Strong Version of Eremenko's Conjecture for Eremenko-Lyubich Class of Finite Order

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Abstract

This report explores basic properties of transcendental entire maps in Eremenko-Lyubich class, $\mathcal{B}$ and discusses a sufficient criterion for these maps to fulfil the strong version of Eremenko's Conjecture.

1 Introduction

A transcendental entire function is an entire function that cannot be expressed as a polynomial. The study of the dynamics of iterated transcendental entire maps is less developed compared to that of polynomials and rational maps. We wish to restrict our study to maps class $\mathcal{B}$ since their bounded singular points make them similar to polynomials. By doing so later results show that these maps still exhibit many interesting features.

In the last section, we will discuss the strong version of Eremenko's Conjecture from [Ere89], which states that every point on the escaping set $I(f) := \{z \in \mathbb{C} \mid \lim_{n\to\infty} f^n(z) = \infty\}$ can be joined by a path in $I(f)$ to ∞. While it is still an open problem in transcendental dynamics, we will prove that the conjecture holds for maps in class $\mathcal{B}$ with finite order.

This report is based on a summer project under the supervision of Dr. Alexandre de Zotti. We will use [Six17] as our main reference throughout the report.

2 Eremenko-Lyubich Class

2.1 Logarithmic Transform

Let f be a transcendental entire function and $z \in \mathbb{C}$. We say that z is a critical point of f if $f'(z) = 0$ and $w \in \mathbb{C}$ is a critical value of f if it is the image of a critical point. The set of all critical values is denoted by $CV(f)$. Moreover, z is a finite asymptotic value if there exists a path $\gamma : [0, \infty) \to \mathbb{C}$ such that $\gamma(t) \to z$ as $t \to \infty$. The set of all finite asymptotic values is denoted by $AV(f)$.

Define the set of all singular values as $S(f) = \overline{CV(f) \cup AV(f)}$. Eremenko and Lyubich aimed to work with functions whose singular values are not too complicated, thus they define the class $\mathcal{B}$ as the following:

$$\mathcal{B} = \{f : \mathbb{C} \to \mathbb{C} \mid f \text{ is transcendental entire and } S(f) \text{ is bounded.}\}$$

Common examples of class $\mathcal{B}$ functions are the exponential family $f_\lambda(z) = \lambda e^z, \lambda \in \mathbb{C}^*$, where $S(f_\lambda) = AV(f_\lambda) = 0$, as well as the cosine family $g_{a,b}(z) = ae^z + be^{-z}, a, b \in \mathbb{C}^*$, where in this case $S(g_{a,b}) = CV(g_{a,b}) = \{\pm 2\sqrt{ab}\}$.

The main technique we will use throughout the report is the logarithmic transform. We will firstly introduce a few preliminary results. More details can be found in [BFRG15]. Suppose that $f \in \mathcal{B}$ and D is a Jordan domain containing $\{0, f(0)\} \cup S(f)$ and $W = \mathbb{C} \backslash \overline{D}$.

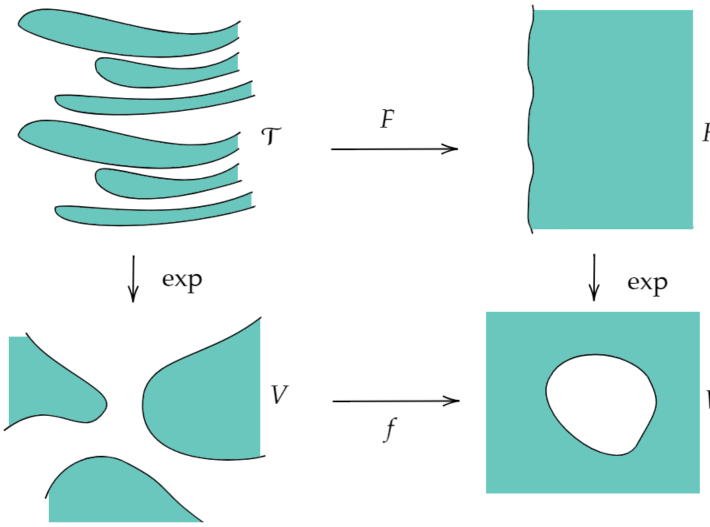
Proposition 2.1. *$f^{-1}(D)$ is simply connected.*

Proof. Pick any 2 points z, w in $f^{-1}(D)$ and any curve γ in $\mathbb{C}$ joining z and w. $f(\gamma)$ has finitely many pieces (say $M \in \mathbb{N}$) which connects $S(f)$ and D and on each of these pieces pick a point z_k for $k = 1, \dots M$ such that by setting $z_0 = f(z)$ and $z_{M+1} = f(w)$ each arc $[z_k, z+k+1]$ on $f(\gamma)$ is either contained in D but intersect $S(f)$, or intersect $\mathbb{C}\backslash D$ but not $S(f)$. Obtain a new curve Γ from $f(\gamma)$ by replacing each arc $[z_k, z+k+1]$ with one that is homotopic to $[z_k, z+k+1]$ relative to endpoints but completely contained in D. The preimage of Γ connecting z and w is indeed a curve in $f^{-1}(D)$. $f^{-1}(D)$ is path-connected, and by the maximum principle, is simply connected.

□

The proposition above leads us to $V = f^{-1}(W)$ having simply connected components, called tracts of f, each of which having an unbounded boundary homeomorphic to $\mathbb{R}$. Indeed, f is a universal covering from each tract to W.

Let $\mathcal{T} = \exp^{-1}(V)$ and $H = \exp^{-1}(W)$. H is simply-connected and both T and H are $2\pi i$ periodic and have real parts which are bounded from below but unbounded from above. It follows that the map $F : \mathcal{T} \to H$, $F(z) = \exp^{-1} \circ f \circ \exp(z)$ is a conformal isomorphism from each component of $\mathcal{T}$, i.e. tracts of F, to H. This is obvious since both $\exp : H \to W$ and $f \circ \exp|_T : T \to W$ where T is any tract of F are universal covering maps of W. We denote by F_T^{-1} by the inverse of F mapping H to a tract T.



We say that such a map $F : \mathcal{T} \to H$ is the logarithmic transform of f. In [EL92], Eremenko and Lyubich proved that maps in class $\mathcal{B}$ are always expanding.

Lemma 2.1 (Expansion Property)**.** *Let $F : \mathcal{T} \to H$ be the logarithmic transform of $f \in \mathcal{B}$ and let $R \in \mathbb{R}$ such that $\{z \in \mathbb{C} \,|\, Re(z) > R\} \subset H$. For any $w \in \mathcal{T}$, if $ReF(w) > R$, then*

$$|F'(w)| \geq \frac{1}{4\pi}(ReF(w) - R).$$

Proof. Let w be contained in some tract T of F and satisfy the assumptions in the lemma. Let $z = F(w)$ and $r = \text{Re}z - R$, then the ball $B(z, r)$ is contained in H. Let $\phi : \mathbb{D} \to B(z, r), \phi(a) = ra + z$, then applying Koebe $\frac{1}{4}$ theorem to $F_T^{-1} \circ \phi : \mathbb{D} \to T$, T contains a ball $B\big(w, \frac{r}{4}|(F_T^{-1})'(z)|\big)$, but since $\mathcal{T}$ is $2\pi i$ periodic, it is necessary that $\frac{r}{4|F'(w)|} \leq \pi$. The lemma holds immediately.

□

We say that a logarithmic transform of $f \in \mathcal{B}$ is normalised if $H = \{z \in \mathbb{C} \,|\, \text{Re}z > 0\}$ and for all $z \in \mathcal{T}$, $|F'(z)| \geq 2$. Indeed, we can normalise any logarithmic transform F in the following way.

Pick $R > 0$ such that $\tilde{H} := \{z \in \mathbb{C} \,|\, \text{Re}z > R\} \subset H$ and $|F'(w)| \geq 2$ for all $w \in \mathcal{T} \cup F^{-1}(\tilde{H})$. Let $\tilde{F}(z) = F(z+R) - R$, then $\tilde{F}$ maps $\{z \in \mathbb{C} \,|\, F(z+R) \in \tilde{H}\}$ to $\{z \in \mathbb{C} \,|\, \text{Re}z > 0\}$. Note that F and $\tilde{F}$ have the same dynamics as they are topologically conjugate. It is often easier to work with normalised forms due to the simplicity of hyperbolic geometry of $\{z \in \mathbb{C} \,|\, \text{Re}z > 0\}$.

2.2 Hyperbolic Metric

In this section, we will review the notion of hyperbolic metric and its fundamental properties.Denote the hyperbolic metric on the unit disk $\mathbb{D}$ by $\rho_{\mathbb{D}} : \mathbb{D} \to \mathbb{R}^+$. For any open simply connected domain $U \subsetneq \mathbb{C}$, the hyperbolic metric on U is defined by

$$\rho_U(z) = \rho_{\mathbb{D}}(\phi(z))|\phi'(z)|,$$

for all $z \in U$, for any Riemann mapping $\phi : U \to \mathbb{D}$. Indeed, it is easy to prove that ρ_U is independent of the choice of the Riemann mapping ϕ. We also define the hyperbolic length of a curve $\gamma : [a, b] \to U$ to be $L_U(\gamma) = \int_a^b \rho(\gamma(t))|\gamma'(t)|dt$, and the hyperbolic distance between 2 points z and w in U to be $d_U(z, w) := \inf\{L_U(\gamma) \mid \gamma \text{ is a curve on } U \text{ joining } z \text{ and } w\}$.

By Schwarz-Pick Lemma, we can generalise this to the case where U and V are any 2 simply connected domains : for any holomorphic map $g : U \to V$ and any $z \in U$, $\rho_U(z) \geq \rho_V(g(z))|g'(z)|$, with equality if and only if g is a biholomorphism. As F is a biholomorphism from each tract T to H, F is indeed an isometry from (T, ρ_T) to (H, ρ_H).

Proposition 2.2 (Standard Estimate). *Let $U \subsetneq \mathbb{C}$ be a simply connected domain equipped with the hyperbolic metric ρ_U. Then, for any $z \in U$, if $R = dist(z, \partial U)$, then $\frac{1}{2R} \leq \rho_U(z) \leq \frac{2}{R}$.*

Proof. Pick any $z \in U$ and Riemann mapping $\phi : U \to \mathbb{D}$ such that $\phi(z) = 0$. Let $R = \text{dist}(z, \partial U)$. By Koebe $\frac{1}{4}$ theorem, U contains the ball $B(z, \frac{1}{4}|(\phi^{-1})'(0)|)$, thus $\frac{1}{4}|(\phi^{-1})'(0)| \leq R$. Since $|(\phi^{-1})'(0)|^{-1} = |\phi'(z)| = \frac{\rho_U(z)}{\rho_{\mathbb{D}}(0)} = \frac{1}{2}\rho_U(z)$, we obtain the upper bound. Now let $V := B(z, R)$. We know that by using the inclusion map from V to U, $\rho_U(z) \leq \rho_V(z)$. The map $\psi : V \to \mathbb{D}$, $w \mapsto \frac{w-z}{R}$ is a biholomorphism mapping z to 0. Then, $\rho_U(z) \leq \rho_V(z) = \rho_{\mathbb{D}}(0)|\psi'(z)| = \frac{2}{R}$.

□

We can apply the standard estimate to any tract T of F. Indeed, $\rho_T(z) \geq \frac{1}{2\pi}$ for all $z \in T$. Moreover, for any curve γ in T and any 2 points z, $w \in T$, $L(\gamma) \leq 2\pi L_T(\gamma)$ and $|z - w| \leq 2\pi d_T(z, w)$.

2.3 The Escaping Set and The Julia Set

We define the escaping set of a transcendental entire function f to be the following:

$$I(f) := \{z \in \mathbb{C} \mid f^n(z) \to \infty \text{ as } n \to \infty\}.$$

The escaping set of a transcendental entire function is shown by [Ere89] to be always non-empty and its boundary also coincides with the Julia set, i.e. $J(f) = \partial I(f)$. In class $\mathcal{B}$, the following theorem shows that the escaping set does not intersect the Fatou set.

Theorem 2.2. *If $f \in \mathcal{B}$, then $J(f) = \overline{I(f)}$.*

Proof. Suppose for a contradiction that there exists some point $z \in F(f) \cap I(f)$. There exists some open neighbourhood U of z in $F(f)$ such that all iterates tend uniformly to ∞ on U, then we have some $k \in \mathbb{N}$ where $U' = f^k(U) \in W$ and $f^n(U') \subset W$ for all $n \geq 0$. Pick a connected component A of $\exp^{-1}(U')$ on $\mathcal{T}$, then it follows that $F^n(A) \subset \mathcal{T}$ for all $n \geq 0$ and for all $w \in A$, $\text{Re}F^n(w) \to +\infty$ as $n \to \infty$. Pick any $w \in A$ and let $r_n = \text{dist}\{F^n(w), \partial F^n(A)\}$. By Koebe $\frac{1}{4}$ theorem, the ball $B(F^{n+1}(w), \frac{r_n}{4}|F'(F^n(w))|)$ is contained in $F^{n+1}(A)$, thus $r_{n+1} \geq \frac{r_n}{4}|F'(F^n(w))|$. By the expanding property, $|F'(F^n(w))| \to \infty$ as $n \to \infty$, thus $r_n \to \infty$. This is a contradiction since the periodicity of $\mathcal{T}$ implies that each $r_n \leq \pi$.

□

We will define the Julia set $J(F)$ and the escaping set $I(F)$ of the logarithmic transform F of a class $\mathcal{B}$ function f as follows.

$$J(F) := \{z \in \overline{\mathcal{T}} \mid O_F^+(z) \subset \overline{\mathcal{T}}\}$$
$$I(F) := \{z \in J(F) \mid \text{Re}F^n(z) \to +\infty \text{ as } n \to \infty\}.$$

Here, $O_F^+(z)$ denotes the forward orbit of z under F. It is easy to prove that $\exp(I(F)) \subset I(f)$. A similar statement for the Julia set is given below.

Proposition 2.3. *If F is a normalised logarithmic transform of $f \in \mathcal{B}$, $\exp J(F) \subset J(f)$.*

Proof. Suppose for a contradiction that there exists a point $z \in J(F)$ where $F^n(z) \in \mathcal{T}$ for all $n \in \mathbb{N}$ $w = e^z \in F(f)$. Then $f^n(w) \in V$ for all $n \in \mathbb{N}$, and there's an open neighborhood U of w in $V \cap W$ in which the sequence $\{f^n\}$ is normal. As F is normalised, $|f^n(w)| > 1$ and $|F'(F^n(z))| \geq 2$ for all $n \in \mathbb{N}$. Pick any $R > 0$ where $U \subset B(0, R)$. By the expanding property, $|f'(w)| \geq \frac{2|f(w)|}{|w|} > \frac{2}{R}$. Then, similarly,

$$|(f^n)'(w)| \geq \frac{2^n|f^n(w)|}{|w|} > \frac{2^n}{R} \to \infty \text{ as } n \to \infty$$

Thus, any subsequence cannot converge uniformly, contradicting normality in U.

□

Each $z \in J(F)$ has a unique external address $\underline{s} = s_0 s_1 s_2 \ldots$ where each s_n is a tract of F and $F^n(z) \in \overline{s_n}$. We denote those with the same external address as $J_{\underline{s}}(F) := \{z \in J(F) \mid z \text{ has external address } \underline{s}\}$ and we say that $\underline{s}$ is admissible if $J_{\underline{s}}(F) \neq \emptyset$.

Setting $F(\infty) = \infty$, let $\underline{s}$ be admissible and define iteratively the sets $S_0 := \overline{s_0} \cup \{\infty\}$, $S_n := \{z \in S_{n-1} \mid F^n(z) \in \overline{s_n} \cup \{\infty\}\}$. Then, $\hat{J}_{\underline{s}}(F) := \cap_{n \in \mathbb{N}} S_n = J_{\underline{s}}(F) \cup \{\infty\}$ is a non-empty, closed, connected subset of $\hat{\mathbb{C}}$, hence a continuum.

It then makes sense to define $\hat{J}_{\underline{s}}(F)$ as the Julia continuum of F of address $\underline{s}$. Note that since each point in $J(F)$ has a unique external address, for any 2 distinct external addresses $\underline{s}$ and $\underline{t}$, $J_{\underline{s}}(F)$ and $J_{\underline{t}}(F)$ are always disjoint. 2 distinct Julia continua will intersect only at ∞.

3 Eremenko's Conjecture on Finite Order Functions

Eremenko conjectured that every point of $I(f)$ can be joined to ∞ by a curve in $I(f)$. This does not hold in general for any transcendental entire function; indeed, there is a function in class $\mathcal{B}$ of which all path-connected components of its escaping set are singletons (see [RRRS11]). Throughout this section, we will discuss how finite order is a sufficient condition for class $\mathcal{B}$ functions to satisfy the conjecture.

The order of an entire function f is defined as $\rho = \limsup_{|z| \to \infty} \frac{\log \log |f(z)|}{\log |z|}$. Equivalently, we can say that the logarithmic transform F of $f \in \mathcal{B}$ has finite order if $\log \mathrm{Re} F(z) = O(\mathrm{Re} z)$ as $\mathrm{Re} z \to \infty$, $z \in \mathcal{T}$. f has finite order exactly when F has finite order. As an example, maps in the exponential family f_λ or the cosine family $g_{a,b}$ have finite order.

From now on, we will assume that F is a normalised logarithmic transform of $f \in \mathcal{B}$. We will use the notation $x^+ := \max\{x, 0\}$ for any $x \in \mathbb{R}$. We will also introduce a few definitions:

- F has bounded slope if there exists constants $\alpha, \beta > 0$ such that for any tract T and z $w \in \overline{T}$ such that $\mathrm{Re} w \geq \mathrm{Re} z$, $|\mathrm{Im} w - \mathrm{Im} z| \leq \alpha(\mathrm{Re} w)^+ + \beta$. This is equivalent to the following: there exists a curve $\gamma : [0, \infty) \to \mathcal{T}$ and some $C > 0$ such that $\gamma(t) \to \infty$ as $t \to \infty$ and $|\mathrm{Im}(\gamma(t))| \leq C \mathrm{Re} \gamma(t)$ for all t.
- F has uniformly bounded wiggling tracts if there are constants $K > 1$ and $\mu > 0$ such that for any tract T of F and $w_0 \in \overline{T}$, each w on the hyperbolic geodesic arc in T joining w_0 to ∞ satisfies $(\mathrm{Re} w)^+ > \frac{1}{K} \mathrm{Re} w_0 - \mu$.
- F satisfies the uniform linear head-start (ULHS) condition if there are constants $K > 1$ and $M > 0$ such that for any tracts T and T' of F and points z, $w \in \overline{T}$, if $F(z)$, $F(w) \in \overline{T}'$, then

$$\mathrm{Re} w > K(\mathrm{Re} z)^+ + M \quad \Rightarrow \quad \mathrm{Re} F(w) > K(\mathrm{Re} F(z))^+ + M.$$

[RRRS11] proved that these 3 definitions are satisfied if F is of finite order. We will discuss these in detail.

Proposition 3.1. *If F is of finite order, F has bounded slope.*

Proof. Let $r = \sup\{\frac{\log \mathrm{Re}F(z)}{\mathrm{Re}z} \mid z \in H \cap \mathcal{T}\}$ be the order of F and let $\gamma : [1, \infty) \to T$, $t \mapsto F_T^{-1}(t)$ be an unbounded geodesic arc in a tract T. For any $t \geq 1$, by the standard estimate,

$$\begin{aligned} |\gamma(t)| - |\gamma(1)| &\leq |\gamma(t) - \gamma(1)| \leq 2\pi d_T(\gamma(t), \gamma(1)) \\ &= 2\pi \log(t) = 2\pi \log \mathrm{Re}F(\gamma(t)) \leq 2\pi r \mathrm{Re}\gamma(t) \end{aligned}$$

Thus, $|\mathrm{Im}\gamma(t)| \leq |\gamma(1)| + 2\pi r \mathrm{Re}\gamma(t) \leq C \mathrm{Re}\gamma(t)$, where C satisfies $C \geq 2\pi r + \frac{|\gamma'(1)|}{\inf\{\mathrm{Re}\gamma(t) \mid t \geq 1\}}$. $\square$

To prove that finite order implies having uniformly bounded wiggling tracts and satisfying the ULHS condition, we may need to apply some concepts of hyperbolic geometry in the following lemma, which we will use repeatedly later on.

Lemma 3.1. *Let T and T' be tracts of F.*

(a) If z, $w \in \overline{T}$, $|F(z) - F(w)| \geq 2|z - w|$.

(b) If F has bounded slope with constants α and β, for all $K \geq 1$, there is a constant $\delta = \delta(\alpha, \beta, K) > 1$ such that for any z, $w \in \overline{T}$ where $F(w)$, $F(z) \in \overline{T}'$ and $\mathrm{Re}F(w) \geq \mathrm{Re}F(z)$,

$$|w - z| > \delta \Rightarrow \mathrm{Re}F(w) > K\mathrm{Re}F(z) + |z - w|.$$

Proof. Let $\Gamma : [0, 1] \to \overline{H}$ be a straight line on H joining $F(z)$ and $F(w)$ and let γ be a line on $\overline{T}$ joining z and w such that $\Gamma = F \circ \gamma$. Then, since $|F'(\gamma(t))| \geq 2$ for all t,

$$|F(z) - |F(w)| = \int_0^1 |\Gamma'(t)|dt = \int_0^1 |F'(\gamma(t))||\gamma'(t)|dt \geq 2\int_0^1 |\gamma'(t)|dt \geq 2|z - w|.$$

We have proven (a). Define constants δ_1 and δ_2 satisfying the following conditions:

- $\delta_1 = \alpha + \beta + 1$,
- For all $x \geq \delta_2$, $e^{x - \frac{1}{2}} - 1 > e^{\frac{x}{2}}$ and $e^{\frac{x}{8\pi}} > \delta_1(x + K + \frac{1}{2})$,
- For any tract T, $\delta_2 > \mathrm{diam}\{x \in T \mid \mathrm{Re}F(x) \leq 1\}$.

Let $\delta = \max\{\delta_1, \delta_2\} + \frac{1}{2}$. Let z and w satisfy the assumptions in (b) and pick $z' \in T$ such that $\mathrm{Re}F(z') = \max\{\mathrm{Re}F(z), 1\}$ and $\mathrm{Im}F(z') = \mathrm{Im}F(z)$. Then from (a), $|z - z'| \leq \frac{1}{2}$ and consequently $|w - z'| > \delta_2$. The third condition implies that $\mathrm{Re}F(w) \geq \mathrm{Re}F(z')$. Pick a point $u \in H$ such that $\mathrm{Re}F(z') = \mathrm{Re}u$ and $d_H(F(z'), u) = d_H(F(z'), F(w))$. Let $s := |F(z') - u|$. By the standard estimate,

$$\frac{|z' - w|}{2\pi} \leq d_T(z', w) \leq d_H(F(z'), F(w) = d_H(F(z'), u) < 2\log\Big(\frac{\mathrm{Re}F(z') + s}{\mathrm{Re}F(z')}\Big) + 1,$$

where the last inequality is obtained by considering the total hyperbolic length of 3 lines joining $F(z')$ and $F(z')+s$, $F(z')+s$ and $u+s$, and $u+s$ and u respectively. Since $|z'-w| > \delta_2$ and $|F(w)-F(z')| \geq s$, we can simplify the inequality as follows:

$$|F(w) - F(z')| \geq s \geq \big(e^{\frac{|w-z'|}{4\pi}} - 1\big)\mathrm{Re}F(z') \geq e^{\frac{|w-z'|}{8\pi}}\mathrm{Re}F(z'). \tag{1}$$

Using (a) and the bounded slope property,

$$\alpha + \beta + 1 < 2|z' - w| \leq |F(w) - F(z')| \leq \mathrm{Re}F(w) + |\mathrm{Im}F(w) - \mathrm{Im}F(z')| \leq (\alpha + 1)\mathrm{Re}F(w) + \beta,$$

thus $\mathrm{Re}F(w) > 1$ and consequently,

$$|F(w) - F(z')| \leq (\alpha + 1)\mathrm{Re}F(w) + \beta < \delta_1 \mathrm{Re}F(w). \tag{2}$$

By the second condition and using both (1) and (2), we can prove (b) as follows.

$$\mathrm{Re}F(w) > \delta_1^{-1} e^{\frac{|w-z'|}{8\pi}}\mathrm{Re}F(z') > \Big(|w - z'| + K + \frac{1}{2}\Big)\mathrm{Re}F(z') \geq K\mathrm{Re}F(z) + |w - z|.$$

$\square$

Proposition 3.2. *If F is of finite order, F has uniformly bounded wiggling tracts.*

Proof. Suppose F has finite order. It is easy to show that there are constants A, $B > 0$ such that for all $z \in \mathcal{T}$, $\log \mathrm{Re}F(z) < A\mathrm{Re}z + B$. Pick any tract T and let $w_0 \in \overline{T}$. Let $\gamma(t) = F_T^{-1}(F(w_0) + t)$, $t \geq 0$ be a geodesic arc in T from w_0. We will consider 2 following cases.

- Suppose $\mathrm{Re}F(w_0) \geq 1$, then for all $t \geq 0$, $d_T(w_0, \gamma(t)) = \log \frac{\mathrm{Re}w_0 + t}{\mathrm{Re}w_0} \leq \log(1+t)$ and additionally,

$$\begin{aligned}\mathrm{Re}w_0 - \mathrm{Re}\gamma(t) &\leq |w_0 - \gamma(t)| \leq 2\pi \log(1+t) \leq 2\pi \log(\mathrm{Re}F(w_0) + t)\\ &\leq 2\pi \log(\mathrm{Re}F(\gamma(t))) \leq 2\pi A \mathrm{Re}\gamma(t) + 2\pi B.\end{aligned}$$

  Then, setting $K := 2\pi A + 1 > 1$, we have $\mathrm{Re}\gamma(t) \geq \frac{1}{K}\mathrm{Re}w_0 - \frac{2\pi B}{K}$.

- Suppose $\mathrm{Re}F(w_0) < 1$. Let $t_0 := 1 - \mathrm{Re}F(w_0) \in (0, 1]$.
  For $t < t_0$, i.e. $\mathrm{Re}F(\gamma(t)) < 1$, by Lemma 3.1($a$),

$$\mathrm{Re}w_0 - \mathrm{Re}\gamma(t) \leq |w_0 - \gamma(t)| \leq \frac{1}{2}|F(w_0) - F(\gamma(t))| \leq \frac{1}{2}.$$

  Then, $\mathrm{Re}\gamma(t) \geq \mathrm{Re}w_0 - \frac{1}{2} \geq \frac{1}{k}\mathrm{Re}w_0 - \frac{1}{2}$.
  For t where $\mathrm{Re}F(\gamma(t)) \geq 1$, $\mathrm{Re}\gamma(t) \geq \frac{1}{K}\mathrm{Re}\gamma(t_0) - \frac{2\pi B}{K} \geq \frac{1}{K}\mathrm{Re}w_0 - \frac{4\pi B + 1}{2K}$.

Let $\mu = \max\{\frac{1}{2}, \frac{4\pi B+1}{2K}\}$, then since constants K and μ are independent of the choice of the tract, the proposition follows.

□

Proposition 3.3. *Let F have bounded slope (with some constants α and β). Then, F has uniformly bounded wiggling tracts if and only if F satisfies the ULHS condition.*

Proof. Suppose F has uniformly bounded wiggling tracts with constants $K > 1$ and $\mu > 0$. Let $\delta = \delta(\alpha, \beta, K)$ and $M = \max\{\delta, K(\mu + 2\pi(\alpha + \beta))\}$. Let z, $w \in \overline{T}$ and $F(z)$, $F(w) \in \overline{T}'$ for some tracts T and T', such that $\mathrm{Re}w > K(\mathrm{Re}z)^+ + M$. This already implies that $|w - z| > M \geq \delta$. If $\mathrm{Re}F(w) \geq \mathrm{Re}F(z)$, then we can conclude by Lemma 3.1(b) that F satisfies the ULHS condition.

Suppose instead that $\mathrm{Re}F(w) < \mathrm{Re}F(z)$, then $\mathrm{Re}F(z) > |w - z| > M > 1$ by the same lemma. Let $\gamma(t) = F_T^{-1}(F(w) + t)$, $t \geq 0$ be a geodesic on T joining w to ∞. By the standard estimate and the bounded slope property,

$$\frac{1}{2\pi}\mathrm{dist}(z, \gamma) \leq \mathrm{dist}_T(z, \gamma) = \mathrm{dist}_H(F(z), F\gamma) \leq \frac{|\mathrm{Im}F(z) - \mathrm{Im}F(w)|}{\mathrm{Re}F(z)} \leq \alpha + \beta.$$

As such, $r_0 := \min\{\mathrm{Re}\gamma(t) \mid t \geq 0\} \leq \mathrm{Re}z + 2\pi(\alpha + \beta)$. As the tracts have uniformly bounded wiggling,

$$\mathrm{Re}w \leq K(r_0)^+ K\mu \leq K(\mathrm{Re}z)^+ + K(\mu + 2\pi(\alpha + \beta) = K(\mathrm{Re}z)^+ + M$$

This contradicts the assumption, so F has to satisfy the ULHS condition.
Now, suppose instead that F satisfies the ULHS condition with constants $K > 1$ and $M > 0$. Let T a tract of F and let $z, w \in \overline{T}$ be such that $F(w) \in \mathcal{T} \cap H$, $|F(z) - F(w)| < K'$ for some value K' independent of z and the choice of the tract T, and the straight line γ_0 joining z and w is contained in $\overline{T}$. From Lemma 3.1(a), $|z - w| < \frac{K'}{2}$.

Pick a curve $\Gamma : [0, \infty) \to T' \cap H$ where $\Gamma(0) = F(w)$, $\mathrm{Re}\Gamma(t) \geq \mathrm{Re}F(w)$ for all $t \geq 0$ and $\Gamma(t) \to \infty$ as $t \to \infty$. Let $\gamma_0 = F_T^{-1} \circ \Gamma$ be the preimage of Γ containing w. By the ULHS condition, for all $t \geq 0$, $\mathrm{Re}w \leq K(\mathrm{Re}\gamma_1(t))^+ + M$. Rewriting, we have $(\mathrm{Re}\gamma_1(t))^+ \geq \frac{1}{K}\mathrm{Re}w - \frac{M}{K}$.

Obtain γ_2 by combining γ_0 and γ_1, so for any point $p \in \gamma_2$,

$$(\mathrm{Re}p)^+ \geq \frac{1}{K}\mathrm{Re}w - \frac{M}{K} - \frac{K'}{2} > \frac{1}{K}\mathrm{Re}z - \mu'$$

where $\mu' = \frac{K'+M}{K} + \frac{K'}{2}$. Then, if γ is the unbounded hyperbolic geodesic on T from z, for any point $p \in \gamma$, $(\mathrm{Re}p)^+ > \frac{1}{K}\mathrm{Re}z - \mu$ where $\mu = \mu' + |\inf \mathrm{Re}\gamma_2 - \inf \mathrm{Re}\gamma|$.

□

Theorem 3.2. *Let $f \in \mathcal{B}$ be of finite order. Every component of $I(f)$ is path-connected and unbounded.*

Proof. Let F be the normalised logarithm transform of f. By previous results, F has bounded slope with some constants α, $\beta > 0$, and satisfies the ULHS condition for some constants $K > 1$ and $M > 0$.

Suppose $\underline{s}$ is an admissible external address. Define the binary relation $\succ$ between points in $\hat{J}_{\underline{s}}(F)$ by $w \succ z$ if and only if $\mathrm{Re}F^n(w) > K(\mathrm{Re}F^n(z))^+ + M$ for some $n \in \mathbb{N}$, and let $\infty \succ z$ for all $z \in J_{\underline{s}}(F)$. $\succ$ is a non-reflexive and transitive relation. For any 2 distinct points z,$w \in J_{\underline{s}}(F)$, it is possible by Lemma 3.1(a) to find $m \in \mathbb{N}$ such that $|F^m(w) - F^m(z)| \geq 2^m|w - z| > \max\{M, \delta(\alpha, \beta, K)\}$, so by Lemma 3.1($b$), assuming w.l.o.g. that $\mathrm{Re}F^m(w) \geq \mathrm{Re}F^m(z)$, $w \succ z$ since

$$\mathrm{Re}F^{m+1}(w) > K\mathrm{Re}F^{m+1}(z) + |F^m(w) - F^m(z)| > K\mathrm{Re}F^{m+1}(z) + M.$$

Thus, $\succ$ is a strict total order on $J_{\underline{s}}(F)$. By continuum theory, $J_{\underline{s}}(F)$ is homeomorphic to a compact interval. There exists only one $\succ$-least point $z_0 \in J_{\underline{s}}(f)$, i.e. for all other points z, $z \succ z_0$. By Lemma 3.1(b), for any $z \in J_{\underline{s}}(F)\backslash\{z_0\}$, it is necessary that $\mathrm{Re}F^n(z) \to +\infty$ as $n \to \infty$, hence $J_{\underline{s}}(F)\backslash\{z_0\} \subset I(F)$. In other words, each path-connected component $I(F)$ is bounded.

Now, pick any point $w \in I(f)$ and take some $k \in \mathbb{N}$ such that $f^n(w) \in W$ for all $n \geq k$. Let $z \in I(F)$ such that $e^z = f^k(w)$. From above, there's a curve $\gamma : [0, \infty) \to I(F)$ where $\gamma(0) = z$ and $\mathrm{Re}\gamma(t) \to +\infty$ as $t \to \infty$. Then, we can always take an unbounded curve $\Gamma : [0, \infty) \to I(f)$ where $\Gamma(0) = w$ and $f^n \circ \Gamma = \exp \circ \gamma$.

$\square$

We have proven that finite order is a sufficient condition for class $\mathcal{B}$ functions to satisfy the strong version of the Eremenko's conjecture. It is not difficult to observe from our proof and generalise that the theorem above also holds for any function in class $\mathcal{B}$ with a corresponding logarithmic transform F which has bounded slope and satisfies the head-start condition.

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