Solutions 1

1. The Cartesian and polar forms are as follows.

$$(a)\ i, e^{i\pi/2}, \qquad (b)\ 1+i, \sqrt{2}e^{i\pi/4}$$
$$(c)\ -16\sqrt{3}+16i, 32e^{5\pi i/6}, \qquad (d)\ -2, 2e^{\pi i}.$$

2. It's sufficient to show $|z|-|w| \leq |z-w|$ and $|w|-|z| \leq |z-w|$. Both come from triangle inequality.

3. Since $\langle z, w\rangle = z\bar{w} = (x+iy)(u-iv) = (ux+vy)+i(uy-vx)$,

$$\begin{aligned}
\mathrm{Re}\langle z, w\rangle &= ux+vy = (x,y)\cdot(u,v),\\
\overline{\langle w, z\rangle} &= \overline{w\bar{z}} = \bar{w}z = \langle z, w\rangle,\\
\langle z, z\rangle &= z\bar{z} = |z|^2 = x^2+y^2 \geq 0.
\end{aligned}$$

Equality on the last line holds if and only if x and y are 0.

4. We can use the identity $|z|^2 = z\bar{z}$. For every $z, w \in \mathbb{C}$,

$$\begin{aligned}
|z\pm w|^2 &= (z\pm w)(\bar{z}\pm\bar{w}) = z\bar{z}+w\bar{w}\pm z\bar{w}\pm w\bar{z}\\
&= |z|^2+|w|^2\pm(z\bar{w}+\overline{z\bar{w}}) = |z|^2+|w|^2\pm 2\mathrm{Re}(z\bar{w}).
\end{aligned}$$

Then,

$$\begin{aligned}
|z+w|^2-|z-w|^2 &= \big(|z|^2+|w|^2+2\mathrm{Re}(z\bar{w})\big)-\big(|z|^2+|w|^2-2\mathrm{Re}(z\bar{w})\big)\\
&= 4\mathrm{Re}(z\bar{w}).
\end{aligned}$$

5. Since $w \neq 1$ and $w^n-1=0$,

$$1+w+\ldots w^{n-1} = \frac{w^n-1}{w-1} = 0.$$

Take the real value of the equation above to get:

$$\cos\left(\frac{2\pi}{n}\right)+\cos\left(\frac{4\pi}{n}\right)+\ldots+\cos\left(\frac{2(n-1)\pi}{n}\right) = 0.$$

6. Since $|-8+8i\sqrt{3}| = 16$ and $\mathrm{Arg}(-8+8i\sqrt{3}) = \frac{2\pi}{3}$, then $z^4 = 2^4e^{2\pi i(3k+1)/3}$ for any integer k. Then,

$$z = 2e^{\pi i(3k+1)/6}, \text{ for } k \in \{0,1,2,3\}.$$

Simplifying the expression, the roots are $\pm(\sqrt{3}+i)$ and $\pm(-1+i\sqrt{3})$.

7. Let $\alpha = \cos(\frac{2\pi}{5})$ and $w = e^{2\pi i/5}$.

   (a) $\alpha = \mathrm{Re}(w) = \frac{w+\bar{w}}{2} = \frac{w+w^4}{2}$ and $\alpha^2 = \frac{w^2+w^3+2}{4}$.

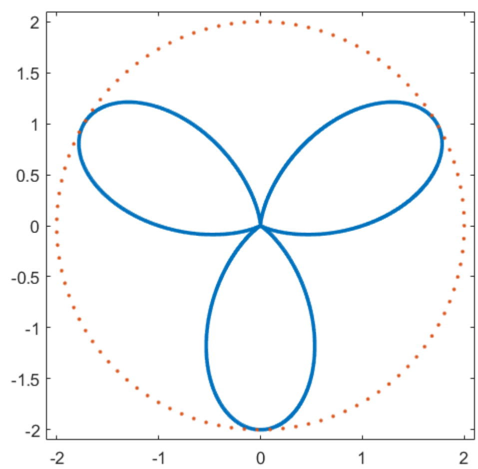
   (b) This is 0 from exercise 5.

   (c) From part (b), we can pick $p = 4$, $q = 2$, and $r = -1$.

   (d) By quadratic formula, $\alpha = \frac{-1\pm\sqrt{5}}{4}$. We pick the $+$ sign since $\alpha > 0$.

8. I will only sketch (a); the rest should be fairly easy to illustrate.

   (a) It's the boundary of a 'flower' with three petals of maximum radius 2 centered at 0. See below.



   (b) When $z = x + iy$, the equation can be rewritten as $x^2 - y^2 = 1$, a hyperbola.

   (c) When $z = x + iy$, multiplying both top and bottom with the complex conjugate $\bar{z} - i$ gives you:

$$\frac{z-i}{z+i} = \frac{x^2 + y^2 - 1 - 2ix}{x^2 + (y+1)^2}.$$

   The denominator is always positive unless $z = -i$, on which the fraction is undefined. The real value is negative exactly when $x^2 + y^2 - 1 < 0$. This gives us the unit disk $\mathbb{D} = \{|z| < 1\}$.

   (d) The imaginary part of the fraction above is 0 when $-2ix = 0$. This gives us the set of purely imaginary numbers $\{iy \mid y \in \mathbb{R}\backslash\{-1\}\}$. We exclude $-i$ since the fractional expression is not defined at that point.

(e) When $z = x + iy$, $\mathrm{Im} z^2 < 0$ exactly when $xy < 0$ and $\mathrm{Im}(z + 1 + i)^2 < 0$ exactly when $(x+1)(y+1) < 0$. This is the set $\{x + iy \mid x < -1, y > 0\} \cup \{x + iy \mid x > 0, y < -1\}$.

9. For each of the five sets in Exercise 9 above, determine whether or not they are open, closed, bounded, connected, simply connected or multiply connected.

   (a) not open, compact, connected, multiply connected.

   (b) not open, closed, unbounded and disconnected.

   (c) open, not closed, bounded, simply connected.

   (d) not open, not closed, unbounded, disconnected.

   (e) open, not closed, unbounded, disconnected.

10. Refer to the definition of convergence of complex numbers.

11. No. Let $r_n = \frac{1}{n}$, $r = 0$, $\theta_n = (-1)^n \frac{\pi}{2}$, and $\theta = 0$. Then, $r_n e^{i\theta_n} = \frac{(-1)^n i}{n}$ converges to $re^{i\theta} = 0$. Even though $r_n \to r$, unfornutately $\theta_n \not\to \theta$.

12. It is easier when f is rewritten as $f(z) = z^2$. Then, for any $a \in \mathbb{C}$, the derivative always exists:

$$f'(a) = \lim_{z \to a} \frac{z^2 - a^2}{z - a} = \lim_{z \to a} z + a = 2a.$$

Alternatively, you may show that Cauchy Riemann equations hold throughout $\mathbb{C}$.

13. Upon computing the derivative at an arbitrary point $a \in \mathbb{C}$,

$$\lim_{z \to 0} \frac{|a + z|^2 - |a|^2}{z} = \lim_{z \to 0} \frac{z\bar{z} + \bar{a}z + a\bar{z}}{z} = \lim_{z \to 0} \bar{z} + \bar{a} + a\frac{\bar{z}}{z} = \bar{a} + a \lim_{z \to 0} \frac{\bar{z}}{z}.$$

When $a = 0$, it is clear that the limit above exists and is equal to 0. However, when $a \neq 0$, the limit does not exist since $\lim_{z \to 0} \frac{\bar{z}}{z}$ does not exist. Since $|z|^2$ is only complex differentiable at one point, it is not holomorphic on any domain.

Solutions 2

1. If $z = x+iy$ and $f(z) = u(x,y)+iv(x,y)$, then $\overline{f(\bar{z})} = p(x,y)+iq(x,y)$ where $p(x,y) = u(x,-y)$ and $q(x,y) = -v(x,-y)$. It remains to show that Cauchy Riemann equations still hold for the pair p and q on the domain $\bar{U} := \{\bar{z} \mid z \in U\}$, which is the reflection of U in the real axis. (Note that the correct domain for $\overline{f(\bar{z})}$ is $\bar{U}$, not U.)

2. This is merely an exercise in multivariable calculus. Use the chain rules:

$$\frac{\partial f}{\partial x} = \frac{x}{r}\frac{\partial f}{\partial r} - \frac{y}{r^2}\frac{\partial f}{\partial \theta}, \qquad \frac{\partial f}{\partial y} = \frac{y}{r}\frac{\partial f}{\partial r} + \frac{x}{r^2}\frac{\partial f}{\partial \theta}.$$

   Log is holomorphic because

$$\frac{d}{d\bar{z}}\text{Log}z = \frac{1}{2\bar{z}}\left(r\frac{\partial}{\partial r} + i\frac{\partial}{\partial \theta}\right)(\ln r + i\theta) = \frac{1}{2\bar{z}}(1-1) = 0.$$

   Its derivative is

$$\frac{d}{dz}\text{Log}z = \frac{1}{2z}\left(r\frac{\partial}{\partial r} - i\frac{\partial}{\partial \theta}\right)(\ln r + i\theta) = \frac{1}{2z}(1+1) = \frac{1}{z}.$$

3. Let $z = x+iy$ and $f(z) = u(x,y)+iv(x,y)$. If $f(z) = \overline{f(z)}$, then $v(x,y) \equiv 0$. By Cauchy Riemann equations, $u_x = v_y \equiv 0$ and $u_y = -v_x \equiv 0$, so then $u(x,y) = c$ for some constant $c \in \mathbb{R}$. Therefore, $f(z) \equiv c$ on U.

4. Let $z = x+iy$. When $x \in \mathbb{R}$ and $|y| < \pi$, $e^{x+iy} = e^x e^{iy}$. The function is surjective because the image is

$$\{e^x e^{iy} \mid x \in \mathbb{R}, |y| < \pi\} = \{re^{iy} \mid r > 0, |y| < \pi\} = \mathbb{C}\backslash(-\infty, 0].$$

   Since e^{iy} is 2π-periodic with respect to y and since the height of the strip is at most 2π, e^z is injective. The inverse of e^z is $\text{Log}(z)$ which is holomorphic on $\mathbb{C}\backslash(-\infty,0]$ with derivative z^{-1}.

5. The preimage is

$$\{z \mid 1 - z^{-1} \in (-\infty, 0]\} = \{(1-x)^{-1} \mid x \in (-\infty, 0]\} = [-1, 0).$$

   This can be taken as the branch cut because its image under $1 - z^{-1}$ is $(-\infty, 0]$, the usual branch cut for $\log(z)$.

6. Let $\tanh^{-1}(z) = w$, then $z = \frac{e^w - e^{-w}}{e^w + e^{-w}}$. This can be rewritten as

$$e^{2w} = \frac{1+z}{1-z}$$

Using logarithm, the expression becomes

$$w = \frac{1}{2}\log\frac{1+z}{1-z}.$$

7. Here, k represents any integer.

   (a) $\frac{1}{2}\ln 2 + i\frac{\pi}{4}(8k-3)$,
   (b) $e^{i\ln\pi}$,
   (c) $\frac{\pi}{2}(1+4k) - i\ln(1+\sqrt{2})$,
   (d) $e^{-\frac{\pi}{8}(1+4k)(\sqrt{3}+i)}$.

8. All are smooth and closed. The only simple ones are $n = 1, 2$.

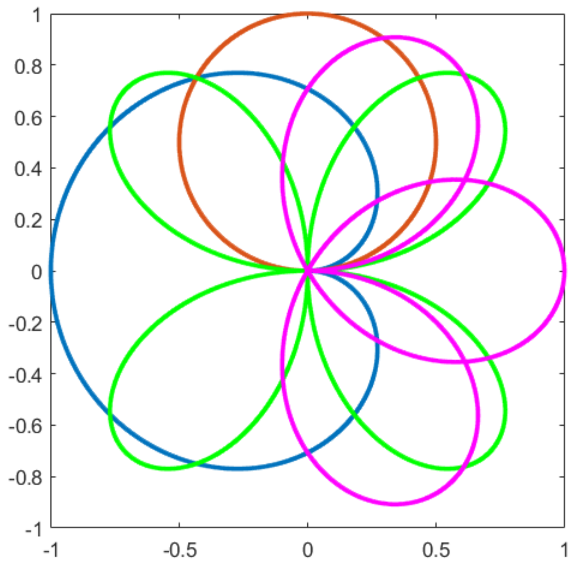


Figure 1: $n = 1$ in blue, $n = 2$ in red, $n = 3$ in pink and $n = 4$ in green

9. The integrals are as follows:

   (a) Use $\gamma(t) = (3+4i)t$ for $0 \le t \le 1$. Since $\gamma'(t) = 5$,

$$\int_\gamma \mathrm{Im} z dz = \int_0^1 4t \cdot |\gamma'(t)| dt = \int_0^1 20t\, dt = 10.$$

(b) Since $\gamma'(t) = 2ie^{it}$,

$$\int_\gamma i\bar{z}+iz^2dz = \int_{\pi/2}^{\pi} (2ie^{-it}+8e^{3it})\cdot 2ie^{it}dt = \int_{\pi/2}^{\pi} -4+16ie^{4it}dt = -2\pi.$$

(c) Since $\gamma'(t) = ie^{it}$,

$$\begin{aligned}\int_\gamma \mathrm{pv}\, z^i dz &= \int_{-\pi/2}^{\pi/2} e^{i\mathrm{Log}(e^{it})} \cdot ie^{it}dt = \int_{-\pi/2}^{\pi/2} ie^{t(-1+i)}dt \\ &= \frac{i}{-1+i}(e^{\frac{\pi}{2}(-1+i)} - e^{\frac{\pi}{2}(1-i)}) = \frac{1-i}{2}(ie^{-\pi/2} + ie^{\pi/2}) \\ &= (1+i)\cosh(\pi/2).\end{aligned}$$

10. Since $\gamma'(t) = (-1+i)e^{(-1+i)t}$, the length of γ is

$$L(\gamma) = \int_0^{2\pi} |-1+i|dt = 2\pi\sqrt{2}.$$

11. The distance between the line segment γ and the point 1 is $2^{-1/2}$, so then

$$\max_{z\in\gamma} |(z-1)^{-3}| = (\min_{z\in\gamma} |z-1|)^{-3} = (2^{-1/2})^{-3} = 2\sqrt{2}.$$

Since $L(\gamma) = |2i-2| = 2\sqrt{2}$, then by ML inequality,

$$\Big|\int_\gamma \frac{1}{(z-1)^3}dz\Big| \le 2\sqrt{2}\cdot L(\gamma) = 8.$$

12. Let $z = x+iy \in \gamma$, then $|e^{\bar{z}}| = e^x \le e^2$ because $0 \le x \le 2$. Therefore,

$$\Big|\int_\gamma e^{\bar{z}}dz\Big| \le L(\gamma)\cdot \max_{z\in\gamma} |e^{\bar{z}}| = 8e^2.$$

13. One primitive is $\frac{z^{i+1}}{i+1}$ because by chain rule, on $\mathbb{C}\backslash(-\infty, 0]$,

$$\frac{dz^{i+1}}{dz} = \frac{de^{(i+1)\mathrm{Log}z}}{dz} = \frac{i+1}{z}\cdot e^{(i+1)\mathrm{Log}z} = (i+1)z^i.$$

The curve γ lies in the domain $\mathbb{C}\backslash(-\infty, 0]$ and it travels from $-i$ to i. Then,

$$\begin{aligned}\int_\gamma \mathrm{pv}\, z^i dz &= \frac{i^{i+1}}{i+1} - \frac{(-i)^{i+1}}{i+1} = \frac{1}{1+i}(ie^{i\mathrm{Log}i} - (-i)e^{i\mathrm{Log}(-i)}) \\ &= \frac{1-i}{2}(ie^{-\pi/2} + ie^{\pi/2}) = (1+i)\cosh(\pi/2).\end{aligned}$$

14. Both integrands are entire functions. As such, the integrals are independent of the choice of the contour.

    (a) The integrand has primitive $iz + z^3/3$. Then,

    $$\int_0^i z^2 + idz = iz + z^3/3|_0^i = -1 - i/3.$$

    (b) The integrand has primitive $i \cosh z$. Then,

    $$\int_{-\pi}^{\pi} \sin(iz) = i \cosh z|_{-\pi}^{\pi} = 0.$$

15. The integrand can be rewritten as $\frac{2}{5}\left(\frac{1}{z-3/2} - \frac{1}{z+1}\right)$. Since -1 is outside of the pentagon γ but $3/2$ is enclosed by γ, we apply Cauchy-Goursat so that the integral is reduced to

    $$\frac{2}{5}\int_\gamma \frac{1}{z-3/2}dz.$$

    By deformation theorem, we can replace γ with any small circle centered at $3/2$. The integral is then reduced to $4\pi i/5$.