quad & y(0) = -10 - 0 + 10 = 0 \\
& y' = 10e^{-t} - 10e^{-t} + 10te^{-t} = 10te^{-t} \\
& \to y'(0) = 0 \\
& y'' = -10e^{-t} + 10e^{-t} + 10e^{-t} - 10te^{-t} \\
& y'' = 10e^{-t} - 10te^{-t} \\
& \to y'' + y' = 10e^{-t} \quad \text{Ok.}
\end{aligned}
$$

6.2.13

$$y' - 6y = 0,\ y(-1) = 4$$

To solve this IVP, we introduce the variable $t_1 = t+1$ such that $y_1(t_1) = y(t)$.

$$
\begin{aligned}
\mathscr{L}\text{ - transform:}\quad & \mathscr{L}\{y_1' - 6y_1\} = 0 \to sY_1 - 4 - 6Y_1 = 0 \\
\text{Solve for } Y\text{:}\quad & Y_1 = \tfrac{4}{s-6} \\
\text{Inverse } \mathscr{L}\text{ -transform:}\quad & y_1(t_1) = \mathscr{L}^{-1}\{\tfrac{4}{s-6}\}(t_1) = 4e^{6t_1} \\
& \to y(t) = 4e^{6(t+1)} = 4e^{6t+6} \\
\text{Control:}\quad & y(-1) = 4e^{0} = 4 \\
& y' = 24e^{6t+6} \\
& \to y'' - 6y = 24e^{6t+6} - 24e^{6t+6} = 0 \quad \text{Ok.}
\end{aligned}
$$

6.3.8

The function

$$f(t) = \begin{cases} t^2 & 1 < t < 2 \\ 0 & \text{Otherwise} \end{cases}$$

can be written using the Heaviside-function as follows;

$$f(t) = t^2[H(t-1) - H(t-2)]$$

such that the leftmost part of the expression evaluates to zero if $t \notin (1,\ 2)$. Oppgave ikke ferdig:)

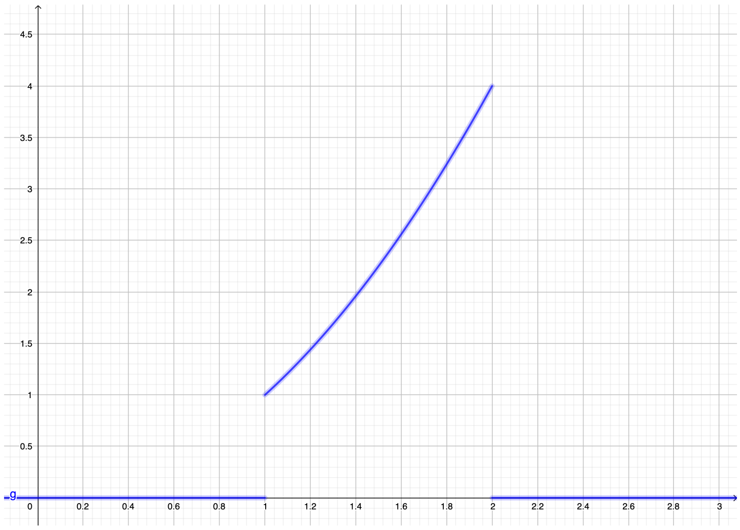

Figure 1: Plot of $f(t)$.

6.3.15

$$F(s) = \frac{e^{-2s}}{s^6} = e^{-2s} \cdot \frac{5!}{s^{5+1}} \cdot \frac{1}{5!}$$

$$\to \mathscr{L}^{-1}\{F\} = H(t-2) \cdot (t-2)^5 \cdot \frac{1}{120} = \frac{(t-2)^5 H(t-2)}{120} \qquad \text{(from table.)}$$

6.3.25

We are solving the IVP

$$\begin{cases} y'' + y = 2t & 0 \le t < 1 \\ 2 & t > 1 \end{cases}$$

with $y(0) = 0,\ y'(0) = -2.$

$$\begin{aligned}
\mathscr{L}\text{ - transform:}\quad & \mathscr{L}\{y''+y\} = \mathscr{L}\{2t\} \to s^2Y + 2 + Y = \tfrac{2}{s^2} \\
\text{Solve for } Y\text{:}\quad & Y = \tfrac{2}{s^2(s^2+1)} - \tfrac{2}{s^2+1} \\
\text{Write as partial fraction:}\quad & 2 = As^3 + (B+C)s^2 + As + B \\
& \to A = 0,\ B = 2,\ C = -2 \\
& \to Y = \tfrac{2}{s^2} - \tfrac{4}{s^2+1} \\
\text{Inverse } \mathscr{L}\text{ -transform:}\quad & y = \mathscr{L}^{-1}\{Y\} = \mathscr{L}^{-1}\left\{\tfrac{2}{s^2}\right\} - \mathscr{L}^{-1}\left\{\tfrac{4}{s^2+1}\right\} \\
& y = 2t - 4\sin t \\
\text{Control:}\quad & y(0) = 0 - 0 = 0 \\
& y' = 2 - 4\cos t \\
& \to y'(0) = 2 - 4 = -2 \\
& y'' = 4\sin t \\
& \to y'' + y = 4\sin t + 2t - 4\sin t = 2t \qquad \text{Ok.}
\end{aligned}$$

Hence,

$$y(t) = \begin{cases} 2t - 4\sin t & 0 \le t < 1 \\ 2 & t \ge 1. \end{cases}$$