TMA4120 - Assignment 5

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September 30, 2022

12.1.14.d

i)

$$
\begin{aligned}
u &= v(x) + w(y) & u_{xy} = 0 \\
u_{xy} &= \frac{\partial^2}{\partial x \partial y}(v(x) + w(y)) \\
&= \frac{\partial}{\partial x}\frac{\partial w(y)}{\partial y} \\
&= 0
\end{aligned}
$$

ii)

$$
\begin{aligned}
u &= v(x)w(y) & uu_{xy} = u_x u_y \\
uu_{xy} &= u\frac{\partial^2}{\partial x \partial y}(vw) \\
&= u\frac{\partial}{\partial x}\left(v\frac{\partial w}{\partial y}\right) \\
&= vw\left(\frac{\partial v}{\partial x}\frac{\partial w}{\partial y}\right) \\
&= u_x u_y
\end{aligned}
$$

iii)

$$u = v(x+2t) + w(x-2t) \qquad\qquad u_{tt} = 4u_{xx}$$

$$\begin{aligned}
u_{tt} &= \frac{\partial^2}{\partial t^2} u \\
&= 2\frac{\partial v}{\partial t} - 2\frac{\partial w}{\partial t} \\
&= 4\frac{\partial^2 v}{\partial t^2} + 4\frac{\partial^2 w}{\partial t^2} \\
u_{xx} &= \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}
\end{aligned}$$

12.1.15

$$\begin{aligned}
&u(x,y) = a\ln(x^2+y^2) + b \\
&u = 110 && x^2+y^2 = 1 \\
&u = 0 && x^2+y^2 = 100 \\
&u_{xx} + u_{yy} = 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{2ax}{(x^2+y^2)}\right) = \frac{2a}{x^2+y^2} - \frac{4ax^2}{(x^2+y^2)^2} \\
\frac{\partial^2 u}{\partial y^2} &= \frac{2a}{x^2+y^2} - \frac{4ay^2}{(x^2+y^2)^2} \\
\to u_{xx} + u_{yy} &= \frac{4a}{x^2+y^2} - \frac{4a(x^2+y^2)}{(x^2+y^2)^2} = 0
\end{aligned}$$

$$\begin{gathered}
x^2+y^2 = 1 \to a\ln 1 + b = 110 \to b = 110 \\
x^2+y^2 = 100 \to a\ln 100 + 110 = 0 \to a = -\frac{110}{\ln 100}
\end{gathered}$$

12.3.5

$$
\begin{aligned}
&u_{tt} = c^2 u_{xx} \\
&u(x,0) = k \sin 3\pi x \\
&u_t(x,0) = 0 \\
&L = 1,\ c^2 = 1,\ k = 0.01
\end{aligned}
$$

Since $u_t(x,0) = g(x) = 0,\ B_n^* = 0.$

$$
\begin{aligned}
B_n &= 2\int_0^1 k \sin 3\pi x \sin n\pi x dx = 0 \ \forall\ n \in \mathbb{N}\backslash\{3\} \\
B_3 &= 2k \int_0^1 \sin^2 3\pi x dx \\
&= k \int_0^1 (1 - \cos 6\pi x) dx \\
&= k
\end{aligned}
$$

Where we have used the otrhonormal properties of the trigonometric basis.

$$u(x,t) = k \cos 3\pi t \sin 3\pi x$$

12.3.7

$$f(x) = kx(1-x)$$

B_n^* is zero as in 12.3.5.

$$
\begin{aligned}
B_n &= 2k \int_0^1 (x - x^2) \sin n\pi x dx \\
&= 2k \left(-\frac{2}{n^3\pi^3} (\cos n\pi - 1) \right) \\
&= k \left(\frac{2}{n\pi} \right)^3 && n \text{ odd}
\end{aligned}
$$

$$u(x,t) = \sum_{n=1,\ \text{odd}}^{\infty} \left(\frac{2}{n\pi} \right)^3 \cos n\pi t \sin n\pi t$$

12.3.14

$$
\begin{aligned}
&L = \pi && c^2 = 1 \\
&u_t(x,0) = 0.01x && x \in [0, \pi/2] \\
&u_t(x,0) = 0.01(\pi - x) && x \in [\pi/2, \pi] \\
&f(x) = 0 \to B_n = 0. &&
\end{aligned}
$$

$$
\begin{aligned}
B_n^* &= \frac{2}{n\pi} \int_0^\pi g(x) \sin nx dx \\
&= \frac{2}{n\pi} \left[\int_0^{\pi/2} 0.01x \sin nx dx + \int_{\pi/2}^{\pi} 0.01(\pi - x) \sin nx dx \right] \\
&= \frac{0.02}{n\pi} \left(\frac{2 \sin n\pi/2}{n^2} \right) \\
&= -\frac{0.04}{n^3\pi} && n \text{ odd}
\end{aligned}
$$

$$
u(x,t) = \sum_{n=1,\ odd}^{\infty} -\frac{0.04}{n^3\pi} \sin nt \sin nx
$$

12.3.15

$$
\begin{aligned}
&u(x,t) = F(x)G(t) \\
&\frac{\partial^2 u}{\partial t^2} = F \cdot G'' \\
&\frac{\partial^4 u}{\partial x^4} = F^{(4)} \cdot G
\end{aligned}
$$

From $u_{tt} = -c^2 u_{x^4}$ we get $FG'' = -c^2 F^{(4)} G$. Rearraning we get $\frac{G''}{-c^2 G} = \frac{F^{(4)}}{F}$ which has to be constant since F and G are functions of different variables. Since $F^{(n)} = (-1)^n \beta^n F$ for n in 2^m (cyclic derivation), we get $F^{(4)}/F = \beta^4$. Similar reasoning can be used on G to get $-G''/c^2 G = \beta^4$.

12.Rev.18

$$
\begin{aligned}
&u_{xx} + u_x = 0 && (1) \\
&u(0, y) = f(y) \\
&u_x(0, y) = g(y)
\end{aligned}
$$

$$
\begin{aligned}
(1) \to \mathcal{L}\{u_x x\} + \mathcal{L}\{u_x\} &= 0 \\
s^2 U - sf(y) - g(y) + sU - f(y) &= 0 \\
\to U = \frac{f \cdot (s+1) + g}{s(s+1)} &= \frac{f}{s} + g\left(\frac{1}{s} - \frac{1}{s+1}\right) \\
\to u(x, y) &= f(y) + (1 + e^{-x})g(y)
\end{aligned}
$$