Fra Kreyszig (10th), avsnitt 6.1

$\boxed{1}$ La $f : [0, \infty) \to \mathbb{R}$ være gitt ved $f(t) = 2t + 8$. Da er

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st}(2t+8)\,dt = 2\int_0^\infty te^{-st}\,dt + 8\int_0^\infty e^{-st}\,dt = \frac{2}{s^2} + \frac{8}{s}$$

ved delvis integrasjon.

$\boxed{12}$ We have

$$f(t) = \begin{cases} t, & \text{for } 0 < t < 1, \\ 1, & \text{for } 1 < t < 2, \\ 0, & \text{for } t > 2. \end{cases}$$

Therefore

$$L(s) = \mathcal{L}(f)(s) = \int_0^1 e^{-st}t dt + \int_1^2 e^{-st} dt.$$

Using integration by parts for the first integral, we get

$$\begin{aligned} L(s) &= -\frac{1}{s}e^{-st}t\Big|_0^1 - \int_0^1 \left(-\frac{1}{s}e^{-st}\right) dt - \frac{1}{s}e^{-st}\Big|_1^2 \\ &= -\frac{1}{s}e^{-s} + \frac{1}{s}\left(-\frac{1}{s}e^{-st}\Big|_0^1\right) - \frac{1}{s}e^{-2s} + \frac{1}{s}e^{-s} \\ &= \frac{1}{s^2}\left(1 - e^{-s}\right) - \frac{1}{s}e^{-2s}. \end{aligned}$$

$\boxed{23}$ Plugger $f(ct)$ inn i (1) side 204:

$$\mathcal{L}(f(ct)) = \int_0^\infty e^{-st} f(ct) dt.$$

Variabelskifte til $\tau = ct$ gir

$$\int_0^\infty e^{-(s/c)\tau} f(\tau)\frac{d\tau}{c} = \frac{1}{c}F\left(\frac{s}{c}\right).$$

Vi har

$$\mathcal{L}(\cos(t)) = \frac{s}{s^2+1}.$$

Formelen vi viste over gir

$$\begin{aligned} \mathcal{L}(\cos(\omega t)) &= \frac{1}{\omega}\frac{s/\omega}{(s/\omega)^2+1} \\ &= \frac{s}{s^2+\omega^2}. \end{aligned}$$

26 Delbrøkoppspaltar og bruker (6) i tabellen side 207.

$$\begin{aligned}\frac{5s+1}{s^2-25} &= \frac{12}{5}\frac{1}{s+5} + \frac{13}{5}\frac{1}{s-5} \\ &= \frac{12}{5}\mathcal{L}(e^{-5t}) + \frac{13}{5}\mathcal{L}(e^{5t}) \\ \Rightarrow f(t) &= \frac{12}{5}e^{-5t} + \frac{13}{5}e^{5t}.\end{aligned}$$

Ein kan også gjer det følgjande, med ekvivalent svar:

$$\begin{aligned}\frac{5s+1}{s^2-25} &= 5\frac{s}{s^2-25} + \frac{1}{5}\frac{5}{s^2-25} \\ &= 5\mathcal{L}(\cosh(5t)) + \frac{1}{5}\mathcal{L}(\sinh(5t)) \\ &= \mathcal{L}(5\ \cosh(5t) + \frac{1}{5}\ \sinh(5t)) \\ \Rightarrow f(t) &= 5\ \cosh(5t) + \frac{1}{5}\ \sinh(5t).\end{aligned}$$

36 Per definisjon av sinus hyperbolicus, har vi

$$f(t) = \sinh t \cos t = \frac{1}{2}(e^t - e^{-t})\cos t = \frac{1}{2}\cos te^t - \frac{1}{2}\cos te^{-t}.$$

Av linearitet er

$$F(s) = \mathcal{L}(f)(s) = \frac{1}{2}\mathcal{L}(\cos te^t)(s) - \frac{1}{2}\mathcal{L}(\cos te^{-t}).$$

Vi kan anvende første forskyvninsteorem (teorem 2, side 208) på hver av Laplace-transformene på høyre side. I notasjonen til teoremet har vi henholdsvis $a = 1$ og $a = -1$ i de to leddene, samt at funksjonen i begge tilfeller er cosinus. Siden

$$\mathcal{L}(\cos(t))(s) = \frac{s}{s^2+1},$$

har vi da

$$F(s) = \frac{1}{2}\mathcal{L}(\cos(t))(s-1) - \frac{1}{2}\mathcal{L}(\cos(t))(s+1) = \frac{1}{2}\left(\frac{s-1}{(s-1)^2+1} - \frac{s+1}{(s+1)^2+1}\right).$$

40 Siden

$$F(s) = \frac{4}{s^2-2s-3} = \frac{4}{(s-1)^2-4},$$

kan vi bruke s-forskyvningsloven, sinh - transformasjonen og linearitet av $\mathcal{L}^{-1}$, til å finne

$$\begin{aligned}e^{-t}f(t) &= \mathcal{L}^{-1}(F(s+1)) \\ &= \mathcal{L}^{-1}(\frac{2\cdot 2}{s^2-2^2}) = 2\sinh 2t.\end{aligned}$$

Dvs.

$$\mathcal{L}^{-1}(F(s)) = f(t) = 2e^t \sinh 2t.$$

Fra Kreyszig (10th), avsnitt 6.2

$\boxed{4}$ Transformerer begge sider av likhetstegnet

$$\mathcal{L}(y'' + 9y) = \mathcal{L}(10e^{-t}).$$

Siden $\mathcal{L}$ er lineær,

$$\begin{aligned}\mathcal{L}(y'') &= s^2Y - sy(0) - y'(0) = s^2Y,\\ \mathcal{L}(e^{-t}) &= \frac{1}{s+1},\end{aligned}$$

finner vi

$$s^2Y + 9Y = 10\frac{1}{s+1},$$

dvs.

$$Y = 10\frac{1}{s+1} \cdot \frac{1}{s^2+9}.$$

Delbrøksoppspaltning (OBS!)

$$\frac{1}{s+1} \cdot \frac{1}{s^2+9} = \frac{A}{s+1} + \frac{Bs+C}{s^2+9}$$

$$\implies 1 = (s^2+9)A + (Bs+C)(s+1)$$

$$\begin{aligned}\implies \mathcal{O}(1):& \qquad 9A + C = 1\\ \mathcal{O}(s):& \qquad B + C = 0\\ \mathcal{O}(s^2):& \qquad A + B = 0\end{aligned}$$

Dvs.

$$A = -B = C = \frac{1}{10}$$

og

$$Y = \frac{1}{s+1} + \frac{1-s}{s^2+9}.$$

Inverstransformerer begge sider av likningen,

$$\begin{aligned}y = \mathcal{L}^{-1}(y) &= \mathcal{L}^{-1}(\frac{1}{s+1}) + \frac{1}{3}\mathcal{L}^{-1}(\frac{3}{s^2+9}) - \mathcal{L}^{-1}(\frac{s}{s^2+9})\\ &= e^{-t} + \frac{1}{3}\sin 3t - \cos 3t.\end{aligned}$$

PS: Sjekk at svaret oppfyller likningen og initialbetingelsene.

$\boxed{13}$

$$y' - 6y = 0, \qquad y(-1) = 4$$

Her er initialbetingelsen gitt for $t_0 = -1 \neq 0$. En måte å løse dette på er å finne en løsning $\tilde{y}(t) = y(t+t_0)$ som er tidsforskyvet i forhold til $y(t)$, slik at initialbetingelsen blir $\tilde{y}(0) = 4$. Differensialligningen for $\tilde{y}(t)$ blir helt lik:

$$\tilde{y}' - 6\tilde{y} = 0, \qquad \tilde{y}(0) = 4.$$

Transformerer og løser for $\tilde{Y}$:

$$\begin{aligned}
\mathcal{L}\{\tilde{y}' - 6\tilde{y}\} &= \mathcal{L}\{0\} \\
s\tilde{Y} - \tilde{y}(0) - 6\tilde{Y} &= 0 \\
\tilde{Y} &= \frac{4}{s-6}
\end{aligned}$$

$$=> \qquad \tilde{y}(t) = 4\mathrm{e}^{6t}.$$

Tidsforskyver tilbake igjen for å få det korrekte svaret:

$$y(t) = 4\mathrm{e}^{6(t+1)}.$$

Fra Kreyszig (10th), avsnitt 6.3

$\boxed{8}$ Since $u(t-1) - u(t-2)$ is 1 for $1 < t < 2$ and 0 otherwise, we can write the given function as

$$f(t) = t^2[u(t-1) - u(t-2)] = t^2u(t-1) - t^2u(t-2).$$

To find its transform, we can use the second shifting theorem. We can write

$$t^2 = (t-1)^2 + 2(t-1) + 1 \quad \text{and} \quad t^2 = (t-2)^2 + 4(t-2) + 4$$

so that

$$\begin{aligned}
f(t) =&(t-1)^2u(t-1) + 2(t-1)u(t-1) + u(t-1) \\
&+ (t-2)^2u(t-2) + 4(t-2)u(t-2) + 4u(t-2).
\end{aligned}$$

We have

$$\begin{aligned}
F(s) = \mathcal{L}(f)(s) =&e^{-s}\mathcal{L}(g)(s) + 2e^{-s}\mathcal{L}(h)(s) + \frac{e^{-s}}{s} \\
&+ e^{-2s}\mathcal{L}(g)(s) + 4e^{-2s}\mathcal{L}(h)(s) + 4\frac{e^{-2s}}{s},
\end{aligned}$$

with $g(t) = t^2$ and $h(t) = t$. Finally, we get

$$F(s) = e^{-s}\Big(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\Big) + 2e^{-2s}\Big(\frac{1}{s^3} + \frac{2}{s^2} + \frac{2}{s}\Big).$$

Given $f(t)$, we could also use Eq. (4**) on page 220:

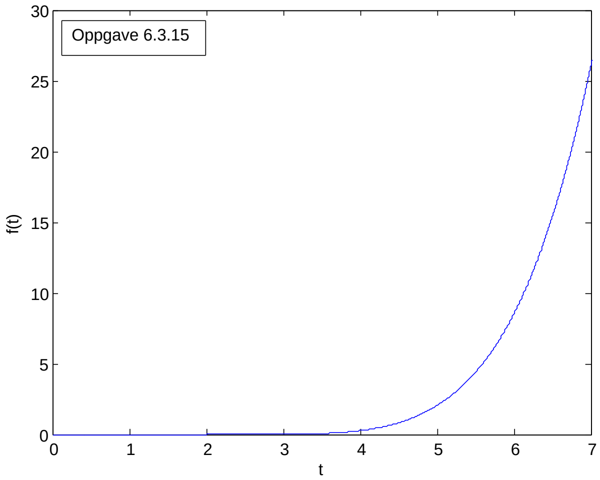
$$\mathcal{L}(f(t)u(t-a)) = e^{-as}\mathcal{L}(f(t+a)).$$

In this case, we obtain the following equivalent result:

$$\begin{aligned}
F(s) = \mathcal{L}(f)(s) &= e^{-s}\mathcal{L}((t+1)^2) + e^{-2s}\mathcal{L}((t+2)^2) \\
&= e^{-s}\Big(\mathcal{L}(t^2) + 2\mathcal{L}(t) + \mathcal{L}(1)\Big) + e^{-2s}\Big(\mathcal{L}(t^2) + 4\mathcal{L}(t) + 4\mathcal{L}(1)\Big) \\
&= e^{-s}\Big(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\Big) + e^{-2s}\Big(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\Big).
\end{aligned}$$

$\boxed{15}$

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}\left(e^{-2s}\frac{1}{s^6}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^6}\right)(t-2)u(t-2) \\
&= \frac{1}{5!}(t-2)^5u(t-2).
\end{aligned}$$



$\boxed{25}$ In this case, the right hand side is

$$f(t) = 2t[1 - u(t-1)] + 2u(t-1) = 2t - 2(t-1)u(t-1)$$

and using the t-shifting formula we find that

$$\begin{aligned} F(s) &= \mathcal{L}(f)(s) = 2\mathcal{L}(t) - 2\mathcal{L}((t-1)u(t-1)) \\ &= \frac{2}{s^2} - \mathrm{e}^{-s}\frac{2}{s^2} = \frac{2}{s^2}(1 - \mathrm{e}^{-s}). \end{aligned}$$

Laplace transforming

$$y'' + y = f(t)\,, \quad t > 0\,, \quad y(0) = 0\,,\ y'(0) = -2$$

yelds

$$\begin{aligned} & s^2Y + 2 + Y = F \\ \Rightarrow \quad & Y = -\frac{2}{s^2+1} + \frac{F}{s^2+1} \\ & \quad = -\frac{2}{s^2+1} + \frac{2}{s^2(s^2+1)} - \mathrm{e}^{-s}\frac{2}{s^2(s^2+1)}. \end{aligned}$$

Let $G(s) = \frac{1}{s^2(s^2+1)}$ and $g(t) = \mathcal{L}^{-1}(G)(t)$.
Partial fractions:

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

Inverse Laplace transform, using $t-$shifting:

$$y(t) = \mathcal{L}^{-1}(Y) = -2\sin t + 2g(t) - 2g(t-1)u(t-1),$$

where

$$g(t) = t - \sin(t).$$

Therefore,

$$y(t) = 2t - 4\sin(t) - 2u(t-1)[(t-1) - \sin(t-1)],$$

i.e.

$$y(t) = \begin{cases} 2t - 4\sin t & \text{for } 0 \le t < 1 \\ 2 - 4\sin(t) + 2\sin(t-1) & \text{for } t \ge 1 \end{cases}$$