TMA4120 - Assignment 1

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6.1.1

$$
\begin{aligned}
f(t) &= 2t + 8 \\
\mathcal{L}\{f\}(s) &= 2 \cdot \mathcal{L}\{t\}(s) + 8 \cdot \mathcal{L}\{1\}(s) && \text{(Linearity)} \\
&= \frac{2}{s^2} + \frac{8}{s} \\
&= \frac{8s + 2}{s^2}
\end{aligned}
$$

6.1.12

From the graph we can obtain the following equation for f;

$$
f(t) = \begin{cases} t & 0 \leq t < 1 \\ 1 & 1 < t < 2 \\ 0 & \quad t \geq 2. \end{cases}
$$

$$
\begin{aligned}
\mathcal{L}\{f\}(s) &= \mathcal{L}\{t[u(t) - u(t-1)]\} + \mathcal{L}\{u(t-1) - u(t-2)\} \\
&= \mathcal{L}\{tu(t)\} - \mathcal{L}\{tu(t-1)\} + \mathcal{L}\{u(t-1)\} - \mathcal{L}\{u(t-2)\} \\
&= \frac{1}{s^2} - \mathcal{L}\{[(t-1)+1]u(t-1)\} + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} \\
&= \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} \\
&= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s}
\end{aligned}
$$

6.1.23

Given $\mathcal{L}\{f(t)\}(s) = F(s)$ and a positive constant $c > 0$, we have

$$\begin{aligned}
\mathcal{L}\{f(ct)\}(s) &= \int_0^\infty e^{-st} f(ct)dt \\
&= \frac{1}{c}\int_0^\infty e^{-sx/c} f(x)dx, \qquad x = ct,\ dt = dx/c \\
&= \frac{1}{c}F(s/c)
\end{aligned}$$

Using $\mathcal{L}\{\cos bt\}(s) = \frac{s}{s^2+b}$ we get

$$\begin{aligned}
\mathcal{L}\{\cos \omega t\}(s) &= \frac{1}{\omega}\frac{s/\omega}{(s/\omega)^2+1} \\
&= \frac{s}{s^2+\omega^2}
\end{aligned}$$

6.1.26

$$\begin{aligned}
F(s) &= \frac{5s+1}{s^2-25} = \frac{5s+1}{(s+5)(s-5)} = \frac{A}{s+5} + \frac{B}{s-5} \to \frac{12}{5(s+5)} + \frac{13}{5(s-5)} \\
\mathcal{L}^{-1}\{F(s)\}(t) &= \mathcal{L}^{-1}\left\{\frac{12}{5(s+5)}\right\} + \mathcal{L}^{-1}\left\{\frac{13}{5(s-5)}\right\} \\
&= \frac{12}{5}e^{-5t} + \frac{13}{5}e^{5t} \\
&= \frac{1}{5}(12e^{-5t} + 13e^{5t})
\end{aligned}$$

6.1.36

$$\begin{aligned}
f(t) &= \sinh t \cos t \\
&= \frac{1}{2}(e^t - e^{-t})\cos t \\
&= \frac{1}{2}e^t \cos t - \frac{1}{2}e^{-t}\cos t \\
\mathcal{L}\{f\} &= \frac{1}{2}\mathcal{L}\{e^t \cos t\} - \frac{1}{2}\mathcal{L}\{e^{-t}\cos t\} \\
&= \frac{1}{2}[F(s-1) - F(s+1)] \\
&= \frac{1}{2}\left[\frac{s-1}{(s-1)^2+1} - \frac{s+1}{(s+1)^2+1}\right]
\end{aligned}$$

6.1.40

$$F(s) = \frac{4}{s^2-2s-3} = \frac{4}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1} \rightarrow \frac{1}{s-3} - \frac{1}{s+1}$$
$$\mathcal{L}^{-1}\{F\} = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{3t} - e^{-t}$$

6.2.4

I wrote down the wrong values from the book so I will be solving the IVP $y'' + \underline{y'} = 10e^{-t}$ with $y(0) = 0 = y'(0)$.

$$\begin{aligned}
&\mathcal{L}\text{ - transform:} && \mathcal{L}\{y''+y'\} = \mathcal{L}\{10e^{-t}\} \rightarrow s^2Y + sY = \tfrac{10}{s+1}\\
&\text{Solve for } Y\text{:} && Y = \tfrac{10}{s(s+1)^2} = \tfrac{A}{s} + \tfrac{B}{s+1} + \tfrac{C}{(s+1)^2}\\
&\text{Write as partial fraction:} && (A+C)s^2 + (A+B+2C)s + C = 10\\
& && \rightarrow A = -10,\ B = -10,\ C = 10\\
& && \rightarrow Y = \tfrac{10}{s} - \tfrac{10}{s+1} - \tfrac{10}{(s+1)^2}\\
&\text{Inverse } \mathcal{L}\text{ -transform:} && y = \mathcal{L}^{-1}\{Y\} = -10e^{-t} - 10te^{-t} + 10 \text{ (from table.)}\\
&\text{Control:} && y(0) = -10 - 0 + 10 = 0\\
& && y' = 10e^{-t} - 10e^{-t} + 10te^{-t} = 10te^{-t}\\
& && \rightarrow y'(0) = 0\\
& && y'' = -10e^{-t} + 10e^{-t} + 10e^{-t} - 10te^{-t}\\
& && y'' = 10e^{-t} - 10te^{-t}\\
& && \rightarrow y'' + y' = 10e^{-t} \quad \text{Ok.}
\end{aligned}$$

6.2.13

$$y' - 6y = 0,\ y(-1) = 4$$

To solve this IVP, we introduce the variable $t_1 = t+1$ such that $y_1(t_1) = y(t)$.

$$\begin{aligned}
&\mathcal{L}\text{ - transform:} && \mathcal{L}\{y_1' - 6y_1\} = 0 \rightarrow sY_1 - 4 - 6Y_1 = 0\\
&\text{Solve for } Y\text{:} && Y_1 = \tfrac{4}{s-6}\\
&\text{Inverse } \mathcal{L}\text{ -transform:} && y_1(t_1) = \mathcal{L}^{-1}\{\tfrac{4}{s-6}\}(t_1) = 4e^{6t_1}\\
& && \rightarrow y(t) = 4e^{6(t+1)} = 4e^{6t+6}\\
&\text{Control:} && y(-1) = 4e^0 = 4\\
& && y' = 24e^{6t+6}\\
& && \rightarrow y' - 6y = 24e^{6t+6} - 24e^{6t+6} = 0 \quad \text{Ok.}
\end{aligned}$$

6.3.8

The function

$$f(t) = \begin{cases} t^2 & 1 < t < 2 \\ 0 & \text{Otherwise} \end{cases}$$

can be written using the Heaviside-function as follows;

$$f(t) = t^2[u(t-1) - u(t-2)]$$

such that the leftmost part of the expression evaluates to zero if $t \notin (1,\ 2)$.

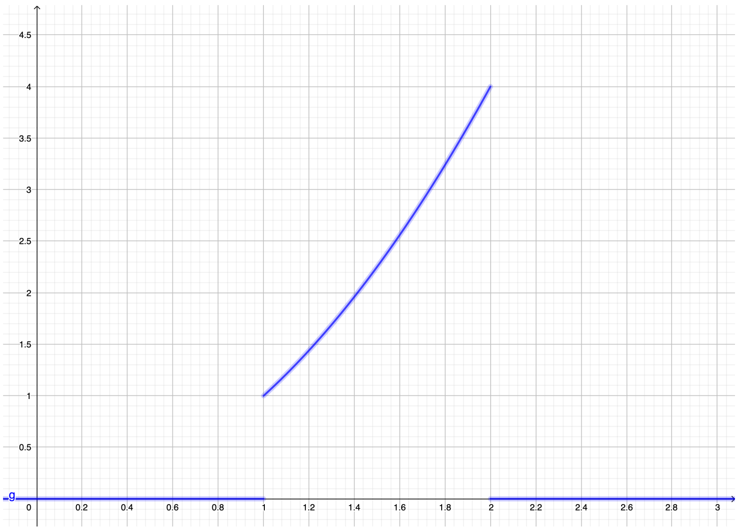

Figure 1: Plot of $f(t)$.

$$\begin{aligned}
\mathcal{L}\{f\} &= \mathcal{L}\{t^2[u(t-1) - u(t-2)]\} \\
&= \mathcal{L}\{[(t-1)^2 + 2(t-1) + 1]u(t-1)\} - \mathcal{L}\{[(t-2)^2 + 4(t-2) + 4]u(t-1)\} \\
&= \mathcal{L}\{(t-1)^2 u(t-1)\} + 2\mathcal{L}\{(t-1)u(t-1)\} + \mathcal{L}\{u(t-1)\} \\
&\qquad - (\mathcal{L}\{(t-2)^2 u(t-2)\} + 4\mathcal{L}\{(t-2)u(t-2)\} + 4\mathcal{L}\{u(t-2)\}) \\
&= \frac{e^{-s}}{s^3}(s^2 + 2s + 2) - \frac{2e^{-2s}}{s^3}(2s^2 + 2s + 1)
\end{aligned}$$

6.3.15

$$\begin{aligned}
F(s) &= \frac{e^{-2s}}{s^6} = e^{-2s} \cdot \frac{5!}{s^{5+1}} \cdot \frac{1}{5!} \\
\to \mathcal{L}^{-1}\{F\} &= u(t-2) \cdot (t-2)^5 \cdot \frac{1}{120} = \frac{(t-2)^5 u(t-2)}{120} \qquad \text{(from table.)}
\end{aligned}$$

6.3.25

We are solving the IVP

$$\begin{cases} y'' + y = 2t & 0 \le t < 1 \\ \quad 2 & t > 1 \end{cases}$$

with $y(0) = 0,\ y'(0) = -2.$

$$\begin{aligned}
\mathcal{L}\text{ - transform:} \quad & \mathcal{L}\{y'' + y\} = \mathcal{L}\{2t\} \to s^2 Y + 2 + Y = \tfrac{2}{s^2} \\
\text{Solve for } Y\text{:} \quad & Y = \tfrac{2}{s^2(s^2+1)} - \tfrac{2}{s^2+1} \\
\text{Write as partial fraction:} \quad & 2 = As^3 + (B+C)s^2 + As + B \\
& \to A = 0,\ B = 2,\ C = -2 \\
& \to Y = \tfrac{2}{s^2} - \tfrac{4}{s^2+1} \\
\text{Inverse } \mathcal{L}\text{ -transform:} \quad & y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\tfrac{2}{s^2}\right\} - \mathcal{L}^{-1}\left\{\tfrac{4}{s^2+1}\right\} \\
& y = 2t - 4\sin t \\
\text{Control:} \quad & y(0) = 0 - 0 = 0 \\
& y' = 2 - 4\cos t \\
& \to y'(0) = 2 - 4 = -2 \\
& y'' = 4\sin t \\
& \to y'' + y = 4\sin t + 2t - 4\sin t = 2t \qquad \text{Ok.}
\end{aligned}$$

Hence,

$$y(t) = \begin{cases} 2t - 4\sin t & 0 < t < 1 \\ \quad 2 & t > 1. \end{cases}$$