Solutions 1

1. The Cartesian and polar forms are as follows.

$$
\begin{aligned}
&(a)\ i, e^{i\pi/2}, && (b)\ 1+i, \sqrt{2}e^{i\pi/4} \\
&(c)\ -16\sqrt{3}+16i, 32e^{5\pi i/6}, && (d)\ -2, 2e^{\pi i}.
\end{aligned}
$$

2. It's sufficient to show $|z| - |w| \leq |z-w|$ and $|w| - |z| \leq |z-w|$. Both come from triangle inequality.

3. Since $\langle z, w\rangle = z\bar{w} = (x+iy)(u-iv) = (ux+vy) + i(uy-vx)$,

$$
\begin{aligned}
\mathrm{Re}\langle z, w\rangle &= ux + vy = (x,y)\cdot(u,v), \\
\overline{\langle w, z\rangle} &= \overline{w\bar{z}} = \bar{w}z = \langle z, w\rangle, \\
\langle z, z\rangle &= z\bar{z} = |z|^2 = x^2 + y^2 \geq 0.
\end{aligned}
$$

   Equality on the last line holds if and only if x and y are 0.

4. We can use the identity $|z|^2 = z\bar{z}$. For every $z, w \in \mathbb{C}$,

$$
\begin{aligned}
|z \pm w|^2 &= (z\pm w)(\bar{z} \pm \bar{w}) = z\bar{z} + w\bar{w} \pm z\bar{w} \pm w\bar{z} \\
&= |z|^2 + |w|^2 \pm (z\bar{w} + \overline{z\bar{w}}) = |z|^2 + |w|^2 \pm 2\mathrm{Re}(z\bar{w}).
\end{aligned}
$$

   Then,

$$
\begin{aligned}
|z+w|^2 - |z-w|^2 &= \big(|z|^2 + |w|^2 + 2\mathrm{Re}(z\bar{w})\big) - \big(|z|^2 + |w|^2 - 2\mathrm{Re}(z\bar{w})\big) \\
&= 4\mathrm{Re}(z\bar{w}).
\end{aligned}
$$

5. Since $w \neq 1$ and $w^n - 1 = 0$,

$$1 + w + \ldots w^{n-1} = \frac{w^n - 1}{w - 1} = 0.$$

   Take the real value of the equation above to get:

$$\cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{4\pi}{n}\right) + \ldots + \cos\left(\frac{2(n-1)\pi}{n}\right) = 0.$$

6. Since $|-8+8i\sqrt{3}| = 16$ and $\mathrm{Arg}(-8+8i\sqrt{3}) = \frac{2\pi}{3}$, then $z^4 = 2^4 e^{2\pi i(3k+1)/3}$ for any integer k. Then,

$$z = 2e^{\pi i(3k+1)/6}, \text{ for } k \in \{0,1,2,3\}.$$

   Simplifying the expression, the roots are $\pm(\sqrt{3}+i)$ and $\pm(-1+i\sqrt{3})$.

7. Let $\alpha = \cos(\frac{2\pi}{5})$ and $w = e^{2\pi i/5}$.

   (a) $\alpha = \mathrm{Re}(w) = \frac{w+\bar{w}}{2} = \frac{w+w^4}{2}$ and $\alpha^2 = \frac{w^2+w^3+2}{4}$.

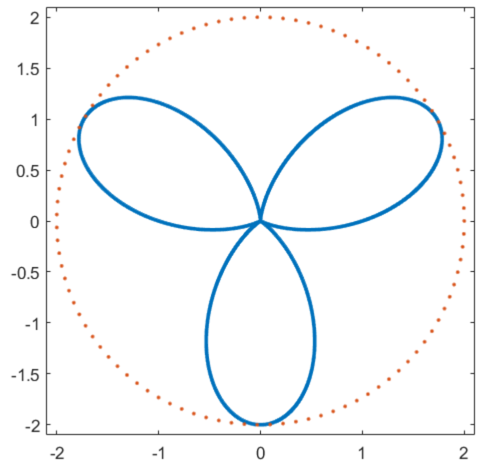
   (b) This is 0 from exercise 5.

   (c) From part (b), we can pick $p = 4$, $q = 2$, and $r = -1$.

   (d) By quadratic formula, $\alpha = \frac{-1\pm\sqrt{5}}{4}$. We pick the $+$ sign since $\alpha > 0$.

8. I will only sketch (a); the rest should be fairly easy to illustrate.

   (a) It's the boundary of a 'flower' with three petals of maximum radius 2 centered at 0. See below.



   (b) When $z = x + iy$, the equation can be rewritten as $x^2 - y^2 = 1$, a hyperbola.

   (c) When $z = x + iy$, multiplying both top and bottom with the complex conjugate $\bar{z} - i$ gives you:

$$\frac{z-i}{z+i} = \frac{x^2+y^2-1-2ix}{x^2+(y+1)^2}.$$

   The denominator is always positive unless $z = -i$, on which the fraction is undefined. The real value is negative exactly when $x^2 + y^2 - 1 < 0$. This gives us the unit disk $\mathbb{D} = \{|z| < 1\}$.

   (d) The imaginary part of the fraction above is 0 when $-2ix = 0$. This gives us the set of purely imaginary numbers $\{iy \mid y \in \mathbb{R}\backslash\{-1\}\}$. We exclude $-i$ since the fractional expression is not defined at that point.

(e) When $z = x + iy$, $\mathrm{Im} z^2 < 0$ exactly when $xy < 0$ and $\mathrm{Im}(z + 1 + i)^2 < 0$ exactly when $(x + 1)(y + 1) < 0$. This is the set $\{x + iy \mid x < -1, y > 0\} \cup \{x + iy \mid x > 0, y < -1\}$.

9. For each of the five sets in Exercise 9 above, determine whether or not they are open, closed, bounded, connected, simply connected or multiply connected.

   (a) not open, compact, connected, multiply connected.

   (b) not open, closed, unbounded and disconnected.

   (c) open, not closed, bounded, simply connected.

   (d) not open, not closed, unbounded, disconnected.

   (e) open, not closed, unbounded, disconnected.

10. Refer to the definition of convergence of complex numbers.

11. No. Let $r_n = \frac{1}{n}$, $r = 0$, $\theta_n = (-1)^n \frac{\pi}{2}$, and $\theta = 0$. Then, $r_n e^{i\theta_n} = \frac{(-1)^n i}{n}$ converges to $re^{i\theta} = 0$. Even though $r_n \to r$, unfornutately $\theta_n \not\to \theta$.

12. It is easier when f is rewritten as $f(z) = z^2$. Then, for any $a \in \mathbb{C}$, the derivative always exists:

$$f'(a) = \lim_{z \to a} \frac{z^2 - a^2}{z - a} = \lim_{z \to a} z + a = 2a.$$

Alternatively, you may show that Cauchy Riemann equations hold throughout $\mathbb{C}$.

13. Upon computing the derivative at an arbitrary point $a \in \mathbb{C}$,

$$\lim_{z \to 0} \frac{|a + z|^2 - |a|^2}{z} = \lim_{z \to 0} \frac{z\bar{z} + \bar{a}z + a\bar{z}}{z} = \lim_{z \to 0} \bar{z} + \bar{a} + a\frac{\bar{z}}{z} = \bar{a} + a \lim_{z \to 0} \frac{\bar{z}}{z}.$$

When $a = 0$, it is clear that the limit above exists and is equal to 0. However, when $a \neq 0$, the limit does not exist since $\lim_{z \to 0} \frac{\bar{z}}{z}$ does not exist. Since $|z|^2$ is only complex differentiable at one point, it is not holomorphic on any domain.

Solutions 2

1. If $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$, then $\overline{f(\bar{z})} = p(x, y) + iq(x, y)$ where $p(x, y) = u(x, -y)$ and $q(x, y) = -v(x, -y)$. It remains to show that Cauchy Riemann equations still hold for the pair p and q on the domain $\bar{U} := \{\bar{z} \mid z \in U\}$, which is the reflection of U in the real axis. (Note that the correct domain for $\overline{f(\bar{z})}$ is $\bar{U}$, not U.)

2. This is merely an exercise in multivariable calculus. Use the chain rules:
$$\frac{\partial f}{\partial x} = \frac{x}{r}\frac{\partial f}{\partial r} - \frac{y}{r^2}\frac{\partial f}{\partial \theta}, \qquad \frac{\partial f}{\partial y} = \frac{y}{r}\frac{\partial f}{\partial r} + \frac{x}{r^2}\frac{\partial f}{\partial \theta}.$$
Log is holomorphic because
$$\frac{d}{d\bar{z}}\mathrm{Log}z = \frac{1}{2\bar{z}}\left(r\frac{\partial}{\partial r} + i\frac{\partial}{\partial \theta}\right)(\ln r + i\theta) = \frac{1}{2\bar{z}}(1-1) = 0.$$
Its derivative is
$$\frac{d}{dz}\mathrm{Log}z = \frac{1}{2z}\left(r\frac{\partial}{\partial r} - i\frac{\partial}{\partial \theta}\right)(\ln r + i\theta) = \frac{1}{2z}(1+1) = \frac{1}{z}.$$

3. Let $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$. If $f(z) = \overline{f(z)}$, then $v(x, y) \equiv 0$. By Cauchy Riemann equations, $u_x = v_y \equiv 0$ and $u_y = -v_x \equiv 0$, so then $u(x, y) = c$ for some constant $c \in \mathbb{R}$. Therefore, $f(z) \equiv c$ on U.

4. Let $z = x + iy$. When $x \in \mathbb{R}$ and $|y| < \pi$, $e^{x+iy} = e^x e^{iy}$. The function is surjective because the image is
$$\{e^x e^{iy} \mid x \in \mathbb{R}, |y| < \pi\} = \{re^{iy} \mid r > 0, |y| < \pi\} = \mathbb{C}\backslash(-\infty, 0].$$
Since e^{iy} is 2π-periodic with respect to y and since the height of the strip is at most 2π, e^z is injective. The inverse of e^z is $\mathrm{Log}(z)$ which is holomorphic on $\mathbb{C}\backslash(-\infty, 0]$ with derivative z^{-1}.

5. The preimage is
$$\{z \mid 1 - z^{-1} \in (-\infty, 0]\} = \{(1-x)^{-1} \mid x \in (-\infty, 0]\} = [-1, 0).$$
This can be taken as the branch cut because its image under $1 - z^{-1}$ is $(-\infty, 0]$, the usual branch cut for $\log(z)$.

6. Let $\tanh^{-1}(z) = w$, then $z = \frac{e^w - e^{-w}}{e^w + e^{-w}}$. This can be rewritten as

$$e^{2w} = \frac{1+z}{1-z}$$

Using logarithm, the expression becomes

$$w = \frac{1}{2}\log\frac{1+z}{1-z}.$$

7. Here, k represents any integer.

   (a) $\frac{1}{2}\ln 2 + i\frac{\pi}{4}(8k-3)$,

   (b) $e^{i\ln\pi}$,

   (c) $\frac{\pi}{2}(1+4k) - i\ln(1+\sqrt{2})$,

   (d) $e^{-\frac{\pi}{8}(1+4k)(\sqrt{3}+i)}$.

8. All are smooth and closed. The only simple ones are $n = 1, 2$.

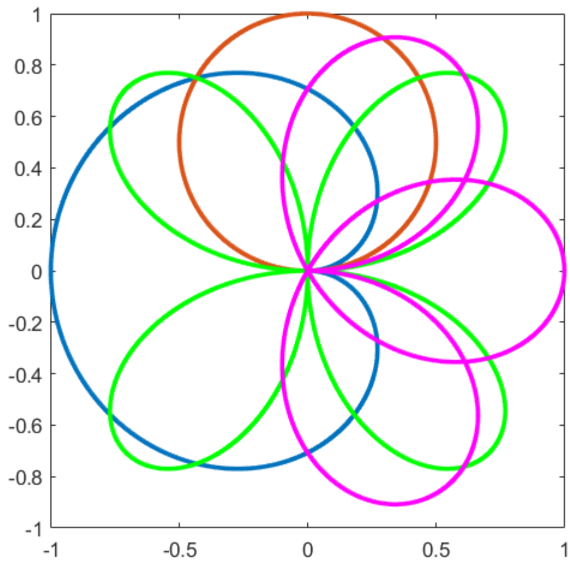


Figure 1: $n = 1$ in blue, $n = 2$ in red, $n = 3$ in pink and $n = 4$ in green

9. The integrals are as follows:

   (a) Use $\gamma(t) = (3+4i)t$ for $0 \le t \le 1$. Since $\gamma'(t) = 5$,

$$\int_\gamma \mathrm{Im} z dz = \int_0^1 4t \cdot |\gamma'(t)| dt = \int_0^1 20t\, dt = 10.$$

(b) Since $\gamma'(t) = 2ie^{it}$,

$$\int_\gamma i\bar{z}+iz^2 dz = \int_{\pi/2}^{\pi} (2ie^{-it}+8e^{3it})\cdot 2ie^{it}dt = \int_{\pi/2}^{\pi} -4+16ie^{4it}dt = -2\pi.$$

(c) Since $\gamma'(t) = ie^{it}$,

$$\begin{aligned}\int_\gamma \mathrm{pv}\, z^i dz &= \int_{-\pi/2}^{\pi/2} e^{i\mathrm{Log}(e^{it})} \cdot ie^{it}dt = \int_{-\pi/2}^{\pi/2} ie^{t(-1+i)}dt \\ &= \frac{i}{-1+i}(e^{\frac{\pi}{2}(-1+i)} - e^{\frac{\pi}{2}(1-i)}) = \frac{1-i}{2}(ie^{-\pi/2} + ie^{\pi/2}) \\ &= (1+i)\cosh(\pi/2).\end{aligned}$$

10. Since $\gamma'(t) = (-1+i)e^{(-1+i)t}$, the length of γ is

$$L(\gamma) = \int_0^{2\pi} |-1+i|dt = 2\pi\sqrt{2}.$$

11. The distance between the line segment γ and the point 1 is $2^{-1/2}$, so then

$$\max_{z\in\gamma} |(z-1)^{-3}| = (\min_{z\in\gamma} |z-1|)^{-3} = (2^{-1/2})^{-3} = 2\sqrt{2}.$$

Since $L(\gamma) = |2i-2| = 2\sqrt{2}$, then by ML inequality,

$$\Big|\int_\gamma \frac{1}{(z-1)^3}dz\Big| \le 2\sqrt{2}\cdot L(\gamma) = 8.$$

12. Let $z = x+iy \in \gamma$, then $|e^{\bar{z}}| = e^x \le e^2$ because $0 \le x \le 2$. Therefore,

$$\Big|\int_\gamma e^{\bar{z}}dz\Big| \le L(\gamma)\cdot \max_{z\in\gamma} |e^{\bar{z}}| = 8e^2.$$

13. One primitive is $\frac{z^{i+1}}{i+1}$ because by chain rule, on $\mathbb{C}\backslash(-\infty, 0]$,

$$\frac{dz^{i+1}}{dz} = \frac{de^{(i+1)\mathrm{Log}z}}{dz} = \frac{i+1}{z}\cdot e^{(i+1)\mathrm{Log}z} = (i+1)z^i.$$

The curve γ lies in the domain $\mathbb{C}\backslash(-\infty, 0]$ and it travels from $-i$ to i. Then,

$$\begin{aligned}\int_\gamma \mathrm{pv}\, z^i dz &= \frac{i^{i+1}}{i+1} - \frac{(-i)^{i+1}}{i+1} = \frac{1}{1+i}(ie^{i\mathrm{Log}i} - (-i)e^{i\mathrm{Log}(-i)}) \\ &= \frac{1-i}{2}(ie^{-\pi/2} + ie^{\pi/2}) = (1+i)\cosh(\pi/2).\end{aligned}$$

14. Both integrands are entire functions. As such, the integrals are independent of the choice of the contour.

    (a) The integrand has primitive $iz + z^3/3$. Then,

    $$\int_0^i z^2 + idz = iz + z^3/3|_0^i = -1 - i/3.$$

    (b) The integrand has primitive $i \cosh z$. Then,

    $$\int_{-\pi}^{\pi} \sin(iz) = i \cosh z|_{-\pi}^{\pi} = 0.$$

15. The integrand can be rewritten as $\frac{2}{5}\left(\frac{1}{z-3/2} - \frac{1}{z+1}\right)$. Since -1 is outside of the pentagon γ but $3/2$ is enclosed by γ, we apply Cauchy-Goursat so that the integral is reduced to

    $$\frac{2}{5}\int_\gamma \frac{1}{z-3/2}dz.$$

    By deformation theorem, we can replace γ with any small circle centered at $3/2$. The integral is then reduced to $4\pi i/5$.

Solutions 3

1. By partial fractions, the integral can be rewritten as
$$\frac{i}{12}\oint_\gamma \frac{dz}{z+1.5i} - \frac{i}{12}\oint_\gamma \frac{dz}{z-1.5i}.$$
The singular points we need to keep our eye on are $\pm 1.5i$.
   (a) The rectangle does not enclose $\pm 1.5i$. Both integrands are holomorphic along and inside γ. By Cauchy-Goursat, the integral is 0.
   (b) The circle only encloses $1.5i$, but not $-1.5i$. The first integral is 0 by Cauchy-Goursat. The second becomes $-\frac{i}{12}\cdot 2\pi i = \frac{\pi}{6}$. In total, the integral is πi.
   (c) Check that γ is a negatively oriented circle centered at 0 of radius π, enclosing both $\pm 1.5i$. Therefore, the integral evaluates to
$$\frac{i}{12}\cdot 2\pi i - \frac{i}{12}\cdot 2\pi i = 0.$$

2. The following functions g are holomorphic along and inside the domain enclosed by $C(0,2)$.
   (a) Apply Cauchy's formula to $g(z) = \frac{z+2}{z-3}$ at the point $z_0 = 1$. The integral is
$$\oint_{C(0,2)} \frac{g(z)}{z-1}dz = 2\pi i g(1) = -3\pi i.$$
   (b) Apply Cauchy's formula to $g(z) = e^{e^z}$ at the point $z_0 = i\pi/2$. The integral is
$$\oint_{C(0,2)} \frac{g(z)}{z-i}dz = 2\pi i g(i) = 2\pi i e^{e^{i\pi/2}} = 2\pi i e^{i}.$$
   (c) Apply Cauchy's differentiation formula to get the 3rd derivative of $g(z) = \sinh(\pi z)$ at the point $z_0 = 0$. The integral is
$$\oint_{C(0,2)} \frac{g(z)}{z^4}dz = \frac{2\pi i}{3!}\frac{d^3}{dz^3}\left(\sinh(\pi z)\right)\bigg|_{z=0} = \frac{\pi^4 i}{3}.$$

3. For any point $z_0 \in \mathbb{C}$, radius $r > 0$ and point w on the circle $C(z_0, r)$, we can apply triangle inequality to get $|w| \leq |w - z_0| + |z_0| = r + |z_0|$ and consequently $|f(w)| \leq \pi(r + |z_0|)$. By Cauchy's inequality,
$$|f''(z_0)| \leq \frac{2\pi(r + |z_0|)}{r^2}.$$
Taking the limit as $r \to \infty$, the right hand side goes to 0. Since $|f''(z_0)|$ is independent of r, $f''(z_0) = 0$ for all z_0. The primitive f' must be some constant a and the primitive f of f' must be of the form $az + b$. However, since $|f(0)| \leq \pi \cdot 0 = 0$, b must be 0.

4. Since $f^{(6)}$ is bounded and entire, it is a constant function of some value a where $|a| > 0$. By taking primitive 6 times, f must be a polynomial of degree 6 because it has a leading term $\frac{a}{6!}z^6$.

5. The inequality implies that $f(z) \neq 0$ for all z, so $1/f(z)$ is a well-defined entire function. Since $|1/f(z)| \leq 1$, it is bounded and therefore constant. f is then constant too.

6. There is some constant $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. By ML inequality,
$$\begin{aligned}\left|\oint_{C(0,R)} \frac{f(z)}{(z - z_0)(z - z_1)} dz\right| &\leq 2\pi R \max_{|z|=R} \left|\frac{f(z)}{(z - z_0)(z - z_1)}\right| \\ &= 2\pi R \frac{M}{\min_{|z|=R} |(z - z_0)(z - z_1)|} \\ &\leq \frac{2\pi M R}{(R - |z_0|)(R - |z_1|)}.\end{aligned}$$
where the final inequality comes from triangle inequality. By taking the limit as $R \to \infty$, this upper bound clearly goes to 0, so then
$$\lim_{R \to \infty} \oint_{C(0,R)} \frac{f(z)}{(z - z_0)(z - z_1)} dz = 0.$$
This integral can be separated by partial fractions and evaluated by Cauchy's integral formula.
$$\begin{aligned}\oint_{C(0,R)} \frac{f(z)}{(z - z_0)(z - z_1)} dz &= \frac{1}{z_0 - z_1}\left[\oint_{C(0,R)} \frac{f(z)}{z - z_0} - \oint_{C(0,R)} \frac{f(z)}{z - z_1} dz\right] \\ &= \frac{f(z_0) - f(z_1)}{2\pi i(z_0 - z_1)}.\end{aligned}$$
This expression is independent of R, so then it must be 0. Therefore $f(z_0) = f(z_1)$.

7. It is holomorphic with derivative $2z$ on $\mathbb{D}$ and 0 on the annulus $\{2 < |z| < 3\}$. It attains maximum on the annulus with $|f(z)| = 2$. The set U is disconnected and therefore the maximum modulus principle does not apply.

8. As f is entire, by maximum modulus principle, it is sufficient to see the behavior of f on the circle $\{|z| = 2\}$ to find maximum points. When $z = 2e^{it}$ where $t \in \mathbb{R}$,

$$\begin{aligned} |z^3 + i| &= |8e^{3it} + 1| = |(8\cos 3t + 1) + i8\sin 3t| \\ &= \left[64\cos^2 3t + 16\cos 3t + 1 + 64\sin^2 3t\right]^{1/2} = [65 + 16\cos 3t]^{1/2} \end{aligned}$$

The real function $\cos 3t$ attains its maximum value 1 at $t = 0, \pm\frac{2\pi}{3}$. At any of these values, we have $|z^3 + 1| = 9$, and this is attained by $z = 2, -1 \pm i\sqrt{3}$.

9. The function $e^{(1+i)z}$ is entire. By the maximum modulus principle, to find the maximum value of $e^{(1+i)z}$ on the closed square $\{x + iy \mid 1 < x, y < \pi\}$, it is sufficient to look at the the function along the boundary of the square. Let $z = x + iy$.

$$|e^{(1+i)z}| = |e^{(x-y)+i(x+y)}| = e^{x-y}.$$

The maximum of $x - y$ is attained on the boundary of the square when $x = \pi$ and $y = 1$. Therefore, the smallest radius is $r = e^{\pi-1}$.

10. Part (a) follows from applying the minimum modulus principle on $\mathbb{D}(z_0, \epsilon)$. If the lemma weren't true, it would in the most direct way contradict the minimum modulus principle. Part (b) follows from triangle inequality:

$$\begin{aligned} |f(z)| &\geq |a_d z^d| - \sum_{n=0}^{d-1} |a_n z^n| \geq |a_d||z|^d - \sum_{n=0}^{d-1} |a_n||z|^{d-1} \\ &\geq |z|^{d-1}\left(|a_d||z| - \sum_{n=0}^{d-1} |a_n|\right) \geq |z|^{d-1} \geq R^{d-1}. \end{aligned}$$

For part (c), $|f|$ must attain minimum on the compact disk $\overline{\mathbb{D}(0, R)}$ where R is from part (b). Let z_0 be a minimum point in this compact disk. If $f(z_0) \neq 0$, then it will contradict part (a). Therefore, $f(z_0) = 0$.

Solutions 4

1. (a) $e^{2\pi z} = e^{4\pi^2} e^{2\pi(z-2\pi)} = e^{4\pi^2} \sum_{n=0}^{\infty} \frac{(2\pi)^n}{n!}(z-2\pi)^n$,
   (b) $\frac{1}{1+z^2} = \sum_{n=0}^{\infty}(-z^2)^n$,
   (c) $\sin z = \cos(z - \frac{\pi}{2}) = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(z - \frac{\pi}{2})^{2n}$.

2. Apply the identity theorem on any sequence of distinct points in V converging to some point in V. Such a sequence always exists because V is non-empty and open.

3. Apply the identity theorem on $\overline{f(\bar{z})}$ and $f(z)$ as both functions agree on $\mathbb{R}$.

4. Yes. Let $z = re^{i\theta}$ where $r > 1$. As $N \to \infty$, $z^{-N-1} \to 0$ because $|z^{-N-1}| = r^{-N-1} \to 0$. Therefore,

$$g(z) = \lim_{N\to\infty} \sum_{n=0}^{N} z^{-n} = \lim_{N\to\infty} \frac{1 - z^{-N-1}}{1 - z^{-1}} = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}.$$

5. (a) About i,

$$\begin{aligned}
\frac{z}{z^2+1} &= (z-i)^{-1}\frac{z}{z+i} = (z-i)^{-1}\left(1 - \frac{i}{z+i}\right) \\
&= (z-i)^{-1}\left(1 - \frac{1}{2\left(1 - \frac{i}{2}(z-i)\right)}\right) \\
&= (z-i)^{-1}\left(1 - \sum_{n=0}^{\infty} \frac{i^n}{2^{n+1}}(z-i)^n\right) \\
&= \frac{1}{2}(z-i)^{-1} - \sum_{n=0}^{\infty} \frac{i^{n+1}}{2^{n+2}}(z-i)^n.
\end{aligned}$$

   This Laurent series is convergent on $\{0 < |z-i| < 2\}$.

   (b) About 0,

$$\begin{aligned}
\frac{2}{z-2} + \frac{1}{4-z} &= \frac{2}{z\left(1 - \frac{2}{z}\right)} + \frac{1}{4\left(1 - \frac{z}{4}\right)} \\
&= \frac{2}{z}\sum_{n=0}^{\infty} 2^n z^{-n} + \frac{1}{4}\sum_{n=0}^{\infty} \frac{z^n}{4^n} \\
&= \sum_{n=-\infty}^{-1} 2^{-n} z^n + \sum_{n=0}^{\infty} 4^{-n-1} z^n.
\end{aligned}$$

This Laurent series is convergent on $\{2 < |z| < 4\}$.

(c) About 1,

$$\begin{aligned}
\frac{3-3z}{2z^2-5z+2} &= \left(\frac{1}{1-2z} + \frac{1}{2-z}\right) \\
&= -\frac{1}{2(z-1)\left(1+\frac{1}{2(z-1)}\right)} + \frac{1}{1-(z-1)} \\
&= -\frac{1}{2(z-1)}\sum_{n=0}^{\infty}\left(-\frac{1}{2(z-1)}\right)^n + \sum_{n=0}^{\infty}(z-1)^n. \\
&= \sum_{n=-\infty}^{-1}(-2)^{-n}(z-1)^n + \sum_{n=0}^{\infty}(z-1)^n.
\end{aligned}$$

This Laurent series is convergent on $\{\frac{1}{2} < |z-1| < 1\}$.

6. (a) The zeros of $\sin z$ are on πn for $n \in \mathbb{Z}$, and none of these are zeros of $\cos z$. Each of them is simple, so then $\cot z$ has simple poles at πn for $n \in \mathbb{Z}$.

   (b) Singularities are at point z such that $\sin z = \sin 2z$. This occurs when $\sin z = 0$, i.e. $z = n\pi$ for $n \in \mathbb{Z}$, or when $\cos z = \frac{1}{2}$, i.e. $z = \pm\frac{\pi}{3} + 2\pi n$ for $n \in \mathbb{Z}$. Each of these are single poles of the function.

   (c) The zeros of the denominator are clearly 0 of order 2 and ± 1 of order 1. The numerator does not have a zero at 0, but it has zeros at ± 1. Therefore, 0 is a double pole and ± 1 are removable singularities.

7. The singularities of f/g are removable because $|f(z)/g(z)| \leq 1$, i.e. bounded. As such, f/g is a bounded entire function, which is a constant function a for some $a \in \mathbb{C}$.

8. (a) Since f has a zero of order $n \geq 1$, $g(z)$ is a well-defined holomorphic function with removable singularity at 0.

   (b) Along $|z| = r$ for any $r < 1$,

$$|g(z)| = \left|\frac{f(z)}{z}\right| < \frac{1}{r}.$$

   As $r \to 1$, the upper bound converges to 1. Thus, the maximum modulus of g along the boundary is 1 and by MMP, $|g(z)| \geq 1$. This implies that $|f(z)| \leq |z|$. Looking at the Taylor series of f should convince you that $|f'(0)| = |g(0)| \leq 1$.

(c) If $|f'(0) = 1$ or $|f(w)| = |w|$ for some point $w \in \mathbb{D}^*$, then $|g(w')| = 1$ where w' is either 0 or w. As g attains maximum in $\mathbb{D}$, it must be a constant function a and therefore $f(z) = az$. Since either $|f'(0) = 1$ or $|f(w)| = |w|$, then $|a| = 1$. This implies that a is of the form $e^{i\theta}$ and clearly $f(z) = e^{i\theta}z$ is a counterclockwise rotation of the unit disk of angle θ.

9. It's easier to look at the image of the four line segments individually. Assume that the orientation of γ is positive. Using Cartesian coordinates $z = x + iy$, $\cos 2z - 1 = (\cos 2x \cosh 2y - 1) - i \sin 2x \sinh 2y$.

   - When $x = \pm\frac{\pi}{4}$, $\cos 2z - 1 = -1 \mp i \sinh 2y$.
     The image of the $x = -\frac{pi}{4}$ side of the square is the same as that of the $x = -\frac{pi}{4}$ side, which is a upward linear curve from $-1 - i \sinh \frac{\pi}{2}$ to $-1 + i \sinh \frac{\pi}{2}$.
   - When $y = \pm\frac{\pi}{4}$, $\cos 2z - 1 = \cos 2x \cosh \frac{\pi}{2} - 1 \mp i \sin 2x \sinh \frac{\pi}{2}$.
     The image of the $y = -\frac{pi}{4}$ side of the square is the same as that of the $y = -\frac{pi}{4}$ side, which is a downward elliptic arc with co-vertices $-1 \pm i \sinh \frac{\pi}{2}$ and rightmost vertex $-1 + \cosh \frac{\pi}{2}$.

   The curve γ has a winding number two about the origin. Since $\cos 2z - 1$ has no poles, it must have exactly two zeros enclosed by γ. (It is in fact a double zero at 0.)

10. When $|z| = 1$, $|e^{z-1}| = e^{x-1} \le 1 < 2 = |2z^n|$. By Rouche's theorem, $e^{z-1} + 2z^n$ has the same number of zeros as $2z^n$, which is n, inside $\mathbb{D}$.

11. When $|z| = 2$, $|5z + 1| \le 5|z| + 1 = 11 < 32 = |z^5|$. By Rouche's theorem, $z^5 + 5z + 1$ has the same number of zeros as z^5, which is 5, in $\mathbb{D}(0, 2)$. When $|z| = 1$, $|z^5| = 1 < 4 = |5z| - 1 \le |5z + 1|$. Therefore, $z^5 + 5z + 1$ has the same number of zeros as $5z + 1$, which is 1, in $\mathbb{D}$. In total, $z^5 + 5z + 1$ has 4 zeros inside $\{1 \le |z| < 2\}$.

Solutions 5

1. (a) The function $f(z) = \cot z$ has a pole of order 1 at 0. Then,

$$\mathrm{Res} f(0) = \frac{1}{0!} \lim_{z\to 0} z\cot z = \lim_{z\to 0} \cos z \frac{z}{\sin z} = 1.$$

(b) $\cos z + 1$ has a double zero at π since its first derivative $-\sin z$ vanishes at π but the second derivative $-\cos z$ does not. The function $f(z) = \frac{z+\pi}{\cos z + 1}$ at has a pole of order 2 at π. Then, using the change of variables $w = z - \pi$,

$$\begin{aligned}
\mathrm{Res} f(\pi) &= \frac{1}{1!} \lim_{z\to\pi} \frac{d}{dz} \frac{(z-\pi)^2}{\cos z + 1} = \lim_{z\to\pi} \frac{d}{dz} \frac{(z-\pi)^2}{\cos z + 1} \\
&= \lim_{z\to\pi} \frac{2(z-\pi)(\cos z + 1) + \sin z (z-\pi)^2}{(\cos z + 1)^2} \\
&= \lim_{w\to 0} \frac{2w(1-\cos w) - w^2 \sin w}{(1-\cos w)^2} \\
&= \lim_{w\to 0} \frac{2w(\frac{w^2}{2} - \frac{w^4}{24} + \ldots) - w^2(w - \frac{w^3}{6} + \ldots)}{(\frac{w^2}{2} - \frac{w^4}{24} + \ldots)^2} \\
&= \lim_{w\to 0} \frac{\frac{w^4}{12} + \ldots}{\frac{w^4}{4} - \ldots} = \frac{1}{3}.
\end{aligned}$$

2. (a) The function $f(z) = \frac{3z+1}{(z+2)(z-1)}$ has single poles at 1 and -2. γ has winding number -1 about 1 and 0 about -2. Thus,

$$\oint_\gamma f(z)dz = -1 \cdot 2\pi i \mathrm{Res} f(1) = -2\pi i \lim_{z\to 1} \frac{3z+1}{z+2} = -\frac{8\pi i}{3}.$$

(b) The function $f(z) = e^{1/z}$ has an essential singularity at 0 and at that point, the residue is 1 since $e^{1/z} = 1 + z^{-1} + \frac{z^{-2}}{2} + \ldots$. Since γ has winding number 1 about the origin,

$$\oint_\gamma f(z)dz = 2\pi i.$$

(c) The function $f(z) = \csc(2\pi z)$ has single poles at every integer. γ has winding numbers 2, 1 and -1 about -1, 0 and 1 respectively.

Therefore,

$$\begin{aligned}\oint_\gamma f(z)dz &= 4\pi i \mathrm{Res} f(-1) + 2\pi i \mathrm{Res} f(0) - 2\pi i \mathrm{Res} f(1)\\ &= 4\pi i \lim_{z\to -1} \frac{z+1}{\sin 2\pi z} + 2\pi i \lim_{z\to 0} \frac{z}{\sin 2\pi z} - 2\pi i \lim_{z\to 1} \frac{z-1}{\sin 2\pi z}\\ &= 4+2-1 = 3.\end{aligned}$$

3. By the change of variables $z = e^{i\theta}$, the integral can be transformed into a contour integral along the unit circle $\gamma(\theta) = e^{i\theta}$ where $0 \le \theta \le 2\pi$.

$$\int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2} = \oint_\gamma \frac{i}{(az-1)(z-a)} dz.$$

The only pole of the integrand enclosed by γ is a and it is a single pole. By residue theorem,

$$\int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2} = 2\pi i \lim_{z\to a} \frac{i}{(az-1)} = \frac{2\pi}{1-a^2}.$$

4. (a) The integrand $f(z)$ is an even function and it has simple poles at $\pm i$ and $\pm 2i$. Use semicircular closed contour γ of radius $R > 2$. The poles enclosed by γ are i and $2i$. By residue theorem,

$$\oint_\gamma f(z)dz = 2\pi i \left(\mathrm{Res} f(i) + \mathrm{Res} f(2i)\right) = \ldots = \frac{\pi}{3}.$$

By ML inequality that the semicircle part γ_2 of γ vanishes to 0 as $R \to \infty$ because

$$\left| \int_{\gamma_2} f(z)dz \right| \le \pi R \cdot \max_{z\in\gamma_2} \left| \frac{z^2}{(z^2+1)(z^2+4)} \right| \le \frac{\pi R^3}{(R^2-1)(R^2-4)} \to 0.$$

This leaves $\pi/3$ as the value of the integral of f on $(-\infty, \infty)$. Therefore,

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)} dx = \frac{\pi}{6}.$$

(b) The integrand

$$f(z) = \frac{ze^{iz}}{(z^2+1)(z^2+4)}$$

has simple poles at $\pm i$ and $\pm 2i$. Use semicircular closed contour γ of radius $R > 2$. The poles enclosed by γ are i and $2i$. By residue theorem,

$$\oint_\gamma f(z)dz = 2\pi i\left(\mathrm{Res}f(i) + \mathrm{Res}f(2i)\right) = \ldots = \frac{\pi i}{3}(e^{-1} - e^{-2}).$$

By Jordan's lemma, the semicircle part γ_2 of γ vanishes to 0 as $R \to \infty$ because

$$\left|\int_{\gamma_2} f(z)dz\right| \leq \pi\cdot\max_{z\in\gamma_2}\left|\frac{z}{(z^2+1)(z^2+4)}\right| \leq \frac{\pi R}{(R^2-1)(R^2-4)} \to 0.$$

Therefore, the integral of f on $(-\infty, \infty)$ is equal to that along γ. By taking the imaginary part,

$$\int_0^\infty \frac{z\sin z}{(z^2+1)(z^2+4)}dx = \frac{\pi}{3}(e^{-1} - e^{-2}).$$

(c) Substitute $y = x - \pi$ so that $\sin x = -\sin y$. From the example in class, this integral is $-\pi$.

(d) You can use the semicircular contour, but I'll use the sector contour γ with angle $\pi/2$ instead. Let $f(z)$ be the integrand; γ will enclose the single pole of f at $e^{i\pi/4}$. Let's use the same notation as in the notes.

$$I_0 = 2\pi i\mathrm{Res}f(e^{i\pi/4}) = \ldots = \frac{\pi}{2\sqrt{2}}(1-i).$$

Use the parametrisation $\gamma_3(r) = ri$ as r varies from R to 0 and obtain that

$$I_3 = \int_R^0 \frac{1}{(ri)^4+1}idr = -i\int_0^R \frac{1}{R^4+1}dr = -iI_1.$$

Also, $I_2 \to 0$ as $R \to \infty$ because by ML inequality

$$\left|\int_{\gamma_2} f(z)dz\right| \leq \frac{\pi R}{2}\max_{z\in\gamma_2}\frac{1}{|z^4+1|} \leq \frac{\pi R}{2(R^4-1)} \to 0.$$

Then, taking the limit $R \to \infty$ and after rearranging, you should obtain

$$\int_0^\infty \frac{1}{1+x^4}dx = \frac{\pi}{2\sqrt{2}}.$$

(e) The function $f(z) = \frac{1}{z^{1/2}(z^2+9)}$ has simple poles at $\pm 3i$. Pick the branch cut to be $\arg z = 0$. Use the keyhole contour to evaluate the given integral I. Using the same notation as in the notes,

$$I_0 = 2\pi i[\mathrm{Res} f(3i) + \mathrm{Res} f(3i)] = \ldots = \frac{\pi}{3}\sqrt{\frac{2}{3}}.$$

Taking $R \to \infty$ and $\epsilon, \delta \to 0$, check that $I_1 \to I$ and that

$$\begin{aligned} I_3 &= \int_{\gamma_3} \frac{1}{z^{1/2}(z^2+9)} dz = \int_R^\rho \frac{1}{\sqrt{r}e^{i(\pi-\epsilon/2)}(r^2 e^{i(4\pi-2\epsilon)}+9)} e^{i(2\pi-\epsilon)} dr \\ &\to -\int_0^\infty \frac{1}{e^{i\pi}\sqrt{r}(r^2+9)} dz = -e^{-\pi i} I = I. \end{aligned}$$

Check that by ML inequality, we have $I_2, I_4 \to 0$. Therefore, this gives $2I = I_0$ and upon simplifying, $I = \frac{\pi}{3\sqrt{6}}$.

(f) Let $f(z)$ be the integrand. It has a triple pole at -1. Pick the branch cut to be $\arg z = 0$. Use the keyhole contour to evaluate the given integral I. Using the same notation as in the notes,

$$I_0 = 2\pi i \mathrm{Res} f(-1) = \ldots = -\frac{2\pi i}{9z^2}\mathrm{p.v.} z^{1/3}\Big|_{z=-1} = \frac{\pi}{9}(\sqrt{3}-i).$$

Taking $R \to \infty$ and $\epsilon, \delta \to 0$, check that $I_1 \to I$ and $I_2 \to \frac{1-i\sqrt{3}}{2} I$. The latter is because

$$\begin{aligned} I_3 &= \int_{\gamma_3} \frac{\sqrt[3]{z}}{(z+1)^3} dz = \int_R^\rho \frac{\sqrt[3]{r}e^{i(2\pi/3-\epsilon/2)}}{(re^{-i\epsilon}+1)^3} e^{i(2\pi-\epsilon)} dr \\ &\to -e^{2\pi i/3} \int_0^\infty \frac{\sqrt[3]{r}}{(r+1)^3} dr = -e^{2\pi i/3} I. \end{aligned}$$

Check that by ML inequality, we have $I_2, I_4 \to 0$. Therefore, this gives $\frac{3-i\sqrt{3}}{2} I = I_0$ and upon simplifying, $I = \frac{2\pi}{9\sqrt{3}}$.

5. This is the trickiest question in the problem set. The usual branch cut for log is $[-\infty, 0]$ and $z^2 + 1 \in [-\infty, 0]$ precisely when $z^2 \in [-\infty, -1]$ and therefore the branch cut is $\{ai \mid a \geq 1, a \leq -1\}$, a union of two vertical rays. To evaluate the integral I asked, it is easier to split the integrand into $f + g$ where

$$f(z) = \frac{\mathrm{Log}(z+i)}{z^2+1}, \qquad g(z) = \frac{\mathrm{Log}(z-i)}{z^2+1}.$$

The branch cut of f can be taken to be $\{ai \mid a \leq= 1\}$ and that of g can be taken to be $\{ai \mid a \geq 1\}$.

The integral of f along $(-\infty, \infty)$ can be evaluated using the usual semicircular contour $\gamma = \gamma_1 \cup \gamma_2$ where $\gamma_1 = [-R, R]$ and γ_2 is an upper semicircle of radius $R > 0$. With the usual argument, you may check that by ML inequality, the integral of f along γ_2 vanishes to 0 as $R \to \infty$. Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\operatorname{Log}(x+i)}{x^2+1} dx &= \lim_{R\to\infty} \int_{\gamma_1} \frac{\operatorname{Log}(z+i)}{z^2+1} dz \\ &= \lim_{R\to\infty} \oint_{\gamma} \frac{\operatorname{Log}(z+i)}{z^2+1} dz \\ &= 2\pi i \operatorname{Res} f(i) = \ldots = \pi \ln 2 + \frac{\pi^2 i}{2}.\end{aligned}$$

To avoid the branch cut of g, we evaluate the integral of g using the lower semicircular contour $\sigma = \sigma_1 \cup \sigma_2$ where σ_1 is the segment from R to $-R$ and $\sigma_2 = \{Re^{i\theta} \mid -\pi \leq \theta \leq 0\}$ is the lower semicircle of radius $R > 0$. With the usual argument, you may check that by ML inequality, the integral of g along σ_2 vanishes to 0 as $R \to \infty$. Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\operatorname{Log}(x-i)}{x^2+1} dx &= -\lim_{R\to\infty} \int_{\sigma_1} \frac{\operatorname{Log}(z-i)}{z^2+1} dz \\ &= -\lim_{R\to\infty} \oint_{\sigma} \frac{\operatorname{Log}(z-i)}{z^2+1} dz \\ &= -2\pi i \operatorname{Res} g(-i) = \ldots = \pi \ln 2 - \frac{\pi^2 i}{2}.\end{aligned}$$

Summing the two integrals together, we obtain

$$\int_{-\infty}^{\infty} \frac{\operatorname{Log}(x-i)}{x^2+1} dx = 2\pi \ln 2.$$

Since the integrand is an even function, we can divide by two and obtain that the integral we wanted to find all along is indeed $\pi \ln 2$.

6. You can check that $U_z = \frac{1}{2}U_x + i\left(-\frac{1}{2}U_y\right)$ satisfies Cauchy-Riemann equations. Alternatively, you may check that the Laplacian can be expressed using Wirtinger derivatives:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial \bar{z}}.$$

This implies that $\frac{\partial}{\partial \bar{z}} U_z = \frac{1}{4}\Delta U = 0$, i.e. U_z is holomorphic.

7. This is another calculus exercise. Compute the Laplacian accordingly and show that it vanishes to 0. At $(0,0)$, the function is not even continuous, since

$$\lim_{x\to 0}\frac{0}{x^2+0^2}=0\neq\infty=\lim_{y\to 0}\frac{y}{0^2+y^2}.$$

8. Let $u(x,y)$ be a bounded harmonic function on $\mathbb{R}^2$. Pick any harmonic conjugate v of u. Then, $f=u+iv$ is an entire function and so is $e^{f(z)}$. Since u is bounded, so is $|e^{f(z)}|=e^{u(x,y)}$. By Liouville, $e^{f(z)}$, $f(z)$ and ultimately u are constant.

9. The difference $u=u_1-u_2$ is harmonic on U and vanishes on the whole subset V. Since u_z is holomorphic on U and vanishes on the whole V, then $u_z\equiv 0$ on U by the identity theorem. Since $2u_z=u_x-iu_y$, then $u_x\equiv u_y\equiv 0$, i.e. u is a constant function, so it must be the zero function.

10. Let $f=u+iv$ where u and v are real-valued functions, then $g=u^2+v^2$. Using harmonicity of u and v,

$$\begin{aligned}\Delta g&=\frac{\partial}{\partial x}(2uu_x+2vv_x)+\frac{\partial}{\partial y}(2uu_y+2vv_y)\\&=2uu_{xx}+2u_x^2+2vv_{xx}+2v_x^2+2uu_{yy}+2u_y^2+2vv_{yy}+2v_y^2\\&=2u(u_{xx}+u_{yy})+2v(v_{xx}+v_{yy})+2(u_x^2+u_y^2+v_x^2+v_y^2)\\&=2(u_x^2+u_y^2+v_x^2+v_y^2).\end{aligned}$$

Since g is harmonic, the expression above is 0 and therefore, $u_x\equiv u_y\equiv v_x\equiv v_y\equiv 0$ on U. This shows that f is constant.

11. (a) $\frac{r}{w-r}=\frac{r}{w}\frac{1}{1-\frac{r}{w}}=\frac{r}{w}\sum_{n\geq 0}\left(\frac{r}{w}\right)=\sum_{n\geq 1}r^nw^{-n}$.

(b) Let $w=z=e^{i\theta}$. Then,

$$\frac{r}{w-r}=\frac{r}{e^{i\theta}-r}=\frac{r(e^{-i\theta}-r)}{|e^{i\theta}-r|^2}=\frac{r(\cos\theta-r)-ir\sin\theta}{1-2r\cos\theta+r^2}$$

and by de Moivre's theorem,

$$\begin{aligned}\sum_{n\geq 1}r^nw^{-n}&=\sum_{n\geq 1}r^n\left(\cos(n\theta)-i\sin(n\theta)\right)\\&=\left(\sum_{n\geq 1}r^n\cos(n\theta)\right)-i\left(\sum_{n\geq 1}r^n\sin(n\theta)\right).\end{aligned}$$

Comparing the real and imaginary parts should give the equations we wanted.

(c) By now, this is just some basic algebraic manipulation inferior to everything else you've done.

12. (a) This example is similar to the one done in class.

$$
\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\int_0^{\pi/2} P(r,t-\theta)dt = 2\tan^{-1}\left(\frac{1+r}{1-r}\tan\frac{t-\theta}{2}\right)\Bigg|_0^{\pi/2} \\
&= \frac{1}{\pi}\tan^{-1}\left(\frac{1+r}{1-r}\tan\frac{\pi-2\theta}{4}\right) + \frac{1}{\pi}\tan^{-1}\left(\frac{1+r}{1-r}\tan\frac{\theta}{2}\right).
\end{aligned}
$$

(b) Use the cosine series on Qn 11 to integrate.

$$
\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} P(r,t-\theta)\cos t\, dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(1+\sum_{n\geq 1} 2r^n\cos(n(t-\theta))\right)\cos t\, dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\cos t\, dt + \frac{1}{\pi}\sum_{n\geq 1} r^n\int_0^{2\pi}\cos(n(t-\theta))\cos t\, dt \\
&= \frac{1}{2\pi}\sum_{n\geq 1} r^n\int_0^{2\pi}\cos(n(t-\theta)+t)+\cos(n(t-\theta)-t)\, dt \\
&= r\cos\theta.
\end{aligned}
$$

(Yes... the corresponding holomorphic function f such that $\mathrm{Re}f = u$ is just the identity function $f(z)=z$.)