Applied Complex Analysis

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Introduction

Denote the set of complex numbers by

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$$

where $i = \sqrt{-1}$ is defined such that $i^2 = -1$.

Complex analysis is the study of functions of a complex variable. In the first few chapters, we shall explore some introductory concepts, such as basic properties of complex numbers and continuity of complex-valued functions. The main emphasis is the concept of *holomorphic* functions, i.e. complex-valued functions which are differentiable in a complex sense, and the many applications of their somewhat magical properties. I used the word 'magical' because holomorphicity is such a rigid condition that many of the results you will see are somewhat unintuitive yet true.

We will start with some motivation. Basic algebra tells us that the number of roots of a polynomial with real coefficients is at most its degree. For example, $x^2 + c$ has two real roots if $c < 0$, one root if $c = 0$, and no roots if $c > 0$. Introducing the imaginary number i provides us with a more elegant way of formulating this idea.

Theorem (Fundamental Theorem of Algebra)**.** *The field $\mathbb{C}$ is algebraically closed, that is, any polynomial with coefficients in $\mathbb{C}$ of degree $d > 1$ has exactly d roots in $\mathbb{C}$, counting multiplicity.*

Many initial attempts of proving the theorem by prominent mathematicians D'Alembert, Euler, Gauss, Lagrange, and Laplace in 1700s were incomplete. In 1806, a Swiss accountant, Parisian bookstore manager and 'amateur' mathematician Jean-Robert Argand completed D'Alembert's ideas and hence became the first person to rigorously prove the fundamental theorem of algebra. We will in fact use properties of holomorphic functions to give 3 different proofs of the theorem, including D'Alembert and Argand's approach.

It is difficult to list the many applications of the fundamental theorem of algebra. The main idea is that the field of complex numbers is the perfect setting to solve equations!

A direct consequence in linear algebra is that every square matrix with entries in $\mathbb{C}$ admits an eigenvalue. When a 2×2 matrix has imaginary eigenvalues, it acts as a rotation of the plane rather than expansion or contraction in certain directions. In the study of continuous dynamics arising from mechanical systems, it is common to use complex numbers in order to capture oscillations in the system.

One of the direct applications of the study of holomorphic functions is contour integration. The integral of a complex function along a closed path is not dependent on the path itself but rather on certain values called *residues* of the function's singularities. This means that it is often easier to integrate a real function of a real variable by converting it into a problem involving a contour integral in the complex plane.

Fourier series and Fourier transforms are useful in decomposing functions into its frequency components. (Think of decomposing nice functions as a sum or an integral of different sine and cosine waves.) Fourier analysis can be easily formulated via complex analysis, and it comes up everywhere: in differential equations, probability, quantum mechanics, signal processing, etc.

Mechanical and electrical engineers as well as computer musicians also encounter complex variables in electrical circuits with alternating current. Digital filters are designed by looking at the locations of *zeros* and *poles* of rational functions called *transfer functions*, which essentially model a device's inputs and outputs.

Iterations of holomorphic functions have long been known to have many applications. Complex polynomials, for example, can be used to model the population of rabbits over time. Powerful basic results in complex analysis, many of which do not apply to generic real differentiable functions, make up one of the many reasons why the study of iterations of holomorphic functions (holomorphic dynamics) is very well developed compared to the other branches of the field of dynamical systems.

Conformal functions are holomorphic functions with strictly non-zero derivative. Such functions have an amazing geometric property of angle preservation at every point and are useful in transforming regions with complicated boundary to those of a much nicer shape (square, disk, etc). You may, for example, want to transform a mechanical problem on a complicated domain into an equivalent problem on a circular disk. In cartography, conformal maps are useful in creating a world map as well as local nautical charts using Mercator and stereographic projections. More recently, conformal functions are applied to the surface of the human brain for brain development study and diagnosis of Alzheimer's disease and schizophrenia.

Chapter 1

Complex Numbers

In this chapter, we will go through the basic algebraic and geometric properties of complex numbers.

1.1 The Algebra of $\mathbb{C}$

The set $\mathbb{C}$ is equipped with the usual arithmetic operators, namely:

- addition $+$: $(x+iy)+(a+ib)=(x+a)+i(y+b)$,
- multiplication $\times$: $(x+iy)\times(a+ib)=(xa-yb)+i(xb+ya)$.

Let's denote by $\mathbb{C}^*$ the set of non-zero complex numbers $\mathbb{C}\backslash\{0\}$. This set is equipped with an additional operator:

- inversion of a non-zero number: $(x+iy)^{-1}=\frac{x-iy}{x^2+y^2}$.

Similar to $\mathbb{R}$, the set of complex numbers $\mathbb{C}$ is a *field*; it satisfies the following axioms:

1. $(\mathbb{C},+)$ is an abelian group:
   - $+$ is associative and commutative,
   - 0 is the identity element of $+$, i.e. $z+0=z$ for all $z\in\mathbb{C}$,
   - Additive inverses exist, i.e. $z+(-z)=0$ for all $z\in\mathbb{C}$;
2. $(\mathbb{C}^*,\times)$ is an abelian group:
   - $\times$ is associative and commutative,
   - 1 is the identity element of $\times$, i.e. $z\times 1=z$ for all $z\in\mathbb{C}^*$,

- Multiplicative inverses exist, i.e. $z \times z^{-1} = 1$ for all $z \in \mathbb{C}^*$;

3. $\times$ is distributive over $+$.

The set $\mathbb{C}$ of complex numbers can be identified with the real vector space $\mathbb{R}^2$ by the vector space isomorphism:

$$\begin{aligned} \mathbb{C} &\to \mathbb{R}^2, \\ z &\mapsto (\operatorname{Re} z, \operatorname{Im} z), \\ x + iy &\mapsto (x, y). \end{aligned}$$

Unlike $\mathbb{C}$, the real plane $\mathbb{R}^2$ is only equipped with addition operator $+$ but not a natural multiplication operator $\times$. Nonetheless, the mapping above allows us to geometrically represent complex numbers as points on the plane. This is typically known as *Argand diagram.*

1.2 The Geometry of $\mathbb{C}$

Every complex number $z = x + iy$ comes with a unique *real part* x and *imaginary part* y. We shall denote them as follows:

$$\operatorname{Re} z = x, \qquad \operatorname{Im} z = y.$$

Geometrically, Re and Im can be thought as functions $\mathbb{C} \to \mathbb{R}$ acting as projections onto the real and imaginary axes respectively.

The *complex conjugate* $\bar{z}$ of a complex number $z = x + iy$ is $\bar{z} = x - iy$. Geometrically, the operation $z \mapsto \bar{z}$ is a reflection over the real axis. The following identity can be thought of as a change of basis from (x, y) to $(z, \bar{z})$.

Proposition 1.1. *For any $z \in \mathbb{C}$, $\operatorname{Re} z = \frac{z+\bar{z}}{2}$ and $\operatorname{Im} z = \frac{z-\bar{z}}{2i}$.*

Another straightforward algebraic exercise also gives us the following basic properties.

Proposition 1.2. *For any $z, w \in \mathbb{C}$, $\overline{z + w} = \bar{z} + \bar{w}$ and $\overline{zw} = \bar{z}\bar{w}$. If $z \neq 0$, $\overline{z^{-1}} = \bar{z}^{-1}$.*

The *absolute value / modulus* of a complex number $z = x + iy$ is

$$|z| = \sqrt{x^2 + y^2}.$$

Phytagoras' theorem indicates that geometrically the modulus $|z|$ of z is equal to the distance between 0 and z.

Proposition 1.3. *For any $z, w \in \mathbb{C}$,*

- $|zw| = |z||w|$,
- $|z + w| \leq |z| + |w|$ *(Triangle inequality).*

The *argument* of z, $\arg(z)$, is defined to be the counterclockwise angle (measured in radians) subtended by the positive real axis $\mathbb{R}^+$ and the line segment joining 0 and z. See figure 1.1.

Notice that arg is a multivalued function. For example, both π and 3π are arguments of i. We can refine this by defining the *principal argument* of z, $\operatorname{Arg}(z)$, to be the unique argument of z lying in $(-\pi, \pi]$.

Remark. The interval $[0, 2\pi)$ is also often chosen to be the codomain of the principal argument.

Proposition 1.4. *For any $z, w \in \mathbb{C}^*$,*

- $\arg(zw) = \arg(z) + \arg(w)$,
- $\operatorname{Arg}(zw) = \operatorname{Arg}(z) + \operatorname{Arg}(w)$ *mod* 2π.

Example 1. Let $z = 1 + i$ and $w = -1 + \sqrt{3}i$. The modulus and arguments of z and w are:

$$|z| = \sqrt{2}, \quad |w| = 2, \quad \arg(z) = \frac{\pi}{4}, \quad \arg w = \frac{2\pi}{3}.$$

Then, the modulus and argument of $(1 + i)(-1 + \sqrt{3}i)$ are $2\sqrt{2}$ and $\frac{11\pi}{12}$ respectively.

For any non-zero complex number $z = x + iy$, if $r = |z|$ and $\theta = \operatorname{Arg}(z)$, then basic trigonometry gives us the following change of variables:

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

The expression $z = r(\cos\theta + i\sin\theta)$ from above is the *polar form* of z.

Theorem 1.5 (Euler's formula)**.** *For any θ, $e^{i\theta} = \cos\theta + i\sin\theta$.*

Proof. We will give two different proofs of the result - one with differential equations, and another with Maclaurin series. The expression $e^{i\theta}$ is a non-zero complex number, so there is a unique $r > 0$ and $\hat{\theta}$ such that

$$e^{i\theta} = r(\cos\hat{\theta} + i\sin\hat{\theta}). \tag{1.1}$$

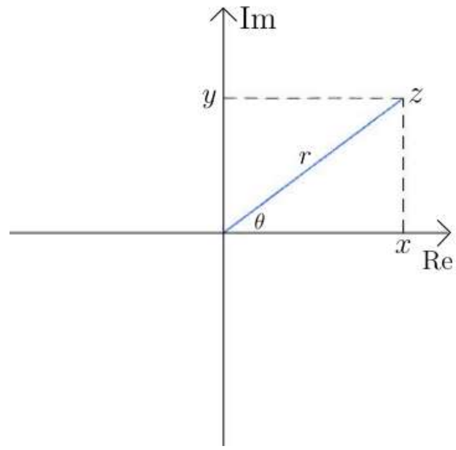

Figure 1.1: A point $z = x + iy = re^{i\theta}$ on the Argand diagram

Here, r and $\hat{\theta}$ are functions of θ. When $\theta = 0$, $r(0) = 1$ and $\hat{\theta}(0) = 0$. Differentiating (1.1) with respect to θ, we obtain

$$\begin{aligned} ie^{i\theta} &= \frac{dr}{d\theta}(\cos\hat{\theta} + i\sin\hat{\theta}) + r\frac{d\hat{\theta}}{d\theta}(-\sin\hat{\theta} + i\cos\hat{\theta}) \\ &= \frac{dr}{d\theta}\frac{e^{i\theta}}{r} + i\frac{d\hat{\theta}}{d\theta}e^{i\theta} \\ &= \left(\frac{dr}{d\theta} + i\frac{d\hat{\theta}}{d\theta}\right)e^{i\theta}, \end{aligned}$$

where the second equality above is obtained from (1.1). From above, we see that $\frac{dr}{d\theta} = 0$ and $\frac{d\hat{\theta}}{d\theta} = 1$. By our initial conditions, we obtain $r(\theta) \equiv 1$ and $\hat{\theta} \equiv \theta$.

Alternatively, recall the following Maclaurin series: $e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$. Using the fact that $i^n = (-1)^{n/2}$ if n is even, and $i^n = (-1)^{n-1/2}i$ if n is odd,

$$\begin{aligned} e^{i\theta} &= \sum_{\text{even } n} \frac{(-1)^n\theta^n}{n!} + \sum_{\text{odd } n} \frac{(-1)^{n-1/2}\theta^n}{n!} \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right) = \cos\theta + i\sin\theta. \end{aligned}$$

□

Example 2. When $\theta = \pi$, we have Euler's identity: $e^{i\pi} = -1$.

The polar form of a complex number z can alternatively be written in the form of $z = re^{i\theta}$. This expression is particularly useful when performing multiplication of complex numbers as we can use the laws of exponent. One particular instance is the following.

Theorem 1.6 (De Moivre's Theorem)**.** *For any θ and integer $n \in \mathbb{Z}$,*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Example 3. To compute and simplify $\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^{10}$, we can use De Moivre's theorem. The term inside the bracket is essentially $\cos\theta + i\sin\theta$ where $\theta = \frac{2\pi}{3}$. Then,

$$\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^{10} = \cos\left(\frac{20\pi}{3}\right) + i\sin\left(\frac{20\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}i}{2}.$$

1.3 Complex Roots

Consider a complex number z_0 and a positive integer n. A complex number w satisfying $w^n = z_0$ is called an n^{th} *root* of z_0.

Suppose $z_0 = 0$. Regardless of n, there is only one root of 0, which is 0 itself. This is due to the fact that $\mathbb{C}$ is an integral domain, i.e. for any two complex numbers z_1 and z_2, if $z_1 z_2 = 0$ then either $z_1 = 0$ or $z_2 = 0$.

Suppose $z \neq 0$ now, then surely any root w is also non-zero. Using their polar forms $z = re^{i\theta}$ and $w = se^{it}$, then the equation becomes:

$$s^n e^{int} = re^{i\theta}$$

Considering the modulus and the argument independently, we obtain two real equations $s^n = r$ and $nt = \theta \bmod 2\pi$. There are therefore n different solutions to w:

$$w_k = r^{1/n} e^{i(\theta + 2\pi k)/n}, \qquad k \in \{0, 1, \dots n-1\}.$$

In the expression above, w_0 is called the *principal root* of z_0. On the complex plane, these roots are evenly spaced on the circle $\{z \in \mathbb{C} \,|\, |z| = r^{1/n}\}$ of radius $r^{1/n}$ centered at the origin.

When $z_0 = 1$, the n^{th} roots of 1 are called the n^{th} *roots of unity*. They all lie on the unit circle and are of the form $e^{2\pi i k/n}$, where $k \in \{0, 1, \dots n-1\}$.

Example 4. The 3rd roots of unity are 1, $e^{2\pi i/3}$ and $e^{4\pi i/3}$. The Cartesian forms of these roots are 1, $\frac{-1+i\sqrt{3}}{2}$, and $\frac{-1-i\sqrt{3}}{2}$.

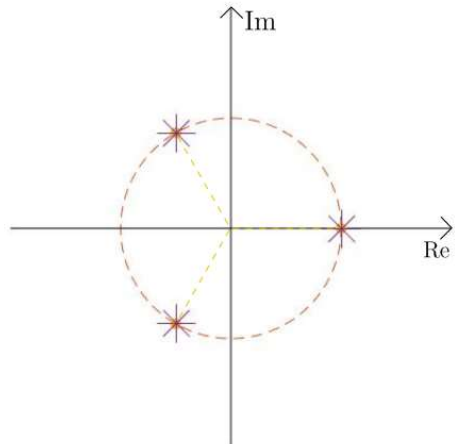

Figure 1.2: 3rd roots of unity

1.4 The Topology of $\mathbb{C}$

An *open disk* of radius $r > 0$ centred at a complex number $a \in \mathbb{C}$ is a subset of $\mathbb{C}$ of the form:

$$\mathbb{D}(z, r) = \{z \in \mathbb{C} \mid |z - a| < r\}.$$

The boundary of this disk is a circle of of radius $r > 0$ centred at a, denoted with a partial sign in front:

$$C(z, r) = \partial\mathbb{D}(z, r) = \{z \in \mathbb{C} \mid |z - a| = r\}.$$

If we include the boundary, we obtain a *closed disk* typically denoted with an overline:

$$\overline{\mathbb{D}(z, r)} = \{z \in \mathbb{C} \mid |z - a| \leq r\}.$$

The geometric and topological properties of the complex plane $\mathbb{C}$ are essentially the same as those of the real plane $\mathbb{R}^2$ since we have the obvious identification $x + iy \mapsto (x, y)$. We will give a brief introduction of necessary topological terminology that we will use in the next few chapters.

Definition 1. A subset $S \subset \mathbb{C}$ is:

- *open* if for every $s \in S$, there is some $r > 0$ such that $\mathbb{D}(s, r) \subset S$,
- *closed* if its complement $\mathbb{C} \backslash S$ is open,
- *bounded* if there is some $r > 0$ where $S \subset \mathbb{D}(0, r)$,

- *compact* if S is closed and bounded.

Example 5. Below are some subsets of $\mathbb{C}$ which we will commonly encounter.

1. The empty set $\emptyset$ is trivially open and compact.
2. The complex plane $\mathbb{C}$ is both open and closed, but not bounded.
3. The punctured plane $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ is open, but not closed nor bounded.
4. The unit disk $\mathbb{D} := \mathbb{D}(0, 1)$ is open and bounded, but not closed.
5. The closed unit disk $\overline{\mathbb{D}}$ and its boundary $\partial\mathbb{D}$ are compact.
6. The real axis $\mathbb{R}$ is closed and unbounded.

Definition 2. An open/closed set $S \subset \mathbb{C}$ is:

- *connected* if S cannot be expressed as a disjoint union of two open/closed non-empty subsets of $\mathbb{C}$,
- *simply connected* if it is connected and it has no "holes", i.e. the complement $\mathbb{C}\backslash S$ has no bounded connected component,
- *multiply connected* if it is connected but not simply connected.

We say that S is a *domain* if it is a non-empty open connected subset of $\mathbb{C}$.

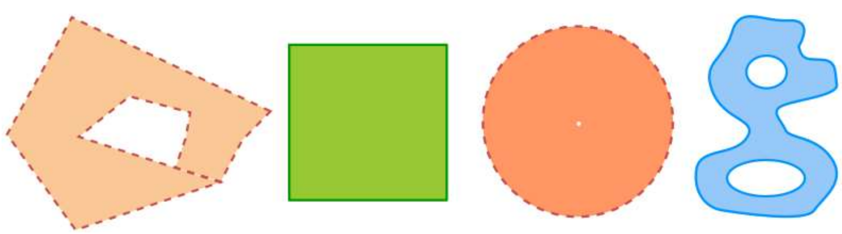

Figure 1.3: Four connected subsets of $\mathbb{C}$. Solid boundary lines are included in the colored set, whereas dashed boundary lines are not included. The first (from the left) is a simply connected domain. The second is closed and simply connected. The third is a punctured disk, which is a multiply connected domain. The last is closed and multiply connected.

Example 6.

1. $\emptyset$, $\mathbb{C}$, $\mathbb{D}$, $\overline{\mathbb{D}}$ and $\mathbb{R}$ are simply connected.

2. The punctured plane $\mathbb{C}^*$, the punctured unit disk $\mathbb{D}^* := \mathbb{D}\backslash\{0\}$, and the unit circle $\partial\mathbb{D}$ are multiply connected.

3. The annulus $\{z \in \mathbb{C} \mid r < |z| < R\}$ of inner radius r and outer radius R is multiply connected.

Short Quiz 1

1. Simplify $\frac{1+i}{i-1}$.
2. Find the modulus of $(3+4i)(-4+3i)$.
3. Find the argument(s) of $\arg(-1+i)$.
4. Express $2e^{-2\pi i/3}$ in the form of $x+iy$.
5. What are the 3^{th} roots of $8i$?
6. Find the value of $(1+i)^6$.

Answers: 1. $-i$, 2. 25, 3. $\frac{3\pi}{4}+2\pi k$, 4. $-1-i\sqrt{3}$, 5. $\pm\sqrt{3}+i$ & $-2i$, 6. $-8i$.