Midterm Solutions

1. (a) Since $-8\pi = 2^3\pi e^{i\pi}$, p.v.$\sqrt[3]{-8\pi} = 2\sqrt[3]{\pi}e^{i\pi/3} = \sqrt[3]{\pi} + i\sqrt[3]{\pi}\sqrt{3}$.

   (b) No, because it's not always true that $\mathrm{Arg}z^2 = 2\mathrm{Arg}z$. (e.g. take $z = e^{2\pi i/3}$.) The equation holds only modulo 2π.

   (c) No. For example, $\frac{z-1}{z} = 1 - \frac{1}{z}$ is holomorphic on $\mathbb{C}^*$ but all its primitives $z - \mathrm{Log}z + c$ for any constant $c \in \mathbb{C}$ are not even continuous nor holomorphic along any choice of branch cut.

2. (a) $u_M(x, y) = ax + by$ and $v_m(x, y) = cx + dy$, then

$$\begin{aligned} f_M(x+iy) &= ax + by + i(cx + dy) = (a + ci)x + (b + di)y \\ &= \frac{a+ci}{2}(z+\bar{z}) + \frac{b+di}{2i}(z-\bar{z}) \\ &= \frac{(a+d)+i(c-b)}{2}z + \frac{(a-d)+i(c+b)}{2}\bar{z}. \end{aligned}$$

   (b) f_M is entire if and only if $w_2 = 0$. That is, $a = d$ and $c = -b$.

3. Both parts can actually be solved simply by showing that the image of f is not dense. Nonetheless, the answers below use more tribal approach. Let $f = u + iv$.

   (a) The function $g = \frac{u}{v}$ is both real and entire. By Cauchy-Riemann, this implies that g is a real constant. Therefore, $u = cv$ for some real c. Applying Cauchy-Riemann on f, this implies that $u_x = cv_x = -cu_y$ and $u_y = cv_y = cu_x$, which imply that $u_x = u_y = v_x = v_y \equiv 0$. Therefore, f is a constant function.

   (b) When u is a bounded function, $|e^f| = e^u$ is bounded. Since e^f is entire, it must be constant by Liouville. Therefore, f is also constant.

4. (a) We wish to find the roots of the denominator in order to find the singularities of p. Check that the roots of the quartic $w^4 + 4$ are $w = \pm1\pm i$. Therefore, the roots of $(z-i)^4+4$ are $z = \pm1, \pm1+2i$. These are the values of $a_1 \ldots a_4$.

   (b) The only singularity enclosed by γ is 1. The rest are outside, so the function $(z+1)^{-1}(z-1-2i)^{-1}(z+1-2i)^{-1}$ is holomorphic along γ and its interior. Apply Cauchy's integral formula at 1.

$$\begin{aligned}\oint_\gamma p(z)dz &= \oint_\gamma \frac{(z+1)^{-1}(z-1-2i)^{-1}(z+1-2i)^{-1}}{z-1}dz \\ &= 2\pi i(1+1)^{-1}(1-1-2i)^{-1}(1+1-2i)^{-1} \\ &= \frac{2\pi}{4(-1+i)} = \frac{\pi}{8}(-1-i).\end{aligned}$$

5. (a) The integrand can be expressed as e^{1-iz}, which is entire. By Cauchy-Goursat, the integral has to be zero.

   (b) The integrand f is holomorphic on $\mathbb{C}\backslash\{\pm 1, \pm i\}$ and has a primitive $F(z) = \frac{1}{2(1-z^4)}$ which is also holomorphic on $\mathbb{C}\backslash\{\pm 1, \pm i\}$. Since the contour γ runs from 0 to $1+i$ avoiding the singularities of f, we can evaluate the integral using the primitive:

$$\int_\gamma f(z)dz = F(i) - F(0) = \frac{1}{2(1-(1+i)^4)} - \frac{1}{2} = -\frac{2}{5}.$$

6. (a) When $|z| = 1$,

$$|B(z)| = \frac{|i+2z|}{|4-2iz|} = \frac{|i+2z|}{|4-2iz||\bar{z}|} = \frac{|i+2z|}{|4\bar{z}-2i|} = \frac{1}{2}\cdot\frac{|i+2z|}{|\overline{2z+i}|} = \frac{1}{2}.$$

   (The above can also be shown using Cartesian $z = x + iy$ or polar coordinates $z = e^{i\theta}$.) $B(z)$ is holomorphic on $\mathbb{C}\backslash\{-2i\}$, and especially on a neighbourhood of the closed unit disk $\bar{\mathbb{D}}$. By the maximum principle, $|B(z)| \leq 1/2$ whenever $z \in \bar{\mathbb{D}}$. Therefore, $M = \frac{1}{2}$.

   (b) Basic trigonometry and Pythagoras gives us $L(\gamma) = 2\sqrt{2+\sqrt{2}}$. The inequality follows from ML inequality.

   (c) $B(z)$ can be expressed as $i + \frac{3}{2z+4i}$. We have a primitive

$$F(z) = iz + \frac{3}{2}\mathrm{Log}(z+2i)$$

   which is holomorphic everywhere except on the branch cut chosen to be $\{x - 2i \,|\, x \leq 0\}$. As γ does not intersect the branch cut, we may use the primitive to evaluate the integral.

$$\begin{aligned}\int_\gamma B(z)dz &= F(1) - F(-i) = i + \frac{3}{2}\mathrm{Log}(1+2i) - 1 - \frac{3}{2}\mathrm{Log}(i) \\ &= -1 + i + \frac{3}{2}\mathrm{Log}(2-i) \\ &= \left(\frac{3}{4}\ln 5 - 1\right) + i\left(1 - \frac{3}{2}\tan^{-1}\frac{1}{2}\right).\end{aligned}$$

Finals Solutions

1. (a) The Laurent series for f valid in $\{\frac{1}{4} < |z| < \frac{1}{2}\}$ is

$$\begin{aligned} f(z) &= \frac{2i}{1-4z} + \frac{i}{1+2z} \\ &= -\frac{i}{2z} \cdot \frac{1}{1-\frac{1}{4z}} + \frac{i}{1+2z} \\ &= -\frac{i}{2z} \sum_{n=0}^{\infty} (4z)^{-n} + i \sum_{n=0}^{\infty} (-2z)^n \\ &= \sum_{n=-\infty}^{-1} (-2^{2n+1} i) z^n + \sum_{n=0}^{\infty} (-2)^n i z^n. \end{aligned}$$

   (b) The residue is zero because f is holomorphic about 0.

   (c) The curve should be the positively oriented circle $C(-0.5, 0.5)$. γ encloses the simple pole -0.5 of f and no zeros of f. By the argument principle, the winding number is $W(f \circ \gamma) = -1$.

2. (a) False. The imaginary part of a constant function is a constant function, which is trivially entire.

   (b) False. The primitive lemma cannot be blindly used since γ intersects with any choice of branch cut of Log. Also, if you do this calculation manually, the value should be $3\pi i$.

   (c) True. For example, $f(z) = \sin(\pi z)$.

   (d) True. Let $f = u + iv$ be holomorphic. By Leibniz,

$$\begin{aligned} (uv)_{xx} &= (u_x v + u v_x)_x = u_{xx} v + 2 u_x v_x + u v_{xx}, \\ (uv)_{yy} &= (u_y v + u v_y)_y = u_{yy} v + 2 u_y v_y + u v_{yy}. \end{aligned}$$

   By harmonicity of u and v and Cauchy-Riemann equations,

$$\begin{aligned} \Delta(uv) &= (u_{xx} + u_{yy}) v + 2(u_x v_x + u_y v_y) + u(v_{xx} + v_{yy}) \\ &= 2(u_x v_x + u_y v_y) = 2(v_y v_x - v_x v_y) = 0. \end{aligned}$$

3. (a) The numerator has simple zeros at $2\pi i n$ for integers n, and the denominator has simple zeros at $\pi i n$ for integers n. In overall, for each integer n, $\pi i n$ is a removable singularity if n is even and a single pole if n is odd.

(b) The function f has a removable singularity at 0. Let's compute the limit

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\cosh\frac{z}{2}}{2e^{2z}} = \frac{1}{2}.$$

Thus, $a_0 = 1/2$ and $k = 0$. The radius of convergence is $R = \pi$.

(c) When $|z| = 1$,

$$|z^{2020} - z^{10} + 2| \le |z|^{2020} + |z|^{10} + 2 = 4 < 5 = |5iz|.$$

When $|z| = \pi$,

$$|-z^{10} + 5iz + 2| \le |z|^{10} + |5iz| + 2 = \pi^{10} + 5\pi + 2 < \pi^{2020} = |z^{2020}|.$$

By Rouche's theorem, the polynomial has the same number of zeros inside $\mathbb{D}$ as $5iz$, which is 1, and it has the same number of zeros inside $\mathbb{D}(0,\pi)$ as z^{2020}, which is 2020. Thus, it has 2019 zeros on the annulus.

4. (a) γ is a rectangle with vertices $\pm R$ and $\pm R + 2i$. Since the singularities of $\cosh \pi z$ are $i(k+\frac{1}{2})$ for all integers k, the only ones enclosed by γ are $\frac{i}{2}$ and $\frac{3i}{2}$.

(b) The integral of f along γ is

$$\begin{aligned}
\oint_\gamma f(z)dz &= 2\pi i \left[\mathrm{Res} f\left(\frac{i}{2}\right) + \mathrm{Res} f\left(\frac{3i}{2}\right)\right] \\
&= 2\pi i \left[\lim_{z\to i/2} \frac{e^{-2\pi i a z}(z - i/2)}{\cosh \pi z} + \lim_{z\to 3i/2} \frac{e^{-2\pi i a z}(z - 3i/2)}{\cosh \pi z}\right] \\
&= 2\pi i \left[e^{\pi a} \lim_{z\to i/2} \frac{1}{\pi \sinh \pi z} + e^{3\pi a} \lim_{z\to 3i/2} \frac{1}{\pi \sinh \pi z}\right] \\
&= 2\pi i \left[\frac{e^{\pi a}}{\pi i} + \frac{e^{3\pi a}}{-\pi i}\right] = 2(e^{\pi a} - e^{3\pi a}).
\end{aligned}$$

(c) Let

$$I = \int_{-\infty}^{\infty} \frac{e^{-2\pi i a x}}{\cosh \pi x} dx$$

and I_j be the integral of f along γ_j for $j = 1, \ldots 4$. As $R \to \infty$, clearly $I_1 \to I$ and $I_3 \to -e^{4\pi a} I$ since

$$I_3 = \int_R^{-R} f(t+2i)dt = -\int_{-R}^{R} \frac{e^{-2\pi i a(t+2i)}}{\cosh \pi(t+2i)} dt = -\int_{-R}^{R} \frac{e^{4\pi a} e^{-2\pi i a t}}{\cosh \pi t} dt.$$

By ML inequality,

$$|I_2| \le L(\gamma_2) \max_{0\le t\le 2} \frac{|e^{-2\pi i a(R+it)}|}{|\cosh \pi(R+it)|} \le 2 \max_{0\le t\le 2} \frac{e^{-2\pi a t}}{\sinh \pi R} = \frac{2e^{4\pi a}}{\sinh \pi R} \to 0.$$

$$|I_4| \le L(\gamma_2) \max_{0\le t\le 2} \frac{|e^{-2\pi i a(-R+it)}|}{|\cosh \pi(-R+it)|} \le 2 \max_{0\le t\le 2} \frac{e^{-2\pi a t}}{\sinh \pi R} = \frac{2e^{4\pi a}}{\sinh \pi R} \to 0.$$

Combining all the integrals together and taking $R \to \infty$, we have

$$2(e^{\pi a} - e^{3\pi a}) = I + 0 - e^{4\pi a} I + 0,$$

which then simplifies to

$$I = \frac{1}{\cosh \pi a}.$$

5. (a) U is open, not closed, unbounded, and disconnected.
   (b) The function $w \mapsto iw - 1$ is entire. When $|z| < 1$, z is enclosed by γ and by Cauchy's differentiation formula,

   $$f(z) = \frac{2\pi i}{1!} \frac{d}{dw} iw - 1|_{w=z} = 2\pi i \cdot i = -2\pi.$$

   When $|z| > 1$, the integrand is holomorphic on the closed disk $\bar{\mathbb{D}}$. By Cauchy-Goursat, $f(z) = 0$. Therefore, the image is $\{0, -2\pi\}$.
   (c) Use the $z = e^{ix}$ substitution. The integral becomes:

   $$\begin{aligned}\int_0^{2\pi} e^{\sin x} \cos(\cos x) dx &= \int_0^{2\pi} e^{\sin x} \frac{e^{i\cos x} + e^{-i\cos x}}{2} dx \\ &= \int_0^{2\pi} \frac{e^{\sin x + i\cos x} + e^{\sin x - i\cos x}}{2} dx \\ &= \int_{C(0,1)} \frac{e^{iz} + e^{-iz}}{2} \frac{dz}{iz} = \int_{C(0,1)} \frac{\cos z}{iz} dz\end{aligned}$$

   Applying residue theorem, this value becomes $2\pi \cos(0) = 2\pi$.

6. (a) Check that the Laplacian is 0.
   (b) Any harmonic conjugate v must satisfy $v_x = -u_y = -2e^{2x}\cos 2y$ and $v_y = 2e^{2x}\sin 2y + 1$. Integrating, v must be of the form $-e^{2x}\cos 2y + y + c$ for some real value c. Therefore,

   $$f(z) = e^{2x}\sin 2y + x + i(-e^{2x}\cos 2y + y + c) = z + i(c - e^{2z}).$$

   To satisfy $f(\pi) = \pi$, $c = e^{2\pi}$.

(c) By MVP,

$$\begin{aligned}
|g(0)| &= \left|\frac{1}{2\pi}\int_0^{2\pi} g(\pi e^{i\theta})\right| \\
&\leq \frac{1}{2\pi}\left|\int_0^{\pi} g(\pi e^{i\theta})d\theta\right| + \frac{1}{2\pi}\left|\int_{\pi}^{2\pi} g(\pi e^{i\theta})d\theta\right| \\
&\leq \frac{1}{2\pi}\int_0^{\pi} |g(\pi e^{i\theta})|d\theta + \frac{1}{2\pi}\int_{\pi}^{2\pi} |g(\pi e^{i\theta})|d\theta \\
&\leq \frac{1}{2\pi}\int_0^{\pi} 1d\theta + \frac{1}{2\pi}\int_{\pi}^{2\pi} 3d\theta = 2.
\end{aligned}$$

(d) $g-h$ is harmonic on S. By the maximum modulus principle, since $g-h \equiv 0$ on the boundary ∂S, then $g-h \equiv 0$ on S.

7. (a) If $z = x+iy$, $|A(z)| = |e^{e^x \cos y + ie^x \sin y}| = e^{e^x \cos y}$. The maximum value is attained when $x = \ln \pi$ and $y = 0$, resulting in $|A(\ln \pi)| = e$. The minimum value is attained when $x = 0$ and $y = \pm\pi$, resulting in $|A(\pm\pi i)| = e^{-1}$.

(b) The derivative is $A'(z) = e^{e^z+z}$. Its modulus is $|A'(z)| = e^{x+e^x \cos y}$. This is clearly maximised when $y = 0$, and $x+e^x$ attains maximum when $x = \ln \pi$.

(c) The primitive is $C(z) = (z+\frac{4}{3})\mathrm{Log}(3z+4) - z$ and we can pick the branch cut to be the ray $\{x - \frac{4}{3} \mid x \leq 0\}$.

(d) The contour γ runs from 0 to 1 in a spiral contained in the closed unit disk $\bar{\mathbb{D}}$ which lies in U. The primitive C can be used to evaluate the integra since γ avoids the branch cut. The endpoints of γ are 0 and $\sin(\pi/2)e^{631\pi i} = -1$. The integral is therefore equal to

$$\begin{aligned}
\int_\gamma B(z)dz &= C(-1) - C(0) \\
&= \left(\frac{1}{3}\mathrm{Log}(-3+4) - (-1)\right) - \left(\frac{4}{3}\mathrm{Log}4 - 0\right) \\
&= 1 - \frac{4}{3}\ln 4.
\end{aligned}$$