Problem Set 1

Note: Assessed problems (and sub-problems) are marked by the asterisk *.

1. * Express and simplify each of the following complex numbers to the forms $x + iy$ for some $x, y \in \mathbb{R}$ and $re^{i\theta}$ for some $r > 0, \theta \in (-\pi, \pi]$.

$$(a)^* \ \frac{1+i}{1-i}, \qquad (b) \ \frac{|3-4i|}{3-4i} + \frac{1}{2-i},$$
$$(c)^* \ \overline{(\sqrt{3}-i)^5}, \qquad (d) \ 2e^{2\pi i/3} + 2e^{4\pi i/3}.$$

2. * Show that for every $z, w \in \mathbb{C}$, $\big||z| - |w|\big| \leq |z - w|$.

3. Define the operator $\langle \cdot, \cdot \rangle : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by $\langle z, w \rangle = z\bar{w}$. Show that if $z = x + iy$ and $w = u + iv$, then

$$\mathrm{Re}\langle z, w \rangle = (x, y) \cdot (u, v),$$

where $\cdot$ denotes the usual dot product of vectors in $\mathbb{R}^2$. Moreover, show that $\langle \cdot, \cdot \rangle$ is a Hermitian inner product on $\mathbb{C}$. That is,

- $\langle z, w \rangle = \overline{\langle w, z \rangle}$,
- $\langle z, z \rangle \geq 0$ and equality holds if and only if $z = 0$.

4. * Show that for every $z, w \in \mathbb{C}$,

$$|z \pm w|^2 = |z|^2 + |w|^2 \pm 2\mathrm{Re}(z\bar{w}).$$

Hence, prove the following identity:

$$|z + w|^2 - |z - w|^2 = 4\mathrm{Re}(z\bar{w}).$$

5. Let n be any integer greater than 2 and $w = e^{2\pi i/n}$.

   (a) Show that $1 + w + w^2 + \ldots w^{n-1} = 0$.

   (b) Hence, prove the following identity:

$$\cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{4\pi}{n}\right) + \ldots + \cos\left(\frac{2(n-1)\pi}{n}\right) = 0.$$

6. * Find all solutions to the equation $z^4 + 8 - 8i\sqrt{3} = 0$.

7. Let's find the exact value of $\alpha = \cos(\frac{2\pi}{5})$ by following the steps below.

(a) Express α and α^2 as a polynomial of $w = e^{2\pi i/5}$.

(b) Compute the value $w^4 + w^3 + w^2 + w + 1$.

(c) Show that $p\alpha^2 + q\alpha + r = 0$ for some integers p, q, and r.

(d) Find α.

8. * Describe and sketch the sets of complex numbers $z = re^{i\theta} \in \mathbb{C}$ determined by the following conditions:

$(a)^*$ $r = \sin(3\theta) + 1$,

(b) $z^2 + \bar{z}^2 = 2$,

$(c)^*$ $\mathrm{Re}\left(\dfrac{z-i}{z+i}\right) < 0$,

(d) $\mathrm{Im}\left(\dfrac{z-i}{z+i}\right) = 0$,

$(e)^*$ $\mathrm{Im} z^2 < 0$ and $\mathrm{Im}(z+1+i)^2 < 0$.

9. * For each of the five sets in Exercise 8 above, determine whether or not they are open, closed, bounded, connected, simply connected or multiply connected.

10. For any sequence of complex numbers z_n, show that $z_n \to 0$ if and only if $|z_n| \to 0$.

11. Is it true that for any sequence of real numbers $r_n > 0$ and $\theta_n \in (-\pi, \pi]$, $r_n e^{i\theta_n} \to re^{i\theta}$ if and only if $r_n \to r$ and $\theta_n \to \theta$? Explain why.

12. * Show that $f(x + iy) = x^2 - y^2 + 2ixy$ is an entire function.

13. * Show that the function $f(z) = |z|^2$ has a derivative at 0 but is not holomorphic on any domain of $\mathbb{C}$.