MAT 342
Applied Complex Analysis
2020 Summer II

Midterm

Question	1	2	3	4	5	6	Total
Grade							

- Instructor: Willie Rush Lim
- Due date: July 31st 2020, 11.59pm EST
- This test has 6 questions, each carrying different weights.
- The use of calculator or other similar aids such as Matlab and Wolfram-alpha is prohibited during the test.
- Credit will be given for all questions attempted with clear explanation.
- Submit your answers as one pdf on blackboard.
- In case of technical difficulties, email lim.willie@stonybrook.edu.

Plagiarism Statement[1]

I certify that my answers are my own work, based on my personal study and/or material from lectures. I also certify that I have not copied in part or whole, or otherwise plagiarised the work of other students and/or persons. I acknowledge that students who plagiarize or otherwise engage in academic dishonesty will face serious consequences, including grade reduction or course failure.

________________________ ________________________ [4]

Signature Date

[1]If you plan on submitting handwritten answers, please copy the plagiarism statement above on your answer sheet.

1. (a) Find real numbers a and b such that $a + bi = \text{p.v.}[-8\pi]^{1/3}$. [4]

   (b) Consider the following statement.

   $$\text{"Log}(-z)^2 = \text{Log}z^2 \text{ because } (-z)^2 = z^2.$$
   $$\text{Therefore, } 2\,\text{Log}(-z) = 2\,\text{Log}z\text{."}$$

   Explain whether or not the statement is true. [4]

   (c) Consider the following statement.

   "The rational function $\frac{p(z)}{q(z)}$, where p and q are co-prime non-constant polynomials, is holomorphic everywhere except at the set of zeros of q."

   Does this explain if any primitive of $\frac{p(z)}{q(z)}$ is also holomorphic everywhere except at the zeros of q? Explain why. [4]

2. Every 2×2 real matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ determines a complex function

$$f_M(x + iy) = u_M(x, y) + iv_M(x + iy),$$

where real-valued functions u_M and v_M are determined by the following equation.

$$\begin{pmatrix} u_M(x, y) \\ v_M(x, y) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(a) Show that there are constants w_1 and $w_2 \in \mathbb{C}$ such that $f_M(z) = w_1 z + w_2 \bar{z}$. What are these constants in terms of a, b, c, d? [8]

(b) Determine an equivalent condition on M such that f_M is an entire function. [4]

3. Suppose f is an entire function. Show that any of the two criteria below imply that f is a constant function.

   (a) $\mathrm{Im}f(z) \neq 0$ for all $z \in \mathbb{C}$ and $\frac{\mathrm{Re}f(z)}{\mathrm{Im}f(z)}$ is an entire function. [10]

   (b) $-1 \leq \mathrm{Re}f(z) \leq 1$ for all $z \in \mathbb{C}$. [8]

4. The rational function

$$p(z) = \frac{1}{(z-i)^4 + 4}$$

is holomorphic on the domain $\mathbb{C} \backslash \{a_1, a_2, a_3, a_4\}$ for some four distinct points a_1, a_2, a_3, and a_4.

(a) Find the values of a_1, a_2, a_3, and a_4. [8]

(b) Use one of the Cauchy's formulas to evaluate the integral of $p(z)$ along γ, a positively oriented closed contour parametrising a rectangle with vertices $\pm i$ and $4 \pm i$. Show that this integral can be expressed in the form of

$$\frac{\pi}{c}(a + ib)$$

where a, b and c are integers. [8]

5. Evaluate the integral of f along a contour γ where f and γ are given as follows.

   (a) $f(x+iy) = e^y e^{1-ix}$ along γ, a positively oriented ellipse determined by the equation $r = \cos(2\theta) + 2$. [6]

   (b) $f(z) = 2z^3(z^4 - 1)^{-2}$ along $\gamma(t) = t + i\sqrt{t}$ where $0 \leq t \leq 1$. [10]

6. Let
$$B(z) = \frac{i + 2z}{4 - 2iz}.$$

   (a) Find the smallest positive real value M such that for every z on the closed unit disk $\bar{\mathbb{D}}$, $|B(z)| \leq M$. [6]

   (b) A particle on the complex plane is trapped within a wall built along the unit circle. It travels straight from $-i$ to $e^{3\pi i/4}$ and then bounces at $e^{3\pi i/4}$ to complete its travel from $e^{3\pi i/4}$ to 1. Denote by γ the curve representing the trajectory of the particle. Without evaluating the integral, show how we can obtain the following estimate. [6]
$$\left| \int_\gamma B(z)dz \right| \leq \sqrt{2 + \sqrt{2}}.$$

   (c) Evaluate the integral
$$\int_\gamma B(z)dz,$$
   leaving your answer in the form of of $a + ib$ for some real numbers a and b. [10]