Problem Set 2

Note: Assessed problems (and sub-problems) are marked by the asterisk *.

1. * Prove that if a function $f(z)$ is holomorphic on some domain U, then $\overline{f(\bar{z})}$ is also holomorphic on U.

2. Show that the Wirtinger derivatives of a differentiable function $f(z)$ in polar coordinates (r, θ), where $z = re^{i\theta}$, are
$$\frac{\partial f}{\partial z} = \frac{1}{2z}\left(r\frac{\partial f}{\partial r} - i\frac{\partial f}{\partial \theta}\right), \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2\bar{z}}\left(r\frac{\partial f}{\partial r} + i\frac{\partial f}{\partial \theta}\right).$$
Hence, show that the principal logarithm Log is holomorphic on $\mathbb{C}\backslash(-\infty, 0]$ and has derivative z^{-1}.

3. * Show that every holomorphic function $f : U \to \mathbb{C}$ on some domain U is a constant function on U if it satisfies $f(z) = \overline{f(z)}$ for all $z \in U$.

4. Show that e^z is a biholomorphism (i.e. holomorphic bijection with holomorphic inverse) from the infinite strip $\{x+iy | x \in \mathbb{R}, -\pi < y < \pi\}$ onto the domain $\mathbb{C}\backslash(-\infty, 0]$.

5. Find the preimage of $(-\infty, 0]$ under the function $f(z) = 1 - \frac{1}{z}$. Hence or otherwise, suggest a branch cut for the function $\log\left(1 - \frac{1}{z}\right)$.

6. * Prove that the inverse of $\tanh^{-1} z$ is the multivalued function
$$\frac{1}{2}\log\frac{1+z}{1-z}.$$

7. * Find all possible values of the following.

   (a)* $\log(-1-i)$, (b) π^i,

   (c) $\cos^{-1} i$, (d) $i^{\frac{-1+i\sqrt{3}}{2}}$.

8. For $n = 1, 2, 3, 4$, sketch the closed curve $\gamma(t) = e^{2it}\sin(nt)$ for $0 \le t \le \pi$ and determine whether or not it is closed, simple and smooth.

9. * For each of the following cases, compute the integral of $f(z)$ along γ.

   (a) $f(z) = \mathrm{Im} z$ and γ is the line segment joining 0 and $3 + 4i$,

   (b) $f(z) = i\bar{z} + z^3$ and $\gamma(t) = 2e^{it}$, $\pi/2 \le t \le \pi$.

(c)* $f(z) = \text{pv}\, z^i$ and $\gamma(t) = e^{it}$, $|t| \leq \pi/2$.

10. Compute the length of the contour defined by

$$\gamma : [0, 2\pi] \to \mathbb{C}, \quad \gamma(t) = e^{-t}(\cos t + i \sin t).$$

11. * Let γ denote the line segment from $2i$ to 2. Show that

$$\Big| \int_\gamma \frac{1}{(z-1)^3} dz \Big| \leq 8.$$

12. Use ML inequality to find an upper bound for the absolute value of

$$\oint_\gamma e^{\bar{z}} dz$$

where γ parametrizes the square with vertices $0, 2, 2+2i$, and $2i$.

13. * Find a primitive of the principal branch of z^i. Hence, use this to calculate the integral in Exercise 9.(c). Show that your answer can be written in the form of $(1+i)\cosh(a)$ for some real number a.

14. By finding a primitive, evaluate each of the following integrals, where the path is taken to be any contour joining the indicated limits of integration.

$$\text{(a)} \int_0^i z^2 + i \, dz, \qquad \text{(b)} \int_{-\pi}^{\pi} \sin(iz) dz.$$

15. * Evaluate the integral

$$\oint_\gamma \frac{2}{2z^2 - z - 3} dz.$$

where γ is a pentagon with vertices 0, $1-i$, 3, $2+i$, and $1+i$.