HYPERBOLICITY OF RENORMALIZATION OF CRITICAL QUASICIRCLE MAPS

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Abstract. There is a well developed renormalization theory of real analytic critical circle maps by de Faria, de Melo, and Yampolsky. In this paper, we extend Yampolsky's result on hyperbolicity of renormalization periodic points to a larger class of dynamical objects, namely critical quasicircle maps, i.e. analytic self homeomorphisms of a quasicircle with a single critical point. Unlike critical circle maps, the inner and outer criticalities of critical quasicircle maps can be distinct. We develop a compact analytic renormalization operator called "Corona Renormalization" with a hyperbolic fixed point whose stable manifold has codimension one and consists of critical quasicircle maps of the same criticality and periodic type rotation number. Our proof is an adaptation of Pacman Renormalization Theory for Siegel disks as well as rigidity results on the escaping dynamics of transcendental entire functions.

Contents

1. Introduction

1.1. **Critical quasicircle maps.** A *critical circle map* is a real analytic self homeomorphism f of the unit circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ with exactly one critical point 0. Yoccoz [Yoc84] showed that if a critical circle map $f : \mathbb{T} \to \mathbb{T}$ has an irrational rotation number θ, then f is topologically conjugate to an irrational rotation. This means that if $\{p_n/q_n\}$ are best rational approximations of θ, then the iterates $\{f^{q_n}(0)\}$ are the closest returns to 0.

Given a critical circle map f of irrational rotation number θ, the n^{th} renormalization $\mathcal{R}^n f$ of f is defined as follows. Consider the commuting pair $p\mathcal{R}^n f = (f^{q_n}|_{I_{n+1}}, f^{q_{n+1}}|_{I_n})$, where $I_n \subset \mathbb{T}$ is the interval between 0 and $f^{q_n}(0)$. Then, $\mathcal{R}^n f$ is the normalized critical commuting pair obtained by rescaling $p\mathcal{R}^n f$ to unit size.

The renormalization theory of critical circle maps serves to justify the universality phenomena empirically observed in smooth families of critical circle maps. Historically, this is one of the two main examples of universality in one-dimensional dynamics, the other being the Feigenbaum universality observed in unimodal maps. The works of Feigenbaum et al. [FKS82] and Östlund et al. [ÖRSS83] translated the universality phenomena into a conjecture on the hyperbolicity of the renormalization operator on the space of critical commuting pairs. The conjecture was later generalized by various authors, in particular Lanford [Lan88] who accounted for more complex universalities.

Theorem 1.1 (Lanford's Program [Yam03])**.** *The renormalization operator $\mathcal{R}$ in the space of critical commuting pairs admits a "horseshoe" attractor $\mathcal{A}$ on which its action is conjugated to the two-sided shift. Moreover, there exists an $\mathcal{R}$-invariant space of critical commuting pairs with the structure of an infinite dimensional smooth manifold, with respect to which $\mathcal{A}$ is a hyperbolic set with one-dimensional expanding direction.*

Given an irrational number $\theta \in (0,1)$ with continued fraction expansion $\theta = [0; a_1, a_2, a_3, \dots]$, we say that θ is *of bounded type* if a_n's are uniformly bounded above, *pre-periodic* if there are positive integers m and p such that $a_n = a_{n+p}$ for all $n \geq m$, and *periodic* if additionally $m = 1$. We will denote corresponding spaces by Θ_{bdd}, Θ_{per} and Θ_{pre} respectively.

De Faria [dF99] introduced the notion of *holomorphic commuting pairs* and proved the universality of scaling ratios and the existence of renormalization horseshoe for critical circle maps of bounded type rotation number. $C^{1+\alpha}$ rigidity was established by de Faria and de Melo [dFdM99] for bounded type rotation number, and later by Khmelev and Yampolsky [KY06] for arbitrary irrational rotation number by studying parabolic bifurcations. Moreover, Yampolsky extended the horseshoe for all irrational rotation numbers in [Yam01], and brought Lanford's program to completion in [Yam02, Yam03] using *cylinder renormalization.*

In this paper, we work with a generalization of critical circle maps, namely critical quasicircle maps.

Definition 1.2. A *critical quasicircle map* is a homeomorphism $f : \mathbf{H} \to \mathbf{H}$ of a quasicircle which extends to a holomorphic map on a neighborhood of $\mathbf{H}$ and has exactly one critical point on $\mathbf{H}$.

Given a critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$, the behaviour at the unique critical point on $\mathbf{H}$ can be encoded by two positive integers, namely the inner criticality d_0 and the outer criticality d_∞. The total local degree of f at the critical point is $d_0 + d_\infty - 1$ and it is at least 2. When the criticalities are specified, we call $f : \mathbf{H} \to \mathbf{H}$ a (d_0, d_∞)*-critical* quasicircle map. See Figure 1 for some examples.

In the bounded type regime, if we assume that either d_0 or d_∞ is one, the quasicircle $\mathbf{H}$ will be the boundary of a rotation domain. By Douady-Ghys surgery [Dou87, Ghy84], $\mathbf{H}$ can be assumed to be the boundary of a *Siegel disk*, i.e. a simply connected rotation domain. Stirnemann [Sti94] first gave a computer-assisted proof of the existence of a renormalization fixed point with a golden mean Siegel disk. McMullen [McM98] applied a measurable deep point argument to prove the existence of renormalization horseshoe for bounded type rotation number. Gaidashev and Yampolsky [Yam08, GY22] gave a computer-assisted proof of the golden mean hyperbolicity of renormalization of Siegel disks using the formalism of *almost*

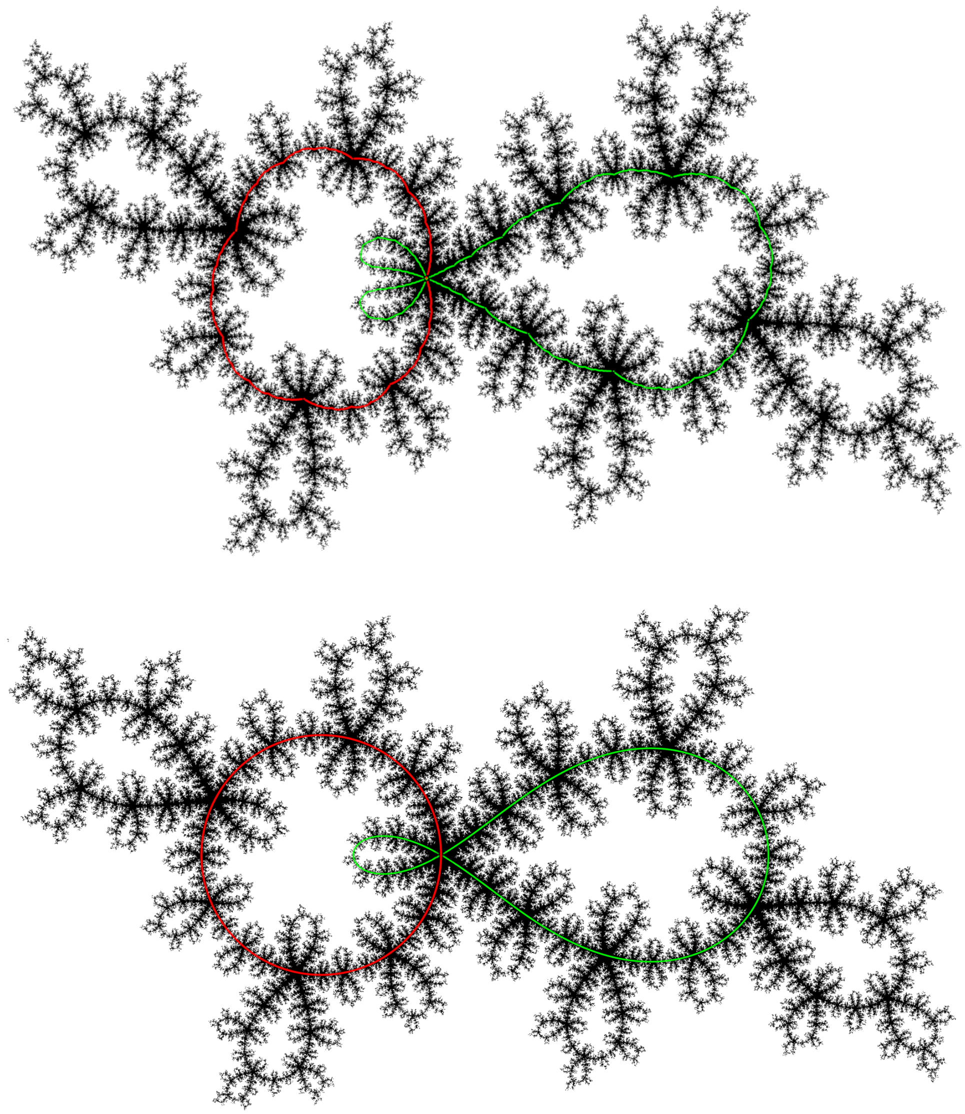

FIGURE 1. The Julia sets of

$$f_{3,2}(z) = bz^3 \frac{4-z}{1-4z+6z^2} \quad \text{and} \quad f_{2,2}(z) = cz^2 \frac{z-3}{1-3z}.$$

The critical values $b \approx -1.144208 - 0.964454i$ and $c \approx -0.755700 - 0.654917i$ are picked such that $f_{3,2} : \mathbf{H} \to \mathbf{H}$ is a $(3,2)$-critical quasicircle map on some quasicircle $\mathbf{H}$, $f_{2,2} : \mathbb{T} \to \mathbb{T}$ is a $(2,2)$-critical circle map, and both have the golden mean rotation number. Both $\mathbf{H}$ and $\mathbb{T}$ are colored red, and their preimages are colored green.

commuting pairs. In [DLS20], Dudko, Lyubich, and Selinger constructed a compact analytic operator, called *Pacman renormalization operator*, with a hyperbolic fixed point whose stable manifold has codimension one and consists of maps with a Siegel disk of a fixed rotation number of periodic type.

From now on, we will be working with critical quasicircle maps $f : \mathbf{H} \to \mathbf{H}$ where $\mathbf{H}$ is a *Herman curve*, that is, $\mathbf{H}$ is not contained in the closure of any rotation domain of f. In the bounded type regime, this is equivalent to the assumption that both d_0 and d_∞ are at least two.

Given any pair of integers $d_0, d_\infty \geq 2$, the problem of realization of (d_0, d_∞)-critical quasicircle maps was solved in our previous work [Lim23a] by studying *a priori bounds* and degeneration of Herman rings of a certain class of rational maps. In [Lim23b], we proved $C^{1+\alpha}$ rigidity and constructed renormalization horseshoe for critical quasicircle maps with bounded type rotation number.

1.2. **Corona renormalization.** In this paper, we continue our study of renormalization of critical quasicircle maps and prove hyperbolicity of renormalization for periodic rotation number. Our approach will follow closely the ideas behind Pacman Renormalization Theory. We design a renormalization operator acting on the space of *coronas*, a doubly-connected version of pacmen.

A corona is a holomorphic map $f : U \to V$ between two nested annuli such that $f : U \backslash \gamma_0 \to V \backslash \gamma_1$ is a unicritical branched covering map where γ_1 is an arc connecting the two boundary components of V. The number of preimages of γ_1 on the boundary components of U determine the inner and outer criticalities $d_0 \geq 2$ and $d_\infty \geq 2$ of a corona; the total degree of f is equal to $d_0 + d_\infty - 1$. When the criticalities are specified, we call f a (d_0, d_∞)-critical corona. See Figure 3 for an illustration.

Similar to pacman renormalization, we define the corona renormalization operator as follows. First, we remove the quadrilateral bounded by γ_1 and its image. The remaining space is a quadrilateral in which the first return map will be called a *pre-corona*. Gluing a pair of opposite sides of this quadrilateral gives us a new corona, which is called the *corona renormalization* $\mathcal{R}f$ of f.

We say that a (d_0, d_∞)-critical corona is *rotational* with rotation number θ if it admits an invariant quasicircle $\mathbf{H}$ on which the map is a (d_0, d_∞)-critical quasicircle map of rotation number θ. The renormalization of a (d_0, d_∞)-critical rotational corona is again a (d_0, d_∞)-critical rotational corona, and the induced action on the rotation number is governed by

$$R_{\mathrm{prm}}(\theta) = \begin{cases} \dfrac{\theta}{1-\theta}, & \text{if } 0 \leq \theta \leq \dfrac{1}{2}, \\ \dfrac{2\theta - 1}{\theta}, & \text{if } \dfrac{1}{2} \leq \theta \leq 1. \end{cases}$$

Theorem A (Hyperbolicity of renormalization)**.** *For any integers $d_0, d_\infty \geq 2$ and any $\theta \in \Theta_{per}$, there exists a corona renormalization operator $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ with the following properties.*

(1) *$\mathcal{U}$ is an open subset of a Banach analytic manifold $\mathcal{B}$ consisting of (d_0, d_∞)-critical coronas.*
(2) *$\mathcal{R}$ is a compact analytic operator with a unique fixed point f_* which is hyperbolic.*

(3) *The local stable manifold $\mathcal{W}^s_{loc}$ of f_* corresponds to the space of rotational coronas with rotation number θ in $\mathcal{B}$.*
(4) *The local unstable manifold $\mathcal{W}^u_{\mathrm{loc}}$ is one-dimensional.*

Similar to [DLS20], the main step is justifying item (4), which will be accomplished via transcendental dynamics. The pre-corona associated to a corona f on the local unstable manifold admits a maximal transcendental extension $\mathbf{F}$. The dynamics of $\mathbf{F}$ can be described as a *cascade*, that is, a collection $\{\mathbf{F}^P\}_{P\in\mathbf{T}}$ of σ-proper maps parametrized by a dense semigroup $\mathbf{T}\subset(\mathbb{R}_{\geq 0},+)$ such that $\mathbf{F}^P\circ\mathbf{F}^Q=\mathbf{F}^{P+Q}$. The second half of this paper is dedicated to the study of the dynamics of $\mathbf{F}$. To justify item (4), we prove the following theorem.

Theorem B (Rigidity of escaping dynamics on $\mathcal{W}^u_{\mathrm{loc}}$)**.** *Let $\mathbf{F}$ be a maximal σ-proper extension of a pre-corona on $\mathcal{W}^u_{\mathrm{loc}}$. The full escaping set*

$$\mathbf{I}(\mathbf{F}) := \left\{ z\in\mathbb{C} : \;\; \textit{either } z\notin\bigcap_P \mathrm{Dom}\left(\mathbf{F}^P\right) \;\textit{or } \mathbf{F}^P(z)\to\infty \;\textit{ as } P\to\infty \right\}$$

moves conformally away from the pre-critical points and supports no invariant line field. Consequently, if $\mathbf{F}$ has an attracting cycle, then the Julia set of $\mathbf{F}$ supports no invariant line field.

One may compare this theorem to Rempe's result [Rem09] on the rigidity of the escaping set of transcendental entire functions. Ultimately,

$$\text{Theorem B} \implies \dim(\mathcal{W}^u_{\mathrm{loc}}) = \text{number of critical orbits} = 1 \implies \text{Theorem A(4)}.$$

We conjecture that this philosophy should hold in a more general setting.

Conjecture C. *Consider a compact analytic renormalization operator with a hyperbolic fixed point such that every map on the unstable manifold $\mathcal{W}^u_{\mathrm{loc}}$ admit a global transcendental extension. Then,*

$$\dim\left(\mathcal{W}^u_{\mathrm{loc}}\right) \leq \text{number of critical orbits.}$$

Remark 1.3. We would like to note a few differences between our case and the pacmen case. Refer to Section 1.3 for a more comprehensive summary.

Firstly, the original proof of item (4) for pacmen does not require such a rigidity theorem. Unlike coronas, every pacman is designed to admit a natural fixed point α associated to it. For a Siegel pacman, the α-fixed point is the center of its Siegel disk. The multiplier of the α-fixed point naturally foliates the Banach neighborhood of the pacman renormalization fixed point. Consequently, hyperbolicity of the pacman renormalization operator and in particular item (4) follows from an application of the λ-lemma along parabolic leaves.

Secondly, the aim of the study of the finite-time escaping set associated to the transcendental extension of pre-pacmen in [DL23] was to attain a puzzle structure, which was ultimately applied to prove the MLC at some infinitely renormalizable satellite parameters. In our case, the full escaping set $\mathbf{I}(\mathbf{F})$ is of interest because, together with the postcritical set, it is the measure-theoretic attractor of $\mathbf{F}^{\geq 0}$ on the Julia set.

Given a critical quasicircle map $f:\mathbf{H}\to\mathbf{H}$, we can define a Banach neighborhood $N(f)$ of f as follows. Pick a skinny annular neighborhood U of $\mathbf{H}$ such that f is holomorphic on a neighborhood of U, and pick a small $\varepsilon>0$. Then, $N(f)$ is the space of unicritical holomorphic maps $g : U \to \mathbb{C}$ such that g extends continuously to the boundary of U and $\sup_{z\in U} |f(z) - g(z)| < \varepsilon$, equipped with the sup norm.

Corollary D. *Consider a small Banach neighborhood $N(f)$ of a (d_0, d_∞)-critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$ with pre-periodic rotation number θ. The space S of maps in $N(f)$ which restrict to a (d_0, d_∞)-critical quasicircle map with rotation number θ forms an analytic submanifold of $N(f)$ of codimension at most one. The corresponding invariant quasicircle moves holomorphically over S.*

We conjecture that the codimension is actually one.

Conjecture E. *The conjugacy class S has codimension one. In particular, critical quasicircle maps are structurally unstable.*

So far, this conjecture is known to be true for periodic type critical quasicircle maps that are close to the renormalization fixed point f_* due to Theorem A, as well as critical circle maps due to standard monotonicity properties of the rotation number. We suspect that the conjecture can be solved via an infinitesimal argument similar to unimodal maps [ALdM03].

Consider a one-dimensional holomorphic family of unicritical holomorphic maps $\{f_\lambda\}_{\lambda\in\Lambda}$. We say that a parameter $\lambda \in \Lambda$ is *hyperbolic* if the forward orbit of the critical point of f_λ tends to an attracting cycle. The set of hyperbolic parameters in Λ is open, and every connected component of such is called a *hyperbolic component*.

Conjecture F (Parameter self-similarity)**.** *Suppose there is a unique parameter $\lambda_* \in \Lambda$ such that f_{λ_*} has a unicritical Herman quasicircle of periodic type rotation number θ. The union of hyperbolic components within Λ is asymptotically self-similar at λ_* with a universal self-similarity factor depending only on θ and the criticality of f_*.*

A version of this conjecture appears in [Lim23a], in which the family $\{f_\lambda\}$ is a family of rational maps. See Figure 2. This conjecture is a generalization of the golden mean universality of critical circle maps [Yam03]. Our hyperbolicity result provides a step forward towards solving this conjecture. However, we suspect that attaining a complete solution would require hyperbolicity of the renormalization horseshoe for bounded type rotation numbers, as well as a thorough study of parameter rays and hyperbolic components of the unstable manifold as a parameter space of transcendental σ-proper maps.

1.3. **Outline.** Sections 2–5 are inspired by the original work on pacman renormalization in [DLS20], and Sections 5–7 are inspired by the detailed study of transcendental dynamics on the unstable manifold in [DL23]. As previously mentioned, the main difference lies in the proof that the local unstable manifold $\mathcal{W}^u_{\text{loc}}$ has dimension one. Once we prove that our renormalization fixed point is hyperbolic, we treat $\mathcal{W}^u_{\text{loc}}$ as a holomorphic family of unicritical transcendental maps of unknown dimension. By adapting some ideas from [Rem09], we deduce the rigidity of escaping dynamics and claim that the deformation space of hyperbolic coronas on the unstable manifold must be supported on the Fatou set, the domain of stability. This implies that $\mathcal{W}^u_{\text{loc}}$ is one-dimensional.

In Section §2, we introduce the definition of *coronas* and *pre-coronas*. We define the corona renormalization operator and show that for any renormalizable corona f, we can always find a compact analytic operator $\mathcal{R}$ on a small Banach neighborhood of f.

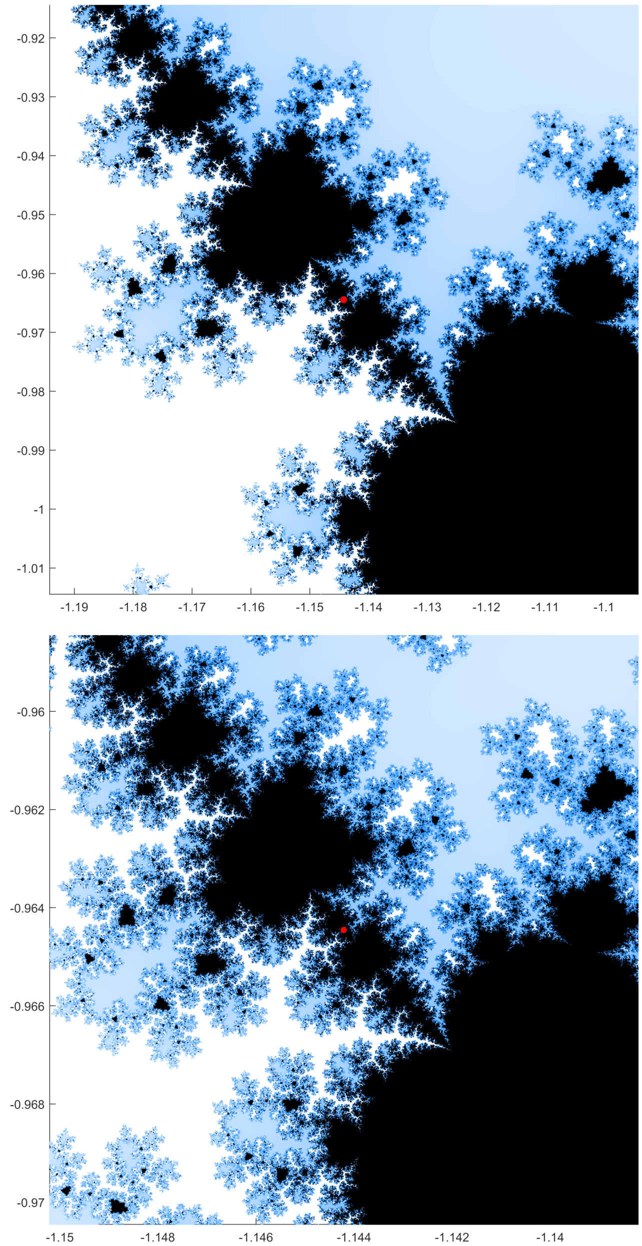

FIGURE 2. Magnifications of the bifurcation locus of the parameter space $\left\{F_c(z) = cz^3\frac{4-z}{1-4z+6z^2}\right\}_{c\in\mathbb{C}^*}$ by different scales about the parameter $c_* \approx -1.144208 - 0.964454i$ marked in red. This family is characterized by critical points 0, ∞, and 1 of local degrees 2, 3, and 4 respectively, where both 0 and ∞ are fixed and $F_c(1) = c$. The point c_* is the unique parameter such that F_{c_*} has a golden mean Herman quasicircle. Figure 1 displays the Julia set of F_{c_*}.

In Section §3, we analyze the structure of a rotational corona f. We prove that any critical quasicircle map can be renormalized to a rotational corona. By applying results in [Lim23b], we also show that rotational coronas are rigid: two rotational coronas are quasiconformally conjugate as long as they have the same criticality and rotation number.

In Section §4, we construct a compact analytic corona renormalization operator $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ and a corona $f_* \in \mathcal{U}$ of periodic rotation number such that $\mathcal{R} f_* = f_*$. In Theorem 4.12, we show that $\mathcal{R}$ and f_* satisfy items (2) and (3) in Theorem A, and that the dimension of the local unstable manifold $\mathcal{W}^u_{\text{loc}}$ is finite and positive. The proof relies on a number of ingredients.

(i) For any corona $f \in \mathcal{U}$ which is many times renormalizable, we can obtain a renormalization tiling which approximates the Herman quasicircle $\mathbf{H}_*$ of f_* by lifting the domain of a high renormalization of f. This tiling is robust under perturbation, and we use them to show in Corollary 4.11 that any infinitely renormalizable rotational corona that stays close to f_* must be a rotational corona.

(ii) By [Lim23b, Theorem K], renormalizations $\mathcal{R}^n f$ of a rotational corona near f_* must converge exponentially fast to f_*.

(iii) In Appendix A, we prove a generalization of Lyubich's Small Orbits Theorem [Lyu99, §2] that works even in the presence of both attracting and repelling eigenvalues. (In the pacman case [DLS20], the foliation induced by the multiplier of the α-fixed point removes the need to generalize the Small Orbits Theorem.)

These three ingredients will imply that $D\mathcal{R}_{f_*}$ has no neutral eigenvalues. To show that a repelling direction exists, we apply [Lim23b, Theorem B], a result on combinatorial rigidity of unicritical Herman quasicircles of a nice class of rational maps.

The second half of the paper is dedicated to proving that $D\mathcal{R}_{f_*}$ has exactly one repelling eigenvalue. In Section §5, we show that for any map f on the local unstable manifold, the maximal extension of the pre-corona associated to f is a commuting pair of σ-proper maps $\mathbf{F} = (\mathbf{f}_\pm : \mathbf{X}_\pm \to \mathbb{C})$. The proof relies on a technical lemma, which we prove separately in Appendix C due to its length. This allows us to identify $\mathcal{W}^u_{\text{loc}}$ with $\boldsymbol{\mathcal{W}}^u_{loc}$, the holomorphic family of transcendental maps $\mathbf{F}$.

Given $\mathbf{F} = (\mathbf{f}_\pm) \in \boldsymbol{\mathcal{W}}^u_{loc}$ and $n \leq 0$, we set $\mathbf{F}_n = \mathcal{R}^n \mathbf{F}$ and denote by $\mathbf{F}_n^\# = (\mathbf{f}_{n,\pm}^\#)$ the rescaled version of $\mathbf{F}_n$ such that $\mathbf{f}_\pm$ are iterates of $\mathbf{f}_{n,\pm}^\#$. We identify $\mathbf{F}$ as a cascade, that is, the semigroup $\left(\mathbf{F}^{\geq 0}, \circ\right)$ generated by $\mathbf{f}_{n,\pm}^\#$ for all $n \leq 0$. The cascade $\mathbf{F}^{\geq 0}$ is isomorphic to a dense sub-semigroup $(\mathbf{T}, +)$ of $\mathbb{R}_{\geq 0}$ and elements of $\mathbf{F}^{\geq 0}$ can be written as $\mathbf{F}^P$ for $P \in \mathbf{T}$. We define the *finite-time escaping set* $\mathbf{I}_{<\infty}(\mathbf{F})$ of $\mathbf{F}$ to be the set of points in the dynamical plane of $\mathbf{F}$ that is not in the domain of $\mathbf{F}^P$ for some $P \in \mathbf{T}$. Most of Section 6 mirrors [DL23, §5] (aside from a number of combinatorial differences), in which we study the structure of the finite-time escaping set of the renormalization fixed point $\mathbf{F}_*$. We construct external rays and deduce its tree structure using their branch points, which are called *alpha-points*. These rays induce a puzzle structure partitioning the whole dynamical plane.

In Section §7, we apply the external structure of $\mathbf{F}_*$ to obtain item (4) in Theorem A. In short, this is done in a number of steps.

(i) We prove that $\mathbf{I}_{<\infty}(\mathbf{F})$ carries no invariant line field and locally moves holomorphically unless it contains a pre-critical point.

(ii) We observe that any map $\mathbf{F}$ close to $\mathbf{F}_*$ inherits most of the external structure of $\mathbf{F}_*$, which we use to study the *infinite-time escaping set*

$$\mathbf{I}_\infty(\mathbf{F}) := \{z \in \mathbb{C}\backslash\mathbf{I}_{<\infty}(\mathbf{F}) \,:\, \mathbf{F}^P(z) \to \infty\}.$$

By adapting the ideas from Rempe [Rem09], we study the motion of points z in the Julia set whose orbit $\mathbf{F}^P(z)$ remain close to ∞ for all P. We then show that $\mathbf{I}_\infty(\mathbf{F})$ also carries no invariant line field and locally moves holomorphically unless it contains a pre-critical point.

(iii) We show that there exist hyperbolic cascades $\mathbf{F}$ arbitrarily close to $\mathbf{F}_*$. When $\mathbf{F}$ is hyperbolic, the Julia set of $\mathbf{F}$ is the union of $\mathbf{I}(\mathbf{F}) := \mathbf{I}_{<\infty}(\mathbf{F}) \cup \mathbf{I}_\infty(\mathbf{F})$ and a zero measure set.

These three ingredients allow us to deduce that the deformation space of a hyperbolic $\mathbf{F}$ can only be supported on the Fatou set. Since $\mathbf{F}$ is unicritical, we conclude that the parameter space $\boldsymbol{\mathcal{W}}^u_{loc}$ is one dimensional.

This paper contains three appendices. Appendix A provides a generalization of Lyubich's Small Orbits Theorem [Lyu99, §2]. The main addition here is the application of two invariant cones rather than just one. Appendix B is a review of results in sector renormalization from [DLS20, DL23], which is a toy model of the induced action of $\mathcal{R}$ on the invariant quasicircle. Appendix C provides the proof of Lemma 5.7, the key towards attaining the transcendental extension; this is an analog of [DLS20, Key Lemma 4.8] in our setting.

1.4. **Acknowledgements.** I would like to thank Dzmitry Dudko for numerous discussions and valuable suggestions on this project. I cannot thank him enough for his kindness and constant encouragement. This project has been partially supported by the NSF grant DMS 2055532 and by Simons Foundation International, LTD.

2. Corona renormalization operator

Throughout this paper, we fix a pair of positive integers $d_0, d_\infty \geq 2$ and set $d := d_0 + d_\infty - 1$.

2.1. **(d_0, d_∞)-critical coronas.** For any open annulus A compactly contained in $\mathbb{C}$, we label the boundary components of A by $\partial^0 A$ and $\partial^\infty A$, and make the convention that $\partial^\infty A$ is the outer boundary, i.e. the one that is closer to ∞. We also say that another annulus A' is *essentially* contained in A if A' is a deformation retract of A.

Definition 2.1. A (d_0, d_∞)-*critical corona* is a map $f : U \to V$ between two bounded open annuli in $\mathbb{C}$ with the following properties.

(1) The boundary components of both U and V are Jordan curves, and U is compactly and essentially contained in V.
(2) There is a proper arc $\gamma_1 \subset V$ connecting $\partial^0 V$ and $\partial^\infty V$ such that the preimage $f^{-1}(\gamma_1)$ is disjoint from γ_1 and is a union of $2d-1$ pairwise disjoint arcs

$$\gamma_0 \subset U, \quad \gamma^0_1, \ldots, \gamma^0_{2(d_0-1)} \subset \partial^0 U, \quad \gamma^\infty_1, \ldots, \gamma^\infty_{2(d_\infty-1)} \subset \partial^\infty U.$$

(3) $f : U \to V$ is holomorphic and $f : U\backslash\gamma_0 \to V\backslash\gamma_1$ is a degree d covering map branched at a unique critical point c_0.

The arc γ_1 is called the *critical arc* of f. See Figure 3 for an illustration.

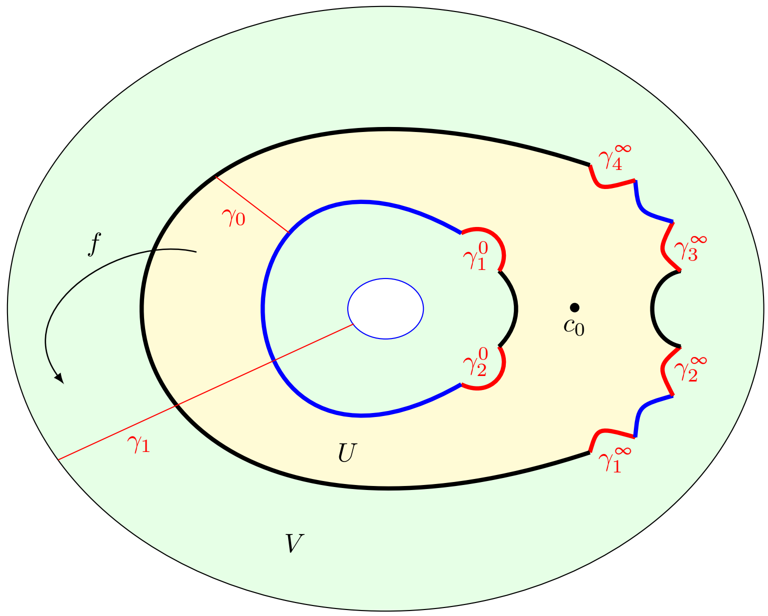

FIGURE 3. A (2,3)-critical corona

Let $f : U \to V$ be a (d_0, d_∞)-critical corona. For any $\bullet \in \{0, \infty\}$, we divide the boundary component $\partial^\bullet U$ into

$$\partial^\bullet_{\mathrm{L}} U := \partial^\bullet U \cap f^{-1}(\partial^\bullet V) \quad \text{ and } \quad \partial^\bullet_{\mathrm{F}} U := \partial^\bullet U \backslash f^{-1}(\partial^\bullet V)$$

according to whether or not it is mapped to the same side the annulus. Each of the above consists of $d_\bullet - 1$ components. Set

$$\partial_{\mathrm{L}} U := \partial^0_{\mathrm{L}} U \cup \partial^\infty_{\mathrm{L}} U \quad \text{ and } \quad \partial_{\mathrm{F}} U := \partial^0_{\mathrm{F}} U \cup \partial^\infty_{\mathrm{F}} U.$$

We call $\partial_{\mathrm{L}} U$ the *legitimate boundary* of U and $\partial_{\mathrm{F}} U$ the *forbidden boundary* of U.

For each $\bullet \in \{0, \infty\}$, we properly embed a collection $\mathscr{R}^\bullet$ of $d_\bullet - 1$ pairwise disjoint rectangles within $V \backslash \overline{U}$ such that the union $B^\bullet$ of their bottom horizontal sides is precisely the legitimate boundary $\partial^\bullet_{\mathrm{L}} U$ and the union $T^\bullet$ of their top horizontal sides is a subset of $\partial^\bullet V$. Let us lift $\mathscr{R}^\bullet$ under f such that their top sides are within the legitimate boundary of U. As we repeat this lifting procedure, we obtain a lamination out of the iterated lifts, and its leaves will be called *external ray segments*.

An infinite chain of external ray segments is called an *external ray* of the corona f. We say that γ is an *inner* external ray if γ intersects B^0, and an *outer* external ray if instead γ intersects B^∞.

For each $\bullet \in \{0, \infty\}$, define the map $\pi_\bullet : B^\bullet \to T^\bullet$ sending the bottom endpoint of each leaf of $\mathscr{R}^\bullet$ to the corresponding top endpoint. Consider the partially defined $d_\bullet$ to one self map $\phi_\bullet := \pi_\bullet^{-1} \circ f$ on $B^\bullet$. Denote by $\mathcal{A}^\bullet$ the set of points of $B^\bullet$ which are invariant under $\phi_\bullet$. Let us identify $\mathbb{T}$ with the quotient $\mathbb{R}/\mathbb{Z}$. There is a semiconjugacy $\theta_\bullet : \mathcal{A}^\bullet \to \mathbb{T}$ between $\phi_\bullet : \mathcal{A}^\bullet \to \mathcal{A}^\bullet$ and the multiplication map

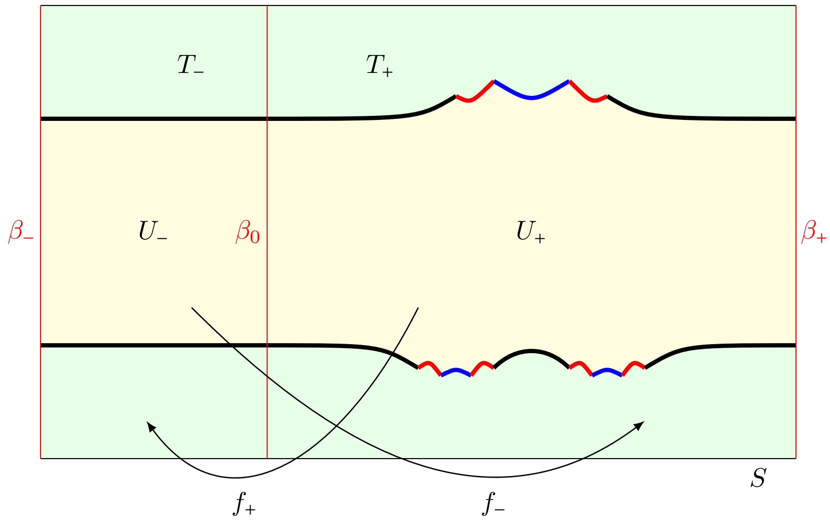

FIGURE 4. A (2,3)-critical pre-corona. It projects to the corona in Figure 3 after gluing β_+ and β_-

$\mathbb{T} \to \mathbb{T}, x \mapsto d_\bullet x \pmod 1$, which is unique up to conjugation with addition by multiples of $\frac{1}{d_\bullet - 1}$.

Given an external ray γ of f, we denote the image by

$$f(\gamma) := f(\gamma \cap U)$$

which is also an external ray of f by definition. The *external angle* of γ is the angle $\theta_\bullet(x)$ where x is the unique point of intersection of γ and $B^\bullet$ for some $\bullet \in \{0, \infty\}$.

2.2. Corona renormalization.

Definition 2.2. A (d_0, d_∞)-*critical pre-corona* is a pair of holomorphic maps

$$F = (f_- : U_- \to S,\ f_+ : U_+ \to S)$$

satisfying the following properties.

(1) S is a topological rectangle with vertical sides β_- and β_+.
(2) β_0 is a vertical arc in S dividing S into subrectangles T_- and T_+, where $\beta_\pm \subset \partial T_\pm$ and $U_\pm$ is a subrectangle of $T_\pm$ with vertical sides contained in $\beta_\pm$ and β_0.
(3) There is a gluing map $\psi : \overline{S} \to \overline{V}$ such that $\psi(\beta_-) = \psi(\beta_+)$, ψ is conformal on a neighborhood of S and injective on $S \backslash (\beta_- \cup \beta_+)$, and ψ projects F into a (d_0, d_∞)-critical corona with critical arc $\psi(\beta_\pm)$.

The gluing map ψ will also be called the *renormalization change of variables* of F. It glues together $f_+(x) \in \beta_-$ and $f_-(x) \in \beta_+$ for every x in $\beta_0 \cap \partial U_\pm$. See Figure 4.

Definition 2.3. A corona $f : U \to V$ is *renormalizable* if there exists a pre-corona

$$F = (f^{k_-} : U_- \to S,\ f^{k_+} : U_+ \to S)$$

on a rectangle $S \subset V$ such that f^{k_-} and f^{k_+} are the first return maps back to S and

$$\mathbf{\Delta}_F = \bigcup_{i=0}^{k_--1} \overline{f^i(U_-)} \cup \bigcup_{j=0}^{k_+-1} \overline{f^j(U_+)}$$

is a closed annulus essentially contained in U. We call F the *pre-renormalization* of f, k_- and k_+ the *return times* of F, and $\mathbf{\Delta}_F$ the *renormalization tiling* of F. The corona obtained by projecting F under its gluing map is called the *renormalization* of f.

Example 2.4 (Prime renormalization)**.** We say that the renormalization of a corona $f : U \to V$ is *prime* if $k_- + k_+ = 3$. Below is an example of a prime corona renormalization.

Assume that the arcs γ_0, γ_1, and $\gamma_2 := f(\gamma_1)$ are pairwise disjoint. Denote by S_1 the open quadrilateral obtained by cutting V along $\gamma_1 \cup \gamma_2$ which does not contain γ_0. Let us assume further that S_1 does not contain the critical value nor the forbidden boundary of U.

Let us remove S_1 from the dynamical plane. We define $\hat{V}$ to be the Riemann surface with boundary obtained from $\overline{V} \backslash S_1$ by gluing $\gamma_1' := f^{-1}(\gamma_2) \cap \gamma_1$ and its image γ_2 along f. In other words, there is a quotient map $\psi : \overline{V} \backslash S_1 \to \hat{V}$ that is conformal on the interior and $\psi(z) = \psi(f(z))$ for all $z \in \gamma_1'$. We embed the abstract Riemann surface $\hat{V}$ into the plane.

The prime renormalization of f is defined by the induced first return map of f on $\hat{V}$. More precisely, consider the lift S_0 of S_1 under f attached to γ_1. The piecewise holomorphic map

$$\begin{cases} f(z), & \text{if } z \in U \backslash (S_1 \cup f^{-1}(S_1)), \\ f^2(z), & \text{if } z \in S_0 \cap f^{-1}(U). \end{cases}$$

descends via ψ into a corona $\hat{f} : \hat{U} \to \hat{V}$ with critical ray $\hat{\gamma}_1 = \psi(\gamma_1')$.

2.3. **Banach neighborhood.** In what follows, every unicritical holomorphic map $f : U \to V$ under consideration will be assumed to admit a slightly larger domain $\tilde{U}$ with piecewise smooth boundary such that $\tilde{U}$ compactly contains U and f extends to a unicritical holomorphic map on $\tilde{U}$ extending continuously to $\partial \tilde{U}$. We define a *Banach neighborhood* of f to be a neighborhood of f of the form $N_{\tilde{U}}(f, \varepsilon)$, which we define to be the space of holomorphic maps $g : \tilde{U} \to \mathbb{C}$ that extend continuously to $\partial \tilde{U}$, admit a single critical point in $c_0(g)$, and

$$\sup_{z \in \tilde{U}} |f(z) - g(z)| < \varepsilon.$$

We equip $N_{\tilde{U}}(f, \varepsilon)$ with the sup norm over $\tilde{U}$.

Lemma 2.5. *Let $f : U \to V$ be a (d_0, d_∞)-critical corona. For sufficiently small $\varepsilon >$ 0, there is a holomorphic motion ∂U_g of ∂U over $g \in N_{\tilde{U}}(f, \varepsilon)$ such that $g : U_g \to V$ is a (d_0, d_∞)-critical corona with the same codomain V and critical arc γ_1.*

Proof. Let A_δ be the δ-neighborhood of ∂U, where $\delta > 0$ is picked small enough such that A_δ contains no critical points of f. For sufficiently small ε, the derivative of $g \in N_{\tilde{U}}(f, \varepsilon)$ is uniformly bounded and non-vanishing on A_δ, and so g has no critical points in A_δ. Thus, we have a well-defined map $\tau_g : \partial U \to A_\delta$ such that $\tau_f = \mathrm{Id}$ and $f = g \circ \tau_g$ on ∂U. Since f has no critical value along ∂U, $\tau_g(z)$ is injective in z and holomorphic in g. Therefore, we have a holomorphic motion of ∂U, and $\tau_g(\partial U)$ bounds an open annulus U_g on which $g : U_g \to V$ is a well-defined (d_0, d_∞)-critical corona with the same critical arc. □

Theorem 2.6. *Suppose a unicritical holomorphic map $f : U \to V$ admits a pre-corona which projects to a corona $\hat{f} : \hat{U} \to \hat{V}$ via a quotient map $\psi_f : S_f \to \hat{V}$. For sufficiently small $\varepsilon > 0$, there is a compact analytic renormalization operator $\mathcal{R}$ on a Banach neighborhood $N_{\tilde{U}}(f, \varepsilon)$ such that $\mathcal{R}f = \hat{f}$ and for each $g \in N_{\tilde{U}}(f, \varepsilon)$,*

1. *g admits a pre-corona which projects to the corona $\mathcal{R}g : \hat{U}_g \to \hat{V}$, and*
2. *the domain $\partial \hat{U}_g$ and the associated gluing map ψ_g depend holomorphically on g.*

Proof. There exists a pre-corona $F = \left(f^{k_\pm} : U_\pm \to S\right)$ and a quotient map ψ_f projecting F to $\hat{f}$. Recall the arcs $\beta_\pm$ and β_0 corresponding to F. For $g \in N_{\tilde{U}}(f, \varepsilon)$, consider the map $\tau_g : \beta_0 \cup \beta_\pm \to \mathbb{C}$ defined by setting τ_g to be the identity map on β_0 and the composition $g^{k_\mp} \circ f^{-k_\mp}$ on $\beta_\pm$; this is an equivariant holomorphic motion of $\beta_0 \cup \beta_\pm$ for sufficiently small $\varepsilon > 0$. By λ-lemma, τ_g extends to a holomorphic motion of S over a neighborhood of f.

Let μ_g be the Beltrami differential of τ_g. Define a global Beltrami differential ν_g by setting $\nu_g = (\psi_f)_* \mu_g$ on $\hat{V}$ and $\nu_g \equiv 0$ outside of $\hat{V}$. Integrate ν_g to obtain a unique quasiconformal map ϕ_g fixing ∞, the critical point of f, and the critical value of f. Then, $\psi_g := \phi_g \circ \psi_f \circ \tau_g^{-1}$ is a conformal map on $S_g := \tau_g(S_f)$ depending holomorphically on g.

The gluing map ψ_g projects the pair $\left(g^{k_-}, g^{k_+}\right)$ on S_g to a map $\hat{g}$ close to $\hat{f}$. By Lemma 2.5, $\hat{g}$ restricts to a corona that has the same range as $\hat{f}$ and depends analytically on g. This yields an analytic operator $g \mapsto \hat{g}$. To make this operator compact, we modify it as follows. Pick another annulus U' where $U \Subset U' \Subset \tilde{U}$. We define $\mathcal{R}$ on $N_{\tilde{U}}(f, \varepsilon)$ to be the renormalization of the restriction of g to U'. □

3. Rotational coronas

Throughout this section, we fix a bounded type irrational $\theta \in \Theta_{bdd}$.

Definition 3.1 (Inner and outer criticalities)**.** Consider a quasicircle $\mathbf{H} \subset \mathbb{C}$ and denote the bounded and unbounded components of $\hat{\mathbb{C}} \backslash \mathbf{H}$ by Y^0 and Y^∞ respectively. We say that $f : \mathbf{H} \to \mathbf{H}$ is a *(d_0, d_∞)-critical quasicircle map* if it is a critical quasicircle map where for any $\bullet \in \{0, \infty\}$ and any point $z \in Y^\bullet$ close to the critical value of f, there are exactly $d_\bullet$ preimages of z in $Y^\bullet$ that are close to the critical point of f.

When a holomorphic map f is given, we also say that an invariant quasicircle $\mathbf{H} \subset \mathbb{C}$ is a *(d_0, d_∞)-critical Herman quasicircle* if $f : \mathbf{H} \to \mathbf{H}$ is a (d_0, d_∞)-critical quasicircle map. The term *Herman quasicircle* originates from [Lim23a] and is meant to acknowledge that its first examples arise from degeneration of Herman rings.

Definition 3.2. A corona $f : U \to V$ is a *rotational corona* if

1. U essentially contains a Herman quasicircle $\mathbf{H}$ that passes through the unique critical point of f;

(2) the critical arc γ_1 intersects $\mathbf{H}$ precisely at one point $m(f)$, which we call the *marked point* of f, which splits γ_1 into an inner external ray R^0 and an outer external ray R^∞.

By design, if a (d_0, d_∞)-critical corona is rotational, then it admits a (d_0, d_∞)-critical Herman quasicircle.

3.1. **Realization of rotational coronas.** Consider the family of degree d rational maps $\{F_c\}_{c\in\mathbb{C}^*}$ where

$$F_c(z) := -c\,\frac{\sum_{j=d_0}^{d}\binom{d}{j}\cdot(-z)^j}{\sum_{j=0}^{d_0-1}\binom{d}{j}\cdot(-z)^j}. \tag{3.1}$$

By [Lim23a, Proposition 10.1], this family is characterized by the property that F_c has critical points at 0, ∞, and 1 with local degrees d_0, d_∞, and d respectively, and that $F_c(0) = 0$, $F_c(\infty) = \infty$, and $F_c(1) = c$.

Theorem 3.3 ([Lim23a, Lim23b])**.** *There exists a unique parameter $c = c(\theta) \in \mathbb{C}^*$ such that F_c admits a (d_0, d_∞)-critical Herman quasicircle $\mathbf{H}$ with rotation number θ which passes through* 1*.*

Consider $f := F_c$ and $\mathbf{H}$ from the theorem above. Examples of the Julia set of f can be found in Figure 1. The map f will serve as our model map. Using external rays, we will show in this subsection that f is corona renormalizable.

For any $n \ge 1$, we refer to the closure of a component of $f^{-n}(\mathbf{H})\backslash f^{-(n-1)}(\mathbf{H})$ as a *bubble* of *generation* n. Every bubble B of generation n is a quasicircle admitting a unique point, which we will call the *root* of B, that lies on the pre-critical set $f^{-(n-1)}(1)$. We call a bubble B of generation n an *outer bubble* (resp. *inner bubble*) if the bubbles B, $f(B), \dots, f^{n-1}(B)$ all lie in the connected component of $\hat{\mathbb{C}}\backslash\mathbf{H}$ containing ∞ (resp. 0).

A *limb* of generation one is the closure of a connected component of $J(f)\backslash\{1\}$ that is disjoint from $\mathbf{H}$. A *filled limb* $\hat{L}$ of generation one is the hull of a limb L of generation one, that is, $\hat{\mathbb{C}}\backslash\hat{L}$ is the unbounded connected component of $\hat{\mathbb{C}}\backslash L$. In general, a (filled) limb of generation $n \ge 1$ is the connected component of the preimage under f^{n-1} of a (filled) limb of generation one. A (filled) limb of generation n contains a unique bubble of generation n, which we will call the *core bubble* of the limb. The *root* of a (filled) limb is the root of its core bubble. We call a (filled) limb an *outer/inner (filled) limb* if its core bubble is an outer/inner bubble.

Let us denote by A_0 and A_∞ the immediate attracting basins of 0 and ∞.

Lemma 3.4. *The boundary of A_0 is the closure of the union of $\mathbf{H}$ and all outer bubbles of f, whereas the boundary of A_0 is the closure of the union of $\mathbf{H}$ and all inner bubbles of f. and A_∞ are locally connected. For any $\varepsilon > 0$, all but finitely many inner and outer limbs of f have diameter at most ε.*

Proof. Denote by Y^0 and Y^∞ the connected components of $\hat{\mathbb{C}}\backslash\mathbf{H}$ containing 0 and ∞ respectively. Perform Douady-Ghys surgery [Ghy84, Dou87] (see also [BF14, §7.2]) along $\mathbf{H}$ to replace the dynamics of f in Y^0 with a rotation disk and obtain a degree d_∞ unicritical polynomial P_∞ whose critical point lies in the boundary of an invariant Siegel disk Z_∞ of P_∞. The maps $f|_{\overline{Y^\infty}}$ and $P_\infty|_{\hat{\mathbb{C}}\backslash Z_\infty}$ are quasiconformally conjugate, and this conjugacy sends A_∞ onto the immediate basin of ∞ of P_∞. In particular, the external boundary of the filled outer limbs of f are quasiconformally equivalent to the limbs of P_∞. The work of [Pet96] (or more generally [WYZZ21]) guarantees that the Julia set of P_∞ is locally connected, and so any infinite sequence of limbs of P_∞ must shrink to a point. Therefore, for any $\varepsilon > 0$, all but finitely many outer limbs of f have diameter at most ε. By swapping the roles of 0 and ∞, we obtain the same result for inner limbs. □

Remark 3.5. The lemma above states that both ∂A_0 and ∂A_∞ are locally connected. In fact, the whole Julia set of f is locally connected. In case $(d_0, d_\infty) = (2,2)$, this was proven by Petersen [Pet96, §4]. For arbitrary criticalities (d_0, d_∞), the availability of complex bounds [Lim23b, §6.3] facilitates a direct generalization of Petersen's proof.

For $\bullet \in \{0,\infty\}$, consider the Böttcher conjugacy $B_\bullet : (A_\bullet, \bullet) \to (\mathbb{D}, \bullet)$ between f and the power map $z^{d_\bullet}$. An *external ray* in $A_\bullet$ of angle $t \in \mathbb{R}/\mathbb{Z}$ is defined by

$$\left\{B_\bullet^{-1}(re^{2\pi i t}) \,:\, 0 < r < 1\right\},$$

and an *equipotential* in $A_\bullet$ of level λ is the analytic Jordan curve defined by

$$\left\{B_\bullet^{-1}(z) \,:\, |z| = e^{-\lambda}\right\}.$$

External rays and equipotentials form a pair of f-invariant transverse foliations of $A_\bullet$. According to Lemma 3.4, every external ray in $A_\bullet$ lands at a point on $\partial A_\bullet$. Every point x in $\partial A_\bullet$ is the landing point of exactly one external ray in $A_\bullet$, except when x is a critical point or its iterated preimage in which case it is the landing point of $d_\bullet$ external rays in $A_\bullet$.

Consider the operator R_{prm} from Appendix B, which encodes how rotation number is transformed under sector renormalization.

Lemma 3.6. *For any point $x \in \mathbf{H}$ that is not a pre-critical point of f, any $\varepsilon > 0$, and any sufficiently high $n \in \mathbb{N}$, there is a rotational pre-corona*

$$P = \left(f_- := f^{k_-} : U_- \to S,\; f_+ := f^{k_+} : U_+ \to S\right)$$

around x such that

1. *P has rotation number $R^n_{prm}(\theta)$;*
2. *every external ray segment of P is within an external ray of P;*
3. *the union $\bigcup_{\diamond\in\{-,+\}} \bigcup_{i=0}^{k_\diamond - 1} f^i(U_\diamond)$ lies in the ε-neighborhood of $\mathbf{H}$.*

Proof. For every integer $i \in \mathbb{Z}$, let $x_i := (f|_{\mathbf{H}})^i(x)$. By Lemma B.2, for all $n \geq 1$, there exist return times $\mathbf{a}_n$, $\mathbf{b}_n$ such that the commuting pair

$$\left(f^{\mathbf{a}_n}|_{[x_{\mathbf{b}_n}, x_0]}, f^{\mathbf{b}_n}|_{[x_0, x_{\mathbf{a}_n}]}\right)$$

is a sector pre-renormalization of $f|_{\mathbf{H}}$ with rotation number $R^n_{\mathrm{prm}}(\theta)$.

Let $k_- := \mathbf{a}_n$ and $k_+ := \mathbf{b}_n$, and let us pick a small constant $\lambda > 0$. For $\bullet \in \{0,\infty\}$, denote by $E^\bullet$ the equipotential in $A_\bullet$ of level λ, and by $R^\bullet_-$, $R^\bullet$, and $R^\bullet_+$ the external rays in $A_\bullet$ that land at x_{k_+}, $x_{k_- + k_+}$, and x_{k_-} respectively. Then, the union $\bigcup_{\bullet\in\{0,\infty\}} R^\bullet_\mp \cup R^\bullet \cup E^\bullet$ encloses a rectangle $S_\pm$ containing the interval $[x_{k_\pm}, x_{k_- + k_+}] \subset \mathbf{H}$.

Let $I_- := [x_{k_+}, x_0]$ and $I_+ := [x_0, x_{k_-}]$. Precisely one of the two intervals, say I_- without loss of generality, contains a critical point of f^{k_-}. The rectangle $S_\pm$ lifts under $f^{k_\pm}$ to a topological disk $\Upsilon_\pm$ containing $I_\pm$, where $f^{k_-} : \Upsilon_- \to S_-$ is a degree

d branched covering map and $f^{k_+} : U_+ \to S_+$ is univalent. Let us denote by U_- the union of Υ_- and all the lifts of S_+ under f^{k_-} that are disjoint from $\mathbf{H}$ and touching Υ_- on the boundary. Set $U_+ := \Upsilon_+$ and $S = S_- \cup S_+$. Then,

$$\left(f^{k_-} : U_- \to S, f^{k_+} : U_+ \to S\right)$$

is a (d_0, d_∞)-critical pre-corona with rotation number $R^n_{\mathrm{prm}}(\theta)$.

Let us embed the restriction of external rays of f in $S \backslash U$ where $U := U_- \cup U_+$. Notice that the boundaries of U_- and U_+ contain equipotential segments of different levels. Assume without loss of generality that the equipotential segments in U_- have higher level. To satisfy (2), we can truncate a pair of small topological triangles near two vertices of the rectangle S_+, one where R^0_+ meets E^0 and the other where R^∞_+ meets E^∞. We will also truncate preimages of these triangles under f^{k_-} in U_-. Replace U and S with the new truncated domains. Then, every point in the legitimate boundary of U is now a landing point of an external ray segment, and (2) follows.

We claim that (3) follows from taking n to be sufficiently large and λ to be sufficiently small. Indeed, if $z \in U_\pm$ intersects an external ray landing at a point $w \in J(f) \cap U_\pm$, then the orbits of z and w remain close under iteration f^i for $i = 1, \ldots, k_\pm$. Suppose $z \in U_\pm$ is outside of $A_0 \cup A_\infty$. Then, it must lie within some filled limb $\hat{L}$ rooted at some pre-critical point $c_{-j} := (f|_{\mathbf{H}})^{-j}(1)$ for some $j \geq 0$. If c_{-j} is not the unique critical point of f^{k_-}, then the forward images $\hat{L}$, $f(\hat{L})$, $\ldots, f^{k_\pm}(\hat{L})$ must remain small due to Lemma 3.4. If c_{-j} is the critical point of f^{k_-} in U_-, then we must have $0 < j < k_-$. In the latter case, the image $f^j(U_-)$ must remain in a small neighborhood of the critical point $c_0 = 1$ of f as we take λ to be small and n to be large. Therefore, the forward orbit z, $f(z)$, $\ldots$, $f^j(z)$ must be close to $\mathbf{H}$. □

In our previous work, we proved a rigidity theorem for critical quasicircle maps.

Theorem 3.7 ([Lim23b, Theorem F]). *Every two (d_0, d_∞)-critical quasicircle maps of the same bounded type rotation number are quasiconformally conjugate on some neighborhood of their Herman curves.*

Together with Lemma 3.6, we have the following result.

Corollary 3.8. *Any (d_0, d_∞)-critical quasicircle map $g : \mathbf{H}_g \to \mathbf{H}_g$ with bounded type rotation number is corona renormalizable, that is, there is a (d_0, d_∞)-critical rotational pre-corona which is an iterate of g near $\mathbf{H}$.*

Proof. Given any (d_0, d_∞)-critical quasicircle map g of bounded type rotation number, Theorem 3.7 asserts that there is a global quasiconformal map ϕ conjugating g on some neighborhood W of its Herman curve with $f := F_c$. By Lemma 3.6, f admits a pre-corona P with range contained within $\phi(W)$. Then, g admits a (d_0, d_∞)-critical pre-corona of the form $\phi^{-1} \circ P \circ \phi$. □

3.2. **Quasiconformal rigidity.** Given a critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$ with critical point $c \in \mathbf{H}$, there is a unique conjugacy $h_f : (\mathbf{H}, c) \to (\mathbb{T}, 1)$ between f and the rigid rotation R_θ sending c to 1. We can endow $\mathbf{H}$ with the *combinatorial metric*, which is the pullback of the normalized Euclidean metric of $\mathbb{T}$ under h_f and thus the unique normalized f-invariant metric of $\mathbf{H}$. For any point $z \in \mathbf{H}$, the *combinatorial position* of z is the point $h_f(z)$ on the unit circle.

We say that two (d_0, d_∞)-critical rotational coronas f_1 and f_2 are *combinatorially equivalent* if

(1) they have the same rotation number,
(2) their marked points $r(f_1)$ and $r(f_2)$ have the same combinatorial position, and
(3) for $\bullet \in \{0, \infty\}$, the external rays $R^\bullet(f_1)$ and $R^\bullet(f_2)$ have the same external angles.

In this subsection, we will prove quasiconformal rigidity of rotational coronas.

Theorem 3.9. *Two combinatorially equivalent (d_0, d_∞)-critical rotational coronas with bounded type rotation number are quasiconformally conjugate.*

The proof below is an application of the pullback argument. Let us make a couple of technical preparations. Recall the model rational map f introduced in the previous subsection.

Definition 3.10. A *bubble chain* of f of generation $l \geq 1$ is an infinite sequence of bubbles $\{B_j\}_{j\geq 1}$ of f where B_1 has generation l and for all $j \geq 1$, B_j contains the root of B_{j+1} and the generation of B_j is strictly increasing in j. We say that a bubble chain $\{B_j\}_{j\geq 1}$

- ▷ is an *outer/inner* bubble chain if each B_j is an outer/inner bubble,
- ▷ is *periodic* of period p if there exists some $k \geq 1$ such that $f^p(B_{j+k}) = B_j$ for all $j \geq k$, and
- ▷ *lands* if the accumulation set $\bigcap_{m\geq n} \overline{\cup_{k\geq m} B_k}$ is a single point, which we call the *landing point* of the bubble chain.

We say that a periodic point z of f is an *outer* (resp. *inner*) periodic point if its orbit is contained in the connected component of $\hat{\mathbb{C}} \backslash \mathbf{H}$ containing ∞ (resp. 0).

Let us fix a rotational pre-corona $P = (f_\pm : U_\pm \to S)$ of f (which exists thanks to Lemma 3.6).

Definition 3.11. We define the *non-escaping set* $K(P)$ of P to be the set of points whose orbit under $f_\pm$ that never escapes $\overline{U_\pm}$. By spreading around $K(P)$, we define the *local non-escaping set* of f relative to P by

$$K^{loc}(f) := \bigcup_{n\geq 0} f^n(K(P)).$$

The set $K^{loc}(f)$ is precisely the set of points which does not escape from the tiling $\mathbf{\Delta}_P$ associated to P.

Lemma 3.12. *The set $K^{loc}(f)$ is a connected compact set, and it is equal to*

- ▷ *the closure of the set of periodic points of f in $K^{loc}(f)$;*
- ▷ *the closure of the set of points of $K^{loc}(f)$ that are contained in $\bigcup_{n\geq 1} f^{-n}(\mathbf{H})$.*

For every outer (resp. inner) periodic point z in $K^{loc}(f)$, there is a unique bubble chain in $K^{loc}(f)$ landing at z.

Proof. The first statement follows from the basic fact that as a rational map, the Julia set $J(f)$ can be characterized as either the set of points in $\hat{\mathbb{C}}$ that do not escape to 0 nor ∞ (which are the only non-repelling periodic points of f) or the closure of the set of repelling periodic points of f, or the closure of the iterated preimages of $\mathbf{H}$. The compactness of $K^{loc}(f)$ is clear, and the connectedness follows from the fact that, if we denote by $\mathbf{H}_P$ the invariant quasiarc of P, then for all $n \geq 1$, $P^{-n}(\mathbf{H}_P)$ is connected (in fact, it admits a tree structure). □

Let $g : U \to V$ be rotational corona that is combinatorially equivalent to f, and let us denote the Herman quasicircle by $\mathbf{H}'$. By Theorem 3.7, there is a quasiconformal conjugacy ϕ between g and f on some neighborhood W' of $\mathbf{H}'$ onto a neighborhood W of $\mathbf{H}$.

By Lemma 3.6, the precorona $P = (f_\pm : U_\pm \to S)$ of f can be assumed such that S is contained in W. The corona g also admits a pre-corona $P' = (g_\pm : U'_\pm \to S')$ contained in W' and it can be selected such that it is conjugate to P via ϕ. As such, we can define the non-escaping set $K(P')$ of P' in a similar way and spread it around to obtain the *local non-escaping set* $K^{loc}(g)$ of g relative to P'. The quasiconformal map ϕ induces a conjugacy between $g|_{K^{loc}(g)}$ and $f|_{K^{loc}(f)}$.

Let us define a *bubble* of g in $K^{loc}(g)$ to be the image under ϕ^{-1} of the intersection of a bubble of f with $K^{loc}(f)$. A *bubble chain* of g in $K^{loc}(g)$ is an infinite sequence of (non-empty) bubbles in $K^{loc}(g)$ defined in a similar way.

Let x be the marked point of g, and let R^∞ and R^0 be the outer and inner external rays of g landing at x. These rays make up the arc $\gamma_1(g)$.

Lemma 3.13. *Every outer (resp. inner) periodic point y of g in $K^{loc}(g)$ is the landing point of a unique periodic outer (resp. inner) bubble chain $\{B_j\}_{j\geq 1}$ in $K^{loc}(g)$ and a unique periodic outer (resp. inner) external ray R_y, which has the same period as y.*

Proof. Suppose y is an outer periodic point of g in $K^{loc}(g)$. As a periodic point, y does not lie on any bubble in $K^{loc}(g)$. By Lemmas 3.4 and 3.12, y must be the landing point of a unique outer bubble chain $\{B_j\}_{j\geq 1}$ in $K^{loc}(g)$.

Let p denote the period of y and let $k \in \mathbb{N}$ be the minimal number such that B_k has generation greater than p. By periodicity, the image of $\{B_j\}_{j\geq k}$ under g^p is also an outer bubble chain that is rooted at a point on $\mathbf{H}$ and lands at y. By Lemma 3.12, the bubble chain $\{B_j\}_{j\geq 1}$ is equal to its image $\{g^p(B_j)\}_{j\geq k}$, and thus it is p-periodic.

Let us pick iterated preimages R_l and R_r of R^∞ landing at points $x_{l,0}$ and $x_{r,0}$ on B_1 respectively such that the union $B_1 \cup R_l \cup R_r \cup \partial V$ bounds a topological rectangle D_0 that contains y and is disjoint from $\mathbf{H}$. Then, D_0 lifts under g^p to a rectangle D_{-1} containing y. Since the vertical sides of D_{-1} are external ray segments with a much smaller external angle difference compared to D_0, then D_{-1} is compactly contained in D. By Schwarz Lemma, $g^p : D_{-1} \to D_0$ is uniformly expanding with respect to the hyperbolic metric of D and y is its unique repelling fixed point.

For every $n \in \mathbb{N}$, let D_{-n} be the lift of D_0 under g^{pn} containing y. Consider the lifts $R_{l,n}$ and $R_{r,n}$ of R_l and R_r under g^{pn} which touch the boundary of D_{-n}; these are external rays landing at points $x_{l,n}$ and $x_{r,n}$ respectively, which are vertices of D_{-n}. By uniform expansion, $x_{l,n}$ and $x_{r,n}$ converge to y and the external rays $R_{l,n}$ and $R_{r,n}$ converge to a limiting external ray R_y, which is a p-periodic outer external ray. By Lemma 3.4, R_y must land at y. □

Let c_0 denote the critical point of g and for $n \in \mathbb{Z}$, let $c_n := (g|_{\mathbf{H}'})^n(c_0)$.

Lemma 3.14. *For any pre-critical point $c_{-t} \in \mathbf{H}'$ of g, there exist an outer periodic point y_t^∞ and an inner periodic point y_t^0 in $K^{loc}(g)$ such that for $\bullet \in \{0, \infty\}$, the unique bubble chain $\mathcal{B}_t$ landing at $y_t^\bullet$ is rooted at c_{-t}.*

Proof. We say that a bubble chain of f is in $K^{loc}(f)$ if its intersection with $K^{loc}(f)$ induces via ϕ a bubble chain in $K^{loc}(g)$. It is sufficient to prove the lemma in the case $g = f$.

Let us denote by $I_\varepsilon \subset \mathbf{H}$ the interval of combinatorial length ε centered at c_1. We will pick $\varepsilon > 0$ to be small enough such that the full preimage under f of I_ε is contained in the tiling $\mathbf{\Delta}_P$. Let us pick the first $s \in \mathbb{N}$ such that c_{-t-s} is contained in I_ε. Below, we will construct the desired outer periodic point y_t^∞, which will have period $p := s + t + 1$. The construction of y_t^0 can be done analogously.

First, let us pick two points x_l and x_r on $\mathbf{H}$ located on the left and right of c_{-t} respectively; we will assume that they are not in the grand orbit of the critical point of f, so there exists a unique pair of external rays R_l and R_r in A_∞ that land on x_l and x_r respectively. Consider the open rectangle D_0 cut out by the union of the interval $J_0 := [x_l, x_r] \subset \mathbf{H}$, the rays $R_l \cup R_r$, and an arc connecting R_l and R_r which is contained in the equipotential in A_∞ of some small level $\lambda > 0$.

Consider the intervals $J_{-j} := (f|_{\mathbf{H}})^{-j}(J_0)$ for $j \geq 1$. We assume that the combinatorial length of J_0 is small enough such that J_{-j} does not contain c_1 for all $j \in \{0, 1, \ldots, s-1\}$, and that J_{-s} is contained in $I_\varepsilon \backslash \{c_1\}$. Let J' be a connected component of $f^{-1}(J_{-s})$ that is contained in an outer bubble S_0 of generation one. Let D_1 be the lift of D_0 under f^{s+1} such that ∂D_1 contains the interval J'. Notice that J' is contained in $\mathbf{\Delta}_P$. By taking sufficiently small $\lambda > 0$, we can guarantee that D_1 is contained in $\mathbf{\Delta}_P$.

Next, consider the outer bubble B_1 rooted at c_{-t} such that $f^t(B_1) = S_0$. Let D_2 be the lift of D_1 under f^t that is attached to B_1. Since $D_1 \subset \mathbf{\Delta}_P$, then $D_2 \subset \mathbf{\Delta}_P$ too. Also, since D_2 is compactly contained in D_0, then $f^p : D_2 \to D_0$ is uniformly expanding with respect to the hyperbolic metric of D_0, and thus admits a unique repelling fixed point y_t^∞.

Let us construct the corresponding outer bubble chain landing at y_t^∞. For $j \geq 1$, we define the outer bubble B_{j+1} inductively to be the unique lift of B_j^∞ under f^p that is rooted at a point in $B_j \cap \overline{D_2}$. By uniform expansion, the roots of B_j converge to y_t^∞. Thus, $\{B_j\}_{j \geq 1}$ is the unique outer bubble chain in $K^{loc}(f)$ that lands at y_t^∞ and is rooted at c_{-t}. □

For each pre-critical point c_{-t} of g, consider the outer and inner periodic bubble chains $\mathcal{B}_t^\infty$ and $\mathcal{B}_t^0$ in $K^{loc}(g)$ given by Lemma 3.14. For each $\bullet \in \{0, \infty\}$, the landing point of $\mathcal{B}_t^\bullet$ is also the landing point of a unique external ray $R_t^\bullet$ of g. Consider

$$\mathcal{T}_t := \mathcal{B}_t \cup R_t \qquad \text{where} \qquad \mathcal{B}_t := \mathcal{B}_t^\infty \cup \mathcal{B}_t^0 \text{ and } R_t := R_t^\infty \cup R_t^0. \tag{3.2}$$

Lemma 3.15 (Rational approximation of $\gamma_1(g)$)**.** *For every $\varepsilon > 0$, there exists a pair of pre-critical points $c_{-t_l}, c_{-t_r} \in \mathbf{H}'$ located on the left and right of x respectively such that $\mathcal{T}_{t_l}$ and $\mathcal{T}_{t_r}$ are both in the ε-neighborhood of $\gamma_1(g)$.*

Proof. Since pre-critical points are dense on $\mathbf{H}'$, there exists a pair of pre-critical points c_{-t_l} and c_{-t_r} on the left and right of x, where the moments t_l and t_r grow as we require them to be arbitrarily close to x. Due to Lemma 3.4, the bubble chains within $\mathcal{T}_{t_l}$ and $\mathcal{T}_{t_r}$ shrink as we get close to x. The outer (resp. inner) external rays within $\mathcal{T}_{t_l}$ and $\mathcal{T}_{t_r}$ are also close to R^∞ (resp. R^0) because their external angles are close to that of R^∞. □

We are now ready to run the pullback argument.

proof of Theorem 3.9. Let $g_1 : U_1 \to V_1$ and $g_2 : U_2 \to V_2$ be two combinatorially equivalent (d_0, d_∞)-critical rotational coronas with rotation number $\theta \in \Theta_{bdd}$. Let f be the rational map from Theorem 3.3 which admits a (d_0, d_∞)-critical Herman quasicircle $\mathbf{H}$ with rotation number θ. From the previous discussion, for $i \in \{1, 2\}$, there is a quasiconformal conjugacy ϕ_i between g_i and f on some neighborhood W_i of the Herman quasicircle $\mathbf{H}_i$ of g_i onto a neighborhood W of $\mathbf{H}$.

We fix a precorona $P = (f_\pm : U_\pm \to S)$ of f where S is contained in W, and let $P_i = (g_\pm : U_{i,\pm} \to S_i)$ be the corresponding pre-corona of g_i conjugate to P via ϕ_i. We consider the local non-escaping set $K^{loc}(g_i)$ of g_i relative to P_i. The quasiconformal map $\phi_2 \circ \phi_1^{-1} : W_1 \to W_2$ restricts to a conjugacy $h : K^{loc}(g_1) \to K^{loc}(g_2)$ between g_1 and g_2.

For $i \in \{1, 2\}$ and $t \in \{t_l, t_r\}$, consider the sets $\mathcal{T}_t(g_i) = \mathcal{B}_t(g_i) \cup R_t(g_i)$ from Lemma 3.15 which approximate the critical arc $\gamma_1(g)$. By design, we can arrange such that for each $t \in \{t_l, t_r\}$, $\phi_2 \circ \phi_1^{-1}$ sends $\mathcal{B}_t(g_1)$ to $\mathcal{B}_t(g_2)$, and the outer/inner rays in $R_t(g_1)$ and $R_t(g_2)$ have the same external angles. For $i \in \{1, 2\}$, consider the union

$$Z_i = K^{loc}(g_i) \cup \bigcup_{n \geq 0} g_i^n (R_{t_l} \cup R_{t_r}).$$

Clearly, Z_i is forward invariant and $V_i \backslash Z_i$ consists of finitely many connected components. Since $R_t(g_1)$ and $R_t(g_2)$ have the same external angles, h extends to a quasiconformal map $h : V_1 \to V_2$ that is equivariant on $Z_i \cup \partial_{\mathrm{L}} U_1$.

Let us define a new domain $\hat{U}_1$ out of U_1 by replacing the forbidden boundary $\partial_{\mathrm{F}} U_1$ with some set $\partial_{\mathrm{F}} \hat{U}_1$ of curves slightly outside of $\partial_{\mathrm{F}} U_1$ such that the image $g_1(\partial_{\mathrm{F}} \hat{U}_1)$ is now contained inside of $\mathbf{H}_1 \cup \mathcal{T}_{t_l}(g_1) \cup \mathcal{T}_{t_r}(g_1)$. In the same manner, we replace U_2 with a slightly larger disk $\hat{U}_2$ such that $h|_{Z_1}$ lifts to a conjugacy between $g_1|_{\partial \hat{U}_1}$ and $g_2|_{\partial \hat{U}_2}$.

We can now run the pullback argument. Set $h_0 := h$ and we inductively construct quasiconformal maps $h_n : V_1 \to V_2$ such that

$$h_n(z) = \begin{cases} h_{n-1}(z), & \text{if } z \notin \hat{U}_1, \\ g_2^{-1} \circ h_{n-1} \circ g_1(z), & \text{if } z \in \hat{U}_1. \end{cases}$$

Each h_n has the same dilatation as h. Since $K^{loc}(g_1)$ is nowhere dense, h_n stabilizes and converges to a quasiconformal conjugacy between g_1 and g_2. □

4. Hyperbolic renormalization fixed point

From now on, let us fix a periodic type irrational $\theta \in \Theta_{per}$. In this section, we will construct the desired corona renormalization fixed point f_* and prove most of Theorem A. The remaining sections §5-7 are dedicated to proving that the local unstable manifold is one-dimensional.

4.1. **Renormalization of critical commuting pairs.** Let us consider a (d_0, d_∞)-critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$ with critical point c and rotation number τ. Let us denote by $\{p_n/q_n\}_{n \in \mathbb{N}}$ the best rational approximations of τ. For every $n \geq 2$, denote by I_n the shortest interval in $\mathbf{H}$ connecting c and $f^{q_n}(c)$. The n^{th} *pre-renormalization* of f is the pair

$$(f^{q_n}|_{I_{n-1}}, f^{q_{n-1}}|_{I_n})$$

and the n^{th} *renormalization* $\mathcal{R}^n f$ of f is the normalized commuting pair obtained by rescaling of the n^{th} pre-renormalization by either the affine map if n is even, or the anti-affine map if n is odd, that sends 0 to c and -1 to $f^{q_{n-1}}(c)$. Each renormalization $\mathcal{R}^n f$ is a (d_0, d_∞)-critical commuting pair.

Let $\mathbb{H}$ and $-\mathbb{H}$ denote the upper and lower half planes of $\mathbb{C}$ respectively.

Definition 4.1. Let $\mathbf{I} \Subset \mathbb{C}$ be a closed quasiarc containing 0 on its interior. A *commuting pair* ζ based on $\mathbf{I}$ is a pair of orientation preserving analytic homeomorphisms

$$\zeta = (f_- : I_- \to f_-(I_-),\ f_+ : I_+ \to f_+(I_+))$$

with the following properties.

- (P_1) I_- and I_+ are closed subintervals of $\mathbf{I}$ of the form $[f_+(0), 0]$ and $[0, f_-(0)]$ respectively such that $\mathbf{I} = I_- \cup I_+ = f_-(I_-) \cup f_+(I_+)$ and $I_- \cap I_+ = \{0\}$.
- (P_2) For all $x \in I_\pm \backslash \{0\}$, $f_\pm'(x) \neq 0$.
- (P_3) Both f_- and f_+ admit holomorphic extensions to a neighborhood B of 0 on which f_- commutes with f_+ and $f_- \circ f_+(\mathbf{I} \cap B) \subset I_-$.

Additionally, a commuting pair ζ is a *critical commuting pair* if

- (P_4) 0 is a critical point of both f_- and f_+.

We say that ζ is *normalized* if $f_+(0) = -1$. A critical commuting pair ζ is called a (d_0, d_∞)-*critical commuting pair* if for any quasiconformal map ϕ mapping I_- and I_+ to real intervals $[-1, 0]$ and $[0, 1]$ respectively and for any sufficiently small round disk D centered at $\phi(f_+(f_-(0)))$, the number of connected components of $\phi(f_+ \circ f_-)^{-1} \phi^{-1}(D \cap -\mathbb{H})$ in $-\mathbb{H}$ is d_∞, whereas the number of connected components of $\phi(f_+ \circ f_-)^{-1} \phi^{-1}(D \cap \mathbb{H})$ in $\mathbb{H}$ is d_0.

We say that a (d_0, d_∞)-critical commuting pair $\zeta = (f_-, f_+)$ is *renormalizable* if there exists a positive integer $\chi = \chi(\zeta)$ that corresponds to the first time $f_-^{\chi+1} \circ f_+(0)$ lies in the interior of I_+. If renormalizable, we call the (d_∞, d_0)-critical commuting pair

$$p\mathcal{R}\zeta := \left(f_-^{\chi} \circ f_+|_{[0, f_-(0)]},\ f_-|_{[f_-^{\chi} f_+(0), 0]} \right)$$

the *pre-renormalization* of ζ, and we call the normalized (d_0, d_∞)-critical commuting pair obtained by conjugating $p\mathcal{R}\zeta$ with the antilinear map $z \mapsto -f_-(0)\bar{z}$ the *renormalization* of $\mathcal{R}\zeta$ of ζ.

If $\mathcal{R}\zeta$ is again renormalizable, we call ζ twice renormalizable, and so on. If ζ is infinitely renormalizable, we define the *rotation number* of ζ to be the irrational number

$$\operatorname{rot}(\zeta) := [0; \chi(\zeta), \chi(\mathcal{R}\zeta), \chi(\mathcal{R}^2\zeta), \ldots].$$

The operator $\mathcal{R}$ acts on the rotation number as follows. If ζ is n times renormalizable,

$$\operatorname{rot}(\mathcal{R}^n \zeta) = G^n(\operatorname{rot}(\zeta)),$$

where $G(\tau) = \left\{ \frac{1}{\tau} \right\}$ is the Gauss map.

One can convert a (d_0, d_∞)-critical commuting pair ζ into a (d_0, d_∞)-critical quasicircle map as follows.

Proposition 4.2. *Let $\zeta = (f_-|_{I_-}, f_+|_{I_+})$ be a commuting pair. Let G_ζ be the gluing map which corresponds to identifying z with $f_+(z)$ for every point z in a neighborhood of $f_-(0)$. Then, G_ζ projects the pair $(f_-|_{[f_+ f_-(0), 0]}, f_+ f_-|_{[0, f_-(0)]})$ into a quasicircle map $f_\zeta : \mathbf{H} \to \mathbf{H}$ having the same rotation number as ζ. If ζ is (d_0, d_∞)-critical, then $f_\zeta : \mathbf{H} \to \mathbf{H}$ is a (d_0, d_∞)-critical quasicircle map.*

Let us denote by p the period of θ under the Gauss map $G(\theta) = \left\{\frac{1}{\theta}\right\}$. By studying rigidity properties of critical commuting pairs, our previous work [Lim23b] culminates in the following result.

Theorem 4.3 ([Lim23b, §7.4-7.5])**.** *There is a unique normalized (d_0, d_∞)-critical commuting pair ζ_* with rotation number θ with the following properties.*

(1) *Renormalization fixed point: There is a linear map $z \mapsto \mu z$, $|\mu| < 1$, which conjugates ζ_* and the pre-renormalization $p\mathcal{R}^p \zeta_*$.*
(2) *Exponential convergence: For any normalized (d_0, d_∞)-critical commuting pair ζ of some rotation number $\tau \in \Theta_{pre}$ where $G^k(\tau) = \theta$ for some $k \in \mathbb{N}$, the renormalizations $\mathcal{R}^{k+np}\zeta$ converge exponentially to ζ_* as $n \to \infty$.*

4.2. **Corona renormalization fixed point.** We say that a rotational corona is *standard* if the arc γ_0 passes through the critical value. Similarly, we say that a rotational pre-corona is *standard* if it is a pre-corona around the critical value.

Theorem 4.4. *There exists a standard (d_0, d_∞)-critical rotational corona $f_* : U_* \to V_*$ with rotation number θ which admits a standard rotational pre-corona*

$$F_* = \left(f_*^a : U_- \to S_*,\ f_*^b : U_+ \to S_*\right)$$

together with a gluing map $\psi_* : S_* \to \overline{V_*}$ projecting F_* back to $f_* : U_* \to V_*$. Moreover, we have an improvement of domain: $\mathbf{\Delta}_{F_*} \Subset U_*$.

Proof. Consider the (d_0, d_∞)-critical commuting pair

$$\zeta_* = (f_- : I_- \to I, f_+ : I_+ \to I)$$

on a quasiarc $I = I_- \cup I_+ = [f_+(0), 0] \cup [0, f_-(0)]$ of rotation number θ from Theorem 4.3. There exists some $\mu \in \mathbb{D}$ such that for any $n \in \mathbb{N}$, there is a pre-renormalization $\zeta_n = (f_{n,-} : J_- \to J, f_{n,+} : J_+ \to J)$ of ζ on a subinterval $J \subset I$ that is conjugate to ζ via the linear map $L^n(z) = \mu^n z$. We will convert this renormalization fixed point in the category of commuting pairs to that in the category of critical quasicircle maps, and then project it to that in the category of rotational coronas.

Consider the gluing map $\phi_1 := G_\zeta$ described in Proposition 4.2. Then, ϕ_1 projects the modified commuting pair $\zeta' := \left(f_-|_{[f_+f_-(0),0]}, f_+f_-|_{[0,f_-(0)]},\right)$ into a (d_0, d_∞)-critical quasicircle map $g : \mathbf{H} \to \mathbf{H}$ having the same rotation number θ.

Denote by $c_0 := \phi_1(0)$ the critical point of g, and let $c_k := g^k(c_0)$ for all $k \in \mathbb{N}$. Consider the modification of ζ_n, which is ζ' rescaled by L^n, and project it to the dynamical plane of g via ϕ_1 to obtain a commuting pair $g_n = \left(g^{\mathbf{a}}|_{[c_{\mathbf{b}}, c_0]}, g^{\mathbf{b}}|_{[c_0, c_{\mathbf{a}}]}\right)$ for some return times $\mathbf{a}$ and $\mathbf{b}$. Then, $\psi_1 := \phi_1 L^n \phi_1^{-1}$ is the gluing map projecting g_n back to g.

To make it standard, we will push g_n forward under one iterate of g. More precisely, we set $\psi_2 := g \circ \psi_1 \circ g^{-1}$. It is well-defined because for every point z close to c_1, the preimage $g^{-1}(z)$ is a set of $d_0 + d_\infty - 1$ points close to c_0 whose images under ψ_1 remain close to c_0 and get mapped to the same point $\psi_2(z)$ under g. The new gluing map ψ_2 sends a small neighborhood of c_1 to a neighborhood of $\mathbf{H}$. Moreover, ψ_2 fixes the critical value c_1 and projects $\tilde{g}_n = \left(g^{\mathbf{a}}|_{[c_{\mathbf{b}+1}, c_1]}, g^{\mathbf{b}}|_{[c_1, c_{\mathbf{a}+1}]}\right)$ back to g.

By Corollary 3.8, g admits a standard pre-corona P defined in a small neighborhood of c_1. The corresponding gluing map ϕ_2 projects P onto a (d_0, d_∞)-critical rotational corona $f_* : U_* \to V_*$. Since θ is periodic, we can prescribe f_* to have rotation number θ. The corresponding Herman quasicircle $\mathbf{H}_*$ of f_* is the image of (an interval in) $\mathbf{H}$ under ϕ_2.

Let us rescale the pre-corona P by ψ_2^{-1} to obtain yet another pre-corona P' in the dynamical plane of g that is much smaller than P. Project P' via ϕ_2 to obtain a pre-corona F_* of f_*. The map $\psi_* := \phi_2 \circ \psi_2 \circ \phi_2^{-1}$ will project the pre-corona F_* back to f_*. The improvement of domain property is satisfied once we take n to be sufficiently high. □

Corollary 4.5. *Let f_* and F_* be from the previous theorem. There exist a pair of small Banach neighborhoods $\mathcal{U}$ and $\mathcal{B}$ of f_* and a compact analytic corona renormalization operator $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ such that $\mathcal{R}f_* = f_*$ and the pre-renormalization of $\mathcal{R}f_*$ is F_*. Moreover, for any rotational corona f in $\mathcal{U}$ with the same rotation number θ, f is infinitely renormalizable and $\mathcal{R}^n f$ converges exponentially fast to f_*.*

Proof. The existence of $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ follows from Theorems 2.6 and 4.4. Exponential convergence is guaranteed by Theorem 4.3 provided that $\mathcal{U}$ is a sufficiently small neighborhood of f_*. □

Lemma 4.6. *For any Banach neighborhood $\mathcal{U}$ of f_* and any (d_0, d_∞)-critical quasicircle map f of rotation number $\tau \in \Theta_{pre}$ where $G^k(\tau) = \theta$ for some $k \in \mathbb{N}$, there is a compact analytic corona renormalization operator $\mathcal{R}_1 : N(f) \to \mathcal{U}$ on a Banach neighborhood $N(f)$ of f.*

Proof. By Theorem 4.3, there is a high $m \in \mathbb{N}$ such that $\mathcal{R}^m f$ is a critical commuting pair of rotation number θ that is arbitrarily close to the critical commuting pair ζ_*. By quasiconformal rigidity, f admits a rotational pre-corona F which projects to a rotational corona g of rotation number θ close to f_*. By Theorem 2.6, there is a compact analytic renormalization operator $\mathcal{R}_1$ on a small neighborhood of f such that $\mathcal{R}_1(f) = g$. □

4.3. **Renormalization tiling.** From now on, let us consider the renormalization operator $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ and the fixed point f_* from Corollary 4.5. Recall that every corona $f \in \mathcal{U}$ has the same codomain V and critical arc γ_1.

Suppose $f \in \mathcal{U}$ is n times renormalizable and $f_1, \ldots, f_n$ all lie in $\mathcal{U}$. This means that f is contained in the neighborhood

$$\mathcal{U}_n := \bigcap_{0 \le k \le n} \mathcal{R}^{-k}(\mathcal{U})$$

of f_*. For $k \in \{0, 1, \ldots, n\}$, denote by $\mathcal{R}^k f = [f_k : U_k \to V]$ the k^{th} renormalization of f, $\psi_k : S_k \to V$ the renormalization change of variables for f_{k-1}, and $\phi_k := \psi_k^{-1}$. Let us cut the dynamical plane of f_k along the critical arc γ_1 and obtain a pre-corona

$$F_k = (f_{k,\pm} : U_{k,\pm} \to V \backslash \gamma_1).$$

Divide $\overline{U_0}$ along the arcs γ_0 and γ_1 to obtain a tiling $\mathbf{\Delta}_0$ of $\overline{U_0}$ consisting of two tiles $\Delta_0(0)$ and $\Delta_0(1)$. We make the convention that $\Delta_0(0)$, γ_0, and $\Delta_0(1)$ are in counterclockwise order. The tiling $\mathbf{\Delta}_0$ is called the *zeroth tiling* associated to f_0.

The map

$$\Phi_n := \phi_1 \circ \phi_2 \circ \ldots \phi_n$$

is well defined on $V\backslash\gamma_1$ and projects F_n to the dynamical plane of f as the pre-corona

$$F_n^{(0)} = \left(f_{n,\pm}^{(0)} : U_{n,\pm}^{(0)} \to S_n^{(0)}\right) \quad \text{where} \quad f_{n,-}^{(0)} = f_0^{\mathbf{a}_n} \text{ and } f_{n,+}^{(0)} = f_0^{\mathbf{b}_n}$$

for some return times $\mathbf{a}_n$ and $\mathbf{b}_n$. Let us also set $\Phi_0 := \mathrm{Id}$.

Define the n^{th} *tiling* $\mathbf{\Delta}_n$ associated to f by spreading around $U_{n,\pm}^{(0)}$ via f. It consists of $f^i\left(U_{n,-}^{(0)}\right)$ for $i \in \{0, 1 \ldots, \mathbf{a}_n - 1\}$ and $f^j\left(U_{n,+}^{(0)}\right)$ for $j \in \{0, 1 \ldots, \mathbf{b}_n - 1\}$. Let us denote by $\Delta_n(0)$ the image of the zeroth tile $\Delta_0(0, f_n)$ of f_n under Φ_n, label the rest of the tiles in $\mathbf{\Delta}_n$ in counterclockwise order by $\Delta_n(i)$ for $i \in \{0, 1, \ldots, \mathbf{a}_n + \mathbf{b}_n - 1\}$.

If f is rotational, then $\mathbf{\Delta}_n$ always forms an annular neighborhood of the Herman quasicircle of f. In general, the map f always acts almost like a rotation on the tiling $\mathbf{\Delta}_n$. There exists $\mathbf{p}_n \in \mathbb{N}_{\geq 1}$ such that f maps $\Delta_n(i)$ univalently onto $\Delta_n(i + \mathbf{p}_n)$ whenever $i \notin \{-\mathbf{p}_n, -\mathbf{p}_n + 1\}$. Moreover, f maps $\Delta_n(-\mathbf{p}_n) \cup \Delta_n(-\mathbf{p}_n + 1)$ back to $S_n^{(n)}$ almost as a degree d covering map branched at its critical point $c_0(f)$.

Lemma 4.7. *The operator $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ can be arranged such that the following holds. For $f \in \mathcal{U}_n$,*

1. *there is a holomorphic motion of $\partial\mathbf{\Delta}_0, \ldots, \partial\mathbf{\Delta}_n$ over $f \in \mathcal{U}_n$ that is equivariant with respect to $f : \partial\Delta_n(i) \to \partial\Delta_n(i + \mathbf{p}_n)$ for $i \notin \{-\mathbf{p}_n, -\mathbf{p}_n + 1\}$;*
2. *for every $f \in \mathcal{U}_n$ and $1 \leq k \leq n$, $\mathbf{\Delta}_m \cup f(\mathbf{\Delta}_m) \Subset \mathbf{\Delta}_{m-1}$;*
3. *the tiling $\mathbf{\Delta}_n(f)$ is close to the Herman curve of f_* in Hausdorff topology.*

Proof. Let us first consider the case where $f = f_*$. By the improvement of domain property in Theorem 4.4, the diameters of the tiles in $\mathbf{\Delta}_n(f_*)$ must shrink to 0 as $n \to \infty$. Consider a tile $\Delta_1(i, f_*)$. There is some $t \geq 0$ and $j \in \{0, 1\}$ such that f_*^t sends $\Delta_1(j, f_*)$ onto $\Delta_1(i, f_*)$. By replacing $\mathcal{R}$ with some high iterate $\mathcal{R}^k$ if necessary, the map

$$\psi_* \circ f_*^{-t} : \Delta_1(i, f_*) \to \Delta_0(j, f_*)$$

expands the Euclidean metric by some high factor $C > 1$. Inductively, (2) and (3) hold for f_*.

By design, it is clear that $\partial\mathbf{\Delta}_0$ moves holomorphically over $f \in \mathcal{U}$. For $1 \leq k \leq n$, we push forward the holomorphic motion $\partial\mathbf{\Delta}_0(f_k)$ via Φ_k and spread it around dynamically to obtain a holomorphic motion of $\partial\mathbf{\Delta}_k(f)$ over $f \in \mathcal{U}_n$.

By continuity, every $f \in \mathcal{U}_n$ also satisfies the following property. For any tile $\Delta_n(i, f)$ within $\mathbf{\Delta}_n$, there is some $t \geq 0$ and $j \in \{0, 1\}$ such that f^t sends $\Delta_n(j, f)$ onto $\Delta_n(i, f)$. We obtain a holomorphic motion of $\partial\mathbf{\Delta}_n(f)$ by pulling back the holomorphic motion of $\partial\mathbf{\Delta}_0(f_n)$ via maps of the form

$$\Psi_{n,i} := \Phi_n^{-1} \circ f^{-t} : \Delta_n(i, f) \to \Delta_0(j, f_n) \tag{4.1}$$

for each tile. This implies (1). Moreover, (2) follows from the observation that each $\Psi_{n,i}$ expands the Euclidean metric by a factor close to C^n. Moreover, (3) follows from (1) as well as the special case of (3) for $f = f_*$. □

Let us extend the tiling $\mathbf{\Delta}_n$ of a subset of $\overline{U_0}$ to a full tiling of $\overline{U_0}$ as follows. Consider

$$\hat{\gamma}_0 := \gamma_0 \backslash f^{-1}(U_0) \quad \text{and} \quad \Gamma := \partial U_0 \cup \hat{\gamma}_0.$$

Observe that $\hat{\gamma}_0$ is a disjoint union of two subarcs $\hat{\gamma}_0^0$ and $\hat{\gamma}_0^\infty$ of γ_0 where each $\hat{\gamma}_0^\bullet$ connects the boundary component $\partial^\bullet U_0$ to $f^{-1}(U_0)$. Consider the maps $\Psi_{n,i}$ from (4.1).

Lemma 4.8. *When $\mathcal{U}$ is sufficiently small, the following holds for all $f \in \mathcal{U}$.*

(1) $\Gamma(f_1)$ *contains* $\Psi_{1,i}\,(\partial\mathbf{\Delta}_1(f) \cap \partial\Delta_1(i,f))$ *for every* i*. Moreover, there is some* i *such that* $\hat{\gamma}_0(f_1)$ *is contained in* $\Psi_{1,i}\,(\partial\mathbf{\Delta}_1(f) \cap \partial\Delta_1(i,f))$*.*
(2) $\Gamma(f)$ *is disjoint from* $\partial\mathbf{\Delta}_1(f)$*.*
(3) *For* $\bullet \in \{0,\infty\}$*, there is an arc* $\xi_0^\bullet$ *such that both* $\xi_0^\bullet \cup \hat{\gamma}_0^\bullet$ *and* $\xi_1^\bullet := f(\xi_0^\bullet)$ *connect* $\partial^\bullet U_0$ *and* $\partial\mathbf{\Delta}_1(f)$*.*

Moreover, $\xi_0 := \xi_0^0 \cup \xi_0^\infty$ *and* $\xi_1 := \xi_1^0 \cup \xi_1^\infty$ *can be chosen such that there is a holomorphic motion of*

$$\Gamma \cup \xi_0 \cup \xi_1 \cup \mathbf{\Delta}_1$$

over $f \in \mathcal{U}$ *that is equivariant with respect to* $f : \xi_0(f) \to \xi_1(f)$*,* $f : \Delta_1(i,f) \to \Delta_1(i+\mathbf{p}_1, f)$ *for* $i \neq \{-\mathbf{p}_1, -\mathbf{p}_1+1\}$*, and each of* $\Psi_{1,i} : \partial\mathbf{\Delta}_1(f) \cap \Delta_1(i,f) \to \Gamma(f_1)$*.*

Proof. Every tile $\Delta_1(i,f)$ is a rectangle. Clearly, each $\Psi_{1,i}$ maps the horizontal sides of $\Delta_1(i,f)$ into ∂U_1. Let us label the vertical sides of $\Delta_1(i,f)$ by $l(i)$ and $r(i)$ such that each $l(i)$ intersects the side $r(i+1)$ of the next tile. Then, the intersection $\partial\mathbf{\Delta}_1(f) \cap \partial\Delta_1(i,f)$ is the union of the horizontal sides of $\Delta_1(i,f)$ and the symmetric difference $l(i) \triangle r(i+1)$ between touching vertical sides across all i's.

It is clear that $l(i) \neq r(i+1)$ for at least one i. For such i, either $l(i)$ is the preimage of $\gamma_0(f_1)$ under $\Psi_{1,i}$ and $r(i+1)$ is the preimage of the arc $\gamma_1(f_1)$ under $\Psi_{1,i+1}$, or vice versa. In this case, $l(i) \triangle r(i+1)$ will be mapped by $\Phi_{1,i}$ or $\Phi_{1,i+1}$ onto $\hat{\gamma}_0(f_1)$. This implies (1).

Item (2) follows directly from Lemma 4.7. Moreover, (2) allows us to find for each $\bullet \in \{0,\infty\}$ a proper arc $\xi_0^\bullet$ in $U_0 \backslash (\hat{\gamma}_0 \cup \mathbf{\Delta}_1)$ in a small neighborhood of γ_0 that connects the tip of $\hat{\gamma}_0^\bullet$ to a point on $\partial\Delta_1(i,f)$ for some $i \neq \{-\mathbf{p}_1, -\mathbf{p}_1+1\}$. This yields (3).

In Lemma 4.7, we already established the equivariant holomorphic motion of $\partial\mathbf{\Delta}_0 \cup \partial\mathbf{\Delta}_1$. By lifting via $\Phi_{1,i}$, this motion immediately extends to an equivariant motion of Γ. We then lift the motion of $\mathbf{\Delta}_0(f_1)$ via $\Psi_{1,i}$ to obtain an equivariant motion of $\partial\mathbf{\Delta}_1 \cup \Gamma$. Finally, by applying the λ-lemma, we extend this motion to $\Gamma \cup \xi_0 \cup \xi_1 \cup \mathbf{\Delta}_1$. □

For $n \in \mathbb{N}$, we define the n^{th} *full renormalization tiling* of U_0 to be the union of the tilings $\mathbf{\Delta}_n$ and $\mathbf{A}_k$ for $k = 0, 1, \ldots, n-1$ where the latter is constructed as follows. Each $\mathbf{A}_k$ is a disjoint union of two tilings $\mathbf{A}_k^0$ and $\mathbf{A}_k^\infty$ where the former is closer to $\partial^0 U_0$ and the latter is closer to $\partial^\infty U_0$. For each $\bullet \in \{0,\infty\}$,

- ▷ $\mathbf{A}_0^\bullet$ is the connected component of $\overline{\mathbf{\Delta}_0 \backslash \mathbf{\Delta}_1}$ that touches $\partial^\bullet U_0$ on the boundary, and it is split by $\hat{\gamma}_0^\bullet \cup \xi_0^\bullet \cup \xi_1^\bullet$ into two tiles $A_0^\bullet(0)$, $A_0^\bullet(1)$. Again, we make the convention that $A_0^\bullet(0)$, $\hat{\gamma}_0^\bullet \cup \xi_0^\bullet$, $A_0^\bullet(1)$ are in counterclockwise order.
- ▷ $\mathbf{A}_k^\bullet$ is the connected component of $\overline{\mathbf{\Delta}_k \backslash \mathbf{\Delta}_{k+1}}$ that touches $\partial^\bullet \mathbf{\Delta}_k$ on the boundary, and it has tiles $\{A_k^\bullet(i)\}_{i=0,1,\ldots,\mathbf{a}_k+\mathbf{b}_k-1}$ obtained by spreading via forward iterates of f the tiles $A_k^\bullet(j,f) := \Phi_k(A_0^\bullet(j,f_k))$ for $j \in \{0,1\}$ and labeled in counterclockwise order.

The first full renormalization tiling is illustrated in Figure 5.

Definition 4.9. A *quasiconformal combinatorial pseudo-conjugacy of level* n between f and f_* is a quasiconformal map $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that sends $\overline{U_0}$ to $\overline{U_\star}$ and preserves the n^{th} renormalization tiling as follows.

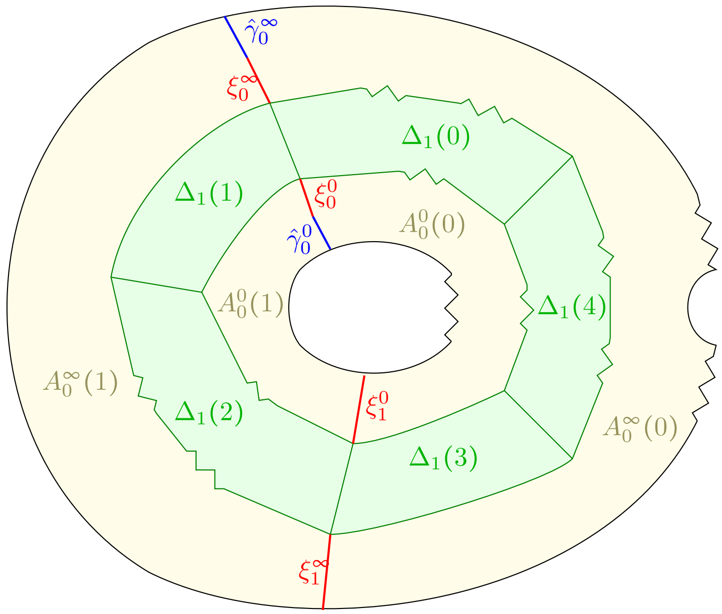

FIGURE 5. The first full renormalization tiling of U_0.

(1) The map h sends $\Delta_n(i,f)$ to $\Delta_n(i,f_*)$ for all i, and is equivariant on $\Delta_n(i,f)$ for all $i \notin \{-\mathbf{p}_n, -\mathbf{p}_n+1\}$;
(2) For all $\bullet \in \{0,\infty\}$ and $k \in \{0,1,\dots n-1\}$, h sends $A_k^\bullet(i,f)$ to $A_k^\bullet(i,f_*)$ for all i, and is equivariant on $A_k^\bullet(i,f)$ for all $i \notin \{-\mathbf{p}_k, -\mathbf{p}_k+1\}$.

Theorem 4.10 (Combinatorial pseudo-conjugacy)**.** *Consider $f \in \mathcal{U}_n$ and let*

$$D := \max_{0\le k\le n} \operatorname{dist}(f_k, f_*).$$

There is a K_D-quasiconformal combinatorial pseudo-conjugacy h of level n between f and f_* such that

$$\sup_{z\in\mathbf{\Delta}_n(f)} |h(z)-z| \le M_D.$$

Moreover, $K_D \to 1$ and $M_D \to 0$ as $D \to 0$.

Proof. Recall that each tile $A_k^\bullet(i,f)$ admits some $t \in \mathbb{N}$ and $j \in \{0,1\}$ such that $\Psi_{k,i} := \Phi_k^{-1} \circ f^{-t}$ univalently maps $A_k^\bullet(i,f)$ onto $A_0^\bullet(j,f_k)$. By Lemma 4.8, we have a holomorphic motion of the first full renormalization tiling over $\mathcal{U}$. Let us pull back this motion via maps of the form $\Psi_{n,i}$ to obtain a holomorphic motion of the full n^{th} renormalization tiling. By equivariance and λ-lemma, this holomorphic motion induces the desired quasiconformal map h. The dilatation K_D of h is bounded by the dilatation of the motion at $f_0, f_1, \dots f_n$, which depends only on D, where $K_D \to 0$ as $D \to \infty$. The estimate M_D follows from the continuity of the holomorphic motion and the compactness of quasiconformal maps. $\square$

Corollary 4.11. *There is some $\varepsilon > 0$ such that the following holds. Suppose $f \in \mathcal{U}$ is infinitely renormalizable and $\mathcal{R}^n f$ is in the ε-neighborhood of f_* for all $n \in \mathbb{N}$. Then, f is a rotational corona with rotation number θ.*

Proof. By Theorem 4.10, we have a $K(\varepsilon)$-quasiconformal combinatorial pseudo-conjugacy h_n of level n between f and f_* for all $n \in \mathbb{N}$. By the compactness of K-quasiconformal maps, h_n converges in subsequence to a quasiconformal map $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, and h^{-1} must be a conjugacy on the Herman quasicircle $\mathbf{H}_*$ of f_*. The image $h^{-1}(\mathbf{H}_*)$ is a Herman quasicircle of f containing the critical point $c_0(f)$ and separating the boundaries of the domain of f. It follows that f must be a rotational corona with rotation number θ. ☐

4.4. **Towards hyperbolicity.**

Theorem 4.12. *The renormalization operator $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ is hyperbolic at the fixed point f_* with a finite positive dimensional local unstable manifold $\mathcal{W}^u_{\mathrm{loc}}$. If $\mathcal{U}$ is sufficiently small, the local stable manifold $\mathcal{W}^s_{loc}$ of f_* consists of the set of (d_0, d_∞)-critical rotational coronas in $\mathcal{U}$ with rotation number θ.*

Proof. Consider a corona f near f_* lying on the local stable manifold $\mathcal{W}^s_{loc}$. For sufficiently small $\mathcal{U}$, $\mathcal{R}^n f$ is in the ε-neighborhood of f_* for all $n \in \mathbb{N}$. By Corollary 4.11, f must be a rotational corona with rotation number θ.

Let us consider the derivative $D\mathcal{R}_{f_*}$ of the renormalization operator at the fixed point f_*. By the compactness of $\mathcal{R}$, the number of neutral and repelling eigenvalues is finite. We claim that neutral eigenvalues do not exist and repelling eigenvalues must exist.

Suppose for a contradiction that there are neutral eigenvalues. By Small Orbits Theorem A.1, there exists an infinitely renormalizable corona f such that its forward orbit lies entirely in the ε-neighborhood of f_* and it satisfies

$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{R}^n f\| = 0. \tag{4.2}$$

By Corollary 4.11, f must be a rotational corona with the same rotation number θ as f_*. By Corollary 4.5, renormalizations $\mathcal{R}^n f$ converge to f_* exponentially fast, which contradicts (4.2). Hence, neutral eigenvalues do not exist.

Consider the family of rational maps F_c from (3.1). By Theorem 3.3, there is a unique parameter c_* such that F_{c*} admits a Herman quasicircle with the same rotation number as f_*. By Lemma 4.6, there is an analytic renormalization operator $\mathcal{R}_1$ on a neighborhood of F_{c_*} such that $\mathcal{R}_1 F_{c_*}$ is a rotational corona with rotation number θ that is sufficiently close to f_*. For any parameter $c \neq c_*$ sufficiently close to c_*, $\mathcal{R}_1 F_c$ is also sufficiently close to f_*. By the uniqueness of c_*, the parameter c can be picked such that F_c is postcritically finite, and so $\mathcal{R}_1 F_c$ is not a rotational corona.

Suppose for a contradiction that $D\mathcal{R}_{f_*}$ has no repelling eigenvalues. Then, $\mathcal{W}^s_{loc}$ is an open neighborhood of f_* and contains $\mathcal{R}_1 F_c$. However, the non-rotationality of $\mathcal{R}_1 F_c$ would contradict Corollary 4.11. ☐

5. Transcendental extension

From now on, we will consider the corona renormalization operator $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ together with its hyperbolic fixed point $f_* : U_* \to V_*$ constructed in Section 4.

Definition 5.1. A map $g : A \to B$ is said to be *σ-proper* if there exist exhaustions A_n, B_n of A, B respectively such that for all n, $g : A_n \to B_n$ is a proper map; equivalently, every connected component of the preimage of a compact set under g is compact.

In [McM98], McMullen proved the existence of maximal σ-proper extensions of holomorphic commuting pairs associated to renormalizations of quadratic Siegel disks. This is generalized in [DLS20, Theorem 5.5] where pre-pacmen on the local unstable manifold are shown to admit maximal σ-proper extension. In this section, we will show that our case is no different. We will study coronas in the local unstable manifold $\mathcal{W}^u_{\text{loc}}$ of f_*, which we will identify as a parameter space (of unknown dimension) of transcendental holomorphic maps onto $\mathbb{C}$.

5.1. **Maximal σ-proper extension.** Consider a corona $f : U \to V$ lying in the local unstable manifold $\mathcal{W}^u_{\text{loc}}$ of f_*. Since f is infinitely anti-renormalizable, it comes with a backward tower of corona renormalizations $\{f_k : U_k \to V\}_{k \le 0}$, where each f_k embeds to U_{k-1} as a pre-corona $F_k = (f_{k,\pm})$ consisting of iterates of f_{k-1}. Let $\psi_k : S_k \to V$ be the renormalization change of variables realizing the renormalization of f_{k-1} and let $\phi_k := \psi_k^{-1} : V \to S_k$.

Let us normalize our coronas such that they have a critical value at 0. For each $k \le 0$, consider the translation $T_k(z) = z - c_1(f_k)$ and denote

$$U_k^\natural = T_k(U_k), \quad V_k^\natural = T_k(V), \quad U_{k,\pm}^\natural = T_{k-1}(S_k), \quad S_k^\natural = T_{k-1}(S_k).$$

The translations T_k's normalize our maps f_k, F_k, and ϕ_k into

$$f_k^\natural : U_k^\natural \to V_k^\natural, \quad F_k^\natural := (f_{k,\pm}^\natural : U_{k,\pm}^\natural \to S_k^\natural), \quad \phi_k^\natural : V_k^\natural \to S_k^\natural$$

respectively. Consider the linear map

$$A_*(z) := \mu_* z$$

where $\mu_* := (\phi_*^\natural)'(0) \in \mathbb{D}$ is the self-similarity factor of f_*.

Lemma 5.2. *The limit*

$$h_f^\natural(z) := \lim_{k \to -\infty} A_*^k \circ \phi_{k+1}^\natural \circ \ldots \circ \phi_1^\natural \circ \phi_0^\natural(z)$$

defines a univalent map on a neighborhood D of 0 where D is independent of f.

Proof. As $\phi_k^\natural \to \phi_*^\natural$ exponentially fast, so is the derivative $\mu_k := (\phi_k^\natural)'(0)$ towards μ_*. There are positive constants ε and δ such that $\varepsilon < 1 - |\mu_*|$ and for all $|z| < \delta$ and $k \le 0$, we have $|\phi_k^\natural(z)| \le (|\mu_*| + \varepsilon)|z|$. Therefore, for all $|z| < \delta$ and $k \le 0$,

$$|\phi_{k+1}^\natural \circ \ldots \circ \phi_0^\natural(z)| \le (|\mu_*| + \varepsilon)^{-k}|z|.$$

The sequence $h^{(k)}(z) := A_*^k \circ \phi_{k+1}^\natural \circ \ldots \circ \phi_0^\natural \circ \phi_0^\natural(z)$ indeed converges to a univalent map on $\{|z| < \delta\}$ since

$$\frac{h^{(k-1)}(z)}{h^{(k)}(z)} = \frac{\phi_k^\natural\left(\phi_{k+1}^\natural \circ \ldots \circ \phi_0^\natural(z)\right)}{\mu_* \phi_{k+1}^\natural \circ \ldots \circ \phi_0^\natural(z)} = \frac{\mu_k}{\mu_*} + O(|\phi_{k+1}^\natural \circ \ldots \circ \phi_0^\natural(z)|) \to 1$$

exponentially fast as $k \to -\infty$. □

For $k \le 0$, let $h_k^\natural := h_{f_k}^\natural$ and denote its rescaling by $h_k^\# := A_*^k \circ h_k^\natural$. The following properties are easy to verify.

Proposition 5.3. *For $k \le 0$,*

$$h^{\natural}_{k-1} \circ \phi^{\natural}_i = A_* \circ h^{\natural}_k \qquad \text{and} \qquad h^{\natural}_0 = h^{\#}_k \circ \phi^{\natural}_{k+1} \circ \ldots \circ \phi^{\natural}_0.$$

Moreover, $h^{\natural}_0$ extends to a univalent map on the interior of $V^{\natural}_0 \backslash \gamma^{\natural}_1$.

The maps $h^{\natural}_k$ act as linear coordinates under which renormalization change of variables are simply linear maps. Objects in linear coordinates will be written in bold:

$$\mathbf{U}_{k,\pm} := h^{\natural}_k(U^{\natural}_{k,\pm}), \quad \mathbf{S}_k := h^{\natural}_k(S^{\natural}_k), \quad \mathbf{F}_k := (\mathbf{f}_{k,\pm} : \mathbf{U}_{k,\pm} \to \mathbf{S}_k).$$

Often, we will also work with the rescaled linear coordinates $h^{\#}_k$ in which we add the symbol "#" as follows:

$$\mathbf{U}^{\#}_{k,\pm} := h^{\#}_k(U^{\natural}_{k,\pm}), \quad \mathbf{S}^{\#}_k := h^{\#}_k(S^{\natural}_k), \quad \mathbf{F}^{\#}_k := \left(\mathbf{f}^{\#}_{k,\pm} : \mathbf{U}^{\#}_{k,\pm} \to \mathbf{S}^{\#}_k\right).$$

By design, it is clear that for all $k \le 0$,

$$\mathbf{f}^{\#}_{k,\pm} = A^k_* \circ \mathbf{f}_{k,\pm} \circ A^{-k}_*. \tag{5.1}$$

Lemma 5.4. *There is a matrix of positive integers $\mathbf{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ such that for every negative integer k,*

$$\mathbf{f}^{\#}_{k+1,-} = \left(\mathbf{f}^{\#}_{k,-}\right)^{m_{11}} \circ \left(\mathbf{f}^{\#}_{k,+}\right)^{m_{12}} \qquad \text{and} \quad \mathbf{f}^{\#}_{k+1,+} = \left(\mathbf{f}^{\#}_{k,-}\right)^{m_{21}} \circ \left(\mathbf{f}^{\#}_{k,+}\right)^{m_{22}}.$$

Proof. The action of renormalization restricted to the Herman quasicircle of f_* is a sector renormalization, and in particular an iterate of prime renormalization. See Appendix B.1. The existence of such a matrix $\mathbf{M}$ follows from Appendix B.2. □

Theorem 5.5 (Maximal extension)**.** *Assume $\mathcal{U}$ is a sufficiently small Banach neighborhood of f_*. For every $f \in \mathcal{W}^u_{\mathrm{loc}}$ and every $k \le 0$, the maps $\mathbf{f}^{\#}_{k,\pm}$ described above extend to σ-proper branched coverings $\mathbf{X}^{\#}_{k,\pm} \to \mathbb{C}$, where $\mathbf{X}^{\#}_{k,\pm}$ are simply connected domains in $\mathbb{C}$.*

Remark 5.6. Actually, $\mathbf{X}^{\#}_{k,\pm}$ are dense subsets of $\mathbb{C}$. For the renormalization fixed point f_*, this property follows from a deep point theorem [Lim23b, §5], which we will apply and explain in more detail in Lemma 7.2. For general $f \in \mathcal{W}^u_{\mathrm{loc}}$, this property will be apparent after we establish Theorem 7.4 on the holomorphic motion of $\partial \mathbf{X}^{\#}_{k,\pm}$.

Proof. For every $k \le 0$, the composition $\phi_{k+1} \circ \ldots \circ \phi_0$ embeds the pre-corona $F_0 = (f_{0,\pm} : U_{0,\pm} \to V \backslash \gamma_1)$ to the dynamical plane of f_k as a pair of iterates

$$\left(f^{\mathbf{a}_k}_k : U^{(k)}_{0,-} \to V^{(k)}_0,\ f^{\mathbf{b}_k}_k : U^{(k)}_{0,+} \to V^{(k)}_0\right). \tag{5.2}$$

Since ϕ_k is contracting at the critical value, the diameter of $U^{(k)}_{0,\pm} \to V^{(k)}_0$ shrinks to 0 as $k \to -\infty$.

To proceed, we need the following technical lemma.

Lemma 5.7. *Assume $\mathcal{U}$ is a sufficiently small Banach neighborhood of f_*. There is an open disk D around the critical value $c_1(f_*)$ of f_* such that for all sufficiently large $n \in \mathbb{N}$, $t \in \{\mathbf{a}_n, \mathbf{b}_n\}$, and $f \in \mathcal{R}^{-n}(\mathcal{U})$, then $f^t(c_1(f))$ is contained in D and D can be pulled back by f^t to a disk $D_0 \subset U_f \backslash \gamma_1$ containing $c_1(f)$ on which $f^t : D_0 \to D$ is a branched covering.*

This lemma initially appears in [DLS20, Key Lemma 4.8] in the context of quadratic Siegel pacmen. Due to its length, the proof will be supplied in Appendix B. The lemma tells us that for sufficiently large $k \ll 0$, the disk D contains $c_1(f_k)$ and the pair in (5.2) extends to a commuting pair of branched coverings

$$\left(f_k^{\mathbf{a}_k} : W_-^{(k)} \to D,\ f_k^{\mathbf{b}_k} : W_+^{(k)} \to D \right), \tag{5.3}$$

where $W_\pm^{(k)} \cup D$ is a subset of $V \backslash \gamma_1$. By conjugating with $h_k^\# \circ T_k$, we transform this pair into the commuting pair of branched coverings

$$\mathbf{f}_{0,\pm} : \mathbf{W}_\pm^{(k)} \to \mathbf{D}^{(k)}.$$

where

$$\mathbf{W}_\pm^{(k)} \coloneqq h_k^\# \circ T_k(W_\pm^{(k)}) \qquad \text{and} \qquad \mathbf{D}^{(k)} \coloneqq h_k^\# \circ T_k(D).$$

For all sufficiently large t and $m \leq 0$,

$$\operatorname{mod}\left(\mathbf{D}^{(tm-t)} \backslash \overline{\mathbf{D}^{(tm)}} \right) \succ 1,$$

and thus

$$\bigcup_{k<0}^{\infty} \mathbf{D}^{(k)} = \mathbb{C}.$$

Therefore, $\mathbf{f}_{0,\pm}$ extends to σ-proper branched coverings from $\mathbf{X}_{0,\pm} \coloneqq \bigcup_{k<0} \mathbf{W}_\pm^{(k)}$ onto $\mathbb{C}$. It is clear from the construction that $\mathbf{X}_{0,\pm}$ is a simply connected domain. □

The proof of the theorem above actually gives us something stronger, which we will use later in Section §7.2.

Lemma 5.8 (Stability of σ-branched structure)**.** *Assume $\mathcal{U}$ is a sufficiently small Banach neighborhood of f_*. For every $f \in \mathcal{W}^u_{\mathrm{loc}}$, there are sequences of nested disks*

$$\mathbf{D}^{(-1)} \subset \mathbf{D}^{(-2)} \subset \mathbf{D}^{(-3)} \subset \dots \qquad \textit{and} \qquad \mathbf{W}_\pm^{(-1)} \subset \mathbf{W}_\pm^{(-2)} \subset \mathbf{W}_\pm^{(-3)} \subset \dots$$

where

$$\bigcup_{k<0} \mathbf{D}^{(k)} = \mathbb{C}, \qquad \textit{and} \qquad \bigcup_{k<0} \mathbf{W}_\pm^{(k)} = \mathbf{X}_{0,\pm}$$

such that for every $k < 0$,

1. *each of $\mathbf{D}^{(k)}$ and $\mathbf{W}_\pm^{(k)}$ depends continuously on f;*
2. *the map $\mathbf{f}_{0,\pm} : \mathbf{W}_\pm^{(k)} \to \mathbf{D}^{(k)}$ is a pair of proper branched coverings of fixed finite degree;*
3. *critical points of $\mathbf{f}_{0,\pm} : \mathbf{W}_\pm^{(k)} \to \mathbf{D}^{(k)}$ move holomorphically over $f \in \mathcal{U}$.*

Proof. The construction of such disks is similar to the proof of the previous theorem. We add the following modification. By Theorem 4.10, we can replace the disk D with a slightly smaller disk $D(f_0, k)$ depending continuously on f_0 such that for all $i \leq \max\{\mathbf{a}_k, \mathbf{b}_k\}$,

$$c_i(f_*) \in D(f_*, k) \quad \text{if and only if} \quad c_i(f_k) \in D(f_0, k).$$

Under this replacement, the domains of branched coverings $(f_k^{\mathbf{a}_k}, f_k^{\mathbf{b}_k})$ from (5.3) become

$$\mathbf{f}_{0,\pm} : W_\pm(f_0, k) \to D(f_0, k),$$

which depend continuously on f_0. By conjugating with $h_k^\# \circ T_k$, we obtain the commuting pair $\mathbf{f}_{0,\pm} : \mathbf{W}_\pm^{(k)} \to \mathbf{D}^{(k)}$ with the desired property. □

5.2. **Cascades.** Consider the anti-renormalization matrix $\mathbf{M}$ from Lemma 5.4. Let us denote by $\mathbf{t} > 1$ and $1/\mathbf{t}$ the eigenvalues of $\mathbf{M}$.

Previously, we see that every corona f in $\mathcal{W}^u_{\mathrm{loc}}$ induces the sequence $\mathbf{F}^\#_n = (\mathbf{f}^\#_{n,\pm})$ for $n \le 0$. For every positive integer n, we can also define $\mathbf{F}^\#_n = (\mathbf{f}^\#_{n,\pm})$ inductively by the relation

$$(\mathbf{f}^\#_{n,-})^a \circ (\mathbf{f}^\#_{n,+})^b = (\mathbf{f}^\#_{n-1,-})^{a'} \circ (\mathbf{f}^\#_{n-1,+})^{b'} \tag{5.4}$$

for any $a, b, a', b' \in \mathbb{Z}_{\ge 0}$ satisfying $(a' \ b') = (a \ b)\mathbf{M}$.

Let us identify the local unstable manifold $\mathcal{W}^u_{\mathrm{loc}}$ with the space $\boldsymbol{\mathcal{W}}^u_{loc}$ of pairs of σ-proper maps $\mathbf{F} = (\mathbf{f}_{0,\pm})$ associated to each $f \in \mathcal{W}^u_{\mathrm{loc}}$. We extend our renormalization operator beyond $\boldsymbol{\mathcal{W}}^u_{loc}$ by setting

$$\mathcal{R}^n \mathbf{F}_0 = \mathbf{F}_n := A_*^{-n} \mathbf{F}^\#_n A^n_n,$$

(compare with (5.1)) and extend $\boldsymbol{\mathcal{W}}^u_{loc}$ to a global unstable manifold $\boldsymbol{\mathcal{W}}^u$ by adding $\mathbf{F}_n$ for all $n \ge 0$ and $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u_{loc}$. The complex manifold structure of $\boldsymbol{\mathcal{W}}^u_{loc}$ extends to $\boldsymbol{\mathcal{W}}^u$ and the renormalization operator $\mathcal{R}$ now acts on $\boldsymbol{\mathcal{W}}^u$ as a biholomorphism with a unique fixed point $\mathbf{F}_*$.

Definition 5.9. We define the space $\mathbf{T}$ of *power-triples* to be the quotient of the semigroup $\mathbb{Z} \times \mathbb{Z}^2_{\ge 0}$ under the equivalence relation $\sim$ where $(n, a, b) \sim (n-1, a', b')$ if and only if $(a' \ b') = (a \ b)\mathbf{M}$.

We will equip $\mathbf{T}$ with the binary operation $+$ defined by

$$(n, a, b) + (n, a', b') = (n, a + a', b + b').$$

With respect to $+$, $\mathbf{T}$ has a unique identity element $0 := (n, 0, 0)$. For $P, Q \in \mathbf{T}$, let us denote by $P \ge Q$ if for all sufficiently large $n \ll 0$, there exist $a, b, a', b' \in \mathbb{N}$ such that $P = (n, a, b)$, $Q = (n, a', b')$, $a \ge a'$, and $b \ge b'$.

From Lemma B.6, $(\mathbf{T}, +, \ge)$ can be identified with a sub-semigroup of $(\mathbb{R}_{\ge 0}, +, \ge)$. Moreover, $\mathbf{T}$ inherits a well-defined scalar multiplication by powers of $\mathbf{t}$ as follows. For every $(n, a, b) \in \mathbf{T}$ and integer k,

$$\mathbf{t}^k (n, a, b) = (n + k, a, b).$$

For every $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$ and every power-triple $P = (n, a, b)$, we will use the notation

$$\mathbf{F}^P := \left(\mathbf{f}^\#_{n,-}\right)^a \circ \left(\mathbf{f}^\#_{n,+}\right)^b.$$

Each $\mathbf{F}^P$ is a σ-proper map from its domain $\mathrm{Dom}\left(\mathbf{F}^P\right)$ onto $\mathbb{C}$. We denote by $\mathbf{F}^{\ge 0}$ the cascade $\left(\mathbf{F}^P\right)_{P \in \mathbf{T}}$ associated to $\mathbf{F}$.

Lemma 5.10. *For every* $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, $P \in \mathbf{T}$, *and* $n \in \mathbb{Z}$,

$$\mathbf{F}^P_0 = \left(\mathbf{F}^\#_{-n}\right)^{\mathbf{t}^n P}.$$

In particular, when $\mathbf{F} = \mathbf{F}_*$,

$$\mathbf{F}^P_* = A_*^{-n} \circ \mathbf{F}_*^{\mathbf{t}^n P} \circ A^n_*. \tag{5.5}$$

5.3. **Critical points and periodic points.** Consider $\mathbf{F} = [\mathbf{f}_\pm : \mathbf{U}_\pm \to \mathbf{S}] \in \boldsymbol{\mathcal{W}}^u_{loc}$ sufficiently close to $\mathbf{F}_*$, and let $\mathbf{F}_n := \mathcal{R}^n\mathbf{F}$ for all $n \in \mathbb{Z}$. Within the cascade $\mathbf{F}^{\geq 0}$, $\mathbf{f}_\pm$ is the first return map of points in $\mathbf{U}_\pm$ back to $\mathbf{S}$. In particular, $\mathbf{U}_- \cup \mathbf{U}_+$ is disjoint from $\mathbf{F}^P(\mathbf{U}_-)$ for all $P < (0,1,0)$ and $\mathbf{F}^P(\mathbf{U}_+)$ for all $P < (0,0,1)$.

Definition 5.11. We define the *zeroth renormalization tiling* $\boldsymbol{\Delta}_0 = \boldsymbol{\Delta}_0(\mathbf{F})$ associated to $\mathbf{F}^{\geq 0}$ to be the tiling consisting of $\Delta_0(0) := \overline{\mathbf{U}_+}$ and $\Delta_0(1) := \overline{\mathbf{U}_-}$, as well as $\mathbf{F}^P(\Delta_0(0))$ for all $P < (0,0,1)$ and $\mathbf{F}^P(\Delta_0(1))$ for all $P < (0,1,0)$. We label the tiles in left-to-right order as $\Delta_0(i)$ for $i \in \mathbb{Z}$. For all $n \in \mathbb{Z}_{<0}$, we define the n^{th} *renormalization tiling* to be the rescaling of the zeroth tiling for $\mathbf{F}_n$, namely

$$\boldsymbol{\Delta}_n(\mathbf{F}) = A_*^n(\boldsymbol{\Delta}_0(\mathbf{F}_n)).$$

Near $\mathbf{F}_*$, the tiling $\boldsymbol{\Delta}_0(\mathbf{F})$ moves holomorphically in $\mathbf{F}$. In general, for $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, the tiling $\boldsymbol{\Delta}_n(\mathbf{F}) := A_*^n(\boldsymbol{\Delta}_0(\mathbf{F}_n))$ is well-defined for all sufficiently large $n \ll 0$. Each tile $\Delta_n(i)$ is a compact disk in $\mathbb{C}$.

Definition 5.12. Consider $[f : U_f \to V] \in \mathcal{W}^u_{\text{loc}}$ and the associated pre-corona $\mathbf{F} = [\mathbf{f}_\pm : \mathbf{U}_\pm \to \mathbf{S}] \in \boldsymbol{\mathcal{W}}^u_{loc}$. Given a subset Z of U_f, the *full lift* $\mathbf{Z}$ of Z to the dynamical plane of $\mathbf{F}$ is defined as

$$\mathbf{Z} := \bigcup_{0 \leq P < (0,0,1)} \mathbf{F}^P(\mathbf{Z}_0) \cup \bigcup_{0 \leq P < (0,1,0)} \mathbf{F}^P(\mathbf{Z}_1),$$

where $\mathbf{Z}_0$ and $\mathbf{Z}_1$ are the embedding of $Z \cap \Delta_0(0,f)$ and $Z \cap \Delta_0(1,f)$ to the dynamical plane of $\mathbf{F}$.

In particular, we will define the *Herman curve* $\mathbf{H}$ of $\mathbf{F}_*$ to be the full lift of the Herman quasicircle of f_*. Observe that $\mathbf{H}$ is an A_*-invariant quasiarc.

Let us fix $\mathbf{F}$ in $\boldsymbol{\mathcal{W}}^u$. For every $x \in \mathbb{C}$ and $T \in \mathbf{T}$, we denote the finite orbit of x up to time T by

$$\text{orb}_x^T(\mathbf{F}) := \left\{\mathbf{F}^P(x) \,:\, 0 \leq P \leq T\right\}.$$

Definition 5.13. For $P \in \mathbf{T}_{>0}$, let us denote by $\text{CP}\left(\mathbf{F}^P\right)$ the set of critical points of $\mathbf{F}^P$ and by $\text{CV}\left(\mathbf{F}^P\right)$ the set of critical values of $\mathbf{F}^P$. We say that a point x is

- ▹ a *critical point* of $\mathbf{F}^{\geq 0}$ if it is in $\text{CP}\left(\mathbf{F}^P\right)$ for some $P \in \mathbf{T}_{>0}$,
- ▹ a *critical value* of $\mathbf{F}^{\geq 0}$ if it is in $\text{CV}\left(\mathbf{F}^P\right)$ for some $P \in \mathbf{T}_{>0}$, and
- ▹ a *periodic point* of $\mathbf{F}^{\geq 0}$ if there is some $P \in \mathbf{T}_{>0}$ such that $\mathbf{F}^P(x) = x$.

Lemma 5.14. *For $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, critical points of $\mathbf{F}^{\geq 0}$ satisfy the following properties.*

(1) *A point x is a critical point of $\mathbf{F}^{\geq 0}$ if and only if $\mathbf{F}^P(x) = 0$ for some $P \in \mathbf{T}_{>0}$.*

(2) *For $P \in \mathbf{T}_{>0}$,*

$$\text{CP}\left(\mathbf{F}^P\right) = \bigcup_{0 < S \leq P} \mathbf{F}^{-S}\{0\} \quad \text{and} \quad \text{CV}\left(\mathbf{F}^P\right) = \{\mathbf{F}^S(0) \,:\, 0 \leq S < P\}.$$

(3) *There is some $K_\mathbf{F} \in \mathbf{T}_{>0}$ such that for every power-triple $P < K_\mathbf{F}$, every critical point of $\mathbf{F}^P$ has local degree d. If 0 is not periodic, this is still true for $P \geq K_\mathbf{F}$. In general, for every $P \in \mathbf{T}$, there is some $k \in \mathbb{N}$ such that the local degree of every critical point of $\mathbf{F}^P$ is at most k.*

Let $T := \min\{(0,1,0),(0,0,1)\}$. If $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u_{loc}$, then for every $P < T$,

(4) $\text{CV}\left(\mathbf{F}^P\right)$ *is a subset of $\boldsymbol{\Delta}_0(\mathbf{F}) \backslash \mathbf{S} \cup \{0\}$ which moves holomorphically with $\mathbf{F}$, and*

(5) *every critical point of* $\mathbf{F}^P$ *has local degree* d.

Proof. Pick a bounded domain $\mathbf{D} \Subset \mathbb{C}$ and select a connected component $\mathbf{D'}$ of $\mathbf{F}^{-P}(\mathbf{D})$. Recall that for sufficiently large $n \ll 0$, the map $\mathbf{F}^P : \mathbf{D'} \to \mathbf{D}$ can be identified via $h_n^{\#}$ with some iterate $f_n^{s_n} : D' \to D$ for some domains $D', D \Subset \mathbb{C}$ and some $s_n \geq 0$. Therefore, x is a critical point of $\mathbf{F}^P$ if and only if $(h_n^{\#})^{-1}(x)$ is a critical point of $f_n^{s_n}$, which happens precisely when $\mathbf{F}^S(x) = 0$ for some $S \leq P$. This leads to (1) and (2).

Suppose $[\mathbf{F} : \mathbf{U}_\pm \to \mathbf{S}] \in \mathcal{W}^u_{loc}$ and $P \leq T$. For all $S < P$, $\mathbf{F}^S(0)$ is contained in some tile $\Delta_0(i, \mathbf{F})$ that is disjoint from $\mathbf{S}$. This implies (4). Also, (5) follows from the fact that for every critical point x of $\mathbf{F}^P$, $\mathrm{orb}_x^P(\mathbf{F})$ passes through the critical value 0 exactly once.

If $\mathbf{F}$ is not close to $\mathbf{F}_*$, then we can take some $n \ll 0$ such that $\mathcal{R}^n\mathbf{F} \in \mathcal{W}^u_{loc}$. Then, (3) follows from (4) and (5) by taking $K_\mathbf{F}$ to be $\mathbf{t}^n T$ and k to be such that $P < (k-1)K_\mathbf{F}$. □

Lemma 5.15 (Discreteness)**.** *For any* $\mathbf{F} \in \mathcal{W}^u$ *and any bounded open subset* D *of* $\mathbb{C}$*, there is some* $Q \in \mathbf{T}_{>0}$ *such that for all* $\mathbf{G} \in \mathcal{W}^u$ *close to* $\mathbf{F}$ *and whenever* $P' < P < Q$*,*

(1) $\mathbf{G}^P$ *is well-defined and univalent on* D*, and*
(2) $\mathbf{G}^P(D)$ *is disjoint from* $\mathbf{G}^{P'}(D)$.

For every $x \in \mathbb{C}$ *and* $T \in \mathbf{T}$*,* $\mathrm{orb}_x^T(\mathbf{F})$ *is discrete in* $\mathbb{C}$.

Proof. There exist some integers $m \leq 0$ and $j \in \{0,1\}$ such that D is compactly contained in some level m tile $\Delta_m(j, \mathbf{G})$ associated to $\mathbf{G}$ for all $\mathbf{G}$ close to $\mathbf{F}$. Set $Q := \mathbf{t}^m \min\{(0,1,0),(0,0,1)\}$. For $P < Q$, the tile $\Delta_m(j,\mathbf{G})$ is mapped by $\mathbf{G}^P$ univalently onto to some other tile $\Delta_m(i,\mathbf{G})$ of level m disjoint from $\Delta_m(0,\mathbf{G}) \cup \Delta_m(1,\mathbf{G})$. This implies (1) and (2).

Let us fix $x \in \mathbb{C}$ and $T \in \mathbf{T}$. Suppose y is an accumulation point of $\mathrm{orb}_x^T(\mathbf{F})$. Pick a small open neighborhood D of y. From the first part, $\mathbf{F}^P(D)$ is disjoint from D for all sufficiently small $P \in \mathbf{T}_{>0}$. This implies that only finitely many points in $\mathrm{orb}_x^T(\mathbf{F})$ are contained in D. □

By a straightforward compactness argument, the lemma above has the following consequence.

Corollary 5.16 (Proper discontinuity)**.** *For any* $P \in \mathbf{T}$*, any compact subset* $\mathbf{Y}$ *of* $\mathrm{Dom}\left(\mathbf{F}^P\right)$*, and any bounded subset* $\mathbf{X}$ *of* $\mathbb{C}$*, there are at most finitely many power-triples* $T \leq P$ *such that* $\mathbf{F}^T(\mathbf{Y})$ *intersects* $\mathbf{X}$.

Corollary 5.17. *Every critical point* x *of* $\mathbf{F}^{\geq 0}$ *admits a minimal* $P \in \mathbf{T}_{>0}$*, called the generation of* x*, such that* $\mathbf{F}^P(x) = 0$.

Proof. By definition, there is some $P \in \mathbf{T}_{>0}$ such that $\mathbf{F}^P(x) = 0$. By Lemma 5.15, $\mathrm{orb}_x^P(\mathbf{F})$ is discrete, so there are at most finitely many power-triples S such that $S < P$ and $\mathbf{F}^S(x) = 0$. □

Corollary 5.18. *Every periodic point of* $\mathbf{F}^{\geq 0}$ *has a minimal period.*

Proof. Suppose x is a periodic point of $\mathbf{F}^{\geq 0}$. The set $\mathbf{T}_x := \{P \in \mathbf{T} : \mathbf{F}^P(x) = x\}$ of periods of x is a sub-semigroup of $\mathbf{T}$. Pick a small neighborhood D of x. By Lemma 5.15, there is some $Q \in \mathbf{T}_{>0}$ such that for all $0 < P < Q$, $\mathbf{F}^P(D)$ is disjoint from D

and thus $P \notin \mathbf{T}_x$. This implies that $\mathbf{T}_x$ is finitely generated, and in particular, of the form $\{nS\}_{n\in\mathbb{N}}$, where $S \in \mathbf{T}_{>0}$ is the minimal period. □

5.4. **The escaping sets.** Consider $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$.

Definition 5.19. Given $P \in \mathbf{T}$, the P^{th} *escaping set* of $\mathbf{F}$ is

$$\mathbf{I}_{\leq P}(\mathbf{F}) := \mathbb{C}\backslash \mathrm{Dom}\left(\mathbf{F}^P\right).$$

The *finite-time escaping set* of $\mathbf{F}$ is the union

$$\mathbf{I}_{<\infty}(\mathbf{F}) := \bigcup_{P\in\mathbf{T}} \mathbf{I}_{\leq P}(\mathbf{F}),$$

the *infinite-time escaping set* of $\mathbf{F}$ is

$$\mathbf{I}_{\infty}(\mathbf{F}) := \{z \in \mathbb{C}\backslash \mathbf{I}_{<\infty}(\mathbf{F}) \,:\, \mathbf{F}^P(z) \to \infty \text{ as } P \to \infty\},$$

and the *full escaping set* of $\mathbf{F}$ is

$$\mathbf{I}(\mathbf{F}) := \mathbf{I}_{<\infty}(\mathbf{F}) \cup \mathbf{I}_{\infty}(\mathbf{F}).$$

Lemma 5.20. *For any $P \in \mathbf{T}$, every connected component of $\mathbf{I}_{\leq P}(\mathbf{F})$ is unbounded.*

Proof. There exists some $n \leq 0$ such that $\mathbf{F}_n := \mathcal{R}^n\mathbf{F}$ is in $\boldsymbol{\mathcal{W}}^u_{loc}$. Since the domains of $\mathbf{f}_{n,\pm}$ are simply connected, then $\mathrm{Dom}\left(\mathbf{F}_n^P\right)$ is simply connected for all $P \in \mathbf{T}$ and so the claim is true for $\mathbf{F}_n$. Since $\mathbf{F}$ is just a rescaling of $\mathbf{F}_n$, the claim is also true for $\mathbf{F}$. □

In Section 6, we will thoroughly study the structure of the finite-time escaping set of the fixed point $\mathbf{F}_*$. In Section 7, we will show that in the hyperbolic case, the finite and infinite-time escaping sets do not carry any invariant line field. This will imply that the unstable manifold indeed has codimension one.

It is clear from the definition that the boundary of $\mathbf{I}_{\leq P}(\mathbf{F})$ coincides with the boundary of $\mathrm{Dom}\left(\mathbf{F}^P\right)$. Points on $\partial\mathbf{I}_{\leq P}(\mathbf{F})$ can be regarded as essential singularities of $\mathbf{F}^P$. The following lemma is an analog of Picard's theorem.

Lemma 5.21. *For every $P \in \mathbf{T}_{>0}$, every point $z \in \partial\mathbf{I}_{\leq P}(\mathbf{F})$, and every scale $r > 0$, the image $\mathbf{F}^P(D)$ of any connected component D of $\mathbb{D}(z,r) \cap \mathrm{Dom}\left(\mathbf{F}^P\right)$ is dense in $\mathbb{C}$.*

This lemma is a direct consequence of σ-properness of $\mathbf{F}^P$. The keen reader may refer to [DL23, Lemma 6.5] for a detailed proof.

Corollary 5.22. *For every $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, $P \in \mathbf{T}_{>0}$, and $x \in \mathbb{C}$, the boundary of $\mathbf{I}_{\leq P}(\mathbf{F})$ is the set of accumulating points of $\mathbf{F}^{-P}(x)$.*

We will later show that $\mathbf{I}_{<\infty}(\mathbf{F})$ has no interior and its closure coincides with the "Julia set" of $\mathbf{F}$, which we will define in the next subsection. This corollary is an analog of the basic result in holomorphic dynamics which states that iterated preimages are dense in the Julia set. The proof below is similar to [DL23, Corollary 6.7].

Proof. By Lemma 5.15, there exists a disk neighborhood B of x such that $B\backslash\{x\} \cap \mathrm{CV}\left(\mathbf{F}^P\right) = \varnothing$. Then, every connected component B' of $\mathbf{F}^{-P}(B)$ contains at most one critical point and the degree of $\mathbf{F}^P : B' \to B$ is at most some uniform constant. Let $\Omega \subset B$ be an even smaller disk neighborhood of x such that $\mathrm{mod}(B\backslash\overline{\Omega}) \asymp 1$. The preimage $\Omega' \subset B'$ of Ω under $\mathbf{F}^P$ is also a disk with $\mathrm{mod}(B'\backslash\overline{\Omega'}) \asymp 1$.

Let us pick a connected component D of $\mathrm{Dom}\left(\mathbf{F}^P\right)$, a point $y \in \partial D$, and a small $\varepsilon > 0$. By Lemma 5.21, there is a connected component $\Omega' \subset D$ of $\mathbf{F}^{-P}(\Omega)$ that is of distance at most ε away from y. Since $\mathrm{mod}\left(B' \backslash \overline{\Omega'}\right) \asymp 1$, then Ω' has a small diameter depending on ε. Since Ω' contains point in $\mathbf{F}^{-P}(x)$, the assertion follows. □

5.5. **Fatou-Julia theory.** Let us formulate a Fatou-Julia theory for our dynamical systems $\mathbf{F}$ in $\boldsymbol{\mathcal{W}}^u$ and state a few analogues of basic results in classical holomorphic dynamics.

Definition 5.23. The *Fatou set* $\mathfrak{F}(\mathbf{F})$ of $\mathbf{F}$ is the set of points z which admit a small neighborhood $D \subset \mathbb{C} \backslash \mathbf{I}_{<\infty}(\mathbf{F})$ such that $\{\mathbf{F}^P|_D\}_{P \in \mathbf{T}}$ forms a normal family. The *Julia set* $\mathfrak{J}(\mathbf{F})$ of $\mathbf{F}$ is the complement $\mathbb{C} \backslash \mathfrak{F}(\mathbf{F})$.

Clearly, $\mathfrak{J}(\mathbf{F})$ contains the closure of $\mathbf{I}_{<\infty}(\mathbf{F})$.

We say that a connected component D of $\mathfrak{F}(\mathbf{F})$ is *periodic* if there is some $P \in \mathbf{T}_{>0}$ such that $\mathbf{F}^P(D) = D$. The smallest such P is called the *period* of D. Moreover, we say that D is *pre-periodic* if there is some $Q \in \mathbf{T}$ such that $\mathbf{F}^Q(D)$ is periodic. The smallest such Q is called the *pre-period* of D. (These quantities exist due to Lemma 5.15. Compare with Corollary 5.18.)

Definition 5.24. The *postcritical set* of $\mathbf{F}$ is

$$\mathfrak{P}(\mathbf{F}) := \overline{\{\mathbf{F}^P(0) \,:\, P \in \mathbf{T}\}}.$$

The postcritical set is is characterized as the smallest forward invariant closed set such that

$$\mathbf{F}^P : \mathrm{Dom}\left(\mathbf{F}^P\right) \backslash \mathbf{F}^{-P}\left(\mathfrak{P}(\mathbf{F})\right) \to \mathbb{C} \backslash \mathfrak{P}(\mathbf{F})$$

is an unbranched covering map which is a local isometry with respect to the hyperbolic metrics.

In the case of $\mathbf{F} = \mathbf{F}_*$, equation (5.5) implies self-similarity of the corresponding dynamical sets.

Lemma 5.25. *The linear map A_* preserves $\mathfrak{F}(\mathbf{F}_*)$, $\mathfrak{J}(\mathbf{F}_*)$, $\mathbf{I}_{<\infty}(\mathbf{F}_*)$, $\mathbf{I}_\infty(\mathbf{F}_*)$, and $\mathfrak{P}(\mathbf{F}_*)$. For all $P \in \mathbf{T}_{>0}$, $A_*(\mathbf{I}_{\leq P}(\mathbf{F}_*)) = \mathbf{I}_{\leq \mathbf{t}P}(\mathbf{F}_*)$.*

Given a periodic point x of (minimal) period P of some $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, we say that x is *superattracting / attracting / parabolic / Siegel / Cremer / repelling* if x is a *superattracting / attracting / parabolic / Siegel / Cremer / repelling* fixed point of $\mathbf{F}^P$.

Proposition 5.26. *Suppose $\mathbf{F}$ admits a periodic point x of some period P.*

1. *If x is attracting or parabolic, then the critical orbit $\{\mathbf{F}^T(0)\}_{T \in \mathbf{T}}$ converges to the periodic orbit $\mathrm{orb}_0^P(\mathbf{F})$.*
2. *If x is Cremer, then $x \in \mathfrak{P}(\mathbf{F})$.*
3. *If x is Siegel, then the boundary of the Siegel disk of $\mathbf{F}^P$ centered at x is contained in $\mathfrak{P}(\mathbf{F})$.*

Proof. (1) follows from a standard analytic continuation argument: if the forward orbit of a periodic Fatou component containing an attracting (resp. superattracting or parabolic) cycle does not containg 0, then the local linearizing (resp. Böttcher or Fatou) coordinates can be extended to a conformal map onto the whole plane, which is impossible. See [Mil06, Lemma 8.5] for details. Suppose $x \notin \mathfrak{P}(\mathbf{F})$. From (1), x is either Cremer or Siegel.

Let us first prove (2) by showing that x must be Siegel. For all $T \in \mathbf{T}$, let us denote by D_T the connected component of $\mathrm{Dom}\left(\mathbf{F}^T\right)\backslash\mathbf{F}^{-T}\left(\mathfrak{P}(\mathbf{F})\right)$ containing x. Suppose first D_P is properly contained in D_0. Then, $\mathbf{F}^P : D_P \to D_0$ is strictly expanding with respect to the hyperbolic metric of D_0, which implies that x must be repelling. Suppose instead $D_P = D_0$. Then, $\left\{\mathbf{F}^{nP}|_{D_0}\right\}_{n\in\mathbb{N}}$ is a normal family of automorphisms of a hyperbolic Riemann surface. By Denjoy-Wolff, the fixed point x must be Siegel.

Denote by Z the Siegel disk cenetered at x. If there exists some minimal $T \in \mathbf{T}$ where $\mathbf{F}^T(0)$ intersects Z, then the intersection $\mathfrak{P}(\mathbf{F}) \cap Z$ is a single $\mathbf{F}^P$-invariant curve on Z. Suppose for a contradiction that $\mathbb{C}\backslash\mathfrak{P}(\mathbf{F})$ intersects the boundary ∂Z. Then, a component E_0 of $\mathbb{C}\backslash\mathfrak{P}(\mathbf{F})$ contains some neighborhood of ∂Z. For $n \in \mathbb{N}$, let E_{nP} be the connected component of $\mathrm{Dom}\left(\mathbf{F}^{nP}\right)\backslash\mathbf{F}^{-nP}(\mathfrak{P}(\mathbf{F}))$ containing $E_0 \cap Z_0$. There are again two cases. If $E_P = E_0$, then $\left\{\mathbf{F}^{nP}|_{E_0}\right\}_{n\in\mathbb{N}}$ forms a normal family and E_0 must be contained in the Fatou set, which is a contradiction. If E_P is a proper subset of E_0, then $\mathbf{F}^P : E_P \to E_0$ is strictly expanding with respect to the hyperbolic metric of E_0, which would contradict the fact that $\mathbf{F}^P$ restricts to a self diffeomorphism of any invariant curve in $Z \cap E_0$. ☐

For any tangent vector v at a point z in $\mathbb{C}\backslash\mathfrak{P}(\mathbf{F})$, denote by $\|v\|$ the norm of v with respect to the hyperbolic metric of $\mathbb{C}\backslash\mathfrak{P}(\mathbf{F})$. If $z \in \mathfrak{P}$, we set $\|v\| = \infty$.

Lemma 5.27 (Julia expansion)**.** *For every point z in $\mathfrak{J}(\mathbf{F})\backslash\mathbf{I}_{<\infty}(\mathbf{F})$,*

$$\left\|\left(\mathbf{F}^P\right)'(z)\right\| \to \infty \quad \textit{as} \quad P \to \infty.$$

Proof. Let us fix a point $z \in \mathfrak{J}(\mathbf{F})\backslash\mathbf{I}_{<\infty}(\mathbf{F})$. Without loss of generality, assume that z does not eventually land on $\mathfrak{P}(\mathbf{F})$.

For any $P \in \mathbf{T}_{>0}$, let

$$\mathfrak{P}_P := \mathbf{I}_{\leq P}(\mathbf{F}) \cup \mathbf{F}^{-P}(\mathfrak{P}(\mathbf{F})).$$

The map $\mathbf{F}^P : \mathbb{C}\backslash\mathfrak{P}_P \to \mathbb{C}\backslash\mathfrak{P}$ is a local isometry with respect to their hyperbolic metrics. Since the union $\bigcup_{P\in\mathbf{T}}\mathfrak{P}_P$ is a dense subset of the Julia set, the distance between $\mathfrak{P}_P$ and z shrinks to 0 as $P \to \infty$. Consequently, the distance r_P between z and $\mathfrak{P}_P$ with respect to the hyperbolic metric of $\mathbb{C}\backslash\mathfrak{P}$ also tends to 0 as $P \to \infty$. The inclusion map $\iota : \mathbb{C}\backslash\mathfrak{P}_P \to \mathbb{C}\backslash\mathfrak{P}$ is contracting by some factor $K(r_P)$ where $K(r) \to 0$ as $r \to 0$. Therefore, as $P \to \infty$, $\left\|\left(\mathbf{F}^P\right)'(z)\right\| \geq K(r_P)^{-1} \to \infty$. ☐

Denote by $\mathrm{dist}_{\hat{\mathbb{C}}}(\cdot,\cdot)$ the spherical distance between two subsets of $\hat{\mathbb{C}}$.

Theorem 5.28 (Measure-theoretic attractor)**.** *If $\mathfrak{J}(\mathbf{F})$ has no interior, then for almost every point z in $\mathfrak{J}(\mathbf{F})\backslash\mathbf{I}_{<\infty}(\mathbf{F})$,*

$$\mathrm{dist}_{\hat{\mathbb{C}}}\left(\mathbf{F}^P(z), \mathfrak{P}(\mathbf{F}) \cup \{\infty\}\right) \to 0 \qquad \textit{as } P \to \infty.$$

In other words, almost every non-escaping point in the Julia set is attracted to the postcritical set.

Proof. Suppose for a contradiction that there exist a positive number $\varepsilon > 0$ and a positive area subset E of $\mathfrak{J}(\mathbf{F})\backslash\mathbf{I}_{<\infty}(\mathbf{F})$ such that for all $z \in E$,

$$\limsup_{P\to\infty} \mathrm{dist}_{\hat{\mathbb{C}}}\left(\mathbf{F}^P(z), \mathfrak{P}(\mathbf{F}) \cup \{\infty\}\right) \geq \varepsilon.$$

Let z be a Lebesgue density point of E. There is a sequence of power-triples P_n such that $P_n \to \infty$ and $y_n := \mathbf{F}^{P_n}(z)$ lies in the compact set

$$K := \{z \in \mathbb{C} : \operatorname{dist}_{\hat{\mathbb{C}}}(z, \mathfrak{P}(\mathbf{F}) \cup \{\infty\}) \geq \varepsilon\}.$$

For each $n \in \mathbb{N}$, consider the spherical ball B_n of radius $\varepsilon/2$ centered at y_n, and let B_n' be the lift of B_n under $\mathbf{F}^{P_n}$ containing z.

By Lemma 5.27, $\left\| \left(\mathbf{F}^{P_n}\right)'(z) \right\| \to \infty$. Since K is compact and $\mathbf{F}^{P_n}|_{B_n'}$ has bounded distortion, the disks B_n' must shrink to a point. Since z is a density point of E,

$$\lim_{n\to\infty} \frac{\operatorname{area}(B_n' \cap E)}{\operatorname{area}(B_n')} = 1.$$

Therefore, we also have

$$\lim_{n\to\infty} \frac{\operatorname{area}(B_n \cap \mathfrak{J}(\mathbf{F}) \backslash \mathbf{I}_{<\infty}(\mathbf{F}))}{\operatorname{area}(B_n)} = 1.$$

Since K is compact, y_n converges in subsequence to some point $y \in K$. Then, the ball B of radius $\varepsilon/2$ centered at y must have the same area as $B \cap \mathfrak{J}(\mathbf{F}) \backslash \mathbf{I}_{<\infty}(\mathbf{F})$. Since $\mathfrak{J}(\mathbf{F})$ is closed, then the ball B has to be contained in $\mathfrak{J}(\mathbf{F})$. This contradicts the assumption that $\mathfrak{J}(\mathbf{F})$ has no interior. □

6. The external structure of $\mathbf{F}_*$

Consider the dynamics of $\mathbf{F} = \mathbf{F}_*$ corresponding to the fixed point f_* of the renormalization operator. We denote by $\mathbf{H}$ the Herman curve of $\mathbf{F}$, which is defined to be the full lift of the Herman curve of f_*. The action of $\mathbf{F}$ along $\mathbf{H}$ can be described as follows. For $a \in \mathbb{C}$, we denote the translation map by a by $T_a(z) := z + a$.

Lemma 6.1. *There is a quasisymmetric map $h : (\mathbf{H}, 0) \to (\mathbb{R}, 0)$ that conjugates the cascade $\mathbf{F}^{\geq 0}|_{\mathbf{H}}$ with the cascade of translations $\left(T^P\right)_{P \in \mathbf{T}}$ defined by $T^{(n,a,b)} := T_{\mathbf{t}^{-n}(b\mathbf{v} - a\mathbf{u})}$, where $\mathbf{u}, \mathbf{v} > 0$ and $\theta = \frac{\mathbf{u}}{\mathbf{u}+\mathbf{v}}$.*

Proof. The pre-corona F_* associated to f_* admits an invariant quasiarc which projects to the Herman curve of f_*. In linear coordinates, this corresponds to an invariant quasiarc $\mathbf{H}_0$ of $\mathbf{F} = (\mathbf{f}_{0,\pm} : \mathbf{U}_\pm \to \mathbf{S})$ which passes through 0 and connects $\mathbf{f}_{0,+}(0)$ and $\mathbf{f}_{0,-}(0)$. The dynamics of $\mathbf{f}_{0,\pm}$ along $\mathbf{H}_0$ is quasisymetrically conjugate to a pair of translations $\left(T_{-\theta}|_{[0,1-\theta]}, T_{1-\theta}|_{[-\theta,0]}\right)$ on the real interval $[-\theta, 1-\theta]$. Set $\mathbf{u} = -\theta$ and $\mathbf{v} = 1 - \theta$. As we extend $\mathbf{f}_{0,\pm}$ to its maximal σ-proper extension via A_*, the quasisymmetric conjugacy h between $(\mathbf{f}_{0,-}, \mathbf{f}_{0,+})$ and $(T_{-\mathbf{u}}, T_{-\mathbf{v}})$ extends to the whole lift $\mathbf{H}$ of $\mathbf{H}_0$. The claim holds because the pairs $(\mathbf{f}_{0,-}, \mathbf{f}_{0,+})$ and $(T_{-\mathbf{u}}, T_{-\mathbf{v}})$ generate the cascades $\mathbf{F}^{\geq 0}|_{\mathbf{H}}$ and $T^{(n,a,b)} := T_{\mathbf{t}^{-n}(b\mathbf{v} - a\mathbf{u})}$ via iteration and rescaling according to (5.5) and Appendix B.2. □

In this section, we will comprehensively describe the dynamics of $\mathbf{F}$ beyond $\mathbf{H}$. We study the structure of preimages of $\mathbf{H}$ in §6.1–6.2, then the structure of the finite-time escaping set $\mathbf{I}_{<\infty} := \mathbf{I}_{<\infty}(\mathbf{F})$ in §6.3–6.4, and lastly the dynamical puzzles cut out by subsets of $\mathbf{I}_{<\infty}$ in §6.5.

6.1. **Lakes.** Let us label the components of $\mathbb{C} \backslash \mathbf{H}$ by $\mathbf{O}^0$ and $\mathbf{O}^\infty$, which we will refer to as the *oceans* of $\mathbf{F}$. The two oceans will be distinguished as follows. For $\bullet \in \{0, \infty\}$ and for any point x in $\mathbf{S} \cap \mathbf{O}^\bullet$ close to 0, we assume that there are $d_\bullet$ preimages of x under $\mathbf{f}_{0,\pm} : \mathbf{U}_\pm \to \mathbf{S}$ that are located near the critical point and inside of $\mathbf{O}^\bullet$.

Definition 6.2. A *lake* $\mathbf{O}$ *of generation* $P \in \mathbf{T}$ is a connected component of $\mathbf{F}^{-P}(\mathbf{O}^\bullet)$ for some $\bullet \in \{0,\infty\}$. Its *coast* is defined as $\partial^c\mathbf{O} \coloneqq \partial\mathbf{O} \cap \operatorname{Dom}\left(\mathbf{F}^P\right)$.

Lemma 6.3 (Chessboard rule)**.** *For every $P \in \mathbf{T}_{>0}$ and $\bullet \in \{0,\infty\}$, the preimage $\mathbf{F}^{-P}(\mathbf{H})$ is a tree in $\operatorname{Dom}\left(\mathbf{F}^P\right)$ and $\mathbf{F}^{-P}(\mathbf{O}^\bullet)$ is disjoint union of lakes $\bigcup_{i\in\mathbb{N}}\mathbf{O}_i$ of generation P such that*

(1) *each lake $\mathbf{O}_i$ is a disk which is unbounded in $\operatorname{Dom}\left(\mathbf{F}^P\right)$ and does not separate $\operatorname{Dom}\left(\mathbf{F}^P\right)$;*
(2) *for $j \neq i$, the intersection $\partial^c\mathbf{O}_i \cap \partial^c\mathbf{O}_j$ is either empty or a singleton consisting of a critical point of $\mathbf{F}^P$.*

Proof. The whole lemma follows immediately from σ-properness of the cascade, e.g. [DL23, Lemma 5.1], and the fact that $\mathrm{CV}(\mathbf{F})$ is contained in $\mathbf{H}$. □

Given any lake $\mathbf{O}$ of some generation $P \in \mathbf{T}_{>0}$, the map $\mathbf{F}^P$ sends $\mathbf{O}$ univalently onto an ocean, and its coast homeomorphically onto $\mathbf{H}$. In general, when $0 < P < Q$, a lake of generation Q is contained in a lake of generation P, and $\mathbf{F}^{Q-P}$ conformally sends any lake of generation Q onto a lake of generation P.

Lemma 6.4. *For every $P \in \mathbf{T}_{>0}$, there is a unique critical point $C_P \in \mathbf{H}$ of $\mathbf{F}^{\geq 0}$ of generation P and a pairwise disjoint collection of lakes*

$$ {}_1\mathbf{O}^0_P, \;\dots,\; {}_{2d_0-3}\mathbf{O}^0_P, \;\; {}_1\mathbf{O}^\infty_P, \;\dots,\; {}_{2d_\infty-3}\mathbf{O}^\infty_P, \tag{6.1}$$

of generation P together with a bouquet of pairwise-disjoint open quasiarcs

$$ {}_1\mathbf{H}^0_P, \;\dots,\; {}_{2d_0-2}\mathbf{H}^0_P, \;\; {}_1\mathbf{H}^\infty_P, \;\dots,\; {}_{2d_\infty-2}\mathbf{H}^\infty_P, \tag{6.2}$$

rooted at C_P such that for each $\bullet \in \{0,\infty\}$ and $j \in \{1,\dots,2d_\bullet - 3\}$,

(1) *the coast of ${}_j\mathbf{O}^\bullet_P$ is ${}_j\mathbf{H}^\bullet_P \cup \{C_P\} \cup {}_{j+1}\mathbf{H}^\bullet_P$;*
(2) *${}_j\mathbf{O}^\bullet_P$ is contained in $\mathbf{O}^\bullet$;*
(3) *${}_j\mathbf{O}^\bullet_P$ is mapped conformally by $\mathbf{F}^P$ onto $\mathbf{O}^\bullet$ if j is is even, and onto $\mathbb{C}\backslash\overline{\mathbf{O}^\bullet}$ if j is odd.*

Proof. The existence and uniqueness of C_P is due to the fact that $\mathbf{F}^P$ restricts to a homeomorphism on $\mathbf{H}$. From the previous lemma, $\mathbf{F}^{-P}(\mathbf{H})$ is a tree. The quasiarcs ${}_j\mathbf{H}^\bullet_P$'s are precisely the components of $\mathbf{F}^{-P}(\mathbf{H})\backslash\{C_P\}$, and the lakes ${}_j\mathbf{O}^\bullet_P$'s in (6.2) are precisely the connected components of $\operatorname{Dom}\left(\mathbf{F}^P\right)\backslash\mathbf{F}^{-P}(\mathbf{H})$ which touch $\mathbf{H}$ at exactly one point, which is C_P. For all $S < P$, the image of each quasiarc ${}_j\mathbf{H}^\bullet_P$ under $\mathbf{F}^S$ is disjoint from 0. Therefore, $\mathbf{F}^P$ maps each of ${}_j\mathbf{H}^\bullet_P$ onto a component of $\mathbf{H}\backslash\{0\}$ homeomorphically. They can be enumerated such that the three claims above hold because C_P has inner and outer criticalities d_0 and d_∞ respectively. □

Each quasiarc in (6.2) is called a *spine* of C_P. The spines in (6.2) are labelled in counterclockwise order about C_P.

Let us pick a pair of power-triples $P, Q \in \mathbf{T}_{>0}$. For any $\bullet \in \{0,\infty\}$ and any $j \in \{1,\dots,d_\bullet - 1\}$, the union of two consecutive spines ${}_{2j-1}\mathbf{H}^\bullet_P \cup {}_{2j}\mathbf{H}^\bullet_P$ are mapped homeomorphically by $\mathbf{F}^P$ onto $\mathbf{H}\backslash\{0\}$ and so it contains a unique critical point ${}_jC^\bullet_{P,Q}$ of generation $P+Q$. Attached to this critical point is a bouquet of lakes

$$ {}_{j,1}\mathbf{O}^{\bullet,0}_{P,Q}, \;\dots,\; {}_{j,2d_0-3}\mathbf{O}^{\bullet,0}_{P,Q}, \;\; {}_{j,1}\mathbf{O}^{\bullet,\infty}_{P,Q}, \;\dots,\; {}_{j,2d_\infty-3}\mathbf{O}^{\bullet,\infty}_{P,Q}, $$

of generation $P+Q$ together with spines

$$ {}_{j,1}\mathbf{H}^{\bullet,0}_{P,Q}, \;\dots,\; {}_{j,2d_0-2}\mathbf{H}^{\bullet,0}_{P,Q}, \; {}_{j,1}\mathbf{H}^{\bullet,\infty}_{P,Q}, \;\dots,\; {}_{j,2d_\infty-2}\mathbf{H}^{\bullet,\infty}_{P,Q}, $$

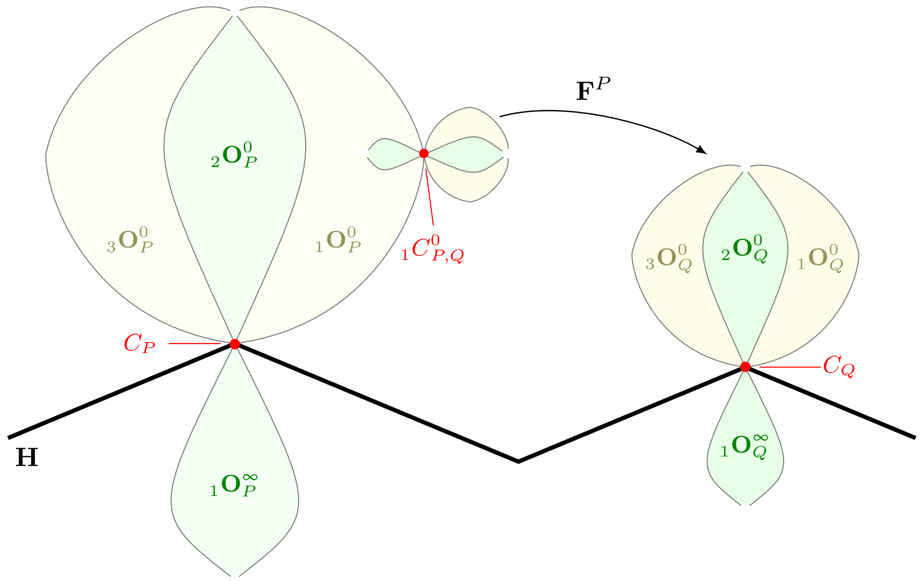

FIGURE 6. The structure of lakes attached to critical points C_P, C_Q, and ${}_1C^0_{P,Q}$ when $(d_0, d_\infty) = (3,2)$.

meeting at ${}_jC^{\bullet}_{P,Q}$ such that each of ${}_{j,k}\mathbf{O}^{\bullet,\circ}_{P,Q}$ has coast ${}_{j,k}\mathbf{H}^{\bullet,\circ}_{P,Q} \cup \{{}_jC^{\bullet}_{P,Q}\} \cup {}_{j,k+1}\mathbf{H}^{\bullet,\circ}_{P,Q}$ and is mapped univalently by $\mathbf{F}^P$ onto ${}_k\mathbf{O}^{\circ}_Q$.

Consider a tuple $S = (P_1, \dots, P_{m+1}) \in \mathbf{T}^{m+1}_{>0}$ of $m+1$ power-triples for some $m \in \mathbb{N}$. We denote the sum by

$$|S| \coloneqq \sum_{i=1}^{m+1} P_i.$$

Given $\blacksquare = (\bullet_1, \dots, \bullet_m) \in \{0,\infty\}^m$ and $J = (j_1, \dots j_m)$ where $j_i \in \{1, \dots, d_{\bullet_i} - 1\}$ for all i, we inductively define a critical point ${}_JC^{\blacksquare}_S$ of generation $|S|$. Attached to this critical point are lakes ${}_{J,i}\mathbf{O}^{\blacksquare,\bullet}_S$ for $\bullet \in \{0,\infty\}$ and $i \in \{1, \dots, 2d_\bullet - 3\}$, and spines ${}_{J,j}\mathbf{H}^{\blacksquare,\bullet}_S$ for $\bullet \in \{0,\infty\}$ and $j \in \{1, \dots, 2d_\bullet - 2\}$.

Definition 6.5. We say that a lake $\mathbf{O}$ is a *middle lake* if it is of the form ${}_{J,j}\mathbf{O}^{\blacksquare,\bullet}_S$ described above. The finite tuple S is called the *itinerary* of $\mathbf{O}$.

Consider a lake $\mathbf{O}$ of generation $P \in \mathbf{T}_{>0}$. Let $Q \in \mathbf{T}$ be the smallest power-triple such that the coast of $\mathbf{O}$ touches $\mathbf{F}^{-Q}(\mathbf{H})$.

Lemma 6.6 (Left and right coasts)**.** *The intersection between $\partial^c\mathbf{O}$ and $\mathbf{F}^{-Q}(\mathbf{H})$ is either a singleton or a closed quasiarc, and the complement $\partial^c\mathbf{O}\backslash\mathbf{F}^{-Q}(\mathbf{H})$ consists of two non-empty open quasiarcs $\partial^c_l\mathbf{O}$ and $\partial^c_r\mathbf{O}$.*

Proof. It is sufficient to consider the case when $Q = 0$. The intersection between $\partial^c\mathbf{O}$ and $\mathbf{F}^{-Q}(\mathbf{H})$ is connected because of the tree structure of $\mathbf{F}^{-P}(\mathbf{H})$. For any point z in $\partial^c\mathbf{O} \cap \mathbf{H}$, every component of $\mathbf{H}\backslash\{z\}$ contains infinitely many critical points of $\mathbf{F}^P$ of generation at most P, and each of these points is a branch point of the tree $\mathbf{F}^{-P}(\mathbf{H})$. Since $\partial^c\mathbf{O} \cap \mathbf{H}$ does not contain such branch points, the claim follows. $\square$

We call $\partial^c_l\mathbf{O}$ and $\partial^c_r\mathbf{O}$ the *left and right coasts* of $\mathbf{O}$, and we always assume that $\partial^c_l\mathbf{O}$, $\partial^c\mathbf{O}\cap\mathbf{F}^{-Q}(\mathbf{H})$, and $\partial^c_r\mathbf{O}$ are oriented counterclockwise relative to $\mathbf{O}$.

The closure of the left coast of $\mathbf{O}$ admits a maximal sequence of critical points $c_{l,1}$, $c_{l,2}$, $c_{l,3}$, ... of $\mathbf{F}^P$, labelled in increasing order of generation. We define the *left itinerary* of $\mathbf{O}$ to be the sequence $I_l := (P_{l,1}, P_{l,2}, \ldots)$ where each $P_{l,i}$ is the generation of $c_{l,i}$. We call the supremum of $P_{l,i}$ across all i's the *left generation* G_l of $\mathbf{O}$. Similarly, we define the *right itinerary* I_r and the *right generation* G_r of $\mathbf{O}$.

Lemma 6.7. *Consider a lake $\mathbf{O}$ of generation $P \in \mathbf{T}_{>0}$.*

(1) *The left and right generations of $\mathbf{O}$ are equal to P.*
(2) *If I_l (resp. I_r) is finite, the left (resp. right) coast of $\mathbf{O}$ contains a spine attached a critical point ${}_JC^{\blacksquare}_S$ of generation $|S| = P$.*
(3) *If both I_l and I_r are finite, then $\mathbf{O}$ is a middle lake attached to the critical point ${}_JC^{\blacksquare}_S$.*
(4) *Either I_l or I_r is a finite sequence.*

Proof. Suppose for a contradiction that $G_l < P$, so then there is some $P' \in \mathbf{T}$ such that $G_l < P' < P$. Then, $\mathbf{F}^{P'}(\mathbf{O})$ is a lake of positive generation with an empty left coast, which is impossible due to Lemma 6.6. Therefore, (1) holds.

Suppose I_l is finite. By (1), there exists a critical point c_l of generation P on $\overline{\partial^c_l\mathbf{O}}$. Removing c_l splits the coast into two open quasiarcs, one of which contains no critical points of $\mathbf{F}^P$ and is thus a spine attached to c_l. This implies (2). Suppose I_r is also finite, so there also exists a critical point c_r of generation P on $\overline{\partial^c_r\mathbf{O}}$. The complement of the interval $[c_l, c_r] \subset \partial^c\mathbf{O}$ is now a pair of spines of generation P attached to c_l and c_r respectively. Recall that $\mathbf{F}^P$ sends each of these spines to a component of $\mathbf{H}\backslash\{0\}$. However, since $\mathbf{F}^P : \partial^c\mathbf{O} \to \mathbf{H}$ is a homeomorphism, we see that $c_l = c_r$ and $\mathbf{O}$ is a middle lake. Hence, (3) holds.

Let us now prove (4). We will again assume without loss of generality that $Q = 0$. Let us pick a point y in $\partial^c\mathbf{O}\cap\mathbf{H}$. If the open interval $(y, C_P) \subset \mathbf{H}$ does not contain any critical point of generation $\le P$, then either $\partial^c_l\mathbf{O}$ or $\partial^c_r\mathbf{O}$ is rooted at C_P and contains no other critical points of generation $\le P$. Otherwise, by Lemma 5.14, there are only finitely many critical points of generation $\le P$ within (y, C_P), and they have some maximum generation $R < P$. We then apply the previous argument to the lake $\mathbf{F}^R(\mathbf{O})$ and the interval $\left(\mathbf{F}^R(y), C_{P-R}\right) \subset \mathbf{H}$. $\square$

Consider a critical point ${}_JC^{\blacksquare}_S$ of $\mathbf{F}^{\ge 0}$. There exist lakes

$$ {}_{J,l}\mathbf{O}^{\blacksquare,0}_S,\ {}_{J,r}\mathbf{O}^{\blacksquare,0}_S,\ {}_{J,l}\mathbf{O}^{\blacksquare,\infty}_S,\ {}_{J,r}\mathbf{O}^{\blacksquare,\infty}_S \tag{6.3}$$

of generation $|S|$ such that

(i) they are disjoint from all the middle lakes rooted at ${}_JC^{\blacksquare}_S$;
(ii) for $\bullet \in \{0, \infty\}$, the right coast of ${}_{J,l}\mathbf{O}^{\blacksquare,\bullet}_S$ contains the spine ${}_{J,2d_\bullet-2}\mathbf{H}^{\blacksquare,\bullet}_S$ and the left coast of ${}_{J,r}\mathbf{O}^{\blacksquare,\bullet}_S$ contains the spine ${}_{J,1}\mathbf{H}^{\blacksquare,\bullet}_S$;
(iii) if $j, j' \in \{l, r\}$ and $j \neq j'$, the coasts of ${}_{J,j}\mathbf{O}^{\blacksquare,0}_S$ and ${}_{J,j'}\mathbf{O}^{\blacksquare,\infty}_S$ intersect on a non-degenerate closed interval in $\mathbf{F}^{-|S|}(\mathbf{H})$ with endpoint ${}_JC^{\blacksquare}_S$.

We will call the lakes in (6.3) the *left/right side lakes* of ${}_JC^{\blacksquare}_S$.

Observe that by (ii),

$$ {}_{J,r}\mathbf{O}^{\blacksquare,0}_S,\ {}_{J,1}\mathbf{O}^{\blacksquare,0}_S, \ldots, {}_{J,2d_0-3}\mathbf{O}^{\blacksquare,0}_S,\ {}_{J,l}\mathbf{O}^{\blacksquare,0}_S,\ {}_{J,r}\mathbf{O}^{\blacksquare,\infty}_S,\ {}_{J,1}\mathbf{O}^{\blacksquare,\infty}_S, \ldots, {}_{J,2d_\infty-3}\mathbf{O}^{\blacksquare,\infty}_S,\ {}_{J,l}\mathbf{O}^{\blacksquare,\infty}_S $$

are in counterclockwise order about ${}^{\blacksquare}_{J}C_S$ and the closure of their union is a neighborhood of ${}^{\blacksquare}_{J}C_S$. By Lemma 6.7 (4), the left itinerary of ${}_{J,l}\mathbf{O}^{\blacksquare,\bullet}_{S}$ and the right itinerary of ${}_{J,r}\mathbf{O}^{\blacksquare,\bullet}_{S}$ are infinite. The following is a consequence of Lemma 6.7 (2)–(4).

Corollary 6.8. *Every lake* $\mathbf{O}$ *is either a middle lake or a side lake of a critical point* ${}_{J}C^{\blacksquare}_{S}$*. In other words,* $\mathbf{O}$ *is of the form* ${}_{J,j}\mathbf{O}^{\blacksquare,\bullet}_{S}$ *where* $j \in \{l, 1, \dots, 2d_\bullet - 3, r\}$.

Given some tuple $S = (P_1, \dots, P_k) \in \mathbf{T}^k_{>0}$, we can perform scalar multiplication by $\mathbf{t}$ and denote $\mathbf{t}S := (\mathbf{t}P_1, \dots, \mathbf{t}P_k)$. The following is a direct consequence of (5.5).

Lemma 6.9. *For any middle or side lake* ${}_{J,j}\mathbf{O}^{\blacksquare,\bullet}_{S}$ *rooted at a critical point* ${}_{J}C^{\blacksquare}_{S}$,

$$A_*\left({}_{J}C^{\blacksquare}_{S}\right) = {}_{J}C^{\blacksquare}_{\mathbf{t}S} \quad \textit{and} \quad A_*\left({}_{J,j}\mathbf{O}^{\blacksquare,\bullet}_{S}\right) = {}_{J,j}\mathbf{O}^{\blacksquare,\bullet}_{\mathbf{t}S}.$$

Proof. Recall from (5.5) that A_* conjugates $\mathbf{F}^P$ and $\mathbf{F}^{\mathbf{t}P}$ for any $P \in \mathbf{T}_{>0}$. Since A_* preserves $\mathbf{H}$, then $A_*(C_P) = C_{\mathbf{t}P}$ and thus $A_*\left({}_{j}\mathbf{O}^{\bullet}_{P}\right) = {}_{j}\mathbf{O}^{\bullet}_{\mathbf{t}P}$ for all $\bullet \in \{0, \infty\}$ and $j \in \{l, 1, \dots, 2d_\bullet - 3, r\}$.

Suppose a spine ${}_{j}\mathbf{H}^{\bullet}_{P}$ attached to C_P contains some critical point ${}_{i}C^{\bullet}_{P,Q}$ where $i = \lceil \frac{j}{2} \rceil$. Since $A_*({}_{i}C^{\bullet}_{P,Q})$ lies on in ${}_{j}\mathbf{H}^{\bullet}_{\mathbf{t}P}$ and is a critical point of generation $\mathbf{t}(P+Q)$, then it is equal to ${}_{i}C^{\bullet}_{\mathbf{t}P,\mathbf{t}Q}$. The rest follows by induction. □

6.2. **Limbs.**

Definition 6.10. A *limb* ${}_{J}\mathbf{L}^{\blacksquare}_{S}$ is the union of the spine ${}_{J}\mathbf{H}^{\blacksquare}_{S}$ together with all spines of the form ${}_{J,j_1,\dots,j_k}\mathbf{H}^{\blacksquare,\bullet_1,\dots,\bullet_k}_{S,P_1,\dots,P_k}$. The *generation* of ${}_{J}\mathbf{L}^{\blacksquare}_{S}$ is $|S|$.

By Lemma 6.9, the linear map A_* sends each limb ${}_{J}\mathbf{L}^{\blacksquare}_{S}$ onto another limb ${}_{J}\mathbf{L}^{\blacksquare}_{\mathbf{t}S}$.

Lemma 6.11. *Every limb is bounded in* $\mathbb{C}$.

The proof we present below is identical to [DL23, Lemma 5.10].

Proof. Recall the rescaled pre-corona $\mathbf{F}^{\#}_{n} = \left(\mathbf{f}^{\#}_{n,\pm} : \mathbf{U}^{\#}_{n,\pm} \to \mathbf{S}^{\#}_{n}\right)$ where $\mathbf{S}^{\#}_{n} := A^n_*(\mathbf{S})$ for all $n \in \mathbb{Z}$. Since $\mathbf{S}$ is compactly contained in $A^{-1}_*(\mathbf{S})$, then $\bigcup_{n\in\mathbb{Z}} \mathbf{S}^{\#}_{n} = \mathbb{C}$. For every integer $n \in \mathbb{Z}$, there is a gluing map $\rho_n : \mathbf{S}^{\#}_{n} \to V$ projecting $\mathbf{F}^{\#}_{n}$ to the corona $f : U \to V$.

Let us fix a large $n \ll 0$. Consider open rectangles

$$X_0 := \rho_n\left(\mathbf{S}^{\#}_{0}\right) \quad \text{and} \quad X_1 := \rho_n\left(\mathbf{S}^{\#}_{-1}\right)$$

living in the dynamical plane of f. Denote by $\mathbf{H}_*$ the Herman curve of f, and consider the interval $I := X_0 \cap \mathbf{H}_*$ and pick a slightly smaller interval $J \subset I$.

Claim 1. There is some $M \in \mathbb{N}$ such that the following holds. For any connected component W of $X_1 \backslash \mathbf{H}$, any $m \geq M$, and any point $x \in J$ with $f^m(x) \in \partial W$, the univalent lift W_{-m} of W under f^m along the orbit $x, \dots, f^m(x)$ is contained in X_0.

Proof. Let Y^0 and Y^∞ denote the inner and outer components of $\mathbb{C}\backslash\mathbf{H}_*$. Assume without loss of generality that W is contained in Y^∞. Since $f^i(x) \in \mathbf{H}_*$ for all $i \geq 0$, then the lift W_{-m} is also contained in Y^∞. We will first claim that W_{-m} is well-defined and $f^m : W_{-m} \to W$ is univalent by ensuring that W_{-k} is disjoint from $\partial_{\mathrm{F}} U$ for all $k \geq 0$.

Let us pick two outer external rays R_l and R_r landing at a pair of points of $\mathbf{H}_*$ such that R_l is slightly on the left of W and R_r is slightly on the right of W. Since $n \ll 0$, the difference δ between the external angles of R_l and R_r is small. For

$k = 1, \ldots, m$, let $R_{l,-k}$ and $R_{r,-k}$ be the preimages of R_l and R_r under f^k such that they are slightly on the left and right of W_{-k} respectively.

By definition, for every arc γ_j^∞ on the forbidden boundary $\partial_{\mathrm{F}} U$ of U, the part that gets mapped to $\gamma_1 \cap Y^\infty$ is an external ray of some definite distance from $\mathbf{H}_*$. The difference between the external angles of $R_{l,-k}$ and $R_{r,-k}$ is δ / d_∞^k, which is even smaller than δ. Therefore, W_{-k} is disjoint from $\partial_{\mathrm{F}} U$ for all k and so $f^m : W_{-m} \to W$ is univalent.

For sufficiently large m, W_{-m} is within a small neigborhood of $\mathbf{H}_*$ and it is sandwiched between the rays $R_{l,-m}$ and $R_{r,-m}$, whose external angles differ by a small constant. By local connectivity (Lemma 3.4), W_{-m} must be contained in a small neighborhood of J, and thus $W_{-m} \subset X_0$. $\square$

The composition $\rho_n \circ A_*^{-n}$ identifies $\mathbf{S}_n^\#$ with X_0. Let $\mathbf{J}_n := A_*^n \circ \rho_n^{-1}(J)$.

Claim 2. There is a power-triple $R \in \mathbf{T}_{>0}$ such that $\mathbf{F}^R(\mathbf{J}_0) \subset \mathbf{J}_{-1}$ and for every point x on $\mathbf{J}_0$, if $\mathbf{F}^P(x) \in \mathbf{S}_{-1}^\#$ for some $P \geq R$, then there is an open subset W_P of $\mathbf{S}_0^\# \backslash \mathbf{H}$ such that $x \in \partial W_P$ and $\mathbf{F}^P$ maps W_P conformally to $\mathbf{S}_{-1}^\# \backslash \mathbf{H}$.

Proof. Since the action of $\mathbf{F}^{\geq 0}$ on $\mathbf{H}$ is combinatorially modelled by the cascade of translations $(T^P)_{P \in \mathbf{T}}$ on $\mathbb{R}$, there is an arbitrarily large $R \in \mathbf{T}$ such that $\mathbf{F}^R(\mathbf{J}_0) \subset \mathbf{J}_{-1}$. Suppose $x \in \mathbf{J}_0$ and $\mathbf{F}^P(x) \in \mathbf{S}_{-1}^\#$ for some $P \geq R$. Since $\mathbf{f}_{-1,\pm}$ is the first return map of the cascade $\mathbf{F}^{\geq 0}$ back to $\mathbf{S}_{-1}^\#$, then $\mathbf{F}^P$ is the m^{th} iterate of the pair $\mathbf{f}_{-1,\pm}$ for some $m \in \mathbb{N}$. If R is chosen to be large enough, then $m \geq M$ and the claim now follows from Claim 1. $\square$

By self-similarity, Claim 2 also holds if we replace $\mathbf{J}_0$, $\mathbf{J}_{-1}$, P, and R with $\mathbf{J}_n$, $\mathbf{J}_{n-1}$, $\mathbf{t}^n P$, and $\mathbf{t}^n R$ respectively.

Claim 3. There is a power-triple $Q \in \mathbf{T}_{>0}$ such that for every $n \ll 0$ and every point $x \in \mathbf{J}_0$, if $\mathbf{F}^P(x) \in \mathbf{S}_n^\#$ for some $P \geq Q$, then there is an open subset W of $\mathbf{S}_0^\# \backslash \mathbf{H}$ such that $x \in \partial W$ and $\mathbf{F}^P$ maps W conformally to $\mathbf{S}_n^\# \backslash \mathbf{H}$.

Proof. Let us choose $Q \in \mathbf{T}_{>0}$ such that $Q > R + R/\mathbf{t} + R/\mathbf{t}^2 + \ldots$. Consider a point $x_0 := x \in \mathbf{J}_0$ such that $\mathbf{F}^P(x) \in \mathbf{S}_n^\#$ for some $P \geq Q$. For $j \in \{0, -1, -2, \ldots, n+2\}$, we set $P_j := \mathbf{t}^j R$ and $x_{j-1} := \mathbf{F}^{P_j}(x_j)$ inductively. Then, we set

$$P_{n+1} := P - P_0 - P_{-1} - \ldots - P_{n+2} \quad \text{ and } x_n := \mathbf{F}^{P_{n+1}}(x_{n+1}).$$

Clearly, $P_{n+1} \geq \mathbf{t}^{n+1} R$. By Claim 2, there exists an open set $W_{n+1} \subset \mathbf{S}_{n+1}^\# \backslash \mathbf{H}$ such that $x_{n+1} \in \partial W_{n+1}$ and $\mathbf{F}^{P_{n+1}}$ maps W_{n+1} conformally to $\mathbf{S}_n^\# \backslash \mathbf{H}$. Inductively, for $j \in \{0, -1, \ldots, n+2\}$, we construct open sets $W_j \subset \mathbf{S}_j^\# \backslash \mathbf{H}$ such that $x_j \in \partial W_j$ and $\mathbf{F}^{P_j}$ maps W_j conformally to W_{j-1}. Therefore, $\mathbf{F}^P$ maps W_0 conformally to $\mathbf{S}_n^\# \backslash \mathbf{H}$. $\square$

To prove the lemma, it is sufficient to consider a limb L of some generation $K \in \mathbb{T}_{>0}$ rooted at the critical point C_K on $\mathbf{H}$. Choose a large $T \in \mathbf{T}$ such that $T \geq Q + K$ and that the critical point C_T is on $\mathbf{J}_0$. There exists some limb L' rooted at C_T such that $\mathbf{F}^{T-K}(L') = L$. Then, the connected component of $\mathbf{S}_n^\# \cap \overline{L}$ containing C_K can be lifted by $\mathbf{F}^{T-K}$ into $\mathbf{S}_0^\#$. As $n \ll 0$ is arbitrary, the lifts of $\mathbf{S}_n^\# \cap \overline{L}$ exhaust L' and so L' is contained in $\mathbf{S}_0^\#$. Hence, L is bounded. $\square$

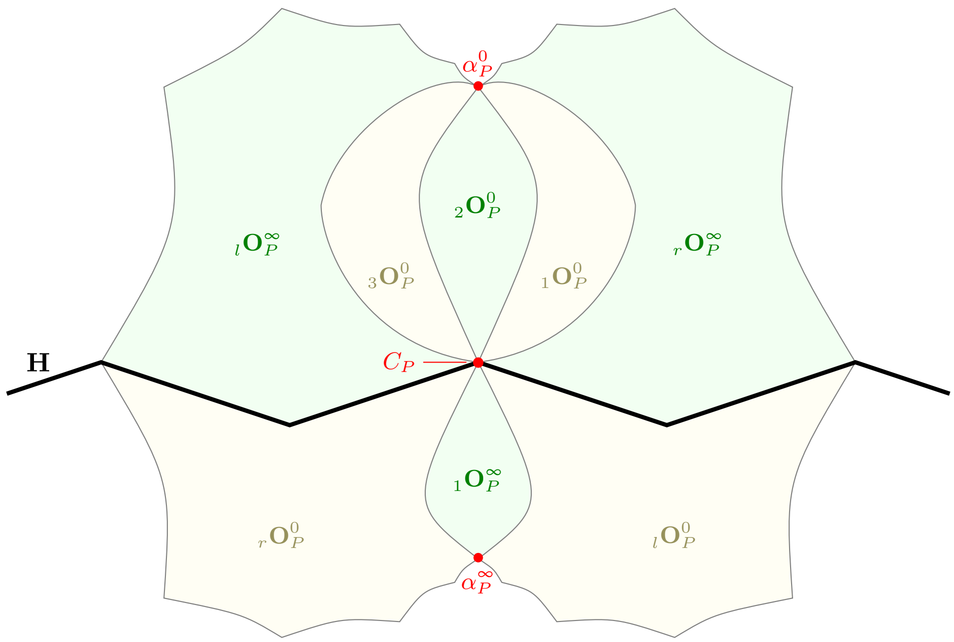

FIGURE 7. The configuration of middle and side lakes rooted at C_P when $(d_0, d_\infty) = (3, 2)$. Their coasts land at α_P^0 and α_P^∞.

6.3. **Alpha-points.** For $P \in \mathbf{T}_{>0}$, let $\mathbf{I}_{\leq P} := \mathbf{I}_{\leq P}(\mathbf{F})$ be the P^{th} escaping set of $\mathbf{F}$.

Lemma 6.12. *Every critical point ${}_JC_S^{\blacksquare}$ admits a pair of points ${}_J\alpha_S^{\blacksquare,0}$ and ${}_J\alpha_S^{\blacksquare,\infty}$ with the following properties. For any $\bullet \in \{0, \infty\}$ and $j \in \{l, 1, \dots, 2d_\bullet - 3, r\}$, both the left and the right coasts of ${}_{J,j}\mathbf{O}_S^{\blacksquare,\bullet}$ land at ${}_J\alpha_S^{\blacksquare,\bullet}$ and*

$$\partial\, {}_{J,j}\mathbf{O}_S^{\blacksquare,\bullet} \setminus \partial^c\, {}_{J,j}\mathbf{O}_S^{\blacksquare,\bullet} = \left\{ {}_J\alpha_S^{\blacksquare,\bullet} \right\}.$$

In particular, every lake is a disk and each of the spines ${}_{J,j}\mathbf{H}_S^{\blacksquare,\bullet}$ attached to ${}_JC_S^{\blacksquare}$ is a quasiarc connecting its common root ${}_JC_S^{\blacksquare}$ to a common landing point ${}_J\alpha_S^{\blacksquare,\bullet}$. Refer to Figure 6.12 for an illustration. We call ${}_J\alpha_S^{\blacksquare,0}$ and ${}_J\alpha_S^{\blacksquare,\infty}$ the *inner* and *outer alpha-points* corresponding to ${}_JC_S^{\blacksquare}$. Moreover, we say that ${}_J\alpha_S^{\blacksquare,\bullet}$ is *the alpha-point* of any lake of the form ${}_{J,j}\mathbf{O}_S^{\blacksquare,\bullet}$.

Proof. By Corollary 6.8, for every lake $\mathbf{O}$, there is some $Q \in \mathbf{T}$ such that $\mathbf{F}^Q(\mathbf{O})$ is either a side lake or a middle lake attached to some critical point on $\mathbf{H}$. Therefore, it is sufficient to prove the lemma for lakes of the form ${}_j\mathbf{O}_P^\bullet$ where $\bullet \in \{0, \infty\}$ and $j \in \{l, 1, \dots, 2d_\bullet - 3, r\}$.

Suppose ${}_j\mathbf{O}_P^\bullet$ is a middle lake. Then, it is contained in some side lake ${}_k\mathbf{O}_{P-P/\mathbf{t}}^\bullet$ of generation $P - P/\mathbf{t}$ where $k \in \{l, r\}$. The composition $\mathbf{F}^{\mathbf{t}P-P} \circ A_*$ sends the pair $\left({}_j\mathbf{O}_P^\bullet, {}_k\mathbf{O}_{P-P/\mathbf{t}}^\bullet\right)$ conformally onto $\left({}_j\mathbf{O}_P^\bullet, \mathbf{O}^\bullet\right)$. In particular, $\mathbf{F}^{\mathbf{t}P-P} \circ A_*$ expands the hyperbolic metric of the ocean $\mathbf{O}^\bullet$. Since $\mathbf{I}_{\leq P} \cap \overline{{}_j\mathbf{O}_P^\bullet}$ is a $\left(\mathbf{F}^{\mathbf{t}P-P} \circ A_*\right)$-invariant compact subset of $\mathbf{O}^\bullet$, then it must be a singleton $\{\alpha_P^\bullet\}$ consisting of the unique repelling fixed point of $\mathbf{F}^{\mathbf{t}P-P} \circ A_*$ located inside of ${}_k\mathbf{O}_{P-P/\mathbf{t}}^\bullet$.

It remains to show that for $j \in \{l, r\}$, the intersection $\overline{\partial^c_j\, {}_j\mathbf{O}^\bullet_P} \cap \mathbf{I}_{\le P}$ is also a compact subset of $\mathbf{O}^\bullet$. By invariance under $\mathbf{F}^{\mathbf{t}P-P} \circ A_*$, this will again imply that $\overline{\partial^c_j\, {}_j\mathbf{O}^\bullet_P} \cap \mathbf{I}_{\le P}$ is the same singleton $\{\alpha^\bullet_P\}$, and we are done.

Let us assume without loss of generality that $j = l$. Denote the left itinerary of ${}_l\mathbf{O}^\bullet_P$ by $(Q_1, Q_2, Q_3, \ldots)$. The left coast of ${}_l\mathbf{O}^\bullet_P$ starts with a segment of the spine ${}_1\mathbf{H}^\bullet_{Q_1}$ connecting C_{Q_1} and ${}_1C^\bullet_{Q_1,Q_2}$. Let us pick a pair of power-triples $R_-, R_+ \in \mathbf{T}_{>0}$ such that the critical points ${}_1C^\bullet_{Q_1,R_-}$ and ${}_1C^\bullet_{Q_1,R_+}$ form a small open interval neighborhood $J \subset {}_1\mathbf{H}^\bullet_{Q_1}$ of ${}_1C^\bullet_{Q_1,Q_2}$. Let $B_\pm$ be spines of generation $Q_1 + R_\pm$ attached to ${}_1C^\bullet_{Q_1,R_\pm}$ that are combinatorially closest to ${}_{1,1}\mathbf{H}^{\bullet,\bullet}_{Q_1,Q_2}$. Let $R := Q_1 + \max\{R_+, R_-\}$. By Lemma 5.20, every connected component of $\mathbf{I}_{\le R}$ is unbounded and thus the union $J \cup B_+ \cup B_- \cup \mathbf{I}_{\le R}$ separates $\partial^c_l\, {}_l\mathbf{O}^\bullet_P \setminus {}_i\mathbf{H}^\bullet_{Q_1}$ from $\mathbf{H}$. This observation imples that $\overline{\partial^c_l\, {}_l\mathbf{O}^\bullet_P} \cap \mathbf{I}_{\le P}$ is indeed compactly contained in $\mathbf{O}^\bullet$. $\square$

The alpha-points ${}_J\alpha^{\blacksquare,\bullet}_S$ can be viewed as preimages of infinity under the map $\mathbf{F}^{|S|}$. They are unique in the following sense.

Lemma 6.13. *Two alpha-points ${}_J\alpha^{\blacksquare,\bullet}_S$ and ${}_{J'}\alpha^{\square,\circ}_{S'}$ coincide if and only if $J = J'$, $\blacksquare = \square$, $\bullet = \circ$, and $S = S'$.*

Proof. Suppose ${}_J\alpha^{\blacksquare,\bullet}_S = {}_{J'}\alpha^{\square,\circ}_{S'}$. Clearly, $|S| = |S'|$. Let us write $S = (P_1, \ldots, P_m)$ and $S' = (Q_1, \ldots, Q_k)$, and pick a power-triple $R \in \mathbf{T}$ such that

$$\max\{P_1 + \ldots + P_{m-1}, Q_1 + \ldots + Q_{k-1}\} < R < |S|.$$

Pushing forward by $\mathbf{F}^R$ yields a pair of alpha-points $\alpha^\bullet_{|S|-R}$ and $\alpha^\circ_{|S'|-R}$ where, since they are equal, $\bullet = \circ$. If $(J, \blacksquare, S) \ne (J', \square, S')$, then this would imply that $\alpha^\bullet_{|S|-R}$ is a critical point of $\mathbf{F}^R$, which is not the case. $\square$

Consequently, if two disjoint spines touch at a common alpha-point, then they are rooted at a common critical point. This guarantees a more precise tree structure of $\mathbf{F}^{-P}(\mathbf{H})$ in terms of spines. For convenience, we will call $\mathbf{H}$ the unique spine of generation 0.

Corollary 6.14. *Consider two distinct spines ${}_{J,j}\mathbf{H}^{\blacksquare,\bullet}_S$ and ${}_{J',j'}\mathbf{H}^{\square,\circ}_{S'}$ with $|S| \ge |S'|$.*

(1) *If the intersection $\overline{{}_{J,j}\mathbf{H}^{\blacksquare,\bullet}_S} \cap \overline{{}_{J',j'}\mathbf{H}^{\square,\circ}_{S'}}$ is non-empty, then it is either the singleton $\{{}_JC^\blacksquare_S\}$ or the set $\{{}_JC^\blacksquare_S, {}_J\alpha^{\blacksquare,\bullet}_S\}$. The former case happens if and only if ${}_{J',j'}\mathbf{H}^{\square,\circ}_{S'}$ contains ${}_JC^\blacksquare_S$, and the latter case happens if and only if $(J, j, \blacksquare, \bullet, S) = (J', j', \square, \circ, S')$.*

(2) *There is a unique sequence of pairwise different spines*

$$B_1 = {}_{J,j}\mathbf{H}^{\blacksquare,\bullet}_S,\ \ B_2,\ \ \ldots,\ \ B_{n-1},\ \ B_n = {}_{J',j'}\mathbf{H}^{\square,\circ}_{S'}$$

such that $\overline{B_i}$ intersects $\overline{B_{i'}}$ if and only if $|i - i'| \le 1$.

Given an alpha-point $\alpha = {}_J\alpha^{\blacksquare,\bullet}_S$, we define

- ▷ a *finite skeleton landing at* α to be the union of a spine ${}_{J,j}\mathbf{H}^{\blacksquare,\bullet}_S$ together with the unique closed quasiarc in $\mathbf{F}^{-|S|}(\mathbf{H})$ connecting ${}_JC^\blacksquare_S$ to 0;
- ▷ an *infinite skeleton landing at* α to be the union of $\partial^c_k\, {}_{J,k}\mathbf{O}^{\blacksquare,\bullet}_S$ for some $k \in \{l, r\}$ together with the unique closed quasiarc in $\mathbf{F}^{-|S|}(\mathbf{H})$ connecting the root of $\partial^c_k\, {}_{J,k}\mathbf{O}^{\blacksquare,\bullet}_S$ to 0.

In short, skeletons landing at α are the shortest paths from 0 to α within the tree of preimages of $\mathbf{H}$. There are exactly $d_\bullet$ skeletons landing at α, and precisely two of them are finite.

The set of skeletons admit a total order “$<$“ defined as follows. Let us fix a ray γ in $\mathbf{H}$ connecting 0 to ∞. Given two distinct skeletons $\mathfrak{S}$ and $\mathfrak{S}'$,

- ▷ we write $\mathfrak{S} < \mathfrak{S}'$ if γ, $\mathfrak{S}$, and $\mathfrak{S}'$ have a counterclockwise orientation around the quasiarc $\mathfrak{S} \cap \mathfrak{S}'$, and
- ▷ we say that $\mathfrak{S}$ and $\mathfrak{S}'$ are $<$-*separated* if there is another skeleton $\mathfrak{S}''$ such that either $\mathfrak{S} < \mathfrak{S}'' < \mathfrak{S}'$ or $\mathfrak{S}' < \mathfrak{S}'' < \mathfrak{S}$.

We say that two alpha-points α and α' in the same ocean $\mathbf{O}^\bullet$ are $<$-*separated* if

- ▷ there exists an alpha point $\alpha'' \in \mathbf{O}^\bullet$ with generation lower than that of α and α', and
- ▷ there exist skeletons $\mathfrak{S}$, $\mathfrak{S}'$, $\mathfrak{S}''$ landing at α, α', α'' respectively such that $\mathfrak{S}$ and $\mathfrak{S}'$ are $<$-separated by $\mathfrak{S}''$.

Let us introduce another partial order on the set of alpha-points. Given two alpha-points α and α' in the same ocean,

- ▷ we write $\alpha \prec \alpha'$ if α' is contained in the closure of a lake attached to α, and
- ▷ we say that α and α' are $\prec$-*separated* if α and α' are contained in two distinct lakes with a common alpha-point.

The following proposition describes the relation between “$<$“ and “$\prec$“.

Proposition 6.15. *Consider two distinct alpha-points α and α' of generations P and P' inside of the ocean $\mathbf{O}^\bullet$ for some $\bullet \in \{0, \infty\}$. Assume $P \le P'$. The following are equivalent.*

(1) *$\alpha \prec \alpha'$;*
(2) *α and α' are not $<$-separated;*
(3) *α and α' are not $\prec$-separated.*

Proof. Suppose (1) holds. Then, α is the alpha-point of a lake $\mathbf{O}$ containing contains α'. This this implies (3). Meanwhile, (2) follows from the observation that any alpha-point α'' $<$-separating α and α' must be contained in a proper sub-lake of $\mathbf{O}$, which necessarily has generation higher than P.

Suppose (1) does not hold, so α' is located outside of every lake with alpha-point α. Let us pick any skeleton $\mathfrak{S}'$ landing at α', and let $\mathfrak{S}_l$ and $\mathfrak{S}_r$ denote the left and right skeletons landing at α respectively. The assumption implies that either $\mathfrak{S}_l$ $<$-separates $\mathfrak{S}_r$ and $\mathfrak{S}'$ or $\mathfrak{S}_r$ $<$-separates $\mathfrak{S}_l$ and $\mathfrak{S}'$. Without loss of generality, let us assume the latter.

Denote by $(c_{r,1}, c_{r,2}, \ldots)$ the infinite sequence of critical points of $\mathbf{F}^P$ of increasing generation that is found along $\mathfrak{S}_r$. Let $\alpha_{r,i}$ denote the alpha-point that is the landing point of the unique spine attached to $c_{r,i}$ that intersects $\mathfrak{S}_k$. It has generation $P_{r,i}$ where $P_{r,i} < P$ and $P_{r,i} \to P$ as $i \to \infty$. Since the intersection $\mathfrak{S}_r \cap \mathfrak{S}'$ is a compact subset of $\operatorname{Dom}\left(\mathbf{F}^P\right)$, then for any sufficiently large $i \gg 0$ and any skeleton $\mathfrak{S}_{r,i}$ landing at $\alpha_{r,i}$, $\mathfrak{S}_r \cap \mathfrak{S}'$ is a proper subset of $\mathfrak{S}_r \cap \mathfrak{S}_{r,i}$. Therefore, $\mathfrak{S}_{r,i}$ $<$-separates $\mathfrak{S}_r$ and $\mathfrak{S}'$, and so α and α' are $<$-separated by $\alpha_{r,i}$.

We have just shown that (1) and (2) are equivalent. Suppose (1) and (2) do not hold. We will now prove that (3) also does not hold.

Let us consider the unique spine B such that α and α' are contained in different components of the closure of $(\mathfrak{S} \cup \mathfrak{S}') \backslash B$. We claim that the generation Q of B is less than P. Indeed, if $Q = P$, then α is the landing point of B and so there exists a lake with alpha-point α which contains both $\mathfrak{S}'\backslash\mathfrak{S}$ and α'. However, this would instead imply (1).

Let $\hat{\mathbf{O}}$ and $\hat{\mathbf{O}}'$ denote the pair of lakes of generation Q such that their coast contains B and $\mathfrak{S}\backslash\mathfrak{S}' \subset \hat{\mathbf{O}}$ and $\mathfrak{S}'\backslash\mathfrak{S} \subset \hat{\mathbf{O}}'$. If $\hat{\mathbf{O}}$ and $\hat{\mathbf{O}}'$ are distinct, they lie on different sides of B and so α and α' are $\prec$-separated by the landing point of B.

Now, suppose instead that $\hat{\mathbf{O}} = \hat{\mathbf{O}}'$. Consider the roots c and c' of $\mathfrak{S}\backslash B$ and $\mathfrak{S}'\backslash B$ respectively. Within the closed interval $[c, c'] \subset B$ (possibly degenerate if $c = c'$), we can find a unique critical point c'' of the smallest generation P'' such that $Q < P'' \leq P$. In fact, $P'' \neq P$ because if otherwise, $\mathfrak{S}'\backslash\mathfrak{S}$ would have been contained in a lake attached to c, and so $\alpha \prec \alpha'$ instead. Since $[c, c']$ does not contain any critical point of generation lower than P'', then $\mathfrak{S}\backslash\mathfrak{S}'$ and $\mathfrak{S}'\backslash\mathfrak{S}$ are contained in distinct side lakes attached to c''. Thus, the alpha-point $\alpha'' \in \hat{\mathbf{O}}$ corresponding to c'' must $\prec$-separate α and α'. □

6.4. **External chains.** Let us pick a power-triple $P \in \mathbf{T}_{>0}$ and $\bullet \in \{0, \infty\}$. Let $\mathbf{O}^\bullet(P)$ denote the unique lake of generation P inside of the ocean $\mathbf{O}^\bullet$ that contains 0 on its boundary. Then, the coast of $\mathbf{O}^\bullet(P)$ intersects $\mathbf{H}$ on some interval $J \subset \mathbf{H}$ containing 0 on its interior. (In fact, J is independent of $\bullet$.) Let us denote by $\alpha^\bullet(P)$ the unique alpha-point in $\partial\mathbf{O}^\bullet(P)$. By self-similarity,

$$\mathbf{O}^\bullet(\mathbf{t}^n P) = A_*^n(\mathbf{O}^\bullet(P)) \qquad \text{for all } n \in \mathbb{Z}$$

and

$$\bigcup_{n<0} \mathbf{O}^\bullet(\mathbf{t}^n P) = \mathbf{O}^\bullet. \tag{6.4}$$

Let us denote by $\mathbf{I}^\bullet_{\leq P}$ the intersection $\mathbf{I}_{\leq P} \cap \mathbf{O}^\bullet$ for $\bullet \in \{0, \infty\}$.

Lemma 6.16. *For every $\bullet \in \{0, \infty\}$ and $P, Q \in \mathbf{T}_{>0}$ with $P < Q$,*

1. *$\mathbf{I}^\bullet_{\leq P}$ is connected;*
2. *$\mathbf{I}^\bullet_{\leq Q}\backslash\mathbf{I}^\bullet_{\leq P}$ is bounded;*
3. *every connected component of $\mathbf{I}^\bullet_{\leq Q}\backslash\mathbf{I}^\bullet_{\leq P}$ is a lift of a component of $\mathbf{I}_{\leq Q-P}$ under $\mathbf{F}^P$; it is contained in a unique lake $\mathbf{O}$ of generation P and its boundary contains the alpha-point of $\mathbf{O}$.*

Proof. Consider a component I of $\mathbf{I}^\bullet_{\leq P}$. It intersects $\mathbf{O}^\bullet(\mathbf{t}^k P)$ for some maximal $k \in \mathbb{Z}$. By Lemma 5.20, since I intersects $\mathbf{O}^\bullet(\mathbf{t}^n P)$ for all $n \leq k$, then it contains the alpha-point $\alpha^\bullet(\mathbf{t}^n P)$ which is the alpha-point of $\mathbf{O}^\bullet(\mathbf{t}^n P)$ for all $n \leq k$. Therefore, $\mathbf{I}^\bullet_{\leq P}$ is connected.

Let us consider a connected component X of $\mathbf{I}^\bullet_{\leq Q}\backslash\mathbf{I}^\bullet_{\leq P}$. Since X avoids $\alpha^\bullet(\mathbf{t}^n P)$ for all $n \ll 0$, it must be contained inside of the lake $\mathbf{O}^\bullet(\mathbf{t}^k P)$ for all $n \ll 0$, and so X is bounded. Since X avoids $\mathbf{F}^{-P}(\mathbf{H})$ and alpha-points of generation P, L is contained in a unique lake $\mathbf{O}$ of generation P. The map $\mathbf{F}^P$ sends $\mathbf{O}$ conformally onto an ocean $\mathbf{O}^\circ$ for some $\circ \in \{0, \infty\}$, hence $\mathbf{F}^P(X) = \mathbf{I}^\circ_{\leq Q-P}$. By unboundedness, X must be attached to the alpha-point of $\mathbf{O}$. □

Definition 6.17. Consider two alpha-points α and α' in the same ocean $\mathbf{O}^\bullet$ with generation P and P' respectively, and suppose $P < P'$ and $\alpha \prec \alpha'$. We define the *external chain* $[\alpha, \alpha']$ to be the set of points in $\mathbf{I}^\bullet_{\leq P'}$ that are inside the closure of the lakes attached to α and outside of any lake that does not contain α'.

Lemma 6.18. *For any triplet of alpha-points $\alpha, \alpha', \alpha''$ with $\alpha \prec \alpha' \prec \alpha''$,*

$$[\alpha, \alpha'] \cap [\alpha', \alpha''] = \{\alpha'\} \quad \textit{and} \quad [\alpha, \alpha'] \cup [\alpha', \alpha''] = [\alpha, \alpha''].$$

Proof. The first equation follows from the fact that α' is a cut point with respect to the "$\prec$" ordering. The inclusion $[\alpha, \alpha'] \cup [\alpha', \alpha''] \subset [\alpha, \alpha'']$ is obvious. Consider a point x in $[\alpha, \alpha'']\backslash[\alpha, \alpha']$. We know that x is within a lake attached to α. If x is inside of a lake that does not contain α', then this lake avoids all lakes attached to α' and in particular does not contain α'' as well, which is a contradiction. Therefore, $x \in [\alpha', \alpha'']$. □

For $P \in \mathbf{T}_{>0}$, we say that the critical point C_P on $\mathbf{H}$ is *dominant* if the interval $[0, C_P] \subset \mathbf{H}$ does not contain any critical point of generation less than P. We will enumerate dominant critical points by $\{C_{P_n}\}_{n\in\mathbb{Z}}$ where $\{P_n\}_{n\in\mathbb{Z}}$ is monotonically increasing in n.

Lemma 6.19. *For $\bullet \in \{0, \infty\}$, $\ldots \prec \alpha^\bullet_{P_{-2}} \prec \alpha^\bullet_{P_{-1}} \prec \alpha^\bullet_{P_0} \prec \alpha^\bullet_{P_1} \prec \alpha^\bullet_{P_2} \prec \ldots$.*

Proof. Suppose for a contradiction that $\alpha^\bullet_{P_n} \not\prec \alpha^\bullet_{P_{n+1}}$ for some $\bullet \in \{0, \infty\}$ and $n \in \mathbb{Z}$. By Propositon 6.15, there is an alpha-point $\alpha \in \mathbf{O}^\bullet$ of some generation P less than P_n which $\prec$-separates $\alpha^\bullet_{P_n}$ and $\alpha^\bullet_{P_{n+1}}$. Then, α is contained in the closure of a lake attached to a critical point $C_Q \in \mathbf{H}$ of some generation $Q \le P$. By $\prec$-separation, C_Q is contained in the interval $(C_{P_n}, C_{P_{n+1}}) \subset \mathbf{H}$. However, this would contradict the assumption that C_{P_n} and $C_{P_{n+1}}$ are dominant. □

Consider the concatenations

$$\mathbf{R}^0 = \bigcup_{n\in\mathbb{Z}} \left[\alpha^0_{P_n}, \alpha^0_{P_{n+1}}\right] \quad \text{and} \quad \mathbf{R}^\infty = \bigcup_{n\in\mathbb{Z}} \left[\alpha^\infty_{P_n}, \alpha^\infty_{P_{n+1}}\right],$$

which we will refer to as the *inner* and *outer zero chains* respectively.

Proposition 6.20. *For $\bullet \in \{0, \infty\}$,*

(1) $\mathbf{R}^\bullet$ *is A_*-invariant;*
(2) $\mathbf{R}^\bullet$ *is an arc landing at* 0*;*
(3) *alpha-points are dense on* $\mathbf{R}^\bullet$*;*
(4) *points on $\mathbf{R}^\bullet$ are continuously parametrized by their escaping time ranging from 0 (near ∞) to $+\infty$ (near 0).*

Let us clarify the last statement. For $P \in \mathbb{R}_{>0}\backslash\mathbf{T}$, we can define the P^{th} escaping set to be

$$\mathbf{I}_{\le P} := \bigcap_{Q\in\mathbf{T}, Q>P} \mathbf{I}_{\le Q}.$$

The *escaping time* of a point x in $\mathbf{I}_{<\infty}$ is the minimum time $P \in \mathbb{R}_{>0}$ such that $x \in \mathbf{I}_{\le P}$.

Proof. To lighten the notation, we will denote $\alpha^\bullet_n := \alpha^\bullet_{P_n}$ and $J^\bullet_n := [\alpha^\bullet_n, \alpha^\bullet_{n+1}]$ for all $\bullet \in \{0, \infty\}$ and $n \in \mathbb{Z}$.

By definition, C_P is dominant if and only if $C_{\mathbf{t}P} = A_*(C_P)$ is dominant, so there is some integer $k \ge 1$ such that $\mathbf{t}P_n = P_{n+k}$ for all $n \in \mathbb{Z}$. Therefore, A_* maps each of $[\alpha^\bullet_{(n-1)k}, \alpha^\bullet_{nk}]$ onto $[\alpha^\bullet_{nk}, \alpha^\bullet_{(n+1)k}]$. This immediately implies items (1) and (2).

Due to self-similarity, it remains for us to show that the external chain $J^\bullet := [\alpha^\bullet_0, \alpha^\bullet_k]$ is an arc that can be continuously parametrized by their escaping time, and that alpha-points are dense on $J^\bullet$. We will do so by constructing nested Markov tilings $\mathcal{P}_r$ for $r \ge 0$ on $J^\bullet$.

Firstly, we set the tiling $\mathcal{P}_0$ of level 0 to be $\{J_i^\bullet\}_{0\le i\le k-1}$. The tiling $\mathcal{P}_1$ of level 1 is constructed as follows. By Lemma B.8, for every chain $J_i \in \mathcal{P}_0$, there exist some $Q_i \in \mathbf{T}_{>0}$ and a pair of integers l_i and r_i such that $0 < l_i < r_i \le i$ and $\mathbf{F}^{Q_i}$ maps $J_i^\bullet$ homeomorphically onto the chain $[\alpha_{l_i}^\bullet, \alpha_{r_i}^\bullet]$. A tile of level 1 in $\mathcal{P}_1$ is the lift of a chain $J_j^\bullet \subset [\alpha_{l_i}^\bullet, \alpha_{r_i}^\bullet]$ under the map $\mathbf{F}^{Q_i} : J_i \to [\alpha_{l_i}^\bullet, \alpha_{r_i}^\bullet]$.

For each tile $I \in \mathcal{P}_1$ in J_i, there exists some $m_I \in \mathbb{N}$ such that $A_*^{m_I}$ sends $\mathbf{F}^{Q_i}(I)$ back to a tile of level 0. Let $\mathbf{O}_i$ denote the lake of generation Q_i which contains $[\alpha_{l_i}^\bullet, \alpha_{r_i}^\bullet]$. The composition

$$\chi_I \coloneqq A_*^{m_I} \circ \mathbf{F}^{Q_i} : \mathbf{O}_i \to \mathbf{O}^\bullet \tag{6.5}$$

expands the hyperbolic metric of $\mathbf{O}^\bullet$.

Inductively, we define tiles in $\mathcal{P}_{n+1}$ of level $n+1$ to be the preimages of tiles of level n under maps of the form (6.5). Since each map χ_I is expanding, the diameter of every tile of level n uniformly exponentially shrinks to zero. Since each tile in $\mathcal{P}_n$ is an external chain containing alpha-points, alpha-points are dense on J.

By Lemma 6.18, we can enumerate our level n tiles by $I_1^n, I_2^n, \dots, I_{s_n}^n \in \mathcal{P}_n$ in increasing order of generation such that I_i^n and I_l^n touch if and only if $|l-i| \le 1$. As tiles shrink, we can extend the "$\prec$" order to a total order on J by defining $x \prec y$ when $x \in I_i^n$ and $y \in I_j^n$ for sufficiently high n and some indices i, j with $i < j$.

Consider a tile I_i^n in $\mathcal{P}_n$ of some high level n, and a composition $\chi \coloneqq \chi_1 \circ \chi_2 \circ \ldots \circ \chi_n$ of n maps of the form (6.5) sending I_i^n onto a tile in $\mathcal{P}_n$. By (5.5), we can write χ as $A_*^{m(n,i)} \circ \mathbf{F}^{Q(n,i)}$ for some $m(n,i) \in \mathbb{N}$ and $Q(n,i) \in \mathbf{T}_{>0}$. Therefore, the difference in the escaping time between the endpoints of I_i^n is at most

$$\mathbf{t}^{-m(r,i)}(P_k - P_0). \tag{6.6}$$

Since $Q_i > 0$ for all $i \in \{0, \dots, k-1\}$, there exists some integer $M \ge 1$ independent of n such that every sequence of M consecutive integers between 1 and n contains an element j_* such that χ_{j_*} has the scaling factor A_* in (6.5). Consequently, as $n \to \infty$, $\min_{1\le i\le s_n} m(n,i) \to \infty$ and thus the quantity in (6.6) tends to zero. Therefore, the escaping time continuously parametrizes points on J. □

In general, for every alpha-point α, there is an infinite sequence of alpha-points $\alpha_0 = \alpha, \alpha_{-1}, \alpha_{-2}, \dots$ of generation decreasing to 0 such that $\ldots \prec \alpha_{-2} \prec \alpha_{-1} \prec \alpha_0$. This allows us to generate the chain

$$(\infty, \alpha] \coloneqq \bigcup_{n\le 0} [\alpha_{n-1}, \alpha_n].$$

Corollary 6.21. *Consider any alpha-point α of some generation $P > 0$. The chain $(\infty, \alpha]$ is an infinite arc continuously parametrized by the escape time from $|P|$ to 0. Moreover, alpha-points are dense in $(\infty, \alpha]$.*

Proof. Suppose first that α is of the form $\alpha_P^\bullet$ for some $P \in \mathbf{T}_{>0}$ and $\bullet \in \{0, \infty\}$. Let us pick a dominant $\alpha_{P_n}^\bullet$ for some $n \in \mathbb{Z}$ such that $P_n \ge P$. There is a unique point $x \in \left(\infty, \alpha_{P_n}^\bullet\right]$ of generation $P_n - P$. Then, $\mathbf{F}^{P_n - P}$ maps the arc $\left(x, \alpha_{P_n}^\bullet\right]$ onto $(\infty, \alpha_P^\bullet]$, which implies the claim.

In general, let $\alpha = {}_J\alpha_S^{\blacksquare,\bullet}$ where $S = (P_1, P_2, \dots, P_k)$ is the corresponding itinerary. There exist alpha-points $\alpha_1, \alpha_2, \dots, \alpha_k = \alpha$ such that $\alpha_1 \prec \alpha_2 \prec \ldots \prec \alpha_k$ and for every i, α_i has itinerary $S_i \coloneqq (P_1, \dots, P_i)$. Therefore, we can split $(\infty, \alpha]$ into $J_1 = (\infty, \alpha_1]$, $J_2 = (\alpha_1, \alpha_2]$, ..., $J_k = (\alpha_{k-1}, \alpha_k]$. When $2 \le i \le k$, the map $\mathbf{F}^{P_1 + \ldots + P_{i-1}}$

sends J_i homeomorphically onto the chain $\left(\infty, \alpha^{\circ}_{P_i}\right]$ for some $\circ \in \{0, \infty\}$. By the previous paragraph, each J_i is an arc continuously parametrized by the landing time. $\square$

As a consequence, whenever $\alpha \prec \alpha'$, the chain $[\alpha, \alpha']$ is a simple arc.

Definition 6.22. An *external ray* is an infinite arc of the form $\mathbf{R} = \bigcup_{n\in\mathbb{Z}}[\alpha_n, \alpha_{n+1}]$ for some sequence of alpha-points $\{\alpha_n\}_{n\in\mathbb{Z}}$ where

- ▷ $\alpha_n \prec \alpha_{n+1}$ for all n;
- ▷ the generation of α_n decreases to 0 as $n \to -\infty$;
- ▷ there is no alpha-point α such that $\alpha_n \prec \alpha$ for all $n \in \mathbb{Z}$.

The *generation* of $\mathbf{R}$ is the limit of the generation of α_n as $n \to +\infty$. We define the image of an external ray $\mathbf{R}$ under $\mathbf{F}^P$ by

$$\mathbf{F}^P(\mathbf{R}) \coloneqq \mathbf{F}^P\left(\mathbf{R} \cap \mathrm{Dom}\left(\mathbf{F}^P\right)\right).$$

We say that $\mathbf{R}$ is *periodic* if $\mathbf{F}^P(\mathbf{R}) = \mathbf{R}$ for some $P \in \mathbf{T}_{>0}$.

The zero chains $\mathbf{R}^0$ and $\mathbf{R}^\infty$ are indeed external rays, which from now on will be referred to as *zero rays*.

The following corollary is an immediate consequence of Proposition 6.15.

Corollary 6.23. *The intersection of any two external rays in the same ocean is non-empty and of the form $(\infty, \alpha]$ for some alpha-point α.*

6.5. **Wakes.** Consider a critical point ${}_J C^{\blacksquare}_S$. For every lake of the form ${}_{J,j}\mathbf{O}^{\blacksquare,\bullet}_S$ where j is either in $\{l, r\}$ or an even number, the map $\mathbf{F}^{|S|}$ sends such a lake conformally onto $\mathbf{O}^\bullet$. The zero ray $\mathbf{R}^\bullet$ lifts under $\mathbf{F}^{|S|} : {}_{J,j}\mathbf{O}^{\blacksquare,\bullet}_S \to \mathbf{O}^\bullet$ to a ray segment, which we will label as ${}_{J,k}\mathbf{R}^{\blacksquare,\bullet}_S$ where

$$k = \begin{cases} 1 & \text{if } j = r, \\ \frac{j}{2} + 1 & \text{if } j \text{ is even}, \\ d_\bullet & \text{if } j = l. \end{cases}$$

Therefore, we obtain $d_\bullet$ ray segments

$$ {}_{J,1}\mathbf{R}^{\blacksquare,\bullet}_S, \quad {}_{J,2}\mathbf{R}^{\blacksquare,\bullet}_S, \quad \ldots, {}_{J,d_\bullet}\mathbf{R}^{\blacksquare,\bullet}_S \tag{6.7}$$

starting from the alpha-point ${}_J\alpha^{\blacksquare,\bullet}_S$ and landing at the critical point ${}_J\mathbf{R}^{\blacksquare}_S$, labelled in an anticlockwise order about ${}_J\mathbf{R}^{\blacksquare}_S$. For $k \in \{1, \ldots, d_\bullet - 1\}$, the closure of ${}_{J,k}\mathbf{R}^{\blacksquare,\bullet}_S \cup {}_{J,k+1}\mathbf{R}^{\blacksquare,\bullet}_S$ bounds a Jordan domain which we denote by ${}_{J,k}\mathbf{W}^{\blacksquare,\bullet}_S$.

Definition 6.24. A wake $\mathbf{W}$ is a Jordan domain of the form ${}_{J,k}\mathbf{W}^{\blacksquare,\bullet}_S$. We call ${}_J C^{\blacksquare}_S$ the *root* of $\mathbf{W}$ and ${}_J\alpha^{\blacksquare,\bullet}_S$ the *alpha-point* of $\mathbf{W}$. The *generation* of $\mathbf{W}$ is $|S|$. If S is a tuple of length m, we say that m is the *level* of $\mathbf{W}$. If $m = 1$, we call $\mathbf{W}$ a *primary* wake. If $m = 2$, we call $\mathbf{W}$ a *secondary* wake.

Due to the tree structure of $\mathbf{I}_{<\infty}$, primary wakes are always pairwise disjoint.

Lemma 6.25. *Consider a wake ${}_{J,j}\mathbf{W}^{\blacksquare,\bullet}_S$ rooted at a critical point ${}_J C^{\blacksquare}_S$.*

1. *If $\mathbf{F}^Q$ sends ${}_J C^{\blacksquare}_S$ to another critical point ${}_{J'} C^{\square}_{S'}$, then $\mathbf{F}^Q : \overline{{}_{J,j}\mathbf{W}^{\blacksquare,\bullet}_S} \to \overline{{}_{J',j}\mathbf{W}^{\square,\bullet}_{S'}}$ is a homeomorphism.*
2. *The map $\mathbf{F}^{|S|}$ conformally sends ${}_{J,j}\mathbf{W}^{\blacksquare,\bullet}_S$ onto $\mathbb{C}\backslash\overline{\mathbf{R}^\bullet}$.*

Proof. (1) follows from the fact that $\mathbf{F}^Q$ maps ${}_{J,j}\mathbf{R}_S^{\blacksquare,\bullet} \cup {}_{J,j+1}\mathbf{R}_S^{\blacksquare,\bullet}$ homeomorphically onto ${}_{J',j}\mathbf{R}_{S'}^{\square,\bullet} \cup {}_{J',j+1}\mathbf{R}_{S'}^{\square,\bullet}$, whereas (2) follows from the fact that $\mathbf{F}^{|S|}$ maps ${}_{J,j}\mathbf{R}_S^{\blacksquare,\bullet}$ for each $j \in \{1,\dots,d_\bullet\}$ homeomorphically onto the zero ray $\mathbf{R}^\bullet$. □

To reduce notation, we consider the *full wake*

$$ {}_J\mathbf{W}_S^{\blacksquare,\bullet} := \bigcup_{j=1}^{d_\bullet - 1} {}_{J,j}\mathbf{W}_S^{\blacksquare,\bullet}(j) $$

which is the union of wakes attached to the critical point ${}_JC_S^{\blacksquare,\bullet}$ on the same side.

Lemma 6.26 (Primary wakes shrink). *For every $n \in \mathbb{Z}$ and every $\varepsilon > 0$, there are at most finitely many primary wakes of diameter at most ε rooted at a point on $\mathbf{H} \cap \mathbf{S}_n^\#$.*

Proof. The proof we present below is similar to [DL23, Lemma 5.29]. By self-similarity, it is sufficient to prove the lemma for $n = 0$. Let

$$ \mathbf{J}_- := \mathbf{U}_- \cap \mathbf{H}, \qquad \mathbf{J}_+ := \mathbf{U}_+ \cap \mathbf{H}, \quad \text{and} \quad \mathbf{J} := \mathbf{J}_- \cup \mathbf{J}_+. $$

The maps $\mathbf{f}_- = \mathbf{F}^{(0,1,0)} : \mathbf{J}_- \to \mathbf{J}$ and $\mathbf{f}_+ = \mathbf{F}^{(0,0,1)} : \mathbf{J}_- \to \mathbf{J}$ are precisely the first return maps of $\mathbf{F}$ back to $\mathbf{J}$.

Consider the semigroup generated by $(0,1,0)$ and $(0,0,1)$ and let us label its elements by $0, Q_0, Q_1, Q_2, \dots$ written in increasing order. Then, every critical point on $\mathbf{J}$ is of the form C_{Q_n} for some $n \geq 0$. Let us fix $\bullet \in \{0,\infty\}$ and consider the full primary wake $\mathbf{W}_n := \mathbf{W}_{Q_n}^\bullet$ attached to C_{Q_n}. For all $n > 0$, $\mathbf{W}_n$ is a preimage under $\mathbf{F}^{Q_n - Q_0}$ of the full wake $\mathbf{W}_0$ with the smallest generation.

Let us pick a curve Γ_0 in $\mathbf{W}_0$ connecting a point in $\mathbf{W}_0$ to the critical point C_{Q_0}. Consider the lift Γ_n of Γ_0 under $\mathbf{F}^{Q_n - Q_0} : \mathbf{W}_n \to \mathbf{W}_0$, which connects a point in $\mathbf{W}_n$ to the critical point C_{Q_n}.

Claim. There is a sequence $\varepsilon_0, \varepsilon_{-1}, \varepsilon_{-2}, \dots$ of positive numbers decreasing to 0 such that the following holds. If the (Euclidean) diameter of Γ_0 is less than ε_0, then the diameter of Γ_n is less than ε_n for all $n \geq 0$.

Proof. It is sufficient to prove the claim in the dynamical plane of the corona f_*. Consider the rational map $g = F_c$ from Theorem 3.3 which admits a (d_0, d_∞)-critical Herman curve $\mathbf{H}_g$ with rotation number equal to that of f_*. By Theorem 3.7, g is quasiconformally conjugate to f_* on a neighborhood of $\mathbf{H}_g$, so it suffices to prove the claim in the dynamical plane of g. We shall do so by applying the local connectivity of the boundary of the immediate basin of attraction of $\bullet$ of g.

Recall that the critical point of g is normalized at $1 \in \mathbf{H}_g$. For $k \geq 0$, we denote $c_k := (g|_{\mathbf{H}_g})^k(1)$. Within the immediate basin of $\bullet$, let us pick two external rays R_l and R_r landing at points on $\mathbf{H}_g$ that are slightly on the left and right of c_0 respectively. Let us pick a disk D_0 of small diameter bounded by $\mathbf{H}_g$, R_l, R_r, and an equipotential within the immediate basin of $\bullet$. Let D_k be the unique lift of D_0 under g^k whose boundary contains c_k. The disk D_k is bounded by $g^{-k}(\mathbf{H})$, a pair of external rays which are preimages of R_l and R_r, and an equipotential of an even smaller level. By local connectivity, the Euclidean diameter of D_k shrinks to zero. □

Let $\mathbf{O}_- \subset \mathbf{O}^\bullet$ be the union of all lakes of generation $(0,1,0)$ whose closure intersects $\mathbf{J}_-$, and let $\mathbf{O}_+ \subset \mathbf{O}^\bullet$ be the union of all lakes of generation $(0,0,1)$ whose closure intersects $\mathbf{J}_+$. The maps $\mathbf{f}_\pm : \mathbf{O}_\pm \to \mathbf{O}^\bullet$ expand the hyperbolic metric of $\mathbf{O}^\bullet$. Note that for all $n \geq 0$, $\mathbf{F}^{Q_{n+1}-Q_n} : \mathbf{W}_{n+1} \to \mathbf{W}_n$ is a restriction of $\mathbf{f}_\pm : \mathbf{O}_\pm \to \mathbf{O}^\bullet$. Then, due to the claim, we conclude that the Euclidean diameter of $\mathbf{W}_n$ shrinks as $n \to \infty$. □

The outer boundary of each of the full wakes attached to ${}_J C^{\blacksquare}_S$ consists of two ray segments, which we will relabel as

$$ {}^+_J\mathbf{R}^{\blacksquare,0}_S := {}_{J,1}\mathbf{R}^{\blacksquare,0}_S, \quad {}^-_J\mathbf{R}^{\blacksquare,0}_S := {}_{J,d_\bullet}\mathbf{R}^{\blacksquare,0}_S, \quad {}^-_J\mathbf{R}^{\blacksquare,\infty}_S := {}_{J,1}\mathbf{R}^{\blacksquare,\infty}_S, \quad {}^+_J\mathbf{R}^{\blacksquare,\infty}_S := {}_{J,d_\bullet}\mathbf{R}^{\blacksquare,\infty}_S. $$

For every $P \in \mathbf{T}_{>0}$, let P^- (resp. P^+) be the first entry of the left (resp. right) itinerary of the side lake ${}_l\mathbf{O}^0_P$ (resp. ${}_r\mathbf{O}^0_P$). Both P^- and P^+ are characterized by the property that $(C_{P^-}, C_{P^+}) \subset \mathbf{H}$ is the maximal open interval in which the only critical point of generation $\leq P$ is C_P.

Lemma 6.27 (Combinatorics of primary wakes)**.** *Given $P \in \mathbf{T}_{>0}$ and $\bullet \in \{0, \infty\}$,*

1. *both ${}^+\mathbf{R}^\bullet_{P^-}$ and ${}^-\mathbf{R}^\bullet_{P^+}$ contain $\alpha^\bullet_P$;*
2. *the closure of $\mathbf{W}^\bullet_P \cup \mathbf{W}^\bullet_{P^-} \cup \mathbf{W}^\bullet_{P^+}$ is a neighborhood of $\alpha^\bullet_P$;*
3. *the ray segments ${}^+\mathbf{R}^\bullet_P$ and ${}^-\mathbf{R}^\bullet_P$ can be presented as infinite concatenations of ray segments*
$$ {}^\pm\mathbf{R}^\bullet_P = \left[\alpha^\bullet_P, \alpha^\bullet_{Q^\pm_1}\right] \cup \left[\alpha^\bullet_{Q^\pm_1}, \alpha^\bullet_{Q^\pm_2}\right] \cup \left[\alpha^\bullet_{Q^\pm_2}, \alpha^\bullet_{Q^\pm_3}\right] \cup \ldots, $$
where
$$ {}^\pm\mathbf{R}^\bullet_P \cap {}^\mp\mathbf{R}^\bullet_{P^\pm} = \left[\alpha^\bullet_P, \alpha^\bullet_{Q^\pm_1}\right], \quad \textit{and for } i \geq 1, \ {}^\pm\mathbf{R}^\bullet_P \cap {}^\mp\mathbf{R}^\bullet_{Q^\pm_i} = \left[\alpha^\bullet_{Q^\pm_i}, \alpha^\bullet_{Q^\pm_{i+1}}\right]; $$
4. *the sequences of alpha-points $\left\{\alpha^\bullet_{Q^+_i}\right\}_{i\geq 1}$ and $\left\{\alpha^\bullet_{Q^-_i}\right\}_{i\geq 1}$ tend to C_P as $i \to \infty$.*

See Figures 8 and 9.

Proof. The left coast of ${}_l\mathbf{O}^0_P$ it is contained in ${}_1\mathbf{W}^0_{P^-}$ because it starts with a segment of the spine ${}_1\mathbf{H}^0_P$ rooted at C_P and is disjoint from the external rays landing at C_{P^-}. Since it lands at the alpha-point α^0_P, the boundary of the wake ${}_1\mathbf{W}^0_{P^-}$ must contain α^0_P. The treatment for the other side lakes of C_P is analogous, and this implies (1).

By Corollary 6.23, the intersection ${}^+\mathbf{R}^\bullet_{P^-} \cap {}^-\mathbf{R}^\bullet_{P^+}$ is a ray segment $[\alpha', \alpha^\bullet_P]$ for some alpha-point α' where $\alpha' \prec \alpha^\bullet_P$. Similarly, we also have that ${}^+\mathbf{R}^\bullet_{P^-} \cap {}^-\mathbf{R}^\bullet_P = [\alpha^\bullet_P, \alpha_-]$ and ${}^+\mathbf{R}^\bullet_P \cap {}^-\mathbf{R}^\bullet_{P^+} = [\alpha^\bullet_P, \alpha_+]$ where $\alpha^\bullet_P \prec \alpha_-$ and $\alpha^\bullet_P \prec \alpha_+$. Therefore, the union of $\mathbf{W}^\bullet_{P^-}$, $\mathbf{W}^\bullet_P$, and $\mathbf{W}^\bullet_{P^+}$ form a neighborhood of $\alpha^\bullet_P$, thus proving (2). More generally, we have just shown that every primary alpha-point is the meeting point of exactly three distinct primary full wakes.

Let us prove (3) and (4) for ${}^-\mathbf{R}^\bullet_P$. The treatment for ${}^+\mathbf{R}^\bullet_P$ is analogous. Let us define $Q^l_1 \in \mathbf{T}$ to be the unique smallest moment greater than P such that $C_{Q^-_1}$ is contained on the interval $(C_{P^-}, C_P) \subset \mathbf{H}$. Then, based on the previous paragraph, the alpha-point α_- must be equal to $\alpha^\bullet_{Q^-_1}$ because it is the meeting point of ${}^-\mathbf{R}^\bullet_P$, ${}^+\mathbf{R}^\bullet_{P^-}$, and the boundary of a primary full wake, which is $\mathbf{W}^\bullet_{Q^-_1}$. Similarly, ${}^+\mathbf{R}^\bullet_{Q^-_1}$ and ${}^-\mathbf{R}^\bullet_P$ meet along a ray segment $\left[\alpha^\bullet_{Q^-_1}, \alpha^\bullet_{Q^-_2}\right]$ for some $Q^-_2 > Q^-_1$. Inductively, we obtain the desired increasing sequence $\{Q^-_i\}_{i\in\mathbb{N}}$ of power-triples. It remains to show that the corresponding sequence of alpha-points $\alpha^\bullet_{Q^-_i}$ indeed converges to C_P.

By Proposition 6.20 (2), there exists an alpha-point $\alpha \in {}^-\mathbf{R}^\bullet_P$ close to C_P, which is is the alpha-point of some primary full wake $\mathbf{W}^\bullet_Q$ where $Q > P$. Since there are at

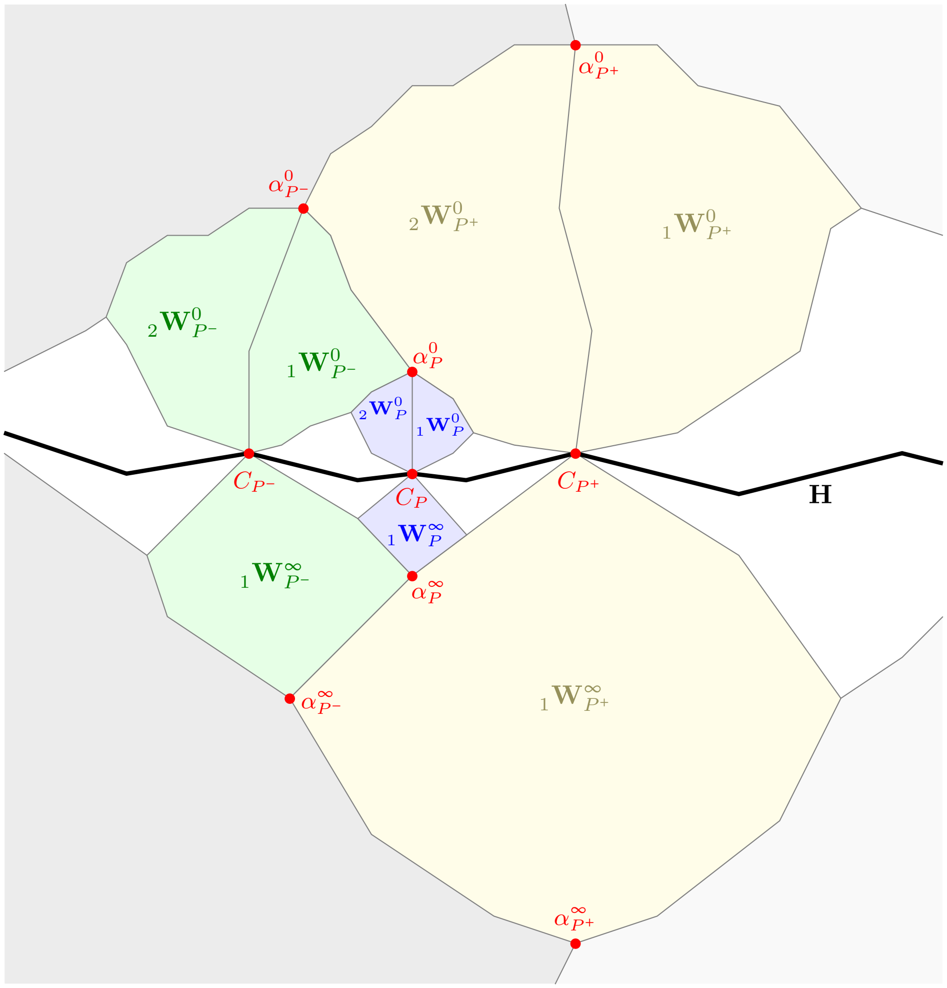

FIGURE 8. A cartoon picture of the structure of wakes when $d_0 = 3$ and $d_\infty = 2$. A more realistic picture can be found in Figure 9.

most finitely many critical points on $\mathbf{H}$ of generation less than Q between C_Q and C_P, the ray segment $[\alpha^\bullet_P, \alpha^\bullet_Q]$ intersects the boundaries of at most finitely many primary wakes. Therefore, $Q = Q_i^-$ for some $i \in \mathbb{N}$. Since α can be picked to be arbitrarily close to C_P, then $\alpha^\bullet_{Q_i^-}$ indeed converges to C_P. $\square$

Corollary 6.28 (Tiling of wakes)**.**

(1) *Primary wakes fill up the ocean: for* $\bullet \in \{0, \infty\}$,

$$\mathbf{O}^\bullet \subset \bigcup_{P \in \mathbf{T}_{>0}} \overline{\mathbf{W}^\bullet_P}.$$

(2) *The closure of a wake* ${}_{J,j}\mathbf{W}^{\blacksquare,\bullet}_S$ *is the union of spines* ${}_{J,2j-1}\mathbf{H}^{\blacksquare,\bullet}_S$ *and* ${}_{J,2j}\mathbf{H}^{\blacksquare,\bullet}_S$ *and the closure of all full wakes rooted at critical points on any of these two spines.*

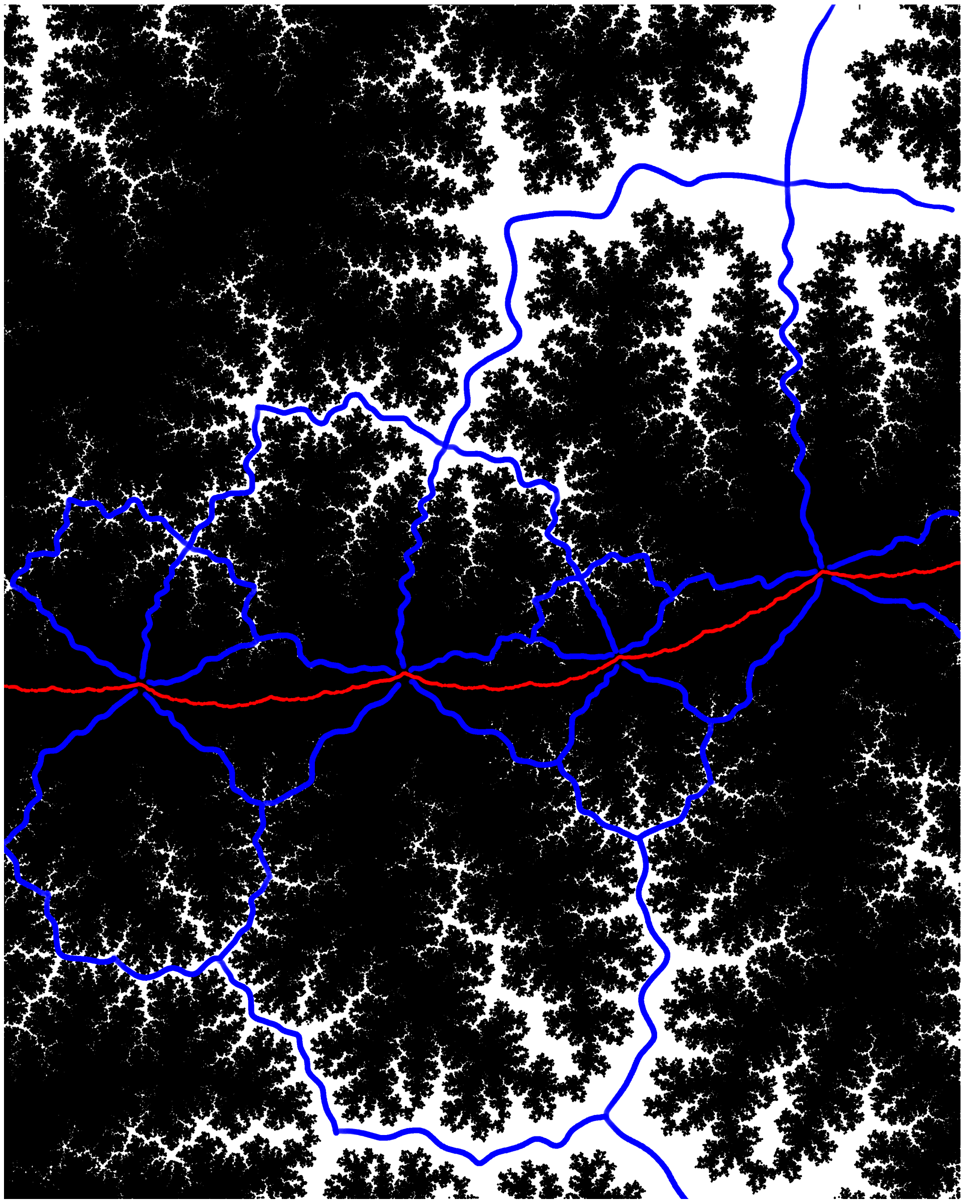

FIGURE 9. An approximate picture of the dynamical plane of $\mathbf{F}_*$ when $d_0 = 3$, $d_\infty = 2$, and θ is the golden mean irrational. This figure is obtained from the magnification of the Julia set of the rational map $f_{3,2}$ in Figure 1 around a point on its Herman curve. The Herman curve $\mathbf{H}$ of $\mathbf{F}_*$ is colored red and some external ray segments are displayed in blue. These external rays are the boundaries of the primary wakes attached to four critical points on $\mathbf{H}$.

(3) *For every $z \in \mathbf{I}_{<\infty}$ and $m \in \mathbb{N}_{\geq 1}$, there are at most three disjoint full wakes of level $\geq m$ containing z on their boundaries. The union of these full wakes forms a neighborhood of z.*

Proof. To prove (1), let us assume for a contradiction that there is a non-empty connected component X of $\mathbf{O}^\bullet \backslash \bigcup_P \overline{\mathbf{W}^\bullet_P}$. By Lemma 6.27, $\overline{X}$ intersects some point x on $\mathbf{H}$. There exists two sequences of power-triples $\{Q_n\}$ and $\{T_n\}$ such that for all $n \in \mathbb{N}$, the primary wakes $\mathbf{W}^\bullet_{Q_n}$ and $\mathbf{W}^\bullet_{T_n}$ touch, the union $\mathbf{H} \cup \overline{\mathbf{W}^\bullet_{Q_n}} \cup \overline{\mathbf{W}^\bullet_{T_n}}$ encloses a unique disk D_n containing X, and the corresponding roots C_{Q_n} and C_{T_n} converge to x as $n \to \infty$. By Lemma 6.26, the diameter of D_n tends to 0 as $n \to \infty$, which implies that such X cannot exist.

Item (2) follows from pulling back the tiling of wakes in (1) by the map $\mathbf{F}^{|S|}$ on $\overline{{}_{J,j}\mathbf{W}^{\blacksquare,\bullet}_S}$. We have thus shown that wakes of a fixed level tile each of the two oceans, and every point in the ocean is contained in the closure of at most three wakes of the same level. This implies (3). □

Lemma 6.29. *Let us equip $\mathbb{C}\backslash\mathbf{H}$ with the hyperbolic metric ρ_0. For every $P \in \mathbf{T}_{>0}$,*

(1) *the map $\mathbf{F}^P : \mathbf{W}^\bullet_P \backslash \mathbf{F}^{-P}(\mathbf{H}) \to \mathbb{C}\backslash\mathbf{H}$ is uniformly expanding (with respect to ρ_0) with a factor independent of P;*

(2) *the hyperbolic diameter of every wake of level two is at most some uniform constant independent of P.*

Proof. For all $P \in \mathbf{T}$, let ρ_P be the hyperbolic metric of $\mathbb{C}\backslash\mathbf{F}^{-P}(\mathbf{H})$. To prove (1), it suffices to show that the inclusion map

$$\iota : \left(\mathbb{C}\backslash\mathbf{F}^{-P}(\mathbf{H}), \rho_P\right) \to (\mathbb{C}\backslash\mathbf{H}, \rho_0)$$

is uniformly contracting on $\mathbf{W}^\bullet_P \backslash \mathbf{F}^{-P}(\mathbf{H})$.

Clearly, ι is uniformly contracting on $\mathbf{W}^\bullet_P$ minus a small neigborhood of C_P because this region is a compact subset of $\mathbf{O}^\bullet$. The uniform contraction of ι on a neighborhood of C_P follows from asymptotic self-similarity of $\mathbf{H}$ and $\partial\mathbf{W}^\bullet_P$ near C_P induced by pulling back A_*-invariance near 0 by $\mathbf{F}^P : C_P \mapsto 0$. One may refer to [DL23, Lemma 5.33] for further details.

Item (2) follows from essentially the same argument. By compactness, every secondary subwake of $\mathbf{W}^\bullet_P$ has uniformly bounded diameter away from a neighborhood of C_P. Near C_P, the claim again follows from the asymptotic self-similarity at C_P.

Lastly, the bounds in both claims are independent of P because every full wake in the same ocean is dynamically related. □

Lemma 6.30. *Any infinite sequence of nested wakes shrinks to a point.*

Proof. Let us define a holomorphic map χ sending level two wakes to level one wakes as follows. Given a critical point c of $\mathbf{F}^{\geq 0}$, let $W(c)$ be the union of all wakes rooted at c. Consider a secondary critical point ${}_jC^\bullet_{P,Q}$, which is contained in $W(C_P)$. The map $\mathbf{F}^P$ sends $W({}_jC^\bullet_{P,Q})$ univalently onto $W(C_Q)$. Let $T \in \mathbf{T}$ be the smallest power-triple such that $Q - T = \mathbf{t}^n P$ for some $n \in \mathbb{Z}$. Then, $\chi := A^{-n} \circ \mathbf{F}^{P+T}$ sends $W({}_jC^\bullet_{P,Q})$ univalently back onto $W(C_P)$. By Lemma 6.29, χ must be uniformly expanding on $W({}_jC^\bullet_{P,Q})$ with expansion factor independent of P.

Now, consider an infinite sequence of nested wakes $W_1 \supset W_2 \supset W_3 \supset \ldots$ where each W_n is of level n. Then, there is a uniform constant $C > 0$ such that for all

$n \geq 3$,

$$\operatorname{diam}_{\rho_0}\left(\chi^{n-2}(W_n)\right) \leq C.$$

Since χ is uniformly expanding, the hyperbolic diameter of W_n tends to 0 exponentially fast as $n \to \infty$. □

6.6. **The structure of $\mathbf{I}_{<\infty}$ and $\mathbb{C}\backslash\mathbf{I}_{<\infty}$.** Using wakes, we will show in this final subsection that the finite-time escaping set consists of topologically tame external rays.

Corollary 6.31. *Every external ray lands at a unique point.*

Proof. Let X be the accumulation set of an external ray. Since the boundary of every wake is made of ray segments, then for every wake W, either $X \subset \overline{W}$ or $X \subset \mathbb{C}\backslash W$.

If X intersects $\mathbf{H}$, then by Corollary 6.28, X must be contained in $\mathbf{H}$. In general, if X intersects $\mathbf{F}^{-P}(\mathbf{H})$ for some $P \in \mathbf{T}$, then $X \subset \mathbf{F}^{-P}(\mathbf{H})$. Since the roots of wakes are dense in $\mathbf{F}^{-P}(\mathbf{H})$, X must be a singleton.

Suppose X is disjoint from $\mathbf{F}^{-P}(\mathbf{H})$ for all P. Then, X is contained in an infinite sequence of nested wakes which, by Lemma 6.30, implies that X is a singleton. □

We say that two points x and y in $\mathbf{I}_{\leq P}$ are *combinatorially equivalent* if there is no alpha-point α such that x and y belong in distinct connected components of $\mathbf{I}_{\leq P}\backslash\{\alpha\}$. Combinatorial equivalence is an equivalence relation in $\mathbf{I}_{<\infty}$.

Corollary 6.32. *Every combinatorial equivalence class in $\mathbf{I}_{<\infty}$ is a singleton. For every $P \in \mathbb{R}_{>0}$,*

$$\mathbf{I}_{\leq P} = \overline{\bigcup_{Q<P} \mathbf{I}_{\leq Q}}. \tag{6.8}$$

Proof. Consider a point $x \in \mathbf{I}_{\leq P}$. There are two cases. If x is contained in some chain $(\infty, \alpha]$ for some alpha-point α, then the triviality of the combinatorial class follows from Corollary 6.21. Now, suppose x is not contained in any external chain. By Corollary 6.28, x is contained in an infinite sequence of nested wakes. Then, the triviality of combinatorial class of x follows from Lemma 6.30. Lastly, equation (6.8) follows directly from the first claim. □

Corollary 6.33. *$\mathbf{I}_{<\infty}$ has empty interior.*

Proof. If the interior of $\mathbf{I}_{<\infty}$ were non-empty, then any connected component of such would be contained in a single combinatorial equivalence class. This would contradict the previous corollary. □

To understand the dynamics of $\mathbf{F}$ outside of $\mathbf{I}_{<\infty}$, it suffices to consider the holomorphic map

$$\hat{\mathbf{F}} : \mathbb{C}\backslash\mathbf{I}_{<\infty} \to \mathbb{C}\backslash\mathbf{I}_{<\infty}, \quad \hat{\mathbf{F}}(z) = \mathbf{F}^P \quad \text{if } z \in \mathbf{W}_P^0 \cup \mathbf{W}_P^\infty. \tag{6.9}$$

The map $\hat{\mathbf{F}}$ is well-defined because of Corollary 6.28.

Definition 6.34. For every point z in $\mathbb{C}\backslash\mathbf{I}_{<\infty}$, the *complete address* of z is an infinite tuple $(\mathbf{W}_0, \mathbf{W}_1, \mathbf{W}_2, \ldots)$ where for every $n \geq 0$, $\mathbf{W}_n$ is the primary wake containing the unique point $\hat{\mathbf{F}}^n(z)$. The *(incomplete) address* of z is the infinite tuple $(P_0, P_1, P_2, \ldots) \in \mathbf{T}_{>0}^{\mathbb{N}}$ where P_n is the generation of $\mathbf{W}_n$.

We say that an element $(P_0, P_1, \dots)$ of $\mathbf{T}_{>0}^{\mathbb{N}}$ is *admissible* if $\sum_{n=0}^{\infty} P_n = \infty$. Moreover, we say that an infinite tuple of primary wakes is *admissible* if the corresponding tuple of generations is admissible.

Proposition 6.35. (1) *An infinite tuple of primary wakes is admissible if and only if it is the complete address of a point in* $\mathbb{C}\backslash\mathbf{I}_{<\infty}$.
(2) *Two different points in* $\mathbb{C}\backslash\mathbf{I}_{<\infty}$ *always have distinct complete addresses.*

Proof. Given a point $z \in \mathbb{C}\backslash\mathbf{I}_{<\infty}$, if the sum of the generation were finite, say $Q \in \mathbb{R}_{>0}$, then z would have escape time Q instead. Conversely, consider any admissible tuple of primary wakes $(\mathbf{W}_0, \mathbf{W}_1, \mathbf{W}_2, \dots)$. Consider a sequence of nested wakes $\mathbf{W}_0' := \mathbf{W}_0 \supset \mathbf{W}_1' \supset \mathbf{W}_2' \supset \dots$ where for $n \geq 0$, $\mathbf{W}_{n+1}'$ is defined inductively by the lift of $\mathbf{W}_n$ under $\hat{\mathbf{F}}^{n+1}|_{\mathbf{W}_n'}$. The intersection of such nested wakes is precisely the set of points admitting the complete address $(\mathbf{W}_0, \mathbf{W}_1, \mathbf{W}_2, \dots)$, and according to Lemma 6.30, it is a singleton. □

Corollary 6.36. $\mathbb{C}\backslash\mathbf{I}_{<\infty}$ *is a dense, totally disconnected subset of* $\mathbb{C}$.

Proof. By Corollary 6.33, $\mathbb{C}\backslash\mathbf{I}_{<\infty}$ must be dense in $\mathbb{C}$. By Proposition 6.35, two distinct points in $\mathbb{C}\backslash\mathbf{I}_{<\infty}$ have different complete itineraries and thus belong in disjoint wakes of sufficiently high generation. This implies the total disconnectivity of $\mathbb{C}\backslash\mathbf{I}_{<\infty}$. □

For $R > 0$, define the large radius non-escaping set of $\mathbf{F}$ by

$$\mathfrak{K}_R := \left\{ z \in \mathbb{C}\backslash\mathbf{I}_{<\infty} \,:\, \left|\mathbf{F}^P(z)\right| \geq R \text{ for all } P \in \mathbf{T} \right\}.$$

We say that a tuple $(P_0, P_1, P_2, \dots) \in \mathbf{T}_{>0}^{\mathbb{N}}$ is *bounded by* $T \in \mathbf{T}$ if $P_n \leq T$ for all n.

Lemma 6.37. (1) *For any high* $R > 0$*, there exists some* $Q_R \in \mathbf{T}_{>0}$ *such that* $Q_R \to 0$ *as* $R \to \infty$ *and that every point* z *in* $\mathfrak{K}_R$ *has address bounded by* Q_R.
(2) *For any* $Q \in \mathbf{T}_{>0}$*, there is some* $R_Q > 0$ *such that every point with address bounded by* Q *is contained in* $\mathfrak{K}_{R_Q}$.

Proof. Consider a point $z \in \mathbb{C}\backslash\mathbf{I}_{<\infty}$ with address $(P_0, P_1, P_2, \dots)$.

Consider $R > 0$, and let $Q_R \in \mathbf{T}_{>0}$ be the smallest power-triple such that all primary wakes of generation Q_R are contained inside of $\mathbb{D}_{Q_R}$. (This quantity exists due to Lemma 5.15.) Suppose $P_n \geq Q_R$ for some $n \in \mathbb{N}$. Then, z is eventually mapped into a wake of generation Q_R, which is contained inside of $\mathbb{D}_R$. This implies (1).

Next, consider $Q \in \mathbf{T}_{>0}$, and let $R_Q > 0$ be such that all primary wakes of generation $\leq Q$ are disjoint from $\mathbb{D}_{R_Q}$. Suppose that $\mathbf{F}^P(z)$ is in $\mathbb{D}_{R_Q}$ for some $P \in \mathbf{T}$. Then, $\mathbf{F}^P(z)$ is contained in a wake of generation greater than Q. This implies (2). □

In the next section, we are interested in the infinite-time escaping set as well. For $\mathbf{F} = \mathbf{F}_*$, this set can be described as follows.

Corollary 6.38. *The infinite-time escaping set* $\mathbf{I}_\infty$ *of* $\mathbf{F}$ *is the set of points in* $\mathbb{C}\backslash\mathbf{I}_{<\infty}$ *whose address* $(P_0, P_1, P_2, \dots)$ *satisfies* $P_n \to 0$ *as* $n \to \infty$.

Proof. Consider a point z in $\mathbb{C}\backslash\mathbf{I}_{<\infty}$ with some address $(P_0, P_1, P_2, \ldots)$. If $z \in \mathbf{I}_\infty$, then given any $R > 0$, $\hat{\mathbf{F}}^n(z)$ must be in $\mathfrak{K}_R$ for all sufficiently high n, which implies that $(P_n, P_{n+1}, \ldots)$ is bounded by Q_R where $Q_R \to 0$ as $R \to \infty$. Conversely, if $P_n \to 0$ as $n \to \infty$, then for all n, $(P_n, P_{n+1}, \ldots)$ is bounded by Q_n where $Q_n \to 0$, and so $\hat{\mathbf{F}}^n(z)$ is contained in $\mathfrak{K}_{Q_n}$ where $Q_n \to \infty$. □

7. Rigidity of escaping dynamics

In Section 5.2, we constructed the global unstable manifold $\boldsymbol{\mathcal{W}}^u$ of the corona renormalization operator $\mathcal{R}$ consisting of cascades of transcendental maps $\mathbf{F}^{\geq 0}$. In this final section, we will conclude the proof of Theorem A by showing that $\boldsymbol{\mathcal{W}}^u$ is one-dimensional. Our approach is to prove Theorem B on the rigidity of escaping dynamics of each $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$. We will apply the external structure of the renormalization fixed point $\mathbf{F}_*$ addressed in Section 6, and adapt an argument by Rempe [Rem09] to show that the set of points in the full escaping set that remain sufficiently close to ∞ under iteration must move holomorphically with dilatation arbitrarily close to zero.

7.1. **Invariant line field.** We say that a corona $f : U \to V$ admits an *invariant line field* supported on a set $E \subset \mathbb{C}$ if there is a measurable Beltrami differential $\mu(z)\frac{d\bar{z}}{dz}$ such that $f^*\mu = \mu$ almost everywhere on U, $|\mu| = 1$ on a positive measure subset of E, and $\mu = 0$ elsewhere.

Similarly, we say that $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$ admits an *invariant line field* supported on a set $E \subset \mathbb{C}$ if there is a measurable Beltrami differential $\mu(z)\frac{d\bar{z}}{dz}$ such that $\left(\mathbf{F}^P\right)^* \mu = \mu$ almost everywhere on $\operatorname{Dom}\left(\mathbf{F}^P\right)$ for all $P \in \mathbf{T}$, $|\mu| = 1$ on a positive measure subset of E, and $\mu = 0$ elsewhere.

In classical holomorphic dynamics, the absence of invariant line fields is equivalent to the lack of deformation space associated to a single holomorphic map. This philosophy remains valid for cascades in the unstable manifold.

Proposition 7.1. *If $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$ admits an invariant line field μ, there is a holomorphic family $\{\mathbf{G}_t\}_{t\in\mathbb{D}}$ in $\boldsymbol{\mathcal{W}}^u$ such that $\mathbf{G}_0 = \mathbf{F}$ and $\mathbf{F}^{\geq 0}$ is quasiconformally conjugate to $\mathbf{G}_t^{\geq 0}$. The conjugacy is conformal outside of the support of μ.*

Proof. A standard application of the measurable Riemann mapping theorem gives us the desired holomorphic family $\{\mathbf{G}_t\}_{t\in\mathbb{D}}$, but a priori we do not know whether this family lives in $\boldsymbol{\mathcal{W}}^u$. To fix this issue, we shall descend back to the realm of coronas.

By anti-renormalizing, let us assume without loss of generality that $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u_{loc}$. Let us project μ to the dynamical plane of f_n for $n \leq 0$ and obtain an invariant line field μ_n of f_n. Then, we integrate μ_n to obtain a Beltrami path $\{f_{n,t}\}_{t\in\mathbb{D}}$ of coronas in a neighborhood of f_*. Let us renormalize to obtain a new path $f_t^{(n)} := \mathcal{R}^{-n} f_{n,t}$ about $f_0^{(n)} \equiv f_0$ for all $n \leq 0$. When $|t| < \frac{1}{2}$, $f_t^{(n)}$ is quasiconformally conjugate to f_0 with uniformly bounded dilatation. Therefore, we can take a limit as $n \to -\infty$ and obtain a holomorphic path g_t of infinitely anti-renormalizable corona. As the limiting path lies in $\mathcal{W}^u_{loc}$, it corresponds to a path in $\boldsymbol{\mathcal{W}}^u$. □

Lemma 7.2. *The renormalization fixed point $\mathbf{F}_*$ admits no invariant line field supported on its full escaping set $\mathbf{I}(\mathbf{F}_*)$.*

Proof. Suppose for a contradiction that $\mathbf{I}(\mathbf{F}_*)$ supports an invariant line field of $\mathbf{F}_*$. By Proposition 7.1, we obtain a family $\{\mathbf{G}_t\}_{t\in\mathbb{D}}$ in $\boldsymbol{\mathcal{W}}^u$ together with quasiconformal maps $h_t : \mathbb{C} \to \mathbb{C}$ conjugating $\mathbf{F}_*$ with $\mathbf{G}_t$ for all $t \in \mathbb{D}$. Each of $\mathbf{G}_t$ induces a rotational corona g_t with rotation number θ, which, by Theorem 4.12, implies that g_t must also be on the local stable manifold. Therefore, $g_t \equiv f_*$ and the family $\mathbf{G}_t$ is trivial. For every t, h_t commutes with $\mathbf{F}_*$ along the Herman quasicircle $\mathbf{H}$. As such, h_t is the identity on $\mathbf{H}$, and so on the grand orbit $\bigcup_P \mathbf{F}^{-P}(\mathbf{H})$ of $\mathbf{H}$ as well.

We claim that the grand orbit of $\mathbf{H}$ is a dense subset of $\mathbb{C}$, and thus $h_t \equiv \mathrm{Id}$. Indeed, by [Lim23b, §5], the critical value $c_1(f_*)$ of f_* is a deep point of the Julia set $J(f_*)$ of f_*. In particular, magnifications of $J(f_*)$ about $c_1(f_*)$ converge to the whole plane. As we pass to the corresponding dynamical plane of the transcendental extension, 0 is a deep point of iterated preimages of $\mathbf{H}$ under $\mathbf{f}_\pm$. By self-similarity, the grand orbit of $\mathbf{H}$ must be dense in $\mathbb{C}$. ☐

7.2. **Rigidity of the finite-time escaping set.** Let us fix

$$T := \min\{(0,1,0),(0,0,1)\}.$$

Lemma 7.3. *There is a unique equivariant holomorphic motion of $\mathbf{I}_{\leq T}(\mathbf{F})$ over some neighborhood $\boldsymbol{\mathcal{U}}$ of $\mathbf{F}_*$.*

Proof. By Lemma 5.14, the set of critical values $\mathrm{CV}\left(\mathbf{F}^T\right)$ of $\mathbf{F}^T$ moves holomorphically within a small neighborhood of $\mathbf{F}_*$. By Lemma 5.8, there is a small neighborhood $\boldsymbol{\mathcal{U}}$ of $\mathbf{F}_*$ and some point $x \in \mathbb{C}$ such that x belongs in the interior of $\mathbf{U}_-(\mathbf{F})$ and does not collide with $\mathrm{CV}\left(\mathbf{F}^T\right)$ for all $\mathbf{F} \in \boldsymbol{\mathcal{U}}$. Moreover, $\mathbf{F}^{-S}(x)$ moves holomorphically with $\mathbf{F} \in \boldsymbol{\mathcal{U}}$ for all $S \leq T$.

If $Q < S \leq T$, then $\mathbf{F}^{-S}(x)$ is disjoint from $\mathbf{F}^{-Q}(x)$ because every point is mapped by $\mathbf{F}^S$ and $\mathbf{F}^Q$ to different tiles of the zeroth renormalization tiling of $\mathbf{F}$. Hence, $\bigcup_{S\leq T} \mathbf{F}^{-S}(x)$ moves holomorphically and equivariantly with $\mathbf{F} \in \boldsymbol{\mathcal{U}}$. By the λ-lemma, this holomorphic motion extends to the closure. Then, by Corollaries 6.33 and 5.22, $\mathbf{I}_{\leq T}(\mathbf{F})$ has no interior and moves holomorphically and equivariantly over $\boldsymbol{\mathcal{U}}$.

Let us show that the motion τ of $\mathbf{I}_{\leq T}(\mathbf{F})$ obtained above is independent of x. Let us pick another point $y = y(\mathbf{F}) \in \mathbb{C}\backslash\mathrm{CV}(\mathbf{F})$ which depends holomorphically on $\mathbf{F} \in \boldsymbol{\mathcal{U}}$. By shrinking $\boldsymbol{\mathcal{U}}$, we can connect x and y by a simple arc $l = l(\mathbf{F})$ which is surrounded by an annulus $A = A(\mathbf{F}) \subset \mathbb{C}\backslash\mathrm{CV}(\mathbf{F})$. Every preimage of l under $\mathbf{F}^T$ is separated from $\mathbf{I}_{\leq T}(\mathbf{F})$ by a conformal preimage of A. Therefore, any sequence of preimages of l under $\mathbf{F}^T$ which accumulates at a point in $\mathbf{I}_{\leq T}(\mathbf{F})$ necessarily shrinks in diameter. As a result, the holomorphic motion τ coincides with the motion of $\overline{\mathbf{F}^{-T}(y(\mathbf{F}))}$.

Finally, let us show that the equivariant holomorphic motion τ of $\mathbf{I}_{\leq T}(\mathbf{F})$ over $\boldsymbol{\mathcal{U}}$ is unique. Suppose there is another equivariant holomorphic motion τ' of $\mathbf{I}_{\leq T}(\mathbf{F})$. Pick any $S \in \mathbf{T}_{>0}$ where $S < T$ and consider the motion $y(\mathbf{F})$ of a point in $\mathbf{I}_{\leq S}(\mathbf{F})$ induced by τ'. By equivariance, $\mathbf{F}^{-(T-S)}(y(\mathbf{F}))$ moves holomorphically by τ'. However, since $\mathbf{I}_{\leq T-S}(\mathbf{F})$ is contained in the closure of $\mathbf{F}^{-(T-S)}(y(\mathbf{F}))$, then τ and τ' coincide on $\mathbf{I}_{\leq T-S}(\mathbf{F})$ for all $S \in \mathbf{T}_{>0}$. By (6.8), $\tau \equiv \tau'$. ☐

We say that a holomorphic motion of a set $E \subset \mathbb{C}$ is a *conformal motion* if its dilatation on E is zero.

Theorem 7.4. *For every $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, $\mathbf{I}_{<\infty}(\mathbf{F})$ has empty interior and supports no invariant line field. For every $P \in \mathbf{T}_{>0}$, on every connected component of the open* *set* $\{\mathbf{F} \in \boldsymbol{\mathcal{W}}^u : 0 \notin \mathbf{I}_{\leq P}(\mathbf{F})\}$*, there is a unique equivariant holomorphic motion of* $\mathbf{I}_{\leq P}$*, and this motion is conformal.*

Proof. Let us fix $P \in \mathbf{T}_{>0}$ and consider the set $\mathbf{D}_P := \{\mathbf{F} \in \boldsymbol{\mathcal{W}}^u : 0 \notin \mathbf{I}_{\leq P}(\mathbf{F})\}$. If $P < T$, then clearly the neighborhood $\boldsymbol{\mathcal{U}}$ of $\mathbf{F}_*$ from Lemma 7.3 is contained in $\mathbf{D}_P$. Else, if $P \geq T$, then $\mathbf{F} \in \mathbf{D}_P \cap \boldsymbol{\mathcal{U}}$ if and only if $\mathbf{F}^{P-T}(0) \notin \mathbf{I}_{\leq T}(\mathbf{F})$, which is an open condition because $\mathbf{I}_{\leq T}$ moves holomorphically over $\boldsymbol{\mathcal{U}}$. Therefore, $\mathbf{D}_P \cap \boldsymbol{\mathcal{U}}$ is open for all P.

If $\mathbf{F} \in \mathbf{D}_P \cap \mathcal{U}$, we can obtain the unique equivariant holomorphic motion of $\mathbf{I}_{\leq P}$ by pulling back the holomorphic motion of $\mathbf{I}_{\leq T}$ via $\mathbf{F}^{P-T}$. In general, for any $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, we can pick a sufficiently large $n \ll 0$ such that $\mathbf{F}_n \in \boldsymbol{\mathcal{U}}$. Clearly, $\mathbf{F} \in \mathbf{D}_P$ if and only if $\mathbf{F}_n \in \mathbf{D}_{\mathbf{t}^{-n}P}$, so $\mathbf{D}_P$ is always an open subset of $\boldsymbol{\mathcal{W}}^u$ on which $\mathbf{I}_{\leq P}(\mathbf{F})$ moves holomorphically and equivariantly. The dilatation of the motion of $\mathbf{I}_{\leq T}(\mathbf{F})$ over $\mathcal{U}$ goes to zero as $\mathbf{F} \to \mathbf{F}_*$. Therefore, for every $\mathbf{F} \in \mathbf{D}_P$, we can take an arbitrarily high $n \ll 0$ to ensure that the dilatation of the motion of $\mathbf{I}_{\leq T}$ at $\mathbf{F}_n$ is arbitrarily small. Pulling back via $\mathbf{F}^{\mathbf{t}^{-n}P-T}$ does not affect the dilatation, so the motion of $\mathbf{I}_{\leq P}$ is indeed conformal over $\mathbf{D}_P$.

By Corollary 6.33 and Lemma 7.2, $\mathbf{I}_{\leq \mathbf{t}^{-n}P}(\mathbf{F}_n)$ has empty interior and supports no invariant line field of $\mathbf{F}_n$. Therefore, $\mathbf{I}_{\leq P}(\mathbf{F})$ also has empty interior and supports no invariant line field of $\mathbf{F}$. ☐

Corollary 7.5. *For all* $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$*, the finite-time escaping set* $\mathbf{I}_{<\infty}(\mathbf{F})$ *is non-empty and* $\mathfrak{J}(\mathbf{F}) = \overline{\mathbf{I}_{<\infty}(\mathbf{F})}$.

Proof. Pick any $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$. From the previous theorem, there exist some small $P \in \mathbf{T}_{>0}$ and some open neighborhood $\boldsymbol{\mathcal{U}} \subset \boldsymbol{\mathcal{W}}^u$ of $\mathbf{F}_*$ containing $\mathbf{F}$ in which the P^{th} escaping set moves holomorphically. Therefore, $\mathbf{I}_{\leq P}(\mathbf{F})$ is clearly non-empty. By Montel's theorem, since $\mathbf{I}_{\leq P}(\mathbf{F})$ contains more than two points, every neighborhood of any point in $\mathfrak{J}(\mathbf{F})$ must contain a point in $\mathbf{I}_{\leq Q}(\mathbf{F})$ where $Q \geq P$. ☐

7.3. **Rigidity of the infinite-time escaping set.** For $R > 0$ and $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, define

$$\mathfrak{J}_R(\mathbf{F}) := \left\{ z \in \mathfrak{J}(\mathbf{F}) \backslash \mathbf{I}_{<\infty}(\mathbf{F}) \ : \ \left|\mathbf{F}^P(z)\right| \geq R \text{ for all } P \in \mathbf{T} \right\}.$$

The forward orbit of every point in the infinite-time escaping set $\mathbf{I}_\infty(\mathbf{F})$ is eventually contained in $\mathfrak{J}_R(\mathbf{F})$. The following lemma is inspired by [Rem09].

Lemma 7.6. *For every* $\mathbf{F}$ *on a neighborhood* $\boldsymbol{\mathcal{U}}$ *of* $\mathbf{F}_*$*, there exists a subset* $\Lambda(\mathbf{F})$ *of* $\mathfrak{J}(\mathbf{F}) \backslash \mathbf{I}_{<\infty}(\mathbf{F})$ *with the following properties.*

1. $\Lambda(\mathbf{F})$ *is forward invariant under* $\mathbf{F}^{\geq 0}$.
2. *There is a unique equivariant holomorphic motion of* Λ *over* $\boldsymbol{\mathcal{U}}$.
3. *There exists some* $R > 1$ *such that* $\Lambda(\mathbf{F})$ *contains* $\mathfrak{J}_R(\mathbf{F})$.

Proof. In the dynamical plane of $\mathbf{F}_*$, every point in in the forward orbit of a point in $\mathfrak{J}_R(\mathbf{F}_*)$ must be contained in a wake of sufficiently low generation in order to avoid the disk $\mathbb{D}_R := \{|z| < R\}$. We consider all such points and define $\Lambda(\mathbf{F}_*)$. In the proof below, we apply the motion of the finite-time escaping set from the previous subsection to show that $\Lambda(\mathbf{F})$ can be defined naturally via a unique holomorphic motion. The proof will be broken down into four steps.

Step 1: Construct truncated wakes which move holomorphically.

Let us pick $r > 0$ such that all primary wakes of $\mathbf{F}_*$ of generation at most $T := \min\{(0,1,0),(0,0,1)\}$ are compactly contained in the domain $\mathbf{V} = \mathbb{C} \backslash \overline{\mathbb{D}_r}$. Let us enumerate primary wakes of generation at most T by $\{\mathbf{W}_i\}_{i\in I}$ for some countable index set I. Denote the generation of each wake $\mathbf{W}_i$ by P_i. For every $i \in I$, consider the truncated wake

$$\hat{\mathbf{W}}_i := \mathbf{W}_i \cap \mathbf{F}_*^{-P_i}(\mathbf{V})$$

obtained by removing from $\mathbf{W}_i$ a small neighborhood of the critical point C_{P_i} that gets mapped to $\overline{\mathbb{D}_r}$

For each $\bullet \in \{0,\infty\}$, there exists a unique point $z^\bullet$ on the intersection of $\partial\mathbf{V}$ and the zero ray $\mathbf{R}^\bullet$ such that the ray segment $\hat{\mathbf{R}}^\bullet = (\infty, z^\bullet)$ is contained in $\mathbf{V}$. The ray segments $\hat{\mathbf{R}}^0$ and $\hat{\mathbf{R}}^\infty$ are contained in $\mathbf{I}_{\leq Q}(\mathbf{F}_*)$ where Q is the maximum of the escaping times of z^0 and z^∞. By Lemma 7.6, the Q^{th} escaping set $\mathbf{I}_{\leq Q}$ moves holomorphically and equivariantly on a small neighborhood $\boldsymbol{\mathcal{U}}$ of $\mathbf{F}_*$. By the λ-lemma, such a motion induces a holomorphic motion of $\hat{\mathbf{R}}^0(\mathbf{F}) \cup \hat{\mathbf{R}}^\infty(\mathbf{F}) \cup \partial\mathbf{V}(\mathbf{F})$, which, by shrinking $\boldsymbol{\mathcal{U}}$ if necessary, can be assumed to not collide with $\mathrm{CV}\left(\mathbf{F}^T\right)$. This allows us to pull back via $\mathbf{F}^P$ for all $P \leq T$ and further extend this motion to a holomorphic motion of

$$\hat{\mathbf{R}}^0(\mathbf{F}) \cup \hat{\mathbf{R}}^\infty(\mathbf{F}) \cup \partial\mathbf{V}(\mathbf{F}) \cup \bigcup_{i\in I} \partial\hat{\mathbf{W}}_i(\mathbf{F})$$

that is equivariant on $\partial\hat{\mathbf{W}}_i(\mathbf{F})$ with respect to $\mathbf{F}^{P_i}$ for every $i \in I$. By λ-lemma, this motion can again be extended to a holomorphic motion Φ_0 on the whole plane that is equivariant with respect to $\mathbf{F}^{P_i}$ on $\partial\hat{\mathbf{W}}_i(\mathbf{F})$ for every $i \in I$.

Step 2: Construct Λ which moves holomorphically and equivariantly.

Consider $\mathbf{V}_0(\mathbf{F}) := \bigcup_{i\in I} \hat{\mathbf{W}}_i(\mathbf{F})$ and define the holomorphic map

$$\hat{\mathbf{F}} : \mathbf{V}_0(\mathbf{F}) \to \mathbf{V}(\mathbf{F}), \quad \hat{\mathbf{F}}(z) = \mathbf{F}^{P_i}(z) \text{ for } z \in \hat{\mathbf{W}}_i(\mathbf{F}).$$

This map satisfies a Markov-like property that $\mathbf{V}_0(\mathbf{F}) \subset \mathbf{V}(\mathbf{F})$ and $\hat{\mathbf{F}}$ sends every connected component of $\mathbf{V}_0(\mathbf{F})$ univalently onto a dense subset of $\mathbf{V}(\mathbf{F})$. (Note that $\hat{\mathbf{F}}_*$ coincides with the map defined in (6.9).)

Consider the non-escaping set $\Lambda(\mathbf{F})$ of $\hat{\mathbf{F}}$ which is defined by

$$\Lambda(\mathbf{F}) := \bigcap_{n\geq 0} \mathbf{V}_{-n}(\mathbf{F}) \quad \text{where} \quad \mathbf{V}_{-n}(\mathbf{F}) := \hat{\mathbf{F}}^{-n}(\mathbf{V}_0(\mathbf{F})).$$

Clearly, $\Lambda(\mathbf{F})$ is forward invariant under $\mathbf{F}^{\geq 0}$. By Lemma 6.30, nested truncated wakes shrink to points, so $\Lambda(\mathbf{F}_*)$ is a closed totally disconnected set.

Let us treat the holomorphic motion $\Phi_0 = \Phi_0(\mathbf{F})$ as a map from the dynamical plane of $\mathbf{F}_*$ to the dynamical plane of $\mathbf{F}$. We will apply the pullback argument to Φ_0 as follows. For $n \geq 0$, let us inductively define the lift of Φ_n to be

$$\Phi_{n+1} := \begin{cases} \Phi_n & \text{on } \mathbb{C}\backslash\mathbf{V}_{-n}(\mathbf{F}_*), \\ \left(\hat{\mathbf{F}}|_{\hat{\mathbf{W}}_i(\mathbf{F})}\right)^{-1} \circ \Phi_n \circ \hat{\mathbf{F}}_* & \text{on } \mathbf{V}_{-n}(\mathbf{F}_*) \cap \hat{\mathbf{W}}_i(\mathbf{F}_*) \text{ for each } i \in I. \end{cases}$$

By equivariance, for all n, Φ_n is quasiconformal on $\mathbb{C}$ with uniformly bounded dilatation and it eventually stabilizes at every point outside of $\Lambda(\mathbf{F}_*)$. Since $\Lambda(\mathbf{F}_*)$ is nowhere dense, Φ_n converges in subsequence to a limiting holomorphic motion Φ which is equivariant on $\Lambda(\mathbf{F})$.

Step 3: Show that the equivariant holomorphic motion of Λ is unique.

Suppose Ψ is another holomorphic motion of $\Lambda(\mathbf{F})$ on some small neighborhood of $\mathbf{F}_*$. We will use the notation $\Psi_{\mathbf{F}}(x)$ to highlight the dependence of $\mathbf{F}$. Let us pick any point $x \in \Lambda(\mathbf{F}_*)$. There is some $(i_0, i_1, \ldots) \in I^{\mathbb{N}}$ such that x is the unique point with address $(i_0, i_1, \ldots)$, that is, $\hat{\mathbf{F}}_*^n(x)$ lies in the truncated wake $\hat{\mathbf{W}}_{i_n}(\mathbf{F}_*)$ for all n.

Suppose for a contradiction that $\Psi_{\mathbf{F}}(x)$ and $\Phi_{\mathbf{F}}(x)$ are distinct. Then, the address of $\Psi_{\mathbf{F}}(x)$ is not equal to $(i_0, i_1, \ldots)$ and, in particular, there is some $n \in \mathbb{N}$ such that $\hat{\mathbf{F}}^n(\Psi_{\mathbf{F}}(x))$ lies in a truncated wake other than $\hat{\mathbf{W}}_{i_n}(\mathbf{F})$. Since the boundary of $\hat{\mathbf{W}}_{i_n}(\mathbf{F})$ moves holomorphically and equivariantly, there exists some $\mathbf{G} \in \boldsymbol{\mathcal{W}}^u$ sufficiently close to $\mathbf{F}_*$ such that $x_n' := \hat{\mathbf{G}}^n \Psi_{\mathbf{G}}(x)$ is on the boundary of $\hat{\mathbf{W}}_{i_n}(\mathbf{G})$. Then, $y_n' := \mathbf{G}^{P_{i_n}}(x_n')$ would lie on $\hat{\mathbf{R}}^0(\mathbf{G}) \cup \hat{\mathbf{R}}^\infty(\mathbf{G}) \cup \partial \mathbf{V}(\mathbf{G})$, which is disjoint from $\Lambda(\mathbf{G})$. However, due to forward invariance, y_n' must be contained in $\Lambda(\mathbf{G})$, hence a contradiction.

Step 4: Show that $\Lambda(\mathbf{F})$ contains $\mathfrak{J}_R(\mathbf{F})$ for some $R > 0$ independent of $\mathbf{F} \in \mathcal{U}$.

It suffices to find R such that for all $\mathbf{F} \in \boldsymbol{\mathcal{U}}$, every point outside of $\mathbf{I}_{<\infty}(\mathbf{F}) \cup \Lambda(\mathbf{F})$ will be sent into the disk $\mathbb{D}_R$ by $\mathbf{F}^P$ for some $P \in \mathbf{T}$.

Let us recall the renormalization tiling $\boldsymbol{\Delta}_n(\mathbf{F})$ defined in §5.3. In the dynamical plane of $\mathbf{F}_*$, there exists some sufficiently large $N < 0$ such that all primary wakes rooted at critical points located in $\Delta_0(0, \mathbf{F}_*) \cup \Delta_0(1, \mathbf{F}_*)$ are contained in the tile $\Delta_N(i, \mathbf{F}_*)$ for some $i \in \{0, 1\}$. Then, every wake of generation greater than T is contained in the tiling $\boldsymbol{\Delta}_N(\mathbf{F}_*)$. In particular, $\mathbb{C} \backslash \overline{\mathbf{V}_0(\mathbf{F}_*)} \subset \boldsymbol{\Delta}_N(\mathbf{F}_*)$.

By shrinking $\boldsymbol{\mathcal{U}}$ if necessary, the tiling $\boldsymbol{\Delta}_N(\mathbf{F})$ moves holomorphically and equivariantly over $\boldsymbol{\mathcal{U}}$ and always contains $\mathbb{C} \backslash \overline{\mathbf{V}_0(\mathbf{F})}$. Therefore, for all $\mathbf{F} \in \boldsymbol{\mathcal{U}}$, every point outside of $\mathbf{I}_{<\infty}(\mathbf{F}) \cup \Lambda(\mathbf{F})$ is eventually mapped to a point in $\mathbb{C} \backslash \overline{\mathbf{V}_0(\mathbf{F})}$, which is then eventually mapped to another point in $\mathbf{F}^{(N,0,1)}(\Delta_N(0, \mathbf{F})) \cup \mathbf{F}^{(N,1,0)}(\Delta_N(1, \mathbf{F}))$, which is contained in the disk $\mathbb{D}_R$ for some large $R > 0$ independent of $\mathbf{F}$. □

Theorem 7.7. *For every $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, $\mathbf{I}_\infty(\mathbf{F})$ supports no invariant line field. Moreover, on every connected component of the interior of $\{\mathbf{F} \in \boldsymbol{\mathcal{W}}^u \ : \ 0 \notin \mathbf{I}_\infty(\mathbf{F})\}$, there is a unique equivariant holomorphic motion of $\mathbf{I}_\infty(\mathbf{F})$, and this motion is conformal.*

Proof. Let $\boldsymbol{\mathcal{U}}$, Λ, and R be from the previous lemma. For every $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, there is some sufficiently large $n \ll 0$ such that $\mathbf{F}_n$ lies in $\boldsymbol{\mathcal{U}}$. Since $\mathbf{F}^P = A_*^n \circ \mathbf{F}_n^{P/\mathbf{t}^n} \circ A_*^{-n}$ for all $P \in \mathbf{T}$, the set $\Lambda_n(\mathbf{F}) := A_*^n(\Lambda(\mathbf{F}_n))$ is forward invariant, contains $\mathfrak{J}_{|\mu_*|^n R}(\mathbf{F})$, and admits a unique equivariant holomorphic motion Φ_n over $\mathcal{R}^{-n}(\boldsymbol{\mathcal{U}})$. The dilatation of Φ_n near $\mathbf{F}$ can be made arbitrarily small by choosing $\mathbf{F}_n$ arbitrarily close to $\mathbf{F}_*$, or equivalently, n to be an arbitraily large negative number. In particular, there is a unique equivariant holomorphic motion of $\mathbf{I}_\infty(\mathbf{F}) \cap \Lambda_n(\mathbf{F})$ and its dilatation near $\mathbf{F}$ shrinks to zero as $n \to \infty$.

On a component of the interior of $\{\mathbf{F} \in \boldsymbol{\mathcal{W}}^u \ : \ 0 \notin \mathbf{I}_\infty(\mathbf{F})\}$, we can extend Φ_n by iteratively pulling back the holomorphic motion of $\mathbf{I}_\infty(\mathbf{F}) \cap \Lambda_n(\mathbf{F})$, yielding a unique equivariant holomorphic motion $\widetilde{\Phi}_n$ on $\mathbf{I}_\infty(\mathbf{F})$. Since we are pulling back by a holomorphic map, the dilatation of $\widetilde{\Phi}_n$ is equal to that of Φ_n. By the uniqueness of the motion, $\widetilde{\Phi} = \widetilde{\Phi}_n$ is independent of n. Moreover, since the dilatation shrinks to zero as $n \to \infty$, then $\widetilde{\Phi}$ is a conformal motion of $\mathbf{I}_\infty$.

Lastly, suppose for a contradiction that $\mathbf{I}_\infty(\mathbf{G})$ supports an invariant line field μ of some $\mathbf{G} \in \boldsymbol{\mathcal{W}}^u$. Since $\mathbf{I}_\infty(\mathbf{F}) \cap \Lambda_n(\mathbf{F})$ moves holomorphically over a neighborhood of $\mathbf{F}_*$ containing $\mathbf{G}$ for some $n \ll 0$, then there is a quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ which has zero dilatation on $\mathbf{I}_\infty(\mathbf{F}_*) \cap \Lambda_n(\mathbf{F})$ and conjugates $\mathbf{F}_*|_{\mathbf{I}_\infty(\mathbf{F}_*) \cap \Lambda_n(\mathbf{F})}$ to

$\mathbf{G}|_{\mathbf{I}_\infty(\mathbf{G})\cap\Lambda_n(\mathbf{F})}$. Consider $\mu' = \phi^*\mu$ on $\mathbf{I}_\infty(\mathbf{F}_*) \cap \Lambda_n(\mathbf{F})$ and pull it back via $\mathbf{F}_*$ to obtain a $\mathbf{F}_*$-invariant Beltrami differential μ' supported on $\mathbf{I}_\infty(\mathbf{F}_*)$. Then, μ' would be an invariant line field of $\mathbf{F}_*$ supported on $\mathbf{I}_\infty(\mathbf{F}_*)$, which is impossible due to Lemma 7.2. □

7.4. **Proof of the main theorems.** We say that $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$ is *hyperbolic* if $\mathbf{F}$ admits an attracting cycle of periodic points. Additionally, we say that $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$ is *superattracting* if 0 is a periodic point of $\mathbf{F}^{\geq 0}$.

Proposition 7.8. *If $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$ is hyperbolic, then $\mathfrak{J}(\mathbf{F})\backslash\mathbf{I}(\mathbf{F})$ has no interior.*

Proof. Suppose $\mathbf{F}$ is hyperbolic. By Proposition 5.26, the postcritical set $\mathfrak{P}(\mathbf{F})$ is contained in the Fatou set. Suppose instead that $\mathfrak{J}(\mathbf{F})$ contains an open ball B. Corollary 7.5 tells us that $\mathbf{I}_{<\infty}(\mathbf{F}) \cap B$ is dense in B. By Lemma 5.21, there is some $P \in \mathbf{T}_{>0}$ such that $\mathbf{F}^P(B\backslash\mathbf{I}_{\leq P}(\mathbf{F}))$ is dense in $\mathbb{C}$. This is impossible because the Fatou set of $\mathbf{F}$ is non-empty. □

Corollary 7.9. *If $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$ is hyperbolic, then $\mathfrak{J}(\mathbf{F})\backslash\mathbf{I}(\mathbf{F})$ has zero Lebesgue measure.*

Proof. This immediately follows from Theorem 5.28 and Proposition 7.8. □

Corollary 7.10. *Consider a hyperbolic component Ω of $\boldsymbol{\mathcal{W}}^u$. There is a unique equivariant holomorphic motion of $\mathfrak{J}(\mathbf{F})$ over $\mathbf{F} \in \Omega$, and such a motion is a conformal motion. If $\mathbf{F} \in \Omega$, then $\mathfrak{J}(\mathbf{F})$ supports no invariant line field of* $\mathbf{F}$.

Proof. For $\mathbf{F} \in \Omega$, the critical value 0 is not contained in $\mathbf{I}(\mathbf{F})$, and so the assertion follows from Corollary 7.9 and Theorems 7.4 and 7.7. □

This completes the proof of Theorem B. To prove Theorem A, we need the following additional ingredient.

Lemma 7.11 (Density of hyperbolicity at $\mathbf{F}_*$). *Every neighborhood $\boldsymbol{\mathcal{U}}$ of the fixed point $\mathbf{F}_*$ contains a superattracting cascade.*

Proof. Suppose for a contradiction that there is a small neighborhood $\boldsymbol{\mathcal{U}}$ of $\mathbf{F}_*$ in which for all $\mathbf{F} \in \boldsymbol{\mathcal{U}}$, we have $\mathbf{F}^{P+Q}(0) \neq \mathbf{F}^Q(0)$ for all $P \in \mathbf{T}_{>0}$, $Q \in \mathbf{T}$. By λ-lemma, this implies that the postcritical set of $\mathbf{F}$ moves holomorphically over $\boldsymbol{\mathcal{U}}$. In the realm of coronas, the corresponding neighborhood $\mathcal{V} \subset \mathcal{W}^u_{\text{loc}}$ of f_* consists of rotational coronas. By Theorem 4.12, $\mathcal{V}$ must lie in the stable manifold, which is impossible.

Therefore, every neighborhood $\boldsymbol{\mathcal{U}}$ of $\mathbf{F}_*$ contains some $\mathbf{G}$ such that $\mathbf{G}^{P+Q}(0) = \mathbf{G}^Q(0)$ for some $P \in \mathbf{T}_{>0}$ and $Q \in \mathbf{T}$. If $Q = 0$, then $\mathbf{G}$ is superattracting and we are done. Hence, let us assume that $Q > 0$. In this case, $\mathbf{G}^Q(0)$ is a periodic point of period P, and by Proposition 5.26, it must be repelling in nature.

Consider any sufficiently small one-dimensional disk $\boldsymbol{\mathcal{U}}'$ about $\mathbf{G}$ embedded in $\boldsymbol{\mathcal{U}}$. By implicit function theorem, every $\mathbf{F} \in \boldsymbol{\mathcal{U}}'$ admits a repelling periodic point $x_{\mathbf{F}}$ of period P such that $x_{\mathbf{G}} = \mathbf{G}^Q(0)$ and $x_{\mathbf{F}}$ depends holomorphically in $\mathbf{F}$. By Corollaries 5.22 and 7.5, there exists a sequence of critical points $x^n_{\mathbf{F}}$ of some generation P_n depending holomorphically on $\mathbf{F} \in \boldsymbol{\mathcal{U}}'$ such that $P_n \to \infty$ and $x^n_{\mathbf{F}} \to x_{\mathbf{F}}$ as $n \to \infty$. By Rouché's theorem, for sufficiently large n, the number of zeros of $\mathbf{F}^{Q+P_n}(x^n_{\mathbf{F}}) - x^n_{\mathbf{F}}$ as a function of $\mathbf{F} \in \boldsymbol{\mathcal{U}}'$ is equal to that of $\mathbf{F}^{Q+P_n}(x^n_{\mathbf{F}}) - x_{\mathbf{F}}$, which is at least one (e.g. $\mathbf{G}$). Therefore, there exist some large $n \in \mathbb{N}$ and some $\mathbf{F} \in \boldsymbol{\mathcal{U}}'$ such that $\mathbf{F}^{Q+P_n}(x^n_{\mathbf{F}}) = x^n_{\mathbf{F}}$ and so $\mathbf{F}^{Q+P_n}(0) = 0$. □

Theorem 7.12. *The global unstable manifold $\boldsymbol{\mathcal{W}}^u$ is biholomorphic to $\mathbb{C}$.*

Proof. We claim that $\boldsymbol{\mathcal{W}}^u$ is one-dimensional. Since $\mathcal{R}$ is an automorphism of $\boldsymbol{\mathcal{W}}^u$ admitting a unique repelling fixed point $\mathbf{F}_*$, then the claim will imply that $\mathcal{R} : \boldsymbol{\mathcal{W}}^u \to \boldsymbol{\mathcal{W}}^u$ is conformally conjugate to the linear map $\mathbb{C} \to \mathbb{C}, z \mapsto \lambda z$ where λ is the repelling eigenvalue of $\mathcal{R}$.

Suppose for a contradiction that $\boldsymbol{\mathcal{W}}^u$ has dimension greater than one. By Lemma 7.11, there exists a superattracting cascade in $\boldsymbol{\mathcal{W}}^u$ of some period P. Since "$\mathbf{F}^P(0) = 0$" is an analytic condition, there exists an embedded holomorphic curve $\mathbb{D} \to \boldsymbol{\mathcal{W}}^u, \lambda \in \mathbb{D} \mapsto \mathbf{F}_\lambda$ such that each $\mathbf{F}_\lambda$ is superattracting of period P.

Let D_λ be the immediate basin of attraction of 0 for the cascade $\mathbf{F}_\lambda$. The only critical point of $\mathbf{F}_\lambda^P$ in D_λ is 0 itself, so by Riemann-Hurwitz formula, D_λ is simply connected. Let $b_\lambda : (D_\lambda, 0) \to (\mathbb{D}, 0)$ be a Böttcher conjugacy, i.e. a Riemann mapping which conjugates $\mathbf{F}_\lambda^P$ with the power map $z \mapsto z^{d_0+d_\infty-1}$. Observe that $B_\lambda := b_\lambda^{-1} \circ b_0 : (D_0, 0) \to (D_\lambda, 0)$ conjugates $\mathbf{F}_0^P$ with $\mathbf{F}_\lambda^P$. The Böttcher conjugacy is unique up to multiplication by some roots of unity. We can select them such that b_λ depends holomorphically on λ and so B_0 is the identity map on D_0.

By Corollary 7.10, the Julia set $\mathfrak{J}(\mathbf{F}_\lambda)$ moves conformally and equivariantly in λ. More precisely, there exists a holomorphic family of quasiconformal maps $\phi_\lambda : \mathbb{C} \to \mathbb{C}$ that have zero dilatation on $\mathfrak{J}(\mathbf{F}_0)$ and conjugates $\mathbf{F}_0|_{\mathfrak{J}(\mathbf{F}_0)}$ and $\mathbf{F}_\lambda|_{\mathfrak{J}(\mathbf{F}_\lambda)}$.

We shall modify ϕ_λ on the Fatou set as follows. For $r \in (0,1)$, let $E_\lambda(r) = b_\lambda^{-1}(\mathbb{D}_r)$ be a disk neighborhood of 0 cut out by an equipotential. Let $\varepsilon = \frac{1}{2}$ and $\varepsilon' = \varepsilon^{d_0+d_\infty-1}$. Define the global quasiconformal map

$$\psi_{\lambda,0}(z) := \begin{cases} \phi_\lambda(z) & \text{if } z \in \mathbb{C} \setminus \bigcup_{0 \le T < P} \mathbf{F}_0^T(E_\lambda(\varepsilon)) \\ \mathbf{F}_\lambda^T \circ B_\lambda \circ \left(\mathbf{F}_0^T|_{E_0(\varepsilon')}\right)^{-1} & \text{if } z \in \mathbf{F}_0^T(E_0(\varepsilon')) \text{ for some } T < P \\ \text{quasiconformal interpolation} & \text{if otherwise.} \end{cases}$$

On $\mathfrak{J}(\mathbf{F}_0)$ and a neighborhood of the periodic cycle $\{\mathbf{F}_0^T(0)\}_T$, $\psi_{\lambda,0}$ conjugates $\mathbf{F}_0^P$ and $\mathbf{F}_\lambda^P$. Inductively, we define for all $n \ge 1$ the quasiconformal map $\psi_{\lambda,n} : \mathbb{C} \to \mathbb{C}$ by lifting $\psi_{\lambda,n-1}$ such that $\mathbf{F}_\lambda^P \circ \psi_{\lambda,n} = \psi_{\lambda,n-1} \circ \mathbf{F}_0^P$. The map $\psi_{\lambda,n}$ has dilatation equal to that of $\psi_{\lambda,0}$ and it agrees with $\psi_{\lambda,n-1}$ on a neighborhood of $\mathfrak{J}(\mathbf{F}_0)$ and on increasingly large part of $\mathfrak{F}(\mathbf{F}_0)$. Moreover $\psi_{\lambda,n}$ is a conformal conjugacy between $\mathbf{F}_0^P$ and $\mathbf{F}_\lambda^P$ on $\mathbf{F}^{-nP+T}(E_0(\varepsilon))$ for all $0 \le T < P$.

As $n \to \infty$, $\psi_{\lambda,n}$ stabilizes and converges to a quasiconformal map ψ_λ conjugating $\mathbf{F}_0^P$ to $\mathbf{F}_\lambda^P$ everywhere. Moreover, ψ_λ is conformal on the Fatou set, and has zero dilatation almost everywhere on the Julia set. By Weyl's lemma, ψ_λ is a linear conjugacy between $\mathbf{F}_0$ and $\mathbf{F}_\lambda$.

Without loss of generality, we can reparametrize λ and assume such that $\psi_\lambda(z) = \lambda z$. Then, within the global parameter space $\boldsymbol{\mathcal{W}}^u$, we have a one-dimensional slice $\mathbf{F}_\lambda = \{\psi_\lambda \circ \mathbf{F}_0 \circ \psi_\lambda^{-1}\}_{\lambda \in \mathbb{C}^*}$. For all $n < 0$, denote the $(-n)^{\text{th}}$ anti-renormalization of $\mathbf{F}_\lambda$ by $\mathbf{F}_{\lambda,n}$. As $n \to -\infty$, we have

$$\mathbf{F}_* = \lim_{n \to -\infty} \mathbf{F}_{\lambda,n} = \lim_{n \to \infty} \psi_\lambda \circ \mathbf{F}_{0,n} \circ \psi_\lambda^{-1} = \psi_\lambda \circ \mathbf{F}_* \circ \psi_\lambda^{-1}.$$

However, the only holomorphic map which commutes with the linear map ψ_λ for all λ is a linear map, and clearly $\mathbf{F}_*^P$ is not a linear map for every $P \in \mathbf{T}_{>0}$. □

At last, we have proven that the corona renormalization fixed point f_* is hyperbolic with one-dimensional local unstable manifold. The proof of Theorem A is finally complete. Let us conclude with a proof of Corollary D.

Corollary 7.13. *Consider a small Banach neighborhood $N(f)$ of a (d_0, d_∞)-critical quasicircle map f of preperiodic type rotation number θ. The space S of maps in $N(f)$ that admit a (d_0, d_∞)-critical Herman quasicircle of rotation number θ forms an analytic submanifold of $N(f)$ of codimension at most one. The Herman quasicircles of maps in S move holomorphically.*

Proof. By Lemma 4.6, there is a compact analytic corona renormalization operator $\mathcal{R}_1$ on a neighborhood of f such that $\mathcal{R}_1 f$ is sufficiently close to the fixed point f_* of $\mathcal{R}$, and thus it lies in the stable manifold of f_*. Then, the preimage $S := \mathcal{R}_1^{-1}(\mathcal{W}^s_{loc})$ is an analytic submanifold of the Banach neighborhood of f consisting of perturbations of f which admit a (d_0, d_∞)-critical Herman quasicircle of rotation number θ. Since the codimension of $\mathcal{W}^s_{loc}$ is one, there is an analytic function $\phi : \mathcal{U} \to \mathbb{C}$ on a Banach neighborhood $\mathcal{U}$ of f_* such that $\mathcal{W}^s_{loc} = \phi^{-1}(0)$. Therefore, S is the zero set of $\phi \circ \mathcal{R}_1$ and so the codimension of S is at most one.

The Herman quasicircle of a corona in $\mathcal{W}^s_{loc}$ moves holomorphically over $\mathcal{W}^s_{loc}$ due to λ-lemma. Since $\mathcal{R}_1$ is analytic, the Herman quasicircles of maps in S also move holomorphically over S. □

Appendix A. Small Orbits Theorem

Consider a complex Banach space $\mathcal{B}$. Given a linear operator $L : \mathcal{B} \to \mathcal{B}$, denote the corresponding set of eigenvalues by $\operatorname{spec}(L)$. We say that an eigenvalue $\lambda \in \operatorname{spec}(L)$ is *attracting* if $|\lambda| < 1$, *neutral* if $|\lambda| = 1$, and *repelling* if $|\lambda| > 1$.

Theorem A.1 (Small Orbits Theorem). *Let $R : (\mathcal{U}, 0) \to (\mathcal{B}, 0)$ be a compact analytic operator on a neighborhood $\mathcal{U}$ of 0 in a complex Banach space $\mathcal{B}$. If the differential $DR_0 : \mathcal{B} \to \mathcal{B}$ has a neutral eigenvalue, then R has slow small orbits: for any neighborhood $\mathcal{V}$ of 0, there is a forward orbit $\{R^n g\}_{n \in \mathbb{N}}$ in $\mathcal{V}$ such that*

$$\lim_{n \to \infty} \frac{1}{n} \log \|R^n g\| = 0.$$

In the absence of repelling eigenvalues of DR_0, the theorem above was proven by Lyubich in [Lyu99, §2]. The original Small Orbits Theorem was a vital ingredient in the proof of hyperbolicity of quadratic-like renormalization horseshoe [Lyu99, Lyu02] and more recently the proof of hyperbolicity of pacman renormalization fixed points [DLS20]. Below we will generalize Lyubich's proof. The key addition is the application of two invariant cones, namely the center-stable cone $\mathcal{C}^{cs}$ and the center-unstable cone $\mathcal{C}^{cu}$.

Proof. Let R be as in the hypothesis. Denote the unit disk in $\mathbb{C}$ by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. We present the Banach space $\mathcal{B}$ as a direct sum

$$\mathcal{B} = E^s \oplus E^c \oplus E^u,$$

where subspaces E^s, E^c, E^u are invariant under DR_0 and

$$\operatorname{spec}(DR_0|_{E^s}) \subset \mathbb{D}, \quad \operatorname{spec}(DR_0|_{E^c}) \subset \partial\mathbb{D}, \quad \operatorname{spec}(DR_0|_{E^u}) \subset \mathbb{C}\backslash\overline{\mathbb{D}}.$$

Note that the spectrum can only accumulate at 0 because R is a compact operator. In particular, the subspace $E^c \oplus E^u$ must be finite dimensional. We will assume that each of the three subspaces have positive dimension. (Else, we are reduced to [Lyu99, §2].)

For $h \in \mathcal{B}$, we will write $h = h^s + h^c + h^u$, where for $a \in \{s, c, u\}$, h^a is the projection of h onto the subspace E^a. We will also denote $h^{cs} := h^c + h^c$ and $h^{cu} := h^c + h^u$.

There exist an adapted norm $\|\cdot\|$ on $\mathcal{B}$ and some positive constants $\mu_s, \mu_{cs}, \mu_{cu}, \mu_u$ such that $\mu_s < 1 < \mu_u$, $\mu_s < \mu_{cu}$, $\mu_{cs} < \mu_u$, and

$$\begin{aligned} &\|DR_0 h\| \le \mu_s \|h\| && \text{for all } h \in E^s, \\ &\|DR_0 h\| \le \mu_{cs} \|h\| && \text{for all } h \in E^{cs}, \\ &\|DR_0 h\| \ge \mu_{cu} \|h\| && \text{for all } h \in E^{cu}, \\ &\|DR_0 h\| \ge \mu_u \|h\| && \text{for all } h \in E^u. \end{aligned}$$

The proof below will involve two fixed constants $\alpha > 1$ and $\delta > 0$ where $\alpha - 1$ and δ are very small. We consider a pair of cone fields C^{cu} and C^{cs} given by

$$C_f^{cu} = \{h \in T_f\mathcal{U} \,:\, \alpha\|h^s\| \le \|h^{cu}\|\} \quad \text{and} \quad C_f^{cs} = \{h \in T_f\mathcal{U} \,:\, \alpha\|h^u\| \le \|h^{cs}\|\} \tag{A.1}$$

for every $f \in \mathcal{U}$. For $a \in \{s, c, u\}$, we denote by $D^a = D^a(\delta)$ the open ball of radius δ centered at 0 in E^a. Let

$$\mathcal{D} := D^s \times D^c \times D^u$$

the corresponding open polydisk centered at 0 in $\mathcal{B}$. The boundary of $\mathcal{D}$ can be decomposed as follows:

$$\partial^s\mathcal{D} := \partial D^s \times D^c \times D^u, \qquad \partial^c\mathcal{D} := D^s \times \partial D^c \times D^u, \qquad \partial^u\mathcal{D} := D^s \times D^c \times \partial D^u.$$

Claim 1. Suppose $\alpha < \min\left\{\frac{\mu_{cu}}{\mu_s}, \frac{\mu_u}{\mu_{cs}}\right\}$. For sufficiently small $\delta > 0$, the following properties hold.

(1) If $f \in \overline{\mathcal{D}}$, then $Rf \notin \partial^s\mathcal{D}$;
(2) If $f \in \partial^u\mathcal{D}$, then $Rf \notin \overline{\mathcal{D}}$;
(3) The cone field C^{cu} is forward invariant: if $f, Rf \in \mathcal{D}$, then

$$DR_f(C_f^{cu}) \subset C_{Rf}^{cs};$$

(4) The cone field C^{cs} is backward invariant: if $f, Rf \in \mathcal{D}$, then

$$(DR_f)^{-1}(C_{Rf}^{cs}) \subset C_f^{cs}.$$

Proof. Fix a small constant $\varepsilon > 0$. We can assume that δ is sufficiently small depending on ε such that the difference

$$Gf := Rf - DR_0 f$$

has C^1 norm on $\overline{\mathcal{D}}$ bounded by ε, that is, for all $f \in \overline{\mathcal{D}}$ and $h \in T_f\mathcal{U}$,

$$\|Gf\| \le \varepsilon\|f\|, \quad \text{ and } \|DG_f h\| \le \varepsilon\|h\|.$$

When f lies in $\overline{\mathcal{D}}$,

$$\|(Rf)^s\| \le \|DR_0|_{E^s}(f^s)\| + \|(Gf)^s\| \le \mu_s\|f^s\| + \varepsilon\|f\|.$$

Assuming $\mu_s + 3\varepsilon < 1$, we then have $\|(Rf)^s\| < \delta$. Additionally, when $\|f^u\| = \delta$,

$$\|(Rf)^u\| \ge \|DR_0|_{E^u}(f^u)\| - \|(Gf)^u\| \ge \mu_u\delta - \varepsilon\|f\|.$$

Assuming $\mu_u - 3\varepsilon > 1$, we then have $\|(Rf)^u\| > \delta$. Hence, (1) and (2) hold.

Suppose both f and Rf are in $\mathcal{D}$. For every $h \in C_f^{cu}$, we have

$$\begin{aligned} \|(DR_f h)^{cu}\| &= \|DR_0|_{E^c \oplus E^u}(h^{cu}) + (DG_f(h))^{cu}\| \\ &\ge \mu_{cu}\|h^{cu}\| - \varepsilon\|h\| \\ &\ge \left(\mu_{cu} - \varepsilon\left(1 + \frac{1}{\alpha}\right)\right)\|h^{cu}\|, \end{aligned}$$

and

$$\begin{aligned}\alpha\|(DR_f h)^s\| &= \alpha\|DR_0|_{E^s}(h^s) + (DF_f(h))^s\| \\ &\leq \alpha\left(\mu_s\|h^s\| + \varepsilon\|h\|\right) \\ &\leq \left(\alpha\mu_s + (\alpha+1)\varepsilon\right)\|h^{cu}\|.\end{aligned}$$

Since $\mu_{cu} - \alpha\mu_s > 0$, we can take ε to be small enough such that $\alpha\|(DR_f h)^s\| \leq \|(DR_f h)^{cu}\|$ and thus $DR_f h \in C^{cu}_{Rf}$. The proof that the cone field C^{cs} is backward invariant works in a similar way, assuming ε is sufficiently small depending on $\mu_u - \alpha\mu_{cs}$. □

Let us consider the perturbation $R_\lambda := \lambda \cdot R$ for $0 < \lambda < 1$. When λ is sufficiently close to 1, R_λ still satisfies all the properties listed in Claim 1. The following claim is a consequence of Lemma A.2, which we will elaborate later separately.

Claim 2. There exists some point $f_\lambda \in \partial^c\mathcal{D}$ such that the orbit $\{R^n_\lambda f_\lambda\}_{n\in\mathbb{N}}$ lies entirely inside of $\overline{\mathcal{D}}$ and $R^n_\lambda f_\lambda \to 0$.

Since R is compact, there exist an increasing sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of positive numbers and some $g \in \overline{\mathcal{D}}$ such that as $n \to \infty$, $\lambda_n \to 1$ and $R_{\lambda_n} f_{\lambda_n} \to g$. Clearly, for all $n \in \mathbb{N}$, the n^{th} iterate $g_n := R^n g$ lies in $\overline{\mathcal{D}}$.

As f_λ is in $\partial^c\mathcal{D}$, f_λ is also inside of the cone $\hat{C}^{cu}_0 = \{\|h^s\| \leq \|h^{cu}\|\}$. Similar to the proof of Claim 1 (3), $\hat{C}^{cu}_0$ is forward invariant under R_λ for $\lambda \leq 1$. Hence, for every $n \in \mathbb{N}$, $\|g^s_n\| \leq \|g^{cu}_n\|$. This implies that for every $n \in \mathbb{N}$,

$$g^{cu}_{n+1} = DR_0|_{E^c\oplus E^u}(g^{cu}_n) + O\left(\|g^{cu}_n\|^2\right). \tag{A.2}$$

At last, we will show that the orbit of g is a slow small orbit. Indeed, suppose for a contradiction that

$$\liminf_{n\to\infty} \frac{1}{n}\log\|g_n\| < -c_0 \tag{A.3}$$

for some constant $c_0 > 0$. Note that this property holds for every norm that is equivalent to $\|\cdot\|$. Pick some $c_1 \in [0, c_0)$. There exists an adapted norm $\|\cdot\|$ equivalent to the original one such that the operator norm of $DR_0|^{-1}_{E^c\oplus E^u}$ is at most e^{c_1}. By (A.2), for sufficiently small $\delta > 0$, there is some $c_2 \in (0, c_0)$ such that

$$\|g^{cu}_{n+1}\| \geq e^{-c_2}\|g^{cu}_n\| \qquad \text{for all } n \in \mathbb{N}.$$

This contradicts (A.3). □

It remains to prove Claim 2, which will follow directly from the lemma below. Again, we suppose $\mathcal{B}$ can be decomposed into $E^s \oplus E^c \oplus E^u$ and consider the cone fields C^{cu} and C^{cs} defined in (A.1). We consider a small neighborhood $\mathcal{U} \subset \mathcal{B}$ of some polydisk $\mathcal{D}$ centered at 0. For any $r > 0$, we denote the open disk $\{z \in \mathbb{C} : |z| < r\}$ by $\mathbb{D}_r$.

Lemma A.2. *Let $R : (\mathcal{U}, 0) \to (\mathcal{B}, 0)$ be a compact analytic operator such that the differential DR_0 preserves the decomposition $\mathcal{B} = E^s \oplus E^c \oplus E^u$ and satisfies the following properties.*

(1) *Hyperbolicity: There exists some $0 < r < 1$ such that*

$$\operatorname{spec}(DR_0|_{E^s}) \subset \mathbb{D}_r, \quad \operatorname{spec}(DR_0|_{E^c}) \subset \mathbb{D}\backslash\mathbb{D}_r, \quad \operatorname{spec}(DR_0|_{E^u}) \subset \mathbb{C}\backslash\overline{\mathbb{D}}.$$

(2) *Boundary behaviour: If $f \in \overline{\mathcal{D}}$, then $Rf \notin \partial^s\mathcal{D}$. If $f \in \partial^u\mathcal{D}$, then $Rf \notin \overline{\mathcal{D}}$.*

(3) *Invariant cone fields: Whenever* $f, Rf \in \overline{\mathcal{D}}$,

$$DR_f(C_f^{cu}) \subset C_{Rf}^{cu}, \qquad (DR_f)^{-1}(C_{Rf}^{cs}) \subset C_f^{cs}.$$

Then, there exists some $f \in \partial^c \mathcal{D}$ *such that* $\{R^n f\}_{n\in\mathbb{N}} \subset \overline{\mathcal{D}}$ *and* $\|R^n f\| \to 0$ *as* $n \to \infty$.

Proof. By the compactness of R, the subspace $E^c \oplus E^u$ is finite dimensional. Let $d_c := \dim(E^c)$ and $d_u := \dim(E^u)$. By (1), the stable manifold

$$\mathcal{A} = \{f \in \overline{\mathcal{D}} : \{R^n f\}_{n\in\mathbb{N}} \subset \overline{\mathcal{D}} \text{ and } \|R^n f\| \to 0\}$$

exists and is a forward invariant analytic submanifold of codimension d_u.

Let us assume for a contradiction that $\mathcal{A}$ is disjoint from $\partial^c\mathcal{D}$.

Claim 1. The set $\mathcal{A}^o := \mathcal{A} \cap \mathcal{D}$ is a forward invariant open submanifold of $\mathcal{A}$.

Proof. The only non-trivial property to prove here is forward invariance. Suppose $f \in \mathcal{A}^o$. As $f \in \mathcal{A}$, then $R^n f \in \overline{\mathcal{D}}$ for all $n \geq 1$. By (2), Rf cannot lie in $\partial^s\mathcal{D} \cup \partial^u\mathcal{D}$. By the assumption, Rf cannot lie in $\partial^c\mathcal{D}$ either. Thus, $Rf \in \mathcal{D}$. □

Claim 2. The set $\partial^c\mathcal{A} := \overline{\mathcal{A}} \backslash (\mathcal{A}^o \cup \partial^s\mathcal{D})$ is also forward invariant.

Proof. Suppose for a contradiction that there is some $f \in \partial^c\mathcal{A}$ such that $Rf \in \mathcal{A}^o \cup \partial^s\mathcal{D}$. By (2), Rf must lie in $\mathcal{A}^o$, which implies that $f \in \mathcal{A} \cap (\partial^c\mathcal{D} \cup \partial^u\mathcal{D})$. However, this is impossible because f does not lie in $\partial^c\mathcal{D}$ by our main assumption, nor in $\partial^u\mathcal{D}$ due to (2). □

Claim 3. The tangent space $T_f A^o$ at every point f in $\mathcal{A}^o$ is contained in C_f^{cs}.

Proof. Let $f \in \mathcal{A}^o$. As A^o is tangent to the subspace $E^s \cup E^c$ at 0, for all sufficiently high n, $R^n f$ is sufficiently close to 0 and so the tangent space $T_{R^n f}\mathcal{A}^o$ lies within $C_{R^n f}^{cs}$. By backward invariance of C^{cs} in (3), the tangent space of $\mathcal{A}^o$ at f also lies within C_f^{cs}. □

Let us consider the family $\mathcal{G}$ of all immersed analytic d_c-dimensional submanifolds Γ of $\mathcal{A}^o$ containing 0 with the following properties.

(a) The tangent space $T_f\Gamma$ at every point $f \in \Gamma$ lies in the cone C_f^{cu};

(b) The accumulation set $\overline{\Gamma}\backslash\Gamma$ lies in $\partial^c\mathcal{A}$.

Note that $\mathcal{G}$ is non-empty: it contains $\mathcal{A}^o \cap (E^c \oplus E^u)$ because, by Claim 3, the intersection between $\mathcal{A}^o$ and the subspace $E^c \oplus E^u$ is transversal. Another consequence of Claim 3 is the following claim.

Claim 4. For every $\Gamma \in \mathcal{G}$ and $h \in T_f\Gamma$, $\|h^c\| \asymp \|h\|$. In particular, the projection $P : \Gamma \to D^c$ is non-singular.

Proof. Let $h \in T_f\Gamma$. By Property (a) and Claim 3, $\alpha\|h^s\| \leq \|h^{cu}\|$ and $\alpha\|h^u\| \leq \|h^{cs}\|$. By triangle inequality, these imply that $(\alpha - 1)\max\{\|h^s\|, \|h^u\|\} \leq \|h^c\|$ and consequently $\|h^c\| \leq \|h\| \leq \frac{\alpha+1}{\alpha-1}\|h^c\|$. □

Recall that the Kobayashi norm of a tangent vector $v \in T_f\Gamma$ at a point f on a complex manifold Γ is defined as

$$\|h\|_\Gamma := \inf\{\|w\|_{\mathbb{D}} : D\phi_f(w) = h \text{ for some holomorphic map } \phi : (\mathbb{D}, 0) \to (\Gamma, f)\}$$

where $\|w\|_{\mathbb{D}}$ denotes the Poincaré metric of $w \in T_0\mathbb{D}$ on the unit disk $\mathbb{D}$. We will supply every $\Gamma \in \mathcal{G}$ with the Kobayashi metric.

Claim 5. There is some $K > 0$ such that for every $\Gamma \in \mathcal{G}$ and $h \in T_0\Gamma$, $\|h\|_\Gamma \leq K\|h\|$.

Proof. By Claim 4, there is some $\delta > 0$ such that for every $\Gamma \in \mathcal{G}$, the component $\Gamma(\delta)$ of $\Gamma \cap D^c(\delta)$ containing 0 is a graph of an analytic map $D^c(\delta) \to D^s \times D^u$. Therefore, for any $h \in T_0\Gamma$,

$$\|h\|_\Gamma \leq \|h\|_{\Gamma(\delta)} = \|h^c\|_{D^c(\delta)}.$$

Clearly, $\|h^c\|_{D^c(\delta)} \asymp \|h^c\|$ (with bounds depending only on δ). By Claim 4, this yields the desired inequality $\|h\|_\Gamma \leq K\|h\|$ for some K independent of Γ. ☐

By Property (3) and Claim 2, the map R induces a well-defined graph transform

$$R_* : \mathcal{G} \to \mathcal{G}, \quad \Gamma \mapsto R\Gamma.$$

Note that $R : \Gamma \to R\Gamma$ is a proper non-singular map, hence a holomorphic covering map. Therefore, for every $\Gamma \in \mathcal{G}$, $n \in \mathbb{N}$, and non-zero tangent vector $h \in T_0\Gamma$,

$$\|h\|_\Gamma = \|(DR^n)_0(h)\|_{R_*^n\Gamma}.$$

By Claim 5,

$$\|h\|_\Gamma \leq K\|(DR^n)_0(h)\|.$$

However, by (1), $\|(DR^n)_0(h)\|$ tends to 0 as $n \to \infty$. This yields a contradiction. ☐

Appendix B. Sector renormalization

B.1. **Renormalization of rotations and translations.** Let us equip the unit circle $\mathbb{T} \subset \mathbb{C}$ with the induced intrinsic metric. Given two points x and y on S^1, we denote by $[x, y] \subset S^1$ the shortest closed interval with endpoints x and y. Consider the rotation

$$\mathbb{L}_\theta : \mathbb{T} \to \mathbb{T}, z \mapsto e^{2\pi i\theta} z$$

for some fixed $\theta \in \mathbb{R}/\mathbb{Z}$. Let us fix a point $x \in \mathbb{T}$ and consider

$$X_- := [\mathbb{L}_\theta^{-1}(x), x], \quad Y := [x, \mathbb{L}_\theta(x)], \quad X_+ := \overline{\mathbb{T}\backslash(Y \cup X_-)}.$$

The first return map on $X_- \cup X_+$ is precisely the commuting pair

$$(\mathbb{L}_\theta|_{X_+},\ \mathbb{L}_\theta^2|_{X_-}),$$

Let us assume that $1 \neq Y$ and denote by ω the length of $X_- \cup X_+$. Then, the map $z \mapsto z^{1/\omega}$ projects the commuting pair to a new rotation $\mathbb{L}_{R_{\text{prm}}(\theta)}$ called the *prime renormalization* of $\mathbb{L}_\theta$. Note that $\mathbb{L}_{R_{\text{prm}}(\theta)}$ is independent of the initial choice of x.

Lemma B.1 ([DLS20, Lemma A.1])**.** *We have*

$$R_{prm}(\theta) = \begin{cases} \dfrac{\theta}{1-\theta}, & \textit{if } 0 \leq \theta \leq \dfrac{1}{2}, \\ \dfrac{2\theta - 1}{\theta}, & \textit{if } \dfrac{1}{2} \leq \theta \leq 1. \end{cases}$$

In general, we define a *sector renormalization* $\mathcal{R}(\mathbb{L}_\theta)$ of $\mathbb{L}_\theta$ as follows. First, consider a pair of intervals X_- and X_+ on $\mathbb{T}$ satisfying $X_- \cap X_+ = \{1\}$. Suppose the first return map on $X := X_- \cup X_+$, which we call a sector pre-renormalization, is a pair of the form

$$(\mathbb{L}_\theta^{\mathbf{a}}|_{X_-},\ \mathbb{L}_\theta^{\mathbf{b}}|_{X_+}), \tag{B.1}$$

for some positive integers $\mathbf{a}$ and $\mathbf{b}$ called the *renormalization return times* of $\mathcal{R}$. The map $z \mapsto z^{1/\omega}$, where ω is the length of X, glues the endpoints of X together and projects the pair (B.1) to a new rotation $\mathbb{L}_\mu = \mathcal{R}(\mathbb{L}_\theta)$.

Lemma B.2 ([DLS20, Lemma A.2])**.** *Sector renormalization $\mathcal{R}$ is an iteration of the prime renormalization. In particular, $\mu = R^m_{prm}(\theta)$ for some $m \geq 1$, and $\mathbb{L}_\theta$ is a fixed point of some sector renormalization if and only if $\theta \in \Theta_{per}$.*

Under the universal cover $\mathbb{R} \to \mathbb{T}, z \mapsto e^{-2\pi i z}$, the rotation $\mathbb{L}_\theta$ can be lifted to the commuting pair of translations

$$T_{-\theta} : z \mapsto z - \theta, \qquad T_{1-\theta} : z \mapsto z + 1 - \theta.$$

The deck transformation $\chi := T_1$ is equal to $T_{1-\theta} \circ T^{-1}_{-\theta}$, and the original rotation $\mathbb{L}_\theta$ can be recovered from the quotient map $T_{-\theta}/\langle\chi\rangle$.

Consider a general commuting pair of translations $(T_{-\mathbf{u}}, T_{\mathbf{v}})$ where $\mathbf{u}, \mathbf{v} \in \mathbb{R}_{\geq 0}$. The prime renormalization $\mathcal{R}_{\text{prm}}$ of $(T_{-\mathbf{u}}, T_{\mathbf{v}})$ is the new commuting pair $(T_{-\mathbf{u}_1}, T_{\mathbf{v}_1})$ where

$$(T_{-\mathbf{u}_1}, T_{\mathbf{v}_1}) := \begin{cases} (T_{-\mathbf{u}} \circ T_{\mathbf{v}}, \quad T_{\mathbf{v}} \quad) & \text{if } \mathbf{u} \geq \mathbf{v}, \\ (\quad T_{-\mathbf{u}} \quad , T_{-\mathbf{u}} \circ T_{\mathbf{v}}) & \text{if } \mathbf{u} < \mathbf{v}. \end{cases} \tag{B.2}$$

Set $\chi := T_{\mathbf{v}} \circ T^{-1}_{-\mathbf{u}}$ and $\chi_1 = T_{\mathbf{v}_1} \circ T^{-1}_{-\mathbf{u}_1}$. The prime renormalization of pairs of translations is equivalent to that of rotations in the following sense.

Lemma B.3. *If $T_{-\mathbf{u}}/\langle\chi\rangle \equiv \mathbb{L}_\theta$, then*

$$\theta = \frac{\mathbf{v}}{\mathbf{u}+\mathbf{v}} \quad \textit{and} \quad T_{-\mathbf{u}_1}/\langle\chi_1\rangle \equiv \mathbb{L}_{R_{prm}(\theta)}.$$

B.2. **Cascade of translations.** By writing $(-\mathbf{u}, \mathbf{v})$ as a column vector, the transformation in (B.2) is represented by either $I^- := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ if $\mathbf{u} \geq \mathbf{v}$ or $I^+ := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ if $\mathbf{u} < \mathbf{v}$. Consider the region $\mathbb{R}_{\leq 0} \times \mathbb{R}_{\geq 0}$, which is split equally into two sectors by the diagonal line $\{x + y = 0\}$. The lower sector is mapped by I^- onto $\mathbb{R}_{\leq 0} \times \mathbb{R}_{\geq 0}$, whereas the upper sector is mapped by I^+ onto $\mathbb{R}_{\leq 0} \times \mathbb{R}_{\geq 0}$.

From now on, suppose θ is of periodic type. There exists some $m > 0$ such that $R^m_{\text{prm}}(\theta) = \theta$. Set $\mathbf{u} = \theta$ and $\mathbf{v} = 1 - \theta$. By (B.2), there is a unique matrix 2×2 matrix $\mathbf{M}$ of the form $I_1 I_2 \dots I_m$, where $I_i \in \{I^+, I^-\}$ for all i, such that the m^{th} prime renormalization $(T_{-\mathbf{u}_1}, T_{\mathbf{v}_1}) := \mathcal{R}^m_{\text{prm}}(T_{-\mathbf{u}}, T_{\mathbf{v}})$ satisfies

$$\begin{pmatrix} -\mathbf{u}_1 \\ \mathbf{v}_1 \end{pmatrix} = \mathbf{M} \begin{pmatrix} -\mathbf{u} \\ \mathbf{v} \end{pmatrix}.$$

The matrix $\mathbf{M}$ is an element of the modular group $\text{SL}_2(\mathbb{Z})$ mapping a sector in $\mathbb{R}_{\leq 0} \times \mathbb{R}_{\geq 0}$ onto $\mathbb{R}_{\leq 0} \times \mathbb{R}_{\geq 0}$. The condition $R^m_{\text{prm}}(\theta) = \theta$ implies that $\begin{pmatrix} -\mathbf{u}_1 \\ \mathbf{v}_1 \end{pmatrix}$ is a scalar multiple of $\begin{pmatrix} -\mathbf{u} \\ \mathbf{v} \end{pmatrix}$. We conclude that $\mathbf{M}$ has two eigenvalues $\mathbf{t} > 1$ and $1/\mathbf{t}$, and that

$$\begin{pmatrix} -\mathbf{u}_1 \\ \mathbf{v}_1 \end{pmatrix} = \frac{1}{\mathbf{t}} \begin{pmatrix} -\mathbf{u} \\ \mathbf{v} \end{pmatrix}.$$

We call $\mathbf{M}$ the *anti-renormalization matrix* associated with θ.

Observe that $\mathbf{M}$ has to be a matrix of positive integers and $\mathbf{t} \notin \mathbb{Q}$. For brevity, set $R := R^m_{\text{prm}}$ and $\mathcal{R} = \mathcal{R}^m_{\text{prm}}$. For $n \in \mathbb{N}$, we write

$$\mathbf{u}_n := \mathbf{t}^{-n}\mathbf{u} \quad \text{and} \quad \mathbf{v}_n := \mathbf{t}^{-n}\mathbf{v}.$$

We then obtain a full pre-renormalization tower $\{(T_{-\mathbf{u}_n}, T_{\mathbf{v}_n})\}_{n\in\mathbb{Z}}$ where

$$\mathcal{R}(T_{-\mathbf{u}_n}, T_{\mathbf{v}_n}) = (T_{-\mathbf{u}_{n+1}}, T_{\mathbf{v}_{n+1}}).$$

Given $(n,a,b) \in \mathbb{Z}\times\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}$, let us write

$$T^{(n,a,b)} := T^a_{-\mathbf{u}_n} \circ T^b_{\mathbf{v}_n} = T_{\mathbf{t}^{-n}(b\mathbf{v}-a\mathbf{u})}.$$

Lemma B.4. *Given a pair of elements (n,a,b) and (n',a',b') of $\mathbb{Z}\times\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}$,*

$$T^{(n,a,b)} = T^{(n',a',b')} \quad \text{if and only if} \quad (a\ b)\mathbf{M}^n = (c\ d)\mathbf{M}^{n'}.$$

Definition B.5. We define the space $\mathbf{T}$ of *power-triples* to be the quotient of the semigroup $\mathbb{Z}\times\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}$ under the equivalence relation $\sim$ where $(n,a,b)\sim(n-1,a',b')$ if and only if $(a'\ b') = (a\ b)\mathbf{M}$.

We will equip $\mathbf{T}$ with the binary operation $+$ defined by

$$(n,a,b)+(n,a',b') = (n,a+a',b+b').$$

With respect to $+$, $\mathbf{T}$ has a unique identity element $0 := (n,0,0)$. Thus, $(\mathbf{T},+)$ still has the structure of a semigroup. According to Lemma B.4, $\mathbf{T}$ acts freely on $\mathbb{R}$ as a cascade of translations $\left(T^P\right)_{P\in\mathbf{T}}$.

Lemma B.6 ([DL23, Lemma 2.2])**.** *There is an embedding $\iota : \mathbf{T}\to\mathbb{R}$ such that $\iota(n-1,a,b) = \mathbf{t}^{-1}\iota(n,a,b)$. Identifying $\mathbf{T}$ with $\iota(\mathbf{T})\subset\mathbb{R}$ equips $\mathbf{T}$ with*

1. *a linear order $\geq$, which can be described as follows: $P\geq Q$ if and only if for sufficiently large $n\ll 0$, we can write $P=(n,a,b)$ and $Q=(n,a',b')$ where $a\geq a'$ and $b\geq b'$;*
2. *subtraction, that is, if $P,T\in\mathbf{T}$ and $P\geq T$, then $P-T\in\mathbf{T}$;*
3. *scalar multiplication by $\mathbf{t}$: $P=(n,a,b)\mapsto \mathbf{t}P = (n+1,a,b)$, which is an automorphism of $\mathbf{T}$.*

Moreover, for $P\in\mathbf{T}$, $n\in\mathbb{Z}$, and $x\in\mathbb{R}$,

$$T^P(x) = \mathbf{t}^n\cdot T^{\mathbf{t}^n P}\left(\mathbf{t}^{-n}x\right).$$

If T^P is a translation by $l>0$, then $T^{\mathbf{t}^n P}$ is a translation by $\mathbf{t}^{-n}l$. The following observation is immediate.

Lemma B.7 (Proper discontinuity)**.** *If $P\in\mathbf{T}_{>0}$ is small, then $|T^P(0)|$ is large.*

For all $P\in\mathbf{T}$, let us denote $b_P := T^{-P}(0)$. We say that b_P is *dominant* if every b_Q on $[0,b_P]$ satisfies $Q\geq P$. By proper discontinuity, we can enumerate all dominant points $\{b_{P_n}\}_{n\in\mathbb{Z}}$ such that $P_n<P_{n+1}$ for all n.

Lemma B.8 ([DL23, Lemma 2.4])**.** *For every $i\in\mathbb{Z}$, there exist some $Q_i\in\mathbf{T}_{>0}$ and some integers m,n such that $n<m\leq i$ and T^{Q_i} maps $[b_{P_i}, b_{P_{i+1}}]$ to $[b_{P_n}, b_{P_m}]$.*

Appendix C. Key lemma for transcendental extension

It remains for us to provide the proof of Lemma 5.7.

Fix $n\in\mathbb{N}$ and a large constant $s\in\mathbb{N}$. We will denote by $\mathbf{a}_n$ and $\mathbf{b}_n$ the n^{th} renormalization return times. Fix a small neighborhood D of the critical value $c_1(f_*)$ of f_*. Consider a corona f that is $m := n+s$ times renormalizable such that $f_i := \mathcal{R}^i f$ is close to the renormalization fixed point f_* for all $i\in\{1,\dots,m\}$. We will denote the critical orbit by $c_j(f) := f^j(c_0(f))$. Our goal is to show that for $t\in\{\mathbf{a}_n,\mathbf{b}_n\}$, $c_{1+t}(f)$ is contained in D and there is a branched covering map

$f^t : (D_0, c_1(f)) \to (D, c_{1+t}(f))$. The proof we present below is similar to the Key Lemma in [DLS20], which is to ensure that pullbacks of D must avoid the forbidden boundary.

Let h be a level m combinatorial pseudo-conjugacy between f and f_*, and consider the renormalization tiling $\mathbf{\Delta}_m(f) := h^{-1}(\mathbf{\Delta}_m(f_*))$ defined in §4.3. Recall that f maps $\Delta_m(f,i)$ univalently onto $\Delta_m(f, i+\mathbf{p}_n)$ whenever $i \notin \{-\mathbf{p}_m, -\mathbf{p}_m+1\}$, and on $\Delta_m(f,-\mathbf{p}_m) \cup \Delta_m(f,-\mathbf{p}_m+1)$, f is almost a degree d covering map branched at its critical point $c_0(f)$ onto its image, which contains $\Delta_m(f,0) \cup \Delta_m(f,1)$. By Theorem 4.10, h is close to the identity map and $\mathbf{\Delta}_m(f)$ approximates the Herman quasicircle $\mathbf{H}_*$ of f_*.

In the dynamical plane of f_*, for sufficiently large $n \gg 0$, both $c_{1+\mathbf{a}_n}(f_*)$ and $c_{1+\mathbf{b}_n}(f_*)$ are contained in D because it is sufficiently close to $c_1(f_*)$. Let us fix $t \in \{\mathbf{a}_n, \mathbf{b}_n\}$. Since s is picked to be large,

$$t \le \max\{\mathbf{a}_n, \mathbf{b}_n\} < \min\{\mathbf{a}_m, \mathbf{b}_m\} - 1.$$

Therefore, the orbit $\{c_j(f_*)\}_{j=1,2,\dots t+1}$ avoids both $\Delta_m(-\mathbf{p}_m, f_*)$ and $\Delta_m(-\mathbf{p}_m+1, f_*)$. Since h is close to the identity, it follows that $c_{1+t}(f)$ is also contained in D.

Let

$$D_1,\ D_2,\ \dots,\ D_{t+1} := D$$

denote the lift of D along the orbit $c_1(f), c_2(f), \dots, c_{1+t}(f)$. We would like to show that for $i \in \{1,2,\dots,t\}$, the disk D_i does not intersect $\partial_{\mathrm{F}} U_f$ so that $f : D_i \to D_{i+1}$ is a branched covering.

C.1. **A new tiling $\mathbf{\Lambda}_m$.** We say that a subset I of $\mathbb{Z}/\mathbf{q}_m\mathbb{Z}$ is an *interval* if it is a sequence of consecutive elements of $\mathbb{Z}/\mathbf{q}_m\mathbb{Z}$ of cardinality less than $\mathbf{p}_m$. For any interval I in $\mathbb{Z}/\mathbf{q}_m\mathbb{Z}$, we will use the notation

$$\Delta_m(I) := \bigcup_{i\in I} \Delta_m(i)$$

and

$$f^{-1}I := \begin{cases} I - \mathbf{p}_m & \text{if } I \cap \{\mathbf{p}_m, \mathbf{p}_m+1, 0, 1\} = \varnothing, \\ (I-\mathbf{p}_m) \cup \{-\mathbf{p}_m, -\mathbf{p}_m+1\} & \text{if } I \cap \{0,1\} \neq \varnothing, \\ (I-\mathbf{p}_m) \cup \{0,1\} & \text{if } I \cap \{\mathbf{p}_m, \mathbf{p}_m+1\} \neq \varnothing. \end{cases}$$

This immediately gives us the following property.

Claim 1. For any interval I in $\mathbb{Z}/\mathbf{q}_m\mathbb{Z}$, the lift of $\Delta_m(I)$ under $f|_{\mathbf{\Delta}_m}$ is contained in $\Delta_m(f^{-1}I)$.

First, consider the dynamical plane of $f_m := \mathcal{R}^m f : U_m \to V$. For $i \in \{0,1\}$, let $\Lambda_0(i, f_m)$ denote the closure of the connected component of $f_m^{-1}(U_m) \backslash (\gamma_0(f_m) \cup \gamma_1)$ contained in $\Delta_m(i)$. By spreading around via iterates of f_m, we obtain a tiling $\mathbf{\Lambda}_0(f_m)$, which is a skinnier version of $\Delta_0(f_m)$. Let us embed it via Φ_m to the dynamical plane of f and spread it around via iterates of f to obtain the tiling $\mathbf{\Lambda}_m = \mathbf{\Lambda}_m(f)$.

Similar to Claim 1, we have:

Claim 2. For any interval I in $\mathbb{Z}/\mathbf{q}_m\mathbb{Z}$, we have

$$\Lambda_m(I) = \mathbf{\Lambda}_m \cap \Delta_m(I)$$

and the lift of $\Lambda_m(I)$ under $f|_{\mathbf{\Lambda}_m}$ is contained in $\Lambda_m(f^{-1}I)$.

The problem with the tiling $\mathbf{\Delta}_m$ is that for $j \in \{1, \dots, t\}$, even when $D_{j+1} \cap \mathbf{\Delta}_m$ is contained in $\Delta_m(I)$ for some interval I, it is possible that $D_j \cap \mathbf{\Delta}_m$ is not contained in $\Delta_m(f^{-1}I)$. However, this issue does not occur for the tiling $\mathbf{\Lambda}_m$.

Claim 3. For any interval I in $\mathbb{Z}/\mathbf{q}_m\mathbb{Z}$, any $j \in \{0, 1, \dots, \min\{\mathbf{a}_m, \mathbf{b}_m\} - 1\}$, and any subset $T \subset V$,

$$T \cap \mathbf{\Delta}_m \subset \Delta_m(I) \implies f^{-j}(T) \cap \mathbf{\Lambda}_m \subset \Lambda_m(f^{-j}I).$$

Proof. By construction, the tiling $\mathbf{\Lambda}$ has the property that $f^j(\mathbf{\Lambda}_m) \subset \mathbf{\Delta}_m$ for all $j < \min\{\mathbf{a}_m, \mathbf{b}_m\}$. Let I, j, and T be as in the hypothesis and suppose $T \cap \mathbf{\Delta}_m$ is contained in $\Delta_m(I)$. Consider a point z in $\mathbf{\Lambda}_m$ such that $f^j(z)$ is contained in T. Clearly, $f^j(z)$ is in $\Delta_m(I)$, and by Claim 1, z is contained in $\mathbf{\Delta}_m(f^{-j}I)$. By Claim 2, the point z is indeed contained in $\Lambda_m(f^{-j}I)$. □

Consider the smallest interval I_{t+1} in $\mathbb{Z}/\mathbf{q}_m\mathbb{Z}$ such that

$$\{0,1\} \subset I_{t+1} \quad \text{and} \quad D_{t+1} \cap \mathbf{\Delta}_m(f) \subset \Delta_m(I_{t+1}).$$

For $j \in \{1, \dots t\}$, let $I_j := f^{-(t+1-j)} I_t$. It is assumed that $D \cap \mathbf{H}_*$ is roughly a level less than n combinatorial interval, so, since $m \succ n$, $|I_j|$ is large for all j.

Let us fix some integer $\eta > 1$. This will be taken to be the maximum of the periods $\eta_k^\bullet$ introduced in the next subsection.

Claim 4. For $j \in \{1, 2, \dots t+1\}$,

(1) $|I_j|/\mathbf{q}_m$ is small and $\Delta_m(I_j, f_*) \cap \mathbf{H}_*$ has a small combinatorial length;
(2) if $j \leq t - 2 - \eta$, the intervals I_j, I_{j+1}, …, $I_{j+\eta+3}$ are pairwise disjoint;
(3) if $1 \leq j \leq \eta + 2$, then I_j is disjoint from $\{-\mathbf{p}_m, -\mathbf{p}_m + 1\}$.

Proof. Since the rotation number is of bounded type, the combinatorial length of the intersection of $\mathbf{H}_*$ with every tile of $\mathbf{\Delta}_m(f_*)$ is comparable. Since D is assumed to be small and s is taken to be sufficiently large, (1) follows.

Since (2) is combinatorial in nature, it suffices to prove (2) in the dynamical plane of f_*, which is obvious from the irrational rotational action of f_* on $\mathbf{H}_*$. If (3) does not hold, then for some integer $j \in [2, \eta+3]$, the interval I_j intersects $\{0, 1\}$, but this contradicts (2) and the fact that I_1 must intersect $\{0, 1\}$. □

C.2. **Spines and pseudo-spines.** Let us first consider the dynamical plane of f_*. Recall that the preimage of $f_*^{-1}(\gamma_1) \backslash \gamma_0$ consists of arcs

$$\gamma_1^0, \dots, \gamma_{2(d_0-1)}^0 \subset \partial^0 U_*, \quad \gamma_1^\infty, \dots, \gamma_{2(d_\infty-1)}^\infty \subset \partial^\infty U_*.$$

The strict preimage $f^{-1}(\mathbf{H}_*) \backslash \mathbf{H}_*$ is a bouquet of pairwise disjoint arcs

$$\sigma_1^0, \ \dots, \ \sigma_{2(d_0-1)}^0, \quad \sigma_1^\infty, \ \dots, \ \sigma_{2(d_\infty-1)}^\infty$$

where each $\sigma_i^\bullet$ connects $c_0(f_*)$ to a point on $\gamma_i^\bullet$. We call each of $\sigma_i^\bullet$ a *spine* of f_* of generation one. In general, a *spine* of generation $g \geq 1$ is a lift of under f^{g-1} of a spine of generation one, and its *root* is the endpoint that is the critical point of f_*^g.

A *spine chain* of generation g is an infinite sequence of spines

$$\Sigma = (S_1, S_2, S_3, \dots)$$

of increasing generation such that S_1 has generation g and for all $i \geq 1$, the root of S_{i+1} is contained in S_i. We say that a spine chain $\mathcal{S}$ is *periodic* with period p if for all $i \geq 1$, $f^p(S_{i+1}) = S_i$.

The following is a direct consequence of Lemma 3.4 and Theorem 3.9.

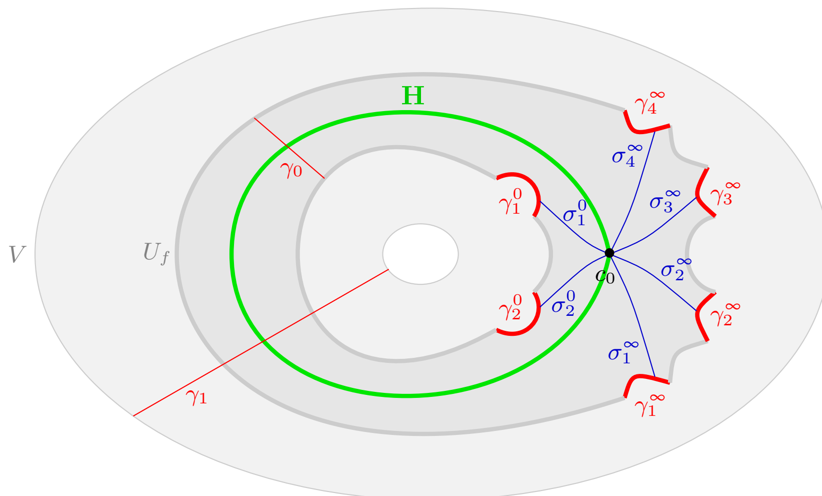

FIGURE 10. The spines of generation one.

Proposition C.1. *Every spine chain lands at a unique point. Different spine chains admits different landing points. The landing point of a periodic spine chain of period p is a periodic point of period p, and it is also the landing point of exactly one periodic external ray of period p.*

When f is rotational with bounded type rotation number, the notion of spines of f can be formulated analogously and the proposition above holds. Below, we will formulate an analog of bubbles for coronas f which are sufficiently close to f_*. This is achieved by replacing $\mathbf{H}_*$ with $\mathbf{\Lambda}_m(f)$.

For f, $\bullet \in \{0,1\}$, and $i \in \{1,\dots,2(d_\bullet - 1)\}$, we define the *pseudo-spine* $\mathbb{S}_i^\bullet$ of generation one to be the closure of the connected component of $f^{-1}(\mathbf{\Lambda}_m)\backslash\mathbf{\Lambda}_m$ that intersects with the spine $\sigma_i^\bullet$ of f_*. Each $\mathbb{S}_i^\bullet$ is connected and

$$\mathbb{S}_i^\bullet \cap \mathbf{\Lambda}_m \subset \Lambda_m(\{-\mathbf{p}_m, -\mathbf{p}_m+1\}), \qquad f(\mathbb{S}_i^\bullet) \subset \mathbf{\Lambda}_m.$$

We say that every pseudo-spine of generation one is attached to $\mathbf{\Lambda}(\{-\mathbf{p}_m, -\mathbf{p}_m+1\})$. In general, a *pseudo-spine* of generation $g \geq 1$ is a lift under f^{g-1} of a pseudo-spine of generation one.

Let us fix a large integer $M \gg 1$. We will assume that f is sufficiently close to f_* depending on M.

Claim 5. Every spine S of f_* of generation up to M is approximated by a pseudo-spine $\mathbb{S}$ of f such that

(1) $\mathbb{S}$ is close to S and $f|_\mathbb{S}$ is close to $f_*|_S$,
(2) if S is attached to another spine S', then $\mathbb{S}$ is attached to the pseudo-bubble corresponding to S';
(3) if S is attached to $\mathbf{H}_*$, then $\mathbb{S}$ is attached to $\Lambda_m(I)$ for some interval I disjoint from $\{0,1\}$.

Proof. This is because $\mathbf{\Lambda}_m$ compactly contains and well approximates $\mathbf{H}_*$. $\square$

Let us fix $\bullet \in \{0,1\}$ and $k \in \{1,\dots,2(d_\bullet - 1)\}$. Let us construct a periodic spine chain

$$\Sigma_k^\bullet = (S_1, S_2, S_3, \dots)$$

for f_* that is very close to $\gamma_k^\bullet$. First, we set $S_1 := \sigma_k^\bullet$. Let us pick $\eta_k^\bullet \geq 1$ such that the pre-critical point $c_{-\eta_k^\bullet+1}(f)$ is close to the critical arc γ_1. Let $c_k^\bullet$ be the preimage of $c_{-\eta_k^\bullet+1}(f)$ located on $\sigma_k^\bullet$ close to $\gamma_k^\bullet$. Then, we set S_2 to be the unique spine rooted at $c_k^\bullet$ that is the lift of S_1 under $f^{\eta_k^\bullet}$. The other spines are then defined by induction, forming a periodic spine chain of period $\eta_k^\bullet$.

Let $x_k^\bullet(f_*)$ be the landing point of $\Sigma_k^\bullet$. It is a repelling periodic point of period $\eta_k^\bullet$ and it is also the landing point of a periodic external ray $R_k^\bullet(f_*)$. Since f is close to f_*, periodic rays $R_k^\bullet(f)$ and repelling periodic points $x_k^\bullet(f)$ exist in the dynamical plane of f.

Let us define a periodic pseudo-spine chain

$$\mathbb{\Sigma}_k^\bullet = (\mathbb{S}_1, \mathbb{S}_2, \mathbb{S}_3, \dots) \tag{C.1}$$

for f landing at $x_k^\bullet(f)$ as follows. Assume $M \gg \eta_k^\bullet$ and let $M' \in \mathbb{N}$ satisfy $\eta_k^\bullet M' \leq M$. For $2 \leq j \leq M'$, we set $\mathbb{S}_j$ to be the pseudo-spine approximating S_j. This can be arranged so that $\mathbb{S}_{M'}$ is within the linearization domain of the repelling periodic point $x_k^\bullet(f)$, and so inductively we define $\mathbb{S}_{M'+j+1}$ to be the unique lift of $\mathbb{S}_{M'+j}$ under $f^{\eta_k^\bullet}$ that is even closer to $x_k^\bullet(f)$.

C.3. **Enlargements of D_j.** Let us inductively define enlargements $\mathcal{D}_j$ and $\mathcal{D}_j'$ of D_j as follows. First, we set $\mathcal{D}_{t+1} = \mathcal{D}_{t+1}' := D$. For $j \leq t$, we set

- ▷ $\mathcal{D}_j'$ = the connected component of $f^{-1}(\mathcal{D}_{j+1})$ containing D_j;
- ▷ $\mathcal{D}_j$ = the smallest topological disk containing $\mathcal{D}_j'$ and the interior of $\Lambda_m(I_j)$.

For all j, $D_j \subset \mathcal{D}_j' \subset \mathcal{D}_j$.

Claim 6. For $j \in \{1,2,\dots,t+1\}$, $\mathcal{D}_j \cap \mathbf{\Lambda}_m$ is connected and its closure is $\Lambda_m(I_j)$.

Proof. This claim is a consequence of Claim 2 and the observation that, due to Claim 3, $D_j \cap \mathbf{\Lambda}_m \subset \Lambda_m(I_j)$ for all j. □

We will assume D to be small enough such that it is disjoint from the rays $f^i(R_k^\bullet)$ for all $i \in \{0,\dots,t\}$, $\bullet \in \{0,\infty\}$, and $k \in \{1,2,\dots,2(d_\bullet - 1)\}$.

Claim 7. For $j \in \{1,2,\dots,t\}$, the disk $\mathcal{D}_j$ is disjoint from all the periodic rays of the form $f^i(R_k^\bullet)$ for all $i \in \{0,\dots,j-1\}$.

Proof. This claim follows from induction. If $\mathcal{D}_j$ intersects $f^i(R_k^\bullet)$, then $\mathcal{D}_j'$ intersects $f^i(R_k^\bullet)$ and so $\mathcal{D}_{j+1}$ intersects $f^{i+1}(R_k^\bullet)$. □

C.4. **Proof of Lemma 5.7.** Let $\mathbf{\Lambda}_m'$ denote the union of all pseudo-spines of f of generation one. To finally show that $f^t : D_1 \to D$ is a branched covering, we will prove by induction the following statements for $j = 1, \dots t+1$.

(a) $\mathcal{D}_j$ intersects $\mathbf{\Lambda}_m'$ if and only if I_j contains $\{-\mathbf{p}_m, -\mathbf{p}_{m+1}\}$;
(b) If $\mathcal{D}_j$ intersects $\mathbf{\Lambda}_m'$, then the intersection is in a small neighborhood of c_0;
(c) If $\mathcal{D}_j$ intersects $\mathbf{\Lambda}_m'$ for $j < t$, then $j < t - \eta$ and $\mathcal{D}_{j+1}, \dots \mathcal{D}_{j+\eta+1}$ are all disjoint from $\mathbf{\Lambda}_m'$;
(d) If $\mathcal{D}_j$ intersects a pseudo-spine chain $\mathbb{\Sigma}_k^\bullet$ from (C.1), then the intersection is within $\mathbf{\Lambda}_m'$;
(e) $\mathcal{D}_j$ is an open disk disjoint from the forbidden boundary $\partial_{\mathrm{F}} U_f$.

Suppose (a)–(e) hold for $j+1, j+2, \dots t+1$. Let us show that they hold for j.

Suppose I_j contains $\{-\mathbf{p}_m, -\mathbf{p}_{m+1}\}$. Then, $\mathcal{D}_{j+1}$ contains either $\Lambda_m(\{-1,0,1\})$ or $\Lambda_m(\{0,1,2\})$, and so the lift $\mathcal{D}'_j$ of $\mathcal{D}_{j+1}$ contains the critical point $c_0(f)$ and intersects $\mathbf{\Lambda}'_m$.

Suppose I_j is disjoint from $\{-\mathbf{p}_m, -\mathbf{p}_{m+1}\}$. Then, $\mathcal{D}_{j+1}$ does not contain the critical value $c_1(f)$ and every point in $\mathcal{D}_{j+1}$ has at most one preimage under f in $\mathcal{D}'_j$. By Claim 6, the preimage of $\mathcal{D}_{j+1} \cap \mathbf{\Lambda}_m$ under $f|_{\mathcal{D}'_j}$ must be contained in $\mathbf{\Lambda}_m$. It follows that $\mathcal{D}'_j$ is disjoint from $\mathbf{\Lambda}'_m$. Since $\mathcal{D}'_j \cup \Lambda_m(I_j)$ does not surround $\mathbf{\Lambda}'_m$, then $\mathcal{D}_j$ is also disjoint from $\mathbf{\Lambda}'_m$.

We just proved (a). Item (b) follows from Claim 6 and the fact that $\Lambda_m(I_{j+1})$ is a small neighborhood of $c_1(f)$ as a result of Claim 4 (i). Item (c) then follows from Claim 4 (2).

Item (e) follows from (b) and (d). Indeed, if $\mathcal{D}_j$ were to intersect $\partial_{\mathrm{F}} U_f$, then by Claim 7, it must intersect some pseudo-spine chain $\mathbb{\Sigma}^\bullet_k$ from (C.1) and because of (d), its intersection is contained in $\mathbf{\Lambda}'_m$. In particular, $\mathcal{D}_k$ can only intersect $\mathbf{\Lambda}'_m$ in a small neighborhood of c_0, which implies that $\mathcal{D}_k$ cannot intersect $\partial_{\mathrm{F}} U_f$.

It remains to prove (d). By continuity, we can assume that (d) holds whenever $j \geq t - \eta$. Let us assume that $j < t - \eta$ and suppose for a contradiction that (d) fails, that is, there is a pseudo-spine chain $\mathbb{\Sigma}^\bullet_k = (\mathbb{S}_1, \mathbb{S}_2, \dots)$ such that $\mathcal{D}_j$ intersects $\mathbb{S}_i$ where $i \geq 2$.

We claim that $\mathcal{D}_j$ intersects $\mathbb{S}_2$. Indeed, suppose otherwise that the smallest possible $i \geq 2$ such that $\mathcal{D}_j$ intersects $\mathbb{S}_i$ satisfies $i > 2$. Since $\mathcal{D}'_j \cap \Lambda_m(I_j)$ is disjoint from the ray $R^\bullet_k$, then the subchain $\mathbb{\Sigma}^{(i)} = (\mathbb{S}_i, \mathbb{S}_{i+1}, \dots)$ intersects $\mathcal{D}'_j$ and its image $f(\mathbb{\Sigma}^{(i)})$ intersects $\mathcal{D}_{j+1}$. By periodicity of $\mathbb{\Sigma}^\bullet_k$, the chain $\mathbb{\Sigma}^{(i-1)}$ intersects $\mathcal{D}_{j+\eta^\bullet_k}$, which is a contradiction to (d) for index $j + \eta^\bullet_k$.

The argument from the previous paragraph results in the intersection of $\mathcal{D}_{j+\eta^\bullet_k}$ and $\mathbb{S}_1$ being non-empty. By (a), the interval $I_{j+\eta^\bullet_k}$ contains $\{-\mathbf{p}_m, -\mathbf{p}_{m+1}\}$, so for $l \in \{1, 2, \dots, \eta^\bullet_k\}$, $f^l(\mathbb{S}_2)$ is attached to $\Lambda_m(I_{j+l})$. Moreover, since the critical value $c_1(f)$ is not contained in $\mathcal{D}_{j+l} \cap \mathbf{\Lambda}_m$, every point in $\mathcal{D}_{j+l}$ has at most one preimage in $\mathcal{D}'_{j+l-1}$.

Consider the lift $\mathbb{S}'_2$ of $f(\mathbb{S}_2)$ under f that is attached to $\Lambda_m(I_j)$. Since $c_1(f)$ is not contained nor surrounded by $\mathcal{D}_{j+1} \cap f(\mathbb{S}_2)$, the lift E of $f(\mathcal{D}_j \cap \mathbb{S}_2)$ under $f|_{\mathcal{D}'_j}$ agrees with the lift under $f|_{\mathbb{S}'_2}$. Therefore, E would be contained in $\mathbb{S}'_2$, not $\mathbb{S}_2$, which is impossible. This concludes the proof of (d).

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