TMA4120 - Assignment 3

William Dugan

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11.1.2

$$
\begin{aligned}
\cos(nx) &\quad p = 2\pi/n \\
\sin(nx) &\quad p = 2\pi/n \\
\cos(2\pi x/k) &\quad p = k \\
\sin(2\pi x/k) &\quad p = k \\
\cos(2\pi nx/k) &\quad p = k/n \\
\sin(2\pi nx/k) &\quad p = k/n
\end{aligned}
$$

11.1.15

$$
\begin{aligned}
f(x) &= x^2,\ 0 < x < 2\pi,\ f(x+2\pi) = f(x) \\
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} x^2 dx = \frac{4\pi^3}{3} \\
a_n &= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos(nx) dx \\
&= \frac{1}{\pi}\left[\frac{2}{n^2} x\cos(nx) - \frac{2-(nx)^2}{n^3}\sin(nx)\right]_0^{2\pi} \\
&= \frac{4}{n^2} \\
b_n &= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin(nx) dx \\
&= \frac{1}{\pi}\left[\frac{2}{n^2} x\sin(nx) + \frac{2-(nx)^2}{n^3}\cos(nx)\right]_0^{2\pi} \\
&= -\frac{4\pi}{n} \\
\to S_f &= 4\left[\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{\cos(nx)}{n^2} - \frac{\pi\sin(nx)}{n}\right)\right]
\end{aligned}
$$

11.1.17

$$f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$$

Since f is an even function, $b_n = 0$.

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^\pi (\pi - x)dx = \frac{\pi}{2} \\
a_n &= \frac{2}{\pi}\int_0^\pi (\pi - x)\cos(nx)dx \\
&= \frac{2}{\pi}\left[(\pi - x)\frac{1}{n}\sin(nx) - \frac{1}{n^2}\cos(nx)\right]_0^\pi \\
&= \frac{2}{\pi n^2}(1 - \cos(n\pi)) = \frac{2(1-(-1)^n)}{\pi n^2} \\
\rightarrow S_f &= \frac{\pi}{2} + \sum_{n=1}^\infty \frac{2(1-(-1)^n)}{\pi n^2}\cos(nx)
\end{aligned}$$

11.1.21

$$f(x) = \begin{cases} -(\pi + x), & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$$

Since f is odd, $a_0,\ a_n = 0$.

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^\pi (\pi - x)\sin(nx)dx \\
&= \frac{2}{\pi}\left[-\frac{\pi - x}{n}\cos(nx) - \frac{1}{n^2}\sin(nx)\right]_0^\pi \\
&= \frac{2}{n} \\
\rightarrow S_f &= \sum_{n=1}^\infty \frac{2}{n}\sin(nx)
\end{aligned}$$

11.2.1

e^x	Neither even nor odd.
$e^{-\lvert x\rvert}$	Even since $e^{-\lvert -x\rvert} = e^{-\lvert x\rvert}$.
$x^3\cos(nx)$	Odd since x^3 is odd and $\cos x$ is even.
$x^2\tan(\pi x)$	Odd since x^2 is even and $\tan x$ is odd.
$\sinh x - \cosh x$	Expands to e^{-z} which is neither odd nor even.

11.2.10

$$f(x) = \begin{cases} -(4+x), & -4 < x < 0 \\ 4-x, & 0 < x < 4 \end{cases}$$

Since f is odd, $a_0,\ a_n = 0$. Using $a = n\pi/4$ we get

$$\begin{aligned} b_n &= \frac{1}{2}\int_0^4 (4-x)\sin(ax)dx \\ &= \frac{1}{2}\left[-\frac{4-x}{a}\cos(ax) - \frac{1}{a^2}\sin(ax)\right]_0^4 \\ &= \frac{2}{a} = \frac{8}{n\pi} \\ \rightarrow S_f &= \sum_{n=1}^{\infty} \frac{8}{n\pi}\sin\left(\frac{n\pi x}{4}\right) \end{aligned}$$

11.2.17

$$f(x) = \begin{cases} 1+x, & -1 < x < 0 \\ 1-x, & 0 < x < 1 \end{cases}$$

Since f is even, $b_n = 0$.

$$\begin{aligned} a_0 &= \int_0^1 (1-x)dx = \frac{1}{2} \\ a_n &= 2\int_0^1 (1-x)\sin(n\pi x)dx \\ &= 2\left[\frac{1-x}{n\pi}\sin(n\pi x) - \frac{1}{(n\pi)^2}\cos(n\pi x)\right]_0^1 \\ &= \frac{2(1-(-1)^n)}{(n\pi)^2} \\ \rightarrow S_f &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{(n\pi)^2}\cos(n\pi x) \end{aligned}$$

11.2.24

$$f(x) = \begin{cases} 0, & 0 < x < 2 \\ 1, & 2 < x < 4 \end{cases}$$

We assume f is periodic with $L = 4$.

a) Even extension

$$\begin{aligned}
a_0 &= \frac{1}{4}\int_0^4 f(x)dx = \frac{1}{2} \\
a_n &= \frac{1}{2}\int_0^4 f(x)\cos\left(\frac{n\pi x}{4}\right)dx \\
&= \frac{1}{2}\left[\frac{4}{n\pi}\sin\left(\frac{n\pi x}{4}\right)\right]_2^4 \\
&= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) \\
\rightarrow S_{f_1} &= \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{4}\right)}{n}
\end{aligned}$$

b) Odd extension

$$\begin{aligned}
b_n &= \frac{1}{2}\int_2^4 \sin\left(\frac{n\pi x}{4}\right)dx \\
&= -\frac{2}{n\pi}[\cos\left(\frac{n\pi x}{4}\right)]_2^4 \\
&= -\frac{2((-1)^n - \cos\left(\frac{n\pi}{2}\right))}{n\pi} \\
\rightarrow S_{f_2} &= -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - \cos\left(\frac{n\pi}{2}\right)}{n}\sin\left(\frac{n\pi x}{4}\right)
\end{aligned}$$

11.2.29

$$f(x) = \sin x,\ 0 < x < \pi$$

We assume f is periodic with $L = \pi$.

a) Even extension

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi} \sin x dx = \frac{2}{\pi} \\
a_n &= \frac{2}{\pi}\int_0^{\pi} \sin x \cos(nx) dx \\
&= \frac{2}{\pi} \cdot \frac{1}{1-1/n^2}\left[-\frac{1}{n}\sin x \sin(nx) + \frac{1}{n^2}\cos x \cos(nx)\right]_0^{\pi} \\
&= \frac{2((-1)^{n+1}-1)}{\pi(n^2-1)} \\
\to S_{f_1} &= \frac{2}{\pi}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}-1}{n^2-1}\sin(nx)\right)
\end{aligned}$$

b) Odd extension

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} \sin x \sin(nx) dx \\
&= \frac{2}{\pi} \cdot \frac{n^2}{n^2-1}\left[-\frac{1}{n}\sin x \cos(nx) - \frac{1}{n^2}\cos x \sin(nx)\right]_0^{\pi} \\
&= 0\ (n \neq 1)
\end{aligned}$$

To avoid dividing by zero we calculate b_1 separately.

$$\begin{aligned}
b_1 &= \frac{2}{\pi}\int_0^{\pi} \sin^2 x dx = \frac{1}{\pi}\int_0^{\pi}(1-\cos(2x))dx = 1 \\
\to S_{f_2} &= \sin x
\end{aligned}$$

11.3.15

$$r(t) = t(\pi^2 - t^2),\ -\pi < x < \pi$$

Since $r(-t) = -r(t)$ we get a Fourier sine series ($a_0, a_n = 0$).

$$b_n = \frac{2}{\pi}\int_{-\pi}^{\pi}(\pi^2 t - t^3)\sin(nx)dx = \frac{12(-1)^{n+1}}{n^3}$$

We can write $y = y_1 + y_2 + \dots$ which yields

$$\begin{aligned} y_n'' + cy_n' + y &= \frac{12(-1)^{n+1}}{n^3}\sin(nt) \\ (-n^2A_n + cnB_n + A_n)\cos(nt) + (-n^2B_n - cnA_n + B_n)\sin(nt) &= \frac{12(-1)^{n+1}}{n^3}\sin(nt) \end{aligned}$$

Rewritten:

$$\begin{aligned} \implies -n^2A_n + cnB_n + A_n = 0 \implies B_n &= \frac{A_n(n^2-1)}{cn} \\ \implies -n^2B_n - cnA_n + B_n &= -n^2\frac{A_n(n^2-1)}{cn} - cnA_n + \frac{A_n(n^2-1)}{cn} \\ &= A_n\left[\frac{-n^2(n^2-1) - (cn)^2 + (n^2-1)}{n}\right] \\ &= A_n\left[\frac{(n^2-1)(1-n^2) - (cn)^2}{cn}\right] \\ &\left(= \frac{12(-1)^{n+1}}{n^3}\right) \\ \implies A_n &= \frac{12(-1)^{n+1}c}{n^2[(n^2-1)(1-n^2) - (cn)^2]} \\ B_n &= \frac{12(-1)^{n+1}(n^2-1)}{n^3[(n^2-1)(1-n^2) - (cn)^2]} \end{aligned}$$