Applied Complex Analysis

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Introduction

Denote the set of complex numbers by

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$$

where $i = \sqrt{-1}$ is defined such that $i^2 = -1$.

Complex analysis is the study of functions of a complex variable. In the first few chapters, we shall explore some introductory concepts, such as basic properties of complex numbers and continuity of complex-valued functions. The main emphasis is the concept of *holomorphic* functions, i.e. complex-valued functions which are differentiable in a complex sense, and the many applications of their somewhat magical properties. I used the word 'magical' because holomorphicity is such a rigid condition that many of the results you will see are somewhat unintuitive yet true.

We will start with some motivation. Basic algebra tells us that the number of roots of a polynomial with real coefficients is at most its degree. For example, $x^2 + c$ has two real roots if $c < 0$, one root if $c = 0$, and no roots if $c > 0$. Introducing the imaginary number i provides us with a more elegant way of formulating this idea.

Theorem (Fundamental Theorem of Algebra)**.** *The field $\mathbb{C}$ is algebraically closed, that is, any polynomial with coefficients in $\mathbb{C}$ of degree $d > 1$ has exactly d roots in $\mathbb{C}$, counting multiplicity.*

Many initial attempts of proving the theorem by prominent mathematicians D'Alembert, Euler, Gauss, Lagrange, and Laplace in 1700s were incomplete. In 1806, a Swiss accountant, Parisian bookstore manager and 'amateur' mathematician Jean-Robert Argand completed D'Alembert's ideas and hence became the first person to rigorously prove the fundamental theorem of algebra. We will in fact use properties of holomorphic functions to give 3 different proofs of the theorem, including D'Alembert and Argand's approach.

It is difficult to list the many applications of the fundamental theorem of algebra. The main idea is that the field of complex numbers is the perfect setting to solve equations!

A direct consequence in linear algebra is that every square matrix with entries in $\mathbb{C}$ admits an eigenvalue. When a 2×2 matrix has imaginary eigenvalues, it acts as a rotation of the plane rather than expansion or contraction in certain directions. In the study of continuous dynamics arising from mechanical systems, it is common to use complex numbers in order to capture oscillations in the system.

One of the direct applications of the study of holomorphic functions is contour integration. The integral of a complex function along a closed path is not dependent on the path itself but rather on certain values called *residues* of the function's singularities. This means that it is often easier to integrate a real function of a real variable by converting it into a problem involving a contour integral in the complex plane.

Fourier series and Fourier transforms are useful in decomposing functions into its frequency components. (Think of decomposing nice functions as a sum or an integral of different sine and cosine waves.) Fourier analysis can be easily formulated via complex analysis, and it comes up everywhere: in differential equations, probability, quantum mechanics, signal processing, etc.

Mechanical and electrical engineers as well as computer musicians also encounter complex variables in electrical circuits with alternating current. Digital filters are designed by looking at the locations of *zeros* and *poles* of rational functions called *transfer functions*, which essentially model a device's inputs and outputs.

Iterations of holomorphic functions have long been known to have many applications. Complex polynomials, for example, can be used to model the population of rabbits over time. Powerful basic results in complex analysis, many of which do not apply to generic real differentiable functions, make up one of the many reasons why the study of iterations of holomorphic functions (holomorphic dynamics) is very well developed compared to the other branches of the field of dynamical systems.

Conformal functions are holomorphic functions with strictly non-zero derivative. Such functions have an amazing geometric property of angle preservation at every point and are useful in transforming regions with complicated boundary to those of a much nicer shape (square, disk, etc). You may, for example, want to transform a mechanical problem on a complicated domain into an equivalent problem on a circular disk. In cartography, conformal maps are useful in creating a world map as well as local nautical charts using Mercator and stereographic projections. More recently, conformal functions are applied to the surface of the human brain for brain development study and diagnosis of Alzheimer's disease and schizophrenia.

Chapter 1

Complex Numbers

In this chapter, we will go through the basic algebraic and geometric properties of complex numbers.

1.1 The Algebra of $\mathbb{C}$

The set $\mathbb{C}$ is equipped with the usual arithmetic operators, namely:

- addition $+$: $(x+iy)+(a+ib)=(x+a)+i(y+b)$,
- multiplication $\times$: $(x+iy)\times(a+ib)=(xa-yb)+i(xb+ya)$.

Let's denote by $\mathbb{C}^*$ the set of non-zero complex numbers $\mathbb{C}\backslash\{0\}$. This set is equipped with an additional operator:

- inversion of a non-zero number: $(x+iy)^{-1}=\frac{x-iy}{x^2+y^2}$.

Similar to $\mathbb{R}$, the set of complex numbers $\mathbb{C}$ is a *field*; it satisfies the following axioms:

1. $(\mathbb{C},+)$ is an abelian group:
   - $+$ is associative and commutative,
   - 0 is the identity element of $+$, i.e. $z+0=z$ for all $z\in\mathbb{C}$,
   - Additive inverses exist, i.e. $z+(-z)=0$ for all $z\in\mathbb{C}$;
2. $(\mathbb{C}^*,\times)$ is an abelian group:
   - $\times$ is associative and commutative,
   - 1 is the identity element of $\times$, i.e. $z\times 1=z$ for all $z\in\mathbb{C}^*$,

- Multiplicative inverses exist, i.e. $z \times z^{-1} = 1$ for all $z \in \mathbb{C}^*$;

3. $\times$ is distributive over $+$.

The set $\mathbb{C}$ of complex numbers can be identified with the real vector space $\mathbb{R}^2$ by the vector space isomorphism:

$$\begin{aligned} \mathbb{C} &\to \mathbb{R}^2, \\ z &\mapsto (\operatorname{Re} z, \operatorname{Im} z), \\ x + iy &\mapsto (x, y). \end{aligned}$$

Unlike $\mathbb{C}$, the real plane $\mathbb{R}^2$ is only equipped with addition operator $+$ but not a natural multiplication operator $\times$. Nonetheless, the mapping above allows us to geometrically represent complex numbers as points on the plane. This is typically known as *Argand diagram*.

1.2 The Geometry of $\mathbb{C}$

Every complex number $z = x + iy$ comes with a unique *real part* x and *imaginary part* y. We shall denote them as follows:

$$\operatorname{Re} z = x, \qquad \operatorname{Im} z = y.$$

Geometrically, Re and Im can be thought as functions $\mathbb{C} \to \mathbb{R}$ acting as projections onto the real and imaginary axes respectively.

The *complex conjugate* $\bar{z}$ of a complex number $z = x + iy$ is $\bar{z} = x - iy$. Geometrically, the operation $z \mapsto \bar{z}$ is a reflection over the real axis. The following identity can be thought of as a change of basis from (x, y) to $(z, \bar{z})$.

Proposition 1.1. *For any $z \in \mathbb{C}$, $\operatorname{Re} z = \frac{z+\bar{z}}{2}$ and $\operatorname{Im} z = \frac{z-\bar{z}}{2i}$.*

Another straightforward algebraic exercise also gives us the following basic properties.

Proposition 1.2. *For any $z, w \in \mathbb{C}$, $\overline{z+w} = \bar{z} + \bar{w}$ and $\overline{zw} = \bar{z}\bar{w}$. If $z \neq 0$, $\overline{z^{-1}} = \bar{z}^{-1}$.*

The *absolute value / modulus* of a complex number $z = x + iy$ is

$$|z| = \sqrt{x^2 + y^2}.$$

Phytagoras' theorem indicates that geometrically the modulus $|z|$ of z is equal to the distance between 0 and z.

Proposition 1.3. *For any $z, w \in \mathbb{C}$,*

- $|zw| = |z||w|$,
- $z\bar{z} = |z|^2$,
- $|z + w| \leq |z| + |w|$ *(Triangle inequality).*

The *argument* of z, $\arg(z)$, is defined to be the counterclockwise angle (measured in radians) subtended by the positive real axis $\mathbb{R}^+$ and the line segment joining 0 and z. See figure 1.1.

Notice that arg is a multivalued function. For example, both π and 3π are arguments of i. We can refine this by defining the *principal argument* of z, $\mathrm{Arg}(z)$, to be the unique argument of z lying in $(-\pi, \pi]$.

Remark. The interval $[0, 2\pi)$ is also often chosen to be the codomain of the principal argument.

Proposition 1.4. *For any $z, w \in \mathbb{C}^*$,*

- $\arg(zw) = \arg(z) + \arg(w)$,
- $\mathrm{Arg}(zw) = \mathrm{Arg}(z) + \mathrm{Arg}(w)$ *mod* 2π.

Example 1. Let $z = 1 + i$ and $w = -1 + \sqrt{3}i$. The modulus and arguments of z and w are:

$$|z| = \sqrt{2}, \quad |w| = 2, \quad \arg(z) = \frac{\pi}{4}, \quad \arg w = \frac{2\pi}{3}.$$

Then, the modulus and argument of $(1 + i)(-1 + \sqrt{3}i)$ are $2\sqrt{2}$ and $\frac{11\pi}{12}$ respectively.

For any non-zero complex number $z = x + iy$, if $r = |z|$ and $\theta = \mathrm{Arg}(z)$, then basic trigonometry gives us the following change of variables:

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

The expression $z = r(\cos\theta + i\sin\theta)$ from above is the *polar form* of z.

Theorem 1.5 (Euler's formula)**.** *For any θ, $e^{i\theta} = \cos\theta + i\sin\theta$.*

Proof. We will give two different proofs of the result - one with differential equations, and another with Maclaurin series. The expression $e^{i\theta}$ is a non-zero complex number, so there is a unique $r > 0$ and $\hat{\theta}$ such that

$$e^{i\theta} = r(\cos\hat{\theta} + i\sin\hat{\theta}). \tag{1.1}$$

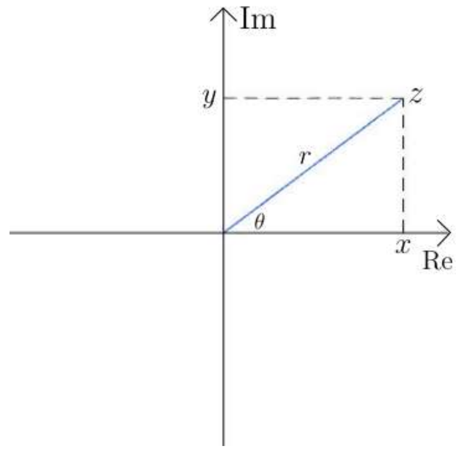

Figure 1.1: A point $z = x + iy = re^{i\theta}$ on the Argand diagram

Here, r and $\hat{\theta}$ are functions of θ. When $\theta = 0$, $r(0) = 1$ and $\hat{\theta}(0) = 0$. Differentiating (1.1) with respect to θ, we obtain

$$\begin{aligned} ie^{i\theta} &= \frac{dr}{d\theta}(\cos\hat{\theta} + i\sin\hat{\theta}) + r\frac{d\hat{\theta}}{d\theta}(-\sin\hat{\theta} + i\cos\hat{\theta}) \\ &= \frac{dr}{d\theta}\frac{e^{i\theta}}{r} + i\frac{d\hat{\theta}}{d\theta}e^{i\theta} \\ &= \left(\frac{dr}{d\theta} + i\frac{d\hat{\theta}}{d\theta}\right)e^{i\theta}, \end{aligned}$$

where the second equality above is obtained from (1.1). From above, we see that $\frac{dr}{d\theta} = 0$ and $\frac{d\hat{\theta}}{d\theta} = 1$. By our initial conditions, we obtain $r(\theta) \equiv 1$ and $\hat{\theta} \equiv \theta$.

Alternatively, recall the following Maclaurin series: $e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$. Using the fact that $i^n = (-1)^{n/2}$ if n is even, and $i^n = (-1)^{n-1/2}i$ if n is odd,

$$\begin{aligned} e^{i\theta} &= \sum_{\text{even } n} \frac{(-1)^n\theta^n}{n!} + \sum_{\text{odd } n} \frac{(-1)^{n-1/2}\theta^n}{n!} \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right) = \cos\theta + i\sin\theta. \end{aligned}$$

□

Example 2. When $\theta = \pi$, we have Euler's identity: $e^{i\pi} = -1$.

The polar form of a complex number z can alternatively be written in the form of $z = re^{i\theta}$. This expression is particularly useful when performing multiplication of complex numbers as we can use the laws of exponent. One particular instance is the following.

Theorem 1.6 (De Moivre's Theorem)**.** *For any θ and integer $n \in \mathbb{Z}$,*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Example 3. To compute and simplify $\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^{10}$, we can use De Moivre's theorem. The term inside the bracket is essentially $\cos\theta + i\sin\theta$ where $\theta = \frac{2\pi}{3}$. Then,

$$\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^{10} = \cos\left(\frac{20\pi}{3}\right) + i\sin\left(\frac{20\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}i}{2}.$$

1.3 Complex Roots

Consider a complex number z_0 and a positive integer n. A complex number w satisfying $w^n = z_0$ is called an n^{th} *root* of z_0.

Suppose $z_0 = 0$. Regardless of n, there is only one root of 0, which is 0 itself. This is due to the fact that $\mathbb{C}$ is an integral domain, i.e. for any two complex numbers z_1 and z_2, if $z_1 z_2 = 0$ then either $z_1 = 0$ or $z_2 = 0$.

Suppose $z \neq 0$ now, then surely any root w is also non-zero. Using their polar forms $z = re^{i\theta}$ and $w = se^{it}$, then the equation becomes:

$$s^n e^{int} = re^{i\theta}$$

Considering the modulus and the argument independently, we obtain two real equations $s^n = r$ and $nt = \theta \mod 2\pi$. There are therefore n different solutions to w:

$$w_k = r^{1/n} e^{i(\theta + 2\pi k)/n}, \qquad k \in \{0, 1, \ldots n-1\}.$$

In the expression above, w_0 is called the *principal root* of z_0. On the complex plane, these roots are evenly spaced on the circle $\{z \in \mathbb{C} \,|\, |z| = r^{1/n}\}$ of radius $r^{1/n}$ centered at the origin.

When $z_0 = 1$, the n^{th} roots of 1 are called the n^{th} *roots of unity*. They all lie on the unit circle and are of the form $e^{2\pi ik/n}$, where $k \in \{0, 1, \ldots n-1\}$.

Example 4. The 3rd roots of unity are 1, $e^{2\pi i/3}$ and $e^{4\pi i/3}$. The Cartesian forms of these roots are 1, $\frac{-1+i\sqrt{3}}{2}$, and $\frac{-1-i\sqrt{3}}{2}$.

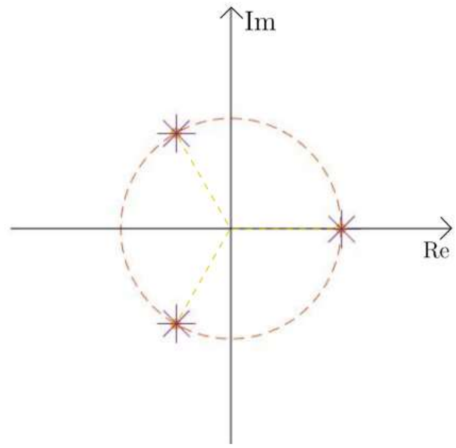

Figure 1.2: 3^{rd} roots of unity

1.4 The Topology of $\mathbb{C}$

An *open disk* of radius $r > 0$ centred at a complex number $a \in \mathbb{C}$ is a subset of $\mathbb{C}$ of the form:

$$\mathbb{D}(z, r) = \{z \in \mathbb{C} \mid |z - a| < r\}.$$

The boundary of this disk is a circle of of radius $r > 0$ centred at a, denoted with a partial sign in front:

$$C(z, r) = \partial\mathbb{D}(z, r) = \{z \in \mathbb{C} \mid |z - a| = r\}.$$

If we include the boundary, we obtain a *closed disk* typically denoted with an overline:

$$\overline{\mathbb{D}(z, r)} = \{z \in \mathbb{C} \mid |z - a| \leq r\}.$$

Example 5. Let's consider the sets

$$A = \{re^{i\theta} \mid r = \sin\theta, \theta \in \mathbb{R}\}, \qquad B = \{re^{i\theta} \mid 0 < r < \sin\theta, \theta \in \mathbb{R}\}.$$

If $z = x + iy$ lies in A, then $x = \sin\theta\cos\theta$ and $y = \sin^2\theta$ for some θ. By double angle formulas,

$$\sin^2(2\theta) + \cos^2(2\theta) = (2x)^2 + (1 - 2y)^2 = 1.$$

This equation represents a circle of radius $\frac{1}{2}$ centered at $\frac{i}{2}$. Therefore, A is the circle $C(\frac{i}{2}, \frac{1}{2})$. For points z on the set B, we only need to consider the case when $\sin\theta > 0$, or principally when $0 < \theta < \pi$. The set B is the open disk $\overline{\mathbb{D}(\frac{i}{2}, \frac{1}{2})}$.

The geometric and topological properties of the complex plane $\mathbb{C}$ are essentially the same as those of the real plane $\mathbb{R}^2$ since we have the obvious identification $x + iy \mapsto (x, y)$. We will give a brief introduction of necessary topological terminology that we will use in the next few chapters.

Definition 1. A subset $S \subset \mathbb{C}$ is:

- *open* if for every $s \in S$, there is some $r > 0$ such that $\mathbb{D}(s, r) \subset S$,
- *closed* if its complement $\mathbb{C}\backslash S$ is open,
- *bounded* if there is some $r > 0$ where $S \subset \mathbb{D}(0, r)$,
- *compact* if S is closed and bounded.

Example 6. Below are some subsets of $\mathbb{C}$ which we will commonly encounter.

1. The empty set $\emptyset$ is trivially open and compact.
2. The complex plane $\mathbb{C}$ is both open and closed, but not bounded.
3. The punctured plane $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ is open, but not closed nor bounded.
4. The unit disk $\mathbb{D} := \mathbb{D}(0, 1)$ is open and bounded, but not closed.
5. The closed unit disk $\overline{\mathbb{D}}$ and its boundary $\partial\mathbb{D}$ are compact.
6. The real axis $\mathbb{R}$ is closed and unbounded.

Definition 2. An open/closed set $S \subset \mathbb{C}$ is:

- *connected* if S cannot be expressed as a disjoint union of two open/closed non-empty subsets of $\mathbb{C}$,
- *simply connected* if it is connected and it has no "holes", i.e. the complement $\mathbb{C}\backslash S$ has no bounded connected component,
- *multiply connected* if it is connected but not simply connected.

We say that S is a *domain* if it is a non-empty open connected subset of $\mathbb{C}$.

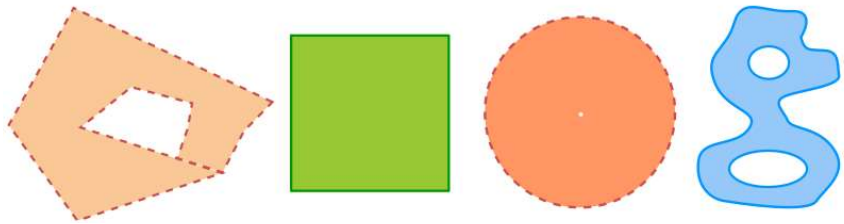

Figure 1.3: Four connected subsets of $\mathbb{C}$. Solid boundary lines are included in the colored set, whereas dashed boundary lines are not included. The first (from the left) is a simply connected domain. The second is closed and simply connected. The third is a punctured disk, which is a multiply connected domain. The last is closed and multiply connected.

Example 7.

1. $\emptyset$, $\mathbb{C}$, $\mathbb{D}$, $\overline{\mathbb{D}}$ and $\mathbb{R}$ are simply connected.

2. The punctured plane $\mathbb{C}^*$, the punctured unit disk $\mathbb{D}^* := \mathbb{D}\backslash\{0\}$, and the unit circle $\partial\mathbb{D}$ are multiply connected.

3. The annulus $\{z \in \mathbb{C} \mid r < |z| < R\}$ of inner radius r and outer radius R is multiply connected.

Example 8. Consider the set $S = \{z \in \mathbb{C} \mid |\mathrm{Im}\left(\frac{1}{z}\right)| < 1\}$. In polar form $z = re^{i\theta}$, the inequality becomes

$$|\mathrm{Im}(r^{-1}e^{-i\theta})| = |r^{-1}\sin(-\theta)| = r^{-1}|\sin\theta| < 1$$

Therefore, $|\sin\theta| < r$. Similar to Example 5, this represents all the complex numbers lying outside two closed disks $\overline{\mathbb{D}(\pm\frac{i}{2}, \frac{1}{2})}$. The set S is illustrated in Figure 1.4; it is open, unbounded, and multiply connected.

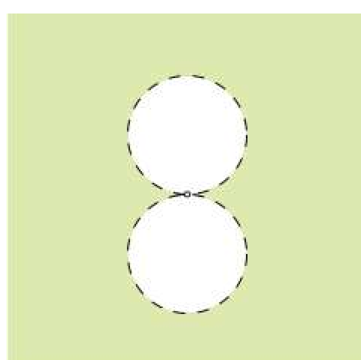

Figure 1.4: The set S.

Short Quiz 1

1. Simplify $\frac{1+i}{i-1}$.
2. Find the modulus of $(3+4i)(-4+3i)$.
3. Find the argument(s) of $\arg(-1+i)$.
4. Express $2e^{-2\pi i/3}$ in the form of $x+iy$.
5. What are the 3^{th} roots of $8i$?
6. Find the value of $(1+i)^6$.
7. If $z \neq 0$, express $\operatorname{Im}\left(\frac{z}{z+\bar{z}}\right)$ in terms of $\theta = \operatorname{Arg}(z)$.

Consider the following subsets:

$$A = \mathbb{D}(2,2) \cup \mathbb{D}(-2,2), \qquad B = \overline{\mathbb{D}(i,1)} \cup \overline{\mathbb{D}(-i,1)},$$
$$C = \mathbb{D}(2,1) \cup \mathbb{D}(-2,1), \qquad D = C(i,1) \cup \overline{\mathbb{D}(-i,1)}.$$

8. Identify subsets which are open.
9. Identify subsets which are connected.
10. Identify subsets which are simply connected.

Answers: 1. $-i$, 2. 25, 3. $\frac{3\pi}{4} + 2\pi k$, 4. $-1 - i\sqrt{3}$, 5. $\pm\sqrt{3} + i$ & $-2i$, 6. $-8i$, 7. $\frac{1}{2}\tan\theta$, 8. A and C, 9. B and D, 10. B.

Chapter 2

Complex Functions

2.1 Convergence and Continuity

Definition 3. A sequence of complex numbers $\{z_n\}_{n\in\mathbb{N}}$ *converges* to a *limit* z if and only if:

$$\text{for all } \epsilon > 0, \text{ there exists } N > 0 \text{ such that for all } n \geq N, |z_n - z| < \epsilon.$$

Convergence of a sequence z_n to z can be denoted by $z_n \to z$, $|z_n - z| \to 0$, or $\lim_{n\to\infty} z_n = z$.

Proposition 2.1. *The limit of a convergent sequence is unique.*

Proof. Suppose for a contradiction that there are distinct limits $z \neq w$ of a sequence z_n. Let $\epsilon = \frac{1}{2}|z-w| > 0$, then for all sufficiently high n, $|z_n - z| < \epsilon$ and $|z_n - w| < \epsilon$. However, by triangle inequality,

$$|z - w| \leq |z - z_n| + |z_n - w| < 2\epsilon = |z - w|.$$

We then have a contradiction. $\square$

Theorem 2.2. *If $z_n \to z$ and $w_n \to w$, then*

- $z_n + w_n \to z + w$,
- $z_n w_n \to zw$.

Proof. Let's pick $\epsilon > 0$. There are some high $N_1, N_2 \in \mathbb{N}$ such that $|z_n - z| < \epsilon/2$ if $n \geq N_1$, and $|w_n - w| < \epsilon/2$ if $n \geq N_2$. By triangle inequality, when $n \geq \max\{N_1, N_2\}$,

$$|z_n + w_n - z - w| \leq |z_n - z| + |w_n - w| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $z_n + w_n \to z + w$.

Set $M = \max\{|w_1|, \dots |w_{N_2}|, |w| + \epsilon\}$. The sequence $\{w_n\}_{n\in\mathbb{N}}$ is bounded because we have the following inclusions:

$$\{w_n\}_{n\in\mathbb{N}} \subset \{w_1, \dots w_{N_2}\} \cup \mathbb{D}(w, \epsilon/2) \subset \mathbb{D}(0, M).$$

There are some $N_3, N_4 \in \mathbb{N}$ such that $|z_n - z| < \epsilon/2M$ if $n \geq N_3$, and $|w_n - w| < \epsilon/2\max\{1, |z|\}$ if $n \geq N_4$. Then, when $n \geq \max\{N_3, N_4\}$,

$$\begin{aligned} |z_n w_n - zw| &\leq |z_n w_n - z w_n| + |z w_n - zw| = |w_n||z_n - z| + |z||w_n - w| \\ &< M \cdot \frac{\epsilon}{2M} + |z| \cdot \frac{\epsilon}{2\max\{1, |z|\}} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This shows that $z_n w_n \to zw$. ☐

In particular, a sequence of complex numbers converges exactly when the real parts and the imaginary parts converge respectively.

Corollary 2.3. *$x_n + iy_n \to x + iy$ if and only if $x_n \to x$ and $y_n \to y$.*

Proof. The $\Leftarrow$ direction is immediate from the previous proposition. The $\Rightarrow$ direction comes from the following inequality:

$$\max\{|x_n - x|, |y_n - y|\} \leq \sqrt{|x_n - x|^2 + |y_n - y|^2} = |x_n + iy_n - x - iy|.$$

As $|x_n + iy_n - x - iy| \to 0$, sandwich rule forces both $|x_n - x|$ and $|y_n - y|$ to converge to 0 too. ☐

Definition 4. Let U and V be non-empty subsets of $\mathbb{C}$. A function $f : U \to V$ is *continuous at $a \in U$* if and only if:

$$\begin{aligned} &\text{for all } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that} \\ &\text{if } z \in U \cap \mathbb{D}(a, \delta), \text{ then } f(z) \in \mathbb{D}(f(a), \epsilon). \end{aligned}$$

We say that f is *continuous* if it is continuous at every point in U.

Proposition 2.4. *A function $f : U \to V$ is continuous at $a \in U$ if and only if for any sequence z_n in U, if $z_n \to a$, then $f(z_n) \to f(a)$.*

Proof. Let f be continuous at a and pick the pair (ϵ, δ) in the definition of continuity. Suppose $z_n \to a$, then there is some high $N \in \mathbb{N}$ such that if $n \geq N$, $|z_n - a| < \delta$. By continuity, if $n \geq N$, $|f(z_n) - f(a)| < \epsilon$. Therefore, $f(z_n) \to f(a)$.

Suppose for any sequence z_n converging to a, $f(z_n) \to f(a)$. Suppose for a contradiction that f is not continuous at a, then there is some $\epsilon > 0$ and sequence of points $z_n \in U \cap \mathbb{D}(a, \frac{1}{n})$ for $n \in \mathbb{N}$ such that $|f(z_n) - f(a)| \geq \epsilon$. Since $|z_n - a| < \frac{1}{n} \to 0$ as $n \to \infty$, then $z_n \to a$. The assumption implies that $f(z_n) \to f(a)$, but this cannot happen because $f(z_n)$ is always at least ϵ away from $f(a)$. This gives the contradiction. ☐

The statement above gives an equivalent way of defining continuity at a point. A shorter way of saying this is

$$\lim_{z \to a} f(z) = f(a).$$

The following is a direct consequence of 2.2.

Proposition 2.5. *Let $f, g : U \to V$ and $h : V \to W$ be continuous functions, then the sum $f+g$, the product $f \cdot g$ and the composition $h \circ f$ are continuous.*

Example 9. Constant functions $f(z) = a$ are trivially continuous. The identity function $\mathrm{Id}(z) = z$ is also continuous. By taking products and sums, we can inductively obtain that every complex polynomial $a_d z^d + a_{d-1} z^{d-1} + \ldots a_1 z + a_0$ is continuous.

Example 10. The modulus function $m(z) = |z|$ is a continuous function on $\mathbb{C}$. Indeed, for any $a \in \mathbb{C}$, if $z \to a$, then by triangle inequality,

$$\big|m(z) - m(a)\big| \leq \big||z| - |a|\big| \leq |z - a| \to 0.$$

By sandwich rule, $m(z) \to m(a)$ too.

By the previous proposition, if $f(z)$ is a continuous function on a subset of $\mathbb{C}$, then so is the composition $|f(z)|$.

Example 11. The functions Re and Im are continuous. (Refer to Corollary 2.3.) Since $\bar{z} = \mathrm{Re}(z) - i\,\mathrm{Im}(z)$, complex conjugation is also continuous.

Continuous functions behave nicely on compact subsets of $\mathbb{C}$.

Theorem 2.6. *Let $f : K \to V$ be a continuous function and K be a compact subset of $\mathbb{C}$. Then, f attains a maximum and a minimum on K, i.e. there are points $a, b \in K$ where $|f(a)| \leq |f(z)| \leq |f(b)|$ for all $z \in K$.*

The theorem above is a consequence of a result from topology. In particular, the image $|f(K)|$ of a compact set K under a continuous function $|f|$ is compact, and any compact subset of $\mathbb{R}$ contains maximum and minimum points.

2.2 Holomorphic Functions

We can define differentiability of complex-valued functions the same way as we define that of functions of one real variable. However, we will emphasise in the next few sections that complex differentiability is actually a much more rigid notion than the usual multivariable real differentiability.

Definition 5. Let $U, V \subset \mathbb{C}$ be open and non-empty. A complex function f is *(complex) differentiable* at a point a if and only if the following limit exists:

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

If so, then $f'(a)$ is called the *(complex) derivative* of f at a. The function f is said to be *holomorphic* on U if it is holomorphic at every point in U, and *entire* if additionally $U = \mathbb{C}$.

Remark. The term "analytic" and "complex differentiable" are often used interchangeably with "holomorphic".

Example 12.

1. Constant functions $f(z) = a$ are entire with derivative 0 everywhere.

2. The identity function $\mathrm{Id}(z) = z$ is an entire function and its derivative is 1 everyhere.

3. The inversion function $\tau(z) = 1/z$ is is holomorphic on $\mathbb{C}^*$ and its derivative is $-z^{-2}$. Indeed, if we choose any angle θ, then if $z = re^{i\theta} + a$,

$$\begin{aligned}\lim_{z \to a} \frac{\tau(z) - \tau(a)}{z - a} &= \lim_{r \to 0} \frac{\frac{1}{re^{i\theta}+a} - \frac{1}{a}}{(re^{i\theta} + a) - a} = \lim_{r \to 0} \frac{\frac{-re^{i\theta}}{a(re^{i\theta}+a)}}{re^{i\theta}} \\ &= \lim_{r \to 0} -\frac{1}{a(re^{i\theta} + a)} = -\frac{1}{a^2}.\end{aligned}$$

4. Complex conjugation $f(z) = \bar{z}$ has no derivative at any point. If we choose any angle θ, then using $z = re^{i\theta} + a$,

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{r \to 0} \frac{(re^{-i\theta} + \bar{a}) - \bar{a}}{(re^{i\theta} + a) - a} = e^{-2i\theta},$$

but the value of this limit is not the same if we choose different values of θ. For example, the limit is 1 when $\theta = 0$, but it is -1 if $\theta = \frac{\pi}{2}$.

Proposition 2.7. *Every holomorphic function is continuous.*

Proof. Let $f : U \to V$ be holomorphic. If $a \in U$,

$$\begin{aligned}\lim_{z \to a} f(z) - f(a) &= \lim_{z \to a} (z - a) \frac{f(z) - f(a)}{z - a} \\ &= \lim_{z \to a} (z - a) \cdot \lim_{z \to a} \frac{f(z) - f(a)}{z - a} \\ &= 0 \cdot f'(a) = 0,\end{aligned}$$

where the second equality follows from Theorem 2.2. Therefore, f is continuous at a. As a is arbitrary, f is continuous on U. $\square$

The rules for differentiation of complex-valued functions is more or less the same as those of functions of one real variable.

Proposition 2.8. *Let $f, g : U \to V$ and $h : V \to W$ be holomorphic. Then,*

(a) the sum $f+g$ is holomorphic and $(f+g)'(z) = f'(z) + g'(z)$,

1. *the product $f \cdot g$ is holomorphic and $(f \cdot g)'(z) = f'(z)g(z) + f(z)g'(z)$,*
2. *the composition $h \circ f$ is holomorphic and $(h \circ f)'(z) = h'(f(z))f'(z)$.*

Proof. (a) follows immediately from Proposition 2.5. For (b),

$$\begin{aligned}(f \cdot g)'(a) &= \lim_{z \to a} \frac{f(z)g(z) - f(a)g(a)}{z-a} \\ &= \lim_{z \to a} \frac{f(z)\big(g(z) - g(a)\big)}{z-a} + \frac{g(a)\big(f(z) - f(a)\big)}{z-a} \\ &= f(a)g'(a) + g(a)f'(a).\end{aligned}$$

For (c), we use the fact that f is continuous:

$$\begin{aligned}(h \circ f)'(a) &= \lim_{z \to a} \frac{h(f(z)) - h(f(a))}{z-a} \\ &= \lim_{z \to a} \frac{h(f(z)) - h(f(a))}{f(z) - f(a)} \cdot \frac{f(z) - f(a)}{z-a} \\ &= \lim_{w \to f(a)} \frac{h(w) - h(f(a))}{w - f(a)} \cdot \lim_{z \to a} \frac{f(z) - f(a)}{z-a} \\ &= h'(f(a))f'(a).\end{aligned}$$

□

Example 13.

1. Every polynomial $p(z) = \sum_{n=0}^{d} a_n z^n$ with complex coefficients $a_n \in \mathbb{C}$ is an entire function. We can show this inductively by product rule above that every monomial $a_n z^n$ is holomorphic with derivative $a_n n z^{n-1}$ and by the addition rule, p is holomorphic.

2. Every rational function, i.e. a function of the form $f(z) = p(z)/q(z)$ where p and q are polynomials, is holomorphic on $\mathbb{C} \backslash \{z \in \mathbb{C} \,|\, q(z) = 0\}$.

Every complex function $f(z)$ admits a unique real-imaginary splitting $f(x+iy) = u(x,y) + iv(x,y)$, where u and v are real-valued functions defined on an open subset of $\mathbb{R}^2$ given by:

$$u(x,y) = \mathrm{Re} f(x+iy), \qquad v(x,y) = \mathrm{Im} f(x+iy).$$

We say that f is *(real) differentiable* if the partial derivatives of u and v with respect to x and y exist. When we are given u and v, we will see that holomorphic functions are precisely solutions of a system of partial differential equations.

Theorem 2.9 (Cauchy-Riemann Equations)**.** *Let $f = u + iv$ be a complex function on an open set $U \subset \mathbb{C}$. Then, f is holomorphic if and only if u and v are continuously differentiable and $u_x = v_y$ and $v_x = -u_y$.*

Proof. Let f be holomorphic at a point $a = a_1 + ia_2 \in U$, then

$$f'(a) = \lim_{h\to 0} \frac{u(a_1 + h_1, a_2 + h_2) - u(a_1, a_2)}{h} + i\frac{v(a_1 + h_1, a_2 + h_2) - v(a_1, a_2)}{h}.$$

The dummy variable $h = h_1 + ih_2$ can converge to zero in various directions, but we will only consider two cases. Suppose $h_2 = 0$, then

$$\begin{aligned} f'(a) &= \lim_{h_1\to 0} \frac{u(a_1 + h_1, a_2) - u(a)}{h_1} + i\frac{v(a_1 + h_1, a_2) - u(a)}{h_1} \\ &= u_x + iv_x. \end{aligned} \tag{2.1}$$

Similarly, if we consider the limit in the direction satisfying $h_1 = 0$,

$$\begin{aligned} f'(a) &= \lim_{ih_2\to 0} \frac{u(a_1, a_2 + h_2) - u(a)}{ih_2} + i\frac{v(a_1, a_2 + ih_2) - v(a)}{ih_2} \\ &= -iu_y + v_y. \end{aligned} \tag{2.2}$$

Comparing (2.1) and (2.2), we obtain $u_x = v_y$ and $v_x = -u_y$. Conversely, suppose u and v are continuously differentiable satisfying $u_x = v_y$ and $v_x = -u_y$. The Taylor series of u and v at a can be expressed as:

$$\begin{aligned} u(a_1 + h_1, a_2 + h_2) &= u(a_1, a_2) + u_x h_1 + u_y h_2 + |h|\psi(h), \\ v(a_1 + h_1, a_2 + h_2) &= v(a_1, a_2) + v_x h_1 + v_y h_2 + |h|\phi(h), \end{aligned}$$

for some functions ψ and ϕ where $\psi(h)$, $\phi(h) \to 0$ as $h \to 0$. All partial derivatives of u and v are evaluated at (a_1, a_2). Then,

$$\begin{aligned} &\lim_{h\to 0} \frac{u(a_1 + h_1, a_2 + h_2) - u(a_1, a_2)}{h} + i\frac{v(a_1 + h_1, a_2 + h_2) - v(a_1, a_2)}{h} \\ &= \lim_{h\to 0} \frac{[u_x h_1 + u_y h_2 + |h|\psi(h)] + i[v_x h_1 + v_y h_2 + |h|\phi(h)]}{h} \\ &= \lim_{h\to 0} \frac{(u_x + iv_x)(h_1 + ih_2) + |h|\big(\psi(h) + i\phi(h)\big)}{h} \\ &= u_x + iv_x. \end{aligned}$$

As the limit converges to $f' = u_x + iv_x$, f is indeed holomorphic. $\square$

Short Quiz 2

1. Does the sequence i^n converge as $n \to \infty$? If so, what is the limit?
2. Does the sequence $\left(\frac{i}{n}\right)^n$ converge as $n \to \infty$? If so, what is the limit?
3. Find the limit of the sequence $\frac{1}{n} + \left(1 + \frac{i}{n}\right)^n$.
4. Which of the following functions are continuous on $\mathbb{C}$?

$$z^3, \quad 1/z, \quad |z-2| + |z+2|, \quad arg(z)$$

5. At which of the following points is $Arg(z)$ continuous?

$$1, \quad i, \quad -1, \quad -i, \quad 0$$

Answers: 1. No, 2. Yes, to 0, 3. e^i, 4. z^3 and $|z-2|+|z+2|$, 5. $1, i, -i$.