Fra Kreyszig (10th), avsnitt 6.4

4 Laplace transforming

$$y'' + 16y = 4\delta(t - 3\pi)\,, \quad y(0) = 2\,,\ y'(0) = 0\,,$$

we get

$$\begin{aligned} & s^2Y - sy(0) - y'(0) + 16Y = 4e^{-3\pi s} \\ \Rightarrow\ & (s^2 + 16)Y = 2s + 4e^{-3\pi s}\,. \end{aligned}$$

Hence,

$$Y(s) = 2\frac{s}{s^2 + 4^2} + e^{-3\pi s}\frac{4}{s^2 + 4^2}\,.$$

Taking the inverse Laplace transform, using $t-$shifting, we obtain

$$\begin{aligned} y(t) = \mathcal{L}^{-1}(Y) &= 2\cos(4t) + \sin(4(t - 3\pi))u(t - 3\pi) \\ &= 2\cos(4t) + \sin(4t)u(t - 3\pi). \end{aligned}$$

10 Let us Laplace transform

$$y'' + 5y' + 6y = \delta(t - 1/2\pi) + u(t - \pi)\cos t, \quad y(0) = 0 = y'(0)\,.$$

Using $t-$shifting, we have that $\mathcal{L}\{u(t - \pi)\cos t\} = -e^{-\pi s}\frac{s}{s^2+1}$, since $\cos t = -\cos(t - \pi)$. Therefore

$$\begin{aligned} & s^2Y - sy(0) - y'(0) + 5sY - y(0) + 6Y = e^{-1/2\pi s} - e^{-\pi s}\frac{s}{s^2 + 1} \\ \Rightarrow\ & \left(s^2 + 5s + 6\right)Y = e^{-1/2\pi s} - e^{-\pi s}\frac{s}{s^2 + 1}\,. \end{aligned}$$

Hence,

$$Y(s) = e^{-1/2\pi s}\frac{1}{s^2 + 5s + 6} - e^{-\pi s}\frac{s}{(s^2 + 1)\,(s^2 + 5s + 6)}.$$

Partial fraction decomposition.

$$\begin{aligned} & \frac{1}{s^2 + 5s + 6} = \frac{1}{(s + 2)(s + 3)} = \frac{A}{s + 2} + \frac{B}{s + 3} \\ \iff\ & 1 = A(s + 3) + B(s + 2) = (A + B)s + 3A + 2B. \end{aligned}$$

This yealds the following linear system for A and B:

$$\begin{cases} A + B = 0 \\ 3A + 2B = 1 \end{cases}\,,$$

with solution $A = 1$ and $B = -1$.

Now

$$\begin{aligned}
&\frac{s}{(s^2+1)(s+2)(s+3)} = \frac{As+B}{s^2+1} + \frac{C}{s+2} + \frac{D}{s+3} \\
\iff\ & As(s+2)(s+3) + B(s+2)(s+3) \\
&\quad + C\left(s^2+1\right)(s+3) + D\left(s^2+1\right)(s+2) \\
&= (A+C+D)s^3 + (5A+B+3C+2D)s^2 \\
&\quad + (6A+5B+C+D)s + 6B+3C+2D\,.
\end{aligned}$$

We obtain the following system of linear equations

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 5 & 1 & 3 & 2 \\ 6 & 5 & 1 & 1 \\ 0 & 6 & 3 & 2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

with solution $A = B = 1/10, C = -2/5$ and $D = 3/10$.

Therefore, we have

$$\begin{aligned}
Y(s) &= e^{-1/2\pi s}\frac{1}{(s+2)(s+3)} - e^{-\pi s}\frac{s}{(s^2+1)(s+2)(s+3)} \\
&= e^{-1/2\pi s}\left(\frac{1}{s+2} - \frac{1}{s+3}\right) - \frac{1}{10}e^{-\pi s}\left(\frac{s}{s^2+1} + \frac{1}{s^2+1} - 4\frac{1}{s+2} + 3\frac{1}{s+3}\right)\,.
\end{aligned}$$

Inverse Laplace transform, using $t-$shifting:

$$\begin{aligned}
y(t) = \mathcal{L}^{-1}(Y) &= \left(e^{-2(t-\pi/2)} - e^{-3(t-\pi/2)}\right)u(t-\pi/2) \\
&\quad - \frac{1}{10}\left(\cos(t-\pi) + \sin(t-\pi) - 4e^{-2(t-\pi)} + 3e^{-3(t-\pi)}\right)u(t-\pi) \\
&= \left(e^{-2(t-\pi/2)} - e^{-3(t-\pi/2)}\right)u(t-\pi/2) \\
&\quad + \frac{1}{10}\left(\cos t + \sin t + 4e^{-2(t-\pi)} - 3e^{-3(t-\pi)}\right)u(t-\pi)\,.
\end{aligned}$$

Fra Kreyszig (10th), avsnitt 6.5

$\boxed{12}$ We can rewrite the equation as

$$y(t) + (y * \cosh)(t) = t + e^t.$$

Laplace transforming and applying the convolution theorem, we obtain

$$\begin{aligned}
Y + Y\frac{s}{s^2-1} &= \frac{1}{s^2} + \frac{1}{s-1}, \\
\Rightarrow\quad Y(s) &= \frac{1}{s} + \frac{1}{s^2},
\end{aligned}$$

with $Y(s) = \mathcal{L}\left(y(t)\right)$. Inverse Laplace transform:

$$y(t) = 1 + t.$$

19 Finn $f(t)$ når

$$\mathcal{L}(f) = \frac{2\pi s}{(s^2+\pi^2)^2}.$$

Skriv

$$\mathcal{L}(f) = 2\frac{\pi}{s^2+\pi^2}\frac{s}{s^2+\pi^2} = 2\mathcal{L}(\sin \pi t)\mathcal{L}(\cos \pi t).$$

Dette gir

$$\begin{aligned} f(t) &= 2\sin \pi t * \cos \pi t \\ &= 2\int_0^t \sin(\pi\tau)\cos(\pi(t-\tau))d\tau. \end{aligned}$$

Ved dei trigonometriske summeformlane og halvvinkelidentitetane finn vi at

$$\begin{aligned} 2\sin \pi\tau \cos(\pi t - \pi\tau) &= 2\sin \pi\tau(\cos \pi t \cos \pi\tau + 2\sin \pi t \sin \pi\tau) \\ &= 2\cos \pi t \sin \pi\tau \cos \pi\tau + 2\sin \pi t \sin^2 \pi\tau \\ &= \cos \pi t \sin 2\pi\tau + \sin \pi t(1-\cos 2\pi\tau) \\ &= \sin \pi t + \sin 2\pi\tau \cos \pi t - \cos 2\pi\tau \sin \pi t \\ &= \sin \pi t + \sin(2\pi\tau - \pi t). \end{aligned}$$

Dette gir

$$\begin{aligned} f(t) &= \int_0^t \sin \pi t + \sin(2\pi\tau - \pi t)d\tau && (1) \\ &= \sin \pi t\Big|_0^t \tau - \frac{1}{2\pi}\Big|_0^t \cos(2\pi\tau - \pi t) && (2) \\ &= t\sin \pi t - \frac{1}{2\pi}(\cos \pi t - \cos(-\pi t)) && (3) \\ &= t\sin \pi t. && (4) \end{aligned}$$

Alternativt kan ein bruke

$$\frac{2\pi s}{(s^2+\pi^2)^2} = \left(-\frac{\pi}{s^2+\pi^2}\right)',$$

smugtitte på delkapittel 6.6, og bruke formelen for den deriverte av Laplacetransformen.

22

$$\begin{aligned} F(s) = \mathcal{L}\{f\} &= e^{-as}\mathcal{L}\{1\}\mathcal{L}\left\{e^{2t}\right\} \\ &= \mathcal{L}\{u(t-a)\}\mathcal{L}\left\{e^{2t}\right\} \\ &= \mathcal{L}\left\{u(t-a) * e^{2t}\right\} \end{aligned}$$

som gir

$$\begin{aligned}
f(t) &= u(t-a) * e^{2t} \\
&= \int_0^t u(\tau-a)e^{2(t-\tau)}\mathrm{d}\tau \\
&= u(t-a)\int_a^t e^{2(t-\tau)}\mathrm{d}\tau \\
&= -\frac{1}{2}u(t-a)e^{2t}e^{-2\tau}\Big|_a^t \\
&= -\frac{1}{2}u(t-a)e^{2t}\left(e^{-2t}-e^{-2a}\right) \\
&= \frac{1}{2}u(t-a)\left(e^{2(t-a)}-1\right).
\end{aligned}$$

Fra Kreyszig (10th), avsnitt 6.6

$\boxed{7}$ Ettersom

$$\begin{aligned}
\mathcal{L}\{t\sinh 2t\} &= -\frac{\mathrm{d}}{\mathrm{d}s}\frac{2}{s^2-4} \\
&= \frac{4s}{\left(s^2-4\right)^2}
\end{aligned}$$

er

$$\begin{aligned}
\mathcal{L}\{f\} &= \mathcal{L}\{t\cdot t\sinh 2t\} \\
&= -\frac{\mathrm{d}}{\mathrm{d}s}\frac{4s}{\left(s^2-4\right)^2} \\
&= -\frac{4\left(s^2-4\right)^2-4s\cdot 2\left(s^2-4\right)2s}{\left(s^2-4\right)^4} \\
&= 4\frac{4+3s^2}{\left(s^2-4\right)^3}.
\end{aligned}$$

$\boxed{15}$ Fra oppgave 6.6.7 ser vi at $f(t)=\frac{1}{4}t\sinh 2t$.

$\boxed{17}$

$$\begin{aligned}
F(s) &= \ln\left(\frac{s}{s-1}\right) \\
F'(s) &= \frac{s-1}{s}\frac{1}{(s-1)^2}(s-1-s) = -\frac{1}{s(s-1)} = \frac{1}{s}-\frac{1}{s-1}.
\end{aligned}$$

$$\mathcal{L}^{-1}(F'(s)) = 1-e^t = -tf(t)$$

$$\Rightarrow f(t) = \frac{e^t-1}{t}.$$

Alternatively, since $\lim_{s\to\infty}F(s)=\lim_{s\to\infty}\ln\left(\frac{s}{s-1}\right)=0$,

$$\begin{aligned}
F(s) = F(s)-F(\infty) = -\int_s^\infty F'(s)ds = -\mathcal{L}\left[\frac{1-e^t}{t}\right](s) = \mathcal{L}\left[\frac{e^t-1}{t}\right](s) \\
\implies \quad f(t) = \frac{e^t-1}{t}
\end{aligned}$$

Fra Kreyszig (10th), avsnitt 6.7

4 Writing $Y_1 = \mathcal{L}(y_1)$, $Y_2 = \mathcal{L}(y_2)$, $G_1 = \mathcal{L}(\cos 4t)$ and $G_2 = \mathcal{L}(\sin 4t)$ we obtain

$$\begin{aligned} sY_1 - y_1(0) &= 4Y_2 - 8G_1 \\ sY_2 - y_2(0) &= -3Y_1 - 9G_2, \end{aligned}$$

with $y_1(0) = 0$ and $y_2(0) = 3$. By collecting Y_1 and Y_2-terms we have

$$\begin{aligned} sY_1 - 4Y_2 &= -8G_1 \\ 3Y_1 + sY_2 &= 3 - 9G_2. \end{aligned}$$

Solving algebraically for Y_1 and Y_2 we get

$$\begin{aligned} Y_1 &= \frac{1}{s^2+12}(12 - 8sG_1 - 36G_2) \\ Y_2 &= \frac{1}{s^2+12}(3s + 24G_1 - 9sG_2). \end{aligned}$$

Substituting $G_1 = \frac{s}{s^2+16}$ and $G_2 = \frac{4}{s^2+16}$ yealds

$$\begin{aligned} Y_1 &= \frac{1}{s^2+12}\left(12 - \frac{8s^2}{s^2+16} - \frac{144}{s^2+16}\right) = \frac{1}{s^2+12}\left(\frac{12s^2+192-8s^2-144}{s^2+16}\right) \\ &= \frac{1}{s^2+12}\left[\frac{4(s^2+12)}{s^2+16}\right] = \frac{4}{s^2+16}. \end{aligned}$$

Inverse transform:

$$y_1(t) = \mathcal{L}^{-1}(Y_1) = \sin(4t).$$

We can proceed in the same way to find $y_2(t)$. We have

$$\begin{aligned} Y_2 &= \frac{1}{s^2+12}\left(3s + \frac{24s}{s^2+16} - \frac{36s}{s^2+16}\right) = \frac{1}{s^2+12}\left(3s - \frac{12s}{s^2+16}\right) \\ &= \frac{1}{s^2+12}\frac{3s(s^2+12)}{s^2+16} = \frac{3s}{s^2+16}, \end{aligned}$$

hence

$$y_2(t) = \mathcal{L}^{-1}(Y_2) = 3\cos(4t).$$

Alternatively, since we had found y_1 already, we could have solved

$$y_2' = -3y_1 - 9\sin 4t = -12\sin 4t\,, \quad y_2(0) = 3,$$

from which

$$y_2(t) = y_2(0) - 12\int_0^t \sin(4\tau)\,d\tau = 3 + 3\cos(4\tau)\Big|_0^t = 3\cos(4t).$$