Quasiconformal Deformations and Sullivan's Theorem of No Wandering Domains

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February 6, 2019

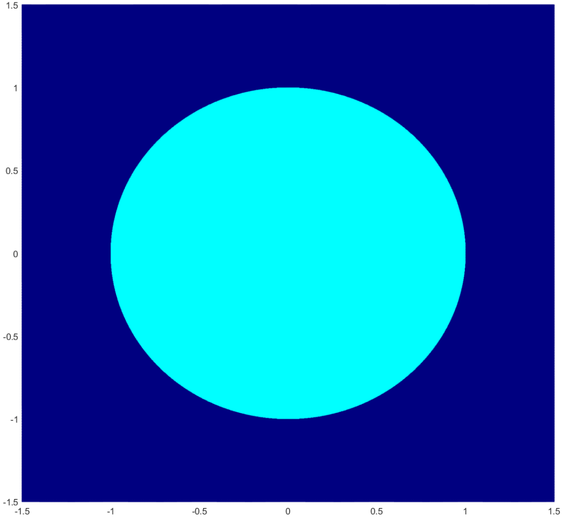
Holomorphic Dynamics

All holomorphic maps from the Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ to itself are rational maps $f(z) = \frac{P(z)}{Q(z)}$. The **degree** of f is $\max\{\deg P, \deg Q\}$.

We are interested in the behaviour of iterations of f. A question we can ask is: for which points $z \in \hat{\mathbb{C}}$ does the sequence of forward iterates $f^n(z)$

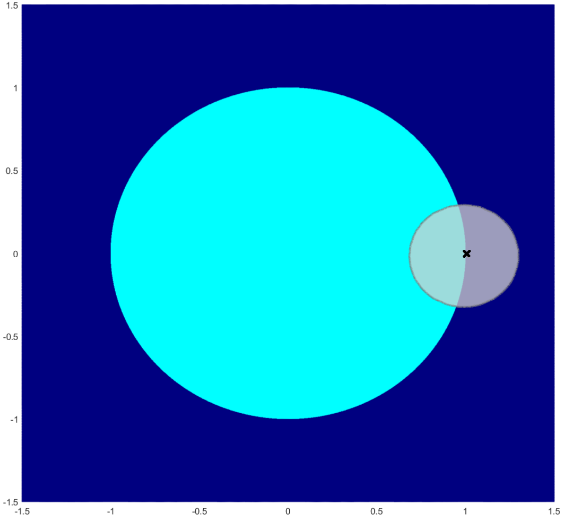
- converge to a point?
- blow up to ∞?
- have a convergent subsequence?
- chaotic, i.e. a small pertubation would result in strikingly different dynamical behaviours?

Simple Example



$$f(z) = z^d$$

Simple Example



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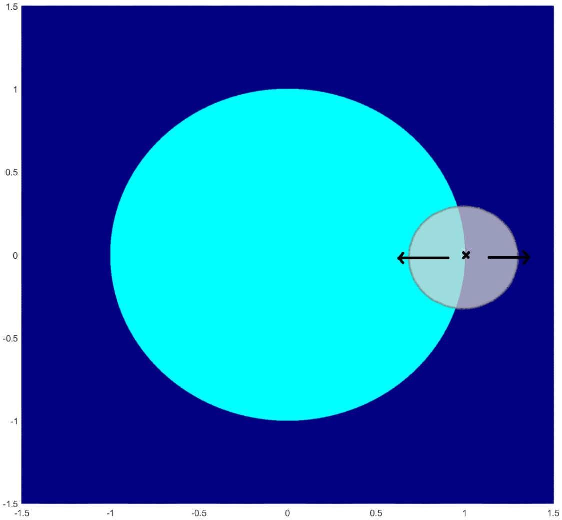


$f(z) = z^d$

Fatou and Julia Sets

The **Fatou set** $F(f)$ is the largest open set with equicontinuous points:

$$\forall w \in F(f), \forall \epsilon > 0, \exists \delta > 0 : \quad \forall n \in \mathbb{N},\ f^n\big(B(w,\delta)\big) \subset B\big(f^n(w),\epsilon\big)$$

The **Julia set** $J(f)$ is the complement of $F(f)$. This is the chaotic set.

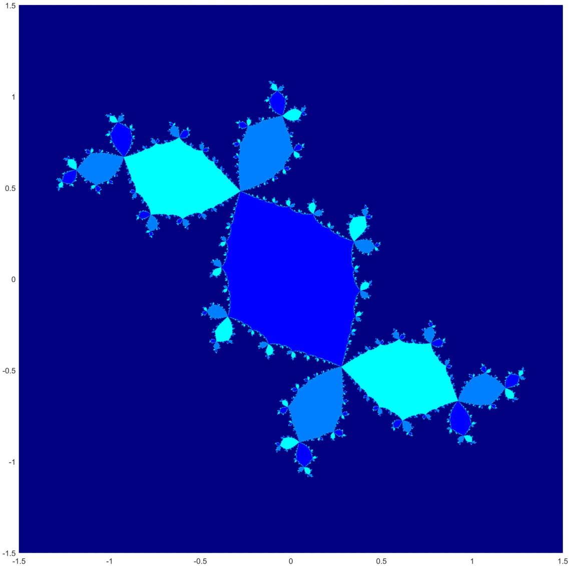
Properties of the Fatou Set

$F(f)$ is completely invariant, i.e. $f^{-1}(F(f)) = F(f)$.

We can classify each connected component U of $F(f)$ as follows:

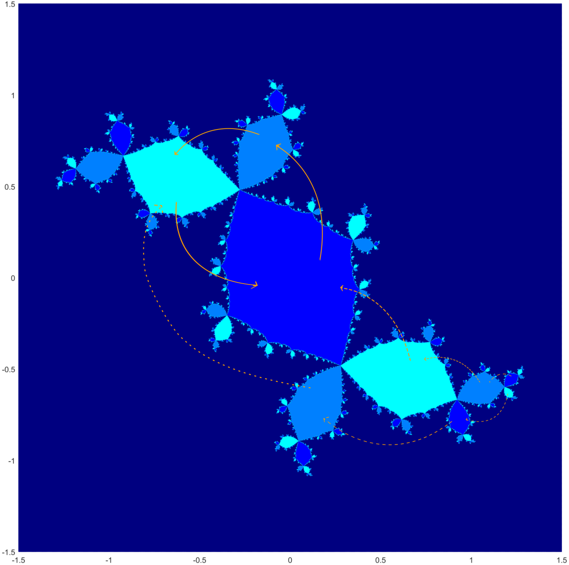
- **periodic**, i.e. $f^p(U) = U$ for some period p,
- **pre-periodic**, i.e. $f^{n+p}(U) = f^n(U)$ for some n, p,
- **wandering**, i.e. $\{f^n(U)\}_{n\in\mathbb{N}}$ are all pairwise disjoint.

Some Examples



$f(z) = z^2 - 0.125 + 0.75i$

Some Examples



$f(z) = z^2 - 0.125 + 0.75i$

Sullivan's No Wandering Domain Theorem

Theorem

All connected components of the Fatou set $F(f)$ of a rational map f of degree ≥ 2 are non-wandering, i.e. periodic or pre-periodic.

Quasiconformal Homeomorphism

Let U, V be non-empty open subsets of $\hat{\mathbb{C}}$.

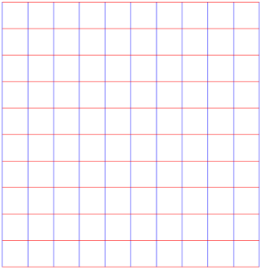
An orientation-preserving homeomorphism $f : U \to V$ is a **quasiconformal(QC) homeomorphism** if

1. f is absolutely continuous on lines, i.e. $x \to f(x+iy)$ and $y \to f(x+iy)$ are differentiable almost everywhere,
2. $\left\|\mu_f(z)\right\|_\infty < 1$.

Here, $\mu_f(z) = \frac{\frac{\partial f}{\partial \bar{z}}(z)}{\frac{\partial f}{\partial z}(z)}$ is the **complex dilatation** of f.

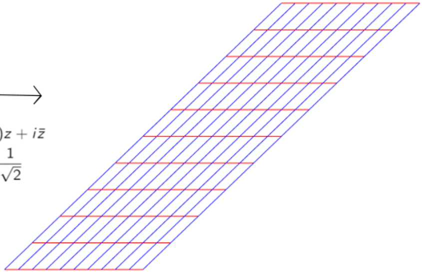
Quasiconformal Homeomorphism

Geometrically, QC maps preserve orientation but it distorts angles.
The distortion level depends on $|\mu_f(z)|$.



$f(z) = (1-i)z + i\bar{z}$

$|\mu_f(z)| = \frac{1}{\sqrt{2}}$



Measurable Riemann Mapping Theorem

A **Beltrami form** on an open subset $U \subset \hat{\mathbb{C}}$ is a measurable $\mu \in L^\infty(U)$ where $\|\mu\|_\infty < 1$.

Theorem (MRMT)

For any Beltrami form μ on $\hat{\mathbb{C}}$, there is a QC homeomorphism $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that μ is the complex dilatation of ϕ, i.e.

$$\mu_\phi(z) = \mu(z)$$

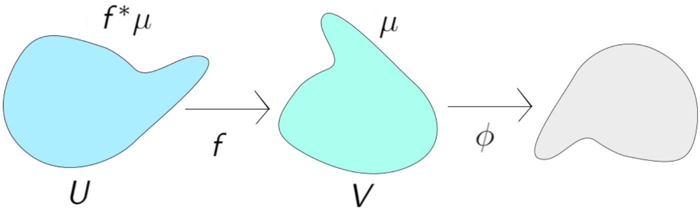
Moreover, ϕ depends analytically on μ and it is unique if we require ϕ to fix 0, 1, *and* ∞.

Remark: MRMT can be used for Beltrami forms on open subsets of $\hat{\mathbb{C}}$. The uniqueness criterion will be different, however.

Pullback of a Beltrami Form

Let $f : U \to V$ be a QC/holomorphic map between open subsets of $\hat{\mathbb{C}}$.
Let μ be a Beltrami form on V and ϕ be the unique QC map with $\mu_\phi = \mu$.

Define the **pullback** of μ via f as $f^*\mu := \mu_{\phi\circ f}$, a Beltrami form on U.



If f is bijective, the **pushforward** operator is $f_* = (f^{-1})^*$.

Deformation Lemma

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map.
Let μ be a Beltrami form on $\hat{\mathbb{C}}$ and ϕ be the unique QC map with $\mu_\phi = \mu$.

If $f^*\mu = \mu$, then $\phi \circ f \circ \phi^{-1}$ is a rational map.

<u>Proof</u>:
Denote by 0 the zero Beltrami form on $\hat{\mathbb{C}}$. By f- inviarance of μ,

$$\begin{aligned}(\phi \circ f \circ \phi^{-1})^* 0 &= \phi^{-1*} f^* \phi^* 0 \\ &= \phi^{-1*} f^* \mu \\ &= \phi^{-1*} \mu \\ &= 0\end{aligned}$$

No Wandering Domain Theorem

Theorem (Sullivan's No Wandering Domains)

All connected components of the Fatou set $F(f)$ of a rational map f of degree ≥ 2 are non-wandering, i.e. periodic or pre-periodic.

Proof of No Wandering Domains

Let f be a rational map of degree d.
Assume a wandering component U of $F(f)$ exists.

Main idea
There's a space M_U of Beltrami forms on U of arbitrarily large dimension.

Meanwhile, the space Rat_d of rational maps of degree d is a complex manifold of dimension $2d + 1$.

If we can construct an analytic map $F : M_U \to Rat_d$, we will obtain a contradiction from noninjectivity.

Construction of $F : M_U \to Rat_d$

Pick any $\mu \in M_U$. We will extend this to a Beltrami form on $\hat{\mathbb{C}}$:

1. Push forward μ via f from U to $\bigcup_{n\in\mathbb{N}} f^n(U)$,
2. Pull back via f to the whole $\bigcup_{m,n\in\mathbb{N}} f^{-m+n}(U)$.
3. Set $\mu(z) = 0$ on the complement $\hat{\mathbb{C}} \setminus \bigcup_{m,n\in\mathbb{N}} f^{-m+n}(U)$,
4. $\|\mu\|_\infty$ is preserved under pullback and pushforward, so $\|\mu\|_\infty < 1$.

By MRMT, we have a unique QC homeomorphism ϕ such that $\mu_\phi = \mu$.

By the construction above, $f^*\mu = \mu$.

By deformation lemma, let $F(\mu) = \phi \circ f \circ \phi^{-1}$ is a rational map of deg d. Also, F is analytic as ϕ depends analytically on μ.

The Contradiction

Step 4: The Contradiction

Since m is arbitrary, we can assume that $2m$ is larger than $4d+2$, which is the real dimension of the smooth complex manifold Rat_d. By Sard's theorem, there exists some element $f_a \in Rat_d$ where the fiber $F^{-1}(\{f_a\})$ is of dimension ≥ 1. In other words, we can take a non-trivial simple curve $\mu_{a(t)}$, where $t \in [0,1]$, in $F^{-1}(\{f_a\})$ connecting 2 distinct Beltrami coefficients $\mu_{a(0)}$ and $\mu_{a(1)}$ with corresponding quasiconformal homeomorphisms $\phi_{a(0)}$ and $\phi_{a(1)}$.

$$\begin{tikzcd}
\hat{\mathbb{C}} \arrow[d, "f_a"'] & \hat{\mathbb{C}} \arrow[l, "\phi_{a(0)}"] \arrow[r, "\phi_{a(1)}"] \arrow[d, "f"] & \hat{\mathbb{C}} \arrow[d, "f_a"'] \\
\hat{\mathbb{C}} & \hat{\mathbb{C}} \arrow[l, "\phi_{a(0)}"] \arrow[r, "\phi_{a(1)}"] & \hat{\mathbb{C}}
\end{tikzcd}$$

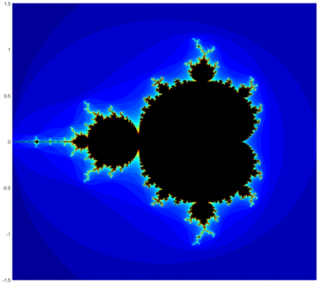
Pick any $t \in [0,1]$. Since $\phi_{a(t)} \circ \phi_{a(0)}^{-1}$ commutes with f_a, for all $n \geq 1$, $\phi_{a(t)} \circ \phi_{a(0)}^{-1}$ restricted to the set periodic points $Per_n(f_a)$ of prime period n is an automorphism. For any n and $z \in Per_n(f_a)$, the map $\phi_{a(t)} \circ \phi_{a(0)}^{-1}(z)$, $t \in [0,1]$ is a continuous path starting from z, but since $Per_n(f_a)$ is finite, it is the identity on $Per_n(f_a)$. We conclude by Lemma 2.0.3 that $\phi_{a(t)} \circ \phi_{a(0)}^{-1}$ is the identity on $J(f_a)$, or in other words $\phi_{a(0)}^{-1} \circ \phi_{a(t)}$ is the identity on $\partial U \subset J(f)$.

Let $V = \phi_{a(0)}(U)$ and $t \in [0,1]$, then as $\phi_{a(t)} \circ \phi_{a(0)}^{-1}$ is the identity on ∂V, $\phi_{a(t)}$ maps U to either V or $\hat{\mathbb{C}} \backslash \overline{V}$. We can assume without loss of generality by conjugation with Möbius maps that U contains ∞, so that $\phi_{a(t)}$ and $\phi_{a(0)}^{-1}$ fixes ∞. Then, $\phi_{a(t)}(U) = V$.

Let $h_{a(t)} := g_{a(t)} \circ R \circ \phi_{a(t)}^{-1} : V \to \mathbb{D}$. By the same argument as in Lemma 2.1.1, we can deduce that $h_{a(t)}$ is a biholomorphism. Thus, $g_{a(1)} \circ g_{a(0)}^{-1} = h_{a(1)} \circ \phi_{a(1)} \circ \phi_{a(0)}^{-1} \circ h_{a(0)}^{-1}$, but on $\partial\mathbb{D}$, $g_{a(1)} \circ g_{a(0)}^{-1}$ simplifies to $h_{a(1)} \circ h_{a(0)}^{-1}$, an automorphism of $\mathbb{D}$. As $g_{a(1)} \circ g_{a(0)}^{-1}$ fixes at least 3 points on $\partial\mathbb{D}$ (namely b_1, b_2 and b_3), $g_{a(1)} \circ g_{a(0)}^{-1} = Id$ on $\overline{\mathbb{D}}$. This is a contradiction because by our construction, $g_{a(0)}$ and $g_{a(1)}$ are distinct.

Other Applications

1. Renormalization Theory and Dynamics
   e.g. proving the quasi-self-similarity of the Mandelbrot Set
2. Kleinian Groups, Hyperbolic 3-manifolds, and Teichmuller Theory
3. Computer graphics, medical image analysis, etc



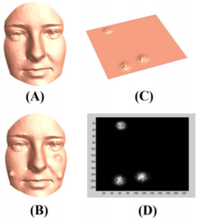


Figure 3.8: (A) shows the original human face and (B) shows a deformed version of the human face with an abnormally swollen area. (C) shows the plot of $|\mu|$ versus the parameter domain. (D) shows the distribution of $|\mu|$ by color.

The End