Solutions 1

1. The Cartesian and polar forms are as follows.

$$(a)\ i, e^{i\pi/2}, \qquad (b)\ 1+i, \sqrt{2}e^{i\pi/4}$$
$$(c)\ -16\sqrt{3}+16i, 32e^{5\pi i/6}, \qquad (d)\ -2, 2e^{\pi i}.$$

2. It's sufficient to show $|z|-|w| \leq |z-w|$ and $|w|-|z| \leq |z-w|$. Both come from triangle inequality.

3. Since $\langle z, w\rangle = z\bar{w} = (x+iy)(u-iv) = (ux+vy)+i(uy-vx)$,

$$\begin{aligned} \mathrm{Re}\langle z, w\rangle &= ux+vy = (x,y)\cdot(u,v), \\ \overline{\langle w, z\rangle} &= \overline{w\bar{z}} = \bar{w}z = \langle z, w\rangle, \\ \langle z, z\rangle &= z\bar{z} = |z|^2 = x^2+y^2 \geq 0. \end{aligned}$$

Equality on the last line holds if and only if x and y are 0.

4. We can use the identity $|z|^2 = z\bar{z}$. For every $z, w \in \mathbb{C}$,

$$\begin{aligned} |z \pm w|^2 &= (z \pm w)(\bar{z} \pm \bar{w}) = z\bar{z} + w\bar{w} \pm z\bar{w} \pm w\bar{z} \\ &= |z|^2 + |w|^2 \pm (z\bar{w} + \overline{z\bar{w}}) = |z|^2 + |w|^2 \pm 2\mathrm{Re}(z\bar{w}). \end{aligned}$$

Then,

$$\begin{aligned} |z+w|^2 - |z-w|^2 &= \big(|z|^2+|w|^2+2\mathrm{Re}(z\bar{w})\big) - \big(|z|^2+|w|^2-2\mathrm{Re}(z\bar{w})\big) \\ &= 4\mathrm{Re}(z\bar{w}). \end{aligned}$$

5. Since $w \neq 1$ and $w^n - 1 = 0$,

$$1 + w + \ldots w^{n-1} = \frac{w^n-1}{w-1} = 0.$$

Take the real value of the equation above to get:

$$\cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{4\pi}{n}\right) + \ldots + \cos\left(\frac{2(n-1)\pi}{n}\right) = 0.$$

6. Since $|-8+8i\sqrt{3}| = 16$ and $\mathrm{Arg}(-8+8i\sqrt{3}) = \frac{2\pi}{3}$, then $z^4 = 2^4 e^{2\pi i(3k+1)/3}$ for any integer k. Then,

$$z = 2e^{\pi i(3k+1)/6}, \text{ for } k \in \{0,1,2,3\}.$$

Simplifying the expression, the roots are $\pm(\sqrt{3}+i)$ and $\pm(-1+i\sqrt{3})$.

7. Let $\alpha = \cos(\frac{2\pi}{5})$ and $w = e^{2\pi i/5}$.

    (a) $\alpha = \mathrm{Re}(w) = \frac{w+\bar{w}}{2} = \frac{w+w^4}{2}$ and $\alpha^2 = \frac{w^2+w^3+2}{4}$.

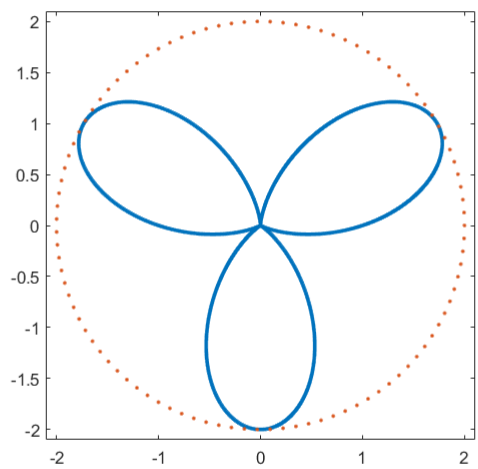
    (b) This is 0 from exercise 5.

    (c) From part (b), we can pick $p = 4$, $q = 2$, and $r = -1$.

    (d) By quadratic formula, $\alpha = \frac{-1\pm\sqrt{5}}{4}$. We pick the $+$ sign since $\alpha > 0$.

8. I will only sketch (a); the rest should be fairly easy to illustrate.

    (a) It's the boundary of a 'flower' with three petals of maximum radius 2 centered at 0. See below.



    (b) When $z = x + iy$, the equation can be rewritten as $x^2 - y^2 = 1$, a hyperbola.

    (c) When $z = x + iy$, multiplying both top and bottom with the complex conjugate $\bar{z} - i$ gives you:

$$\frac{z-i}{z+i} = \frac{x^2+y^2-1-2ix}{x^2+(y+1)^2}.$$

    The denominator is always positive unless $z = -i$, on which the fraction is undefined. The real value is negative exactly when $x^2 + y^2 - 1 < 0$. This gives us the unit disk $\mathbb{D} = \{|z| < 1\}$.

    (d) The imaginary part of the fraction above is 0 when $-2ix = 0$. This gives us the set of purely imaginary numbers $\{iy \mid y \in \mathbb{R}\backslash\{-1\}\}$. We exclude $-i$ since the fractional expression is not defined at that point.

(e) When $z = x + iy$, $\text{Im}z^2 < 0$ exactly when $xy < 0$ and $\text{Im}(z + 1 + i)^2 < 0$ exactly when $(x+1)(y+1) < 0$. This is the set $\{x + iy \mid x < -1, y > 0\} \cup \{x + iy \mid x > 0, y < -1\}$.

9. For each of the five sets in Exercise 9 above, determine whether or not they are open, closed, bounded, connected, simply connected or multiply connected.

   (a) not open, compact, connected, multiply connected.

   (b) not open, closed, unbounded and disconnected.

   (c) open, not closed, bounded, simply connected.

   (d) not open, not closed, unbounded, disconnected.

   (e) open, not closed, unbounded, disconnected.

10. Refer to the definition of convergence of complex numbers.

11. No. Let $r_n = \frac{1}{n}$, $r = 0$, $\theta_n = (-1)^n \frac{\pi}{2}$, and $\theta = 0$. Then, $r_n e^{i\theta_n} = \frac{(-1)^n i}{n}$ converges to $re^{i\theta} = 0$. Even though $r_n \to r$, unfornutately $\theta_n \not\to \theta$.

12. It is easier when f is rewritten as $f(z) = z^2$. Then, for any $a \in \mathbb{C}$, the derivative always exists:

$$f'(a) = \lim_{z \to a} \frac{z^2 - a^2}{z - a} = \lim_{z \to a} z + a = 2a.$$

Alternatively, you may show that Cauchy Riemann equations hold throughout $\mathbb{C}$.

13. Upon computing the derivative at an arbitrary point $a \in \mathbb{C}$,

$$\lim_{z \to 0} \frac{|a + z|^2 - |a|^2}{z} = \lim_{z \to 0} \frac{z\bar{z} + \bar{a}z + a\bar{z}}{z} = \lim_{z \to 0} \bar{z} + \bar{a} + a\frac{\bar{z}}{z} = \bar{a} + a \lim_{z \to 0} \frac{\bar{z}}{z}.$$

When $a = 0$, it is clear that the limit above exists and is equal to 0. However, when $a \neq 0$, the limit does not exist since $\lim_{z \to 0} \frac{\bar{z}}{z}$ does not exist. Since $|z|^2$ is only complex differentiable at one point, it is not holomorphic on any domain.