Fra Kreyszig (10th), avsnitt 12.6

11 Vis at løsningen til problemet

$$u_t = c^2 u_{xx},$$
$$u_x(0,t) = 0, \quad u_x(L,t) = 0,$$
$$u(x,0) = f(x)$$

er

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t}$$

der

$$A_0 = \frac{1}{L}\int_0^L f(x)\mathrm{d}x, \quad A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x.$$

12

$$\begin{aligned} u_x(0,t) &= 0 \\ u_x(L,t) &= 0 \\ u(x,0) &= x \\ L &= \pi \\ c &= 1 \end{aligned}$$

$$\begin{aligned} &\overset{Eks.4}{\Longrightarrow} u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) e^{-n^2 t} \\ &\text{med} \quad A_0 = \frac{1}{\pi}\int_0^\pi f(x)dx = \frac{1}{\pi}\int_0^\pi x dx = \frac{x^2}{2\pi}\Big|_0^\pi = \frac{\pi}{2} \\ &\qquad A_n = \frac{2}{\pi}\int_0^\pi x\cos(nx)dx = \frac{2}{\pi}x\frac{\sin(nx)}{n}\Big|_0^\pi - \frac{2}{\pi}\int_0^\pi \frac{\sin(nx)}{n}dx \\ &\qquad = \frac{2}{\pi}\frac{\cos(nx)}{n^2}\Big|_0^\pi = \frac{2}{\pi n^2}((-1)^n - 1) \end{aligned}$$

$\boxed{14}$

$$\begin{aligned}
A_0 &= \frac{1}{L}\int_0^L f(x)\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^\pi \cos 2x \ \mathrm{d}x \\
&= \frac{1}{\pi}\Big|_0^\pi \frac{\sin 2x}{2} = 0. \\
A_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L} \ \mathrm{d}x \\
&= \frac{2}{\pi}\int_0^\pi \cos 2x \cos nx \ \mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^\pi \cos 2x \cos nx \ \mathrm{d}x \\
&= 0
\end{aligned}$$

for $n \neq 2$ (se side 479 i boken)

$$\begin{aligned}
A_2 &= \frac{2}{\pi}\int_0^\pi \cos^2 2x \ \mathrm{d}x \\
&= \frac{2}{\pi}\int_0^\pi \left(\frac{1+\cos 4x}{2}\right)\mathrm{d}x \\
&= \frac{1}{\pi}\left(\int_0^\pi \mathrm{d}x + |_0^\pi \frac{\sin 4x}{4}\right) \\
&= 1
\end{aligned}$$

Dette gir løsningen

$$\begin{aligned}
u(x,t) &= A_0 + \sum_{n=1}^\infty A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t} \\
&= e^{-4t}\cos 2x
\end{aligned}$$

$\boxed{16}$

$$\begin{aligned}
&u_t = c^2 u_{xx} + H \\
&L = \pi \\
&u(0,t) = u(\pi,t) = 0 \qquad t \geq 0 \\
&u(x,0) = f(x)
\end{aligned}$$

La $u(x,t) = v(x,t) - Hx\frac{x-\pi}{2c^2}$. Dette gir

$$\begin{aligned}
u_t(x,t) &= v_t(x,t) \\
u_{xx}(x,t) &= v_{xx}(x,t) - \frac{H}{c^2} \\
\implies v_t(x,t) &= u_t(x,t) = c^2 u_{xx}(x,t) + H = c^2 v_{xx} - H + H = c^2 v_{xx}(x,t)
\end{aligned}$$

Vi har dermed

$$
\begin{aligned}
v_t(x,t) &= c^2 v_{xx}(x,t) \\
v(0,t) &= u(0,t) = 0 \\
v(\pi,t) &= u(\pi,t) = 0 \\
v(x,0) &= u(x,0) + Hx\frac{x-\pi}{2c^2} = f(x) + Hx\frac{x-\pi}{2c^2}
\end{aligned}
$$

$$
\overset{(9)-(10)}{\Longrightarrow} v(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx) e^{-\lambda_n^2 t}
$$

med

$$
\lambda_n = cn \qquad \text{og} \qquad B_n = \frac{2}{\pi}\int_0^{\pi}\left(f(x) + Hx\frac{x-\pi}{2c^2}\right)\sin(nx)dx
$$

Dermed er

$$
u(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx) e^{-\lambda_n^2 t} - Hx\frac{x-\pi}{2c^2}
$$

med

$$
\lambda_n = cn \qquad \text{og} \qquad B_n = \frac{2}{\pi}\int_0^{\pi}\left(f(x) + Hx\frac{x-\pi}{2c^2}\right)\sin(nx)dx
$$

$\boxed{21}$ Oppgitte betingelser:

$$
u(0,y) = u(a,y) = u(x,0) = 0, \qquad \text{og} \qquad u(x,a) = 25, \qquad \text{med } a = 24
$$

Steady-state temperatur vil si at temperaturen ikke lenger endrer seg med tiden: $u_t = 0$. Varmeledningsligningen i to dimensjoner blir dermed

$$
\begin{aligned}
\nabla^2 u &= 0 \\
u_{xx} + u_{yy} &= 0, \qquad (1)
\end{aligned}
$$

som er Laplaces ligning. Bruker seperasjon av variable:

$$
u(x,y) = F(x)G(y)
$$

Innsatt i (1):

$$
\begin{aligned}
\frac{\mathrm{d}^2F}{\mathrm{d}x^2}G + F\frac{\mathrm{d}^2G}{\mathrm{d}y^2} &= 0 \\
\frac{1}{F}\frac{\mathrm{d}^2F}{\mathrm{d}x^2} &= -\frac{1}{G}\frac{\mathrm{d}^2G}{\mathrm{d}y^2} = k \qquad \text{(en konstant)}
\end{aligned}
$$

Vi har dermed to ordinære differensialligninger:

$$
F'' - kF = 0, \qquad\qquad G'' + kG = 0
$$

Med $k = \mu^2 > 0$:

$$
F(x) = C_1\mathrm{e}^{\mu x} + C_2\mathrm{e}^{-\mu x}
$$

Initialbetingelsene $u(0,y) = u(a,y) = 0$ gir $C_1 = C_2 = 0$.

Med $k = 0$:

$$F(x) = C_3 x + C_4$$

Initialbetingelsene $u(0,y) = u(a,y) = 0$ gir $C_3 = C_4 = 0$.

Med $k = -\mu^2 < 0$:

$$F(x) = C_5 \cos(\mu x) + C_6 \sin(\mu x)$$

Initialbetingelsen $u(0,y) = 0$ gir:

$$\begin{aligned} C_5 \cos 0 + C_6 \sin 0 &= 0 \\ C_5 &= 0 \end{aligned}$$

Initialbetingelsen $u(a,y) = 0$ gir:

$$\begin{aligned} C_6 \sin(\mu a) &= 0 \\ \mu a &= n\pi \\ \mu &= \frac{n\pi}{a} \qquad n = 0, \pm 1, \pm 2, ... \end{aligned}$$

$$F(x) = C_6 \sin\left(\frac{n\pi x}{a}\right)$$

Med $k = -\mu^2$ blir ligningen for $G(y)$:

$$G'' - \mu^2 G = 0$$

Med løsning

$$G(y) = C_7 \mathrm{e}^{\mu y} + C_8 \mathrm{e}^{-\mu y}$$

Initialbetingelsen $u(x,0) = 0$ gir $C_8 = -C_7$:

$$\begin{aligned} G(y) &= C_7 \left(\mathrm{e}^{\mu y} - \mathrm{e}^{-\mu y}\right) \\ &= 2C_7 \sinh(\mu y) \\ &= 2C_7 \sinh\left(\frac{n\pi y}{a}\right) \end{aligned}$$

For hver eneste n får vi dermed en løsning:

$$u_n(x,y) = C_9 \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$$

Nå er fortsatt $n = 0, \pm 1, \pm 2$, men siden u_n er odde mtp n (som betyr at u_{-n} er proporsjonal med u_n) holder det å summere $u(x,y)$ for bare positive n. Løsningen for $n = 0$ er $u = 0$, som ikke er særlig interessant.

Generell løsning:

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$$

Bruker nå siste betingelse: $u(x,a) = 25$

$$25 = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)\sinh(n\pi)$$

$$\begin{aligned}
B_n \sinh(n\pi) &= \frac{2}{a}\int\limits_0^a 25\sin\left(\frac{n\pi x}{a}\right)\,\mathrm{d}x \\
\frac{B_n \sinh(n\pi) a}{50} &= \left[\frac{-a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right]_0^a \\
\frac{B_n \sinh(n\pi)}{50} &= \frac{1}{n\pi}(1-(-1)^n) \\
B_n &= \frac{50}{n\pi\sinh(n\pi)}(1-(-1)^n)
\end{aligned}$$

Som gir løsningen:

$$u(x,y) = \sum_{n=1}^{\infty} \frac{50}{n\pi\sinh(n\pi)}(1-(-1)^n)\sinh\left(\frac{n\pi y}{24}\right)\sin\left(\frac{n\pi x}{24}\right)$$