Midterm Solutions

1. (a) Since $-8\pi = 2^3\pi e^{i\pi}$, p.v.$\sqrt[3]{-8\pi} = 2\sqrt[3]{\pi}e^{i\pi/3} = \sqrt[3]{\pi} + i\sqrt[3]{\pi}\sqrt{3}$.

   (b) No, because it's not always true that $\text{Arg}z^2 = 2\text{Arg}z$. (e.g. take $z = e^{2\pi i/3}$.) The equation holds only modulo 2π.

   (c) No. For example, $\frac{z-1}{z} = 1 - \frac{1}{z}$ is holomorphic on $\mathbb{C}^*$ but all its primitive $z - \text{Log}z + c$ for any constant $c \in \mathbb{C}$ not even continuous nor holomorphic along any choice of branch cut.

2. (a) $u_M(x, y) = ax + by$ and $v_m(x, y) = cx + dy$, then

$$\begin{aligned} f_M(x + iy) &= ax + by + i(cx + dy) = (a + ci)x + (b + di)y \\ &= \frac{a + ci}{2}(z + \bar{z}) + \frac{b + di}{2i}(z - \bar{z}) \\ &= \frac{(a + d) + i(c - b)}{2}z + \frac{(a - d) + i(c + b)}{2}\bar{z}. \end{aligned}$$

   (b) f_M is entire if and only if $w_2 = 0$. That is, $a = d$ and $c = -b$.

3. Both parts can actually be solved simply by showing that the image of f is not dense. Nonetheless, the answers below use more tribal approach. Let $f = u + iv$.

   (a) The function $g = \frac{u}{v}$ is both real and entire. By Cauchy-Riemann, this implies that g is a real constant. Therefore, $u = cv$ for some real c. Applying Cauchy-Riemann on f, this implies that $u_x = cv_x = -cu_y$ and $u_y = cv_y = cu_x$, which imply that $u_x = u_y = v_x = v_y \equiv 0$. Therefore, f is a constant function.

   (b) When u is a bounded function, $|e^f| = e^u$ is bounded. Since e^f is entire, it must be constant by Liouville. Therefore, f is also constant.

4. (a) We wish to find the roots of the denominator in order to find the singularities of p. Check that the roots of the quartic $w^4 + 4$ are $w = \pm 1 \pm i$. Therefore, the roots of $(z - i)^4 + 4$ are $z = \pm 1, \pm 1 + 2i$. These are the values of $a_1 \ldots a_4$.

   (b) The only singularity enclosed by γ is 1. The rest are outside, so the function $(z + 1)^{-1}(z - 1 - 2i)^{-1}(z + 1 - 2i)^{-1}$ is holomorphic along γ and its interior. Apply Cauchy's integral formula at 1.

$$\begin{aligned}\oint_\gamma p(z)dz &= \oint_\gamma \frac{(z+1)^{-1}(z-1-2i)^{-1}(z+1-2i)^{-1}}{z-1}dz \\ &= 2\pi i(1+1)^{-1}(1-1-2i)^{-1}(1+1-2i)^{-1} \\ &= \frac{2\pi}{4(-1+i)} = \frac{\pi}{8}(-1-i).\end{aligned}$$

5. (a) The integrand can be expressed as e^{1-iz}, which is entire. By Cauchy-Goursat, the integral has to be zero.

   (b) The integrand f is holomorphic on $\mathbb{C}\backslash\{\pm 1, \pm i\}$ and has a primitive $F(z) = \frac{1}{2(1-z^4)}$ which is also holomorphic on $\mathbb{C}\backslash\{\pm 1, \pm i\}$. Since the contour γ runs from 0 to $1+i$ avoiding the singularities of f, we can evaluate the integral using the primitive:

$$\int_\gamma f(z)dz = F(i) - F(0) = \frac{1}{2(1-(1+i)^4)} - \frac{1}{2} = -\frac{2}{5}.$$

6. (a) When $|z| = 1$,

$$|B(z)| = \frac{|i+2z|}{|4-2iz|} = \frac{|i+2z|}{|4-2iz||\bar{z}|} = \frac{|i+2z|}{|4\bar{z}-2i|} = \frac{1}{2}\cdot\frac{|i+2z|}{|\overline{2z+i}|} = \frac{1}{2}.$$

   (The above can also be shown using Cartesian $z = x+iy$ or polar coordinates $z = e^{i\theta}$.) $B(z)$ is holomorphic on $\mathbb{C}\backslash\{-2i\}$, and especially on a neighbourhood of the closed unit disk $\bar{\mathbb{D}}$. By the maximum principle, $|B(z)| \leq 1$ whenever $z \in \bar{\mathbb{D}}$. Therefore, $M = \frac{1}{2}$.

   (b) Basic trigonometry and Pythagoras us $L(\gamma) = 2\sqrt{2+\sqrt{2}}$. The inequality follows from ML inequality.

   (c) $B(z)$ can be expressed as $i + \frac{3}{2z+4i}$. We have a primitive

$$F(z) = iz + \frac{3}{2}\mathrm{Log}(z+2i)$$

   which is holomorphic everywhere except on the branch cut chosen to be $\{x-2i \mid x \leq 0\}$. As γ does not intersect the branch cut, we may use the primitive to evaluate the integral.

$$\begin{aligned}\int_\gamma B(z)dz &= F(1) - F(-i) = i + \frac{3}{2}\mathrm{Log}(1+2i) - 1 - \frac{3}{2}\mathrm{Log}(i) \\ &= -1 + i + \frac{3}{2}\mathrm{Log}(2-i) \\ &= \left(\frac{3}{4}\ln 5 - 1\right) + i\left(1 - \frac{3}{2}\tan^{-1}\frac{1}{2}\right).\end{aligned}$$