Quasiconformal Deformations in Holomorphic Dynamics: Sullivan's No Wandering Domain Theorem, and Fatou-Shishikura Inequality

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Abstract

This report discusses the role of quasiconformal methods in proving two fundamental theorems in the study of the iterations of rational maps: Sullivan's No Wandering Domain Theorem and the Fatou-Shishikura Inequality.

1 Introduction

The theory of holomorphic dynamics has been flourishing in the last few decades. In the advent of the study of iteration of rational maps, Pierre Fatou conjectured that all components of the Fatou set $F(f)$ of a rational map f are non-wandering, i.e. all are eventually periodic. It is a significant statement on the dynamics of the stable regions of f, which was only proven by Dennis Sullivan in [Sul85] using a technique involving quasiconformal homeomorphisms in 1985. It was in the early 1980s when new techniques of quasiconformal deformation and specifically quasiconformal surgery were developed.

Theorem 1.1 (No Wandering Domain Theorem)**.** *Any rational map f has no wandering Fatou components.*

Two years later, Mitsuhiro Shishikura also used quasiconformal surgery to give a sharp upper bound $(2d-2)$ on the number of non-repelling periodic cycles in [Shi87]. This was a considerable improvement from Fatou's inequality, which states that the sum of the number of attracting periodic cycles and half the number of indifferent periodic cycles is bounded by $2d-2$.

Theorem 1.2 (Fatou-Shishikura Inequality)**.** *Every rational map f of degree ≥ 2 has at most $2d-2$ non-repelling periodic cycles.*

Quasiconformal deformation refers to the construction of holomorphic maps with prescribed dynamics by deforming quasiregular maps in the Riemann sphere $\hat{\mathbb{C}}$ via conjugation with quasiconformal homeomorphisms. We also wish to discuss quasiconformal surgery, a method which additionally involves gluing two maps associated to two disjoint dynamical systems in the plane into one quasiregular map possessing the dynamical properties of both maps. Aside from the Fatou-Shishikura inequality, the surgery method has been used to prove many other results in holomorphic dynamics (see [BF14]).

This report is based on a research project under the supervision of Dr. Fabrizio Bianchi. In section 2, we will review some standard properties of iterations of rational maps and present the key lemma on quasiconformal deformation. Sections 3 and 4 discuss the theorem of no wandering domains and the Fatou-Shishikura inequality, serving as examples of how quasiconformal deformation serves as an invaluable tool in rational dynamics.

2 Foundational Results

2.1 Dynamics of Rational Maps

Recall that a point w is a critical point of a holomorphic map f if $f'(w) = 0$. Some knowledge of critical points of a rational map proves to be vital in understanding Fatou components. Some good references for these results include [Mil06, CG93].

Proposition 2.1. *A rational map f of degree $d \geq 2$ has at most $2d - 2$ critical points.*

Proof. By the Riemann-Hurwitz formula, we obtain the equation $\chi(\hat{\mathbb{C}}) + \sum_{z \in \hat{\mathbb{C}}} \delta_z = d \cdot \chi(\hat{\mathbb{C}})$, where the Euler characteristic of the Riemann sphere is $\chi(\hat{\mathbb{C}}) = 2$ and $\delta_z = d - |f^{-1}(\{z\})|$ is the deficiency of a point $z \in \hat{\mathbb{C}}$. If z is not a critical point, z has d pre-images and thus $\delta_z = 0$, otherwise z has less than d pre-images and $\delta_z \geq 1$. Evaluating the formula gives us $\sum_{z \in \hat{\mathbb{C}}} \delta_z = 2d - 2$, hence the lemma holds. □

Lemma 2.0.1. *Every immediate basin of attraction of an attracting periodic point of a rational map f of degree ≥ 2 contains at least one critical point.*

Proof. Let p be an attracting fixed point of f with multiplier $\lambda \in (0, 1)$. Let $\mathcal{A}$ be the immediate basin of attraction of p and assume that it has no critical points. As p is topologically attracting, there's an open neighbourhood $V_0 \subset \mathcal{A}$ of p such that $f(V_0) < V_0$. Then, pick a surjective inverse branch $f^{-1} : V_0 \to V_1$ where V_1 is simply connected, $f(V_1) = V_0 \Subset V_1 \subset \mathcal{A}$. Similarly, as V_1 contains no critical points, there's a well-defined surjective inverse branch $f^{-1} : V_1 \to V_2$ where $V_1 \Subset V_2 \subset \mathcal{A}$. Inductively, we can define a sequence $f^{-n} : V_0 \to V_n$ where $V_{n-1} \Subset V_n \subset \mathcal{A}$ for all $n \in \mathbb{N}$.

This family is normal by Montel's Theorem since each V_n does not intersect with $J(f)$. There exists a subsequence f^{-n_j} converging uniformly to g but then $(f^{-n_j})'(p) = \lambda^{-n_j} \to \infty$ as $j \to \infty$. This is a contradiction. If p is an attracting periodic point with period k, we can apply the same argument above to f^k. □

Corollary 2.0.1. *A rational map of degree ≥ 2 has at most $2d - 2$ attracting periodic cycles.*

A result similar to the lemma above on the existence of critical points in parabolic basins is given below. The proof will require Leau-Fatou Parabolic Flower Theorem (See [CG93] §II 5 and [Mil06] §10).

Lemma 2.0.2. *Every immediate parabolic basin of attraction of a rationally indifferent periodic point p a rational map f of degree ≥ 2 contains at least ν critical points of f, where $\nu \in \mathbb{N}_{\geq 0}$ is the parabolic multiplicity of p.*

Proof. Assume that the multiplier of p is 1 by replacing f with f^q where q is the denominator which appears in the original multiplier. The immediate parabolic basin of the parabolic cycle of p has ν connected components. Pick one component $\mathcal{A}$ and let P be an attracting petal contained in $\mathcal{A}$. Let $\phi : P \to \{z \in \hat{\mathbb{C}} \mid \mathrm{Re} z > 0\}$ be the corresponding Fatou coordinate, a biholomorphism. Using the Abel functional equation $\phi \circ (z) = \phi(z) + 1$, we can extend ϕ maps $\mathcal{A}$ onto the whole $\mathbb{C}$. If $\mathcal{A}$ contains no critical points of ϕ, then $\phi^{-1} : \mathbb{C} \to \mathcal{A}$ is well-defined, but then it omits the Julia set $J(f)$ and contradicts Picard's Theorem. Thus, there's a critical point w of ϕ. Pick the smallest positive integer n such that $f^n(w) \in P$, then as P contains no critical points of ϕ, the Abel functional equation leads us to the following.

$$\phi'(f^n(w))(f^n)'(w) = \phi'(w) = 0 \qquad \to \qquad (f^n)'(w) = 0$$

Thus, f must have a critical point of the form $f^m(w)$, for some m, on $\mathcal{A}$. The same argument on the other components of the immediate parabolic basin gives us ν critical points of f. □

Corollary 2.0.2. *The total number of attracting and rationally indifferent cycles of a rational map of degree ≥ 2 has at most $2d - 2$.*

Lemma 2.0.3. *$J(f)$ is contained in the closure of $Per(f)$, the set of all periodic points of f.*

Proof. Let $z_0 \in J(f)$ be a non-critical point with some non-critical pre-images z_1 and z_2. By the Inverse Function Theorem, for each $k \in \{0, 1, 2\}$ we can find open neighbourhoods U_k of z_k such that $f : U_k \to U_0$ is a homeomorphism with the inverse g_j.

Suppose that for all $m \in \mathbb{N}$ and $z \in U_0$, $f^m(z) \notin \{z, g_1(z), g_2(z)\}$. For each $m \geq 1$, let $h_m : U_0 \to \hat{\mathbb{C}}$ be the cross ratio of 4 distinct points defined by

$$h_m(z) = \frac{f^m(z) - g_1(z)}{f^m(z) - g_2(z)} \cdot \frac{z - g_2(z)}{z - g_1(z)}.$$

Then, h_m is a well-defined holomorphic map which omits 0, 1 and ∞. Hence, the family $\{h_m\}$ is normal in U_0 and so is $\{f_m\}$, but this is a contradiction. Thus, there exists some $z \in U_0$ and $m \in \mathbb{N}$ such that $f^m(z) \in \{z, g_1(z), g_2(z)\}$, i.e. z is a periodic point of f. As U_0 is an arbitrary neighbourhood of z_0, each point z_0 is arbitrarily close to $Per(f)$, hence $J(f)$ is in the closure of $Per(f)$. □

2.2 Quasiconformal Maps

A quasiconformal map is an orientation-preserving homeomorphism with a bounded deviation from a conformal isomorphism. There are various ways - either geometrically or analytically - to define quasiconformal maps (refer to [Ahl66]). One of the most important results in the theory of quasiconformal maps is the Measurable Riemann Mapping Theorem, proven by Lars Ahlfors and Lipman Bers (Chapter 5 in [Ahl66]).

Theorem 2.1 (Measurable Riemann Mapping Theorem)**.** *Suppose $\mu : \hat{\mathbb{C}} \to \mathbb{D}$ is a measurable function such that $\|\mu\|_\infty < 1$. Then, there exists a quasiconformal homeomorphism $f : \hat{\mathbb{C}} \to \mathbb{D}$ such that it solves the Beltrami equation $f_{\bar{z}} = \mu f_z$. Moreover, if we require that the map fixes three distinct points, e.g. 0, 1, and ∞, then the map f is unique.*

We define f to be a K-quasiregular map if it is the composition of a holomorphic map and a K-quasiconformal homeomorphism. We will now present the key lemma.

Lemma 2.1.1. *Suppose $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiregular map of degree d and $\mu : \hat{\mathbb{C}} \to \mathbb{D}$ is an f-invariant bounded measurable function, i.e. $f^*\mu = \mu$ such that $\|\mu\|_\infty < 1$. If $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is the unique quasiconformal homeomorphism with Beltrami coefficient μ fixing 0, 1, and ∞, then $g = \phi \circ f \circ \phi^{-1} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map of degree d.*

Proof. The existence of ϕ is guaranteed by the Measurable Riemann Mapping Theorem. Let μ_0 be the trivial Beltrami coefficient, i.e. $\mu_0 \equiv 0$. We wish to see that the pullback of μ_0 by g is μ_0. Indeed, knowing that $\phi^*\mu_0 = \phi_{\bar{z}}/\phi_z = \mu_0$,

$$g^*\mu_0 = (\phi^{-1})^* f^* \phi^* \mu_0 = (\phi^{-1})^* f^* \mu = (\phi^{-1})^* \mu = (\phi^{-1})^* \phi^* \mu_0 = \mu_0.$$

Thus, g is a holomorphic rational map in $\hat{\mathbb{C}}$ and as f is a rational map of degree d, g is also of degree d. □

If the assumptions are met, we say that g is a quasiconformal deformation of f. When we apply the key lemma, we generally wish to find the appropriate f-invariant Beltrami coefficient μ hoping to obtain a quasiconformal deformation g with known dynamics.

3 No Wandering Domain Theorem

We will now present the proof of Theorem 1.1 which states that any rational map f has no wandering Fatou components.

Firstly assume that a rational map f has a wandering component U and we let $U_n = f^n(U)$ for all $n \in \mathbb{N}$ and $U_0 = U$. As there are only finitely many critical points of f we can assume that each U_n does not contain any critical point. We will then divide the proof into 4 steps:

1. Prove that U is simply connected,
2. Define a large family of Beltrami coefficients M on U of arbitrary real dimension $2m$,
3. Construct an analytic map $F : M \to Rat_d$, where Rat_d is the set of all rational maps of degree d, via quasiconformal deformation,
4. Pick a simple finite path in M where the image under F is constant and obtain a contradiction by .

Step 1: U is Simply Connected

The assumption means that the each U_n are disjoint from each other. In fact, the grand orbit of U, $GO(U) = \cup_{m=0}^{\infty} \cup_{n=0}^{\infty} f^{-m}U_n$ is a disjoint union of infinitely many domains. By Lemma 2.1, we can assume that for all $n \in \mathbb{N}$ each U_n contains no critical points. The following lemma by Baker will significantly simplify the subsequent steps of the proof.

Lemma 3.0.1 (Baker's Lemma). *Suppose f has a wandering Fatou component U such that the forward orbit of U under f contains no critical points. U is simply connected.*

Proof. Each map $f : U_n \to U_{n+1}$ is a covering map as it is surjective and has no critical points. The induced map $f_* : \pi(U_n) \to \pi(U_{n+1})$ is then injective. By conjugation with some complex Möbius map, assume without loss of generality that $\infty \in U$, so $f^{-1}(U)$ contains all poles of f. As U is wandering, it is necessary that $area(U_n) \to 0$ as $n \to \infty$. Any convergent subsequence $\{f^n : U \to U_n\}_{n\in\mathbb{N}}$ must converge uniformly locally to a constant function since if there exists a non-constant limit $g : U \to \hat{\mathbb{C}}$ then g is open and $area(g(U)) > 0$, which is a contradiction.

Set $\alpha = \inf\{\text{dist}(p, \hat{\mathbb{C}} \backslash f^{-1}(U)) \mid f(p) = \infty\}$ such that whenever p is a pole, then $B(p, \alpha) \subset f^{-1}(U)$. Pick any simple closed curve γ_0 in U and let $\gamma_n = f^n(\gamma_0) \subset U_n$. Let V_n be the union of γ_n and the bounded component of $\mathbb{C}\backslash U_n$. Since $\text{diam}(V_n) \to 0$ as $n \to \infty$, there exists some $N \in \mathbb{N}$ such that for $n \geq N$, the diameter of V_n is smaller than α, so V_n contains no poles of f.

Claim now that for $n \geq N$, f maps each V_n to V_{n+1}. Indeed, if there exists some $k \geq N$ which map some point in the interior $int(V_k)$ to $\hat{\mathbb{C}}\backslash V_{k+1}$, then as $f(\gamma_k) = \gamma_{k+1}$, it is necessary that $f(V_k) = \hat{\mathbb{C}}\backslash int(V_k)$. However, this implies that $f(V_k$ contains ∞ and it would contradict the fact that V_k contains no poles of f.

As such, the family $\{f^n : V_N \to \hat{\mathbb{C}}\}_{n\in\mathbb{N}}$ is normal and V_N is contained in the Fatou set, in particular U_N. Thus, γ_N is homotopic to a point. By injectivity of f_*, $\pi(U)$ is trivial. □

Step 2: Constructing the Large Family M

Let $R : U \to \mathbb{D}$ be a biholomorphism. Fatou's theorem tells us that for almost every boundary point z on the unit circle $\partial\mathbb{D}$, the radial limit $\lim_{r\to 1} R^{-1}(rz) \in \partial U$ exist. Pick $2m+3$ of such distinct points on $\partial\mathbb{D}$, namely $\{b_1, b_2, b_3, A_1, \ldots, A_{2m}\}$. For each $k \in \{1, \ldots 2m\}$ pick an open interval I_k in $\partial\mathbb{D}$ containing A_k as its midpoint and not containing b_1, b_2, or b_3 such that all intervals I_k are pairwise disjoint. Denote by J_k the closed interval contained in I_k such that it has half the Euclidean length of I_k and it also has midpoint A_k.

For each $a = (a_1, \ldots a_{2m}) \in J_1 \times J_2 \times \ldots J_{2m}$ we construct a piecewise linear diffeomorphism g_a of $\partial\mathbb{D}$ to itself such that g_a fixes all points outside all intervals I_k for $k = 1, \ldots, 2m$ and on each I_k, g_a is linear and $g_a(A_k) = a_k$. Extend it to the unit disk $\mathbb{D}$ using the formula $g_a(rz) = rg_a(z)$ for all $r \in [0, 1)$ and $z \in \partial\mathbb{D}$. See Figure 1 for illustration.

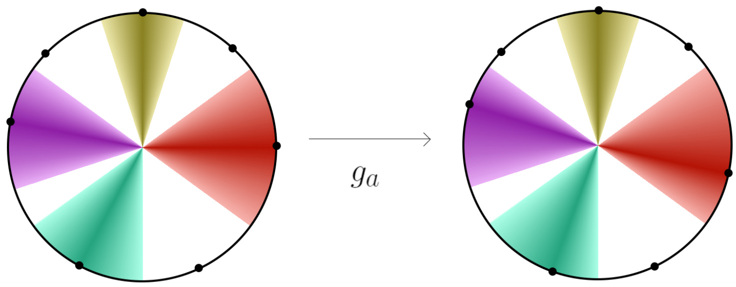

Figure 1: g_a on $\overline{\mathbb{D}}$ where $m = 2$; the coloured sectors represent the extension of the intervals I_k and the white sectors are fixed by g_α.

Let μ_a be the Beltrami coefficient of $g_a \circ R$ defined on U. To prove this, it is sufficient to consider the following simplest case:

Let $A = 1$ be contained in the closed arc I in the unit disk joining $e^{-2\pi i\epsilon}$ and $e^{2\pi i\epsilon}$ where $\epsilon \in (0, \frac{1}{2})$. Then, pick $T \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$ and let $a = e^{2\pi i T}$. Then, the corresponding map g_a and its complex dilatation then defined as follows:

$$g_a(z) = \begin{cases} az^{1-\frac{T}{\epsilon}}|z|^{\frac{T}{\epsilon}}, & \text{if } \arg z \in [0, 2\pi i\epsilon], \\ az^{1+\frac{T}{\epsilon}}|z|^{-\frac{T}{\epsilon}}, & \text{if } \arg z \in [-2\pi i\epsilon, 0], \\ z, & \text{otherwise.} \end{cases} \qquad \frac{\frac{dg_a(z)}{d\bar{z}}}{\frac{dg_a(z)}{dz}} = \begin{cases} \frac{z}{\bar{z}} \cdot \frac{T}{2\epsilon - T}, & \text{if } \arg z \in [0, 2\pi i\epsilon], \\ \frac{z}{\bar{z}} \cdot \frac{-T}{2\epsilon + T}, & \text{if } \arg z \in [-2\pi i\epsilon, 0], \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, for each $T \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$, the complex dilatation of g_a is measurable and its maximum norm is never greater than half. Thus, g_a is a quasiconformal homeomorphism. We can then find the explicit expression the Beltrami coefficient μ_a of $g_a \circ R$. For $z \in U$, if $w = R(z) \neq 0$,

$$\mu_a(z) = \begin{cases} \frac{w\overline{R'(z)}}{\overline{w}R'(z)} \cdot \frac{T}{2\epsilon - T}, & \text{if } \arg w \in [0, 2\pi i\epsilon], \\ \frac{w\overline{R'(z)}}{\overline{w}R'(z)} \cdot \frac{-T}{2\epsilon + T}, & \text{if } \arg w \in [-2\pi i\epsilon, 0], \\ 0, & \text{otherwise.} \end{cases}$$

We can observe that $T \mapsto \mu_a(z)$ is analytic for all $z \in U$.

In the general case, on each sector of the unit disk extending to I_k, if $a_k = e^{2\pi i T_k}$, μ_a will also depend analytically on T_k and thus a_k. We have created a manifold of Beltrami coefficients M of real dimension $2m$ analytically parameterized by $a \in J_1 \times J_2 \times \ldots J_{2m}$. Our construction is such that the Beltrami coefficient extends to 0 on ∂U if and only if g_a is conformal, i.e. $a = A$ and $g_a = Id$.

Step 3: Quasiconformal Deformation

Recall that by the wandering property of U, U_n have to be pairwise disjoint. Pick any $\mu_a \in M$ and for all $n \in \mathbb{N}$, push it forward to U_n via $f^n|_U : U \to U_n$ such that for all $z \in U_n$, $\mu_\lambda(z) = \mu_a(f|_U^{-n}(z))$. Then, pull it back from the forward orbit of U to the whole grand orbit via f such that for any $n_1, n_2 \in \mathbb{N}$, if $z \in f^{-n_1}(U_{n_2})$, $\mu_a(z) = (f^{n_1})^*\mu_a(z)$. Setting μ_a to be 0 outside the grand orbit, we have then extended μ_a to $\hat{\mathbb{C}}$.

From above, $\mu_a : \hat{\mathbb{C}} \to \mathbb{D}$ is f-invariant. Denote by ϕ_λ the unique quasiconformal homeomorphism with Beltrami coefficient μ_a and fixed points 0, 1 and ∞. By Lemma 2.1.1, the deformation $f_a := \phi_a \circ f \circ \phi_a^{-1} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map of degree d, i.e. $f_a \in Rat_d$. The map $F : M \to Rat_d$, $\mu_a \mapsto f_a$ is well-defined and analytic.

Step 4: The Contradiction

Since m is arbitrary, we can assume that $2m$ is larger than $4d+2$, which is the real dimension of the smooth complex manifold Rat_d. By Sard's theorem, there exists some element $f_a \in Rat_d$ where the fiber $F^{-1}(\{f_a\})$ is of dimension ≥ 1. In other words, we can take a non-trivial simple curve $\mu_{a(t)}$, where $t \in [0,1]$, in $F^{-1}(\{f_a\})$ connecting 2 distinct Beltrami coefficients $\mu_{a(0)}$ and $\mu_{a(1)}$ with corresponding quasiconformal homeomorphisms $\phi_{a(0)}$ and $\phi_{a(1)}$.

$$\begin{tikzcd} \hat{\mathbb{C}} \arrow[d, "f_a"'] & \hat{\mathbb{C}} \arrow[l, "\phi_{a(0)}"'] \arrow[r, "\phi_{a(1)}"] \arrow[d, "f"] & \hat{\mathbb{C}} \arrow[d, "f_a"] \\ \hat{\mathbb{C}} & \hat{\mathbb{C}} \arrow[l, "\phi_{a(0)}"] \arrow[r, "\phi_{a(1)}"] & \hat{\mathbb{C}} \end{tikzcd}$$

Pick any $t \in [0,1]$. Since $\phi_{a(t)} \circ \phi_{a(0)}^{-1}$ commutes with f_a, for all $n \geq 1$, $\phi_{a(t)} \circ \phi_{a(0)}^{-1}$ restricted to the set periodic points $Per_n(f_a)$ of prime period n is an automorphism. For any n and $z \in Per_n(f_a)$, the map $\phi_{a(t)} \circ \phi_{a(0)}^{-1}(z)$, $t \in [0,1]$ is a continuous path starting from z, but since $Per_n(f_a)$ is finite, it is the identity on $Per_n(f_a)$. We conclude by Lemma 2.0.3 that $\phi_{a(t)} \circ \phi_{a(0)}^{-1}$ is the identity on $J(f_a)$, or in other words $\phi_{a(0)}^{-1} \circ \phi_{a(t)}$ is the identity on $\partial U \subset J(f)$.

Let $V = \phi_{a(0)}(U)$ and $t \in [0,1]$, then as $\phi_{a(t)} \circ \phi_{a(0)}^{-1}$ is the identity on ∂V, $\phi_{a(t)}$ maps U to either V or $\hat{\mathbb{C}} \backslash \overline{V}$. We can assume without loss of generality by conjugation with Möbius maps that U contains ∞, so that $\phi_{a(t)}$ and $\phi_{a(0)}^{-1}$ fixes ∞. Then, $\phi_{a(t)}(U) = V$.

Let $h_{a(t)} := g_{a(t)} \circ R \circ \phi_{a(t)}^{-1} : V \to \mathbb{D}$. By the same argument as in Lemma 2.1.1, we can deduce that $h_{a(t)}$ is a biholomorphism. Thus, $g_{a(1)} \circ g_{a(0)}^{-1} = h_{a(1)} \circ \phi_{a(1)} \circ \phi_{a(0)}^{-1} \circ h_{a(0)}^{-1}$, but on $\partial\mathbb{D}$, $g_{a(1)} \circ g_{a(0)}^{-1}$ simplifies to $h_{a(1)} \circ h_{a(0)}^{-1}$, an automorphism of $\mathbb{D}$. As $g_{a(1)} \circ g_{a(0)}^{-1}$ fixes at least 3 points on $\partial\mathbb{D}$ (namely b_1, b_2 and b_3), $g_{a(1)} \circ g_{a(0)}^{-1} = Id$ on $\overline{\mathbb{D}}$. This is a contradiction because by our construction, $g_{a(0)}$ and $g_{a(1)}$ are distinct.

4 Fatou-Shishikura Inequality

In this section, we will prove Theorem 1.2 which states that every rational map f of degree $d \geq 2$ has at most $2d-2$ non-repelling periodic cycles.

Again, we will break down the proof into several steps:

1. Deform f by quasiconformal conjugation into a quasiregular map g_ϵ for some small $\epsilon > 0$ such that all non-repelling periodic cycles become attracting periodic cycles of g_ϵ.
2. Construct a region E which is disjoint from $V_\epsilon := \{z \in \mathbb{C} \mid |z| \geq \epsilon^{-\frac{1}{k}}\}$ for some $k \in \mathbb{N}$ such that $g_\epsilon(E \cup V_\epsilon) \subset E$.
3. Use E to perform quasiconformal surgery to deform g_ϵ into a rational map of degree d with only attracting periodic cycles.

Step 1: From Non-Repelling to Attracting

We assume by prior conjugation that all non-repelling periodic cycles of f do not contain ∞. Let $\{w_j\}_{j=0,1,2\dots m-1}$, $m \in \mathbb{N}$, be the set of all non-repelling periodic points of f and denote the multiplier of each with $\lambda_j \leq 1$. Construct a polynomial $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that

$$h(w_j) = 0, \qquad h'(w_j) = -1, \qquad \text{for all } j = 1, 2, \dots m.$$

Let $k = \deg(h)$. Then, pick any $\epsilon \in (0,1)$ and define $f_\epsilon(z) = f(z + \epsilon h(z))$. f_ϵ is a polynomial of degree dk and each w_j is an attracting periodic point of f_ϵ since $f_\epsilon(w_j) = f(w_j)$ and the multiplier of w_j for f_ϵ is $(1-\epsilon)^m \lambda_j < 1$.

Define a smooth non-increasing curve $\rho : [0,\infty) \to [0,1]$ such that $\rho(x) = 1$ for $x < 1$ and $\rho(x) = 0$ for $x \geq 2$. We will use ρ to suppress the deviation of f_ϵ from f in the region of high modulus. As such, we define the following functions h_ϵ and g_ϵ on $\hat{\mathbb{C}}$.

$$\begin{aligned} h_\epsilon(z) &:= z + \epsilon\rho(\epsilon^{\frac{1}{k}}|z|)h(z), \\ g_\epsilon(z) &:= f \circ h_\epsilon(z). \end{aligned}$$

Notice that $g_\epsilon \equiv f_\epsilon$ in the ball $B(0, \epsilon^{-\frac{1}{k}})$ and $g_\epsilon \equiv f$ outside $B(0, 2\epsilon^{-\frac{1}{k}})$. We will prove that h_ϵ is indeed a quasiconformal homeomorphism for sufficiently small ϵ.

Pick a large value $M > 1$ satisfying the following:

- $|h(z)| \leq M \max\{1, |z|^k\}$,
- For $|z| > 1$, $|h'(z)| \leq M|z|^{k-1}$,
- For all $x \in \mathbb{R}$, $|\rho'(x)| \leq M$.

We will now look at $\mu(z) = \frac{(h_\epsilon)_{\bar{z}}(z)}{(h_\epsilon)_z(z)}$. Outside the annulus $[\epsilon^{-\frac{1}{k}}, 2\epsilon^{-\frac{1}{k}}]$, μ vanishes since h_ϵ is holomorphic, so we will assume now that z lies within. By the assumptions above,

$$\begin{aligned} |(h_\epsilon)_{\bar{z}}(z)| &= \left|\epsilon^{1+\frac{1}{k}} h(z)\rho'(\epsilon^{-\frac{1}{k}}|z|)\frac{\partial |z|}{\partial \bar{z}}\right| \leq \epsilon^{1+\frac{1}{k}} M^2 |z|^k 2^{-1} \leq 2^{k-1} M^2 \epsilon^{\frac{1}{k}} \\ |(h_\epsilon)_z(z) - 1| &\leq |(h_\epsilon)_z(z) - 1| \leq \epsilon\Big|\rho(\epsilon^{\frac{1}{k}})h'(z) + \epsilon^{\frac{1}{k}}\rho'(\epsilon^{\frac{1}{k}})\frac{\partial |z|}{\partial z}h(z)\Big| \\ &\leq \epsilon\Big(M|z|^{k-1} + \frac{1}{2}\epsilon^{\frac{1}{k}} M^2 |z|^k\Big) \leq 2^k M \epsilon^{\frac{1}{k}} \end{aligned}$$

We then know that for sufficiently small ϵ, the Jacobian of h_ϵ is positive and $\|\mu\|_\infty < 1$. We can conclude that h_ϵ is a quasiconformal homeomorphism and thus g_ϵ is a quasiregular map of $\hat{\mathbb{C}}$ to itself of degree d.

We can also make sure that ϵ is sufficiently small such that each w_j is contained in $\{z \in \mathbb{C} \,|\, |z| < \epsilon^{-\frac{1}{k}} - 1\}$ and thus an attracting periodic point of g_ϵ.

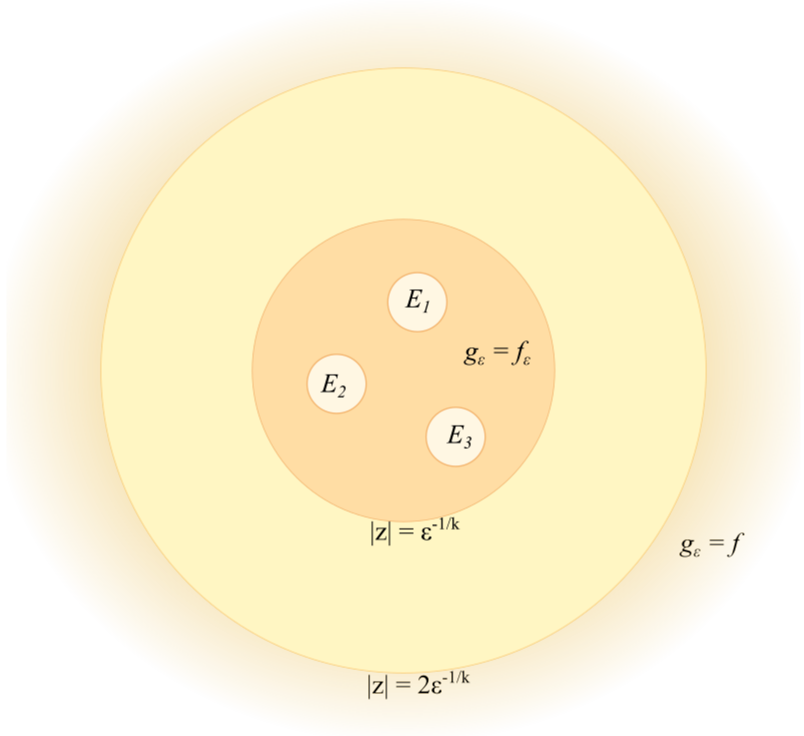

Figure 2: The case where there are only 3 non-repelling periodic points. E consists of E_1, E_2, and E_3.

Step 2: The Region E

Let $V_\epsilon := \{z \in \hat{\mathbb{C}} \mid |z| \geq \epsilon^{-\frac{1}{k}}\}$. We would like to obtain an open region E (dependent of ϵ) consisting of a disjoint union of open neighbourhoods E_j of w_j (see Figure 2) satisfying the following conditions:

(a) $E_j \subset B(w_j, 1)$,

(b) $g_\epsilon(E \cup V_\epsilon) \subset E$.

Recall the assumption that for all non-repelling periodic points w_j, $|w_j| + 1 < \epsilon^{-\frac{1}{k}}$. Thus, condition (a) automatically implies that E is disjoint from V_ϵ. The 2 conditions are crucial in finding an appropriate Beltrami coefficient in Step 3. Upon discussing the construction of each neighbourhood E_j, we will only consider some cases in detail.

We will first estimate the deviation of g_ϵ from f using the chordal metric d where $d(z, w) = \frac{2|z-w|}{\sqrt{(1+|z|^2)(1+|w|^2)}}$. This deviation is obviously 0 in $\{|z| \geq 2\epsilon^{-\frac{1}{k}}\}$, and for $|z| < 2\epsilon^{-\frac{1}{k}}$, Since any rational map is Lipschitz with respect to d ([Bea91] §2.3), if the Lipschitz constant of f is M_f, then let $\tilde{M} := \max\{M_f, M\}$ and we obtain the following inequality.

$$
\begin{aligned}
d(g_\epsilon(z), f(z)) \leq M_f d(h_\epsilon(z), z) &= \tilde{M} \frac{2|h_\epsilon(z) - z|}{\sqrt{(1 + |h_\epsilon(z)|^2)(1 + |w|^2)}} \\
&\leq \tilde{M} \frac{2\epsilon|h(z)|}{\sqrt{1 + |z|^2}} \leq 2\epsilon\tilde{M}^2 \frac{\max\{1, |z|^k\}}{\max\{1, |z|\}} \\
&\leq 2^k \epsilon^{\frac{1}{k}} \tilde{M}^2 \qquad (1)
\end{aligned}
$$

Without loss of generality, we pick a cycle $\{w_0, w_1, \ldots, w_{p-1}\}$ of prime period p. We will consider 3 distinct cases:

Case 1: The periodic cycle is attracting.

Since w_0 is topologically attracting, there exists some open Jordan domain W containing w_0 such that $f^p(W) \Subset W$. Pick a sequence of domains W_j for $j = 1, 2, \dots p-1$ satisfying

$$f^p(W) \Subset W_1 \Subset W_2 \Subset \dots \Subset W_{p-1} \Subset W.$$

Then, let $E_0 = f^p(W)$ and $E_j = f^j(W_j)$ for each $j = 1, 2, \dots, p-1$. For each j, we then have

$$f(E_{j-1}) \Subset E_j, \qquad f(E_{p-1}) \Subset E_0.$$

We can always pick a smaller W such that condition (a) is satisfied. By prior conjugation, we can assume without loss of generality that $g_\epsilon(\infty) \in E_0$. By inequality (1), we can pick a sufficiently small ϵ such that $g_\epsilon(V_\epsilon) \subset E_0$ and $g_\epsilon(E) \subset E$.

Case 2: The periodic cycle is irrationally indifferent

Lemma 4.0.1. *There exists a well-defined holomorphic map ϕ in a neighbourhood of w_0 such that $\phi(w_0) = 0$, $\phi'(w_0) = 1$ and as $z \to 0$,*

$$\phi \circ g_\epsilon^p \circ \phi^{-1}(z) = \lambda z[(1-\epsilon)^p + \mathcal{O}(\epsilon z) + \mathcal{O}(z^{k+2})].$$

Proof. Let α be the translation $z \mapsto z - w_0$. Then, we have the expansion $\alpha \circ f^p \circ \alpha^{-1}(z) = \lambda z + a_n z^n + \dots$ for some integer $n \geq 2$. Suppose that $n < k+3$. The holomorphic map $\beta(z) = z + \frac{a_n}{\lambda^n - \lambda} z^n$ is well defined since λ is not a root of unity. It is easy to check that $\beta^{-1} \circ \alpha \circ f^p \circ \alpha^{-1} \circ \beta = \lambda z + \mathcal{O}(z^{n+1})$ near 0. Apply the above conjugation repeatedly until we have a holomorphic function ϕ defined near w_0 such that $\phi(w_0) = 0$, $\phi'(w_0) = 1$ and $\phi \circ f^p \circ \phi^{-1}(z) = \lambda z + \mathcal{O}(z^{k+3})$ as $z \to 0$.

By definition of g_ϵ, the composition $\phi^{-1} \circ g_\epsilon^p \circ \phi$ is analytic in the (z, ϵ) variable in the neighbourhood of $(0, 0)$, so $\phi^{-1} \circ g_\epsilon^p \circ \phi(z) = \sum_{m=0}^\infty \sum_{n=0}^\infty a_{m,n} z^m \epsilon^n$ for some constants $a_{m,n}$, where $a_{0,0} = 0$ and $a_{1,0} = \lambda$. Then, as $z \to 0$,

$$\begin{aligned}
\phi \circ g_\epsilon^p \circ \phi^{-1}(z) &= \sum_{m=0}^\infty a_{m_0} z^m - a_{0,0} - a_{1,0} z + \sum_{n=0}^\infty a_{0,n} \epsilon^n + \sum_{n=0}^\infty a_{1,n} z \epsilon^n + \sum_{m=2}^\infty \sum_{n=1}^\infty a_{m,n} z^m \epsilon^n \\
&= \phi \circ f^p \circ \phi^{-1}(z) - 0 - \lambda z + \phi \circ g_\epsilon^p \circ \phi^{-1}(0) + z \frac{\partial}{\partial z} \phi \circ g_\epsilon^p \circ \phi^{-1}(z) + \mathcal{O}(\epsilon z^2) \\
&= \mathcal{O}(z^{k+3}) + \lambda z (1-\epsilon)^p + \mathcal{O}(\epsilon z^2)
\end{aligned}$$

Rearranging, we have proven the lemma. □

Pick a sufficiently small ϵ such that ϕ is a biholomorphism from some neighbourhood E_0 of E_0 onto the disk $B(0, \epsilon^{\frac{1}{k+1}})$. If $|z| < \epsilon^{\frac{1}{k+1}}$, then from the lemma above,

$$\phi \circ g_\epsilon^p \circ \phi^{-1}(z) = \lambda z[1 - p\epsilon + \mathcal{O}(\epsilon^{\frac{k+2}{k+1}})].$$

So for sufficiently small ϵ, $|\phi \circ g_\epsilon^p \circ \phi^{-1}(z)| \leq |\lambda z| < \epsilon^{\frac{1}{k+1}}$ and consequently, $g_\epsilon^p(E_0) \subset E_0$. Define a neighbourhood $E_j := g_\epsilon^j(R_0)$ of w_j for $j = 1, 2 \dots p-1$, then if $E = \cup_{j=0}^{p-1} E_j$, $g_\epsilon(E) \subset E$. ϵ can be made sufficiently small such that condition (a) holds.

Suppose f has no attracting cycle, then by prior conjugation, we can again assume without loss of generality that $g\epsilon(\infty) \in E_0$ and argue by (1) that $g_\epsilon(V_\epsilon) \subset E$.

Case 3: The periodic cycle is rationally indifferent

The construction of such neighbourhoods E_j where the cycle is parabolic combines the techniques used in the previous two cases and and will not be discussed in detail. See [Bea91] §9.6.

Step 3: Quasiconformal Surgery

By construction, the regions E and $\hat{\mathbb{C}}\backslash g_\epsilon^{-1}(E)$ are disjoint from V_ϵ and in both regions, g_ϵ is holomorphic. As $g_\epsilon(E) \subset E$, we have that for all $n \geq 0$, $g_\epsilon^{-n}(E) \subset g_\epsilon^{-n-1}(E)$.

Define a Beltrami coefficient μ in the following way. Let $\mu = 0$ on E and pull back μ from E via g_ϵ to all its preimages $g^{-n}(E)$ for all n. Specifically, if $\psi : E \to D$ is a conformal homeomorphism to some region D, we have that in $\in g^{-1}(E)\backslash E$,

$$\mu = \frac{(\psi \circ g_\epsilon)_{\bar{z}}}{(\psi \circ g_\epsilon)_z} = \frac{(\phi_z \circ g_\epsilon)(g_\epsilon)_{\bar{z}} + (\phi_{\bar{z}} \circ g_\epsilon)\overline{(g_\epsilon)_z}}{(\phi_z \circ g_\epsilon)(g_\epsilon)_z + (\phi_{\bar{z}} \circ g_\epsilon)\overline{(g_\epsilon)_{\bar{z}}}} = \frac{(g_\epsilon)_{\bar{z}}}{(g_\epsilon)_z}$$

and in $g^{-2}(E)\backslash g^{-1}(E)$, since $(g_\epsilon)_{\bar{z}} = 0$,

$$\mu = \frac{(\psi \circ g_\epsilon^2)_{\bar{z}}}{(\psi \circ g_\epsilon^2)_z} = \frac{(g_\epsilon^2)_{\bar{z}}}{(g_\epsilon^2)_z} = \frac{\big((g_\epsilon)_z \circ g_\epsilon\big)(g_\epsilon)_{\bar{z}} + \big((g_\epsilon)_{\bar{z}} \circ g_\epsilon\big)\overline{(g_\epsilon)_z}}{\big((g_\epsilon)_z \circ g_\epsilon\big)(g_\epsilon)_z + \big((g_\epsilon)_{\bar{z}} \circ g_\epsilon\big)\overline{(g_\epsilon)_{\bar{z}}}} = \frac{\overline{(g_\epsilon)_z}}{(g_\epsilon)_z}\mu \circ g_\epsilon$$

Inductively, we find that for any $w \in g_\epsilon^{-n-1}(E)\backslash g_\epsilon^{-n}(E)$, then $g_\epsilon^n(w) \in g^{-1}(E)\backslash E$ and $|\mu(w)| = |\mu(g_\epsilon^n(w))|$. Set $\mu = 0$ as well on $\hat{\mathbb{C}}\backslash \cup_{n=0}^{\infty} g_\epsilon^{-n}(E)$. Thus, $\|\mu\|_\infty$ is determined by the complex dilatation of g_ϵ on $g^{-1}(E)\backslash E$ and this is necessarily less than 1.

μ is a well-defined g_ϵ-invariant Beltrami coefficient so that by Lemma 2.1.1, we have a quasiconformal homeomorphism ϕ solving $\phi_{\bar{z}} = \mu\phi_z$ such that the conjugation $\phi \circ g_\epsilon \circ \phi^{-1}$ is a rational map of degree d with attracting periodic points $\phi(w_j)$. By Lemma 2.0.1, we have obtained the Shishikura inequality.

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