

Solutions 1

1. The Cartesian and polar forms are as follows.

$$\begin{array}{ll} (a) \ i, e^{i\pi/2}, & (b) \ 1 + i, \sqrt{2}e^{i\pi/4} \\ (c) \ -16\sqrt{3} + 16i, 32e^{5\pi i/6}, & (d) \ -2, 2e^{\pi i}. \end{array}$$

2. It's sufficient to show $|z| - |w| \leq |z - w|$ and $|w| - |z| \leq |z - w|$. Both come from triangle inequality.

3. Since $\langle z, w \rangle = z\bar{w} = (x + iy)(u - iv) = (ux + vy) + i(uy - vx)$,

$$\begin{aligned} \operatorname{Re}\langle z, w \rangle &= ux + vy = (x, y) \cdot (u, v), \\ \overline{\langle w, z \rangle} &= \overline{w\bar{z}} = \bar{w}z = \langle z, w \rangle, \\ \langle z, z \rangle &= z\bar{z} = |z|^2 = x^2 + y^2 \geq 0. \end{aligned}$$

Equality on the last line holds if and only if x and y are 0.

4. We can use the identity $|z|^2 = z\bar{z}$. For every $z, w \in \mathbb{C}$,

$$\begin{aligned} |z \pm w|^2 &= (z \pm w)(\bar{z} \pm \bar{w}) = z\bar{z} + w\bar{w} \pm z\bar{w} \pm w\bar{z} \\ &= |z|^2 + |w|^2 \pm (z\bar{w} + \overline{z\bar{w}}) = |z|^2 + |w|^2 \pm 2\operatorname{Re}(z\bar{w}). \end{aligned}$$

Then,

$$\begin{aligned} |z + w|^2 - |z - w|^2 &= (|z|^2 + |w|^2 + 2\operatorname{Re}(z\bar{w})) - (|z|^2 + |w|^2 - 2\operatorname{Re}(z\bar{w})) \\ &= 4\operatorname{Re}(z\bar{w}). \end{aligned}$$

5. Since $w \neq 1$ and $w^n - 1 = 0$,

$$1 + w + \dots + w^{n-1} = \frac{w^n - 1}{w - 1} = 0.$$

Take the real value of the equation above to get:

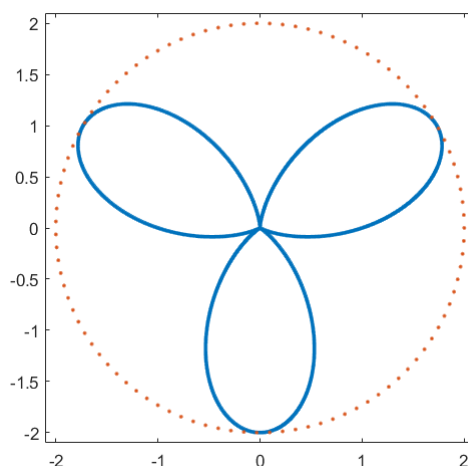
$$\cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{4\pi}{n}\right) + \dots + \cos\left(\frac{2(n-1)\pi}{n}\right) = 0.$$

6. Since $|-8 + 8i\sqrt{3}| = 16$ and $\operatorname{Arg}(-8 + 8i\sqrt{3}) = \frac{2\pi}{3}$, then $z^4 = 2^4 e^{2\pi i(3k+1)/3}$ for any integer k . Then,

$$z = 2e^{\pi i(3k+1)/6}, \text{ for } k \in \{0, 1, 2, 3\}.$$

Simplifying the expression, the roots are $\pm(\sqrt{3} + i)$ and $\pm(-1 + i\sqrt{3})$.

7. Let $\alpha = \cos(\frac{2\pi}{5})$ and $w = e^{2\pi i/5}$.
- (a) $\alpha = \operatorname{Re}(w) = \frac{w+\bar{w}}{2} = \frac{w+w^4}{2}$ and $\alpha^2 = \frac{w^2+w^3+2}{4}$.
 - (b) This is 0 from exercise 5.
 - (c) From part (b), we can pick $p = 4$, $q = 2$, and $r = -1$.
 - (d) By quadratic formula, $\alpha = \frac{-1 \pm \sqrt{5}}{4}$. We pick the $+$ sign since $\alpha > 0$.
8. I will only sketch (a); the rest should be fairly easy to illustrate.
- (a) It's the boundary of a 'flower' with three petals of maximum radius 2 centered at 0. See below.



- (b) When $z = x + iy$, the equation can be rewritten as $x^2 - y^2 = 1$, a hyperbola.
- (c) When $z = x + iy$, multiplying both top and bottom with the complex conjugate $\bar{z} - i$ gives you:

$$\frac{z - i}{z + i} = \frac{x^2 + y^2 - 1 - 2ix}{x^2 + (y + 1)^2}.$$

The denominator is always positive unless $z = -i$, on which the fraction is undefined. The real value is negative exactly when $x^2 + y^2 - 1 < 0$. This gives us the unit disk $\mathbb{D} = \{|z| < 1\}$.

- (d) The imaginary part of the fraction above is 0 when $-2ix = 0$. This gives us the set of purely imaginary numbers $\{iy \mid y \in \mathbb{R} \setminus \{-1\}\}$. We exclude $-i$ since the fractional expression is not defined at that point.

- (e) When $z = x + iy$, $\operatorname{Im} z^2 < 0$ exactly when $xy < 0$ and $\operatorname{Im}(z + 1 + i)^2 < 0$ exactly when $(x + 1)(y + 1) < 0$. This is the set $\{x + iy \mid x < -1, y > 0\} \cup \{x + iy \mid x > 0, y < -1\}$.
9. For each of the five sets in Exercise 9 above, determine whether or not they are open, closed, bounded, connected, simply connected or multiply connected.
- not open, compact, connected, multiply connected.
 - not open, closed, unbounded and disconnected.
 - open, not closed, bounded, simply connected.
 - not open, not closed, unbounded, disconnected.
 - open, not closed, unbounded, disconnected.
10. Refer to the definition of convergence of complex numbers.
11. No. Let $r_n = \frac{1}{n}$, $r = 0$, $\theta_n = (-1)^n \frac{\pi}{2}$, and $\theta = 0$. Then, $r_n e^{i\theta_n} = \frac{(-1)^n i}{n}$ converges to $re^{i\theta} = 0$. Even though $r_n \rightarrow r$, unfortunately $\theta_n \not\rightarrow \theta$.
12. It is easier when f is rewritten as $f(z) = z^2$. Then, for any $a \in \mathbb{C}$, the derivative always exists:

$$f'(a) = \lim_{z \rightarrow a} \frac{z^2 - a^2}{z - a} = \lim_{z \rightarrow a} z + a = 2a.$$

Alternatively, you may show that Cauchy Riemann equations hold throughout \mathbb{C} .

13. Upon computing the derivative at an arbitrary point $a \in \mathbb{C}$,

$$\lim_{z \rightarrow 0} \frac{|a + z|^2 - |a|^2}{z} = \lim_{z \rightarrow 0} \frac{z\bar{z} + \bar{a}z + a\bar{z}}{z} = \lim_{z \rightarrow 0} \bar{z} + \bar{a} + a\frac{\bar{z}}{z} = \bar{a} + a \lim_{z \rightarrow 0} \frac{\bar{z}}{z}.$$

When $a = 0$, it is clear that the limit above exists and is equal to 0. However, when $a \neq 0$, the limit does not exist since $\lim_{z \rightarrow 0} \frac{\bar{z}}{z}$ does not exist. Since $|z|^2$ is only complex differentiable at one point, it is not holomorphic on any domain.

Solutions 2

1. If $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$, then $\overline{f(\bar{z})} = p(x, y) + iq(x, y)$ where $p(x, y) = u(x, -y)$ and $q(x, y) = -v(x, -y)$. It remains to show that Cauchy Riemann equations still hold for the pair p and q on the domain $\bar{U} := \{\bar{z} \mid z \in U\}$, which is the reflection of U in the real axis. (Note that the correct domain for $\overline{f(\bar{z})}$ is \bar{U} , not U .)

2. This is merely an exercise in multivariable calculus. Use the chain rules:

$$\frac{\partial f}{\partial x} = \frac{x}{r} \frac{\partial f}{\partial r} - \frac{y}{r^2} \frac{\partial f}{\partial \theta}, \quad \frac{\partial f}{\partial y} = \frac{y}{r} \frac{\partial f}{\partial r} + \frac{x}{r^2} \frac{\partial f}{\partial \theta}.$$

Log is holomorphic because

$$\frac{d}{d\bar{z}} \text{Log} z = \frac{1}{2\bar{z}} \left(r \frac{\partial}{\partial r} + i \frac{\partial}{\partial \theta} \right) (\ln r + i\theta) = \frac{1}{2\bar{z}} (1 - 1) = 0.$$

Its derivative is

$$\frac{d}{dz} \text{Log} z = \frac{1}{2z} \left(r \frac{\partial}{\partial r} - i \frac{\partial}{\partial \theta} \right) (\ln r + i\theta) = \frac{1}{2z} (1 + 1) = \frac{1}{z}.$$

3. Let $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$. If $f(z) = \overline{f(z)}$, then $v(x, y) \equiv 0$. By Cauchy Riemann equations, $u_x = v_y \equiv 0$ and $u_y = -v_x \equiv 0$, so then $u(x, y) = c$ for some constant $c \in \mathbb{R}$. Therefore, $f(z) \equiv c$ on U .
4. Let $z = x + iy$. When $x \in \mathbb{R}$ and $|y| < \pi$, $e^{x+iy} = e^x e^{iy}$. The function is surjective because the image is

$$\{e^x e^{iy} \mid x \in \mathbb{R}, |y| < \pi\} = \{re^{iy} \mid r > 0, |y| < \pi\} = \mathbb{C} \setminus (-\infty, 0].$$

Since e^{iy} is 2π -periodic with respect to y and since the height of the strip is at most 2π , e^z is injective. The inverse of e^z is $\text{Log}(z)$ which is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$ with derivative z^{-1} .

5. The preimage is

$$\{z \mid 1 - z^{-1} \in (-\infty, 0]\} = \{(1 - x)^{-1} \mid x \in (-\infty, 0]\} = [-1, 0).$$

This can be taken as the branch cut because its image under $1 - z^{-1}$ is $(-\infty, 0]$, the usual branch cut for $\log(z)$.

6. Let $\tanh^{-1}(z) = w$, then $z = \frac{e^w - e^{-w}}{e^w + e^{-w}}$. This can be rewritten as

$$e^{2w} = \frac{1+z}{1-z}$$

Using logarithm, the expression becomes

$$w = \frac{1}{2} \log \frac{1+z}{1-z}.$$

7. Here, k represents any integer.

- (a) $\frac{1}{2} \ln 2 + i\frac{\pi}{4}(8k-3)$,
- (b) $e^{i \ln \pi}$,
- (c) $\frac{\pi}{2}(1+4k) - i \ln(1+\sqrt{2})$,
- (d) $e^{-\frac{\pi}{8}(1+4k)(\sqrt{3}+i)}$.

8. All are smooth and closed. The only simple ones are $n = 1, 2$.

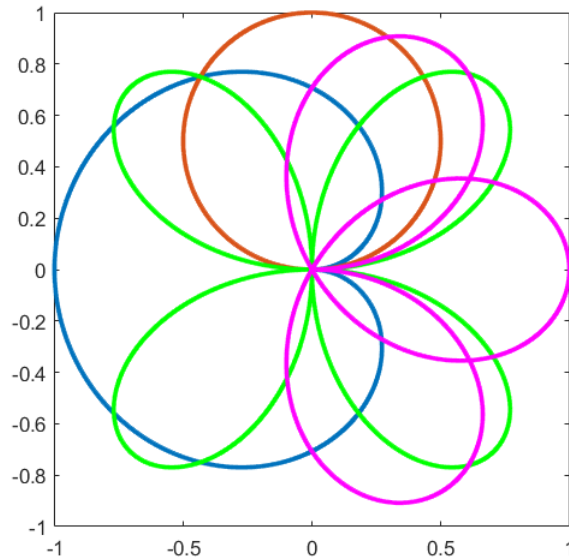


Figure 1: $n = 1$ in blue, $n = 2$ in red, $n = 3$ in pink and $n = 4$ in green

9. The integrals are as follows:

- (a) Use $\gamma(t) = (3+4i)t$ for $0 \leq t \leq 1$. Since $\gamma'(t) = 5$,

$$\int_{\gamma} \operatorname{Im} z dz = \int_0^1 4t \cdot |\gamma'(t)| dt = \int_0^1 20t dt = 10.$$

(b) Since $\gamma'(t) = 2ie^{it}$,

$$\int_{\gamma} i\bar{z} + iz^2 dz = \int_{\pi/2}^{\pi} (2ie^{-it} + 8e^{3it}) \cdot 2ie^{it} dt = \int_{\pi/2}^{\pi} -4 + 16ie^{4it} dt = -2\pi.$$

(c) Since $\gamma'(t) = ie^{it}$,

$$\begin{aligned} \int_{\gamma} \text{pv } z^i dz &= \int_{-\pi/2}^{\pi/2} e^{i\text{Log}(e^{it})} \cdot ie^{it} dt = \int_{-\pi/2}^{\pi/2} ie^{t(-1+i)} dt \\ &= \frac{i}{-1+i} (e^{\frac{\pi}{2}(-1+i)} - e^{\frac{\pi}{2}(1-i)}) = \frac{1-i}{2} (ie^{-\pi/2} + ie^{\pi/2}) \\ &= (1+i) \cosh(\pi/2). \end{aligned}$$

10. Since $\gamma'(t) = (-1+i)e^{(-1+i)t}$, the length of γ is

$$L(\gamma) = \int_0^{2\pi} |-1+i| dt = 2\pi\sqrt{2}.$$

11. The distance between the line segment γ and the point 1 is $2^{-1/2}$, so then

$$\max_{z \in \gamma} |(z-1)^{-3}| = (\min_{z \in \gamma} |z-1|)^{-3} = (2^{-1/2})^{-3} = 2\sqrt{2}.$$

Since $L(\gamma) = |2i-2| = 2\sqrt{2}$, then by ML inequality,

$$\left| \int_{\gamma} \frac{1}{(z-1)^3} dz \right| \leq 2\sqrt{2} \cdot L(\gamma) = 8.$$

12. Let $z = x + iy \in \gamma$, then $|e^{\bar{z}}| = e^x \leq e^2$ because $0 \leq x \leq 2$. Therefore,

$$\left| \int_{\gamma} e^{\bar{z}} dz \right| \leq L(\gamma) \cdot \max_{z \in \gamma} |e^{\bar{z}}| = 8e^2.$$

13. One primitive is $\frac{z^{i+1}}{i+1}$ because by chain rule, on $\mathbb{C} \setminus (-\infty, 0]$,

$$\frac{dz^{i+1}}{dz} = \frac{de^{(i+1)\text{Log}z}}{dz} = \frac{i+1}{z} \cdot e^{(i+1)\text{Log}z} = (i+1)z^i.$$

The curve γ lies in the domain $\mathbb{C} \setminus (-\infty, 0]$ and it travels from $-i$ to i . Then,

$$\begin{aligned} \int_{\gamma} \text{pv } z^i dz &= \frac{i^{i+1}}{i+1} - \frac{(-i)^{i+1}}{i+1} = \frac{1}{1+i} (ie^{i\text{Log}i} - (-i)e^{i\text{Log}(-i)}) \\ &= \frac{1-i}{2} (ie^{-\pi/2} + ie^{\pi/2}) = (1+i) \cosh(\pi/2). \end{aligned}$$

14. Both integrands are entire functions. As such, the integrals are independent of the choice of the contour.

(a) The integrand has primitive $iz + z^3/3$. Then,

$$\int_0^i z^2 + idz = iz + z^3/3 \Big|_0^i = -1 - i/3.$$

(b) The integrand has primitive $i \cosh z$. Then,

$$\int_{-\pi}^{\pi} \sin(iz) = i \cosh z \Big|_{-\pi}^{\pi} = 0.$$

15. The integrand can be rewritten as $\frac{2}{5} \left(\frac{1}{z-3/2} - \frac{1}{z+1} \right)$. Since -1 is outside of the pentagon γ but $3/2$ is enclosed by γ , we apply Cauchy-Goursat so that the integral is reduced to

$$\frac{2}{5} \int_{\gamma} \frac{1}{z - 3/2} dz.$$

By deformation theorem, we can replace γ with any small circle centered at $3/2$. The integral is then reduced to $4\pi i/5$.

Solutions 3

1. By partial fractions, the integral can be rewritten as

$$\frac{i}{12} \oint_{\gamma} \frac{dz}{z + 1.5i} - \frac{i}{12} \oint_{\gamma} \frac{dz}{z - 1.5i}.$$

The singular points we need to keep our eye on are $\pm 1.5i$.

- (a) The rectangle does not enclose $\pm 1.5i$. Both integrands are holomorphic along and inside γ . By Cauchy-Goursat, the integral is 0.
- (b) The circle only encloses $1.5i$, but not $-1.5i$. The first integral is 0 by Cauchy-Goursat. The second becomes $-\frac{i}{12} \cdot 2\pi i = \frac{\pi}{6}$. In total, the integral is πi .
- (c) Check that γ is a negatively oriented circle centered at 0 of radius π , enclosing both $\pm 1.5i$. Therefore, the integral evaluates to

$$\frac{i}{12} \cdot 2\pi i - \frac{i}{12} \cdot 2\pi i = 0.$$

2. The following functions g are holomorphic along and inside the domain enclosed by $C(0, 2)$.

- (a) Apply Cauchy's formula to $g(z) = \frac{z+2}{z-3}$ at the point $z_0 = 1$. The integral is

$$\oint_{C(0,2)} \frac{g(z)}{z-1} dz = 2\pi i g(1) = -3\pi i.$$

- (b) Apply Cauchy's formula to $g(z) = e^{e^z}$ at the point $z_0 = i\pi/2$. The integral is

$$\oint_{C(0,2)} \frac{g(z)}{z-i} dz = 2\pi i g(i) = 2\pi i e^{e^{i\pi/2}} = 2\pi i e^i.$$

- (c) Apply Cauchy's differentiation formula to get the 3rd derivative of $g(z) = \sinh(\pi z)$ at the point $z_0 = 0$. The integral is

$$\oint_{C(0,2)} \frac{g(z)}{z^4} dz = \frac{2\pi i}{3!} \frac{d^3}{dz^3} (\sinh(\pi z)) \Big|_{z=0} = \frac{\pi^4 i}{3}.$$

3. For any point $z_0 \in \mathbb{C}$, radius $r > 0$ and point w on the circle $C(z_0, r)$, we can apply triangle inequality to get $|w| \leq |w - z_0| + |z_0| = r + |z_0|$ and consequently $|f(w)| \leq \pi(r + |z_0|)$. By Cauchy's inequality,

$$|f''(z_0)| \leq \frac{2\pi(r + |z_0|)}{r^2}.$$

Taking the limit as $r \rightarrow \infty$, the right hand side goes to 0. Since $|f''(z_0)|$ is independent of r , $f''(z_0) = 0$ for all z_0 . The primitive f' must be some constant a and the primitive f of f' must be of the form $az + b$. However, since $|f(0)| \leq \pi \cdot 0 = 0$, b must be 0.

4. Since $f^{(6)}$ is bounded and entire, it is a constant function of some value a where $|a| > 0$. By taking primitive 6 times, f must be a polynomial of degree 6 because it has a leading term $\frac{a}{6!}z^6$.
5. The inequality implies that $f(z) \neq 0$ for all z , so $1/f(z)$ is a well-defined entire function. Since $|1/f(z)| \leq 1$, it is bounded and therefore constant. f is then constant too.
6. There is some constant $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. By ML inequality,

$$\begin{aligned} \left| \oint_{C(0,R)} \frac{f(z)}{(z - z_0)(z - z_1)} dz \right| &\leq 2\pi R \max_{|z|=R} \left| \frac{f(z)}{(z - z_0)(z - z_1)} \right| \\ &= 2\pi R \frac{M}{\min_{|z|=R} |(z - z_0)(z - z_1)|} \\ &\leq \frac{2\pi MR}{(R - |z_0|)(R - |z_1|)}. \end{aligned}$$

where the final inequality comes from triangle inequality. By taking the limit as $R \rightarrow \infty$, this upper bound clearly goes to 0, so then

$$\lim_{R \rightarrow \infty} \oint_{C(0,R)} \frac{f(z)}{(z - z_0)(z - z_1)} dz = 0.$$

This integral can be separated by partial fractions and evaluated by Cauchy's integral formula.

$$\begin{aligned} \oint_{C(0,R)} \frac{f(z)}{(z - z_0)(z - z_1)} dz &= \frac{1}{z_0 - z_1} \left[\oint_{C(0,R)} \frac{f(z)}{z - z_0} - \oint_{C(0,R)} \frac{f(z)}{z - z_1} dz \right] \\ &= \frac{f(z_0) - f(z_1)}{2\pi i(z_0 - z_1)}. \end{aligned}$$

This expression is independent of R , so then it must be 0. Therefore $f(z_0) = f(z_1)$.

7. It is holomorphic with derivative $2z$ on \mathbb{D} and 0 on the annulus $\{2 < |z| < 3\}$. It attains maximum on the annulus with $|f(z)| = 2$. The set U is disconnected and therefore the maximum modulus principle does not apply.
8. As f is entire, by maximum modulus principle, it is sufficient to see the behavior of f on the circle $\{|z| = 2\}$ to find maximum points. When $z = 2e^{it}$ where $t \in \mathbb{R}$,

$$\begin{aligned} |z^3 + i| &= |8e^{3it} + i| = |(8 \cos 3t + 1) + i8 \sin 3t| \\ &= [64 \cos^2 3t + 16 \cos 3t + 1 + 64 \sin^2 3t]^{1/2} = [65 + 16 \cos 3t]^{1/2} \end{aligned}$$

The real function $\cos 3t$ attains its maximum value 1 at $t = 0, \pm \frac{2\pi}{3}$. At any of these values, we have $|z^3 + i| = 9$, and this is attained by $z = 2, -1 \pm i\sqrt{3}$.

9. The function $e^{(1+i)z}$ is entire. By the maximum modulus principle, to find the maximum value of $e^{(1+i)z}$ on the closed square $\{x + iy \mid 1 < x, y < \pi\}$, it is sufficient to look at the the function along the boundary of the square. Let $z = x + iy$.

$$|e^{(1+i)z}| = |e^{(x-y)+i(x+y)}| = e^{x-y}.$$

The maximum of $x - y$ is attained on the boundary of the square when $x = \pi$ and $y = 1$. Therefore, the smallest radius is $r = e^{\pi-1}$.

10. Part (a) follows from applying the minimum modulus principle on $\mathbb{D}(z_0, \epsilon)$. If the lemma weren't true, it would in the most direct way contradict the minimum modulus principle. Part (b) follows from triangle inequality:

$$\begin{aligned} |f(z)| &\geq |a_d z^d| - \sum_{n=0}^{d-1} |a_n z^n| \geq |a_d| |z|^d - \sum_{n=0}^{d-1} |a_n| |z|^{d-1} \\ &\geq |z|^{d-1} \left(|a_d| |z| - \sum_{n=0}^{d-1} |a_n| \right) \geq |z|^{d-1} \geq R^{d-1}. \end{aligned}$$

For part (c), $|f|$ must attain minimum on the compact disk $\overline{\mathbb{D}(0, R)}$ where R is from part (b). Let z_0 be a minimum point in this compact disk. If $f(z_0) \neq 0$, then it will contradict part (a). Therefore, $f(z_0) = 0$.

Solutions 4

1. (a) $e^{2\pi z} = e^{4\pi^2} e^{2\pi(z-2\pi)} = e^{4\pi^2} \sum_{n=0}^{\infty} \frac{(2\pi)^n}{n!} (z-2\pi)^n$,
 (b) $\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-z^2)^n$,
 (c) $\sin z = \cos(z - \frac{\pi}{2}) = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (z - \frac{\pi}{2})^{2n}$.
2. Apply the identity theorem on any sequence of distinct points in V converging to some point in V . Such a sequence always exists because V is non-empty and open.
3. Apply the identity theorem on $\overline{f(\bar{z})}$ and $f(z)$ as both functions agree on \mathbb{R} .
4. Yes. Let $z = re^{i\theta}$ where $r > 1$. As $N \rightarrow \infty$, $z^{-N-1} \rightarrow 0$ because $|z^{-N-1}| = r^{-N-1} \rightarrow 0$. Therefore,

$$g(z) = \lim_{N \rightarrow \infty} \sum_{n=0}^N z^{-n} = \lim_{N \rightarrow \infty} \frac{1 - z^{-N-1}}{1 - z^{-1}} = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}.$$

5. (a) About i ,

$$\begin{aligned} \frac{z}{z^2+1} &= (z-i)^{-1} \frac{z}{z+i} = (z-i)^{-1} \left(1 - \frac{i}{z+i}\right) \\ &= (z-i)^{-1} \left(1 - \frac{1}{2(1 - \frac{i}{2}(z-i))}\right) \\ &= (z-i)^{-1} \left(1 - \sum_{n=0}^{\infty} \frac{i^n}{2^{n+1}} (z-i)^n\right) \\ &= \frac{1}{2} (z-i)^{-1} - \sum_{n=0}^{\infty} \frac{i^{n+1}}{2^{n+2}} (z-i)^n. \end{aligned}$$

This Laurent series is convergent on $\{0 < |z-i| < 2\}$.

- (b) About 0,

$$\begin{aligned} \frac{2}{z-2} + \frac{1}{4-z} &= \frac{2}{z(1-\frac{2}{z})} + \frac{1}{4(1-\frac{z}{4})} \\ &= \frac{2}{z} \sum_{n=0}^{\infty} 2^n z^{-n} + \frac{1}{4} \sum_{n=0}^{\infty} \frac{z^n}{4^n} \\ &= \sum_{n=-\infty}^{-1} 2^{-n} z^n + \sum_{n=0}^{\infty} 4^{-n-1} z^n. \end{aligned}$$

This Laurent series is convergent on $\{2 < |z| < 4\}$.

(c) About 1,

$$\begin{aligned}
 \frac{3-3z}{2z^2-5z+2} &= \left(\frac{1}{1-2z} + \frac{1}{2-z} \right) \\
 &= -\frac{1}{2(z-1)\left(1+\frac{1}{2(z-1)}\right)} + \frac{1}{1-(z-1)} \\
 &= -\frac{1}{2(z-1)} \sum_{n=0}^{\infty} \left(-\frac{1}{2(z-1)} \right)^n + \sum_{n=0}^{\infty} (z-1)^n. \\
 &= \sum_{n=-\infty}^{-1} (-2)^{-n} (z-1)^n + \sum_{n=0}^{\infty} (z-1)^n.
 \end{aligned}$$

This Laurent series is convergent on $\{\frac{1}{2} < |z-1| < 1\}$.

6. (a) The zeros of $\sin z$ are on πn for $n \in \mathbb{Z}$, and none of these are zeros of $\cos z$. Each of them is simple, so then $\cot z$ has simple poles at πn for $n \in \mathbb{Z}$.
- (b) Singularities are at point z such that $\sin z = \sin 2z$. This occurs when $\sin z = 0$, i.e. $z = n\pi$ for $n \in \mathbb{Z}$, or when $\cos z = \frac{1}{2}$, i.e. $z = \pm\frac{\pi}{3} + 2\pi n$ for $n \in \mathbb{Z}$. Each of these are single poles of the function.
- (c) The zeros of the denominator are clearly 0 of order 2 and ± 1 of order 1. The numerator does not have a zero at 0, but it has zeros at ± 1 . Therefore, 0 is a double pole and ± 1 are removable singularities.
7. The singularities of f/g are removable because $|f(z)/g(z)| \leq 1$, i.e. bounded. As such, f/g is a bounded entire function, which is a constant function a for some $a \in \mathbb{C}$.
8. (a) Since f has a zero of order $n \geq 1$, $g(z)$ is a well-defined holomorphic function with removable singularity at 0.
- (b) Along $|z| = r$ for any $r < 1$,

$$|g(z)| = \left| \frac{f(z)}{z} \right| < \frac{1}{r}.$$

As $r \rightarrow 1$, the upper bound converges to 1. Thus, the maximum modulus of g along the boundary is 1 and by MMP, $|g(z)| \geq 1$. This implies that $|f(z)| \leq |z|$. Looking at the Taylor series of f should convince you that $|f'(0)| = |g(0)| \leq 1$.

(c) If $|f'(0)| = 1$ or $|f(w)| = |w|$ for some point $w \in \mathbb{D}^*$, then $|g(w')| = 1$ where w' is either 0 or w . As g attains maximum in \mathbb{D} , it must be a constant function a and therefore $f(z) = az$. Since either $|f'(0)| = 1$ or $|f(w)| = |w|$, then $|a| = 1$. This implies that a is of the form $e^{i\theta}$ and clearly $f(z) = e^{i\theta}z$ is a counterclockwise rotation of the unit disk of angle θ .

9. It's easier to look at the image of the four line segments individually. Assume that the orientation of γ is positive. Using Cartesian coordinates $z = x + iy$, $\cos 2z - 1 = (\cos 2x \cosh 2y - 1) - i \sin 2x \sinh 2y$.

- When $x = \pm \frac{\pi}{4}$, $\cos 2z - 1 = -1 \mp i \sinh 2y$.
The image of the $x = -\frac{\pi}{4}$ side of the square is the same as that of the $x = -\frac{\pi}{4}$ side, which is a upward linear curve from $-1 - i \sinh \frac{\pi}{2}$ to $-1 + i \sinh \frac{\pi}{2}$.
- When $y = \pm \frac{\pi}{4}$, $\cos 2z - 1 = \cos 2x \cosh \frac{\pi}{2} - 1 \mp i \sin 2x \sinh \frac{\pi}{2}$.
The image of the $y = -\frac{\pi}{4}$ side of the square is the same as that of the $y = -\frac{\pi}{4}$ side, which is a downward elliptic arc with co-vertices $-1 \pm i \sinh \frac{\pi}{2}$ and rightmost vertex $-1 + \cosh \frac{\pi}{2}$.

The curve γ has a winding number two about the origin. Since $\cos 2z - 1$ has no poles, it must have exactly two zeros enclosed by γ . (It is in fact a double zero at 0.)

10. When $|z| = 1$, $|e^{z-1}| = e^{x-1} \leq 1 < 2 = |2z^n|$. By Rouché's theorem, $e^{z-1} + 2z^n$ has the same number of zeros as $2z^n$, which is n , inside \mathbb{D} .
11. When $|z| = 2$, $|5z + 1| \leq 5|z| + 1 = 11 < 32 = |z^5|$. By Rouché's theorem, $z^5 + 5z + 1$ has the same number of zeros as z^5 , which is 5, in $\mathbb{D}(0, 2)$. When $|z| = 1$, $|z^5| = 1 < 4 = |5z| - 1 \leq |5z + 1|$. Therefore, $z^5 + 5z + 1$ has the same number of zeros as $5z + 1$, which is 1, in \mathbb{D} . In total, $z^5 + 5z + 1$ has 4 zeros inside $\{1 \leq |z| < 2\}$.

Solutions 5

1. (a) The function $f(z) = \cot z$ has a pole of order 1 at 0. Then,

$$\operatorname{Res} f(0) = \frac{1}{0!} \lim_{z \rightarrow 0} z \cot z = \lim_{z \rightarrow 0} \cos z \frac{z}{\sin z} = 1.$$

- (b) $\cos z + 1$ has a double zero at π since its first derivative $-\sin z$ vanishes at π but the second derivative $-\cos z$ does not. The function $f(z) = \frac{z+\pi}{\cos z+1}$ at has a pole of order 2 at π . Then, using the change of variables $w = z - \pi$,

$$\begin{aligned} \operatorname{Res} f(\pi) &= \frac{1}{1!} \lim_{z \rightarrow \pi} \frac{d}{dz} \frac{(z - \pi)^2}{\cos z + 1} = \lim_{z \rightarrow \pi} \frac{d}{dz} \frac{(z - \pi)^2}{\cos z + 1} \\ &= \lim_{z \rightarrow \pi} \frac{2(z - \pi)(\cos z + 1) + \sin z(z - \pi)^2}{(\cos z + 1)^2} \\ &= \lim_{w \rightarrow 0} \frac{2w(1 - \cos w) - w^2 \sin w}{(1 - \cos w)^2} \\ &= \lim_{w \rightarrow 0} \frac{2w(\frac{w^2}{2} - \frac{w^4}{24} + \dots) - w^2(w - \frac{w^3}{6} + \dots)}{(\frac{w^2}{2} - \frac{w^4}{24} + \dots)^2} \\ &= \lim_{w \rightarrow 0} \frac{\frac{w^5}{12} + \dots}{\frac{w^4}{4} - \dots} = 0. \end{aligned}$$

2. (a) The function $f(z) = \frac{3z+1}{(z+2)(z-1)}$ has single poles at 1 and -2 . γ has winding number -1 about 1 and 0 about -2 . Thus,

$$\oint_{\gamma} f(z) dz = -1 \cdot 2\pi i \operatorname{Res} f(1) = -2\pi i \lim_{z \rightarrow 1} \frac{3z+1}{z+2} = -\frac{8\pi i}{3}.$$

- (b) The function $f(z) = e^{1/z}$ has an essential singularity at 0 and at that point, the residue is 1 since $e^{1/z} = 1 + z^{-1} + \frac{z^{-2}}{2} + \dots$. Since γ has winding number 1 about the origin,

$$\oint_{\gamma} f(z) dz = 2\pi i.$$

- (c) The function $f(z) = \csc(2\pi z)$ has single poles at every integer. γ has winding numbers 2, 1 and -1 about -1 , 0 and 1 respectively.

Therefore,

$$\begin{aligned}
\oint_{\gamma} f(z)dz &= 4\pi i \operatorname{Res} f(-1) + 2\pi i \operatorname{Res} f(0) - 2\pi i \operatorname{Res} f(1) \\
&= 4\pi i \lim_{z \rightarrow -1} \frac{z+1}{\sin 2\pi z} + 2\pi i \lim_{z \rightarrow 0} \frac{z}{\sin 2\pi z} - 2\pi i \lim_{z \rightarrow 1} \frac{z-1}{\sin 2\pi z} \\
&= 4 + 2 - 1 = 3.
\end{aligned}$$

3. By the change of variables $z = e^{i\theta}$, the integral can be transformed into a contour integral along the unit circle $\gamma(\theta) = e^{i\theta}$ where $0 \leq \theta \leq 2\pi$.

$$\int_0^{2\pi} \frac{d\theta}{1 - 2a \cos \theta + a^2} = \oint_{\gamma} \frac{i}{(az - 1)(z - a)} dz.$$

The only pole of the integrand enclosed by γ is a and it is a single pole. By residue theorem,

$$\int_0^{2\pi} \frac{d\theta}{1 - 2a \cos \theta + a^2} = 2\pi i \lim_{z \rightarrow a} \frac{i}{(az - 1)} = \frac{2\pi}{1 - a^2}.$$

4. (a) The integrand $f(z)$ is an even function and it has simple poles at $\pm i$ and $\pm 2i$. Use semicircular closed contour γ of radius $R > 2$. The poles enclosed by γ are i and $2i$. By residue theorem,

$$\oint_{\gamma} f(z)dz = 2\pi i (\operatorname{Res} f(i) + \operatorname{Res} f(2i)) = \dots = \frac{\pi}{3}.$$

By ML inequality that the semicircle part γ_2 of γ vanishes to 0 as $R \rightarrow \infty$ because

$$\left| \int_{\gamma_2} f(z)dz \right| \leq \pi R \cdot \max_{z \in \gamma_2} \left| \frac{z^2}{(z^2 + 1)(z^2 + 4)} \right| \leq \frac{\pi R^3}{(R^2 - 1)(R^2 - 4)} \rightarrow 0.$$

This leaves $\pi/3$ as the value of the integral of f on $(-\infty, \infty)$. Therefore,

$$\int_0^{\infty} \frac{x^2}{(x^2 + 1)(x^2 + 4)} dx = \frac{\pi}{6}.$$

- (b) The integrand

$$f(z) = \frac{ze^{iz}}{(z^2 + 1)(z^2 + 4)}$$

has simple poles at $\pm i$ and $\pm 2i$. Use semicircular closed contour γ of radius $R > 2$. The poles enclosed by γ are i and $2i$. By residue theorem,

$$\oint_{\gamma} f(z)dz = 2\pi i (\text{Res}f(i) + \text{Res}f(2i)) = \dots = \frac{\pi i}{3}(e^{-1} - e^{-2}).$$

By Jordan's lemma, the semicircle part γ_2 of γ vanishes to 0 as $R \rightarrow \infty$ because

$$\left| \int_{\gamma_2} f(z)dz \right| \leq \pi \cdot \max_{z \in \gamma_2} \left| \frac{z}{(z^2 + 1)(z^2 + 4)} \right| \leq \frac{\pi R}{(R^2 - 1)(R^2 - 4)} \rightarrow 0.$$

Therefore, the integral of f on $(-\infty, \infty)$ is equal to that along γ . By taking the imaginary part,

$$\int_0^{\infty} \frac{z \sin z}{(z^2 + 1)(z^2 + 4)} dx = \frac{\pi}{3}(e^{-1} - e^{-2}).$$

- (c) Substitute $y = x - \pi$ so that $\sin x = -\sin y$. From the example in class, this integral is $-\pi$.
- (d) You can use the semicircular contour, but I'll use the sector contour γ with angle $\pi/2$ instead. Let $f(z)$ be the integrand; γ will enclose the single pole of f at $e^{i\pi/4}$. Let's use the same notation as in the notes.

$$I_0 = 2\pi i \text{Res}f(e^{i\pi/4}) = \dots = \frac{\pi}{2\sqrt{2}}(1 - i).$$

Use the parametrisation $\gamma_3(r) = ri$ as r varies from R to 0 and obtain that

$$I_3 = \int_R^0 \frac{1}{(ri)^4 + 1} i dr = -i \int_0^R \frac{1}{R^4 + 1} dr = -iI_1.$$

Also, $I_2 \rightarrow 0$ as $R \rightarrow \infty$ because by ML inequality

$$\left| \int_{\gamma_2} f(z)dz \right| \leq \frac{\pi R}{2} \max_{z \in \gamma_2} \frac{1}{|z^4 + 1|} \leq \frac{\pi R}{2(R^4 - 1)} \rightarrow 0.$$

Then, taking the limit $R \rightarrow \infty$ and after rearranging, you should obtain

$$\int_0^{\infty} \frac{1}{1 + x^4} dx = \frac{\pi}{2\sqrt{2}}.$$

- (e) The function $f(z) = \frac{1}{z^{1/2}(z^2+9)}$ has simple poles at $\pm 3i$. Pick the branch cut to be $\arg z = 0$. Use the keyhole contour to evaluate the given integral I . Using the same notation as in the notes,

$$I_0 = 2\pi i [\text{Res} f(3i) + \text{Res} f(-3i)] = \dots = \frac{\pi}{3} \sqrt{\frac{2}{3}}.$$

Taking $R \rightarrow \infty$ and $\epsilon, \delta \rightarrow 0$, check that $I_1 \rightarrow I$ and that

$$\begin{aligned} I_3 &= \int_{\gamma_3} \frac{1}{z^{1/2}(z^2+9)} dz = \int_R^\rho \frac{1}{\sqrt{r} e^{i(\pi-\epsilon/2)} (r^2 e^{i(4\pi-2\epsilon)} + 9)} e^{i(2\pi-\epsilon)} dr \\ &\rightarrow - \int_0^\infty \frac{1}{e^{i\pi} \sqrt{r} (r^2+9)} dz = -e^{-\pi i} I = I. \end{aligned}$$

Check that by ML inequality, we have $I_2, I_4 \rightarrow 0$. Therefore, this gives $2I = I_0$ and upon simplifying, $I = \frac{\pi}{3\sqrt{6}}$.

- (f) Let $f(z)$ be the integrand. It has a triple pole at -1 . Pick the branch cut to be $\arg z = 0$. Use the keyhole contour to evaluate the given integral I . Using the same notation as in the notes,

$$I_0 = 2\pi i \text{Res} f(-1) = \dots = -\frac{2\pi i}{9z^2} \text{p.v.} z^{1/3} \Big|_{z=-1} = \frac{\pi}{9} (\sqrt{3} - i).$$

Taking $R \rightarrow \infty$ and $\epsilon, \delta \rightarrow 0$, check that $I_1 \rightarrow I$ and $I_2 \rightarrow \frac{1-i\sqrt{3}}{2} I$. The latter is because

$$\begin{aligned} I_3 &= \int_{\gamma_3} \frac{\sqrt[3]{z}}{(z+1)^3} dz = \int_R^\rho \frac{\sqrt[3]{r} e^{i(2\pi/3-\epsilon/2)}}{(r e^{-i\epsilon} + 1)^3} e^{i(2\pi-\epsilon)} dr \\ &\rightarrow -e^{2\pi i/3} \int_0^\infty \frac{\sqrt[3]{r}}{(r+1)^3} dr = -e^{2\pi i/3} I. \end{aligned}$$

Check that by ML inequality, we have $I_2, I_4 \rightarrow 0$. Therefore, this gives $\frac{3-i\sqrt{3}}{2} I = I_0$ and upon simplifying, $I = \frac{2\pi}{9\sqrt{3}}$.

5. This is the trickiest question in the problem set. The usual branch cut for \log is $[-\infty, 0]$ and $z^2 + 1 \in [-\infty, 0]$ precisely when $z^2 \in [-\infty, -1]$ and therefore the branch cut is $\{ai \mid a \geq 1, a \leq -1\}$, a union of two vertical rays. To evaluate the integral I asked, it is easier to split the integrand into $f + g$ where

$$f(z) = \frac{\text{Log}(z+i)}{z^2+1}, \quad g(z) = \frac{\text{Log}(z-i)}{z^2+1}.$$

The branch cut of f can be taken to be $\{ai \mid a \leq 1\}$ and that of g can be taken to be $\{ai \mid a \geq 1\}$.

The integral of f along $(-\infty, \infty)$ can be evaluated using the usual semicircular contour $\gamma = \gamma_1 \cup \gamma_2$ where $\gamma_1 = [-R, R]$ and γ_2 is an upper semicircle of radius $R > 0$. With the usual argument, you may check that by ML inequality, the integral of f along γ_2 vanishes to 0 as $R \rightarrow \infty$. Therefore,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\operatorname{Log}(x+i)}{x^2+1} dx &= \lim_{R \rightarrow \infty} \int_{\gamma_1} \frac{\operatorname{Log}(z+i)}{z^2+1} dz \\ &= \lim_{R \rightarrow \infty} \oint_{\gamma} \frac{\operatorname{Log}(z+i)}{z^2+1} dz \\ &= 2\pi i \operatorname{Res} f(i) = \dots = \pi \ln 2 + \frac{\pi^2 i}{2}. \end{aligned}$$

To avoid the branch cut of g , we evaluate the integral of g using the lower semicircular contour $\sigma = \sigma_1 \cup \sigma_2$ where σ_1 is the segment from R to $-R$ and $\sigma_2 = \{Re^{i\theta} \mid -\pi \leq \theta \leq 0\}$ is the lower semicircle of radius $R > 0$. With the usual argument, you may check that by ML inequality, the integral of g along σ_2 vanishes to 0 as $R \rightarrow \infty$. Therefore,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\operatorname{Log}(x-i)}{x^2+1} dx &= - \lim_{R \rightarrow \infty} \int_{\sigma_1} \frac{\operatorname{Log}(z-i)}{z^2+1} dz \\ &= - \lim_{R \rightarrow \infty} \oint_{\sigma} \frac{\operatorname{Log}(z-i)}{z^2+1} dz \\ &= -2\pi i \operatorname{Res} g(-i) = \dots = \pi \ln 2 - \frac{\pi^2 i}{2}. \end{aligned}$$

Summing the two integrals together, we obtain

$$\int_{-\infty}^{\infty} \frac{\operatorname{Log}(x-i)}{x^2+1} dx = 2\pi \ln 2.$$

Since the integrand is an even function, we can divide by two and obtain that the integral we wanted to find all along is indeed $\pi \ln 2$.

6. You can check that $U_z = \frac{1}{2}U_x + i\left(-\frac{1}{2}U_y\right)$ satisfies Cauchy-Riemann equations. Alternatively, you may check that the Laplacian can be expressed using Wirtinger derivatives:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z}.$$

This implies that $\frac{\partial}{\partial \bar{z}} U_z = \frac{1}{4} \Delta U = 0$, i.e. U_z is holomorphic.

7. This is another calculus exercise. Compute the Laplacian accordingly and show that it vanishes to 0. At $(0,0)$, the function is not even continuous, since

$$\lim_{x \rightarrow 0} \frac{0}{x^2 + 0^2} = 0 \neq \infty = \lim_{y \rightarrow 0} \frac{y}{0^2 + y^2}.$$

8. Let $u(x, y)$ be a bounded harmonic function on \mathbb{R}^2 . Pick any harmonic conjugate v of u . Then, $f = u + iv$ is an entire function and so is $e^{f(z)}$. Since u is bounded, so is $|e^{f(z)}| = e^{u(x,y)}$. By Liouville, $e^{f(z)}$, $f(z)$ and ultimately u are constant.
9. The difference $u = u_1 - u_2$ is harmonic on U and vanishes on the whole subset V . Since u_z is holomorphic on U and vanishes on the whole V , then $u_z \equiv 0$ on U by the identity theorem. Since $2u_z = u_x - iu_y$, then $u_x \equiv u_y \equiv 0$, i.e. u is a constant function, so it must be the zero function.
10. Let $f = u + iv$ where u and v are real-valued functions, then $g = u^2 + v^2$. Using harmonicity of u and v ,

$$\begin{aligned} \Delta g &= \frac{\partial}{\partial x}(2uu_x + 2vv_x) + \frac{\partial}{\partial y}(2uu_y + 2vv_y) \\ &= 2uu_{xx} + 2u_x^2 + 2vv_{xx} + 2v_x^2 + 2uu_{yy} + 2u_y^2 + 2vv_{yy} + 2v_y^2 \\ &= 2u(u_{xx} + u_{yy}) + 2v(v_{xx} + v_{yy}) + 2(u_x^2 + u_y^2 + v_x^2 + v_y^2) \\ &= 2(u_x^2 + u_y^2 + v_x^2 + v_y^2). \end{aligned}$$

Since g is harmonic, the expression above is 0 and therefore, $u_x \equiv u_y \equiv v_x \equiv v_y \equiv 0$ on U . This shows that f is constant.

11. (a) $\frac{r}{w-r} = \frac{r}{w} \frac{1}{1-\frac{r}{w}} = \frac{r}{w} \sum_{n \geq 0} \left(\frac{r}{w}\right)^n = \sum_{n \geq 1} r^n w^{-n}$.
- (b) Let $w = z = e^{i\theta}$. Then,

$$\frac{r}{w-r} = \frac{r}{e^{i\theta}-r} = \frac{r(e^{-i\theta}-r)}{|e^{i\theta}-r|^2} = \frac{r(\cos \theta - r) - ir \sin \theta}{1 - 2r \cos \theta + r^2}$$

and by de Moivre's theorem,

$$\begin{aligned} \sum_{n \geq 1} r^n w^{-n} &= \sum_{n \geq 1} r^n (\cos(n\theta) - i \sin(n\theta)) \\ &= \left(\sum_{n \geq 1} r^n \cos(n\theta) \right) - i \left(\sum_{n \geq 1} r^n \sin(n\theta) \right). \end{aligned}$$

Comparing the real and imaginary parts should give the equations we wanted.

- (c) By now, this is just some basic algebraic manipulation inferior to everything else you've done.

12. (a) This example is similar to the one done in class.

$$\begin{aligned} u(r, \theta) &= \frac{1}{2\pi} \int_0^{\pi/2} P(r, t - \theta) dt = 2 \tan^{-1} \left(\frac{1+r}{1-r} \tan \frac{t - \theta}{2} \right) \Big|_0^{\pi/2} \\ &= \frac{1}{\pi} \tan^{-1} \left(\frac{1+r}{1-r} \tan \frac{\pi - 2\theta}{4} \right) + \frac{1}{\pi} \tan^{-1} \left(\frac{1+r}{1-r} \tan \frac{\theta}{2} \right). \end{aligned}$$

- (b) Use the cosine series on Qn 11 to integrate.

$$\begin{aligned} u(r, \theta) &= \frac{1}{2\pi} \int_0^{2\pi} P(r, t - \theta) \cos t \, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(1 + \sum_{n \geq 1} 2r^n \cos(n(t - \theta)) \right) \cos t \, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \cos t \, dt + \frac{1}{\pi} \sum_{n \geq 1} r^n \int_0^{2\pi} \cos(n(t - \theta)) \cos t \, dt \\ &= \frac{1}{2\pi} \sum_{n \geq 1} r^n \int_0^{2\pi} \cos(n(t - \theta) + t) + \cos(n(t - \theta) - t) \, dt \\ &= r \cos \theta. \end{aligned}$$

(Yes... the corresponding holomorphic function f such that $\operatorname{Re} f = u$ is just the identity function $f(z) = z$.)