HYPERBOLICITY OF RENORMALIZATION OF CRITICAL QUASICIRCLE MAPS (DRAFT)

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ABSTRACT. There is a well developed renormalization theory of real analytic critical circle maps by de Faria, de Melo, and Yampolsky. In this paper, we extend Yampolsky's result on hyperbolicity of renormalization periodic points to a larger class of dynamical objects, namely critical quasicircle maps, i.e. analytic self homeomorphisms of a quasicircle with a single critical point. Unlike critical circle maps, the inner and outer criticalities of critical quasicircle maps can be distinct. We develop a compact analytic renormalization operator called "Corona Renormalization" with a hyperbolic fixed point whose stable manifold has codimension one and consists of critical quasicircle maps of the same criticality and periodic type rotation number. Our proof is an adaptation of Pacman Renormalization Theory for Siegel disks as well as rigidity results on the escaping dynamics of transcendental entire functions.

CONTENTS

1. INTRODUCTION

1.1. **Critical quasicircle maps.** A *critical circle map* is a real analytic self homeomorphism f of the unit circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ with exactly one critical point 0. Yoccoz [Yoc84] showed that if a critical circle map $f : \mathbb{T} \to \mathbb{T}$ has an irrational rotation number θ, then f is topologically conjugate to an irrational rotation. This means that if $\{p_n/q_n\}$ are best rational approximations of θ, then the iterates $\{f^{q_n}(0)\}$ are the closest returns to 0.

Given a critical circle map f of irrational rotation number θ, the n^{th} renormalization $\mathcal{R}^n f$ of f is defined as follows. Consider the commuting pair $p\mathcal{R}^n f =$

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$(f^{q_n}|_{I_{n+1}}, f^{q_{n+1}}|_{I_n})$, where $I_n \subset \mathbb{T}$ is the interval between 0 and $f^{q_n}(0)$. Then, $\mathcal{R}^n f$ is the normalized critical commuting pair obtained by rescaling $p\mathcal{R}^n f$ to unit size.

The renormalization theory of critical circle maps serves to justify the universality phenomena empirically observed in smooth families of critical circle maps. Historically, this is one of the two main examples of universality in one-dimensional dynamics, the other being the Feigenbaum universality observed in unimodal maps. The works of Feigenbaum et al. [FKS82] and Oslund et al. [ÖRSS83] translated the universality phenomena into a conjecture on the hyperbolicity of the renormalization operator on the space of critical commuting pairs. The conjecture was later generalized by various authors, in particular Lanford [Lan88] who accounted for more complex universalities.

Theorem 1.1 (Lanford's Program [Yam03])**.** *The renormalization operator $\mathcal{R}$ in the space of critical commuting pairs possesses a "horseshoe" attractor A on which its action is conjugated to the two-sided shift. Moreover, there exists an $\mathcal{R}$-invariant space of critical commuting pairs with the structure of an infinite dimensional smooth manifold, with respect to which A is a hyperbolic set with one-dimensional expanding direction.*

Given an irrational number $\theta \in (0,1)$ with continued fraction expansion $\theta = [0; a_1, a_2, a_3, \ldots]$, we say that θ is *of bounded type* if a_n's are uniformly bounded above, *pre-periodic* if there are positive integers m and p such that $a_n = a_{n+p}$ for all $n \geq m$, and *periodic* if additionally $m = 1$. We will denote corresponding spaces by Θ_{bdd}, Θ_{per} and Θ_{pre} respectively.

De Faria [dF99] introduced the notion of *holomorphic commuting pairs* and proved the universality of scaling ratios and the existence of renormalization horseshoe for critical circle maps of bounded type rotation number. $C^{1+\alpha}$ rigidity was established by de Faria and de Melo [dFdM99] for bounded type rotation number, and later by Khmelev and Yampolsky [KY06] for arbitrary irrational rotation number by studying parabolic bifurcations. Moreover, Yampolsky extended the horseshoe for all irrational rotation numbers in [Yam01], and brought Lanford's program to completion in [Yam02, Yam03] using *cylinder renormalization*.

In this paper, we work with a generalization of critical circle maps, namely critical quasicircle maps.

Definition 1.2. A *critical quasicircle map* is a homeomorphism $f : \mathbf{H} \to \mathbf{H}$ of a quasicircle which extends to a holomorphic map on a neighborhood of $\mathbf{H}$ and has exactly one critical point on $\mathbf{H}$.

Given a critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$, the behaviour at the unique critical point on $\mathbf{H}$ can be encoded by two positive integers, namely the inner criticality d_0 and the outer criticality d_∞. The total local degree of f at the critical point is $d_0 + d_\infty - 1$ and it is at least 2. When the criticalities are specified, we call $f : \mathbf{H} \to \mathbf{H}$ a (d_0, d_∞)-*critical* quasicircle map. See Figure 1 for some examples.

In the bounded type regime, if we assume that either d_0 or d_∞ is one, the quasicircle $\mathbf{H}$ will be the boundary of a rotation domain. By Douady-Ghys surgery, $\mathbf{H}$ can be assumed to be the boundary of a *Siegel disk*, i.e. a simply connected rotation domain. Stirnemann [Sti94] first gave a computer-assisted proof of the existence of a renormalization fixed point with a golden-mean Siegel disk. McMullen [McM98] applied a measurable deep point argument to prove the existence of renormalization horseshoe for bounded type rotation number. Gaidashev and Yampolsky

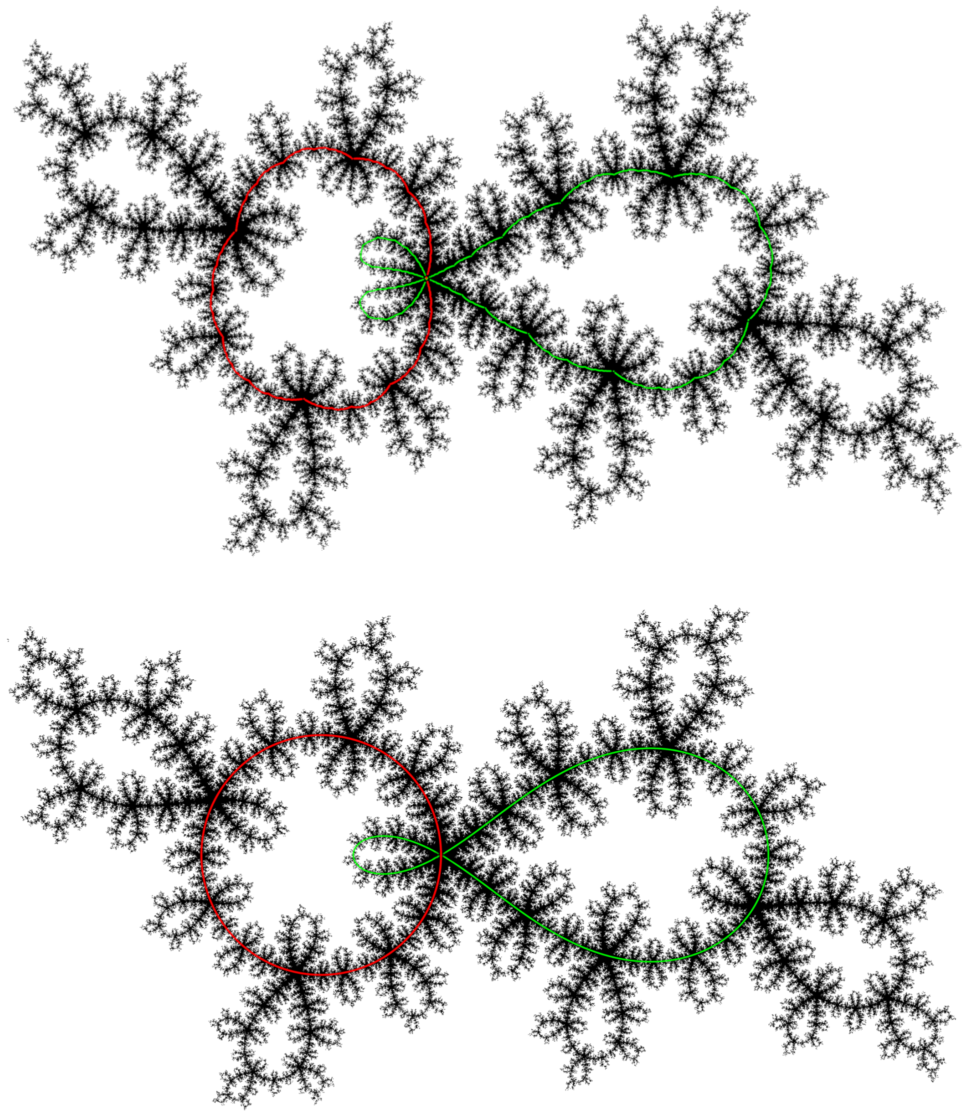

FIGURE 1. The Julia sets of

$$f_{3,2}(z) = bz^3 \frac{4-z}{1-4z+6z^2} \quad \text{and} \quad f_{2,2}(z) = cz^2 \frac{z-3}{1-3z}.$$

The critical values $b \approx -1.144208 - 0.964454i$ and $c \approx -0.755700 - 0.654917i$ are picked such that $f_{3,2} : \mathbf{H} \to \mathbf{H}$ is a $(3,2)$-critical quasicircle map on some quasicircle $\mathbf{H}$, $f_{2,2} : \mathbb{T} \to \mathbb{T}$ is a $(2,2)$-critical circle map, and both have the golden mean rotation number. Both $\mathbf{H}$ and $\mathbb{T}$ are colored red, and their preimages are colored green.

[Yam08, GY22] gave a computer-assisted proof of the golden mean hyperbolicity of renormalization of Siegel disks using the formalism of *almost commuting pairs*. In [DLS20], Dudko, Lyubich, and Selinger constructed a compact analytic operator, called *Pacman renormalization operator*, with a hyperbolic fixed point whose stable manifold has codimension one and consists of maps with a Siegel disk of a fixed rotation number of periodic type.

From now on, we will be working with critical quasicircle maps $f : \mathbf{H} \to \mathbf{H}$ where $\mathbf{H}$ is a *Herman curve*, that is, $\mathbf{H}$ is not contained in the closure of any rotation domain of f. In the bounded type regime, this is equivalent to the assumption that both d_0 and d_∞ are at least two.

Given any pair of integers $d_0, d_\infty \geq 2$, the problem of realization of (d_0, d_∞)-critical quasicircle maps was solved in our previous work [Lim23a] by studying *a priori bounds* and degeneration of Herman rings of a certain class of rational maps. In [Lim23b], we proved $C^{1+\alpha}$ rigidity and constructed renormalization horseshoe for critical quasicircle maps with bounded type rotation number.

1.2. **Corona renormalization.** The main aim of this paper is to continue our study of renormalization of critical quasicircle maps and prove hyperbolicity of renormalization for periodic rotation number. Our approach will follow closely the ideas behind Pacman Renormalization Theory. We design a renormalization operator acting on the space of *coronas*, a doubly-connected version of pacmen.

A corona is a holomorphic map $f : U \to V$ between two nested annuli such that $f : U\backslash\gamma_0 \to V\backslash\gamma_1$ is a unicritical branched covering map where γ_1 is an arc connecting the two boundary components of V. The number of preimages of γ_1 on the boundary components of U determine the inner and outer criticalities d_0 and d_∞ of a corona. When the criticalities are specified, we call f a (d_0, d_∞)-critical corona. See Figure 3 for an illustration.

Similar to pacman renormalization, we define the corona renormalization operator as follows. First, we remove the quadrilateral bounded by γ_1 and its image. The remaining space is a quadrilateral in which the first return map will be called a *pre-corona*. Gluing a pair of opposite sides of this quadrilateral gives us a new corona, which is called the *corona renormalization* $\mathcal{R}f$ of f.

We say that a (d_0, d_∞)-critical corona is *rotational* with rotation number θ if it admits an invariant quasicircle $\mathbf{H}$ on which the map is a (d_0, d_∞)-critical quasicircle map of rotation number θ. The renormalization of a (d_0, d_∞)-critical rotational corona is again a (d_0, d_∞)-critical rotational corona, and the induced action on the rotation number is governed by

$$R_{prm}(\theta) = \begin{cases} \dfrac{\theta}{1-\theta}, & \text{if } 0 \leq \theta \leq \dfrac{1}{2}, \\ \dfrac{2\theta - 1}{\theta}, & \text{if } \dfrac{1}{2} \leq \theta \leq 1. \end{cases}$$

Theorem A (Hyperbolicity of renormalization)**.** *For any integers $d_0, d_\infty \geq 2$ and any $\theta \in \Theta_{per}$, there exists a corona renormalization operator $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ with the following properties.*

1. *$\mathcal{U}$ is an open subset of a complex Banach manifold $\mathcal{B}$ consisting of (d_0, d_∞)-critical coronas of criticality $d_0 + d_\infty - 1$.*
2. *$\mathcal{R}$ has a unique hyperbolic fixed point $f_* \in \mathcal{U}$.*

(3) *The local stable manifold* $\mathcal{W}^s_{loc}$ *of* f_* *corresponds to the space of rotational coronas with rotation number* θ *in* $\mathcal{B}$.
(4) *The local unstable manifold* $\mathcal{W}^u_{\mathrm{loc}}$ *is one-dimensional.*

Similar to [DLS20], the main step is justifying item (4), which will be accomplished via transcendental dynamics. The pre-corona associated to a corona f on the local unstable manifold admits a maximal transcendental extension $\mathbf{F}$. The dynamics of $\mathbf{F}$ can be described as a *cascade*, that is, a collection $\{\mathbf{F}^P\}_{P\in\mathbf{T}}$ of σ-proper maps parametrized by a dense semigroup $\mathbf{T}\subset(\mathbb{R}_{\geq 0},+)$ such that $\mathbf{F}^P\circ\mathbf{F}^Q=\mathbf{F}^{P+Q}$. The second half of this paper is dedicated to the study of the dynamics of $\mathbf{F}$. To justify item (4), we prove the following theorem.

Theorem B (Rigidity of escaping dynamics on $\mathcal{W}^u_{\mathrm{loc}}$)**.** *Let* $\mathbf{F}$ *be a maximal* σ*-proper extension of a pre-corona on* $\mathcal{W}^u_{\mathrm{loc}}$. *The full escaping set*

$$\mathbf{I}(\mathbf{F}) \coloneqq \left\{ z\in\mathbb{C}: \quad \textit{either } z\notin\bigcap_P \mathrm{Dom}\left(\mathbf{F}^P\right) \textit{ or } \mathbf{F}^P(z)\to\infty \textit{ as } P\to\infty \right\}$$

moves conformally away from the pre-critical points and supports no invariant line field. Consequently, if $\mathbf{F}$ *has an attracting cycle, then the Julia set of* $\mathbf{F}$ *supports no invariant line field.*

One may compare this theorem to Rempe's result [Rem09] on the rigidity of the escaping set of transcendental entire functions. Ultimately,

$$\text{Theorem B} \implies \dim\mathcal{W}^u_{\mathrm{loc}} = \text{number of critical orbits} = 1 \implies \text{Theorem A(4)}.$$

Remark 1.3. We would like to note a few differences between our case and the pacmen case. Refer to Section 1.3 for a more comprehensive summary.

Firstly, the original proof of item (4) for pacmen does not require such a rigidity theorem. Unlike coronas, every pacman is designed to admit a natural fixed point α associated to it. For a Siegel pacman, the α-fixed point is the center of its Siegel disk. The multiplier of the α-fixed point naturally foliates the Banach neighborhood of the pacman renormalization fixed point. Consequently, hyperbolicity of the pacman renormalization operator and in particular item (4) follows from an application of the λ-lemma along parabolic leaves.

Secondly, the study of the finite-time escaping set associated to transcendental extension of pre-pacmen was conducted in [DL23] to attain a puzzle structure, which was ultimately applied to prove the MLC at some infinitely renormalizable satellite parameters. In our case, the full escaping set $\mathbf{I}(\mathbf{F})$ is of interest because, together with the postcritical set, it is the measure-theoretic attractor of $\mathbf{F}^{\geq 0}$ on the Julia set.

Given a critical quasicircle map $f:\mathbf{H}\to\mathbf{H}$, we can define a Banach neighborhood $N(f)$ of f as follows. Pick a small neighborhood U of $\mathbf{H}$ such that f is holomorphic on a neighborhood of U, and pick a small $\varepsilon>0$. Then, $N(f)$ is the space of unicritical holomorphic maps $g:U\to\mathbb{C}$ such that $\sup_{z\in U}|f(z)-g(z)|<\varepsilon$, equipped with the sup norm.

Corollary C. *Consider a small Banach neighborhood* $N(f)$ *of a* (d_0,d_∞)*-critical quasicircle* $f:\mathbf{H}\to\mathbf{H}$ *with pre-periodic rotation number* θ. *The space* S *of maps in* $N(f)$ *which restrict to a* (d_0,d_∞)*-critical quasicircle map with rotation number* θ *forms an analytic submanifold of* $N(f)$ *of codimension at most one. The corresponding quasicircles move holomorphically over* S.

We conjecture that the codimension is actually one.

Conjecture D. *The conjugacy class S has codimension one. In particular, critical quasicircle maps are structurally unstable.*

So far, this conjecture is known to be true for periodic type critical quasicircle maps that are close to the renormalization fixed point f_* due to Theorem A, as well as critical circle maps due to standard monotonicity properties of the rotation number. We suspect that the conjecture can be solved via an infinitesimal argument similar to unimodal maps [ALdM03].

Consider a one-dimensional holomorphic family of unicritical holomorphic maps $\{f_\lambda\}_{\lambda\in\Lambda}$. We say that a parameter $\lambda \in \Lambda$ is *hyperbolic* if the forward orbit of the critical point of f_λ tends to an attracting cycle. The space of hyperbolic parameters in Λ is open, and every connected component of such is called a *hyperbolic component*.

Conjecture E (Parameter self-similarity)**.** *Suppose there is a unique parameter $\lambda_* \in \Lambda$ such that f_{λ_*} has a unicritical Herman quasicircle of periodic type rotation number θ. The union of hyperbolic components within Λ is asymptotically self-similar at λ_* with a universal self-similarity factor depending only on θ and the criticality of f_*.*

A version of this conjecture appears in [Lim23a], in which the family $\{f_\lambda\}$ is a family of rational maps. See Figure 2. This conjecture is a generalization of parameter golden-mean universality of critical circle maps. Our hyperbolicity result provides a step forward towards solving this conjecture. However, we suspect that attaining a complete solution would require hyperbolicity of the renormalization horseshoe for bounded type rotation numbers, as well as a thorough study of parameter rays and hyperbolic components of the unstable manifold as a parameter space of transcendental σ-proper maps.

1.3. **Outline.** Sections 2–5 are inspired by the original work on pacman renormalization in [DLS20], and Sections 5–7 are inspired by the detailed study of transcendental dynamics on the unstable manifold in [DL23]. As previously mentioned, the main difference lies in the proof that the local unstable manifold $\mathcal{W}^u_{\text{loc}}$ has dimension one. Once we prove that our renormalization fixed point is hyperbolic, we treat $\mathcal{W}^u_{\text{loc}}$ as a holomorphic family of unicritical transcendental maps of unknown dimension. By adapting some ideas from [Rem09], we deduce the rigidity of escaping dynamics and claim that the deformation space of hyperbolic coronas on the unstable manifold must be supported on the Fatou set, the domain of stability. This implies that $\mathcal{W}^u_{\text{loc}}$ is one dimensional.

In Section §2, we introduce the definition of *coronas* and *pre-coronas*. We define the corona renormalization operator and show that for any renormalizable corona f, we can always find a compact holomorphic operator $\mathcal{R}$ on a small Banach neighborhood of f.

In Section §3, we analyze the structure of a rotational corona f. We prove that any critical quasicircle map can be renormalized to a rotational corona. By applying results in [Lim23b], we also show that rotational coronas are rigid: two rotational coronas are quasiconformally conjugate as long as they have the same criticality and rotation number.

In Section §4, we construct a compact analytic corona renormalization operator $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ and a corona $f_* \in \mathcal{U}$ of periodic rotation number such that $\mathcal{R} f_* = f_*$. In

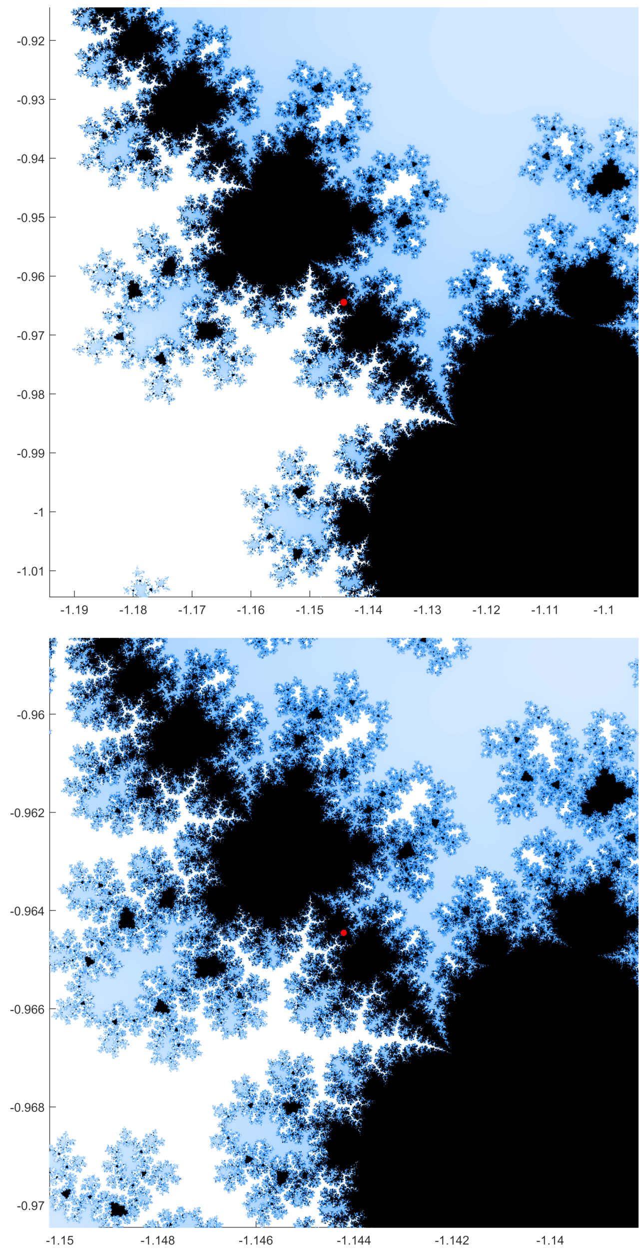

FIGURE 2. Magnifications of the bifurcation locus of the parameter space $\left\{F_c(z) = cz^3 \frac{4-z}{1-4z+6z^2}\right\}_{c\in\mathbb{C}^*}$ by different scales about the parameter $c_* \approx -1.144208 - 0.964454i$ marked in red. This family is characterized by critical points 0, ∞, and 1 of local degrees 2, 3, and 4 respectively, where both 0 and ∞ are fixed and $F_c(1) = c$. The point c_* is the unique parameter such that F_{c_*} has a golden mean Herman quasicircle. Figure 1 displays the Julia set of F_{c_*}.

Theorem 4.12, we show that $\mathcal{R}$ and f_* satisfy items (2) and (3) in Theorem A, and that the dimension of the local unstable manifold $\mathcal{W}^u_{\mathrm{loc}}$ is finite and positive. The proof relies on a number of ingredients.

(i) For any corona $f \in \mathcal{U}$ which is many times renormalizable, we can obtain a renormalization tiling which approximates the Herman quasicircle $\mathbf{H}_*$ of f_* by lifting the domain of a high renormalization of f. This tiling is robust under perturbations, and we use them to show in Corollary 4.11 that any infinitely renormalizable rotational corona that stays close to f_* must be a rotational corona.

(ii) By [Lim23b, Theorem K], renormalizations $\mathcal{R}^n f$ of a rotational corona near f_* must converge exponentially fast to f_*.

(iii) In Appendix A, we prove a generalization of Lyubich's Small Orbits Theorem [Lyu99, §2] that works even in the presence of both attracting and repelling eigenvalues. (In the pacman case [DLS20], the foliation induced by the multiplier of the α-fixed point removes the need to generalize the Small Orbits Theorem.)

These three ingredients will imply that $D\mathcal{R}_{f_*}$ has no neutral eigenvalues. To show that a repelling direction exists, we apply [Lim23b, Theorem B], a result on combinatorial rigidity of unicritical Herman quasicircles of a nice class of rational maps.

The second half of the paper is dedicated to proving that $D\mathcal{R}_{f_*}$ has exactly one repelling eigenvalue. In Section §5, we show that for any map f on the local unstable manifold, the maximal extension of the pre-corona associated to f is a commuting pair of σ-proper map $\mathbf{F} = (\mathbf{f}_\pm : \mathbf{X}_\pm \to \mathbb{C})$. The proof relies on a technical lemma, which we prove separately in Appendix B due to its length. This allows us to identify $\mathcal{W}^u_{\mathrm{loc}}$ with $\boldsymbol{\mathcal{W}}^u_{loc}$, the holomorphic family of transcendental maps $\mathbf{F}$.

Given $\mathbf{F} = (\mathbf{f}_\pm) \in \boldsymbol{\mathcal{W}}^u_{loc}$ and $n \le 0$, we set $\mathbf{F}_n = \mathcal{R}^n\mathbf{F}$ and denote by $\mathbf{F}^\#_n = (\mathbf{f}^\#_{n,\pm})$ the rescaled version of $\mathbf{F}_n$ such that $\mathbf{f}_\pm$ are iterates of $\mathbf{f}^\#_{n,\pm}$. We identify $\mathbf{F}$ as a cascade, that is, the semigroup $\left(\mathbf{F}^{\ge 0}, \circ\right)$ generated by $\mathbf{f}^\#_{n,\pm}$ for all $n \le 0$. The cascade $\mathbf{F}^{\ge 0}$ is isomorphic to a dense sub-semigroup $(\mathbf{T}, +)$ of $\mathbb{R}_{\ge 0}$ and elements of $\mathbf{F}^{\ge 0}$ can be written as $\mathbf{F}^P$ for $P \in \mathbf{T}$. We define the *finite-time escaping set* $\mathbf{I}_{<\infty}(\mathbf{F})$ of $\mathbf{F}$ to be the set of points in the dynamical plane of $\mathbf{F}$ that is not in the domain of $\mathbf{F}^P$ for some $P \in \mathbf{T}$. In Section §6, we study the structure of the escaping set of the renormalization fixed point $\mathbf{F}_*$. Similar to [DL23, §5], we construct external rays and deduce its tree structure using their branch points, which are called *alpha-points*. These escaping rays produces a puzzle structure which partitions the whole dynamical plane.

In Section §7, we apply the external structure of $\mathbf{F}_*$ to obtain item (4) in Theorem A. In short, this is done in a number of steps.

(i) We prove that $\mathbf{I}_{<\infty}(\mathbf{F})$ carries no invariant line field and locally moves holomorphically unless it contains a pre-critical point.

(ii) We observe that any map $\mathbf{F}$ close to $\mathbf{F}_*$ inherits most of the external structure of $\mathbf{F}_*$, which we use to study the *infinite-time escaping set*

$$\mathbf{I}_\infty(\mathbf{F}) := \{z \in \mathbb{C}\backslash\mathbf{I}_{<\infty}(\mathbf{F}) \,:\, \mathbf{F}^P(z) \to \infty\}.$$

By adapting the ideas from Rempe [Rem09], we show that $\mathbf{I}_\infty(\mathbf{F})$ also carries no invariant line field and locally moves holomorphically unless it contains a pre-critical point.

(iii) We show that there exist hyperbolic cascades $\mathbf{F}$ arbitrarily close to $\mathbf{F}_*$. When $\mathbf{F}$ is hyperbolic, the Julia set of $\mathbf{F}$ is the union of $\mathbf{I}(\mathbf{F}) := \mathbf{I}_{<\infty}(\mathbf{F}) \cup \mathbf{I}_\infty(\mathbf{F})$ and a zero measure set.

These three ingredients allow us to deduce that the deformation space of a hyperbolic $\mathbf{F}$ can only be supported on the Fatou set. Since $\mathbf{F}$ is unicritical, this implies that the parameter space $\boldsymbol{\mathcal{W}}^u_{loc}$ is one dimensional.

This paper contains three appendices. Appendix A provides a generalization of Lyubich's Small Orbits Theorem [Lyu99, §2]. The key addition here is the application of two invariant cones rather than just one. Appendix B is a review of results in sector renormalization from [DLS20, DL23] essential in our paper. Appendix C provides the proof of Lemma 5.6, the key towards attaining the transcendental extension; this is an analog of [DLS20, Key Lemma 4.8] in our setting.

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2. Corona renormalization operator

Let $d_0, d_\infty \geq 2$ be a pair of positive integers and let $d := d_0 + d_\infty - 1$.

2.1. **(d_0, d_∞)-critical coronas.** For any open annulus A compactly contained in $\mathbb{C}$, we label the boundary components of A by $\partial^0 A$ and $\partial^\infty A$, and make the convention that $\partial^\infty A$ is the outer boundary, i.e. the one that is closer to ∞. We also say that another annulus A' is *essentially* contained in A if $A' \subset A$ and the inclusion map $A' \hookrightarrow A$ induces an isomorphism of fundamental groups.

Definition 2.1. A (d_0, d_∞)*-critical corona* is a map $f : U \to V$ between two bounded open annuli in $\mathbb{C}$ with the following properties.

(1) The boundary components of both U and V are Jordan curves, and U is compactly and essentially contained in V.
(2) There is a proper arc $\gamma_1 \subset V$ connecting $\partial^0 V$ and $\partial^\infty V$ such that the preimage $f^{-1}(\gamma_1)$ is disjoint from γ_1 and is a union of $2d-1$ pairwise disjoint arcs
$$\gamma_0 \subset U, \quad \gamma_1^0, \ldots, \gamma^0_{2(d_0-1)} \subset \partial^0 U, \quad \gamma_1^\infty, \ldots, \gamma^\infty_{2(d_\infty-1)} \subset \partial^\infty U.$$
(3) $f : U \to V$ is holomorphic and $f : U \backslash \gamma_0 \to V \backslash \gamma_1$ is a degree d covering map branched at a unique critical point c_0.

The arc γ_1 is called the *critical arc* of f. See Figure 3 for an illustration.

Let $f : U \to V$ be a (d_0, d_∞)-critical corona. For any $\bullet \in \{0, \infty\}$, we divide the boundary component $\partial^\bullet U$ into
$$\partial^\bullet_{\mathrm{L}} U := \partial^\bullet U \cap f^{-1}(\partial^\bullet V) \quad \text{and} \quad \partial^\bullet_{\mathrm{F}} U := \partial^\bullet U \backslash f^{-1}(\partial^\bullet V)$$
according to whether or not it is mapped to the same side the annulus. Each of the above consists of $d_\bullet - 1$ components. Set
$$\partial_{\mathrm{L}} U := \partial^0_{\mathrm{L}} U \cup \partial^\infty_{\mathrm{L}} U \quad \text{and} \quad \partial_{\mathrm{F}} U := \partial^0_{\mathrm{F}} U \cup \partial^\infty_{\mathrm{F}} U.$$
We call $\partial_{\mathrm{L}} U$ the *legitimate boundary* of U and $\partial_{\mathrm{F}} U$ the *forbidden boundary* of U.

For each $\bullet \in \{0, \infty\}$, we properly embed a collection $\mathscr{R}^\bullet$ of $d_\bullet - 1$ pairwise disjoint rectangles within $V \backslash \overline{U}$ such that the union $B^\bullet$ of their bottom horizontal sides is

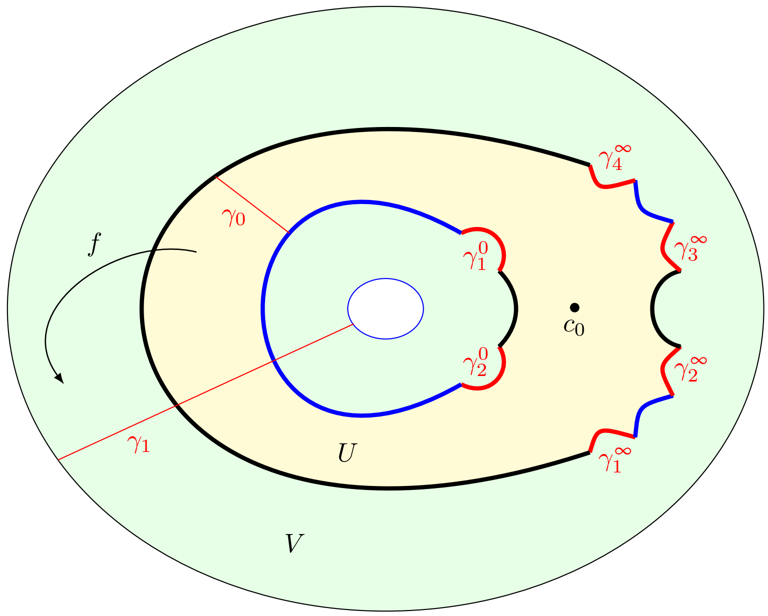

FIGURE 3. A (2,3)-critical corona

precisely the legitimate boundary $\partial^\bullet_{\mathrm{L}} U$ and the union $T^\bullet$ of their top horizontal sides is a subset of $\partial^\bullet V$. Let us lift $\mathscr{R}^\bullet$ under f such that their top sides are within the legitimate boundary of U. As we repeat this lifting procedure, we obtain a lamination out of the iterated lifts, and its leaves will be called *external ray segments*.

An infinite chain of external ray segments is called an *external ray* of the corona f. We say that γ is an *inner* external ray if γ intersects B^0, and an *outer* external ray if instead γ intersects B^∞.

For each $\bullet \in \{0, \infty\}$, define the map $\pi_\bullet : B^\bullet \to T^\bullet$ sending the bottom endpoint of each leaf of $\mathscr{R}^\bullet$ to the corresponding top endpoint. Consider the partially defined $d_\bullet$ to one map $\phi_\bullet := \pi_\bullet^{-1} \circ f$ on $B^\bullet$. Denote by $\mathcal{A}^\bullet$ the set of points of $B^\bullet$ which are invariant under $\phi_\bullet$. Let us identify S^1 with the quotient $\mathbb{R}/\mathbb{Z}$. There is a semiconjugacy $\mathcal{A}^\bullet \to S^1$ between $\phi_\bullet : \mathcal{A}^\bullet \to \mathcal{A}^\bullet$ and the multiplication map $S^1 \to S^1, x \mapsto d_\bullet x \pmod 1$, which is unique up to conjugation with addition by multiples of $\frac{1}{d_\bullet - 1}$.

Given an external ray γ of f, we denote the image by

$$f(\gamma) := f(\gamma \cap U)$$

which is also an external ray of f by definition. The *external angle* of γ is the angle $\theta_\bullet(x)$ where x is the unique point of intersection of γ and $B^\bullet$ for some $\bullet \in \{0, \infty\}$.

2.2. **Corona renormalization.**

Definition 2.2. A (d_0, d_∞)-*critical pre-corona* is a pair of holomorphic maps

$$F = (f_- : U_- \to S,\ f_+ : U_+ \to S)$$

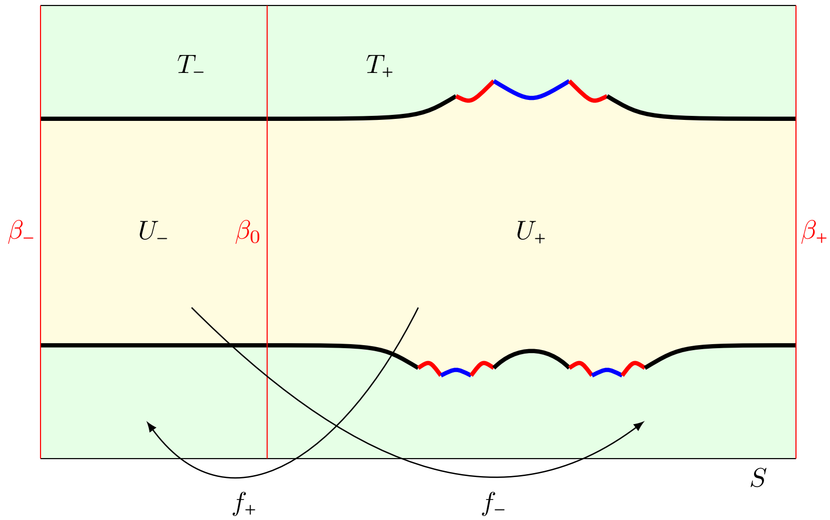

Figure 4. A (2,3)-critical pre-corona. It projects to the corona in Figure 3 after gluing β_+ and β_-

satisfying the following properties.

(1) S is a topological rectangle with vertical sides β_- and β_+.
(2) β_0 is a vertical arc in S dividing S into subrectangles T_- and T_+, where $\beta_\pm \subset \partial T_\pm$ and $U_\pm$ is a subrectangle of $T_\pm$ with vertical sides contained in $\beta_\pm$ and β_0.
(3) There is a gluing map $\psi : \overline{S} \to \overline{V}$ such that $\psi(\beta_-) = \psi(\beta_+)$, ψ is conformal on a neighborhood of S and injective on $S \backslash (\beta_- \cup \beta_+)$, and ψ projects F into a (d_0, d_∞)-critical corona with critical arc $\psi(\beta_\pm)$.

The gluing map ψ will also be called the *renormalization change of variables* of F. It glues together $f_+(x) \in \beta_-$ and $f_-(x) \in \beta_+$ for every x in $\beta_0 \cap \partial U_\pm$. See Figure 4.

Definition 2.3. A corona $f : U \to V$ is *renormalizable* if there exists a pre-corona

$$F = (f^{k_-} : U_- \to S,\ f^{k_+} : U_+ \to S)$$

on a rectangle $S \subset V$ such that f^{k_-} and f^{k_+} are the first return maps back to S and

$$\boldsymbol{\Delta}_F = \bigcup_{i=0}^{k_- -1} \overline{f^i(U_-)} \cup \bigcup_{j=0}^{k_+ -1} \overline{f^j(U_+)}$$

is a closed annulus essentially contained in U. We call F the *pre-renormalization* of f, k_- and k_+ the *return times* of F, and $\boldsymbol{\Delta}_F$ the *renormalization tiling* of F. The corona obtained by projecting F under its gluing map is called the *renormalization* of f.

Example 2.4 (Prime renormalization)**.** We say that the renormalization of a corona $f : U \to V$ is *prime* if $k_- + k_+ = 3$. Below is an example of a prime corona renormalization.

Assume that the arcs γ_0, γ_1, and $\gamma_2 := f(\gamma_1)$ are pairwise disjoint. Denote by S_1 the open quadrilateral obtained by cutting V along $\gamma_1 \cup \gamma_2$ which does not contain γ_0. Let us assume further that S_1 does not contain the critical value nor the forbidden boundary of U.

Let us remove S_1 from the dynamical plane. We define $\hat{V}$ to be the Riemann surface with boundary obtained from $\overline{V}\backslash S_1$ by gluing $\gamma_1' := f^{-1}(\gamma_2) \cap \gamma_1$ and its image γ_2 along f. In other words, there is a quotient map $\psi : \overline{V}\backslash S_1 \to \hat{V}$ that is conformal on the interior and $\psi(z) = \psi(f(z))$ for all $z \in \gamma_1'$. We embed the abstract Riemann surface $\hat{V}$ into the plane.

The prime renormalization of f is defined by the induced first return map of f on $\hat{V}$. More precisely, consider the lift S_0 of S_1 under f attached to γ_1. The piecewise holomorphic map

$$\begin{cases} f(z), & \text{if } z \in U\backslash(S_1 \cup f^{-1}(S_1)), \\ f^2(z), & \text{if } z \in S_0 \cap f^{-1}(U). \end{cases}$$

descends via ψ into a corona $\hat{f} : \hat{U} \to \hat{V}$ with critical ray $\hat{\gamma}_1 = \psi(\gamma_1')$.

2.3. **Banach neighborhood.** In what follows, every unicritical holomorphic map $f : U \to V$ under consideration will be assumed to admit a slightly larger domain $\tilde{U}$ with piecewise smooth boundary such that $\tilde{U}$ compactly contains U and f extends to a unicritical holomorphic map on $\tilde{U}$ extending continuously to $\partial\tilde{U}$. We define a *Banach neighborhood* of f to be a neighborhood of f of the form $N_{\tilde{U}}(f,\varepsilon)$, which we define to be the space of holomorphic maps $g : \tilde{U} \to \mathbb{C}$ that extend continuously to $\partial\tilde{U}$, admit a single critical point in $c_0(g)$, and

$$\sup_{z\in\tilde{U}} |f(z) - g(z)| < \varepsilon.$$

We equip $N_{\tilde{U}}(f,\varepsilon)$ with the sup norm over $\tilde{U}$.

Lemma 2.5. *Let $f : U \to V$ be a (d_0, d_∞)-critical corona. For sufficiently small $\varepsilon > 0$, there is a holomorphic motion ∂U_g of ∂U over $g \in N_{\tilde{U}}(f,\varepsilon)$ such that $g : U_g \to V$ is a (d_0, d_∞)-critical corona with the same codomain V and critical arc γ_1.*

Proof. Let A_δ be the δ-neighborhood of ∂U, where $\delta > 0$ is picked small enough such that A_δ contains no critical points of f. For sufficiently small ε, the derivative of $g \in N_{\tilde{U}}(f,\varepsilon)$ is uniformly bounded and non-vanishing on A_δ, and so g has no critical points in A_δ. Thus, we have a well-defined map $\tau_g : \partial U \to A_\delta$ such that $\tau_f = \mathrm{Id}$ and $f = g \circ \tau_g$ on ∂U. Since f has no critical value along ∂U, $\tau_g(z)$ is injective in z and holomorphic in g. Therefore, we have a holomorphic motion of ∂U, and $\tau_g(\partial U)$ bounds an open annulus U_g on which $g : U_g \to V$ is a well-defined (d_0, d_∞)-critical corona with the same critical arc. □

Theorem 2.6. *Suppose a unicritical holomorphic map $f : U \to V$ admits a pre-corona which projects to a corona $\hat{f} : \hat{U} \to \hat{V}$ via a quotient map $\psi_f : S_f \to \hat{V}$. For sufficiently small $\varepsilon > 0$, there is a compact analytic renormalization operator $\mathcal{R}$ on a Banach neighborhood $N_{\tilde{U}}(f,\varepsilon)$ such that $\mathcal{R}f = \hat{f}$ and for each $g \in N_{\tilde{U}}(f,\varepsilon)$,*

1. *g admits a pre-corona which projects to the corona $\mathcal{R}g : \hat{U}_g \to \hat{V}$, and*
2. *the domain $\partial\hat{U}_g$ and the associated gluing map ψ_g depend holomorphically on g.*

Proof. There exists a pre-corona $F = \left(f^{k_\pm} : U_\pm \to S\right)$ and a quotient map ψ_f projecting F to $\hat{f}$. Recall the arcs $\beta_\pm$ and β_0 corresponding to F. For $g \in N_{\tilde{U}}(f,\varepsilon)$, consider the map $\tau_g : \beta_0 \cup \beta_\pm \to \mathbb{C}$ defined by setting τ_g to be the identity map on β_0 and the composition $g^{k_\mp} \circ f^{-k_\mp}$ on $\beta_\pm$; this is an equivariant holomorphic motion of $\beta_0 \cup \beta_\pm$ for sufficiently small $\varepsilon > 0$. By λ-lemma, τ_g extends to a holomorphic motion of S over a neighborhood of f.

Let μ_g be the Beltrami differential of τ_g. Define a global Beltrami differential ν_g by setting $\nu_g = (\psi_f)_* \mu_g$ on $\hat{V}$ and $\nu_g \equiv 0$ outside of $\hat{V}$. Integrate ν_g to obtain a unique quasiconformal map ϕ_g fixing ∞, the critical point of f, and the critical value of f. Then, $\psi_g := \phi_g \circ \psi_f \circ \tau_g^{-1}$ is a conformal map on $S_g := \tau_g(S_f)$ depending holomorphically on g.

The gluing map ψ_g projects the pair $\left(g^{k_-}, g^{k_+}\right)$ on S_g to a map $\hat{g}$ close to $\hat{f}$. By Lemma 2.5, $\hat{g}$ restricts to a corona that has the same range as $\hat{f}$ and depends analytically on g. This yields an analytic operator $g \mapsto \hat{g}$. To make this operator compact, we modify it as follows. Pick another annulus U' where $U \Subset U' \Subset \tilde{U}$. We define $\mathcal{R}$ on $N_{\tilde{U}}(f, \varepsilon)$ to be the renormalization of the restriction of g to U'. □

3. Rotational coronas

Throughout this section, let us fix a pair of integers $d_0, d_\infty \geq 2$ and a bounded type irrational $\theta \in \Theta_{bdd}$.

Definition 3.1 (Inner and outer criticalities)**.** Consider a quasicircle $\mathbf{H} \subset \mathbb{C}$ and denote the bounded and unbounded components of $\hat{\mathbb{C}} \backslash \mathbf{H}$ by Y^0 and Y^∞ respectively. We say that $f : \mathbf{H} \to \mathbf{H}$ is a (d_0, d_∞)*-critical quasicircle map* if it is a critical quasicircle map where for any $\bullet \in \{0, \infty\}$ and any point $z \in Y^\bullet$ close to the critical value of f, there are exactly $d_\bullet$ preimages of z in $Y^\bullet$ that are close to the critical point of f.

When a holomorphic map f is given, we also say that an invariant quasicircle $\mathbf{H} \subset \mathbb{C}$ is a (d_0, d_∞)*-critical Herman quasicircle* if $f : \mathbf{H} \to \mathbf{H}$ is a (d_0, d_∞)-critical quasicircle map. The term *Herman quasicircle* originates from [Lim23a] and is meant to acknowledge that its first examples arise from degeneration of Herman rings.

Definition 3.2. A corona $f : U \to V$ is a *rotational corona* if

(1) U essentially contains a Herman quasicircle $\mathbf{H}$ that passes through the unique critical point of f;
(2) the critical arc γ_1 intersects $\mathbf{H}$ precisely at one point $r(f)$ which splits γ_1 into an inner external ray R^0 and an outer external ray R^∞.

If $\mathbf{H}$ is a (d_0, d_∞)*-critical* Herman quasicircle, we also say that f is a (d_0, d_∞)*-critical rotational corona.*

We call $r(f)$ the *marked point* of f. Let us define the *non-escaping set* of a corona $f : U \to V$ to be

$$K(f) := \bigcap_{n \geq 0} f^{-n}\left(\overline{U}\right).$$

If f is rotational, we also define the *Julia set* of f to be the closure of the iterated preimages of the Herman quasicircle $\mathbf{H}$ in U, that is

$$J(f) := \overline{\bigcup_{n \geq 0} f^{-n}(\mathbf{H})}.$$

We say that a rotational pre-corona F is *around* a point x if the arc β_0 intersects the Herman quasicircle of F at the point x, i.e. $r(f) = x$.

3.1. **Realization of rotational coronas.** Consider the family of degree d rational maps $\{F_c\}_{c\in\mathbb{C}^*}$ defined by

$$F_c(z) := -c\,\frac{\sum_{j=d_0}^{d}\binom{d}{j}\cdot(-z)^j}{\sum_{j=0}^{d_0-1}\binom{d}{j}\cdot(-z)^j}. \tag{3.1}$$

By [Lim23a, Proposition 10.1], this family is characterized by the property that F_c has critical points at 0, ∞, and 1 with local degrees d_0, d_∞, and d respectively, and that $F_c(0) = 0$, $F_c(\infty) = \infty$, and $F_c(1) = c$.

Theorem 3.3 ([Lim23a, Lim23b])**.** *There exists a unique parameter $c = c(\theta) \in \mathbb{C}^*$ such that F_c admits a (d_0, d_∞)-critical Herman quasicircle $\mathbf{H}$ with rotation number θ which contains a unique critical point of F_c at* 1*.*

Consider $f := F_c$ and $\mathbf{H}$ from the theorem above. For any $n \geq 1$, we refer to the closure of a component of $f^{-n}(\mathbf{H})\backslash f^{-(n-1)}(\mathbf{H})$ as a *bubble* of *generation* n. Every bubble B of generation n is a quasicircle admitting a unique point, which we will call the *root* of B, that lies on the pre-critical set $f^{-(n-1)}(1)$. We call a bubble B of generation n an *outer bubble* (resp. *inner bubble*) if the bubbles B, $f(B), \ldots,$ $f^{n-1}(B)$ all lie in the connected component of $\hat{\mathbb{C}}\backslash\mathbf{H}$ containing ∞ (resp. 0).

A *limb* of generation one is the closure of a connected component of $J(f)\backslash\{1\}$ that is disjoint from $\mathbf{H}$. A *filled limb* $\hat{L}$ of generation one is the hull of a limb L of generation one, that is, $\hat{\mathbb{C}}\backslash\hat{L}$ is the unbounded connected component of $\hat{\mathbb{C}}\backslash L$. In general, a (filled) limb of generation $n \geq 1$ is the connected component of the preimage under f^{n-1} of a (filled) limb of generation one. A (filled) limb of generation n contains a unique bubble of generation n, which we will call the *core bubble* of the limb. The *root* of a (filled) limb is the root of its core bubble. We call a (filled) limb an *outer/inner (filled) limb* if its core bubble is an outer/inner bubble.

Lemma 3.4. *The immediate basins of* 0 *and* ∞ *of f have locally connected boundaries. For any $\varepsilon > 0$, all but finitely many inner and outer limbs of f have diameter at most ε.*

Proof. Denote by Y^0 and Y^∞ the connected components of $\hat{\mathbb{C}}\backslash\mathbf{H}$ containing 0 and ∞ respectively. Perform Douady-Ghys surgery [Ghy84, Dou87] (see also [BF14, §7.2]) along $\mathbf{H}$ to replace the dynamics of f in Y^0 with a rotation disk and obtain a degree d_∞ unicritical polynomial P_∞ whose critical point lies in the boundary of an invariant Siegel disk Z_∞ of P_∞. The maps $f|_{\overline{Y^\infty}}$ and $P_\infty|_{\hat{\mathbb{C}}\backslash Z_\infty}$ are quasiconformally conjugate, so the filled outer limbs of f are quasiconformally equivalent to the limbs of P_∞. The work of [WYZZ21] guarantees that the Julia set of P_∞ is locally connected, and so any infinite sequence of limbs of P_∞ must shrink to a point. Therefore, for any $\varepsilon > 0$, all but finitely many outer limbs of f have diameter at most ε. By swapping the roles of 0 and ∞, we obtain the same result for inner limbs. □

Remark 3.5. In the proof above, we used the result that the external boundaries of the Julia set of F_c are locally connected. In fact, the whole Julia set of F_c is actually locally connected. In case $(d_0, d_\infty) = (2, 2)$, this was proven by Petersen [Pet96, §4]. For arbitrary criticalities (d_0, d_∞), the availability of complex bounds [Lim23b, §6.3] facilitates a direct generalization of Petersen's proof.

Consider the operator R_{prm} from Appendix B, which encodes how rotation number is transformed under sector renormalization.

Lemma 3.6. *For any point $x \in \mathbf{H}$ that is not a pre-critical point of f, any $\varepsilon > 0$, and any sufficiently high $n \in \mathbb{N}$, there is a rotational pre-corona*

$$P = \big(f_- := f^{k_-} : U_- \to S,\ f_+ := f^{k_+} : U_+ \to S\big)$$

around x such that

(1) *P has rotation number $R^n_{prm}(\theta)$;*
(2) *every external ray segment of P is within an external ray of P;*
(3) *the union $\bigcup_{\diamond \in \{-,+\}} \bigcup_{i=0}^{k_\diamond - 1} f^i(U_\diamond)$ lies in the ε-neighborhood of $\mathbf{H}$.*

Proof. For each $i \in \mathbb{Z}$, denote $x_i := (f|_{\mathbf{H}})^i(x)$. By Lemma B.2, for all $n \geq 1$, there exist return times $\mathbf{a}_n$, $\mathbf{b}_n$ such that the commuting pair

$$\big(f^{\mathbf{a}_n}|_{[x_{\mathbf{b}_n}, x_0]}, f^{\mathbf{b}_n}|_{[x_0, x_{\mathbf{a}_n}]}\big)$$

is a sector pre-renormalization of $f|_{\mathbf{H}}$ with rotation number $R^n_{prm}(\theta)$.

Let $k_- = \mathbf{a}_n$ and $k_+ = \mathbf{b}_n$, and let us pick a small constant $\lambda > 0$. For $\bullet \in \{0, \infty\}$, denote by $E^\bullet$ the equipotential in the immediate basin of $\bullet$ of level λ, and by $R^\bullet_-$, $R^\bullet$, and $R^\bullet_+$ the external rays in the immediate basin of $\bullet$ which land at the points x_{k_+}, $x_{k_- + k_+}$, and x_{k_-} respectively. These external rays and equipotentials bound a pair of rectangles S_- and S_+, where S_+ contains the segment $J_+ := [x_{k_+}, x_{k_-+k_+}] \subset \mathbf{H}$ and S_- contains the segment $J_- := [x_{k_-+k_+}, x_{k_-}] \subset \mathbf{H}$.

Let $I_- := [x_{k_+}, x_0]$ and $I_+ := [x_0, x_{k_-}]$. Precisely one of the two intervals, say I_- without loss of generality, contains a critical point of f^{k_-}. The rectangle $S^\pm$ lifts under $f^{k_\pm}$ to a topological disk $\Upsilon_\pm$ containing $I_\pm$, and $f^{k_-} : \Upsilon_- \to S_-$ is a degree d branched covering map and $f^{k_+} : U_+ \to S_+$ is univalent. Set U_- to be the union of Υ_- and all the lifts of S_+ under f^{k_-} that are disjoint from $\mathbf{H}$ and touching Υ_- on the boundary. Set $U_+ := \Upsilon_+$ and $S = S_- \cup S_+$. Then,

$$\big(f^{k_-} : U_- \to S, f^{k_+} : U_+ \to S\big)$$

is a (d_0, d_∞)-critical pre-corona with rotation number $R^n_{prm}(\theta)$.

Let us embed the restriction of external rays of f in $S \backslash U$ where $U := U_- \cup U_+$. Notice that the boundaries of U_- and U_+ contain equipotential segments of different levels. Assume without loss of generality that the equipotential segments in U_- have higher level. To satisfy (2), we can truncate a pair of small topological triangles near two vertices of the rectangle S_+, one where R^0_+ meets E^0 and the other where R^∞_+ meets E^∞. We will also truncate preimages of these triangles under f^{k_-} in U_-. Replace U and S with the new truncated domains. Then, every point in the legitimate boundary of U is now a landing point of an external ray segment, and (2) follows.

We claim that (3) follows from taking n to be sufficiently large and λ to be sufficiently small. Indeed, if $z \in U_\pm$ intersects an external ray in the basin of $\bullet \in \{0, \infty\}$ which lands at a point $w \in J(f) \cap U_\pm$, then the orbits of z and w remain close under iteration f^i for $i = 1, \ldots, k_\diamond$. Suppose $z \in U_\pm$ does not lie in the immediate attracting basin of 0 nor ∞. Then, it must lie within some filled limb $\hat{L}$ rooted at some pre-critical point $c_j \in \mathbf{H}$ for some $j \leq 0$. If c_j is not the unique critical point of f^{k_-}, then the forward images $\hat{L}$, $f(\hat{L})$, $\ldots, f^{k_\pm}(\hat{L})$ must remain small due to Lemma 3.4. If c_j happens to be the critical point of f^{k_-} in U_-, then we must have $-k_- < j < 0$. The image $f^j(U_-)$ must remain in a small neighborhood of the critical point $c_0 = 1$ of f. Therefore, the forward orbit z, $f(z)$, $\ldots$, $f^j(z)$ must be close to $\mathbf{H}$. □

In our previous work, we proved a rigidity theorem for critical quasicircle maps.

Theorem 3.7 ([Lim23b, Theorem F]). *Any two (d_0, d_∞)-critical quasicircle maps of the same bounded type rotation number are quasiconformally conjugate on some neighborhood of their Herman curves.*

Combining this with Lemma 3.6 gives us the following.

Corollary 3.8. *Any (d_0, d_∞)-critical quasicircle map $g : \mathbf{H}_g \to \mathbf{H}_g$ with bounded type rotation number is corona renormalizable, that is, there is a (d_0, d_∞)-critical rotational pre-corona which is an iterate of g near $\mathbf{H}$.*

Proof. Given any (d_0, d_∞)-critical quasicircle map g of bounded type rotation number, Theorem 3.7 asserts that there is a global quasiconformal map ϕ conjugating g on some neighborhood W of its Herman curve with $f := F_c$. By Lemma 3.6, f admits a pre-corona P with range contained within $\phi(W)$. Then, g admits a (d_0, d_∞)-critical pre-corona of the form $\phi^{-1} \circ P \circ \phi$. □

3.2. **Quasiconformal rigidity.** Given a critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$ with critical point $c \in \mathbf{H}$, there is a unique conjugacy $h_f : (\mathbf{H}, c) \to (\mathbb{T}, 1)$ between f and the rigid rotation R_θ sending c to 1. For any point $z \in \mathbf{H}$, the *combinatorial position* of z is the point $h_f(z)$ on the unit circle.

We say that two (d_0, d_∞)-critical rotational coronas f_1 and f_2 are *combinatorially equivalent* if

(1) they have the same rotation number,
(2) their marked points $r(f_1)$ and $r(f_2)$ have the same combinatorial position, and
(3) for $\bullet \in \{0, \infty\}$, the external rays $R^\bullet(f_1)$ and $R^\bullet(f_2)$ have the same external angles.

Theorem 3.9. *Two combinatorially equivalent (d_0, d_∞)-critical rotational coronas are quasiconformally conjugate.*

We will use the pullback argument to prove this theorem. Let us make a couple of technical preparations.

Let $f : U \to V$ be a (d_0, d_∞)-critical rotational corona with rotation number θ, which is a renormalization of the map $F_{c(\theta)}$ in Theorem 3.3. The strict preimage $f^{-1}(\mathbf{H}) \backslash \mathbf{H}$ in U consists of $2(d_0 + d_\infty - 2)$ arcs which we will label as

$$\sigma_1^0,\ \ldots,\ \sigma_{2d_0-2}^0,\ \sigma_1^\infty,\ \ldots,\ \sigma_{2d_\infty-2}^\infty$$

where each $\sigma_i^\bullet$ connects the critical point c_0 to the segment $\gamma_i^\bullet \subset \partial^\bullet U$, labelled in cyclic order around c_0. See to Figure 5.

A *bubble* of generation one of f is a closed arc of the form $\overline{\sigma_{2i-1}^\bullet \cup \sigma_{2i}^\bullet}$, and its root is the critical point of f. For $n \geq 2$, a bubble B of generation n is a connected preimage under f^{n-1} of a bubble B' of generation one, and its root is the corresponding preimage of the root of B'. We call a bubble B of generation n an *outer* (resp. *inner*) bubble if B, $f(B)$, $\ldots$, $f^{n-1}(B)$ all lie in the connected component of $\hat{\mathbb{C}} \backslash \mathbf{H}$ containing ∞ (resp. 0).

A *limb* L of generation n is the union of a bubble B of generation n, which we will call the *core* of L, together with all the connected components of $\overline{J(f) \backslash B}$

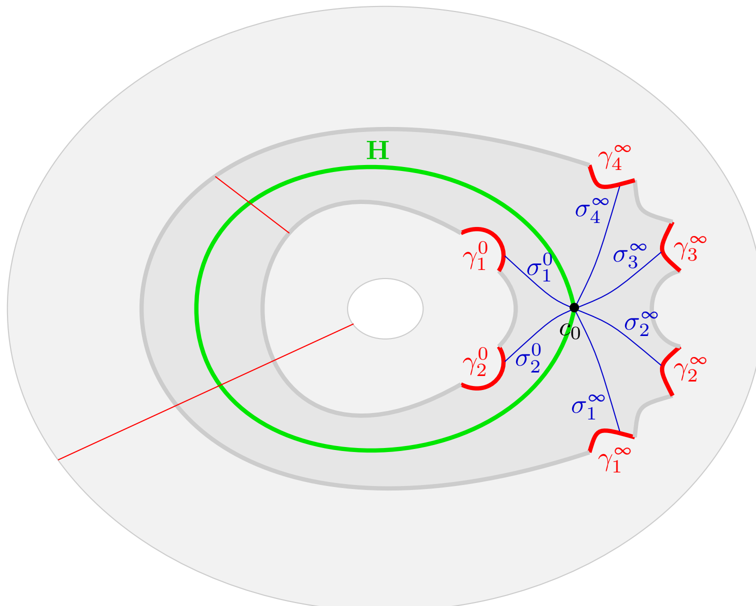

FIGURE 5. The arcs $\sigma_i^\bullet$'s make up the strict preimage of $\mathbf{H}$.

that intersects B but not $\mathbf{H}$. Again, we call L an *outer/inner limb* if its core is an outer/inner bubble.

A *bubble chain* of generation n is an infinite union of bubbles $B_n, B_{n+1}, B_{n+2}, \ldots$ where for all $j \geq n$, the bubble B_j has generation j and contains the root of B_{j+1}. We say that a bubble chain $(B_k)_{k\geq n}$

- ▷ is an *outer/inner* bubble chain if it does not contain any inner/outer bubbles,
- ▷ is *periodic* of period p if there is some minimal $p \geq 1$ such that $f^p(B_{k+p}) = B_k$ for all $k \geq n$, and
- ▷ *lands* if the accumulation set $\bigcap_{m\geq n} \overline{\cup_{k\geq m} B_k}$ is a single point, which we call the *landing point* of the bubble chain.

The following lemma follows immediately from Lemma 3.4.

Lemma 3.10. *Any outer/inner bubble chain lands.*

By Lemma 3.6, f is renormalizable. Consider a pre-corona P of f around some point $x \in \mathbf{H}$. Let us denote by R^0 and R^∞ the unique inner and outer external rays of f landing at x.

Consider the non-escaping set $K(P)$ of P. Let us define the *local non-escaping set* and the *local Julia set* of f relative to P respectively by

$$K^{loc}(f) := \bigcup_{n\geq 0} f^n(K(P)) \quad \text{and} \quad J^{loc}(f) := \overline{\bigcup_{n\geq 0} f|_{K^{loc}(f)}^{-n}(\mathbf{H})}.$$

Again, we denote by Y^0 and Y^∞ the bounded and unbounded connected components of $\hat{\mathbb{C}} \backslash \mathbf{H}$ respectively.

Lemma 3.11. *Let $y \in J^{loc}(f)$ be a periodic point of f. If the forward orbit of y lies on Y^∞ (resp. Y^0), then there exist a unique periodic outer (resp. inner) external ray R_y and a unique periodic outer (resp. inner) bubble chain $\{B_k\}_{k\geq 1}$ of generation one that land at y and have the same period as y.*

Proof. Suppose the forward orbit of a periodic point y in $J^{loc}(f)$ lies entirely in Y^∞. Under this assumption, y does not lie on any bubble and cannot be separated from $\mathbf{H}$ by any bubble. Therefore, y must be the landing point of a unique outer bubble chain of generation one, which we will label as $\{B_k\}_{k\geq 1}$.

Let p denote the period of y. By periodicity, the image of $\{B_k\}_{k\geq p+1}$ under f^p is also a bubble chain of generation one landing at y. By uniqueness, the bubble chain $\{B_k\}_{k\geq 1}$ coincides with the image $\{f^p(B_k)\}_{k\geq p+1}$, hence it is p-periodic.

Let us pick iterated preimages R_r and R_l of the ray R^∞ such that both R_r and R_l land at a point on B_1 and that the union $B_1 \cup R_l \cup R_r \cup \partial V$ bounds a topological rectangle $D \subset V$ that contains y and is disjoint from $\mathbf{H}$. Then, D lifts under f^p to a topological rectangle D_{-1} containing y. Since the vertical sides of D_{-1} are external ray segments with a much smaller external angle difference compared to D, then D_{-1} is compactly contained in D. By Schwarz Lemma, $f^p : D_{-1} \to D$ uniformly expands the hyperbolic metric of D.

For every $n \in \mathbb{N}$, let D_{-n} be the lift of D under f^{pn} containing y. Denote by $x_{r,0}$ and $x_{l,0}$ the landing points of R_r and R_l. Consider the lifts $R_{r,n}$ and $R_{l,n}$ of R_r and R_l under f^{pn} which touch the boundary of D_{-n}; these are external rays landing at points $x_{r,n}$ and $x_{l,n}$ respectively, which are vertices of D_{-n}. By Schwarz Lemma, $x_{r,n}$ and $x_{l,n}$ converge to the unique repelling fixed point y of $f^p : D_{-1} \to D$, and the external rays $R_{r,n}$ and $R_{l,n}$ converge to a limiting external ray R_y, which is a p-periodic outer external ray. By Lemma 3.4, R_y must land at y. □

Denote any pre-critical point on $\mathbf{H}$ by c_{-t} where $t \geq 1$ is such that $f^t(c_{-t}) = c_0$.

Lemma 3.12. *For any pre-critical point $c_{-t} \in \mathbf{H}$, there is a periodic outer bubble chain $\mathcal{B}_t^\infty = \{B_{t,k}^\infty\}_{k\geq 1}$, a periodic inner bubble chain $\mathcal{B}_t^0 = \{B_{t,k}^0\}_{k\geq 1}$, a periodic outer external ray R_t^∞, and a periodic inner external ray R_t^0 such that both $\mathcal{B}_t^\infty$ and $\mathcal{B}_t^0$ are rooted at c_{-t}, and for each $\bullet \in \{0,\infty\}$, both $\mathcal{B}_t^\bullet$ and $R_t^\bullet$ land at the same periodic point $y_t^\bullet$.*

Proof. Let us pick two iterated preimages x_l and x_r of x on $\mathbf{H}$ located on the left and right of c_{-t} respectively. Let R_l and R_r denote the corresponding preimages of R^∞ that land on x_l and x_r respectively. Then, $[x_l, x_r] \cap R_l \cap R_r \cap \partial V$ bounds a disk D_1.

We assume that $[x_l, x_r]$ is small enough such that it does not contain x nor the critical value of f. Then, there exists a univalent lift D_0 of D_1 under f such that ∂D_0 consists of two outer external ray segments, a subset of $\partial_L U$, and an interval within some outer bubble of generation one. Then, we take the univalent lift D_{-t} of D_0 under f^t such that ∂D_{-t} contains an interval on an outer bubble B^∞ of generation $t+1$ attached to c_{-t}.

Since D_{-t} is contained in D_1, $f^{t+1} : D_{-t} \to D_1$ expands the hyperbolic metric of D_{-t}. Consequently, there exists a unique periodic point y_t^∞ of period $t+1$ inside of D_{-t}. By Lemma 3.11, y_t^∞ is the landing point of a unique $(t+1)$-periodic outer external ray R_t^∞ and a unique $(t+1)$-periodic outer bubble chain $\mathcal{B}_t^\infty = \{B_{t,k}^\infty\}_{k\geq 1}$ of generation one. By design, it is clear the bubble $B_{t,1}^\infty$ of generation one coincides with B^∞, and for $k \geq 1$, $B_{t,k+1}^\infty$ is inside of D_{-t} and is mapped by f^{t+1} onto $B_{t,k}^\infty$.

The construction of y_t^0, R_t^0 and $\mathcal{B}_t^0$ is similar. □

For each pre-critical point c_{-t}, we pick $\mathcal{B}_t^0$, R_t^0, y_t^0, $\mathcal{B}_t^\infty$, R_t^∞, y_t^∞ as in the lemma above, and consider the union

$$\mathcal{S}_t := \{c_{-t}\} \bigcup_{\bullet\in\{0,\infty\}} \left(\mathcal{B}_t^\bullet \cup \{y_t^\bullet\} \cup R_t^\bullet\right). \tag{3.2}$$

Lemma 3.13 (Rational approximation of R^0 and R^∞)**.** *For every $\varepsilon > 0$, there exists a pair of pre-critical points $c_{-t_l}, c_{-t_r} \in \mathbf{H}$ located on the left and right of x respectively such that $\mathcal{S}_{-t_l}$ and $\mathcal{S}_{-t_r}$ are both in the ε-neighborhood of the external rays $R^0 \cup R^\infty$.*

Proof. Since pre-critical points are dense on $\mathbf{H}$, there exists a pair of pre-critical points c_{-t_l} and c_{-t_r} on the left and right of x, where the moments t_l and t_r grow as we require them to be arbitrarily close to x. Due to Lemma 3.4, the bubble chains within $\mathcal{S}_{t_l}$ and $\mathcal{S}_{t_r}$ shrink as we get close to x. The outer (resp. inner) external rays within $\mathcal{S}_{t_l}$ and $\mathcal{S}_{t_r}$ are also close to R^∞ (resp. R^0) because their external angles are close to that of R^∞. □

We are now ready to run the pullback argument.

proof of Theorem 3.9. Consider the corona $f : U \to V$ from the discussion above, and let $f_1 : U_1 \to V_1$ and $f_2 : U_2 \to V_2$ be two (d_0, d_∞)-critical rotational coronas with the same bounded type rotation number θ. For each $i \in \{1, 2\}$, there is a global quasiconformal map ϕ_i that conjugates f_i on some neighborhood W_i of its Herman curve $\mathbf{H}_i$ to f restricted to some neighborhood W of its Herman curve. By Lemma 3.6, the map f admits a pre-corona P with range contained in W. Then, for each $i \in \{1, 2\}$, f_i admits a pre-corona $P_i : U_i \to S_i$ that is conjugate to P via ϕ_i.

The quasiconformal map $h := \phi_2^{-1} \circ \phi_1$ conjugates the pre-corona P_1 and P_2 and clearly conjugates f_1 and f_2 on their respective local non-escaping sets $K^{loc}(f_1)$ and $K^{loc}(f_2)$.

For $i \in \{1, 2\}$, consider $\mathcal{S}_{t_l}(f_i) = \phi_i^{-1}(\mathcal{S}_{t_l}(f))$ and $\mathcal{S}_{t_r}(f_i) = \phi_i^{-1}(\mathcal{S}_{t_r}(f))$, which approximate the external rays $R^0(f_i) \cup R^\infty(f_i)$ from the previous lemma. For $i \in \{1, 2\}$, consider the union T_i of the local non-escaping set $K^{loc}_{f_i}$ relative to P_i and the forward orbit of every external ray inside of $\mathcal{S}_{t_l}(f_i) \cup \mathcal{S}_{t_r}(f_i)$. Since T_i is forward invariant under f_i and splits V_i into a finite number of components, we can replace h with a global quasiconformal map that extends $h|_{K^{loc}(f_1)}$ and is equivariant on $T_1 \cup \partial_{\mathrm{L}} U_1$.

Let us move $\partial_{\mathrm{F}} U_1$ slightly outwards to obtain a new disk $\hat{U}_1$ such that $f_1(\partial_{\mathrm{F}} \hat{U}_1)$ is now contained inside of $\mathbf{H}_1 \cup \mathcal{S}_{t_l}(f_1) \cup \mathcal{S}_{t_r}(f_1)$. In the same manner, we replace U_2 with a slightly larger disk $\hat{U}_2$ such that $h|_{T_{f_1}}$ lifts to a conjugacy between $f_1|_{\partial \hat{U}_1}$ and $f_2|_{\partial \hat{U}_2}$.

We can now run the pullback argument. Set $h_0 := h$ and we inductively construct quasiconformal maps $h_n : V_1 \to V_2$ such that

$$h_n(z) = \begin{cases} h_{n-1}(z), & \text{if } z \notin \hat{U}_1, \\ f_2^{-1} \circ h_{n-1} \circ f_1(z), & \text{if } z \in \hat{U}_1. \end{cases}$$

Each h_n is isotopic to and has the same dilatation as h. Note that $K^{loc}(f_1)$ is nowhere dense because so is $K^{loc}(f)$. Therefore, as $n \to \infty$, h_n stabilizes and converges to a quasiconformal conjugacy between f_1 and f_2. □

4. Hyperbolic renormalization fixed point

From now on, let us fix a pair of positive integers $d_0, d_\infty \geq 2$ and a periodic type irrational $\theta \in \Theta_{per}$. In this section, we will construct the desired corona renormalization fixed point f_* and prove most of Theorem A. The remaining sections §5-7 are dedicated to proving that the local unstable manifold is one dimensional.

4.1. Renormalization of critical commuting pairs. Consider a $(d_0, d_\infty$-critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$ with critical point c and rotation number θ'. For each $n \in \mathbb{N}$, denote by I_n the shortest interval in $\mathbf{H}$ connecting c and $f^{q_n}(c)$. The n^{th} *pre-renormalization* of f is the pair

$$(f^{q_n}|_{I_{n+1}}, f^{q_{n+1}}|_{I_n})$$

and the n^{th} *renormalization* $\mathcal{R}^n f$ of f is the normalized commuting pair obtained by rescaling of the n^{th} pre-renormalization by either the affine map if n is even, or the anti-affine map if n is odd, that sends 0 to c and 1 to $f^{q_n}(c)$. Each renormalization $\mathcal{R}^n f$ is a (d_0, d_∞)-critical commuting pair.

Definition 4.1. Let $\mathbf{I} \Subset \mathbb{C}$ be a closed quasiarc containing 0 on its interior. A *critical commuting pair* ζ based on $\mathbf{I}$ is a pair of orientation preserving analytic homeomorphisms

$$\zeta = (f_- : I_- \to f_-(I_-),\ f_+ : I_+ \to f_+(I_+))$$

with the following properties.

- (P_1) I_- and I_+ are closed subintervals of $\mathbf{I}$ of the form $[f_+(0), 0]$ and $[0, f_-(0)]$ respectively such that $\mathbf{I} = I_- \cup I_+ = f_-(I_-) \cup f_+(I_+)$ and $I_- \cap I_+ = \{0\}$.
- (P_2) For all $x \in I_\pm \backslash \{0\}$, $f'_\pm(x) \neq 0$.
- (P_3) Both f_- and f_+ admit a holomorphic extension to a neighborhood B of 0 such that 0 is a critical point of both f_- and f_+, f_- commutes with f_+ on B, and $f_- \circ f_+(\mathbf{I} \cap B) \subset I_-$.

We say that ζ is *normalized* if $f_+(0) = -1$. Additionally, we call ζ a (d_0, d_∞)*-critical commuting pair* if for any quasiconformal map ϕ mapping I_- and I_+ to real intervals $[-1, 0]$ and $[0, 1]$ respectively and for any sufficiently small round disk D centered at $\phi(f_+(f_-(0)))$, the number of connected components of $\phi(f_+ \circ f_-)^{-1}\phi^{-1}(D \cap -\mathbb{H})$ in $-\mathbb{H}$ is d_∞, whereas the number of connected components of $\phi(f_+ \circ f_-)^{-1}\phi^{-1}(D \cap \mathbb{H})$ in $\mathbb{H}$ is d_0.

We say that a (d_0, d_∞)-critical commuting pair $\zeta = (f_-, f_+)$ is *renormalizable* if there exists a positive integer $\chi = \chi(\zeta)$ that corresponds to the first time $f_-^{\chi+1} \circ f_+(0)$ lies in the interior of I_-. If renormalizable, we call the (d_∞, d_0)-critical commuting pair

$$p\mathcal{R}\zeta := \left(f_-^{\chi} \circ f_+|_{[f_-(0), 0]},\ f_-|_{[0, f_-^{\chi} f_+(0)]}\right)$$

the *pre-renormalization* of ζ, and we call the normalized (d_0, d_∞)-critical commuting pair obtained by conjugating $p\mathcal{R}\zeta$ with the antilinear map $z \mapsto -f_-(0)\bar{z}$ the *renormalization* of $\mathcal{R}\zeta$ of ζ.

If $\mathcal{R}\zeta$ is again renormalizable, we call ζ twice renormalizable, and so on. If ζ is infinitely renormalizable, we define the *rotation number* of ζ to be the irrational number

$$\rho(\zeta) := [0; \chi(\zeta), \chi(\mathcal{R}\zeta), \chi(\mathcal{R}^2\zeta), \ldots].$$

One can convert a (d_0, d_∞)-critical commuting pair ζ into a (d_0, d_∞)-critical quasicircle map as follows.

Proposition 4.2. *Let G_ζ be the gluing map which corresponds to identifying z with $f_+(z)$ for every point z in a small neighborhood of $f_-(0)$. Then, G_ζ projects the pair $\left(f_-|_{[f_+f_-(0),0]}, f_+f_-|_{[0,f_-(0)]},\right)$ into a (d_0,d_∞)-critical quasicircle map $f_\zeta : \mathbf{H} \to \mathbf{H}$ having the same rotation number as ζ.*

Let us denote by p the period of θ under the Gauss map $G(\theta) = \left\{\frac{1}{\theta}\right\}$.

Theorem 4.3 ([Lim23b, §7.4-7.5])**.** *There is a unique normalized (d_0,d_∞)-critical commuting pair ζ_* with rotation number θ with the following property. For any normalized (d_0,d_∞)-critical commuting pair ζ' of some rotation number θ' where $G^k(\theta') = \theta$ for some $k \in \mathbb{N}$, the renormalizations $\mathcal{R}^{k+np}\zeta'$ converge exponentially to ζ as $n \to \infty$. Moreover, there is a linear map $z \mapsto \mu z$, $|\mu| < 1$, which conjugates ζ and the pre-renormalization $p\mathcal{R}^p\zeta$.*

4.2. **Corona renormalization fixed point.** We say that a rotational corona is *standard* if the arc γ_0 passes through the critical value. Similarly, we say that a rotational pre-corona is *standard* if it is a pre-corona around the critical value.

Theorem 4.4. *For any $\theta \in \Theta_{per}$ and any pair $d_0, d_\infty \geq 2$, there is a standard (d_0,d_∞)-critical rotational corona $f_* : U_* \to V_*$ with rotation number θ which admits a standard rotational pre-corona*

$$F_* = (f_*^a : U_- \to S_*,\ f_*^b : U_+ \to S_*)$$

together with a gluing map $\psi_* : S_* \to \overline{V_*}$ projecting F_* back to $f_* : U_* \to V_*$. Moreover, we have an improvement of domain: $\mathbf{\Delta}_{F_*} \Subset U_*$.

Proof. Consider the (d_0,d_∞)-critical commuting pair $\zeta_* = (f_- : I_- \to I, f_+ : I_+ \to I)$ on a quasiarc $I = I_- \cup I_+ = [f_+(0),0] \cup [0,f_-(0)]$ of rotation number θ from Theorem 4.3. There exists some $\mu \in \mathbb{D}$ such that for any $n \in \mathbb{N}$, there is a pre-renormalization $\zeta_n = (f_{n,-} : J_- \to J, f_{n,+} : J_+ \to J)$ of ζ on a subinterval $J \subset I$ that is conjugate to ζ via the linear map $L^n(z) = \mu^n z$. We will convert this renormalization fixed point in the category of commuting pairs to that in the category of critical quasicircle maps, and then project it to that in the category of rotational coronas.

Consider the gluing map $\phi_1 := G_\zeta$ described in Proposition 4.2. Then, ϕ_1 projects the modified commuting pair $\zeta' := \left(f_-|_{[f_+f_-(0),0]}, f_+f_-|_{[0,f_-(0)]},\right)$ into a (d_0,d_∞)-critical quasicircle map $g : \mathbf{H} \to \mathbf{H}$ having the same rotation number θ.

Denote by $c_0 := \phi_1(0)$ the critical point of g, and let $c_k := g^k(c_0)$ for all $k \in \mathbb{N}$. Consider the modification of ζ_n, which is ζ' rescaled by L^n, and project it to the dynamical plane of g via ϕ_1 to obtain a commuting pair $g_n = \left(g^{\mathbf{a}}|_{[c_{\mathbf{b}},c_0]}, g^{\mathbf{b}}|_{[c_0,c_{\mathbf{a}}]}\right)$ for some return times $\mathbf{a}$ and $\mathbf{b}$. Then, $\psi_1 := \phi_1 L^n \phi_1^{-1}$ is the gluing map projecting g_n back to g.

To make it standard, we will push g_n forward under one iterate of g. More precisely, we set $\psi_2 := g \circ \psi_1 \circ g^{-1}$. It is well-defined because for every point z close to c_1, the preimage $g^{-1}(z)$ is a set of $d_0 + d_\infty - 1$ points close to c_0 whose images under ψ_1 remain close to c_0 and get mapped to the same point $\psi_2(z)$ under g. The new gluing map ψ_2 sends a small neighborhood of c_1 to a neighborhood of $\mathbf{H}$. Moreover, ψ_2 fixes the critical value c_1 and projects $\tilde{g}_n = \left(g^{\mathbf{a}}|_{[c_{\mathbf{b}+1},c_1]}, g^{\mathbf{b}}|_{[c_1,c_{\mathbf{a}+1}]}\right)$ back to g.

By Corollary 3.8, g admits a standard pre-corona P defined in a small neighborhood of c_1. The corresponding gluing map ϕ_2 projects P onto a (d_0,d_∞)-critical rotational corona $f_* : U_* \to V_*$. Since θ is periodic, we can prescribe f_* to have rotation number θ. The corresponding Herman quasicircle $\mathbf{H}_*$ of f_* is the image of (an interval in) $\mathbf{H}$ under ϕ_2.

Let us rescale the pre-corona P by ψ_2^{-1} to obtain yet another pre-corona P' in the dynamical plane of g that is much smaller than P. Project P' via ϕ_2 to obtain a pre-corona F_* of f_*. The map $\psi_* := \phi_2 \circ \psi_2 \circ \phi_2^{-1}$ will project the pre-corona F_* back to f_*. The improvement of domain property is satisfied once we take n to be sufficiently high. □

Corollary 4.5. *Let f_* and F_* be from the previous theorem. There exist a pair of small Banach neighborhoods $\mathcal{U}$ and $\mathcal{B}$ of f_* and a compact analytic corona renormalization operator $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ such that $\mathcal{R} f_* = f_*$ and the pre-renormalization of $\mathcal{R} f_*$ is F_*. Moreover, for any rotational corona f in $\mathcal{U}$ with the same rotation number θ, f is infinitely renormalizable and $\mathcal{R}^n f$ converges exponentially fast to f_*.*

Proof. The existence of $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ follows from Theorems 2.6 and 4.4. Exponential convergence is guaranteed by Theorem 4.3 provided that $\mathcal{U}$ is a sufficiently small neighborhood of f_*. □

Lemma 4.6. *For any Banach neighborhood $\mathcal{U}$ of f_* and any (d_0, d_∞)-critical quasicircle map f of pre-periodic rotation number θ' where $G^k(\theta') = \theta$ for some $k \in \mathbb{N}$, there is a compact analytic corona renormalization operator $\mathcal{R}_1 : N(f) \to \mathcal{U}$ on a Banach neighborhood $N(f)$ of f.*

Proof. By Theorem 4.3, there is a high $m \in \mathbb{N}$ such that $\mathcal{R}^m f$ is a critical commuting pair of rotation number θ that is arbitrarily close to the critical commuting pair ζ_*. By quasiconformal rigidity, f admits a rotational pre-corona F which projects to a rotational corona g of rotation number θ close to f_*. By Theorem 2.6, there is a compact analytic renormalization operator $\mathcal{R}_1$ on a small neighborhood of f such that $\mathcal{R}_1(f) = g$. □

4.3. **Renormalization tiling.** Consider the renormalization operator $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ and the fixed point f_* from Corollary 4.5. Suppose a corona f in $\mathcal{U}$ is n times renormalizable. For $k \in \{0, 1, \dots, n\}$, denote by $\mathcal{R}^k f = [f_k : U_k \to V]$ the k^{th} renormalization of f, $\psi_k : S_k \to V$ the renormalization change of variables for f_{k-1}, and $\phi_k := \psi_k^{-1}$. Let us cut the dynamical plane of f_k along the critical arc γ_1 and obtain a pre-corona

$$F_k = (f_{k,\pm} : U_{k,\pm} \to V \backslash \gamma_1).$$

Set $\Phi_0 := \text{Id}$. Divide $\overline{U_0}$ along the arcs γ_0 and γ_1 to obtain a tiling $\mathbf{\Delta}_0$ of $\overline{U_0}$ consisting of two tiles $\Delta_0(0)$ and $\Delta_0(1)$. We make the convention that $\Delta_0(0)$, γ_0, and $\Delta_0(1)$ are in counterclockwise order. The tiling $\mathbf{\Delta}_0$ is called the *zeroth tiling* associated to f_0.

The map

$$\Phi_n := \phi_1 \circ \phi_2 \circ \dots \phi_n$$

is well defined on $V \backslash \gamma_1$ and projects F_n to the dynamical plane of f as the pre-corona

$$F_n^{(0)} = \left(f_{n,\pm}^{(0)} : U_{n,\pm}^{(0)} \to S_n^{(0)} \right) \quad \text{where} \quad f_{n,-}^{(0)} = f_0^{\mathbf{a}_n} \text{ and } f_{n,+}^{(0)} = f_0^{\mathbf{b}_n}$$

for some return times $\mathbf{a}_n$ and $\mathbf{b}_n$.

Define the n^{th} *tiling* $\mathbf{\Delta}_n$ associated to f by spreading around $U^{(0)}_{n,\pm}$ via f. It consists of $f^i\left(U^{(0)}_{n,-}\right)$ for $i \in \{0,1\ldots,\mathbf{a}_n-1\}$ and $f^j\left(U^{(0)}_{n,+}\right)$ for $j \in \{0,1\ldots,\mathbf{b}_n-1\}$. Let us denote by $\Delta_n(0)$ the image of the zeroth tile $\Delta_0(0,f_n)$ of f_n under Φ_n, label the rest of the tiles in $\mathbf{\Delta}_n$ in counterclockwise order by $\Delta_n(i)$ for $i \in \{0,1,\ldots,\mathbf{a}_n+\mathbf{b}_n-1\}$.

The map f always acts almost like a rotation on the tiling $\mathbf{\Delta}_n$. There exists $\mathbf{p}_n \in \mathbb{N}_{\geq 1}$ such that f maps $\Delta_n(i)$ univalently onto $\Delta_n(i+\mathbf{p}_n)$ whenever $i \notin \{-\mathbf{p}_n,-\mathbf{p}_n+1\}$. Moreover, f maps $\Delta_n(-\mathbf{p}_n)\cup\Delta_n(-\mathbf{p}_n+1)$ back to $S^{(n)}_n$ almost as a degree $d := d_0+d_\infty-1$ covering map branched at its critical point $c_0(f)$.

Lemma 4.7. *The operator $\mathcal{R}:\mathcal{U}\to\mathcal{B}$ can be arranged such that the following holds. Suppose $f\in\mathcal{U}$ is n times renormalizable and $f_1,\ldots,f_n$ all lie in $\mathcal{U}$.*

1. *There is a holomorphic motion of $\partial\mathbf{\Delta}_0$, $\ldots$, $\partial\mathbf{\Delta}_n$ over $f\in\mathcal{U}_n$ that is equivariant with respect to the maps $f:\partial\Delta_n(i)\to\partial\Delta_n(i+\mathbf{p}_n)$ for $i\notin\{-\mathbf{p}_n,-\mathbf{p}_n+1\}$,*
2. *For each $f\in\mathcal{U}_n$ and $1\leq k\leq n$, $\mathbf{\Delta}_m\cup f(\mathbf{\Delta}_m)\Subset\mathbf{\Delta}_{m-1}$,*
3. *$\mathbf{\Delta}_n(f)$ is close to the Herman curve $\mathbf{H}_*$ of f_* in Hausdorff topology.*

Proof. Let us first consider the case where $f=f_*$. By the improvement of domain property in Theorem 4.4, the diameters of the tiles in $\mathbf{\Delta}_n(f_*)$ must shrink to 0 as $n\to\infty$. Consider a tile $\Delta_1(i,f_*)$. There is some $t\geq 0$ and $j\in\{0,1\}$ such that f_*^t sends $\Delta_1(j,f_*)$ onto $\Delta_1(i,f_*)$. By replacing $\mathcal{R}$ with some high iterate $\mathcal{R}^k$ if necessary, the map

$$\psi_*\circ f_*^{-t}:\Delta_1(i,f_*)\to\Delta_0(j,f_*)$$

expands the Euclidean metric by some high factor $C>1$. Inductively, (2) and (3) hold for f_*.

Now, consider a small neighborhood $\mathcal{U}_n=\bigcap_{0\leq k\leq n}\mathcal{R}^{-k}(\mathcal{U})$ of f_*. By design, it is clear that $\partial\mathbf{\Delta}_0$ moves holomorphically over $f\in\mathcal{U}$. For $1\leq k\leq n$, we push forward the holomorphic motion $\partial\mathbf{\Delta}_0(f_k)$ via Φ_k and spread it around dynamically to obtain a holomorphic motion of $\partial\mathbf{\Delta}_k(f)$ over $f\in\mathcal{U}_n$.

By continuity, every $f\in\mathcal{U}_n$ also satisfies the following property. For any tile $\Delta_n(i,f)$ within $\mathbf{\Delta}_n$, there is some $t\geq 0$ and $j\in\{0,1\}$ such that f^t sends $\Delta_n(j,f)$ onto $\Delta_n(i,f)$. We obtain a holomorphic motion of $\partial\mathbf{\Delta}_n(f)$ by pulling back the holomorphic motion of $\partial\mathbf{\Delta}_0(f_n)$ via maps of the form

$$\Psi_{n,i}:=\Phi_n^{-1}\circ f^{-t}:\Delta_n(i,f)\to\Delta_0(j,f_n) \tag{4.1}$$

for each tile. This implies (1). Moreover, (2) follows from the observation that each $\Psi_{n,i}$ expands the Euclidean metric by a factor close to C^n. Moreover, (3) follows from (1) as well as the special case of (3) for $f=f_*$. $\square$

We will extend the tiling $\mathbf{\Delta}_n$ of a subset of $\overline{U_0}$ to a full tiling of $\overline{U_0}$ as follows. Let

$$\hat{\gamma}_0:=\gamma_0\backslash f^{-1}(U_0)\quad\text{and}\quad\Gamma:=\partial U_0\cup\hat{\gamma}_0.$$

Note that $\hat{\gamma}_0$ is a disjoint union of two subarcs $\hat{\gamma}_0^0$ and $\hat{\gamma}_0^\infty$ of γ_0 where each $\hat{\gamma}_0^\bullet$ connects $\partial^\bullet U_0$ to $f^{-1}(U_0)$. Consider the maps $\Psi_{n,i}$ from (4.1).

Lemma 4.8. *When $\mathcal{U}$ is sufficiently small, the following holds for all $f\in\mathcal{U}$.*

1. *$\Gamma(f_1)$ contains $\Psi_{1,i}(\partial\mathbf{\Delta}_1(f)\cap\partial\Delta_1(i,f))$ for every i. Moreover, there is some i such that $\hat{\gamma}_0(f_1)$ is contained in $\Psi_{1,i}(\partial\mathbf{\Delta}_1(f)\cap\partial\Delta_1(i,f))$.*
2. *$\Gamma(f)$ is disjoint from $\partial\mathbf{\Delta}_1(f)$.*

(3) *For* $\bullet \in \{0, \infty\}$*, there is an arc* $\xi_0^\bullet$ *such that both* $\xi_0^\bullet \cup \hat{\gamma}_0^\bullet$ *and* $\xi_1^\bullet := f(\xi_0^\bullet)$ *connect* $\partial^\bullet U_0$ *and* $\partial \mathbf{\Delta}_1(f)$*.*

Moreover, $\xi_0 := \xi_0^0 \cup \xi_0^\infty$ *and* $\xi_1 := \xi_1^0 \cup \xi_1^\infty$ *can be chosen such that there is a holomorphic motion of*

$$\Gamma \cup \xi_0 \cup \xi_1 \cup \mathbf{\Delta}_1$$

over $f \in \mathcal{U}$ *that is equivariant with respect to* $f : \xi_0(f) \to \xi_1(f)$*,* $f : \Delta_1(i, f) \to \Delta_1(i + \mathbf{p}_1, f)$ *for* $i \neq \{-\mathbf{p}_1, -\mathbf{p}_1 + 1\}$*, and each of* $\Psi_{1,i} : \partial \mathbf{\Delta}_1(f) \cap \Delta_1(i, f) \to \Gamma(f_1)$*.*

Proof. Every tile $\Delta_1(i, f)$ is a rectangle. Clearly, each $\Psi_{1,i}$ maps the horizontal sides of $\Delta_1(i, f)$ to ∂U_1. Let us label the vertical sides by $l(i)$ and $r(i)$ such that each $l(i)$ intersects the side $r(i+1)$ of the next tile. Then, the intersection $\partial \mathbf{\Delta}_1(f) \cap \partial \Delta_1(i, f)$ is the union of the horizontal sides of $\Delta_1(i, f)$ and the symmetric difference $l(i) \vartriangle r(i+1)$ between touching sides across all i's.

It is clear that $l(i) \neq r(i+1)$ for at least one i. For such i, either $l(i)$ is the preimage of $\gamma_0(f_1)$ under $\Psi_{1,i}$ and $r(i+1)$ is the preimage of $\gamma_1(f_1)$ under $\Psi_{1,i+1}$, or vice versa. In this case, $l(i) \vartriangle r(i+1)$ will be mapped by $\Phi_{1,i}$ or $\Phi_{1,i+1}$ onto $\hat{\gamma}_0(f_1)$. This implies (1).

Item (2) follows directly from Lemma 4.7. Moreover, (2) allows us to find for each $\bullet \in \{0, \infty\}$ a proper arc $\xi_0^\bullet$ in $U_0 \backslash (\hat{\gamma}_0 \cup \mathbf{\Delta}_1)$ in a small neighborhood of γ_0 that connects the tip of $\hat{\gamma}_0^\bullet$ and a point on $\partial \Delta_1(i, f)$ for some $i \neq \{-\mathbf{p}_1, -\mathbf{p}_1 + 1\}$. This yields (3).

In Lemma 4.7, we already established the equivariant holomorphic motion of $\partial \mathbf{\Delta}_0 \cup \partial \mathbf{\Delta}_1$. By lifting via $\Phi_{1,i}$, this immediately extends to an equivariant motion of Γ. Lift the motion of $\mathbf{\Delta}_0(f_1)$ via $\Psi_{1,i}$ to obtain an equivariant motion of $\partial \mathbf{\Delta}_1 \cup \Gamma$. Finally, we apply λ-lemma to extend this motion to $\Gamma \cup \xi_0 \cup \xi_1 \cup \mathbf{\Delta}_1$. ☐

For $n \in \mathbb{N}$, we define the n^{th} *full renormalization tiling* of U_0 to be the union of the tilings $\mathbf{\Delta}_n$ and $\mathbf{A}_k$ for $k = 0, 1, \dots, n-1$ where the latter is constructed as follows. Each $\mathbf{A}_k$ is a disjoint union of two tilings $\mathbf{A}_k^0$ and $\mathbf{A}_k^\infty$ where the former is closer to $\partial^0 U_0$ and the latter is closer to $\partial^\infty U_0$. For each $\bullet \in \{0, \infty\}$,

- ▷ $\mathbf{A}_0^\bullet$ is the connected component of $\overline{\mathbf{\Delta}_0 \backslash \mathbf{\Delta}_1}$ that touches $\partial^\bullet U_0$ on the boundary, and it is split by $\hat{\gamma}_0^\bullet \cup \xi_0^\bullet \cup \xi_1^\bullet$ into two tiles $A_0^\bullet(0)$, $A_0^\bullet(1)$. Again, we make the convention that $A_0^\bullet(0)$, $\hat{\gamma}_0^\bullet \cup \xi_0^\bullet$, $A_0^\bullet(1)$ are in counterclockwise order.
- ▷ $\mathbf{A}_k^\bullet$ is the connected component of $\overline{\mathbf{\Delta}_k \backslash \mathbf{\Delta}_{k+1}}$ that touches $\partial^\bullet \mathbf{\Delta}_k$ on the boundary, and it has tiles $\{A_k^\bullet(i)\}_{i=0,1,\dots \mathbf{a}_k + \mathbf{b}_k - 1}$ obtained by spreading via forward iterates of f the tiles $A_k^\bullet(j, f) := \Phi_k(A_0^\bullet(j, f_n))$ for $j \in \{0, 1\}$ and labeled in counterclockwise order.

The first full renormalization tiling is illustrated in Figure 6.

Definition 4.9. A *quasiconformal combinatorial pseudo-conjugacy of level* n between f and f_* is a quasiconformal map $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that maps $\overline{U_0}$ to $\overline{U_\star}$ and preserves the n^{th} renormalization tiling as follows.

(1) The map h sends $\Delta_n(i, f)$ to $\Delta_n(i, f_*)$ for all i, and is equivariant on $\Delta_n(i, f)$ for all $i \notin \{-\mathbf{p}_n, -\mathbf{p}_n + 1\}$;

(2) For all $\bullet \in \{0, \infty\}$ and $k \in \{0, 1, \dots n-1\}$, h sends $A_k^\bullet(i, f)$ to $A_k^\bullet(i, f_*)$ for all i, and is equivariant on $A_k^\bullet(i, f)$ for all $i \notin \{-\mathbf{p}_k, -\mathbf{p}_k + 1\}$.

Theorem 4.10 (Combinatorial pseudo-conjugacy)**.** *If* $D := \max_{0 \le k \le n} \operatorname{dist}(f_k, f_*)$ *is sufficiently small, there is a* K_D*-quasiconformal combinatorial pseudo-conjugacy*

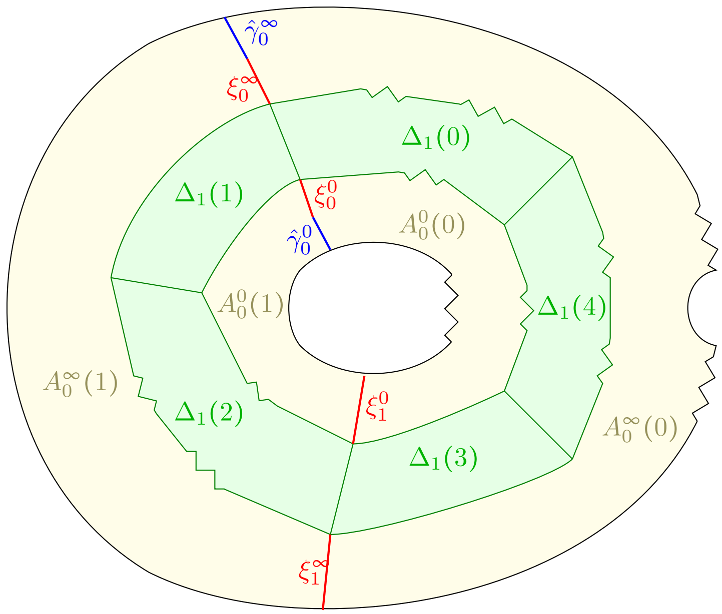

FIGURE 6. The first full renormalization tiling of U_0.

h of level n between f and f_* such that $\sup_{z\in\mathbf{\Delta}_n(f)}|h(z)-z| \leq M_D$. *Moreover,* $K_D \to 1$ *and* $M_D \to 0$ *as* $D \to 0$.

Proof. By Lemma 4.8, we have a holomorphic motion of the first full renormalization tiling over $\mathcal{U}$. Assume D is sufficiently small so that $f_1, \dots, f_n$ all lie in $\mathcal{U}$. Each tile $A_k^\bullet(i, f)$ admits some $t \in \mathbb{N}$ and $j \in \{0, 1\}$ such that $\Psi_{k,i} := \Phi_k^{-1} \circ f^{-t}$ univalently maps $A_k^\bullet(i, f)$ onto $A_0^\bullet(j, f_k)$. We keep pulling back via maps of the form $\Psi_{n,i}$ to obtain a holomorphic motion of the full n^{th} renormalization tiling. By equivariance and λ-lemma, the holomorphic motion induces the desired quasiconformal map h. The dilatation K_D of h is bounded by the dilatation of the motion at $f_0, f_1, \dots f_n$, which depends only on D, where $K_D \to 0$ as $D \to \infty$. The estimate M_D follows from the continuity of the holomorphic motion and the compactness of quasiconformal maps. ☐

Corollary 4.11. *There is some $\varepsilon > 0$ such that the following holds. Suppose $f \in \mathcal{U}$ is infinitely renormalizable and $\mathcal{R}^n f$ is in the ε-neighborhood of f_* for all $n \in \mathbb{N}$. Then, f is a rotational corona.*

Proof. By Theorem 4.10, we have a $K(\varepsilon)$-quasiconformal combinatorial pseudo-conjugacy h_n of level n between f and f_* for all $n \in \mathbb{N}$. By the compactness of K-quasiconformal maps, h_n converges in subsequence to a quasiconformal map $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, and h^{-1} must be a conjugacy on the Herman quasicircle $\mathbf{H}_*$ of f_*. The image $h^{-1}(\mathbf{H}_*)$ is a Herman quasicircle of f containing the critical point $c_0(f)$ and separating the boundaries of the domain of f. It follows that f must be a rotational corona. ☐

4.4. **Towards hyperbolicity.**

Theorem 4.12. *The renormalization operator $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ is hyperbolic at the fixed point f_* with a finite positive dimensional local unstable manifold $\mathcal{W}^u_{\text{loc}}$. If $\mathcal{U}$ is sufficiently small, the local stable manifold $\mathcal{W}^s_{loc}$ of f_* consists of the set of (d_0, d_∞)-critical rotational coronas in $\mathcal{U}$ with the same rotation number as f_*.*

Proof. Consider a corona f near f_* lying on the local stable manifold $\mathcal{W}^s_{loc}$. For sufficiently small $\mathcal{U}$, $\mathcal{R}^n f$ is in the ε-neighborhood of f_* for all $n \in \mathbb{N}$. By Corollary 4.11, f must be a rotational corona.

Let us consider the derivative $D\mathcal{R}_{f_*}$ of the renormalization operator at the fixed point f_*. By the compactness of $\mathcal{R}$, the number of neutral and repelling eigenvalues is finite. We claim that neutral eigenvalues do not exist and repelling eigenvalues must exist.

Suppose for a contradiction that there are neutral eigenvalues. By Small Orbits Theorem A.1, there exists an infinitely renormalizable corona f such that its forward orbit lies entirely in the ε-neighborhood of f_* and it satisfies

$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{R}^n f\| = 0. \tag{4.2}$$

By Corollary 4.11, f must be a rotational corona with the same rotation number as f_*. By Corollary 4.5, renormalizations $\mathcal{R}^n f$ converge to f_* exponentially fast, which contradicts (4.2). Hence, neutral eigenvalues do not exist.

Consider the family of rational maps F_c from (3.1). By Theorem 3.3, there is a unique parameter c_* such that F_{c*} admits a Herman quasicircle with the same rotation number as f_*. By Lemma 4.6, there is an analytic renormalization operator $\mathcal{R}_1$ on a neighborhood of F_{c_*} such that $\mathcal{R}_1 F_{c_*}$ is a rotational corona that is sufficiently close to f_* and has the same rotation number as f_*. For any parameter $c \neq c_*$ sufficiently close to c_*, $\mathcal{R}_1 F_c$ is also sufficiently close to f_*. By the uniqueness of c_*, the parameter c can be picked such that F_c is postcritically finite, and so $\mathcal{R}_1 F_c$ is not a rotational corona.

Suppose for a contradiction that $D\mathcal{R}_{f_*}$ has no repelling eigenvalues. Then, $\mathcal{W}^s_{loc}$ is an open neighborhood of f_* and contains $\mathcal{R}_1 F_c$. However, the non-rotationality of $\mathcal{R}_1 F_c$ would contradict Corollary 4.11. ☐

5. Transcendental extension

From now on, we will consider the corona renormalization operator $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ together with its hyperbolic fixed point $f_* : U_* \to V_*$ constructed in Section 4.

Definition 5.1. A map $g : A \to B$ is said to be *σ-proper* if there exist exhaustions A_n, B_n of A, B respectively such that for all n, $g : A_n \to B_n$ is a proper map; equivalently, every connected component of the preimage of a compact set under g is compact.

In [McM98], McMullen proved the existence of maximal σ-proper extensions of holomorphic commuting pairs associated to renormalizations of quadratic Siegel disks. This is generalized in [DLS20, Theorem 5.5] where pre-pacmen on the local unstable manifold are shown to admit maximal σ-proper extension. In this section, we will show that our case is no different. We will study coronas in the local unstable manifold $\mathcal{W}^u_{\text{loc}}$ of f_*, which we will identify as a holomorphic parameter space of transcendental holomorphic maps onto $\mathbb{C}$.

5.1. **Maximal σ-proper extension.** Consider a corona $f : U \to V$ lying in the local unstable manifold $\mathcal{W}^u_{\mathrm{loc}}$ of f_*. Since f is infinitely anti-renormalizable, it comes with a backward tower of corona renormalizations $\{f_k : U_k \to V\}_{k \le 0}$, where each f_k embeds to U_{k-1} as a pre-corona $F^{(k-1)}_k = \left(f^{(k-1)}_{k,\pm}\right)$ consisting of a pair of iterates of f_{k-1}. Let $\psi_k : S_k \to V$ be the renormalization change of variables realizing the renormalization of f_{k-1} and let $\phi_k := \psi_k^{-1} : V \to S_k$.

Let us normalize our coronas such that they have a critical value at 0. For each $k \le 0$, consider the translation $T_k(z) = z - c_1(f_k)$ and denote

$$U^\natural_k = T_k(U_k), \quad V^\natural_k = T_k(V), \quad U^\natural_{k,\pm} = T_{k-1}(S_k), \quad S^\natural_k = T_{k-1}(S_k).$$

By conjugating with T_k, we modify our maps f_k, F_k, and ϕ_k into

$$f^\natural_k : U^\natural_k \to V^\natural_k, \quad F^\natural_k := (f^\natural_{k,\pm} : U^\natural_{k,\pm} \to S^\natural_k), \quad \phi^\natural_k : V^\natural_k \to S^\natural_k$$

respectively. Consider the linear map

$$A_*(z) := \mu_* z$$

where $\mu_* := (\phi^\natural_*)'(0)$ is the self-similarity factor of f_*.

Lemma 5.2. *The limit*

$$h^\natural_f(z) := \lim_{k \to -\infty} A^k_* \circ \phi^\natural_{k+1} \circ \ldots \circ \phi^\natural_1 \circ \phi^\natural_0(z)$$

defines a univalent map on a neighborhood D of 0 where D is independent of f.

Proof. As $\phi^\natural_k \to \phi^\natural_*$ exponentially fast, so is the derivative $\mu_k := (\phi^\natural_k)'(0)$ towards μ_*. There are positive constants ε and δ such that $\varepsilon < 1 - |\mu_*|$ and for all $|z| < \delta$ and $k \le 0$, $|\phi^\natural_k(z)| \le (|\mu_*| + \varepsilon)|z|$. Therefore, for all $|z| < \delta$ and $k \le 0$,

$$|\phi^\natural_{k+1} \circ \ldots \circ \phi^\natural_0(z)| \le (|\mu_*| + \varepsilon)^{-k}|z|.$$

The sequence $h^{(k)}(z) := A^k_* \circ \phi^\natural_{k+1} \circ \ldots \circ \phi^\natural_0 \circ \phi^\natural_0(z)$ indeed converges to a univalent map on $\{|z| < \delta\}$ since

$$\frac{h^{(k-1)}(z)}{h^{(k)}(z)} = \frac{\phi^\natural_k\left(\phi^\natural_{k+1} \circ \ldots \circ \phi^\natural_0(z)\right)}{\mu_* \phi^\natural_{k+1} \circ \ldots \circ \phi^\natural_0(z)} = \frac{\mu_k}{\mu_*} + O(|\phi^\natural_{k+1} \circ \ldots \circ \phi^\natural_0(z)|) \to 1$$

exponentially fast as $k \to -\infty$. □

For $k \le 0$, let $h^\natural_k := h^\natural_{f_k}$ and denote its rescaling by $h^\#_k := A^k_* \circ h^\natural_k$.

Proposition 5.3. *For $k \le 0$,*

$$h^\natural_{k-1} \circ \phi^\natural_i = A_* \circ h^\natural_k \qquad \textit{and} \qquad h^\natural_0 = h^\#_k \circ \phi^\natural_{k+1} \circ \ldots \circ \phi^\natural_0.$$

Moreover, $h^\natural_0$ extends to a univalent map on the interior of $V^\natural_0 \backslash \gamma^\natural_1$.

The maps $h^\natural_k$ act as linear coordinates under which renormalization change of variables are simply linear maps. Objects in linear coordinates will be written in bold:

$$\mathbf{U}_{k,\pm} := h^\natural_k(U^\natural_{k,\pm}), \quad \mathbf{S}_k := h^\natural_k(S^\natural_k), \quad \mathbf{F}_k := (\mathbf{f}_{k,\pm} : \mathbf{U}_{k,\pm} \to \mathbf{S}_k).$$

Often, we will also work with the rescaled linear coordinates $h^\#_k$ in which we add the symbol "#" as follows:

$$\mathbf{U}^\#_{k,\pm} := h^\#_k(U^\natural_{k,\pm}), \quad \mathbf{S}^\#_k := h^\#_k(S^\natural_k), \quad \mathbf{F}^\#_k := \left(\mathbf{f}^\#_{k,\pm} : \mathbf{U}^\#_{k,\pm} \to \mathbf{S}^\#_k\right).$$

By design, it is clear that for all $k \le 0$,

$$\mathbf{f}^\#_{k,\pm} = A^k_* \circ \mathbf{f}_{k,\pm} \circ A^{-k}_*.$$

Lemma 5.4. *There is a matrix of positive integers* $\mathbf{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ *such that for every negative integer* k*,*

$$\mathbf{f}^{\#}_{k+1,-} = \left(\mathbf{f}^{\#}_{k,-}\right)^{m_{11}} \circ \left(\mathbf{f}^{\#}_{k,+}\right)^{m_{12}} \quad \text{and} \quad \mathbf{f}^{\#}_{k+1,+} = \left(\mathbf{f}^{\#}_{k,-}\right)^{m_{21}} \circ \left(\mathbf{f}^{\#}_{k,+}\right)^{m_{22}}.$$

Proof. The action of renormalization restricted to the Herman quasicircle of f_* is a sector renormalization, and in particular an iterate of prime renormalization. See Appendix B.1. The existence of such a matrix $\mathbf{M}$ follows from Appendix B.2. □

Theorem 5.5 (Maximal extension)**.** *Assume $\mathcal{U}$ is a sufficiently small Banach neighborhood of f_*. For all $k \leq 0$, the maps $\mathbf{f}^{\#}_{k,\pm}$ extend to σ-proper branched coverings $\mathbf{X}^{\#}_{k,\pm} \to \mathbb{C}$, where $\mathbf{X}^{\#}_{k,\pm}$ are open connected subsets of $\mathbb{C}$.*

Proof. For each $k \leq 0$, the composition $\phi_{k+1} \circ \ldots \circ \phi_0$ embeds the pre-corona $F_0 = (f_{0,\pm} : U_{0,\pm} \to V \backslash \gamma_1)$ to the dynamical plane of f_k as a pair of iterates

$$\left(f_k^{\mathbf{a}_k} : U^{(k)}_{0,-} \to V^{(k)}_0,\ f_k^{\mathbf{b}_k} : U^{(k)}_{0,+} \to V^{(k)}_0 \right). \tag{5.1}$$

Since ϕ_k is contracting at the critical value, the diameter of $U^{(k)}_{0,\pm} \to V^{(k)}_0$ shrinks to 0 as $k \to -\infty$.

To proceed, we need the following technical lemma.

Lemma 5.6. *Assume $\mathcal{U}$ is a sufficiently small Banach neighborhood of f_*. There is an open disk D around the critical value $c_1(f_*)$ of f_* such that for all sufficiently large $n \in \mathbb{N}$, $t \in \{\mathbf{a}_n, \mathbf{b}_n\}$, and $f \in \mathcal{R}^{-n}(\mathcal{U})$, then $f^t(c_1(f))$ is contained in D and D can be pulled back by f^t to a disk $D_0 \subset U_f \backslash \gamma_1$ containing $c_1(f)$ on which $f^t : D_0 \to D$ is a branched covering.*

This lemma initially appears in [DLS20, Key Lemma 4.8] in the context of quadratic Siegel pacmen. Due to its length, the proof will be supplied in Appendix B. The lemma tells us that for sufficiently large $k \ll 0$, the disk D contains $c_1(f_k)$ and the pair in (5.1) extends to a commuting pair of branched coverings

$$\left(f_k^{\mathbf{a}_k} : W^{(k)}_{-} \to D,\ f_k^{\mathbf{b}_k} : W^{(k)}_{+} \to D \right), \tag{5.2}$$

where $W^{(k)}_{\pm} \cup D \subset V \backslash \gamma_1$. By conjugating with $h^{\#}_k \circ T_k$, we transform this pair into the commuting pair of branched coverings

$$\mathbf{f}_{0,\pm} : \mathbf{W}^{(k)}_{\pm} \to \mathbf{D}^{(k)}.$$

Consider the rescaled disk $\mathbf{D}^{(k)} := h^{\#}_k \circ T_k(D)$. For sufficiently large t and $m \leq 0$,

$$\operatorname{mod}\left(\mathbf{D}^{(tm-t)} \backslash \overline{\mathbf{D}^{(tm)}}\right) > 1,$$

and thus

$$\bigcup_{k<0}^{\infty} \mathbf{D}^{(k)} = \mathbb{C}.$$

Therefore, $\mathbf{f}_{0,\pm}$ extends to σ-proper branched coverings from $\mathbf{X}_{0,\pm} := \bigcup_{k<0} \mathbf{W}^{(k)}_{\pm}$ onto $\mathbb{C}$, and clearly $\mathbf{X}_{0,\pm}$ is open and connected. □

The proof of the theorem above actually gives us something stronger, which we will use later in Section §7.2.

Lemma 5.7 (Stability of σ-branched structure)**.** *Assume $\mathcal{U}$ is a sufficiently small Banach neighborhood of f_*. For every $f \in \mathcal{W}^u_{\text{loc}}$, there are sequences of disks $\mathbf{D}^{(-1)} \subset \mathbf{D}^{(-2)} \subset \mathbf{D}^{(-3)} \subset \dots$ and $\mathbf{W}^{(-1)}_\pm \subset \mathbf{W}^{(-2)}_\pm \subset \mathbf{W}^{(-3)}_\pm \subset \dots$ such that*

(1) $\bigcup_{k<0} \mathbf{D}^{(k)} = \mathbb{C}$ *and* $\bigcup_{k<0} \mathbf{W}^{(k)}_\pm = \mathbf{X}_{0,\pm}$*;*
(2) *each of* $\mathbf{D}^{(k)}$ *and* $\mathbf{W}^{(k)}_\pm$ *depends continuously on* f*;*
(3) *the map* $\mathbf{f}_{0,\pm} : \mathbf{W}^{(k)}_\pm \to \mathbf{D}^{(k)}$ *is a pair of proper branched coverings of fixed finite degree;*
(4) *critical points of* $\mathbf{f}_{0,\pm} : \mathbf{W}^{(k)}_\pm \to \mathbf{D}^{(k)}$ *move holomorphically over* $f \in \mathcal{U}$*.*

Proof. The construction of such disks is similar to the proof of the previous theorem. We add the following modification. By Theorem 4.10, we can replace the disk D with a slightly smaller disk $D(f_0, k)$ depending continuously on f_0 such that for all $i \leq \max\{\mathbf{a}_k, \mathbf{b}_k\}$,

$$c_i(f_*) \in D(f_*, k) \quad \text{if and only if} \quad c_i(f_k) \in D(f_0, k).$$

Under this replacement, the domains of branched coverings $(f_k^{\mathbf{a}_k}, f_k^{\mathbf{b}_k})$ from (5.2) become

$$\mathbf{f}_{0,\pm} : W_\pm(f_0, k) \to D(f_0, k),$$

which depend continuously on f_0. By conjugating with $h^\#_k \circ T_k$, we obtain the commuting pair $\mathbf{f}_{0,\pm} : \mathbf{W}^{(k)}_\pm \to \mathbf{D}^{(k)}$ with the desired property. □

5.2. **Cascades.** Recall the anti-renormalization matrix $\mathbf{M}$ from Lemma 5.4. We shall denote by $\mathbf{t} > 1$ and $1/\mathbf{t}$ the eigenvalues of $\mathbf{M}$.

Let us identify the local unstable manifold $\mathcal{W}^u_{\text{loc}}$ with the space $\boldsymbol{\mathcal{W}}^u_{loc}$ of σ-proper maps $\mathbf{F} = (\mathbf{f}_{0,\pm})$ associated to each $f \in \mathcal{W}^u_{\text{loc}}$. For all $n \in \mathbb{N}$, we define $\mathbf{F}^\#_n = (\mathbf{f}_{n,\pm})$ inductively by the relation

$$(\mathbf{f}^\#_{n,-})^a \circ (\mathbf{f}^\#_{n,+})^b = (\mathbf{f}^\#_{n-1,-})^{a'} \circ (\mathbf{f}^\#_{n-1,+})^{b'} \tag{5.3}$$

for any $a, b, a', b' \in \mathbb{N}$ satisfying $(a' \ b') = (a \ b)\mathbf{M}$. We extend our renormalization operator by setting

$$\mathcal{R}^n \mathbf{F}_0 = \mathbf{F}_n := A_*^{-n} \mathbf{F}^\#_n A^n_n,$$

and extend $\boldsymbol{\mathcal{W}}^u_{loc}$ to a global unstable manifold $\boldsymbol{\mathcal{W}}^u$ by adding $\mathbf{F}_n$ for all $n \geq 0$ and $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u_{loc}$. The complex manifold structure of $\boldsymbol{\mathcal{W}}^u_{loc}$ extends to $\boldsymbol{\mathcal{W}}^u$ and the renormalization operator $\mathcal{R}$ acts on $\boldsymbol{\mathcal{W}}^u$ as a biholomorphism with a unique fixed point $\mathbf{F}_*$.

Definition 5.8. We define the space $\mathbf{T}$ of *power-triples* to be the quotient space of $\mathbb{Z} \times \mathbb{N}^2$ under the equivalence relation $\sim$ where $(n, a, b) \sim (n-1, a', b')$ if and only if $(a' \ b') = (a \ b)\mathbf{M}$.

We will equip $\mathbf{T}$ with the binary operation $+$ defined by

$$(n, a, b) + (n, a', b') = (n, a + a', b + b').$$

With respect to $+$, $\mathbf{T}$ has a unique identity element $0 := (n, 0, 0)$. For $P, Q \in \mathbf{T}$, let us denote by $P \geq Q$ if for all sufficiently large $n \ll 0$, there exist $a, b, a', b' \in \mathbb{N}$ such that $P = (n, a, b)$, $Q = (n, a', b')$, $a \geq a'$, and $b \geq b'$.

From Lemma B.6, $(\mathbf{T},+,\geq)$ can be identified with a sub-semigroup of $(\mathbb{R}_{\geq 0},+,\geq)$. $\mathbf{T}$ inherits a well-defined scalar multiplication by powers of $\mathbf{t}$ as follows. For every $(n,a,b)\in\mathbf{T}$ and integer k,

$$\mathbf{t}^k(n,a,b)=(n+k,a,b).$$

For every $\mathbf{F}\in\boldsymbol{\mathcal{W}}^u$ and every power-triple $P=(n,a,b)$, we will use the notation

$$\mathbf{F}^P:=\left(\mathbf{f}^{\#}_{n,-}\right)^a\circ\left(\mathbf{f}^{\#}_{n,+}\right)^b.$$

Each $\mathbf{F}^P$ is a σ-proper map onto $\mathbb{C}$. We denote by $\mathbf{F}^{\geq 0}$ the cascade $\left(\mathbf{F}^P\right)_{P\in\mathbf{T}}$ associated to $\mathbf{F}$.

Lemma 5.9. *For every $\mathbf{F}\in\boldsymbol{\mathcal{W}}^u$, $P\in\mathbf{T}$, and $n\in\mathbb{Z}$,*

$$\mathbf{F}_0^P=\left(\mathbf{F}^{\#}_{-n}\right)^{\mathbf{t}^nP}.$$

In particular, when $\mathbf{F}=\mathbf{F}_*$,

$$\mathbf{F}_*^P=A_*^{-n}\circ\mathbf{F}_*^{\mathbf{t}^nP}\circ A_*^n. \tag{5.4}$$

5.3. **Critical points and periodic points.** Consider $\mathbf{F}=[\mathbf{f}_\pm:\mathbf{U}_\pm\to\mathbf{S}]\in\boldsymbol{\mathcal{W}}^u_{loc}$ sufficiently close to $\mathbf{F}_*$, and let $\mathbf{F}_n=\mathcal{R}^n\mathbf{F}$ for all $n\in\mathbb{Z}$. Within the cascade $\mathbf{F}^{\geq 0}$, $\mathbf{f}_\pm$ is the first return map of points in $\mathbf{U}_\pm$ back to $\mathbf{S}$. In particular, $\mathbf{U}_-\cup\mathbf{U}_+$ is disjoint from $\mathbf{F}^P(\mathbf{U}_-)$ for all $P<(0,1,0)$ and $\mathbf{F}^P(\mathbf{U}_+)$ for all $P<(0,0,1)$.

Definition 5.10. We define the *zeroth renormalization tiling* $\boldsymbol{\Delta}_0=\boldsymbol{\Delta}_0(\mathbf{F})$ associated to $\mathbf{F}^{\geq 0}$ to be the tiling consisting of $\Delta_0(0):=\overline{\mathbf{U}_+}$ and $\Delta_0(1):=\overline{\mathbf{U}_-}$, as well as $\mathbf{F}^P(\Delta_0(0))$ for all $P<(0,0,1)$ and $\mathbf{F}^P(\Delta_0(1))$ for all $P<(0,1,0)$. We label the tiles in left-to-right order as $\Delta_0(i)$ for $i\in\mathbb{Z}$. For all $n\in\mathbb{Z}_{<0}$, we define the n^{th} *renormalization tiling* to be the rescaling of the zeroth tiling for $\mathbf{F}_n$, namely

$$\boldsymbol{\Delta}_n(\mathbf{F})=A_*^n(\boldsymbol{\Delta}_0(\mathbf{F}_n)).$$

Near $\mathbf{F}_*$, the tiling $\boldsymbol{\Delta}_0(\mathbf{F})$ moves holomorphically in $\mathbf{F}$. In general, for $\mathbf{F}\in\boldsymbol{\mathcal{W}}^u$, the tiling $\boldsymbol{\Delta}_n(\mathbf{F}):=A_*^n(\boldsymbol{\Delta}_0(\mathbf{F}_n))$ is well-defined for all sufficiently large $n\ll 0$. Each tile $\Delta_n(i)$ is a compact disk in $\mathbb{C}$.

Definition 5.11. Consider $[f:U_f\to V]\in\mathcal{W}^u_{\mathrm{loc}}$ and the associated pre-corona $\mathbf{F}=[\mathbf{f}_\pm:\mathbf{U}_\pm\to\mathbf{S}]\in\boldsymbol{\mathcal{W}}^u_{loc}$. Given a subset Z of U_f, the *full lift* $\mathbf{Z}$ of Z to the dynamical plane of $\mathbf{F}$ is defined as

$$\mathbf{Z}:=\bigcup_{0\leq P<(0,0,1)}\mathbf{F}^P(\mathbf{Z}_0)\cup\bigcup_{0\leq P<(0,1,0)}\mathbf{F}^P(\mathbf{Z}_1),$$

where $\mathbf{Z}_0$ and $\mathbf{Z}_1$ are the embedding of $Z\cap\Delta_0(0,f)$ and $Z\cap\Delta_0(1,f)$ to the dynamical plane of $\mathbf{F}$.

In particular, we will define the *Herman quasicircle* $\mathbf{H}$ of $\mathbf{F}_*$ to be the full lift of the Herman quasicircle of f_*. Observe that $\mathbf{H}$ is an A_*-invariant quasicircle passing through 0 and ∞.

Let us fix $\mathbf{F}$ in $\boldsymbol{\mathcal{W}}^u$. For every $x\in\mathbb{C}$ and $T\in\mathbf{T}$, we denote the finite orbit of x up to time T by

$$\mathrm{orb}_x^T(\mathbf{F}):=\left\{\mathbf{F}^P(x)\,:\,0\leq P\leq T\right\}.$$

Definition 5.12. We say that a point x is a *critical point* of $\mathbf{F}^{\geq 0}$ if it is a critical point for $\mathbf{F}^P$ for some $P\in\mathbf{T}_{>0}$, and x is a *periodic point* if there is some $P\in\mathbf{T}_{>0}$ such that $\mathbf{F}^P(x)=x$.

Let $d := d_0 + d_\infty - 1$.

Lemma 5.13. *Critical points of* $\mathbf{F}^{\geq 0}$ *satisfy the following properties.*

(1) *A point* x *is a critical point of* $\mathbf{F}^{\geq 0}$ *if and only if* $\mathbf{F}^P(x) = 0$ *for some* $P \in \mathbf{T}_{>0}$.
(2) *The set* $\mathrm{CP}\left(\mathbf{F}^P\right)$ *of critical points of* $\mathbf{F}^P$ *consists of* $\mathbf{F}^{-S}(0)$ *for all* $S \in \mathbf{T}$ *such that* $0 < S \leq P$*, whereas the set* $\mathrm{CV}\left(\mathbf{F}^P\right)$ *of critical values is* $\{\mathbf{F}^S(0) : S < P\}$.
(3) *There is some* $K_{\mathbf{F}} \in \mathbf{T}_{>0}$ *such that for every power-triple* $P < K_{\mathbf{F}}$*, every critical point of* $\mathbf{F}^P$ *has local degree* d*. If* 0 *is not periodic, this is still true for* $P \geq K_{\mathbf{F}}$*. In general, for every* $P \in \mathbf{T}$*, there is some* $k \in \mathbb{N}$ *such that the local degree of every critical point of* $\mathbf{F}^P$ *is at most* k.

Let $T := \min\{(0,1,0),(0,0,1)\}$*. If* $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u_{loc}$*, then for every* $P < T$,

(4) $\mathrm{CV}\left(\mathbf{F}^P\right)$ *is a subset of* $\boldsymbol{\Delta}_0(\mathbf{F})\backslash\mathbf{S} \cup \{0\}$ *which moves holomorphically with* $\mathbf{F}$*, and*
(5) *every critical point of* $\mathbf{F}^P$ *has local degree* d.

Proof. Pick a bounded domain $\mathbf{D} \Subset \mathbb{C}$ and select a connected component $\mathbf{D}'$ of $\mathbf{F}^{-P}(\mathbf{D})$. Recall that for sufficiently large $n \ll 0$, the map $\mathbf{F}^P : \mathbf{D}' \to \mathbf{D}$ can be identified via $h_n^{\#}$ with $f_n^{s_n} : D' \to D$ for some domains $D', D \Subset \mathbb{C}$ and some $s_n \geq 0$. Therefore, x is a critical point of $\mathbf{F}^P$ if and only if $(h_n^{\#})^{-1}(x)$ is a critical point of $f_n^{s_n}$, which happens precisely when $\mathbf{F}^S(x) = 0$ for some $S \leq P$. This leads to (1) and (2).

Suppose $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u_{loc}$ and $P \leq T$. For all $S < P$, $\mathbf{F}^S(0)$ is contained in some tile $\Delta_0(i,\mathbf{F})$ that is disjoint from $\mathbf{S}$. This implies (4). Also, (5) follows from the fact that for every critical point x of $\mathbf{F}^P$, $\mathrm{orb}_x^P(\mathbf{F})$ passes through the critical value 0 exactly once.

If $\mathbf{F}$ is not close to $\mathbf{F}_*$, then we can take some $n \ll 0$ such that $\mathcal{R}^n\mathbf{F} \in \boldsymbol{\mathcal{W}}^u_{loc}$. Then, (3) follows from (4) and (5) by taking $K_{\mathbf{F}}$ to be $\mathbf{t}^n T$ and k to be such that $P < (k-1)K_{\mathbf{F}}$. □

Lemma 5.14 (Discreteness)**.** *For any bounded open subset* D *of* $\mathbb{C}$*, there is some* $Q \in \mathbf{T}_{>0}$ *such that for all* $\mathbf{G} \in \boldsymbol{\mathcal{W}}^u$ *close to* $\mathbf{F}$ *and whenever* $P' < P < Q$,

(1) $\mathbf{G}^P$ *is well-defined and univalent on* D*, and*
(2) $\mathbf{G}^P(D)$ *is disjoint from* $\mathbf{G}^{P'}(D)$.

For every $x \in \mathbb{C}$ *and* $T \in \mathbf{T}$, $\mathrm{orb}_x^T(\mathbf{F})$ *is discrete in* $\mathbb{C}$.

Proof. There exist some integers $m \leq 0$ and i such that D is compactly contained in some level m tile $\Delta_m(j,\mathbf{G})$ associated to $\mathbf{G}$ where $j \in \{0,1\}$ for all $\mathbf{G}$ close to $\mathbf{F}$. Set $Q := \mathbf{t}^m \min\{(0,1,0),(0,0,1)\}$. For $P < Q$, the tile $\Delta_m(j,\mathbf{G})$ is mapped by $\mathbf{G}^P$ to some other tile $\Delta_m(i,\mathbf{G})$ of level m. This implies (1) and (2).

Given $x \in \mathbb{C}$ and $T \in \mathbf{T}$, suppose y is an accumulation point of $\mathrm{orb}_x^T(\mathbf{F})$. Pick a small open neighborhood D of y. From the first part, $\mathbf{F}^P(D)$ is disjoint from D for all sufficiently small power-triple P. This implies that only finitely many points in $\mathrm{orb}_x^T(\mathbf{F})$ are contained in D. □

By a straightforward compactness argument, the lemma above has the following consequence.

Corollary 5.15 (Proper discontinuity)**.** *For any* $P \in \mathbf{T}$*, any compact subset* $\mathbf{Y}$ *of* $\mathrm{Dom}\left(\mathbf{F}^P\right)$*, and any bounded subset* $\mathbf{X}$ *of* $\mathbb{C}$*, there are at most finitely many power-triples* $T \le P$ *such that* $\mathbf{F}^T(\mathbf{Y})$ *intersects* $\mathbf{X}$*.*

Corollary 5.16. *Every critical point* x *of* $\mathbf{F}^{\ge 0}$ *admits a minimal* $P \in \mathbf{T}_{>0}$*, called the generation of* x*, such that* $\mathbf{F}^P(x) = 0$*.*

Proof. By definition, there is some $P \in \mathbf{T}_{>0}$ such that $\mathbf{F}^P(x) = 0$. By Lemma 5.14, $\mathrm{orb}_x^P(\mathbf{F})$ is discrete, so there are at most finitely many power triples S such that $S < P$ and $\mathbf{F}^S(x) = 0$. □

Corollary 5.17. *Every periodic point of* $\mathbf{F}^{\ge 0}$ *has a minimal period.*

Proof. Suppose x is a periodic point of $\mathbf{F}^{\ge 0}$. The set $\mathbf{T}_x := \{P \in \mathbf{T} : \mathbf{F}^P(x) = x\}$ of periods of x is a sub-semigroup of $\mathbf{T}$. Pick a small neighborhood D of x. By Lemma 5.14, there is some $Q \in \mathbf{T}_{>0}$ such that for all $0 < P < Q$, $\mathbf{F}^P(D)$ is disjoint from D and thus $P \notin \mathbf{T}_x$. This implies that $\mathbf{T}_x$ is finitely generated, and in particular, of the form $\{nS\}_{n\in\mathbb{N}}$, where $S > 0$ is the minimal period. □

5.4. **Fatou, Julia, and escaping sets.** Consider $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$.

Definition 5.18. Given $P \in \mathbf{T}$, the P^{th} *escaping set* of $\mathbf{F}$ is

$$\mathbf{I}_{\le P}(\mathbf{F}) := \mathbb{C}\backslash\mathrm{Dom}\left(\mathbf{F}^P\right).$$

The *finite-time escaping set* of $\mathbf{F}$ is the union

$$\mathbf{I}_{<\infty}(\mathbf{F}) := \bigcup_{P\in\mathbf{T}} \mathbf{I}_{\le P}(\mathbf{F}),$$

the *infinite-time escaping set* of $\mathbf{F}$ is

$$\mathbf{I}_{\infty}(\mathbf{F}) := \{z \in \mathbb{C}\backslash\mathbf{I}_{<\infty}(\mathbf{F}) : \mathbf{F}^P(z) \to \infty \text{ as } P \to \infty\},$$

and the *full escaping set* of $\mathbf{F}$ is

$$\mathbf{I}(\mathbf{F}) := \mathbf{I}_{<\infty}(\mathbf{F}) \cup \mathbf{I}_{\infty}(\mathbf{F}).$$

Lemma 5.19. *For any* $P \in \mathbf{T}$*, every connected component of* $\mathbf{I}_{\le P}(\mathbf{F})$ *is unbounded.*

Proof. There exists some $n \le 0$ such that $\mathbf{F}_n := \mathcal{R}^n\mathbf{F}$ is in $\boldsymbol{\mathcal{W}}^u_{loc}$. Since the domains of $\mathbf{f}_{n,\pm}$ are simply connected, then $\mathrm{Dom}\left(\mathbf{F}_n^P\right)$ is simply connected for all $P \in \mathbf{T}$ and so the claim is true for $\mathbf{F}_n$. Since $\mathbf{F}$ is just a rescaling of $\mathbf{F}_n$, the claim is also true for $\mathbf{F}$. □

In Section 6, we will thoroughly study the structure of the finite-time escaping set of the fixed point $\mathbf{F}_*$. In Section 7, we will show that in the hyperbolic case, the finite and infinite-time escaping sets does not carry any invariant line field. This will imply that the unstable manifold indeed has codimension one. For the rest of this subsection, we will formulate a Fatou-Julia theory for our dynamical systems $\mathbf{F}$ in $\boldsymbol{\mathcal{W}}^u$ and state a few analogues of basic results in classical holomorphic dynamics.

Definition 5.20. The *Fatou set* $\mathfrak{F}(\mathbf{F})$ of $\mathbf{F}$ is be the set of points z which admit a small neighborhood $X \subset \mathbb{C}\backslash\mathbf{I}_{<\infty}(\mathbf{F})$ such that $\{\mathbf{F}^P|_X\}_{P\in\mathbf{T}}$ forms a normal family. The *Julia set* $\mathfrak{J}(\mathbf{F})$ of $\mathbf{F}$ is the complement $\mathbb{C}\backslash\mathfrak{F}(\mathbf{F})$.

Clearly, $\mathfrak{J}(\mathbf{F})$ contains the closure of $\mathbf{I}_{<\infty}(\mathbf{F})$.

We say that a connected component X of $\mathfrak{F}(\mathbf{F})$ is *periodic* if there is some $P \in \mathbf{T}_{>0}$ such that $\mathbf{F}^P(X) = X$. The smallest such P is called the *period* of X. Moreover, we say that X is *pre-periodic* if there is some $Q \in \mathbf{T}$ such that $\mathbf{F}^Q(X)$ is periodic. The smallest such Q is called the *pre-period* of X. (These quantities exist due to Lemma 5.14. Compare with Corollary 5.17.)

Definition 5.21. The *postcritical set* of $\mathbf{F}$ is

$$\mathfrak{P}(\mathbf{F}) := \overline{\{\mathbf{F}^P(0) \,:\, P \in \mathbf{T}\}}.$$

The postcritical set is is characterized as the smallest forward invariant closed set such that

$$\mathbf{F}^P : \mathrm{Dom}\left(\mathbf{F}^P\right) \backslash \mathbf{F}^{-P}\left(\mathfrak{P}(\mathbf{F})\right) \to \mathbb{C} \backslash \mathfrak{P}(\mathbf{F})$$

is an unbranched covering map which is a local isometry with respect to the hyperbolic metrics.

In the case of $\mathbf{F} = \mathbf{F}_*$, equation (5.4) implies self-similarity of the corresponding dynamical sets.

Lemma 5.22. *The linear map A_* preserves $\mathfrak{F}(\mathbf{F}_*)$, $\mathfrak{J}(\mathbf{F}_*)$, $\mathbf{I}_{<\infty}(\mathbf{F}_*)$, $\mathbf{I}_\infty(\mathbf{F}_*)$, and $\mathfrak{P}(\mathbf{F}_*)$. For all $P \in \mathbf{T}_{>0}$, $A_*(\mathbf{I}_{\leq P}(\mathbf{F}_*)) = \mathbf{I}_{\leq \mathbf{t}P}(\mathbf{F}_*)$.*

Given a periodic point x of (minimal) period P of some $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, we say that x is *superattracting* / *attracting* / *Siegel* / *Cremer* / *repelling* if x is a *superattracting* / *attracting* / *Siegel* / *Cremer* / *repelling* fixed point of $\mathbf{F}^P$.

Proposition 5.23. *Suppose $\mathbf{F}$ admits a periodic point x of some period P.*

(1) *If x is attracting or parabolic, then the critical orbit $\{\mathbf{F}^T(0)\}_{T\in\mathbb{T}}$ converges to the periodic orbit $\mathrm{orb}_0^P(\mathbf{F})$.*
(2) *If x is Cremer, then $x \in \mathfrak{P}(\mathbf{F})$.*
(3) *If x is Siegel, then the boundary of the Siegel disk of $\mathbf{F}^P$ centered at x is contained in $\mathfrak{P}(\mathbf{F})$.*

Proof. (1) follows from a standard analytic continuation argument. See [Mil06, Lemma 8.5].

Suppose $x \notin \mathfrak{P}(\mathbf{F})$. For all $T \in \mathbf{T}$, let us denote by D_T the connected component of $\mathrm{Dom}\left(\mathbf{F}^T\right) \backslash \mathbf{F}^{-T}\left(\mathfrak{P}(\mathbf{F})\right)$ containing x. We claim that x cannot be Cremer. Indeed, suppose first D_P is properly contained in D_0. Then, $\mathbf{F}^P : D_P \to D_0$ is strictly expanding with respect to the hyperbolic metric of D_0, which implies that x must be repelling. Suppose instead $D_P = D_0$. Then, $\left\{\mathbf{F}^{nP}|_{D_0}\right\}_{n\in\mathbb{N}}$ is a normal family of self maps of a hyperbolic Riemann surface. By Denjoy-Wolff, the fixed point x is either attracting or Siegel.

Let us assume that x is Siegel. Denote by Z the Siegel disk cenetered at x. If there exists some minimal $T \in \mathbf{T}$ where $\mathbf{F}^T(0)$ intersects Z, then the intersection $\mathfrak{P}(\mathbf{F}) \cap Z$ is a single $\mathbf{F}^P$-invariant curve on Z. Suppose for a contradiction that $\mathbb{C} \backslash \mathfrak{P}(\mathbf{F})$ intersects the boundary ∂Z. Then, a component E_0 of $\mathbb{C} \backslash \mathfrak{P}(\mathbf{F})$ contains some neighborhood of ∂Z. For $n \in \mathbb{N}$, let E_{nP} be the connected component of $\mathrm{Dom}\left(\mathbf{F}^{nP}\right) \backslash \mathbf{F}^{-nP}(\mathfrak{P}(\mathbf{F}))$ containing $E_0 \cap Z_0$.

There are again two cases. If $E_P = E_0$, then $\left\{\mathbf{F}^{nP}|_{E_0}\right\}_{n\in\mathbb{N}}$ forms a normal family and E_0 must be contained in the Fatou set, which is a contradiction. If E_P is a proper subset of E_0, then $\mathbf{F}^P : E_P \to E_0$ is strictly expanding with respect to the hyperbolic metric of E_0, which would contradict the fact that $\mathbf{F}^P$ restricts to a self diffeomorphism of any invariant curve in $Z \cap E_0$. $\square$

For any tangent vector v at a point z in $\mathbb{C}\backslash\mathfrak{P}(\mathbf{F})$, denote by $\|v\|$ the norm of v with respect to the hyperbolic metric of $\mathbb{C}\backslash\mathfrak{P}(\mathbf{F})$. If $z \in \mathfrak{P}$, we set $\|v\| = \infty$.

Lemma 5.24. *Every point $z \in \mathfrak{J}(\mathbf{F})\backslash\mathbf{I}_{<\infty}(\mathbf{F})$ satisfies $\|\left(\mathbf{F}^P\right)'(z)\| \to \infty$ as $P \to \infty$.*

Proof. Let us fix a point $z \in \mathfrak{J}(\mathbf{F})\backslash\mathbf{I}_{<\infty}(\mathbf{F})$. Without loss of generality, assume that z does not eventually land on $\mathfrak{P}(\mathbf{F})$.

For any $P \in \mathbf{T}_{>0}$, let $\mathfrak{P}_P := \mathbf{I}_{\leq P}(\mathbf{F}) \cup \mathbf{F}^{-P}(\mathfrak{P}(\mathbf{F}))$. The map $\mathbf{F}^P : \mathbb{C}\backslash\mathfrak{P}_P \to \mathbb{C}\backslash\mathfrak{P}$ is a local isometry with respect to their hyperbolic metrics. Since the union $\bigcup_{P\in\mathbf{T}}\mathfrak{P}_P$ is a dense subset of the Julia set, the distance between $\mathfrak{P}_P$ and z shrinks to 0 as $P \to \infty$. Consequently, the hyperbolic distance r_P between z and $\mathfrak{P}_P$ inside $\mathbb{C}\backslash\mathfrak{P}$ also tends to 0 as $P \to \infty$. The inclusion map $\iota : \mathbb{C}\backslash\mathfrak{P}_P \to \mathbb{C}\backslash\mathfrak{P}$ is contracting by some factor $K(r_P)$ where $K(r) \to 0$ as $r \to 0$. Therefore, as $P \to \infty$, $\|\left(\mathbf{F}^P\right)'(z)\| \geq K(r_P)^{-1} \to \infty$. $\square$

Denote by $\mathrm{dist}_{\hat{\mathbb{C}}}(\cdot,\cdot)$ the spherical distance between two subsets of $\hat{\mathbb{C}}$.

Theorem 5.25. *If $\mathfrak{J}(\mathbf{F})$ has no interior, then for almost every $z \in \mathfrak{J}(\mathbf{F})\backslash\mathbf{I}_{<\infty}(\mathbf{F})$,*

$$\mathrm{dist}_{\hat{\mathbb{C}}}\left(\mathbf{F}^P(z), \mathfrak{P}(\mathbf{F}) \cup \{\infty\}\right) \to 0 \qquad \text{as } P \to \infty.$$

In other words, almost every non-escaping point in the Julia set is attracted to the postcritical set.

Proof. Suppose for a contradiction that there exist a positive number $\varepsilon > 0$ and a positive area subset E of $\mathfrak{J}(\mathbf{F})\backslash\mathbf{I}_{<\infty}(\mathbf{F})$ such that for all $z \in E$,

$$\limsup_{P\to\infty} \mathrm{dist}\left(\mathbf{F}^P(z), \mathfrak{P}(\mathbf{F}) \cup \{\infty\}\right) \geq \varepsilon.$$

Let x be a Lebesgue density point of E. There is a sequence of power-triples P_n such that $P_n \to \infty$ and $y_n := \mathbf{F}^{P_n}(z)$ lies in the compact set

$$K := \{z \in \mathbb{C} : \mathrm{dist}(z, \mathfrak{P}(\mathbf{F}) \cup \{\infty\}) \geq \varepsilon\}.$$

For each $n \in \mathbb{N}$, consider the spherical ball B_n of radius $\varepsilon/2$ centered at y_n, and let B_n' be the lift of B_n under $\mathbf{F}^{P_n}$ containing z.

By Lemma 5.24, $\left\|\left(\mathbf{F}^{P_n}\right)'(z)\right\| \to \infty$. Since K is compact and $\mathbf{F}^{P_n}|_{B_n'}$ has bounded distortion, the disks B_n' must shrink to a point. Since z is a density point of E,

$$\lim_{n\to\infty} \frac{\mathrm{area}(B_n' \cap E)}{\mathrm{area}(B_n')} = 1.$$

Therefore, we also have

$$\lim_{n\to\infty} \frac{\mathrm{area}(B_n \cap \mathfrak{J}(\mathbf{F})\backslash\mathbf{I}_{<\infty}(\mathbf{F}))}{\mathrm{area}(B_n)} = 1.$$

Since K is compact, y_n converges in subsequence to some point $y \in K$. Then, the ball B of radius $\varepsilon/2$ centered at y must satisfy $\mathrm{area}\left(B\backslash\left(\mathfrak{J}(\mathbf{F})\backslash\mathbf{I}_{<\infty}(\mathbf{F})\right)\right) = 0$. Since $\mathfrak{J}(\mathbf{F})$ is closed, then the ball B is contained in $\mathfrak{J}(\mathbf{F})$. This contradicts the assumption that $\mathfrak{J}(\mathbf{F})$ has no interior. $\square$

6. The external structure of $\mathbf{F}_*$

Let us study in detail the dynamics of the cascade $\mathbf{F}^{\geq 0}$ corresponding to the fixed point f_* of the renormalization operator. We denote by $\mathbf{H}$ the Herman quasicircle of $\mathbf{F}$, which is defined to be the full lift of the Herman quasicircle of f_*.

6.1. Lakes.

The dynamics of $\mathbf{F}$ along $\mathbf{H}$ can be described as follows. For $a \in \mathbb{C}$, we denote the translation map by a by $T_a(z) := z + a$.

Lemma 6.1. *There is a quasiconformal map $h : \mathbb{C} \to \mathbb{C}$ with the following properties.*

1. *h sends $(\mathbf{H}, 0)$ to $(\mathbb{R}, 0)$;*
2. *h conjugates the cascade $\mathbf{F}^{\geq 0}|_{\mathbf{H}}$ with the cascade of translations $(T^P)_{P \in \mathbf{T}}$ defined by $T^{(n,a,b)} := T_{\mathbf{t}^{-n}(b\mathbf{v} - a\mathbf{u})}$ where $\mathbf{u}, \mathbf{v} > 0$ and $\theta = \frac{\mathbf{u}}{\mathbf{u}+\mathbf{v}}$.*

Proof. The pre-corona F_* associated to f_* admits an invariant quasiarc which projects to the Herman quasicircle of f_*. In linear coordinates, this corresponds to an invariant quasiarc $\mathbf{H}_0$ of $\mathbf{F} = (\mathbf{f}_\pm : \mathbf{U}_\pm \to \mathbf{S})$ which passes through 0 and connects $\mathbf{f}_+(0)$ and $\mathbf{f}_-(0)$. The dynamics $\mathbf{f}_\pm$ along $\mathbf{H}_0$ is quasisymetrically conjugate to a pair of translations $\left(T_{-\theta}|_{[0,1-\theta]}, T_{1-\theta}|_{[-\theta,0]}\right)$ on the real interval $[-\theta, 1-\theta]$. Set $\mathbf{u} = -\theta$ and $\mathbf{v} = 1 - \theta$. As we extend $\mathbf{f}_\pm$ to its maximal σ-proper extension via A_*, the quasisymmetric conjugacy h between $(\mathbf{f}_-, \mathbf{f}_+)$ and $(T_{-\mathbf{u}}, T_{-\mathbf{v}})$ extends to the whole lift $\mathbf{H}$ of $\mathbf{H}_)$. See Appendix B.2. □

Definition 6.2. Let us label the components of $\mathbb{C}\backslash\mathbf{H}$ by $\mathbf{O}^0$ and $\mathbf{O}^\infty$, which we will refer to as the *oceans* of $\mathbf{F}$. A *lake* $\mathbf{O}$ *of generation* $P \in \mathbf{T}$ is a connected component of $\mathbf{F}^{-P}(\mathbf{O}^\bullet)$ for some $\bullet \in \{0, \infty\}$, and its *coast* is $\partial^c\mathbf{O} := \partial\mathbf{O} \cap \operatorname{Dom}\left(\mathbf{F}^P\right)$.

The following lemma is a direct consequence of σ-properness of the cascade.

Lemma 6.3 (Chessboard rule)**.** *For every $P \in \mathbf{T}_{>0}$ and $\bullet \in \{0, \infty\}$, the preimage $\mathbf{F}^{-P}(\mathbf{H})$ is a tree in $\operatorname{Dom}\left(\mathbf{F}^P\right)$ and $\mathbf{F}^{-P}(\mathbf{O}^\bullet)$ is disjoint union of lakes $\bigcup_{i\in\mathbb{N}} \mathbf{O}_i$ of generation P such that*

1. *each lake $\mathbf{O}_i$ is a disk which is unbounded in $\operatorname{Dom}\left(\mathbf{F}^P\right)$ and does not separate $\operatorname{Dom}\left(\mathbf{F}^P\right)$;*
2. *for $j \neq i$, the intersection $\partial^c\mathbf{O}_i \cap \partial^c\mathbf{O}_j$ is either empty or a singleton consisting of a critical point of $\mathbf{F}^P$.*

Proof. The whole lemma follows immediately from [DL23, Lemma 5.1] and the fact that $\operatorname{CV}(\mathbf{F})$ is contained in $\mathbf{H}$. □

Given any lake $\mathbf{O}$ of some generation $P > 0$, the map $\mathbf{F}^P$ sends $\mathbf{O}$ univalently onto an ocean, and its coast homeomorphically onto $\mathbf{H}$. In general, when $0 < P < Q$, a lake of generation Q is contained in a lake of generation P, and $\mathbf{F}^{Q-P}$ conformally sends any lake of generation Q onto a lake of generation P.

Lemma 6.4. *For every $P \in \mathbf{T}_{>0}$, there is a unique critical point $C_P \in \mathbf{H}$ of $\mathbf{F}^{\geq 0}$ of generation P and a pairwise disjoint collection of lakes*

$$ {}_1\mathbf{O}^0_P, \ \ldots, {}_{2d_0-3}\mathbf{O}^0_P, \ {}_1\mathbf{O}^\infty_P, \ \ldots, {}_{2d_\infty-3}\mathbf{O}^\infty_P, \tag{6.1} $$

of generation P together with a bouquet of pairwise-disjoint open quasiarcs

$$ {}_1\mathbf{H}^0_P, \ \ldots, {}_{2d_0-2}\mathbf{H}^0_P, \ {}_1\mathbf{H}^\infty_P, \ \ldots, {}_{2d_\infty-2}\mathbf{H}^\infty_P, \tag{6.2} $$

rooted at C_P such that for each $\bullet \in \{0, \infty\}$ and $j \in \{1, \ldots, 2d_\bullet - 3\}$,

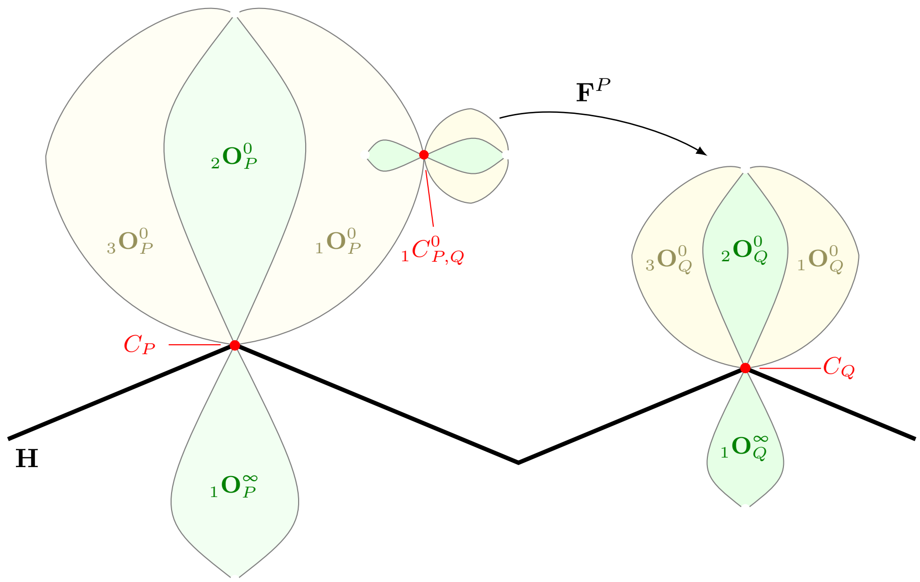

FIGURE 7. The structure of lakes attached to critical points C_P, C_Q, and ${}_1C^0_{P,Q}$ when $d_0 = 3$ and $d_\infty = 2$

(1) *the coast of* ${}_j\mathbf{O}^\bullet_P$ *is* ${}_j\mathbf{H}^\bullet_P \cup \{C_P\} \cup {}_{j+1}\mathbf{H}^\bullet_P$*;*
(2) ${}_j\mathbf{O}^\bullet_P$ *is contained in* $\mathbf{O}^\bullet$*;*
(3) ${}_j\mathbf{O}^\bullet_P$ *is mapped conformally by* $\mathbf{F}^P$ *onto* $\mathbf{O}^\bullet$ *if* j *is is even, and onto* $\mathbb{C}\backslash\overline{\mathbf{O}^\bullet}$ *if* j *is odd.*

Proof. The existence and uniqueness of C_P is due to the fact that $\mathbf{F}^P$ restricts to a homeomorphism on $\mathbf{H}$. From the previous lemma, $\mathbf{F}^{-P}(\mathbf{H})$ is a tree. The quasiarcs ${}_j\mathbf{H}^\bullet_P$'s are precisely components of $\mathbf{F}^{-P}(\mathbf{H})\backslash\{C_P\}$, and the lakes ${}_j\mathbf{O}^\bullet_P$'s in (6.2) are precisely connected components of $\mathrm{Dom}\left(\mathbf{F}^P\right)\backslash\mathbf{F}^{-P}(\mathbf{H})$ which touch $\mathbf{H}$ at exactly one point, which is C_P. For all $S < P$, the image of each quasiarc ${}_j\mathbf{H}^\bullet_P$ under $\mathbf{F}^S$ is disjoint from 0. Therefore, $\mathbf{F}^P$ maps each of ${}_j\mathbf{H}^\bullet_P$ onto a component of $\mathbf{H}\backslash\{0\}$ homeomorphically. They can be enumerated such that all the three claims hold because C_P has inner and outer criticalities d_0 and d_∞ respectively. □

Each quasiarc in (6.2) is called a *spine* of C_P. The spines (6.2) are labelled in counterclockwise order about C_P.

Let us pick a pair of power-triples $P, Q \in \mathbf{T}_{>0}$. For any $\bullet \in \{0, \infty\}$ and any $j \in \{1, \dots d_\bullet - 1\}$, the union of two consecutive spines ${}_{2i-1}\mathbf{H}^\bullet_P \cup {}_{2i}\mathbf{H}^\bullet_P$ are mapped homeomorphically by $\mathbf{F}^P$ onto $\mathbf{H}\backslash\{0\}$ and so it contains a unique critical point ${}_jC^\bullet_{P,Q}$ of generation $P + Q$. Attached to this critical point is a bouquet of lakes

$$ {}_{j,1}\mathbf{O}^{\bullet,0}_{P,Q},\ \dots,\ {}_{j,2d_0-3}\mathbf{O}^{\bullet,0}_{P,Q},\ {}_{j,1}\mathbf{O}^{\bullet,\infty}_{P,Q},\ \dots,\ {}_{j,2d_\infty-3}\mathbf{O}^{\bullet,\infty}_{P,Q}, $$

of generation $P + Q$ together with spines

$$ {}_{j,1}\mathbf{H}^{\bullet,0}_{P,Q},\ \dots,\ {}_{j,2d_0-2}\mathbf{H}^{\bullet,0}_{P,Q},\ {}_{j,1}\mathbf{H}^{\bullet,\infty}_{P,Q},\ \dots,\ {}_{j,2d_\infty-2}\mathbf{H}^{\bullet,\infty}_{P,Q}, $$

meeting at ${}_jC^{\bullet}_{P,Q}$ such that each of ${}_{j,k}\mathbf{O}^{\bullet,\circ}_{P,Q}$ has coast ${}_{j,k}\mathbf{H}^{\bullet,\circ}_{P,Q} \cup {}_{j,k+1}\mathbf{H}^{\bullet,\circ}_{P,Q}$ and is mapped univalently by $\mathbf{F}^P$ onto ${}_k\mathbf{O}^{\circ}_Q$.

Consider a tuple $S = (P_1, \dots, P_{m+1}) \in \mathbf{T}^{m+1}_{>0}$ of power-triples. We denote the sum by

$$|S| \coloneqq \sum_{i=1}^{m+1} P_i.$$

Given $\blacksquare = (\bullet_1, \dots, \bullet_m) \in \{0, \infty\}^m$, $J = (j_1, \dots j_m)$ where $j_i \in \{1, \dots, d_{\bullet_i} - 1\}$ for all i, we inductively define a critical point ${}_JC^{\blacksquare}_S$ of generation $|S|$. Attached to this critical point are lakes ${}_{J,i}\mathbf{O}^{\blacksquare,\bullet}_S$ for $\bullet \in \{0, \infty\}$ and $i \in \{1, \dots, 2d_\bullet - 3\}$, and spines ${}_{J,j}\mathbf{H}^{\blacksquare,\bullet}_S$ for $\bullet \in \{0, \infty\}$ and $j \in \{1, \dots, 2d_\bullet - 2\}$.

Definition 6.5. We say that a lake $\mathbf{O}$ is a *middle lake* if it is of the form ${}_{J,j}\mathbf{O}^{\blacksquare,\bullet}_S$. The finite tuple S is called the *itinerary* of $\mathbf{O}$.

Consider a lake $\mathbf{O}$ of generation $P \in \mathbf{T}_{>0}$. Let $Q \in \mathbf{T}$ be the smallest power triple such that the coast of $\mathbf{O}$ touches $\mathbf{F}^{-Q}(\mathbf{H})$.

Lemma 6.6 (Left and right coasts)**.** *The intersection $\partial^c\mathbf{O} \cap \mathbf{F}^{-Q}(\mathbf{H})$ is a closed connected quasiarc and the complement $\partial^c\mathbf{O} \backslash \mathbf{F}^{-Q}(\mathbf{H})$ consists of two non-empty open quasiarcs $\partial^c_l\mathbf{O}$ and $\partial^c_r\mathbf{O}$.*

Proof. It is sufficient to consider the case when $Q = 0$. For any point z in $\partial^c\mathbf{O} \cap \mathbf{H}$, every component of $\mathbf{H}\backslash\{z\}$ contains infinitely many critical points of $\mathbf{F}^P$ of generation at most P, and each of these points is a branch point of the tree $\mathbf{F}^{-P}(\mathbf{H})$. Since $\partial^c\mathbf{O} \cap \mathbf{H}$ does not contain such branch points, the claim follows. □

We call $\partial^c_l\mathbf{O}$ and $\partial^c_r\mathbf{O}$ the *left and right coasts* of $\mathbf{O}$. We always assume that $\partial^c_l\mathbf{O}$, $\partial^c\mathbf{O} \cap \mathbf{F}^{-Q}(\mathbf{H})$, and $\partial^c_r\mathbf{O}$ are oriented counterclockwise relative to $\mathbf{O}$. (This distinction is consistent once we know from Lemma 6.12 that $\mathbf{O}$ is a disk.)

The closure $\overline{\partial^c_l\mathbf{O}}$ of the left coast admits a maximal sequence of critical points $c_{l,1}, c_{l,2}, \dots$, labelled in increasing order of generation, such that for every i, the arc $(c_{l,i}, c_{l,i+1}) \subset \partial^c_l\mathbf{O}$ admits no critical points of $\mathbf{F}^P$. We define the *left itinerary* of $\mathbf{O}$ to be the sequence $I_l \coloneqq (P_{l,1}, P_{l,2}, \dots)$ where each $P_{l,i}$ is the generation of $c_{l,i}$. We call the supremum of $P_{l,i}$ across all i's the *left generation* G_l of $\mathbf{O}$. Similarly, we can define the *right itinerary* I_r and the *right generation* G_r of $\mathbf{O}$.

Lemma 6.7. *Consider a lake $\mathbf{O}$ of generation $P \in \mathbf{T}_{>0}$.*

1. *The left and right generations of $\mathbf{O}$ are equal to P.*
2. *If I_l (resp. I_r) is finite, the left (resp. right) coast of $\mathbf{O}$ contains a spine attached a critical point ${}_JC^{\blacksquare}_S$ of generation $|S| = P$.*
3. *If both I_l and I_r are finite, then $\mathbf{O}$ is a middle lake attached to the critical point ${}_JC^{\blacksquare}_S$.*
4. *Either I_l or I_r is a finite sequence.*

Proof. Suppose instead that $G_l < P$, so then there is some $P' \in \mathbf{T}$ such that $G_l < P' < P$. This implies that $\mathbf{F}^{P'}(\mathbf{O})$ is a lake of positive generation with an empty left coast, which is a contradiction to Lemma 6.6. Therefore, the left generation must be equal to P. By the same argument, so is the right generation of $\mathbf{O}$. Thus, (1) holds.

Suppose I_l is finite. By (1), there exists a critical point c_l of generation P on $\overline{\partial^c_l\mathbf{O}}$. Removing c_l splits the coast into two open quasiarcs, one of which contains no critical points of $\mathbf{F}^P$ and is thus a spine attached to c_l. This implies (2). Suppose I_r is also finite, so there also exists a critical point c_r of generation P on $\overline{\partial^c_r \mathbf{O}}$. The complement of the interval $[c_l, c_r]$ within $\partial^c \mathbf{O}$ is now a pair of spines of generation P attached to c_l and c_r respectively. The map $\mathbf{F}^P$ sends each of these spines to a component of $\mathbf{H}\backslash\{0\}$. However, since $\mathbf{F}^P : \partial^c \mathbf{O} \to \mathbf{H}$ is a homeomorphism, we see that $c_l = c_r$ and $\mathbf{O}$ is a middle lake. Hence, (3) holds.

Let us now prove (4). We will again assume without loss of generality that $Q = 0$. Let us pick a point y in $\partial^c \mathbf{O} \cap \mathbf{H}$. If the open interval $(y, C_P) \subset \mathbf{H}$ does not contain any critical point of generation $\leq P$, then either $\partial^c_l \mathbf{O}$ or $\partial^c_r \mathbf{O}$ is rooted at C_P and contains no other critical points of generation $\leq P$. Otherwise, by Lemma 5.13, there are only finitely many critical points of generation $\leq P$ within (y, C_P), and they have some maximum generation $R < P$. We then apply the previous argument to the lake $\mathbf{F}^R(\mathbf{O})$ and the interval $\left(\mathbf{F}^R(y), C_{P-R}\right) \subset \mathbf{H}$. □

Consider a critical point ${}_JC_S^{\blacksquare}$ of $\mathbf{F}^{\geq 0}$. There exist lakes

$$ {}_{J,l}\mathbf{O}_S^{\blacksquare,0},\ {}_{J,r}\mathbf{O}_S^{\blacksquare,0},\ {}_{J,l}\mathbf{O}_S^{\blacksquare,\infty},\ {}_{J,r}\mathbf{O}_S^{\blacksquare,\infty} \tag{6.3}$$

of generation $|S|$ such that

(i) they are disjoint from ${}_{J,j}\mathbf{O}_S^{\blacksquare,\bullet}$ for all $\bullet \in \{0, \infty\}$ and $j \in \{1, \ldots, 2d_\bullet - 3\}$;
(ii) for $\bullet \in \{0, \infty\}$, the right coast of ${}_{J,l}\mathbf{O}_S^{\blacksquare,\bullet}$ contains the spine ${}_{J,2d_\bullet-2}\mathbf{H}_S^{\blacksquare,\bullet}$ and the left coast of ${}_{J,r}\mathbf{O}_S^{\blacksquare,\bullet}$ contains the spine ${}_{J,1}\mathbf{H}_S^{\blacksquare,\bullet}$;
(iii) if $\{j, j'\} \in \{l, r\}$ and $j \neq j'$, the coasts of ${}_{J,j}\mathbf{O}_S^{\blacksquare,0}$ and ${}_{J,j'}\mathbf{O}_S^{\blacksquare,\infty}$ intersect on a non-degenerate closed interval in $\mathbf{F}^{-|S|}(\mathbf{H})$ with endpoint ${}_JC_S^{\blacksquare}$.

We will call the lakes in (6.3) the *left/right side lakes* of ${}_JC_S^{\blacksquare}$.

Observe that by (ii),

$$ {}_{J,r}\mathbf{O}_S^{\blacksquare,0}, {}_{J,1}\mathbf{O}_S^{\blacksquare,0}, \ldots, {}_{J,2d_0-3}\mathbf{O}_S^{\blacksquare,0}, {}_{J,l}\mathbf{O}_S^{\blacksquare,0}, {}_{J,r}\mathbf{O}_S^{\blacksquare,\infty}, {}_{J,1}\mathbf{O}_S^{\blacksquare,\infty}, \ldots, {}_{J,2d_\infty-3}\mathbf{O}_S^{\blacksquare,\infty}, {}_{J,l}\mathbf{O}_S^{\blacksquare,\infty} $$

are in counterclockwise order about ${}^{\blacksquare}_JC_S$ and their union is a neighborhood of ${}^{\blacksquare}_JC_S$. By Lemma 6.7 (4), the left itinerary of ${}_{J,l}\mathbf{O}_S^{\blacksquare,\bullet}$ and the right itinerary of ${}_{J,r}\mathbf{O}_S^{\blacksquare,\bullet}$ are infinite. The following is a consequence of Lemma 6.7 (2)–(4).

Corollary 6.8. *Every lake* $\mathbf{O}$ *is either a middle lake or a side lake of a critical point* ${}_JC_S^{\blacksquare}$. *In other words,* $\mathbf{O}$ *is of the form* ${}_{J,j}\mathbf{O}_S^{\blacksquare,\bullet}$ *where* $j \in \{l, 1, \ldots, 2d_\bullet - 3, r\}$.

Given some tuple $S = (P_1, \ldots, P_k) \in \mathbf{T}^k_{>0}$, we can perform scalar multiplication by $\mathbf{t}$ and denote $\mathbf{t}S := (\mathbf{t}P_1, \ldots, \mathbf{t}P_k)$. The following is a direct consequence of (5.4).

Lemma 6.9. *For any critical point of the form* ${}_JC_S^{\blacksquare}$ *and a bubble* ${}_{J,j}\mathbf{O}_S^{\blacksquare,\bullet}$ *where* $j \in \{l, 1, \ldots, 2d_\bullet - 1, r\}$,

$$ A_*\left({}_JC_S^{\blacksquare}\right) = {}_JC_{\mathbf{t}S}^{\blacksquare} \quad \text{and} \quad A_*\left({}_{J,j}\mathbf{O}_S^{\blacksquare,\bullet}\right) = {}_{J,j}\mathbf{O}_{\mathbf{t}S}^{\blacksquare,\bullet}. $$

Proof. Recall from (5.4) that A_* conjugates $\mathbf{F}^P$ and $\mathbf{F}^{\mathbf{t}P}$ for any $P \in \mathbf{T}_{>0}$. Since A_* preserves $\mathbf{H}$, then $A_*(C_P) = C_{\mathbf{t}P}$ and thus $A_*\left({}_j\mathbf{O}_P^{\bullet}\right) = {}_j\mathbf{O}_{\mathbf{t}P}^{\bullet}$ for all $\bullet \in \{0, \infty\}$ and $j \in \{l, 1, \ldots, 2d_\bullet - 3, r\}$.

Suppose a spine ${}_j\mathbf{H}_P^{\bullet}$ attached to C_P contains some critical point ${}_iC_{P,Q}^{\bullet}$ where $i = \lceil \frac{j}{2} \rceil$. Since $A_*({}_iC_{P,Q}^{\bullet})$ lies on in ${}_j\mathbf{H}_{\mathbf{t}P}^{\bullet}$ and is a critical point of generation $\mathbf{t}(P+Q)$, then it is equal to ${}_iC_{\mathbf{t}P,\mathbf{t}Q}^{\bullet}$. The rest follows by induction. □

6.2. **Limbs.**

Definition 6.10. A *limb* ${}_J\mathbf{L}^{\blacksquare}_S$ is the union of the spine ${}_J\mathbf{H}^{\blacksquare}_S$ together with all spines of the form ${}_{J,j_1,\dots,j_k}\mathbf{H}^{\blacksquare,\bullet_1,\dots,\bullet_k}_{S,P_1,\dots,P_k}$.

By Lemma 6.9, the linear map A_* sends each limb ${}_J\mathbf{L}^{\blacksquare}_S$ onto ${}_J\mathbf{L}^{\blacksquare}_{\mathbf{t}S}$.

Lemma 6.11. *Every limb is pre-compact.*

The proof we present below is identical to [DL23, Lemma 5.10].

Proof. Recall the rescaled pre-corona $\mathbf{F}^{\#}_n = \left(\mathbf{f}^{\#}_{n,\pm} : \mathbf{U}^{\#}_{n,\pm} \to \mathbf{S}^{\#}_n\right)$ where $\mathbf{S}^{\#}_n := A^n_*(\mathbf{S})$ for all $n \in \mathbb{Z}$. Since $\mathbf{S}$ is compactly contained in $A^{-1}_*(\mathbf{S})$, then $\bigcup_{n\in\mathbb{Z}} \mathbf{S}^{\#}_n = \mathbb{C}$. For each $n \in \mathbb{Z}$, there is a gluing map $\rho_n : \mathbf{S}^{\#}_n \to V$ projecting $\mathbf{F}^{\#}_n$ to the corona $f : U \to V$.

Let us choose a large $n \ll 0$. Consider open rectangles $X = \rho_n(\mathbf{S}^{\#}_0)$ and $X' = \rho_n(\mathbf{S}^{\#}_{-1})$ living in the dynamical plane of f. Denote by $\mathbf{H}_*$ the Herman quasicircle of f, and consider the interval $I := X \cap \mathbf{H}_*$ and a slightly smaller interval $J \subset I$.

Claim 1. There is some $M \in \mathbb{N}$ such that the following holds. For any connected component W of $X'\backslash\mathbf{H}$, any $m \geq M$, and any point $x \in J$ with $f^m(x) \in \partial W$, the domain W univalently lifts to a domain W_{-m} along the orbit $x, f(x), \dots, f^m(x)$ such that $W_{-m} \subset X$.

Proof. Let Y^0 and Y^∞ denote the inner and outer components of $\mathbb{C}\backslash\mathbf{H}$. Let W_{-m} be the corresponding lift of W along the orbit $x, f(x), \dots, f^m(x)$. Assume without loss of generality that W is contained in Y^∞. Since $f^i(x) \in \mathbf{H}_*$ for all $i \geq 0$, then the lift W_{-m} is also within Y^∞.

Let us pick two outer external rays R_l and R_r landing at a pair of points of $\mathbf{H}_*$ such that R_l is slightly on the left of W and R_r is slightly on the right of W. Since $n \ll 0$, the difference δ between the external angles of R_l and R_r is small.

For $k = 1, 1, \dots, m$, let $R_{l,-k}$ and $R_{r,-k}$ be the preimages of R_l and R_r under f^k such that they are slightly on the left and right of W_{-k} respectively. By definition, for each arc γ^∞_j on the forbidden boundary $\partial_{\mathrm{F}}U$ of U, the part which gets mapped to $\gamma_1 \cap Y^\infty$ is an external ray of some definite distance from $\mathbf{H}$. The difference between the external angles of $R_{l,-k}$ and $R_{r,-k}$ is δ/d^k_∞, which is even smaller than δ. Therefore, W_{-k} is disjoint from $\partial_{\mathrm{F}}U$ for all k. Therefore, $f^m : W_{-m} \to W$ is univalent.

For sufficiently large m, W_{-m} is within a small neigborhood of $\mathbf{H}$ and it is sandwiched between the rays $R_{-m,l}$ and $R_{-m,r}$, whose external angles differ by a small constant. By local connectivity, W_{-m} must be contained in a small neighborhood of J, and thus $W_{-m} \subset X$. □

The composition $\rho_n \circ A^{-n}_*$ identifies $\mathbf{S}^{\#}_n$ with X. Let $\mathbf{J}_n := A^n_* \circ \rho^{-1}_n(J)$.

Claim 2. There is a power-triple $R \in \mathbf{T}_{>0}$ such that $\mathbf{F}^R(\mathbf{J}_0) \subset \mathbf{J}_{-1}$ and for every point x on $\mathbf{J}_0$, if $\mathbf{F}^P(x) \in \mathbf{S}^{\#}_{-1}$ for some $P \geq R$, then there is an open set $W_P \subset \mathbf{S}^{\#}_0\backslash\mathbf{H}$ such that $x \in \partial W_P$ and $\mathbf{F}^P$ maps W_P conformally to $\mathbf{S}^{\#}_{-1}\backslash\mathbf{H}$.

Proof. Since the action of $\mathbf{F}^{\geq 0}$ on $\mathbf{H}$ is combinatorially modelled by the cascade of translations $(T^P)_{P\in\mathbf{T}}$ on $\mathbb{R}$, there is an arbitrarily large $R \in \mathbf{T}$ such that $\mathbf{F}^R(\mathbf{J}_0) \subset \mathbf{J}_{-1}$. Suppose $x \in \mathbf{J}_0$ and $\mathbf{F}^P(x) \in \mathbf{S}^{\#}_{-1}$ for some $P \geq R$. Since $\mathbf{f}_{-1,\pm}$ is the first return map of the cascade $\mathbf{F}^{\geq 0}$ back to $\mathbf{S}^{\#}_{-1}$, then $\mathbf{F}^P$ is the m^{th} iterate of the pair $\mathbf{f}_{-1,\pm}$

for some $m \in \mathbb{N}$. If R is chosen to be large enough, then $m \geq M$ and the claim now follows from Claim 1. $\square$

By self-similarity, Claim 2 also holds if we replace $\mathbf{J}_0$, $\mathbf{J}_{-1}$, P, and R by $\mathbf{J}_n$, $\mathbf{J}_{n-1}$, $\mathbf{t}^n P$, and $\mathbf{t}^n R$ respectively.

Claim 3. There is a power-triple $Q \in \mathbf{T}_{>0}$ such that for every $n \ll 0$ and every point $x \in \mathbf{J}_0$, if $\mathbf{F}^P(x) \in \mathbf{S}_n^{\#}$ for some $P \geq Q$, then there is an open set $W \subset \mathbf{S}_0^{\#} \backslash \mathbf{H}$ such that $x \in \partial W$ and $\mathbf{F}^P$ maps W conformally to $\mathbf{S}_n^{\#} \backslash \mathbf{H}$.

Proof. Let us choose $Q \in \mathbf{T}_{>0}$ such that $Q > R + R/\mathbf{t} + R/\mathbf{t}^2 + \dots$. Consider a point $x_0 := x \in \mathbf{J}_0$ such that $\mathbf{F}^P(x) \in \mathbf{S}_n^{\#}$ for some $P \geq Q$. For $j \in \{0, -1, -2, \dots, n+2\}$, we set $P_j := \mathbf{t}^j R$ and $x_{j-1} := \mathbf{F}^{P_j}(x_j)$ inductively. Then, we set

$$P_{n+1} := P - P_0 - P_{-1} - \dots - P_{n+2} \quad \text{and } x_n := \mathbf{F}^{P_{n+1}}(x_{n+1}).$$

Clearly, $P_{n+1} \geq \mathbf{t}^{n+1} R$. By Claim 2, there exists an open set $W_{n+1} \subset \mathbf{S}_{n+1}^{\#} \backslash \mathbf{H}$ such that $x_{n+1} \in \partial W_{n+1}$ and $\mathbf{F}^{P_{n+1}}$ maps W_{n+1} conformally to $\mathbf{S}_n^{\#} \backslash \mathbf{H}$. Inductively, for $j \in \{0, -1, \dots, n+2\}$, we construct open sets $W_j \subset \mathbf{S}_j^{\#} \backslash \mathbf{H}$ such that $x_j \in \partial W_j$ and $\mathbf{F}^{P_j}$ maps W_j conformally to W_{j-1}. Therefore, $\mathbf{F}^P$ maps W_0 conformally to $\mathbf{S}_n^{\#} \backslash \mathbf{H}$. $\square$

To prove the lemma, it is sufficient to consider limbs rooted at a critical point along $\mathbf{H}$. Pick such a limb L and let K be its generation. Choose a large $T \in \mathbf{T}$ such that $T \geq Q + K$ such that the critical point C_T is in $\mathbf{J}_0$. There exists some limb L' rooted at C_T such that $\mathbf{F}^{T-K}(L') = L$. Then, the connected component of $\mathbf{S}_n^{\#} \cap \overline{L}$ containing C_K can be lifted by $\mathbf{F}^{T-K}$ into $\mathbf{S}_0^{\#}$. As $n \ll 0$ is arbitrary, the lifts of $\mathbf{S}_n^{\#} \cap \overline{L}$ exhaust L' and so L' is contained in $\mathbf{S}_0^{\#}$. This implies that L is bounded. $\square$

6.3. **Alpha-points.** For $P \in \mathbf{T}_{>0}$, let $\mathbf{I}_{\leq P} = \mathbf{I}_{\leq P}(\mathbf{F})$ denote the P^{th} escaping set of $\mathbf{F}$.

Lemma 6.12. *Every critical point ${}_J C_S^{\blacksquare}$ admits a pair of points ${}_J\alpha_S^{\blacksquare,0}$ and ${}_J\alpha_S^{\blacksquare,\infty}$ with the following properties. For any $\bullet \in \{0, \infty\}$ and $j \in \{l, 1, \dots, 2d_\bullet - 3, r\}$, both the left and the right coasts of ${}_{J,j}\mathbf{O}_S^{\blacksquare,\bullet}$ land at ${}_J\alpha_S^{\blacksquare,\bullet}$ and*

$$\partial\, {}_{J,j}\mathbf{O}_S^{\blacksquare,\bullet} \Big\backslash \partial^c\, {}_{J,j}\mathbf{O}_S^{\blacksquare,\bullet} = \left\{ {}_J\alpha_S^{\blacksquare,\bullet} \right\}.$$

In particular, every lake is a disk and each of the spines ${}_{J,j}\mathbf{H}^\bullet$ attached to ${}_J C_S^{\blacksquare}$ is a quasiarc connecting its common root ${}_J C_S^{\blacksquare}$ to a common landing point ${}_J\alpha_S^{\blacksquare,\bullet}$. We call ${}_J\alpha_S^{\blacksquare,0}$ and ${}_J\alpha_S^{\blacksquare,\infty}$ the *inner* and *outer alpha-points* corresponding to ${}_J C_S^{\blacksquare}$.

Proof. By Corollary 6.8, there is some $Q \in \mathbf{T}$ such that $\mathbf{F}^Q(\mathbf{O})$ is either a side lake or a middle lake attached to some critical point on $\mathbf{H}$. Therefore, it is sufficient to prove the lemma for lakes of the form ${}_j\mathbf{O}_P^\bullet$ where $\bullet \in \{0, \infty\}$ and $j \in \{l, 1, \dots, 2d_\bullet - 3, r\}$.

Observe that a middle lake of the form ${}_j\mathbf{O}_P^\bullet$ is contained in some side lake ${}_k\mathbf{O}_{P-P/\mathbf{t}}^\bullet$ of generation $P - P/\mathbf{t}$ where $k \in \{l, r\}$. The composition $\mathbf{F}^{\mathbf{t}P-P} \circ A_*$ sends the pair $\left({}_j\mathbf{O}_P^\bullet, {}_k\mathbf{O}_{P-P/\mathbf{t}}^\bullet\right)$ conformally onto $({}_j\mathbf{O}_P^\bullet, \mathbf{O}^\bullet)$. In particular, $\mathbf{F}^{\mathbf{t}P-P} \circ A_*$ expands the hyperbolic metric of the ocean $\mathbf{O}^\bullet$. Since $\mathbf{I}_{\leq P} \cap \overline{{}_j\mathbf{O}_P^\bullet}$ is a $\left(\mathbf{F}^{\mathbf{t}P-P} \circ A_*\right)$-invariant compact subset of $\mathbf{O}^\bullet$, then it must be a singleton $\{\alpha_P^\bullet\}$ consisting of the unique repelling fixed point of $\mathbf{F}^{\mathbf{t}P-P} \circ A_*$ inside of ${}_k\mathbf{O}_{P-P/\mathbf{t}}^\bullet$.

We will now claim that for $j \in \{l, r\}$, the intersection $\overline{\partial^c_j\, {}_j\mathbf{O}^\bullet_P} \cap \mathbf{I}_{\le P}$ is also a compact subset of $\mathbf{O}^\bullet$. By invariance under $\mathbf{F}^{\mathbf{t}P-P} \circ A_*$, this will again imply that $\overline{\partial^c_j\, {}_j\mathbf{O}^\bullet_P} \cap \mathbf{I}_{\le P}$ is the same singleton $\{\alpha^\bullet_P\}$, and we are done.

Let us assume without loss of generality that $j = l$. Denote the left itinerary of ${}_l\mathbf{O}^\bullet_P$ by $(Q_1, Q_2, Q_3, \ldots)$. The left coast of ${}_j\mathbf{O}^\bullet_P$ by $(Q_1, Q_2, Q_3, \ldots)$ contains a unique critical point of the form ${}_kC^\bullet_{Q_1,Q_2}$. Consider power-triples $R_-, R_+ \in \mathbf{T}_{>0}$ such that the critical points ${}_kC^\bullet_{Q_1,R_-}$ and ${}_kC^\bullet_{Q_1,R_+}$ form a small interval neighborhood J of ${}_jC^\bullet_{Q_1,Q_2}$ within some spine ${}_i\mathbf{H}^\bullet_{Q_1}$ of generation Q_1. Let $B_\pm$ be spines of generation $Q_1 + R_\pm$ attached to ${}_kC^\bullet_{Q_1,R_\pm}$ that are combinatorially closest to ${}_kC^\bullet_{Q_1,Q_2}$. Let $R := Q_1 + \max\{R_+, R_-\}$. By Lemma 5.19, every connected component of $\mathbf{I}_{\le R}$ is unbounded, so the union $J \cup B_+ \cup B_- \cup \mathbf{I}_{\le R}$ separates $\partial^c_l\, {}_l\mathbf{O}^\bullet_P \setminus {}_i\mathbf{H}^\bullet_{Q_1}$ from $\mathbf{H}$. Hence, $\overline{\partial^c_l\, {}_l\mathbf{O}^\bullet_P} \cap \mathbf{I}_{\le P}$ is indeed compactly contained in $\mathbf{O}^\bullet$. □

The alpha-points ${}_J\alpha^{\blacksquare,\bullet}_S$ can be viewed as preimages of infinity under the map $\mathbf{F}^{|S|}$. They are unique in the following sense.

Lemma 6.13. *Two alpha-points ${}_J\alpha^{\blacksquare,\bullet}_S$ and ${}_{J'}\alpha^{\square,\circ}_{S'}$ coincide if and only if $J = J'$, $\blacksquare = \square$, $\bullet = \circ$, and $S = S'$.*

Proof. Suppose ${}_J\alpha^{\blacksquare,\bullet}_S = {}_{J'}\alpha^{\square,\circ}_{S'}$. Clearly, $|S| = |S'|$. Let us write $S = (P_1, \ldots, P_m)$ and $S' = (Q_1, \ldots, Q_k)$, and pick a power triple $R \in \mathbf{T}$ such that $\max\{P_1 + \ldots + P_{m-1}, Q_1 + \ldots + Q_{k-1}\} < R < |S|$. Pushing forward by $\mathbf{F}^R$ yields a pair of alpha-points $\alpha^\bullet_{|S|-R}$ and $\alpha^\circ_{|S'|-R}$, where, since they are equal, $\bullet = \circ$. If $(J, \blacksquare, S) \neq (J', \square, S')$, then this would imply that $\alpha^\bullet_{|S|-R}$ is a critical point of $\mathbf{F}^R$, which is not the case. □

By the lemma above, if two disjoint spines touch at a common alpha-point, then they are rooted at a common critical point. This yields a more precise tree structure of $\mathbf{F}^{-P}(\mathbf{H})$ in terms of spines. For convenience, we will call $\mathbf{H}$ the unique spine of generation 0.

Corollary 6.14. *Consider two distinct spines ${}_{J,j}\mathbf{H}^{\blacksquare,\bullet}_S$ and ${}_{J',j'}\mathbf{H}^{\square,\circ}_{S'}$ with $|S| \ge |S'|$.*

(1) *If the intersection $\overline{{}_{J,j}\mathbf{H}^{\blacksquare,\bullet}_S} \cap \overline{{}_{J',j'}\mathbf{H}^{\square,\circ}_{S'}}$ is non-empty, then it is a singleton consisting of the critical point ${}_JC^\blacksquare_S$.*
(2) *There is a unique sequence of pairwise different spines $B_1, \ldots, B_n$ such that $B_1 = {}_J\mathbf{H}^\blacksquare_S$, $B_n = {}_{J'}\mathbf{H}^\square_{S'}$, and $\overline{B_i}$ intersects $\overline{B_{i+1}}$ for all $i < n$.*

Let us equip the set of alpha-points with partial ordering defined as follows. Given two alpha-points α and α',

▷ if α' is within the closure of a lake attached to α, we write $\alpha \wedge \alpha' = \alpha$ and $\alpha \preceq \alpha'$;
▷ if α and α' lie in two distinct lakes attached to some alpha-point α'', we write $\alpha'' = \alpha \wedge \alpha'$ and say that α and α' are $\prec$-separated.

Given an alpha-point $\alpha = {}_J\alpha^{\blacksquare,\bullet}_S$, we define

▷ a *finite skeleton landing at* α to be the union of a spine ${}_{J,j}\mathbf{H}^{\blacksquare,\bullet}_S$ together with the unique closed quasiarc in $\mathbf{F}^{-|S|}(\mathbf{H})$ connecting ${}_JC^\blacksquare_S$ to 0;
▷ an *infinite skeleton landing at* α to be the union of $\partial^c_k\, {}_{J,k}\mathbf{O}^{\blacksquare,\bullet}_S$ for some $k \in \{l, r\}$ together with the unique closed quasiarc in $\mathbf{F}^{-|S|}(\mathbf{H})$ connecting the root of $\partial^c_k\, {}_{J,k}\mathbf{O}^{\blacksquare,\bullet}_S$ to 0.

In short, skeletons of α are the shortest paths from 0 to α within the tree of preimages of $\mathbf{H}$. Each α has $d_\bullet$ skeletons landing at α, and precisely two of them are finite.

The set of skeletons admit a total order “$<$“ which is defined as follows. Let us fix a ray γ in $\mathbf{H}$ connecting 0 to ∞. Given two distinct skeletons $\mathfrak{S}$ and $\mathfrak{S}'$,

- ▷ if γ, $\mathfrak{S}$, and $\mathfrak{S}'$ have a counterclockwise orientation around the quasiarc $\mathfrak{S} \cap \mathfrak{S}'$, we write $\mathfrak{S} < \mathfrak{S}'$;
- ▷ we say that $\mathfrak{S}$ and $\mathfrak{S}'$ are $<$-separated if there is another skeleton $\mathfrak{S}''$ such that either $\mathfrak{S} < \mathfrak{S}'' < \mathfrak{S}'$ or $\mathfrak{S}' < \mathfrak{S}'' < \mathfrak{S}$.

We say that two alpha-points α and α' are $<$-separated by an alpha-point α'' if there exists a skeleton $\mathfrak{S}''$ landing at α'' which separates every pair of skeletons $\mathfrak{S}$ and $\mathfrak{S}'$ landing at α and α' respectively.

We have equipped the set of alpha-points with two notions of ordering, namely “$<$“ and “$\prec$“. They are related as follows.

Proposition 6.15. *Consider two distinct alpha-points α and α' of generations P and P' inside of the ocean $\mathbf{O}^\bullet$ for some $\bullet \in \{0, \infty\}$. Assume $P \leq P'$. The following are equivalent.*

1. *$\alpha \prec \alpha'$;*
2. *α and α' are not $<$-separated by any alpha-point in $\mathbf{O}^\bullet$ of generation less than P;*
3. *α and α' are not $\prec$-separated.*

Proof. Suppose (1) holds. There is a lake $\mathbf{O}$ attached to α which contains α'. Clearly, this implies (3). (2) follows from the following observation. If an alpha-point $\alpha'' \in \mathbf{O}^\bullet$ $<$-separates α and α', then α'' must be contained in a proper sub-lake of $\mathbf{O}$, which necessarily has generation more than P.

Suppose (1) does not hold. Then α' is located outside of every lake attached to α. Therefore, either $\mathfrak{S}_l$ $<$-separates $\mathfrak{S}_r$ and $\mathfrak{S}'$ or $\mathfrak{S}_r$ $<$-separates $\mathfrak{S}_l$ and $\mathfrak{S}'$. Let us assume the latter.

Denote by $(c_{r,1}, c_{r,2}, \ldots)$ the infinite sequence of critical points of $\mathbf{F}^P$ of increasing generation that is found along $\mathfrak{S}_r$. Let $\alpha_{r,i}$ denote the alpha-point that is the landing point of the unique spine attached to $c_{r,i}$ that intersects $\mathfrak{S}_k$. It has generation $P_{r,i}$ where $P_{r,i} < P$ and $P_{r,i} \to P$ as $i \to \infty$. The intersection $\mathfrak{S}' \cap \mathfrak{S}_r$ is a compact subset of $\mathrm{Dom}\left(\mathbf{F}^P\right)$. Let us pick $i \gg 0$ such that for any skeleton $\mathfrak{S}_{r,i}$ landing at $\alpha_{r,i}$, $\mathfrak{S}_r \cap \mathfrak{S}'$ is a proper subset of $\mathfrak{S}_r \cap \mathfrak{S}_{r,i}$. Therefore, $\mathfrak{S}_{r,i}$ $<$-separates $\mathfrak{S}_r$ and $\mathfrak{S}'$, and so α and α' are $<$-separated by $\alpha_{r,i}$.

We have just shown that (1) and (2) are equivalent. Suppose (1) and (2) do not hold. We will prove that (3) also does not hold.

Let c and c' be the critical points which are roots $\mathfrak{S} \backslash \mathfrak{S}'$ and $\mathfrak{S}' \backslash \mathfrak{S}$ respectively; they lie on a common spine B of some generation Q. Suppose for a contradiction that $Q = P$. Then, α is the landing point of B and so $\mathfrak{S}' \backslash \mathfrak{S}$ as well as α' would be contained inside a lake attached to α. However, this would instead imply (1). Hence, $Q < P$.

Let $\hat{\mathbf{O}}$ and $\hat{\mathbf{O}}'$ denote the pair of lakes of generation Q such that their coast contains B and $\mathfrak{S} \backslash \mathfrak{S}' \subset \hat{\mathbf{O}}$ and $\mathfrak{S}' \backslash \mathfrak{S} \subset \hat{\mathbf{O}}'$. If $\hat{\mathbf{O}}$ and $\hat{\mathbf{O}}'$ are distinct, they lie on different sides of B and so α and α' is $\prec$-separated by the landing point of B.

Now, suppose instead that $\hat{\mathbf{O}} = \hat{\mathbf{O}}'$. Within the closed interval $[c, c'] \subset B$ (possibly degenerate if $c = c'$), we can find a unique critical point c'' of the smallest generation P'' where $Q < P'' \le P$. Suppose for a contradiction that $P'' = P$. Then, $\mathfrak{S}'\backslash\mathfrak{S}$ would have been contained in a lake attached to c, and so $\alpha \prec \alpha'$ which would imply (1) again. Hence, $P'' < P$.

Since $[c, c']$ does not contain any critical points of generation lower than P'', then $\mathfrak{S}\backslash\mathfrak{S}'$ and $\mathfrak{S}'\backslash\mathfrak{S}$ are contained in distinct lakes attached to c''. Consequently, the alpha-point $\alpha'' \in \hat{\mathbf{O}}$ corresponding to c'' $\prec$-separates α and α'. □

6.4. **External chains.** Let us pick a power-triple $P \in \mathbf{T}_{>0}$ and $\bullet \in \{0, \infty\}$. Let $\mathbf{O}^\bullet(P)$ denote the unique lake of generation P inside of the ocean $\mathbf{O}^\bullet$ which contains 0 on its boundary. Then, the coast of $\mathbf{O}^\bullet(P)$ intersects $\mathbf{H}$ on some interval $J \subset \mathbf{H}$ containing 0 on its interior. (In fact, J is independent of $\bullet$.) Let us denote by $\alpha^\bullet(P)$ the unique alpha-point in $\partial\mathbf{O}^\bullet(P)$. By self-similarity, $\mathbf{O}^\bullet(\mathbf{t}^n P) = A_*^n(\mathbf{O}^\bullet(P))$ for all n and

$$\bigcup_{n<0} \mathbf{O}^\bullet(\mathbf{t}^n P) = \mathbf{O}^\bullet. \tag{6.4}$$

Let us denote by $\mathbf{I}_P^\bullet$ the intersection $\mathbf{I}_{\le P} \cap \mathbf{O}^\bullet$ for $\bullet \in \{0, \infty\}$.

Lemma 6.16. *For every $\bullet \in \{0,\infty\}$ and $P, Q \in \mathbf{T}_{>0}$ such that $P < Q$,*

(1) *$\mathbf{I}_P^\bullet$ is connected;*
(2) *$\mathbf{I}_Q^\bullet \backslash \mathbf{I}_P^\bullet$ is bounded;*
(3) *every connected component of $\mathbf{I}_Q^\bullet \backslash \mathbf{I}_P^\bullet$ is a lift of a component of $\mathbf{I}_{\le Q-P}$ under $\mathbf{F}^P$, contained in a unique lake $\mathbf{O}$ of generation P, and attached to the alpha-point of $\mathbf{O}$.*

Proof. Suppose a component I of $\mathbf{I}_{\le P}$ intersects $\mathbf{O}^\bullet(\mathbf{t}^k P)$ for some maximal $k \in \mathbb{N}$. By Lemma 5.19, since I intersects $\mathbf{O}^\bullet(\mathbf{t}^n P)$ for all $n \le k$, then it intersects the alpha-point $\alpha^\bullet(\mathbf{t}^n P)$ for all $n \le k$. Therefore, $\mathbf{I}_{\le P}$ is connected.

Let us consider a connected component L of $\mathbf{I}_Q^\bullet \backslash \mathbf{I}_P^\bullet$. Since L avoids $\alpha^\bullet(\mathbf{t}^n P)$ for all $n \ll 0$, it must be contained inside of the lake $\mathbf{O}^\bullet(\mathbf{t}^k P)$ for all $n \ll 0$, and so L is bounded. Since L avoids $\mathbf{F}^{-P}$ and alpha-points of generation P, L is contained in a unique lake $\mathbf{O}$ of generation P. Since $\mathbf{F}^P$ sends $\mathbf{O}$ conformally onto the ocean $\mathbf{O}^\bullet$ for some $\bullet \in \{0, \infty\}$, then $\mathbf{F}^P(L) = \mathbf{I}^\bullet_{\le Q-P}$. By unboundedness, L must be attached to the alpha-point of $\mathbf{O}$. □

Consider two alpha-points α and α' in the same ocean $\mathbf{O}^\bullet$ with generation P and P' respectively and suppose $P < P'$ and $\alpha \prec \alpha'$. We can define the *external chain* $[\alpha, \alpha']$ to be the set of points in $\mathbf{I}^\bullet_{\le P'}$ that are inside the closure of the lakes attached to α and outside of any lake that does not contain α'.

Lemma 6.17. *For any alpha-points $\alpha, \alpha', \alpha''$ satisfying $\alpha \prec \alpha' \prec \alpha''$,*

$$[\alpha, \alpha'] \cap [\alpha', \alpha''] = \{\alpha'\} \quad \textit{and} \quad [\alpha, \alpha'] \cup [\alpha', \alpha''] = [\alpha, \alpha''].$$

Proof. The first equation follows from the fact that α' is a cut point with respect to the "$\prec$" ordering. The inclusion $[\alpha, \alpha'] \cup [\alpha', \alpha''] \subset [\alpha, \alpha'']$ is obvious. Consider a point x in $[\alpha, \alpha'']\backslash[\alpha, \alpha']$. We know that x is within a lake attached to α. If x is inside of a lake that does not contain α', then this lake avoids all lakes attached to α' and in particular does not contain α'' as well, which is a contradiction. Therefore, $x \in [\alpha', \alpha'']$. □

For $P \in \mathbf{T}_{>0}$, we say that the critical point $C_P \in \mathbf{H}$ is *dominant* if the interval $[0, C_P] \subset \mathbf{H}$ does not contain any critical point of generation less than P. We will enumerate dominant critical points by $\{C_{P_n}\}_{n\in\mathbb{Z}}$ where $\{P_n\}_{n\in\mathbb{Z}}$ is monotonically increasing in n.

Lemma 6.18. *For* $\bullet \in \{0,\infty\}$, $\dots \prec \alpha^\bullet_{P_{-2}} \prec \alpha^\bullet_{P_{-1}} \prec \alpha^\bullet_{P_0} \prec \alpha^\bullet_{P_1} \prec \alpha^\bullet_{P_2} \prec \dots$.

Proof. Suppose for a contradiction that $\alpha^\bullet_{P_n} \nprec \alpha^\bullet_{P_{n+1}}$ for some $\bullet \in \{0,\infty\}$ and $n \in \mathbb{Z}$. By Propositon 6.15, there is an alpha-point $\alpha \in \mathbf{O}^\bullet$ of some generation P less than P_n which $\prec$-separates $\alpha^\bullet_{P_n}$ and $\alpha^\bullet_{P_{n+1}}$. Suppose α is contained in the closure of a lake attached to the critical point $C_Q \in \mathbf{H}$ of some generation $Q \le P$. Any skeleton landing at α is disjoint from the closure of every spine landing at either $\alpha^\bullet_{P_n}$ or $\alpha^\bullet_{P_{n+1}}$, so then C_Q is contained in the interval $(C_{P_n}, C_{P_{n+1}}) \subset \mathbf{H}$. However, this would contradict the assumption that C_{P_n} and $C_{P_{n+1}}$ are dominant. ☐

For $P \in \mathbb{R}_{>0}\backslash\mathbf{T}$, we can define $\mathbf{I}_{\le P}$ to be $\mathbf{I}_{\le P} := \bigcap_{Q\in\mathbf{T}, Q>P} \mathbf{I}_{\le Q}$. Given any point $x \in \mathbf{I}_{<\infty}$, we define the *escaping time* of x to be the minimum time $P \in \mathbb{R}_{>0}$ such that $x \in \mathbf{I}_{\le P}$.

We define the *inner* and *outer zero chains* to be

$$\mathbf{R}^0 = \bigcup_{n\in\mathbb{Z}} [\alpha^0_{P_n}, \alpha^0_{P_{n+1}}] \quad \text{and} \quad \mathbf{R}^\infty = \bigcup_{n\in\mathbb{Z}} [\alpha^\infty_{P_n}, \alpha^\infty_{P_{n+1}}]$$

respectively.

Proposition 6.19. *For* $\bullet \in \{0,\infty\}$,

(1) $\mathbf{R}^\bullet$ *is* A_**-invariant;*
(2) $\mathbf{R}^\bullet$ *is an arc landing at* 0*;*
(3) *alpha-points are dense on* $\mathbf{R}^\bullet$*;*
(4) *points on* $\mathbf{R}^\bullet$ *are continuously parametrized by their escaping time ranging from* 0 *(near* ∞*) to* $+\infty$ *(near* 0*).*

Proof. To lighten the notation, we will denote $\alpha^\bullet_n := \alpha^\bullet_{P_n}$ for all $\bullet \in \{0,\infty\}$ and $n \in \mathbb{Z}$.

By definition, C_P is dominant if and only if $C_{\mathbf{t}P} = A_*(C_P)$ is dominant, so there is some integer $k \ge 1$ such that $\mathbf{t}P_n = P_{n+k}$ for all $n \in \mathbb{Z}$. As a consequence, A_* maps each of $[\alpha^\bullet_{(n-1)k}, \alpha^\bullet_{nk}]$ onto $[\alpha^\bullet_{nk}, \alpha^\bullet_{(n+1)k}]$. This implies A_*-invariance and that $\mathbf{R}^\bullet$ accumulates at 0.

Due to self-similarity, it is sufficient for us to show that the external chain $J := [\alpha_0, \alpha_k]$ is an arc that can be continuously parametrized by their escaping time, and that alpha-points are dense on J. We will do so by constructing nested Markov tiling $\mathcal{P}_r$ for $r \ge 0$ on J.

The tiling $\mathcal{P}_0$ of level 0 consists of external chains $J_i := [\alpha^\bullet_i, \alpha^\bullet_{i+1}]$ for all $i \in \{0, 1, \dots k-1\}$. The tiling $\mathcal{P}_1$ of level 1 is constructed as follows. By Lemma B.8, for every chain $J_i \in \mathcal{P}_0$, there exist some $Q_i \in \mathbf{T}_{>0}$ and a pair of integers l_i and r_i such that $0 < l_i < r_i \le i$ and $\mathbf{F}^{Q_i}$ maps J_i homeomorphically onto the chain $[\alpha^\bullet_{l_i}, \alpha^\bullet_{r_i}]$. A tile of level 1 in $\mathcal{P}_1$ is the preimage of a chain of the form $[\alpha^\bullet_j, \alpha^\bullet_{j+1}]$ under the map $\mathbf{F}^{Q_i} : J_i \to [\alpha^\bullet_{l_i}, \alpha^\bullet_{r_i}]$.

For each tile $I \in \mathcal{P}_1$ in J_i, there exists some $m_I \in \mathbb{N}$ such that $A_*^{m_I}$ sends $\mathbf{F}^{Q_i}(I)$ back to a tile of level 0. Let $\mathbf{O}_i$ denote the lake of generation Q_i which contains $[\alpha^\bullet_{l_i}, \alpha^\bullet_{r_i}]$. The composition

$$\chi_I := A_*^{m_I} \circ \mathbf{F}^{Q_i} : \mathbf{O}_i \to \mathbf{O}^\bullet \tag{6.5}$$

expands the hyperbolic metric of $\mathbf{O}^\bullet$.

Inductively, we define tiles in $\mathcal{P}_{n+1}$ of level $n+1$ to be the preimages of tiles of level n under maps of the form (6.5). Since each map χ_I is expanding, the diameter of every tile of level n uniformly exponentially shrinks to zero. Since each tile in $\mathcal{P}_n$ is an external chain containing alpha-points, alpha-points are dense on J.

By Lemma 6.17, we can enumerate our level n tiles by $I^n_1, I^n_2, \dots, I^n_{s_n} \in \mathcal{P}_n$ in increasing order of generation such that I^n_i and I^n_l touch if and only if $|l-i| \le 1$. As tiles shrink, we can extend the "$\prec$" order to a total order on J by defining $x \prec y$ when $x \in I^n_i$ and $y \in I^n_j$ for sufficiently high n where $i < j$.

Consider a tile I^n_i in $\mathcal{P}_n$ of some high level n. Since Consider a composition $\chi := \chi_1 \circ \chi_2 \circ \dots \circ \chi_n$ of n maps of the form (6.5) sending I^n_i onto a tile of level 0. By (5.4), we can write χ as $A_*^{m(n,i)} \circ \mathbf{F}^{Q(n,i)}$ for some $m(n,i) \in \mathbb{N}$ and $Q(n,i) \in \mathbf{T}_{>0}$. Therefore, the difference in the escaping time between the endpoints of I^n_i is at most

$$\mathbf{t}^{-m(r,i)}(P_k - P_0). \tag{6.6}$$

Since $Q_i > 0$ for all $i \in \{0, \dots, k-1\}$, there exists some uniform $M \ge 1$ such that every sequence of M consecutive integers between 1 and s_n admits an element j_* such that χ_{j_*} has the scaling factor A_* in (6.5). As a consequence, as $n \to \infty$, $\min_{1 \le i \le s_n} m(n,i) \to \infty$ and thus the quantity in (6.6) tends to zero. Therefore, the escaping time continuously parametrizes points on J. □

In general, for every alpha-point α, there is an infinite sequence of alpha-points $\alpha_0 = \alpha, \alpha_{-1}, \alpha_{-2}, \dots$ of generation decreasing to 0 such that $\dots \prec \alpha_{-2} \prec \alpha_{-1} \prec \alpha_0$. This allows us to generate the chain

$$(\infty, \alpha] := \bigcup_{n \le 0} [\alpha_{n-1}, \alpha_n].$$

Corollary 6.20. *Consider any alpha-point α of some generation $P > 0$. The chain $(\infty, \alpha]$ is an infinite arc continuously parametrized by the escape time from $|P|$ to 0. Moreover, alpha-points are dense in $(\infty, \alpha]$.*

Proof. Suppose first that α is of the form $\alpha^\bullet_P$ for some $P \in \mathbf{T}_{>0}$ and $\bullet \in \{0, \infty\}$. Let us pick a dominant $\alpha^\bullet_{P_n}$ for some $n \in \mathbb{Z}$ such that $P_n \ge P$. There is a unique point $x \in \left(\infty, \alpha^\bullet_{P_n}\right]$ of generation $P_n - P$. Then, $\mathbf{F}^{P_n - P}$ maps the arc $\left(x, \alpha^\bullet_{P_n}\right]$ onto $(\infty, \alpha^\bullet_P]$, which implies the claim.

In general, let $\alpha = {}_J\alpha^{\blacksquare, \bullet}_S$ where $S = (P_1, P_2, \dots, P_k)$ is the corresponding itinerary. There exist alpha-points $\alpha_1, \alpha_2, \dots, \alpha_k = \alpha$ such that $\alpha_1 \prec \alpha_2 \prec \dots \prec \alpha_k$ and for each i, α_i has itinerary $S_i := (P_1, \dots, P_i)$. Therefore, we can split $(\infty, \alpha]$ into $J_1 = (\infty, \alpha_1]$, $J_2 = (\alpha_1, \alpha_2]$, $\dots$, $J_k = (\alpha_{k-1}, \alpha_k)$. When $i \ge 2$, the map $\mathbf{F}^{P_1 + \dots + P_{i-1}}$ maps J_i homeomorphically onto the chain $\left(\infty, \alpha^\bullet_{P_i}\right]$. By the previous paragraph, each J_i is an arc continuously parametrized by the landing time. □

As a consequence, whenever $\alpha \prec \alpha'$, then the chain $[\alpha, \alpha']$ is a simple arc.

Definition 6.21. An *external ray* is an infinite arc of the form $\mathbf{R} = \bigcup_{n \in \mathbb{Z}} [\alpha_n, \alpha_{n+1}]$ for some sequence of alpha-points $\{\alpha_n\}_{n \in \mathbb{Z}}$ where

- ▷ $\alpha_n \prec \alpha_{n+1}$ for all n;
- ▷ the generation of α_n decreases to 0 as $n \to -\infty$;
- ▷ there is no alpha-point α such that $\alpha_n \prec \alpha$ for all $n \in \mathbb{Z}$.

The *generation* of $\mathbf{R}$ is the limit of the generation of α_n as $n \to +\infty$. We define the image of an external ray $\mathbf{R}$ under $\mathbf{F}^P$ by

$$\mathbf{F}^P(\mathbf{R}) := \mathbf{F}^P\left(\mathbf{R} \cap \mathrm{Dom}\left(\mathbf{F}^P\right)\right).$$

We say that $\mathbf{R}$ is *periodic* if $\mathbf{F}^P(\mathbf{R}) = \mathbf{R}$ for some $P \in \mathbf{T}_{>0}$.

The zero chains $\mathbf{R}^0$ and $\mathbf{R}^\infty$ are indeed external rays, which from now on will be referred to as *zero rays*.

The following is an immediate consequence of Proposition 6.15.

Corollary 6.22. *The intersection of any two external rays in the same ocean is non-empty and of the form $(\infty, \alpha]$ for some alpha-point α.*

6.5. **Wakes.** Consider a zero ray $\mathbf{R}^\bullet$ where $\bullet \in \{0, \infty\}$. As $\mathbf{R}^\bullet$ lands at the critical value 0, for any power-triple $P \in \mathbf{T}_{>0}$, there are precisely $d_\bullet$ external rays

$$_1\mathbf{R}^\bullet_P,\ _2\mathbf{R}^\bullet_P,\ \ldots,\ _{d_\bullet}\mathbf{R}^\bullet_P \tag{6.7}$$

in $\mathbf{O}^\bullet$ landing at the critical point C_P which are preimages of $\mathbf{R}^\bullet$ under $\mathbf{F}^P$. We assume that the rays in (6.7) are labelled in counterclockwise order about C_P. The intersection of each of these rays is precisely the external ray segment $(\infty, \alpha^\bullet_P]$.

For $j \in \{1, \ldots, d_\bullet - 1\}$, denote the truncated arc

$$_j\hat{\mathbf{R}}^\bullet_P := {_j\mathbf{R}^\bullet_P} \backslash (\infty, \alpha^\bullet_P].$$

The union $\{C_P, \alpha^\bullet_P\} \cup {_j\hat{\mathbf{R}}^\bullet_P} \cup {_{j+1}\hat{\mathbf{R}}^\bullet_P}$ is a Jordan curve bounding an open Jordan disk $_j\mathbf{W}^\bullet_P$, which we will refer to as a *primary wake* of generation P rooted at C_P. Due to the tree structure of $\mathbf{I}_{<\infty}$, primary wakes are always pairwise disjoint.

In general, given a critical point $_JC^{\blacksquare}_S$, we define for each $j \in \{1, \ldots, d_\bullet - 1\}$ the arc $_{J,j}\hat{\mathbf{R}}^{\blacksquare,\bullet}_S$ to be the unique lift under the map $\mathbf{F}^{|S|-P}$, for sufficiently small $P \in \mathbf{T}_{>0}$, which connects $_JC^{\blacksquare}_S$ and $_J\alpha^{\blacksquare,\bullet}_S$. The union $\left\{{_JC^{\blacksquare}_S}, {_J\alpha^{\blacksquare,\bullet}_S}\right\} \cup {_{J,j}\hat{\mathbf{R}}^{\blacksquare,\bullet}_S} \cup {_{J,j+1}\hat{\mathbf{R}}^{\blacksquare,\bullet}_S}$ is a Jordan curve bounding an open disk $_{J,j}\mathbf{W}^{\blacksquare,\bullet}_S$, called a *wake*. We call the alpha-point $_J\alpha^{\blacksquare,\bullet}_S$ the *top point* of the wake $_{J,j}\mathbf{W}^{\blacksquare,\bullet}_S$. If S is a tuple of length $m \in \mathbb{N}$, we say that m is the *level* of the wake $_{J,j}\mathbf{W}^{\blacksquare,\bullet}_S$.

Lemma 6.23. *Consider a wake $_{J,j}\mathbf{W}^{\blacksquare,\bullet}_S$ rooted at a critical point $_JC^{\blacksquare}_S$.*

(1) *If $\mathbf{F}^Q$ sends $_JC^{\blacksquare}_S$ to another critical point $_{J'}C^{\square}_{S'}$, then $\mathbf{F}^Q : \overline{_{J,j}\mathbf{W}^{\blacksquare,\bullet}_S} \to \overline{_{J',j}\mathbf{W}^{\square,\bullet}_{S'}}$ is a homeomorphism.*
(2) *The map $\mathbf{F}^{|S|}$ conformally sends $_{J,j}\mathbf{W}^{\blacksquare,\bullet}_S$ onto $\mathbb{C} \backslash \overline{\mathbf{R}^\bullet}$.*

Proof. The first claim follows from the fact that $\mathbf{F}^Q$ maps $_{J,j}\hat{\mathbf{R}}^{\blacksquare,\bullet}_S \cup {_{J,j+1}\hat{\mathbf{R}}^{\blacksquare,\bullet}_S}$ homeomorphically onto $_{J',j}\hat{\mathbf{R}}^{\square,\bullet}_{S'} \cup {_{J',j+1}\hat{\mathbf{R}}^{\square,\bullet}_{S'}}$, and the second claim follows from the fact that $\mathbf{F}^{|S|}$ maps $_{J,j}\hat{\mathbf{R}}^{\blacksquare,\bullet}_S$ for each $j \in \{1, \ldots, d_\bullet\}$ homeomorphically onto the zero ray $\mathbf{R}^\bullet$. □

To reduce notation, we consider the *full wake*

$$_J\mathbf{W}^{\blacksquare,\bullet}_S := \bigcup_{j=1}^{d_\bullet - 1} {_{J,j}\mathbf{W}^{\blacksquare,\bullet}_S}(j)$$

and present its boundary as

$$\left\{{_JC^{\blacksquare}_S}\right\} \cup {^l_J\mathbf{E}^{\blacksquare,\bullet}_S} \cup \left\{{_J\alpha^{\blacksquare,\bullet}_S}\right\} \cup {^r_J\mathbf{E}^{\blacksquare,\bullet}_S}$$

where ${}^{r}_{J}\mathbf{E}^{\blacksquare,\bullet}_{S} := {}_{J,1}\hat{\mathbf{R}}^{\blacksquare,\bullet}_{S}$ and ${}^{l}_{J}\mathbf{E}^{\blacksquare,\bullet}_{S} := {}_{J,d_\bullet}\hat{\mathbf{R}}^{\blacksquare,\bullet}_{S}$.

Let us denote $\mathbf{J}_n := \mathbf{H} \cap \mathbf{S}^{\#}_n$ for $n \in \mathbb{Z}$.

Lemma 6.24 (Primary wakes shrink)**.** *For every $n \in \mathbb{Z}$ and every $\varepsilon > 0$, there are at most finitely many primary wakes of diameter at most ε rooted at a point on $\mathbf{J}_n$.*

Proof. The proof we present below is similar to [DL23, Lemma 5.29]. By self-similarity, it is sufficient to prove the lemma for $n = 0$. Let $\mathbf{J}_- := \mathbf{U}_- \cap \mathbf{H}$ and $\mathbf{J}_+ := \mathbf{U}_+ \cap \mathbf{H}$. Then, $\mathbf{f}_- = \mathbf{F}^{(0,1,0)} : \mathbf{J}_- \to \mathbf{J}$ and $\mathbf{f}_+ = \mathbf{F}^{(0,0,1)} : \mathbf{J}_- \to \mathbf{J}$ are precisely the first return maps of $\mathbf{F}$ back to $\mathbf{J}$.

Consider the semigroup generated by $(0,1,0)$ and $(0,0,1)$ and let us label its elements by $0, Q_0, Q_{-1}, Q_{-2}, \ldots$ written in increasing order. Then, every critical point on $\mathbf{J}$ is of the form C_{Q_n} for some $n \le 0$. Let us fix $\bullet \in \{0,\infty\}$ and consider the full wake $\mathbf{W}_n := \mathbf{W}^{\bullet}_{Q_n}$ attached to C_{Q_n}. For all $n < 0$, $\mathbf{W}_n$ is a preimage under $\mathbf{F}^{Q_n - Q_0}$ of the wake $\mathbf{W}_0$ with the smallest generation.

Let $\mathbf{O}_- \subset \mathbf{O}^\bullet$ be the union of all lakes of generation $(0,1,0)$ whose closure intersects $\mathbf{J}_-$, and let $\mathbf{O}_+ \subset \mathbf{O}^\bullet$ be the union of all lakes of generation $(0,0,1)$ whose closure intersects $\mathbf{J}_+$. The maps $\mathbf{f}_\pm : \mathbf{O}_\pm \to \mathbf{O}^\bullet$ expand the hyperbolic metric of $\mathbf{O}^\bullet$.

Let us pick a curve Γ_0 on $\mathbf{W}_0$ connecting a point $y_0 \in \mathbf{W}_0$ to the critical point C_{Q_0}. Consider the lift Γ_n of Γ_0 under $\mathbf{F}^{Q_n - Q_0}$ connecting the point $y_n \in \mathbf{W}_n$ to the critical point C_{Q_n}. Similar to [DL23, Lemma 5.29], it is sufficient to prove the following claim.

Claim 1. There is a sequence $\varepsilon_0, \varepsilon_{-1}, \varepsilon_{-2}, \ldots$ of positive numbers decreasing to 0 such that the following holds. If the Euclidean diameter of Γ_0 is less than ε_0, then the Euclidean diameter of Γ_n is less than ε_n for all $n \in \mathbb{N}$.

Proof. It is sufficient to prove the claim in the dynamical plane of the corona f_*. Consider the rational map g from Theorem 3.3 which admits a (d_0, d_∞)-critical Herman quasicircle $\mathbf{H}_g$ with rotation number equal to that of f_*. By Theorem 3.7, g is quasiconformally conjugate to f_* on a neighborhood of $\mathbf{H}_g$, so it suffices to prove the claim in the dynamical plane of g. We shall do so using local connectivity of the boundary of the immediate basin of attraction of $\bullet$ of g.

For $k \ge 0$, let us denote by $c_k := (g|_{\mathbf{H}_g})^k(1)$ the critical point of g^{k+1} on $\mathbf{H}_g$. Within the immediate basin of $\bullet$, let us pick two external rays R_l and R_r landing at points on $\mathbf{H}_g$ that are slightly on the left and right of c_0 respectively. Let us pick a disk D_0 of small diameter bounded by $\mathbf{H}_g$, R_l, R_r, and an equipotential within the immediate basin of $\bullet$. Let D_k be the unique lift of D_0 under g^k whose boundary contains c_k. The disk D_k is bounded by $g^{-k}(\mathbf{H})$, a pair of external rays which are preimages of R_l and R_r, and an equipotential of an even smaller level. By local connectivity, the Euclidean diameter of D_k shrinks to zero. □

□

For every $P \in \mathbf{T}_{>0}$, let $P^l \in \mathbf{T}_{<0}$ denote the unique power-triple less than P such that C_{P^l} is on the left of C_P and that the open interval in $\mathbf{H}$ between C_{P^l} and C_P contains no critical points of $\mathbf{F}^P$. In other words, the left itinerary of any left side lake attached to C_P starts with P^l. Similarly, we denote by P^r the first entry of the right itinerary of any right side lake attached to C_P.

Lemma 6.25 (Combinatorics of primary wakes)**.** *Given $P \in \mathbf{T}_{>0}$ and $\bullet \in \{0,\infty\}$,*

(1) *both ${}^{l}\mathbf{E}^{\bullet}_{P^l}$ and ${}^{r}\mathbf{E}^{\bullet}_{P^r}$ contain $\alpha^{\bullet}_{P}$;*

(2) *the union of the closures of* $\mathbf{W}^\bullet_P$, ${}_1\mathbf{W}^\bullet_{P^l}$, *and* ${}_{d_\bullet-1}\mathbf{W}^\bullet_{P^r}$ *is a neighborhood of* $\alpha^\bullet_P$;

(3) *the ray segments* ${}^l\mathbf{E}^\bullet_P$ *and* ${}^r\mathbf{E}^\bullet_P$ *can be presented as infinite concatenations of ray segments*

$$
\begin{aligned}
{}^l\mathbf{E}^\bullet_P &= \left[\alpha^\bullet_P, \alpha^\bullet_{Q^l_1}\right] \cup \left[\alpha^\bullet_{Q^l_1}, \alpha^\bullet_{Q^l_2}\right] \cup \left[\alpha^\bullet_{Q^l_2}, \alpha^\bullet_{Q^l_3}\right] \cup \dots, \\
{}^r\mathbf{E}^\bullet_P &= \left[\alpha^\bullet_P, \alpha^\bullet_{Q^r_1}\right] \cup \left[\alpha^\bullet_{Q^r_1}, \alpha^\bullet_{Q^r_2}\right] \cup \left[\alpha^\bullet_{Q^r_2}, \alpha^\bullet_{Q^r_3}\right] \cup \dots,
\end{aligned}
$$

where

$$
{}^l\mathbf{E}^\bullet_P \cap {}^r\mathbf{E}^\bullet_{P^l} = \left[\alpha^\bullet_P, \alpha^\bullet_{Q^l_1}\right], \qquad {}^r\mathbf{E}^\bullet_P \cap {}^l\mathbf{E}^\bullet_{P^r} = \left[\alpha^\bullet_P, \alpha^\bullet_{Q^r_1}\right],
$$

and for all $i \geq 1$,

$$
{}^l\mathbf{E}^\bullet_P \cap {}^r\mathbf{E}^\bullet_{Q^l_i} = \left[\alpha^\bullet_{Q^l_i}, \alpha^\bullet_{Q^l_{i+1}}\right], \qquad {}^r\mathbf{E}^\bullet_P \cap {}^l\mathbf{E}^\bullet_{Q^r_i} = \left[\alpha^\bullet_{Q^r_i}, \alpha^\bullet_{Q^r_{i+1}}\right];
$$

(4) *the sequences of alpha-points* $\left\{\alpha^\bullet_{Q^l_i}\right\}_{i\geq 1}$ *and* $\left\{\alpha^\bullet_{Q^r_i}\right\}_{i\geq 1}$ *tend to* C_P *as* $i \to \infty$.

See Figures 8 and 9.

Proof. The left coast of the left side lake of C_P is rooted at C_{P^l} and is thus contained in ${}_1\mathbf{W}^\bullet_{P^l}$. Since it lands at the alpha-point $\alpha^\bullet_P$, the boundary of the wake ${}_1\mathbf{W}^\bullet_{P^l}$ must contain $\alpha^\bullet_{P^l}$. The treatment for the wake ${}_{d_\bullet-1}\mathbf{W}^\bullet_{P^r}$ is analogous.

By Corollary 6.22, the intersection ${}^r\mathbf{E}^\bullet_{P^l} \cap {}^l\mathbf{E}^\bullet_{P^r}$ is a ray segment $[\alpha', \alpha^\bullet_P]$ for some $\alpha' \prec \alpha^\bullet_P$. Similarly, we also have that ${}^r\mathbf{E}^\bullet_{P^l} \cap {}^l\mathbf{E}^\bullet_P = [\alpha^\bullet_P, \alpha(l)]$ and ${}^r\mathbf{E}^\bullet_P \cap {}^l\mathbf{E}^\bullet_{P^r} = [\alpha^\bullet_P, \alpha(r)]$ where $\alpha^\bullet_P \prec \alpha(l)$ and $\alpha^\bullet_P \prec \alpha(r)$. Therefore, the union of ${}_1\mathbf{W}^\bullet_{P^l}$, $\mathbf{W}^\bullet_P$, and ${}_{d_\bullet-1}\mathbf{W}^\bullet_{P^r}$ form a neighborhood of $\alpha^\bullet_P$. This implies that every primary alpha-point is the meeting point of exactly three distinct primary full wakes.

Let us prove (3) and (4) for ${}^l\mathbf{E}^\bullet_P$. The treatment for ${}^r\mathbf{E}^\bullet_P$ is analogous. By the previous paragraph, the alpha-point $\alpha(l)$ must be of the form $\alpha^\bullet_{Q^l_1}$ for some $Q^l_1 > P$, as it is the meeting point of ${}^l\mathbf{E}^\bullet_P$, ${}^r\mathbf{E}^\bullet_{P^l}$, and the boundary of a primary full wake, which is $\mathbf{W}^\bullet_{Q^l_1}$. Similarly, ${}^r\mathbf{E}^\bullet_{Q^l_1}$ and ${}^l\mathbf{E}^\bullet_P$ meet along a ray segment $\left[\alpha^\bullet_{Q^l_1}, \alpha^\bullet_{Q^l_2}\right]$ for some $Q^l_2 > Q^l_1$. Inductively, we obtain the desired increasing sequence $\left\{Q^l_i\right\}_{i\in\mathbb{N}}$ of power-triples. It remains to show that the corresponding sequence of alpha-points $\alpha^\bullet_{Q^l_i}$ converges to C_P.

By Proposition 6.19 (2), there exists an alpha-point $\alpha \in {}^l\mathbf{E}^\bullet_P$ close to C_P. From the above, α is the top of some primary full wake $\mathbf{W}^\bullet_Q$ where $Q > P$. Since there are at most finitely many critical points on $\mathbf{H}$ of generation less than Q between C_Q and C_P, the arc $[\alpha^\bullet_P, \alpha^\bullet_Q]$ intersects the boundaries of at most finitely many primary wakes. Therefore, $Q = Q^l_i$ for some $i \in \mathbb{N}$. Since α can be picked to be arbitrarily close to C_P, then $\alpha^\bullet_{Q^l_i}$ indeed converges to C_P. □

Corollary 6.26 (Tiling of wakes)**.**

(1) *Primary wakes fill up the ocean. More precisely, for* $\bullet \in \{0, \infty\}$,

$$
\mathbf{O}^\bullet \subset \bigcup_{P \in \mathbf{T}_{>0}} \overline{\mathbf{W}^\bullet_P}.
$$

(2) *The closure* $\overline{{}_{J,j}\mathbf{W}^{\blacksquare,\bullet}_S}$ *of a wake is the union of spines* ${}_{J,2j-1}\mathbf{H}^{\blacksquare,\bullet}_S$ *and* ${}_{J,2j}\mathbf{H}^{\blacksquare,\bullet}_S$ *and the closure of all full wakes rooted at critical points on any of these two spines.*

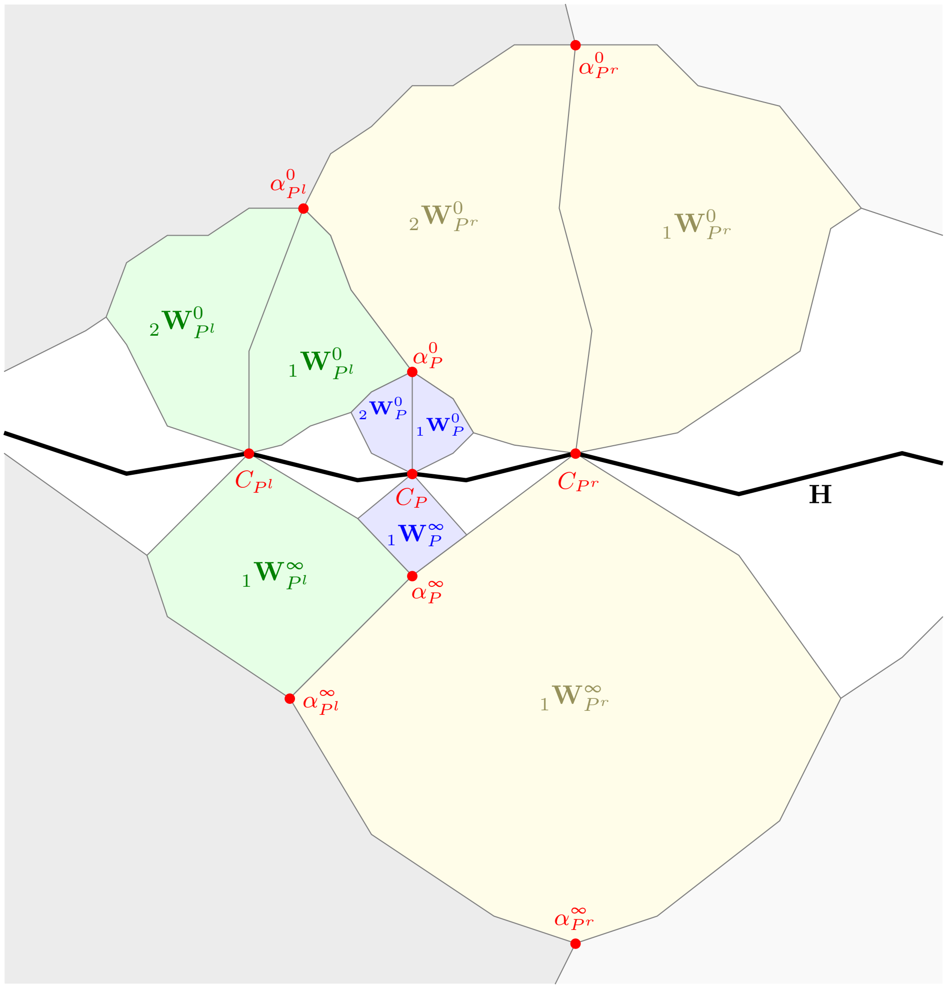

FIGURE 8. A cartoon picture of the structure of wakes when $d_0 = 3$ and $d_\infty = 2$. A more realistic picture can be found in Figure 9.

(3) *For any finite-time escaping point $z \in \mathbf{I}_{<\infty}$ and any $m \in \mathbb{N}_{\geq 1}$, there are at most three disjoint full wakes of level $\geq m$ containing z on their boundaries. The union of the closure of these full wakes forms a neighborhood of z.*

Proof. Suppose for a contradiction that there is a non-empty connected component Y of $\mathbf{O}^\bullet \setminus \bigcup_P \overline{\mathbf{W}^\bullet_P}$. By Lemma 6.25, the closure of Y intersects some point x on $\mathbf{H}$. There exists two sequences of primary full wakes $\mathbf{W}^\bullet_{Q_n}$ and $\mathbf{W}^\bullet_{T_n}$ such that for all $n \in \mathbb{N}$, $\mathbf{W}^\bullet_{Q_n}$ and $\mathbf{W}^\bullet_{T_n}$ touch, Y is contained in the unique bounded connected component D_n of $\mathbb{C}\setminus\left(\mathbf{H} \cup \overline{\mathbf{W}^\bullet_{Q_n}} \cup \overline{\mathbf{W}^\bullet_{T_n}}\right)$, and the corresponding roots C_{Q_n} and C_{T_n} converge to x as $n \to \infty$. By Lemma 6.24, the diameter of D_n tends to 0 as $n \to \infty$, which implies that such Y does not exist.

Item (2) follows from pulling back the tiling of wakes in (1) by the map $\mathbf{F}^{|S|}$ on $\overline{{}_{J,j}\mathbf{W}^{\blacksquare,\bullet}_S}$. We have thus shown that wakes of a fixed level tile each of the two oceans,

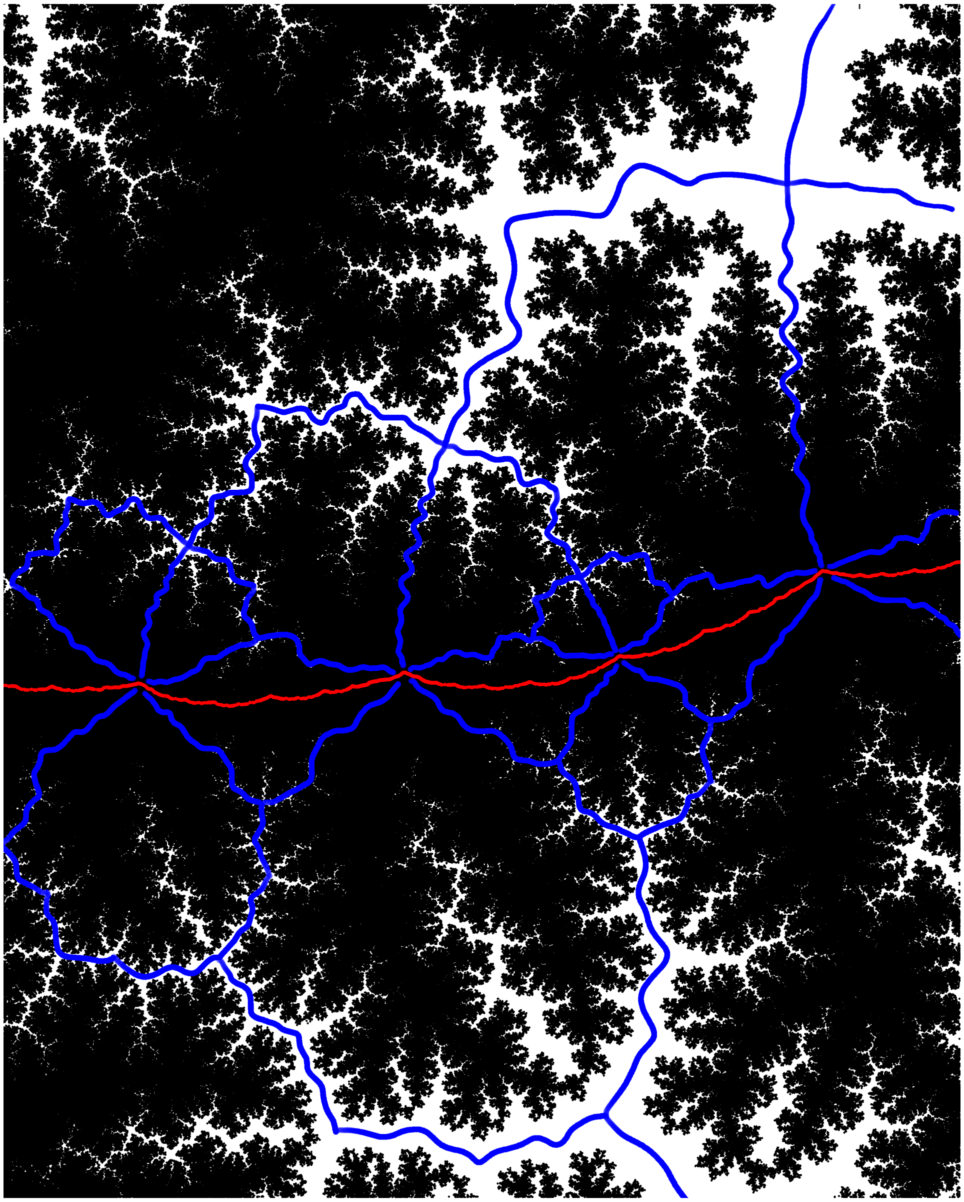

FIGURE 9. An approximate picture of the dynamical plane of $\mathbf{F}_*$ when $d_0 = 3$, $d_\infty = 2$, and θ is the golden mean irrational. This figure is obtained from the magnification of the Julia set of the rational map $f_{3,2}$ in Figure 1 around a point on its Herman quasicircle. The Herman quasicircle $\mathbf{H}$ of $\mathbf{F}_*$ is colored red and some external ray segments are displayed in blue. These external rays are the boundaries of the primary wakes attached to four critical points on $\mathbf{H}$.

and every point in the ocean is contained in the closure of at most three wakes of the same level. This implies (3). ☐

Lemma 6.27. *For every primary full wake $\mathbf{W}^\bullet_P$,*

1. *the map $\mathbf{F}^P : \mathbf{W}^\bullet_P \backslash \mathbf{F}^{-P}(\mathbf{H}) \to \mathbb{C}\backslash\mathbf{H}$ is uniformly expanding with respect to the hyperbolic metric of $\mathbb{C}\backslash\mathbf{H}$; the expansion factor is at least some constant independent of P;*
2. *the hyperbolic diameter (with respect to the metric of $\mathbb{C}\backslash\mathbf{H}$) of every wake of level two is at most some uniform constant independent of P.*

Proof. For all $P \in \mathbf{T}$, let ρ_P be the hyperbolic metric of $\mathbb{C}\backslash\mathbf{F}^{-P}(\mathbf{H})$. To prove (2), it suffices to show that the inclusion map $\iota : \left(\mathbb{C}\backslash\mathbf{F}^{-P}(\mathbf{H}), \rho_P\right) \to (\mathbb{C}\backslash\mathbf{H}, \rho_0)$ is uniformly contracting on $\mathbf{W}^\bullet_P \backslash \mathbf{F}^{-P}(\mathbf{H})$.

Clearly, ι is uniformly contracting on $\mathbf{W}^\bullet_P$ minus a small neigborhood of C_P because this region is a compact subset of $\mathbf{O}^\bullet$. The uniform contraction of ι on a neighborhood of C_P follows from asymptotic self-similarity of $\mathbf{H}$ and $\partial\mathbf{W}^\bullet_P$ near C_P induced by pulling back A_*-invariance near 0 by $\mathbf{F}^P : C_P \mapsto 0$. See [DL23, Lemma 5.33] for further details.

The second claim follows from essentially the same argument. By compactness, every secondary subwake of $\mathbf{W}^\bullet_P$ has uniformly bounded diameter away from a neighborhood of C_P. Near C_P, the claim again follows from the asymptotic self-similarity at C_P.

Lastly, the bounds in both claims are independent of P because every full wake in the same ocean is dynamically related. ☐

Lemma 6.28. *Any infinite sequence of nested wakes shrinks to a point.*

Proof. Let us define a holomorphic map χ sending level two wakes to level one wakes as follows. Given a critical point c of $\mathbf{F}^{\geq 0}$, let $W(c)$ be the union of the two full wakes attached to c. Consider a secondary critical point ${}_jC^\bullet_{P,Q}$, which is contained in $W(C_P)$. The map $\mathbf{F}^P$ sends $W({}_jC^\bullet_{P,Q})$ univalently onto $W(C_Q)$. Let $T \in \mathbf{T}$ be the smallest power-triple such that $\mathbf{F}^T$ sends C_Q to $C_{\mathbf{t}^n P}$ for some $n \in \mathbb{Z}$. Then, $\chi := A^{-n} \circ \mathbf{F}^{P+T}$ sends $W({}_jC^\bullet_{P,Q})$ univalently back onto $W(C_P)$. By Lemma 6.27, χ must be uniformly expanding on $W({}_jC^\bullet_{P,Q})$ with expansion factor independent of P.

Now, consider an infinite sequence of nested wakes $W_1 \supset W_2 \supset W_3 \supset \ldots$ where each W_n is of level n. Then, there is a uniform constant $C > 0$ such that for all $n \geq 3$,

$$\operatorname{diam}_{\rho_0}\left(\chi^{n-2}(W_n)\right) \leq C.$$

Since χ is uniformly expanding, the hyperbolic diameter of W_n tends to 0 exponentially fast as $n \to \infty$. ☐

Corollary 6.29. *Every external ray lands at a unique point.*

Proof. Let Y be the accumulation set of an external ray. Since the boundary of every wake is made of ray segments, for every wake W, either $Y \subset \overline{W}$ or $Y \subset \mathbb{C}\backslash W$.

If Y intersects $\mathbf{H}$, then by Corollary 6.26, Y must be contained in $\mathbf{H}$. In general, if Y intersects $\mathbf{F}^{-P}(\mathbf{H})$ for some $P \in \mathbf{T}$, then $Y \subset \mathbf{F}^{-P}(\mathbf{H})$. Since the roots of wakes are dense in $\mathbf{F}^{-P}(\mathbf{H})$, Y must be a singleton.

Suppose Y is disjoint from $\mathbf{F}^{-P}(\mathbf{H})$ for all P. Then, Y is contained in an infinite sequence of nested wakes which, by Lemma 6.28, implies that Y is a singleton. ☐

We say that two points x and y in $\mathbf{I}_{\le P}$ are *combinatorially equivalent* if there is no alpha-point α such that x and y belong in distinct connected components of $\mathbf{I}_{\le P}\backslash\{\alpha\}$. This generates an equivalence relation on $\mathbf{I}_{<\infty}$.

Corollary 6.30. *Every combinatorial equivalence class in $\mathbf{I}_{<\infty}$ is a singleton. For every $P \in \mathbb{R}_{>0}$,*

$$\mathbf{I}_{\le P} = \overline{\bigcup_{Q<P} \mathbf{I}_{\le Q}}. \tag{6.8}$$

Proof. Consider a point $x \in \mathbf{I}_{\le P}$. There are two cases. Suppose x is contained in some chain $(\infty, \alpha]$ for some alpha-point α. In this case, the combinatorial class is a singleton because of Corollary 6.20. Now, suppose x is not contained in any external chain. By Corollary 6.26, x is contained in an infinite sequence of nested wakes. Then, the triviality of combinatorial class of x follows from Lemma 6.28. Lastly, equation (6.8) follows directly from the first claim. □

Corollary 6.31. $\mathbf{I}_{<\infty}$ *has empty interior.*

Proof. If the interior of $\mathbf{I}_{<\infty}$ were non-empty, then any connected component of such would be contained in a single combinatorial equivalence class. This is impossible due to the previous corollary. □

7. Rigidity of escaping dynamics

In Section 5.2, we constructed the global unstable manifold $\boldsymbol{\mathcal{W}}^u$ of the corona renormalization operator $\mathcal{R}$ consisting of cascades of transcendental maps $\mathbf{F}^{\ge 0}$. In this final section, we will conclude the proof of Theorem A by showing that $\boldsymbol{\mathcal{W}}^u$ is one-dimensional. Our approach is to prove Theorem B on the rigidity of escaping dynamics of each $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$. We will apply the external structure of the renormalization fixed point $\mathbf{F}_*$ addressed in Section 6, and adapt an argument by Rempe [Rem09] to show that the set of points in the infinite-time escaping set $\mathbf{I}_\infty(\mathbf{F})$ that remain sufficiently close to ∞ under iteration must move holomorphically with dilatation arbitrarily close to zero.

7.1. Invariant line field.

We say that a corona $f : U \to V$ admits an *invariant line field* supported on a set $E \subset \mathbb{C}$ if there is a measurable Beltrami differential $\mu(z)\frac{d\bar{z}}{dz}$ such that $f^*\mu = \mu$ almost everywhere on U, $|\mu| = 1$ on a positive measure subset of E, and $\mu = 0$ elsewhere.

Similarly, we say that $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$ admits an *invariant line field* supported on a set $E \subset \mathbb{C}$ if there is a measurable Beltrami differential $\mu(z)\frac{d\bar{z}}{dz}$ such that $\left(\mathbf{F}^P\right)^* \mu = \mu$ almost everywhere on $\operatorname{Dom}\left(\mathbf{F}^P\right)$ for all $P \in \mathbf{T}$, $|\mu| = 1$ on a positive measure subset of E, and $\mu = 0$ elsewhere.

The absence of invariant line fields is equivalent to the lack of deformation space associated to a single holomorphic map. This philosophy holds still holds for cascades in the unstable manifold.

Proposition 7.1. *If $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$ admits an invariant line field μ, there is a holomorphic family $\{\mathbf{G}_t\}_{t\in\mathbb{D}}$ in $\boldsymbol{\mathcal{W}}^u$ such that $\mathbf{G}_0 = \mathbf{F}$ and $\mathbf{F}^{\ge 0}$ is quasiconformally conjugate to $\mathbf{G}_t^{\ge 0}$. The conjugacy is conformal outside of the support of μ.*

Proof. A standard application of the measurable Riemann mapping theorem gives us the desired holomorphic family $\{\mathbf{G}_t\}_{t\in\mathbb{D}}$, but a priori we do not know whether this family lives in $\boldsymbol{\mathcal{W}}^u$. To fix this issue, we shall descend back to the realm of coronas.

By anti-renormalizing, let us assume without loss of generality that $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u_{loc}$. Let us project μ to the dynamical plane of f_n for $n \leq 0$ and obtain an invariant line field μ_n of f_n. Then, we integrate μ_n to obtain a Beltrami path $\{f_{n,t}\}_{t\in\mathbb{D}}$ of coronas in a neighborhood of f_*. Let us anti-renormalize to obtain a new path $f_t^{(n)} := \mathcal{R}^{-n} f_{n,t}$ where $f_0^{(n)} \equiv f_0$ for all $n \leq 0$. When $|t| < \frac{1}{2}$, $f_t^{(n)}$ is quasiconformally conjugate to f_0 with uniformly bounded dilatation. Therefore, we can take a limit as $n \to -\infty$ and obtain a holomorphic path g_t of infinitely anti-renormalizable corona. As the limiting path lies in $\mathcal{W}^u_{loc}$, it corresponds to a path in $\boldsymbol{\mathcal{W}}^u$. ☐

Lemma 7.2. *The renormalization fixed point $\mathbf{F}_*$ admits no invariant line field supported on its full escaping set $\mathbf{I}(\mathbf{F}_*)$.*

Proof. Suppose for a contradiction that $\mathbf{I}(\mathbf{F}_*)$ supports an invariant line field of $\mathbf{F}_*$. By Proposition 7.1, we obtain a family $\{\mathbf{G}_t\}_{t\in\mathbb{D}}$ in $\boldsymbol{\mathcal{W}}^u$ together with quasiconformal maps $h_t : \mathbb{C} \to \mathbb{C}$ conjugating $\mathbf{F}_*$ with $\mathbf{G}_t$ for all $t \in \mathbb{D}$. Each of $\mathbf{G}_t$ induces a rotational corona g_t with rotation number θ, which, by Theorem 4.12, implies that g_t must also be on the local stable manifold. Therefore, $g_t \equiv f_*$ and the family $\mathbf{G}_t$ is trivial. Therefore, the family of quasiconformal conjugacies h_t commutes with $\mathbf{F}_*$ along the Herman quasicircle $\mathbf{H}$. As such, h_t is the identity on $\mathbf{H}$, and so on the grand orbit $\bigcup_P \mathbf{F}^{-P}(\mathbf{H})$ of $\mathbf{H}$ as well.

We claim that the grand orbit of $\mathbf{H}$ is a dense subset of $\mathbb{C}$, and thus $h_t \equiv \mathrm{Id}$. Indeed, by [Lim23b, §5], the critical value $c_1(f_*)$ of f_* is a deep point of the Julia set $J(f_*)$ of f_*. In particular, magnifications of $J(f_*)$ about $c_1(f_*)$ converge to the whole plane. Therefore, as we pass to the corresponding dynamical plane of the transcendental extension, 0 is a deep point of iterated preimages of $\mathbf{H}$ under $\mathbf{f}_\pm$. By self-similarity, the grand orbit of $\mathbf{H}$ must be dense in $\mathbb{C}$. ☐

7.2. **Rigidity of the finite-time escaping set.**

Lemma 7.3. *For any moment $P \in \mathbf{T}_{>0}$, any point $z \in \partial\mathrm{Dom}\left(\mathbf{F}^P\right)$, and any scale $r > 0$, the image $\mathbf{F}^P(D)$ of any connected component D of $\mathbb{D}(z,r) \cap \mathrm{Dom}\left(\mathbf{F}^P\right)$ is dense in $\mathbb{C}$.*

Proof. This follows directly from the σ-properness of $\mathbf{F}$. Refer to [DL23, Lemma 6.5] for details. ☐

Corollary 7.4. *For every $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, $P \in \mathbf{T}_{>0}$, and $x \in \mathbb{C}$, the boundary of $\mathrm{Dom}\left(\mathbf{F}^P\right)$ is the set of accumulating points of $\mathbf{F}^{-P}(x)$.*

The proof below is similar to [DL23, Corollary 6.7].

Proof. By Lemma 5.14, there exists a disk neighborhood B of x such that $B\backslash\{x\} \cap \mathrm{CV}\left(\mathbf{F}^P\right) = \varnothing$. Then, every connected component B' of $\mathbf{F}^{-P}(B)$ contains at most one critical point and the degree of $\mathbf{F}^P : B' \to B$ is at most some uniform constant. Let $\Omega \subset B$ be an even smaller disk neighborhood of x such that $\mathrm{mod}(B\backslash\overline{\Omega}) \asymp 1$. The preimage $\Omega' \subset B'$ of Ω under $\mathbf{F}^P$ is also a disk with $\mathrm{mod}(B'\backslash\overline{\Omega'}) \asymp 1$.

Let us pick a connected component D of $\mathrm{Dom}\left(\mathbf{F}^P\right)$, a point $y \in \partial D$, and a small $\varepsilon > 0$. By Lemma 7.3, there is a connected component $\Omega' \subset D$ of $\mathbf{F}^{-P}(\Omega)$ that is of distance at most ε away from y. Since $\mathrm{mod}\left(B'\backslash\overline{\Omega'}\right) \asymp 1$, then Ω' has a small diameter depending on ε. Since Ω' contains point in $\mathbf{F}^{-P}(x)$, the assertion follows. $\square$

We say that a holomorphic motion of a set $E \subset \mathbb{C}$ is a *conformal motion* if its dilatation on E is zero.

Set
$$T := \min\{(0,1,0),(0,0,1)\}.$$

Lemma 7.5. *There is a unique equivariant holomorphic motion of $\mathbf{I}_{\le T}(\mathbf{F})$ over some neighborhood $\boldsymbol{\mathcal{U}}$ of $\mathbf{F}_*$.*

Proof. By Lemma 5.13, the set of critical values $\mathrm{CV}\left(\mathbf{F}^T\right)$ of $\mathbf{F}^T$ moves holomorphically within a small neighborhood of $\mathbf{F}_*$. By Lemma 5.7, there is a small neighborhood $\boldsymbol{\mathcal{U}}$ of $\mathbf{F}_*$ and some point $x \in \mathbb{C}$ such that x belongs in the interior of $\mathbf{U}_-(\mathbf{F})$ and does not collide with $\mathrm{CV}(\mathbf{F}^T)$ for all $\mathbf{F} \in \boldsymbol{\mathcal{U}}$. Moreover, $\mathbf{F}^{-S}(x)$ moves holomorphically with $\mathbf{F} \in \boldsymbol{\mathcal{U}}$ for all $S \le T$.

If $Q < S \le T$, then $\mathbf{F}^{-S}(x)$ is disjoint from $\mathbf{F}^{-Q}(x)$ because every point is mapped by $\mathbf{F}^S$ and $\mathbf{F}^Q$ to different tiles of the zeroth renormalization tiling of $\mathbf{F}$. Hence, $\bigcup_{S \le T} \mathbf{F}^{-S}(x)$ moves holomorphically and equivariantly with $\mathbf{F} \in \boldsymbol{\mathcal{U}}$. By the λ-lemma, this holomorphic motion extends to the closure. By Corollaries 6.31 and 7.4, $\mathbf{I}_{\le T}(\mathbf{F})$ has no interior and moves holomorphically and equivariantly over $\boldsymbol{\mathcal{U}}$.

Let us show that the motion τ of $\mathbf{I}_{\le T}(\mathbf{F})$ obtained above is independent of x. Let us pick another point $y = y(\mathbf{F}) \in \mathbb{C}\backslash \mathrm{CV}(\mathbf{F})$ which depends holomorphically on $\mathbf{F} \in \boldsymbol{\mathcal{U}}$. By shrinking $\boldsymbol{\mathcal{U}}$, we can connect x and y by a simple arc $l = l(\mathbf{F})$ which is surrounded by an annulus $A = A(\mathbf{F}) \subset \mathbb{C}\backslash\mathrm{CV}(\mathbf{F})$. Every preimage of l under $\mathbf{F}^T$ is separated from $\mathbf{I}_{\le T}(\mathbf{F})$ by a conformal preimage of A. Therefore, any sequence of preimages of l under $\mathbf{F}^T$ which accumulates at a point in $\mathbf{I}_{\le T}(\mathbf{F})$ necessarily shrinks in diameter. Therefore, the holomorphic motion coincides with the motion of $\overline{\mathbf{F}^{-T}(y(\mathbf{F}))}$.

Finally, let us show that the equivariant holomorphic motion τ of $\mathbf{I}_{\le T}(\mathbf{F})$ over $\boldsymbol{\mathcal{U}}$ is unique. Suppose there is another equivariant holomorphic motion τ' of $\mathbf{I}_{\le T}(\mathbf{F})$. Pick any $S \in \mathbf{T}_{>0}$ where $S < T$ and consider the motion $y(\mathbf{F})$ of a point in $\mathbf{I}_{\le S}(\mathbf{F})$ induced by τ'. By equivariance, $\mathbf{F}^{-(T-S)}(y(\mathbf{F}))$ moves holomorphically by τ'. However, since $\mathbf{I}_{\le T-S}(\mathbf{F})$ is contained in the closure of $\mathbf{F}^{-(T-S)}(y(\mathbf{F}))$, we see that τ and τ' coincide on $\mathbf{I}_{\le T-S}(\mathbf{F})$ for all $S \in \mathbf{T}_{>0}$. By (6.8), $\tau \equiv \tau'$. $\square$

Theorem 7.6. *For every $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, $\mathbf{I}_{<\infty}(\mathbf{F})$ has empty interior and supports no invariant line field. Moreover, for every $P \in \mathbf{T}_{>0}$, on every connected component of the open set $\{\mathbf{F} \in \boldsymbol{\mathcal{W}}^u : 0 \notin \mathbf{I}_{\le P}(\mathbf{F})\}$, there is a unique equivariant holomorphic motion of $\mathbf{I}_{\le P}(\mathbf{F})$, and this motion is conformal.*

Proof. Let us fix $P \in \mathbf{T}_{>0}$ and consider the set $\mathbf{D}_P := \{\mathbf{F} \in \boldsymbol{\mathcal{W}}^u : 0 \notin \mathbf{I}_{\le P}(\mathbf{F})\}$. If $P < T$, then clearly the neighborhood $\boldsymbol{\mathcal{U}}$ of $\mathbf{F}_*$ from Lemma 7.5 is contained in $\mathbf{D}_P$. Else, if $P \ge T$, then $\mathbf{F} \in \mathbf{D}_P \cap \boldsymbol{\mathcal{U}}$ if and only if $\mathbf{F}^{P-T}(0) \notin \mathbf{I}_{\le T}(\mathbf{F})$, which is an open condition because $\mathbf{I}_{\le T}(\mathbf{F})$ moves holomorphically over $\boldsymbol{\mathcal{U}}$. Therefore, $\mathbf{D}_P \cap \boldsymbol{\mathcal{U}}$ is open for all P.

If $\mathbf{F} \in \mathbf{D}_P \cap \mathbf{U}$, we can obtain the unique equivariant holomorphic motion of $\mathbf{I}_{\le P}(\mathbf{F})$ by pulling back the holomorphic motion of $\mathbf{I}_{\le T}(\mathbf{F})$. In general, for any $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, we can pick a sufficiently large $n \ll 0$ such that $\mathbf{F}_n \in \boldsymbol{\mathcal{U}}$. Clearly, $\mathbf{F} \in \mathbf{D}_P$ if and only if $\mathbf{F}_n \in \mathbf{D}_{\mathbf{t}^{-n}P}$, so $\mathbf{D}_P$ is always an open subset of $\boldsymbol{\mathcal{W}}^u$ on which $\mathbf{I}_{\le P}(\mathbf{F})$ moves holomorphically and equivariantly. The dilatation of the motion of $\mathbf{I}_{\le P}(\mathbf{F})$

can be made arbitrarily small by selecting n to be arbitrarily large and $\mathbf{F}_n$ to be arbitrarily close to $\mathbf{F}_*$. This shows that the motion is conformal.

By Corollary 6.31 and Lemma 7.2, $\mathbf{I}_{\leq \mathbf{t}^{-n}P}(\mathbf{F}_n)$ has empty interior and supports no invariant line field of $\mathbf{F}_n$. Therefore, $\mathbf{I}_{\leq P}(\mathbf{F})$ also has empty interior and supports no invariant line field of $\mathbf{F}$. □

Corollary 7.7. *For all $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, the finite-time escaping set $\mathbf{I}_{<\infty}(\mathbf{F})$ is non-empty and $\mathfrak{J}(\mathbf{F}) = \overline{\mathbf{I}_{<\infty}(\mathbf{F})}$.*

Proof. Pick any $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$. From the previous theorem, there are some $P \in \mathbf{T}_{>0}$ and some open neighborhood $\boldsymbol{\mathcal{U}} \subset \boldsymbol{\mathcal{W}}^u$ of $\mathbf{F}_*$ containing $\mathbf{F}$ in which the P^{th} escaping set moves holomorphically. Therefore, $\mathbf{I}_{\leq P}(\mathbf{F})$ is clearly non-empty. By Montel's theorem, since $\mathbf{I}_{\leq P}(\mathbf{F})$ contains more than two points, for any $z \in \mathfrak{J}(\mathbf{F})$, every neighborhood of any point in $\mathfrak{J}(\mathbf{F})$ must contain a point in $\mathbf{I}_{\leq Q}(\mathbf{F})$ where $Q \geq P$. □

7.3. **Rigidity of the infinite-time escaping set.** For $R > 0$ and $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, define

$$\mathfrak{J}_R(\mathbf{F}) := \left\{ z \in \mathfrak{J}(\mathbf{F}) \backslash \mathbf{I}_{<\infty}(\mathbf{F}) \,:\, \left|\mathbf{F}^P(z)\right| \geq R \text{ for all } P \in \mathbf{T} \right\}.$$

Clearly, the forward orbit of every point in the infinite-time escaping set of $\mathbf{F}$ is eventually contained in $\mathfrak{J}_R(\mathbf{F})$. The following lemma is inspired by [Rem09].

Lemma 7.8. *For every $\mathbf{F}$ on a neighborhood $\boldsymbol{\mathcal{U}}$ of $\mathbf{F}_*$, there exists a subset $\Lambda(\mathbf{F})$ of $\mathfrak{J}(\mathbf{F}) \backslash \mathbf{I}_{<\infty}(\mathbf{F})$ with the following properties.*

1. *$\Lambda(\mathbf{F})$ is forward invariant under $\mathbf{F}^{\geq 0}$.*
2. *There is a unique equivariant holomorphic motion of $\Lambda(\mathbf{F})$ over $\mathbf{F} \in \boldsymbol{\mathcal{U}}$.*
3. *There exists some $R > 1$ such that $\Lambda(\mathbf{F})$ contains $\mathfrak{J}_R(\mathbf{F})$.*

Proof. In the dynamical plane of $\mathbf{F}_*$, for every point x in $\mathfrak{J}_R(\mathbf{F}_*)$, every point $\mathbf{F}_*^P(x)$ in the forward orbit must be contained in a wake of sufficiently low generation in order to avoid $\mathbb{D}_R$. We consider all such points and define $\Lambda(\mathbf{F}_*)$. In the proof below, we apply the motion of the finite escaping set from the previous section to show that $\Lambda(\mathbf{F})$ can be defined naturally via a unique holomorphic motion. The proof will be broken down to four steps.

Step 1: Construct truncated wakes which move holomorphically.

Let us pick $r > 0$ such that all primary wakes of $\mathbf{F}_*$ of generation at most $T := \min\{(0,1,0),(0,0,1)\}$ are compactly contained in the domain $\mathbf{V} = \mathbb{C} \backslash \overline{\mathbb{D}_r}$. Let us enumerate primary wakes of generation at most T by $\{\mathbf{W}_i\}_{i \in I}$ for some countable index set I. Denote the generation of each wake $\mathbf{W}_i$ by P_i. For each $i \in I$, consider the truncated wake $\hat{\mathbf{W}}_i := \mathbf{W}_i \cap \mathbf{F}_*^{-P_i}(\mathbf{V})$.

For each $\bullet \in \{0, \infty\}$, there exists a unique point $z^\bullet$ on the intersection of $\partial \mathbf{V}$ and the zero ray $\mathbf{R}^\bullet$ such that the ray segment $\hat{\mathbf{R}}^\bullet = (\infty, z^\bullet)$ is contained in V. Let Q be the maximum of the escaping times of z^0 and z^∞. By Lemma 7.8, the Q^{th} escaping set $\mathbf{I}_{\leq Q}(\mathbf{F})$ moves holomorphically and equivariantly on a small neighborhood $\boldsymbol{\mathcal{U}}$ of $\mathbf{F}_*$. By the λ-lemma, such motion induces a holomorphic motion of $\hat{\mathbf{R}}^0(\mathbf{F}) \cup \hat{\mathbf{R}}^\infty(\mathbf{F}) \cup \partial \mathbf{V}(\mathbf{F})$, which, by shrinking $\boldsymbol{\mathcal{U}}$ if necessary, can be assumed to not collide with $\mathrm{CV}\left(\mathbf{F}^T\right)$. This allows us to pull back via $\mathbf{F}^P$ for all $P \leq T$ and further extend this motion to a holomorphic motion of

$$\hat{\mathbf{R}}^0(\mathbf{F}) \cup \hat{\mathbf{R}}^\infty(\mathbf{F}) \cup \partial \mathbf{V}(\mathbf{F}) \cup \bigcup_{i \in I} \partial \hat{\mathbf{W}}_i(\mathbf{F})$$

that is equivariant on $\partial\hat{\mathbf{W}}_i(\mathbf{F})$ with respect to $\mathbf{F}^{P_i}$ for each $i \in I$. By λ-lemma, this motion can again be extended to a holomorphic motion Φ_0 on the whole plane that is equivariant with respect to $\mathbf{F}^{P_i}$ on $\partial\hat{\mathbf{W}}_i(\mathbf{F})$ for each $i \in I$.

Step 2: Construct $\Lambda(\mathbf{F})$ which move holomorphically and equivariantly.

Let $\mathbf{V}_0(\mathbf{F}) := \bigcup_{i\in I} \hat{\mathbf{W}}_i(\mathbf{F})$ and define the map

$$\hat{\mathbf{F}} : \mathbf{V}_0(\mathbf{F}) \to \mathbf{V}(\mathbf{F}), \quad \hat{\mathbf{F}}(z) = \mathbf{F}^{P_i}(z) \text{ for } z \in \hat{\mathbf{W}}_i(\mathbf{F}).$$

This map satisfy a Markov-like property that $\mathbf{V}_0(\mathbf{F}) \subset \mathbf{V}(\mathbf{F})$ and $\hat{\mathbf{F}}$ sends every connected component of $\mathbf{V}_0(\mathbf{F})$ univalently onto a dense subset of $\mathbf{V}(\mathbf{F})$.

Let $\mathbf{V}_{-n}(\mathbf{F}) := \hat{\mathbf{F}}^{-n}(\mathbf{V}_0(\mathbf{F}))$. Consider the non-escaping set $\Lambda(\mathbf{F})$ of $\hat{\mathbf{F}}$ which is defined by

$$\Lambda(\mathbf{F}) := \bigcap_{n\geq 0} \mathbf{V}_{-n}(\mathbf{F}).$$

By design, $\Lambda(\mathbf{F})$ is forward invariant under $\mathbf{F}^{\geq 0}$. By Lemma 6.28, nested truncated wakes shrink to points, and thus $\Lambda(\mathbf{F}_*)$ is a closed totally disconnected set.

Let us treat $\Phi_0 = \Phi_0(\mathbf{F})$ as a map from the dynamical plane of $\mathbf{F}_*$ to the dynamical plane of $\mathbf{F}$. We will apply the pullback argument to the holomorphic motion Φ_0 as follows. For $n \geq 0$, we inductively define the lift of Φ_n to be

$$\Phi_{n+1} := \begin{cases} \Phi_n & \text{on } \mathbb{C}\backslash \mathbf{V}_{-n}(\mathbf{F}_*), \\ \left(\hat{\mathbf{F}}|_{\hat{\mathbf{W}}_i(\mathbf{F})}\right)^{-1} \circ \Phi_n \circ \hat{\mathbf{F}}_* & \text{on } \mathbf{V}_{-n}(\mathbf{F}_*) \cap \hat{\mathbf{W}}_i(\mathbf{F}_*) \text{ for each } i \in I. \end{cases}$$

Each Φ_n is quasiconformal on $\mathbb{C}$ with uniformly bounded dilatation and it eventually stabilizes at every point outside of $\Lambda(\mathbf{F}_*)$. Since $\Lambda(\mathbf{F}_*)$ is nowhere dense, Φ_n converges in subsequence to a limiting holomorphic motion Φ which is equivariant on $\Lambda(\mathbf{F})$.

Step 3: The equivariant holomorphic motion of $\Lambda(\mathbf{F})$ is unique.

Suppose Ψ is another holomorphic motion of $\Lambda(\mathbf{F})$ on some small neighborhood of $\mathbf{F}_*$. We will use the notation $\Psi_{\mathbf{F}}(x)$ to highlight the dependence of $\mathbf{F}$. Let us pick any point $x \in \Lambda(\mathbf{F}_*)$. There is some $(i_0, i_1, \ldots) \in I^{\mathbb{N}}$ such that x is the unique point with itinerary $(i_0, i_1, \ldots)$, that is, $\hat{\mathbf{F}}_*^n(x)$ lies in the truncated wake $\hat{\mathbf{W}}_{i_n}(\mathbf{F}_*)$ for all n. Suppose for a contradiction that $\Psi_{\mathbf{F}}(x)$ and $\Phi_{\mathbf{F}}(x)$ are distinct. Then, the itinerary of $\Psi_{\mathbf{F}}(x)$ is not equal to $(i_0, i_1, \ldots)$ and, in particular, there is some $n \in \mathbb{N}$ such that $\hat{\mathbf{F}}^n(\Psi_{\mathbf{F}}(x))$ lies in a truncated wake other than $\hat{\mathbf{W}}_{i_n}(\mathbf{F})$. Since the boundary of $\hat{\mathbf{W}}_{i_n}(\mathbf{F})$ moves holomorphically and equivariantly, there is some $\mathbf{G}$ sufficiently close to $\mathbf{F}_*$ such that $x'_n := \tilde{\mathbf{G}}^n \Psi_{\mathbf{G}}(x)$ is on the boundary of $\hat{\mathbf{W}}_{i_n}(\mathbf{G})$. Then, $y'_n := \mathbf{G}^{P_{i_n}}(x'_n)$ would lie on $S^0(\mathbf{G}) \cup S^\infty(\mathbf{G}) \cup \partial D(\mathbf{G})$, which is disjoint from $\Lambda(\mathbf{G})$. However, due to forward invariance, y'_n must be contained in $\Lambda(\mathbf{G})$, hence a contradiction.

Step 4: $\mathfrak{J}_R(\mathbf{F})$ is contained in $\Lambda(\mathbf{F})$ for some $R \gg 0$.

It suffices to find R such that for all $\mathbf{F} \in \boldsymbol{\mathcal{U}}$, every point outside of $\mathbf{I}_{<\infty}(\mathbf{F}) \cup \Lambda(\mathbf{F})$ will be sent into the disk $\mathbb{D}_R$ by $\mathbf{F}^P$ for some $P \in \mathbf{T}$.

In the dynamical plane of $\mathbf{F}_*$, there exists some sufficiently large $N < 0$ such that all primary wakes rooted at critical points located in $\Delta_0(0, \mathbf{F}_*) \cup \Delta_0(1, \mathbf{F}_*)$ are contained in the tile $\Delta_N(i, \mathbf{F}_*)$ for some $i \in \{0, 1\}$. Then, every wake of generation greater than T is contained in the tiling $\boldsymbol{\Delta}_N(\mathbf{F}_*)$ and so $\mathbb{C}\backslash\overline{\mathbf{V}_0(\mathbf{F}_*)} \subset \boldsymbol{\Delta}_N(\mathbf{F}_*)$.

By shrinking $\boldsymbol{\mathcal{U}}$ if necessary, the tiling $\mathbf{\Delta}_N(\mathbf{F})$ moves holomorphically and equivariantly over $\boldsymbol{\mathcal{U}}$ and always contains $\mathbb{C}\backslash\overline{\mathbf{V}_0(\mathbf{F})}$. Therefore, for all $\mathbf{F} \in \boldsymbol{\mathcal{U}}$, every point outside of $\mathbf{I}_{<\infty}(\mathbf{F}) \cup \Lambda(\mathbf{F})$ is eventually mapped by $\mathbf{F}^P$ for some $P \in \mathbf{T}$ to a point in $\mathbb{C}\backslash\overline{\mathbf{V}_0(\mathbf{F})}$, which is then sent by $\mathbf{F}^Q$ for some $Q \in \mathbf{T}$ to a point in $\mathbf{F}^{(N,0,1)}(\Delta_N(0,\mathbf{F})) \cup \mathbf{F}^{(N,1,0)}(\Delta_N(1,\mathbf{F}))$, which is contained in the disk $\mathbb{D}_R$ for some large $R > 0$ independent of $\mathbf{F}$. □

Theorem 7.9. *For every $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, $\mathbf{I}_\infty(\mathbf{F})$ supports no invariant line field. Moreover, on every connected component of the interior of $\{\mathbf{F} \in \boldsymbol{\mathcal{W}}^u : 0 \notin \mathbf{I}_\infty(\mathbf{F})\}$, there is a unique equivariant holomorphic motion of $\mathbf{I}_\infty(\mathbf{F})$, and this motion is conformal.*

Proof. Let $\boldsymbol{\mathcal{U}}$, Λ, and R be from the previous lemma. For every $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, there is some sufficiently large $n \ll 0$ such that $\mathbf{F}_n$ lies in $\boldsymbol{\mathcal{U}}$. Since $\mathbf{F}^P = A_*^n \circ \mathbf{F}_n^{P/\mathbf{t}^n} \circ A_*^{-n}$ for all $P \in \mathbf{T}$, the set $\Lambda_n(\mathbf{F}) \coloneqq A_*^n(\Lambda(\mathbf{F}_n))$ is forward invariant, contains $\mathfrak{J}_{|\mu_*|^n R}(\mathbf{F})$, and admits a unique equivariant holomorphic motion Φ_n over $\mathcal{R}^{-n}(\boldsymbol{\mathcal{U}})$. The dilatation of Φ_n near $\mathbf{F}$ can be made arbitrarily small by choosing $\mathbf{F}_n$ arbitrarily close to $\mathbf{F}_*$, or equivalently, n to be an arbitraily large negative number. In particular, there is a unique equivariant holomorphic motion of $\mathbf{I}_\infty(\mathbf{F}) \cap \Lambda_n(\mathbf{F})$ and its dilatation near $\mathbf{F}$ shrinks to zero as $n \to \infty$.

On a component of the interior of $\{\mathbf{F} \in \boldsymbol{\mathcal{W}}^u : 0 \notin \mathbf{I}_\infty(\mathbf{F})\}$, we can extend Φ_n by iteratively pulling back the holomorphic motion of $\mathbf{I}_\infty(\mathbf{F}) \cap \Lambda_n(\mathbf{F})$, yielding a unique equivariant holomorphic motion $\tilde{\Phi}_n$ on $\mathbf{I}_\infty(\mathbf{F})$. Since we are pulling back by a holomorphic map, the dilatation of $\tilde{\Phi}_n$ is equal to that of Φ_n. By the uniqueness of the motion, $\tilde{\Phi}_n$ is independent of n. Moreover, since the dilatation shrink to zero as $n \to \infty$, then the dilatation of $\tilde{\Phi}_n$ must be zero.

Lastly, suppose for a contradiction that $\mathbf{I}_\infty(\mathbf{G})$ supports an invariant line field μ of some $\mathbf{G} \in \boldsymbol{\mathcal{W}}^u$. Since $\mathbf{I}_\infty(\mathbf{F}) \cap \Lambda_n(\mathbf{F})$ moves holomorphically over a neighborhood of $\mathbf{F}_*$ containing $\mathbf{G}$ for some $n \ll 0$, then there is a quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ which has zero dilatation on $\mathbf{F}_*|_{\mathbf{I}_\infty(\mathbf{F}_*) \cap \Lambda_n(\mathbf{F})}$ and conjugates $\mathbf{F}_*|_{\mathbf{I}_\infty(\mathbf{F}_*) \cap \Lambda_n(\mathbf{F})}$ to $\mathbf{G}|_{\mathbf{I}_\infty(\mathbf{G}) \cap \Lambda_n(\mathbf{F})}$. Consider $\mu' = \phi^*\mu$ on $\mathbf{I}_\infty(\mathbf{F}_*) \cap \Lambda_n(\mathbf{F})$ and pull it back via $\mathbf{F}_*$ to obtain a $\mathbf{F}_*$-invariant Beltrami differential μ' supported on $\mathbf{I}_\infty(\mathbf{F}_*)$. Then, μ' would be an invariant line field of $\mathbf{F}_*$ supported on $\mathbf{I}_\infty(\mathbf{F}_*)$, which is impossible due to Lemma 7.2. □

7.4. **Proof of the main theorems.** We say that $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$ is *hyperbolic* if $\mathbf{F}$ admits an attracting cycle of periodic points. Additionally, we say that $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$ is *superattracting* if 0 is a periodic point of $\mathbf{F}^{\geq 0}$.

Proposition 7.10. *If $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$ is hyperbolic, then $\mathfrak{J}(\mathbf{F})\backslash\mathbf{I}(\mathbf{F})$ has zero Lebesgue measure.*

Proof. Suppose $\mathbf{F}$ is hyperbolic. By Proposition 5.23, the critical point is within the basin of an attracting cycle and thus $\mathfrak{P}(\mathbf{F})$ is a subset of the Fatou set. By Theorem 5.25, it suffices to show that $\mathfrak{J}(\mathbf{F})$ has no interior. Suppose instead that $\mathfrak{J}(\mathbf{F})$ contains an open ball B. Corollary 7.7 tells us that $\mathbf{I}_{<\infty}(\mathbf{F}) \cap B$ is dense in B. By Lemma 7.3, there is some $P \in \mathbf{T}_{>0}$ such that $\mathbf{F}^P(B\backslash\mathbf{I}_{\leq P}(\mathbf{F}))$ is dense in $\mathbb{C}$. This is impossible because the Fatou set of $\mathbf{F}$ is non-empty. □

Corollary 7.11. *Consider a hyperbolic component $\mathcal{H}$ of $\boldsymbol{\mathcal{W}}^u$. There is a unique equivariant holomorphic motion of $\mathfrak{J}(\mathbf{F})$ over $\mathbf{F} \in \mathcal{H}$, and such a motion is a conformal motion. If $\mathbf{F} \in \mathcal{H}$, then $\mathfrak{J}(\mathbf{F})$ supports no invariant line field of $\mathbf{F}$.*

Proof. For $\mathbf{F} \in \mathcal{H}$, the critical value 0 is not contained in $\mathbf{I}(\mathbf{F})$, and so the assertion follows from Proposition 7.10 and Theorems 7.6 and 7.9. □

This completes the proof of Theorem B. The following lemma guarantees the existence of hyperbolic components.

Lemma 7.12. *Every neighborhood $\boldsymbol{\mathcal{U}}$ of the fixed point $\mathbf{F}_*$ contains a superattracting element.*

Proof. Assume for a contradiction that there is a small neighborhood $\boldsymbol{\mathcal{U}}$ of $\mathbf{F}_*$ in which for all $\mathbf{F} \in \boldsymbol{\mathcal{U}}$, we have $\mathbf{F}^{P+Q}(0) \neq \mathbf{F}^Q(0)$ for all $P \in \mathbf{T}_{>0}$, $Q \in \mathbf{T}$. By λ-lemma, this implies that the postcritical set of $\mathbf{F}$ moves holomorphically over $\boldsymbol{\mathcal{U}}$. Consequently, the corresponding neighborhood $\mathcal{V} \subset \mathcal{W}^u_{\text{loc}}$ of f_* consists of rotational coronas. By Theorem 4.12, $\mathcal{V}$ must lie in the stable manifold, which is a contradiction.

Therefore, every neighborhood $\boldsymbol{\mathcal{U}}$ of $\mathbf{F}_*$ contains some $\mathbf{G}$ such that $\mathbf{G}^{P+Q}(0) = \mathbf{G}^Q(0)$ for some $P \in \mathbf{T}_{>0}$ and $Q \in \mathbf{T}$. If $Q = 0$, then $\mathbf{G}$ is superattracting and we are done. Hence, let us assume that $Q > 0$. In this case, $\mathbf{G}^Q(0)$ is a periodic point of period Q, and by Proposition 5.23, it must be repelling in nature.

Consider any sufficiently small embedded one-dimensional disk $\boldsymbol{\mathcal{U}}'$ in $\boldsymbol{\mathcal{U}}$ which contains $\mathbf{G}$. By implicit function theorem, every $\mathbf{F} \in \boldsymbol{\mathcal{U}}'$ admits a repelling periodic point $w_\mathbf{F}$ of period P such that $w_\mathbf{G} = \mathbf{G}^Q(0)$ and $w_\mathbf{F}$ depends holomorphically in $\mathbf{F}$. By Corollaries 7.4 and 7.7, there exists a sequence of critical points $w^n_\mathbf{F}$ of some generation P_n depending holomorphically in $\mathbf{F} \in \boldsymbol{\mathcal{U}}'$ such that $P_n \to \infty$ and $w^n_\mathbf{F} \to w_\mathbf{F}$ as $n \to \infty$. By Rouché's theorem, for sufficiently large n, the number of zeros of $\mathbf{F}^{Q+P_n}(w^n_\mathbf{F}) - w^n_\mathbf{F}$ as a function of $\mathbf{F} \in \boldsymbol{\mathcal{U}}'$ is equal to that of $\mathbf{F}^{Q+P_n}(w^n_\mathbf{F}) - w_\mathbf{F}$, which is at least one (e.g. $\mathbf{G}$). Therefore, there is some large $n \in \mathbb{N}$ and some $\mathbf{F} \in \boldsymbol{\mathcal{U}}'$ such that $\mathbf{F}^{Q+P_n}(w^n_\mathbf{F}) = w^n_\mathbf{F}$ and so $\mathbf{F}^{Q+P_n}(0) = 0$. □

Theorem 7.13. *The global unstable manifold $\boldsymbol{\mathcal{W}}^u$ is biholomorphic to $\mathbb{C}$.*

Proof. We claim that $\boldsymbol{\mathcal{W}}^u$ is one-dimensional. Since $\mathcal{R}$ is an automorphism of $\boldsymbol{\mathcal{W}}^u$ admitting a unique repelling fixed point $\mathbf{F}_*$, then the claim will imply that $\mathcal{R} : \boldsymbol{\mathcal{W}}^u \to \boldsymbol{\mathcal{W}}^u$ is conformally conjugate to a linear map $\mathbb{C} \to \mathbb{C}, z \mapsto \lambda z$ where λ is the repelling eigenvalue of $\mathcal{R}$.

Suppose for a contradiction that $\boldsymbol{\mathcal{W}}^u$ has dimension greater than one. By Lemma 7.12, there exists a superattracting cascade in $\boldsymbol{\mathcal{W}}^u$. By the assumption, there exists an embedded holomorphic curve $\mathbb{D} \to \boldsymbol{\mathcal{W}}^u, \lambda \in \mathbb{D} \mapsto \mathbf{F}_\lambda$ such that each $\mathbf{F}_\lambda$ is superattracting.

Let P be the period of 0 of each $\mathbf{F}_\lambda$. Denote the immediate basin of attraction of 0 for $\mathbf{F}_\lambda$ by D_λ and let $b_\lambda : (D_\lambda, 0) \to (\mathbb{D}, 0)$ be a Böttcher conjugacy, i.e. a Riemann mapping which conjugates $\mathbf{F}^P_\lambda$ with the power map $z \mapsto z^{d_0+d_\infty-1}$. Observe that $B_\lambda := b_\lambda^{-1} \circ b_0 : (D_0, 0) \to (D_\lambda, 0)$ conjugates $\mathbf{F}^P_0$ with $\mathbf{F}^P_\lambda$. The Böttcher conjugacy is unique up to multiplication by some roots of unity. We can select them such that b_λ depends holomorphically on λ and so B_0 is the identity map on D_0.

By Corollary 7.11, the Julia set $\mathfrak{J}(\mathbf{F}_\lambda)$ moves conformally in λ. More precisely, there exists a holomorphic family of quasiconformal maps $\phi_\lambda : \mathbb{C} \to \mathbb{C}$ that have zero dilatation on $\mathfrak{J}(\mathbf{F}_0)$ and conjugates $\mathbf{F}_0|_{\mathfrak{J}(\mathbf{F}_0)}$ and $\mathbf{F}_\lambda|_{\mathfrak{J}(\mathbf{F}_\lambda)}$.

We shall modify ϕ_λ on the Fatou set as follows. For $r \in (0,1)$, let $E_\lambda(r) = b_\lambda^{-1}(\mathbb{D}_r)$ be a disk neighborhood of 0 cut out by an equipotential. Let $\varepsilon = \frac{1}{2}$ and $\varepsilon' = \varepsilon^{d_0+d_\infty-1}$. Define the global quasiconformal map

$$\psi_{\lambda,0}(z) := \begin{cases} \phi_\lambda(z) & \text{if } z \in \mathbb{C} \backslash \bigcup_{0\leq T<P} \mathbf{F}_0^T(E_\lambda(\varepsilon)) \\ \mathbf{F}_\lambda^T \circ B_\lambda \circ \left(\mathbf{F}_0^T|_{E_0(\varepsilon')}\right)^{-1} & \text{if } z \in \mathbf{F}_0^T(E_0(\varepsilon')) \text{ for some } T < P \\ \text{quasiconformal interpolation} & \text{if otherwise.} \end{cases}$$

On $\mathfrak{J}(\mathbf{F}_0)$ and a neighborhood of the periodic cycle $\{\mathbf{F}_0\}$, $\psi_{\lambda,0}$ conjugates $\mathbf{F}_0^P$ and $\mathbf{F}_\lambda^P$. Inductively, we define for all $n \geq 1$ the quasiconformal map $\psi_{\lambda,n} : \mathbb{C} \to \mathbb{C}$ by lifting $\psi_{\lambda,n-1}$ such that $\mathbf{F}_\lambda^P \circ \psi_{\lambda,n} = \psi_{\lambda,n-1} \circ \mathbf{F}_0^P$. The map $\psi_{\lambda,n}$ has dilatation equal to that of $\psi_{\lambda,0}$ and it agrees with $\psi_{\lambda,n-1}$ on a neighborhood of $\mathfrak{J}(\mathbf{F}_0)$ and on increasingly large part of $\mathfrak{F}(\mathbf{F}_0)$. Moreover $\psi_{\lambda,n}$ is a conformal conjugacy between $\mathbf{F}_0^P$ and $\mathbf{F}_\lambda^P$ on $\cup_{0\leq T<P}\mathbf{F}^{-nP+T}(E_0(\varepsilon))$.

As $n \to \infty$, $\psi_{\lambda,n}$ stabilizes and converges to a quasiconformal map ψ_λ conjugating $\mathbf{F}_0^P$ to $\mathbf{F}_\lambda^P$ everywhere. Moreover, ψ_λ is conformal on the Fatou set, and has zero dilatation almost everywhere on the Julia set. By Weyl's lemma, ψ_λ is a linear conjugacy between $\mathbf{F}_0$ and $\mathbf{F}_\lambda$.

Suppose for a contradiction that the family $\{\mathbf{F}_\lambda\}$ is trivial. Without loss of generality, we can change λ such that $\psi_\lambda(z) = \lambda z$. Then, within the global parameter space $\boldsymbol{\mathcal{W}}^u$, we have a one-dimensional slice $\mathbf{F}_\lambda = \{\psi_\lambda \circ \mathbf{F}_0 \circ \psi_\lambda^{-1}\}_{\lambda\in\mathbb{C}^*}$. For all $n < 0$, denote the n^{th} anti-renormalization of $\mathbf{F}_\lambda$ by $\mathbf{F}_{\lambda,n}$. AS $n \to -\infty$, we have $\mathbf{F}_{\lambda,n} \to \mathbf{F}_*$

$$\mathbf{F}_* = \lim_{n\to-\infty} \mathbf{F}_{\lambda,n} = \lim_{n\to\infty} \psi_\lambda \circ \mathbf{F}_{0,n} \circ \psi_\lambda^{-1} = \psi_\lambda \circ \mathbf{F}_* \circ \psi_\lambda^{-1}.$$

However, the only holomorphic map which commutes with the linear map ψ_λ for all λ is a linear map, but $\mathbf{F}_*^P$ is not a linear map for any $P \in \mathbf{T}_{>0}$. This is a contradiction. □

At last, we have proven that the corona renormalization fixed point f_* is hyperbolic with one-dimensional local unstable manifold. The proof of Theorem A is finally complete. Let us conclude with a proof of Corollary C.

Corollary 7.14. *Consider a small Banach neighborhood $N(f)$ of a (d_0, d_∞)-critical quasicircle map f of preperiodic type rotation number θ. The space S of maps in $N(f)$ which admits a (d_0, d_∞)-critical Herman quasicircle of rotation number θ forms an analytic submanifold of $N(f)$ of codimension at most one. The Herman quasicircles of maps in S move holomorphically.*

Proof. By Lemma 4.6, there is a compact analytic corona renormalization operator $\mathcal{R}_1$ on a neighborhood of f such that $\mathcal{R}_1 f$ is sufficiently close to f_*, and thus it lies in the stable manifold of f_*. Then, the preimage $S := \mathcal{R}_1^{-1}(\mathcal{W}^s_{loc})$ is an analytic submanifold of the Banach neighborhood of f consisting of perturbations of f which admit a (d_0, d_∞)-critical Herman quasicircle of rotation number θ. Since the codimension of $\mathcal{W}^s_{loc}$ is one, there is an analytic function $\phi : \mathcal{U} \to \mathbb{C}$ on a Banach neighborhood $\mathcal{U}$ of f_* such that $\mathcal{W}^s_{loc} = \phi^{-1}(0)$. Therefore, S is the zero set of $\phi \circ \mathcal{R}_1$ and so the codimension of S is at most one.

The Herman quasicircle of a corona in $\mathcal{W}^s_{loc}$ moves holomorphically over $\mathcal{W}^s_{loc}$ due to λ-lemma. Since $\mathcal{R}_1$ is analytic, the Herman quasicircles of maps in S also move holomorphically over S. □

Appendix A. Small Orbits Theorem

Consider a complex Banach space $\mathcal{B}$. Given a linear operator $L : \mathcal{B} \to \mathcal{B}$, denote the corresponding set of eigenvalues by $\text{spec}\,(L)$. We say that an eigenvalue $\lambda \in \text{spec}\,(L)$ is *attracting* if $|\lambda| < 1$, *neutral* if $|\lambda| = 1$, and *repelling* if $|\lambda| > 1$. In this appendix, we prove the following theorem.

Theorem A.1 (Small Orbits Theorem)**.** *Let $R : (\mathcal{U}, 0) \to (\mathcal{B}, 0)$ be a compact analytic operator on a neighborhood $\mathcal{U}$ of 0 in a complex Banach space $\mathcal{B}$. If the differential $DR_0 : \mathcal{B} \to \mathcal{B}$ has a neutral eigenvalue, then R has slow small orbits: for any neighborhood $\mathcal{V}$ of 0, there is an orbit $\{R^n g\}_{n \in \mathbb{N}}$ in $\mathcal{V}$ such that*

$$\lim_{n \to \infty} \frac{1}{n} \log \|R^n g\| = 0.$$

In the absence of repelling eigenvalues of DR_0, the theorem above was proven by Lyubich in [Lyu99, §2]. The original Small Orbits Theorem was a vital ingredient in the proof of hyperbolicity of quadratic-like renormalization horseshoe [Lyu99, Lyu02] and more recently the proof of hyperbolicity of pacman renormalization fixed points [DLS20]. Below we will generalize Lyubich's proof. The key addition is the application of two invariant cones, namely the center-stable cone $\mathcal{C}^{cs}$ and the center-unstable cone $\mathcal{C}^{cu}$.

Proof. Let R be as in the hypothesis. The only non-trivial case left to consider is when DR_0 has both attracting and repelling eigenvalues as well. Denote the unit disk in $\mathbb{C}$ by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. We present the Banach space $\mathcal{B}$ as a direct sum

$$\mathcal{B} = E^s \oplus E^c \oplus E^u,$$

where subspaces E^s, E^c, E^u are invariant under DR_0 and

$$\text{spec}\,(DR_0|_{E^s}) \subset \mathbb{D}, \quad \text{spec}\,(DR_0|_{E^c}) \subset \partial\mathbb{D}, \quad \text{spec}\,(DR_0|_{E^u}) \subset \mathbb{C} \backslash \overline{\mathbb{D}}.$$

Note that the spectrum can only accumulate at 0 because R is a compact operator. In particular, the subspace $E^c \oplus E^u$ must be finite dimensional.

For $h \in \mathcal{B}$, we will write $h = h^s + h^c + h^u$, where for $a \in \{s, c, u\}$, h^a is the projection of h onto the subspace E^a. We will also denote by h^{cs} and h^{cu} the projections of h onto the subspaces $E^s \oplus E^c$ and $E^c \oplus E^u$ respectively.

Fix a small constant $\delta > 0$. For $a \in \{s, c, u\}$, we denote by $D^a = D^a(\delta)$ the open ball of radius δ centered at 0 in E^a. Let

$$D = D^s \times D^c \times D^u$$

the corresponding open polydisk of side length $\delta > 0$ centered at 0 in $\mathcal{B}$. We will decompose the boundary of the polydisk $D = D(\delta)$ as follows:

$$\partial^s D := \partial D^s \times D^c \times D^u, \qquad \partial^c D := D^s \times \partial D^c \times D^u, \qquad \partial^u D := D^s \times D^c \times \partial D^u.$$

There exist an adapted norm $\|\cdot\|$ on $\mathcal{B}$ and some positive constants $\mu_s, \mu_{cs}, \mu_{cu}, \mu_u$ such that $\mu_s < 1 < \mu_u$, $\mu_s < \mu_{cu}$, $\mu_{cs} < \mu_u$, and

$$\begin{aligned}
&\|DR_0 h\| \leq \mu_s \|h\| && \text{for all } h \in E^s,\\
&\|DR_0 h\| \leq \mu_{cs} \|h\| && \text{for all } h \in E^{cs},\\
&\|DR_0 h\| \geq \mu_{cu} \|h\| && \text{for all } h \in E^{cu},\\
&\|DR_0 h\| \geq \mu_u \|h\| && \text{for all } h \in E^u.
\end{aligned}$$

Fix $\alpha > 1$. Consider a pair of cone fields C^{cu} and C^{cs} given by

$$C_f^{cu} = \{h \in T_f\mathcal{U} \,:\, \alpha\|h^s\| \leq \|h^{cu}\|\}, \quad C_f^{cs} = \{h \in T_f\mathcal{U} \,:\, \alpha\|h^u\| \leq \|h^{cs}\|\}$$

for each $f \in \mathcal{U}$.

Claim 1. Suppose $\alpha < \min\left\{\frac{\mu_{cu}}{\mu_s}, \frac{\mu_u}{\mu_{cs}}\right\}$. For sufficiently small $\delta > 0$, the following properties hold.

(1) If $f \in \overline{D}$, then $Rf \notin \partial^s D$;
(2) If $f \in \partial^u D$, then $Rf \notin \overline{D}$;
(3) The cone field C^{cu} is forward invariant: if $f, Rf \in D$, then
$$DR_f(C_f^{cu}) \subset C_{Rf}^{cs};$$
(4) The cone field C^{cs} is backward invariant: if $f, Rf \in D$, then
$$(DR_f)^{-1}(C_{Rf}^{cs}) \subset C_f^{cs}.$$

Proof. Fix a small constant $\varepsilon > 0$. We can assume that δ is sufficiently small depending on ε such that the difference

$$Gf := Rf - DR_0 f$$

on $\overline{D}$ has C^1 norm bounded by ε, that is, for all $f \in \overline{D}$ and $h \in T_f\mathcal{U}$,

$$\|Gf\| \leq \varepsilon\|f\|, \quad \text{ and } \|DG_f h\| \leq \varepsilon\|h\|.$$

When f lies in $\overline{D}$,

$$\|(Rf)^s\| \leq \|DR_0|_{E^s}(f^s)\| + \|(Gf)^s\| \leq \mu_s\|f^s\| + \varepsilon\|f\|.$$

Assuming $\mu_s + 3\varepsilon < 1$, we then have $\|(Rf)^s\| < \delta$. Additionally, when $\|f^u\| = \delta$,

$$\|(Rf)^u\| \geq \|DR_0|_{E^u}(f^u)\| - \|(Gf)^u\| \geq \mu_u\delta - \varepsilon\|f\|.$$

Assuming $\mu_u - 3\varepsilon > 1$, we then have $\|(Rf)^u\| > \delta$. Hence, (1) and (2) hold.

Suppose $f, Rf \in D$. If $h \in C_f^{cu}$, then

$$\begin{aligned}\|(DR_f h)^{cu}\| &= \|DR_0|_{E^c\oplus E^u}(h^{cu}) + (DF_f(h))^{cu}\| \\ &\geq \mu_{cu}\|h^{cu}\| - \varepsilon\|h\| \\ &\geq \left(\mu_{cu} - \varepsilon\left(1 + \frac{1}{\alpha}\right)\right)\|h^{cu}\|,\end{aligned}$$

and

$$\begin{aligned}\alpha\|(DR_f h)^s\| &= \alpha\|DR_0|_{E^s}(h^s) + (DF_f(h))^s\| \\ &\leq \alpha\left(\mu_s\|h^s\| + \varepsilon\|h\|\right) \\ &\leq \left(\alpha\mu_s + (\alpha+1)\varepsilon\right)\|h^{cu}\|.\end{aligned}$$

Since $\mu_{cu} - \alpha\mu_s > 0$, for sufficiently small $\varepsilon > 0$ depending on α, we have $DR_f h \in C_{Rf}^{cu}$. The proof that the cone field C^{cs} is backward invariant works in a similar way, assuming ε is sufficiently small depending on μ_{cs} and μ_u. □

Let us consider the perturbation $R_\lambda := \lambda \cdot R$ for $0 < \lambda < 1$. Clearly, for λ sufficiently close to 1, R_λ also satisfies all the properties listed in Claim 1. The following claim is a consequence of Lemma A.2.

Claim 2. There exists some point $f_\lambda \in \partial^c D$ such that the orbit $\{R_\lambda^n f_\lambda\}_{n\in\mathbb{N}}$ lies entirely inside of $\overline{D}$ and $R_\lambda^n f_\lambda \to 0$.

Since R is compact, there exist an increasing sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of positive numbers and some $g \in \overline{D}$ such that as $n \to \infty$, $\lambda_n \to 1$ and $R_{\lambda_n} f_{\lambda_n} \to g$. Clearly, for all $n \in \mathbb{N}$, the n^{th} iterate $g_n := R^n g$ lies in $\overline{D}$.

As f_λ lies in $\partial^c D$, f_λ lies in the cone $C_0^{cu} = \{\|h^s\| \leq \|h^{cu}\|\}$. Similar to the proof of Claim 1 (3), C_0^{cu} is forward invariant under R_λ for $\lambda \leq 1$. Hence, for every $n \in \mathbb{N}$, $\|g_n^s\| \leq \|g_n^{cu}\|$. This implies that for every $n \in \mathbb{N}$,

$$g_{n+1}^{cu} = DR_0|_{E^c\oplus E^u}(g_n^{cu}) + O\left(\|g_n^{cu}\|^2\right). \tag{A.1}$$

At last, we will show that the orbit of g is a slow small orbit. Indeed, suppose for a contradiction that

$$\liminf_{n\to\infty} \frac{1}{n} \log \|g_n\| < -c \tag{A.2}$$

for some constant $c > 0$. Note that this property holds for every norm that is equivalent to $\|\cdot\|$. Pick some $\kappa' \in [1, e^c)$. There exists an adapted norm $\|\cdot\|$ equivalent to the original one such that the operator norm of $DR_0|_{E^c\oplus E^u}^{-1}$ is at most κ'. By (A.1), for sufficiently small $\delta > 0$, there is some $\kappa \in (1, e^c)$ such that

$$\|g_{n+1}^{cu}\| \geq \kappa^{-n} \|g_n^{cu}\| \qquad \text{for all } n \in \mathbb{N}.$$

This contradicts (A.2). □

It remains to prove Claim 2. This follows directly from the lemma below. For any $r > 0$, we denote the open disk $\{z \in \mathbb{C} : |z| < r\}$ by $\mathbb{D}_r$.

Lemma A.2. *Let $R : (U, 0) \to (\mathcal{B}, 0)$ be a compact analytic map on a neighborhood U of the polydisk D such that the differential DR_0 preserves the decomposition $\mathcal{B} = E^s \oplus E^c \oplus E^u$ and satisfies the following properties.*

(1) *Hyperbolicity: There exists some $0 < r < 1$ such that*

$$\operatorname{spec}(DR_0|_{E^s}) \subset \mathbb{D}_r, \quad \operatorname{spec}(DR_0|_{E^c}) \subset \mathbb{D}\backslash\mathbb{D}_r, \quad \operatorname{spec}(DR_0|_{E^u}) \subset \mathbb{C}\backslash\overline{\mathbb{D}}.$$

(2) *Boundary behaviour: If $f \in \overline{D}$, then $Rf \notin \partial^s D$. If $f \in \partial^u D$, then $Rf \notin \overline{D}$.*
(3) *Invariant cone fields: Whenever $f, Rf \in \overline{D}$,*

$$DR_f(C_f^{cu}) \subset C_{Rf}^{cu}, \qquad (DR_f)^{-1}(C_{Rf}^{cs}) \subset C_f^{cs}.$$

Then, there exists some $f \in \partial^c D$ such that $\{R^n f\}_{n\in\mathbb{N}} \subset \overline{D}$ and $\|R^n f\| \to 0$ as $n \to \infty$.

Proof. By the compactness of R, the subspace $E^c \oplus E^u$ is finite dimensional. Let $d_c := \dim(E^c)$ and $d_u := \dim(E^u)$. By (1), the stable manifold

$$A = \{f \in \overline{D} : \{R^n f\}_{n\in\mathbb{N}} \subset \overline{D} \text{ and } \|R^n f\| \to 0\}$$

exists and is a forward invariant analytic submanifold of codimension d_u. Suppose for a contradiction that A is disjoint from $\partial^c D$.

Claim 1. The set $A^o := A \cap D$ is a forward invariant open submanifold of A.

Proof. The only non-trivial property to prove here is forward invariance. Suppose $f \in A^o$. As $f \in A$, then $Rf \in \overline{D}$. By (2), Rf cannot lie in $\partial^s D \cup \partial^u D$. By assumption, Rf cannot lie in $\partial^c D$ either. Thus, $Rf \in D$. □

Claim 2. The tangent space $T_f A^o$ at every point f in A^o lies within the cone C_f^{cs}.

Proof. Let $f \in A^o$. As A^o is tangent to the subspace $E^s \cup E^c$ at 0, for all sufficiently high n, $R^n f$ is sufficiently close to 0 and so the tangent space $T_{R^n f} A^o$ lies within $C^{cs}_{R^n f}$. By backward invariance of C^{cs} in (3), the tangent space of A^o at f also lies within C^{cs}_f. □

Claim 3. The set $\partial^c A := \overline{A} \backslash (A^o \cup \partial^s D)$ is forward invariant.

Proof. Suppose for a contradiction that there is some $f \in \partial^c A$ such that $Rf \in A^o \cup \partial^s D$. By (2), Rf must lie in A^o, which implies that $f \in A \cap (\partial^c D \cup \partial^u D)$. However, this is impossible because f cannot lie in $\partial^c D$ by our main assumption, nor $\partial^u D$ due to (2). □

Let's consider the family $\mathcal{G}$ of all immersed analytic d_c-dimensional submanifolds Γ of A^o containing 0 with the following properties.

(a) The tangent space $T_f \Gamma$ at every point $f \in \Gamma$ lies in the cone C^{cu}_f;

(b) The accumulation set $\overline{\Gamma} \backslash \Gamma$ lies in $\partial^c A$.

Dima Dudko pointed out that $\mathcal{G}$ is non-empty: it contains $A^o \cap (E^c \oplus E^u)$. Indeed, by Claim 2, the intersection between A^o and the subspace $E^c \oplus E^u$ is transversal. Another consequence of Claim 2 is the following claim.

Claim 4. For every $\Gamma \in \mathcal{G}$ and $h \in T_f \Gamma$, $\|h^c\| \asymp \|h\|$. In particular, the projection $P : \Gamma \to D^c$ is non-singular.

Proof. Let $h \in T_f \Gamma$. By Property (a) and Claim 2, $\alpha \|h^s\| \le \|h^{cu}\|$ and $\alpha \|h^u\| \le \|h^{cs}\|$. By triangle inequality, these imply that $(\alpha - 1) \max\{\|h^s\|, \|h^u\|\} \le \|h^c\|$ and consequently $\|h^c\| \le \|h\| \le \frac{\alpha+1}{\alpha-1} \|h^c\|$. □

Recall that the Kobayashi norm of a tangent vector $v \in T_f \Gamma$ at a point f on a complex manifold Γ is defined as

$$\|h\|_\Gamma := \inf \{\|w\|_{\mathbb{D}} \,:\, D\phi_f(w) = h \text{ for some holomorphic map } \phi : (\mathbb{D}, 0) \to (\Gamma, f)\}$$

where $\|w\|_{\mathbb{D}}$ denotes the Poincaré metric of $w \in T_0 \mathbb{D}$ on the unit disk $\mathbb{D}$. We will supply every $\Gamma \in \mathcal{G}$ with the Kobayashi metric.

Claim 5. There is some $K > 0$ such that for every $\Gamma \in \mathcal{G}$ and $h \in T_0 \Gamma$, $\|h\|_\Gamma \le K \|h\|$.

Proof. By Claim 4, there is some $\delta > 0$ such that for every $\Gamma \in \mathcal{G}$, the component $\Gamma(\delta)$ of $\Gamma \cap D^c(\delta)$ containing 0 is a graph of an analytic map $D^c(\delta) \to D^s \times D^u$. Therefore, for any $h \in T_0 \Gamma$,

$$\|h\|_\Gamma \le \|h\|_{\Gamma(\delta)} = \|h^c\|_{D^c(\delta)}.$$

Clearly, $\|h^c\|_{D^c(\delta)} \asymp \|h^c\|$ (with bounds depending only on δ). By Claim 4, this yields the desired inequality $\|h\|_\Gamma \le K \|h\|$ for some K independent of Γ. □

Property (3) and Claim 3 imply that the map R induces a well-defined graph transform

$$R_* : \mathcal{G} \to \mathcal{G}, \quad \Gamma \mapsto R\Gamma.$$

Note that $R : \Gamma \to R\Gamma$ is a proper non-singular map, hence a holomorphic covering map. Therefore, for any $\Gamma \in \mathcal{G}$, $n \in \mathbb{N}$, and non-zero tangent vector $h \in T_0 \Gamma$,

$$\|h\|_\Gamma = \|(DR^n)_0(h)\|_{R^n_* \Gamma}.$$

By Claim 5,

$$\|h\|_\Gamma \le K \|(DR^n)_0(h)\|.$$

However, by (1), $\|(DR^n)_0(h)\|$ tends to 0 as $n \to \infty$. This yields a contradiction. □

Appendix B. Sector renormalization

B.1. **Renormalization of rotations and translations.** Let us equip the unit circle $\mathbb{T} \subset \mathbb{C}$ with the normalized Euclidean metric. Consider the rotation

$$\mathbb{L}_\theta : \mathbb{T} \to \mathbb{T}, z \mapsto e^{2\pi i\theta} z$$

by an angle $2\pi\theta \in \mathbb{T}$. Pick any point $x \in \mathbb{T}$ and consider the shortest interval $Y \subset \mathbb{T}$ between x and $\mathbb{L}_\theta(x)$. Consider the pair of intervals

$$X_- := \mathbb{L}_\theta^{-1}(Y), \quad X_+ := \overline{\mathbb{T}\backslash(Y \cup X_-)}.$$

Then, the first return map of $X_- \cup X_+$ is precisely the commuting pair

$$(\mathbb{L}_\theta|_{X_+}, \mathbb{L}_\theta^2|_{X_-}),$$

Let us assume that $1 \neq Y$ and denote by ω the length of $X_- \cup X_+$. Then, the map $z \mapsto z^{1/\omega}$ projects the commuting pair to a new rotation $\mathbb{L}_{R_{prm}(\theta)}$ called the *prime renormalization* of $\mathbb{L}_\theta$. Note that $\mathbb{L}_{R_{prm}(\theta)}$ is independent of the initial choice of x.

Lemma B.1 ([DLS20, Lemma A.1])**.** *We have*

$$R_{prm}(\theta) = \begin{cases} \dfrac{\theta}{1-\theta}, & \text{if } 0 \leq \theta \leq \dfrac{1}{2}, \\ \dfrac{2\theta - 1}{\theta}, & \text{if } \dfrac{1}{2} \leq \theta \leq 1. \end{cases}$$

In general, we define a *sector renormalization* $\mathcal{R}(\mathbb{L}_\theta)$ of $\mathbb{L}_\theta$ as follows. First, consider a pair of intervals X_- and X_+ on $\mathbb{T}$ satisfying $X_- \cap X_+ = \{1\}$. The first return map on $X := X_- \cup X_+$, which we call a sector pre-renormalization, will be a pair of forward iterates of the form

$$(\mathbb{L}_\theta^{\mathbf{a}}|_{X_-}, \mathbb{L}_\theta^{\mathbf{b}}|_{X_+}). \tag{B.1}$$

The positive integers $\mathbf{a}$ and $\mathbf{b}$ are called the *renormalization return times* of $\mathcal{R}$. Denote by ω the length of X, then the map $z \mapsto z^{1/\omega}$ glues the endpoints of X together and projects the pair (B.1) to a new rotation $\mathbb{L}_\mu = \mathcal{R}(\mathbb{L}_\theta)$.

Lemma B.2 ([DLS20, Lemma A.2])**.** *Sector renormalization $\mathcal{R}$ is an iteration of the prime renormalization. In particular, $\mu = R_{prm}^m(\theta)$ for some $m \geq 1$, and $\mathbb{L}_\theta$ is a fixed point of some sector renormalization if and only if $\theta \in \Theta_{per}$.*

Under the universal cover $\mathbb{R} \to \mathbb{T}, z \mapsto e^{-2\pi i z}$, the rotation $\mathbb{L}_\theta$ can be lifted to the commuting pair of translations

$$T_{-\theta} : z \mapsto z - \theta, \qquad T_{1-\theta} : z \mapsto z + 1 - \theta.$$

Notice that the deck transformation $\chi := T_1$ is equal to $T_{1-\theta} \circ T_{-\theta}^{-1}$, and the original rotation $\mathbb{L}_\theta$ can be recovered from $T_{-\theta}/\langle\chi\rangle$.

Consider a general commuting pair $(T_{-\mathbf{u}}, T_{\mathbf{v}})$ where $\mathbf{u}, \mathbf{v} \in \mathbb{R}_{\geq 0}$. The prime renormalization $\mathcal{R}_{prm}$ of $(T_{-\mathbf{u}}, T_{\mathbf{v}})$ is the new commuting pair $(T_{-\mathbf{u}_1}, T_{\mathbf{v}_1})$ defined as follows.

$$(T_{-\mathbf{u}_1}, T_{\mathbf{v}_1}) := \begin{cases} (T_{-\mathbf{u}} \circ T_{\mathbf{v}}, \quad T_{\mathbf{v}} \quad) & \text{if } \mathbf{u} \geq \mathbf{v}, \\ (\quad T_{-\mathbf{u}} \quad, T_{-\mathbf{u}} \circ T_{\mathbf{v}}) & \text{if } \mathbf{u} < \mathbf{v}. \end{cases} \tag{B.2}$$

Set $\chi := T_{\mathbf{v}} \circ T_{-\mathbf{u}}^{-1}$ and $\chi_1 = T_{\mathbf{v}_1} \circ T_{-\mathbf{u}_1}^{-1}$.

Lemma B.3. *If $T_{-\mathbf{u}}/\langle\chi\rangle \equiv \mathbb{L}_\theta$, then $\theta = \frac{\mathbf{v}}{\mathbf{u}+\mathbf{v}}$ and $T_{-\mathbf{u}_1}/\langle\chi_1\rangle \equiv \mathbb{L}_{R_{prm}(\theta)}$.*

B.2. **Cascade of translations.** Suppose θ is periodic, that is, $R^m_{prm}(\theta) = \theta$ for some $m \in \mathbb{N}$. Set $\mathbf{u} = \theta$ and $\mathbf{v} = 1 - \theta$. By B.2, there is a unique matrix 2×2 matrix $\mathbf{M}$ of the form $I_1 I_2 \dots I_m$, where each I_i is either $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, such that the m^{th} prime renormalization $(T_{-\mathbf{u}_1}, T_{\mathbf{v}_1}) \coloneqq \mathcal{R}_{prm}(T_{-\mathbf{u}}, T_{\mathbf{v}})$ satisfies

$$\begin{pmatrix} -\mathbf{u}_1 \\ \mathbf{v}_1 \end{pmatrix} = \mathbf{M} \begin{pmatrix} -\mathbf{u} \\ \mathbf{v} \end{pmatrix}.$$

Note that $\mathbf{M}$ lies in the modular group $\mathrm{SL}_2(\mathbb{Z})$ mapping a sector in $\mathbb{R}_{\le 0} \times \mathbb{R}_{\ge 0}$ onto $\mathbb{R}_{\le 0} \times \mathbb{R}_{\ge 0}$. The condition $R^m_{prm}(\theta) = \theta$ implies that $\begin{pmatrix} -\mathbf{u}_1 \\ \mathbf{v}_1 \end{pmatrix}$ is a scalar multiple of $\begin{pmatrix} -\mathbf{u} \\ \mathbf{v} \end{pmatrix}$. We see that $\mathbf{M}$ has two eigenvalues $\mathbf{t} > 1$ and $1/\mathbf{t} < 0$, and that

$$\begin{pmatrix} -\mathbf{u}_1 \\ \mathbf{v}_1 \end{pmatrix} = \frac{1}{\mathbf{t}} \begin{pmatrix} -\mathbf{u} \\ \mathbf{v} \end{pmatrix}.$$

We call $\mathbf{M}$ the *anti-renormalization matrix* associated with θ.

Observe that $\mathbf{M}$ has to be a matrix of positive integers and $\mathbf{t} \notin \mathbb{Q}$. We set $R \coloneqq R^m_{prm}$ and $\mathcal{R} = \mathcal{R}^m_{prm}$. For $n \in \mathbb{N}$, we write

$$\mathbf{u}_n = \mathbf{t}^{-n} \mathbf{u} \quad \text{and} \quad \mathbf{v}_n = \mathbf{t}^{-n} \mathbf{v}.$$

We then have a full pre-renormalization tower $\{(T_{-\mathbf{u}_n}, T_{\mathbf{v}_n})\}_{n \in \mathbb{Z}}$ where

$$\mathcal{R}(T_{-\mathbf{u}_n}, T_{\mathbf{v}_n}) = (T_{-\mathbf{u}_{n+1}}, T_{\mathbf{v}_{n+1}}).$$

Given a *power-triple* $(n, a, b) \in \mathbb{Z} \times Z_{\ge 0} \times \mathbb{Z}_{\ge 0}$, we write

$$T^{(n,a,b)} \coloneqq T^a_{-\mathbf{u}_n} \circ T^b_{\mathbf{v}_n} = T_{\mathbf{t}^{-n}(b\mathbf{v} - a\mathbf{u})}.$$

Lemma B.4. *For any pair of power-triples (n, a, b) and (n', a', b'), $T^{(n,a,b)} = T^{n', a', b'}$ if and only if $(a, b)\mathbf{M}^n = (c, d)\mathbf{M}^{n'}$.*

If we write $\sigma_n(n, (a, b)) \coloneqq (n - 1, (a, b)\mathbf{M})$, then

$$T^{(n,a,b)} = T^{\sigma_n(n,a,b)}.$$

Definition B.5. We define the *semigroup of power-triples* $\mathbf{T}$ as the quotient $\mathbb{Z} \times \mathbb{Z}^2_{\ge 0}/_\sim$ under the equivalence relation $(n, a, b) \sim \sigma_n(n, a, b)$. It acts naturally on $\mathbb{R}$ by translations as a *cascade* $(T^P)_{P \in \mathbf{T}}$.

The previous lemma tells us that the cascade of translations $(T^P)_{P \in \mathbf{T}}$ acts freely on $\mathbb{R}$.

For any $c \in \mathbb{C}$, denote the corresponding linear map by $A_c : \mathbb{C} \to \mathbb{C}, z \mapsto cz$.

Lemma B.6 ([DL23, Lemma 2.2])**.** *There is an embedding $\iota : \mathbf{T} \to \mathbf{R}$ such that $\iota(n - 1, a, b) = \mathbf{t}^{-1} \iota(n, a, b)$. Identifying $\mathbf{T}$ with $\iota(\mathbf{T}) \subset \mathbb{R}$ equips $\mathbf{T}$ with*

1. *a linear order $\ge$;*
2. *subtraction, that is, if $P, T \in \mathbf{T}$ and $P \ge T$, then $P - T \in \mathbf{T}$;*
3. *scalar multiplication by $\mathbf{t}$: $P = (n, a, b) \mapsto \mathbf{t}P = (n + 1, a, b)$, which is an automorphism of $\mathbf{T}$.*

Moreover, for any $P \in \mathbf{T}$ and $n \in \mathbb{Z}$,

$$T^P = A_{\mathbf{t}^n} \circ T^{\mathbf{t}^n P} \circ A_{\mathbf{t}^{-n}}.$$

Lemma B.7 (Proper discontinuity, [DL23, Lemma 2.3])**.** *If $P \in \mathbf{T}_{>0}$ is small, then $|T^P(0)|$ is large.*

For all $P \in \mathbf{T}$, let us denote $b_P := T^{-P}(0)$. We say that b_P is *dominant* if every b_Q on $[0, b_P]$ satisfies $Q \geq P$. By proper discontinuity, we can enumerate all dominant points $\{b_{P_n}\}_{n\in\mathbb{Z}}$ such that $P_n < P_{n+1}$ for all n.

Lemma B.8 ([DL23, Lemma 2.4])**.** *For every $i \in \mathbb{Z}$, there exist some $Q_i \in \mathbf{T}_{>0}$ and some integers m, n such that $n < m \leq i$ and T^{Q_i} maps $[b_{P_i}, b_{P_{i+1}}]$ to $[b_{P_n}, b_{P_m}]$.*

Appendix C. Key lemma for transcendental extension

In this appendix, we will provide the proof of Lemma 5.6. The proof present below is similar to the Key Lemma in [DLS20], which is to ensure that pullbacks of D avoid the forbidden boundary.

Proof of Lemma 5.6. Pick a large $s \in \mathbb{N}$ and choose the neighborhood $\mathcal{U}$ of f_* such that every $f \in \mathcal{R}^{-n}(\mathcal{U})$ is $m := n+s$ times renormalizable, and for each $i \in \{1, \ldots, m\}$, $f_i := \mathcal{R}^i f$ is close to f_*.

Pick $f \in \mathcal{R}^{-n}(\mathcal{U})$. Let h be a level m combinatorial pseudo-conjugacy between f and f_*, and consider the renormalization tiling $\boldsymbol{\Delta}_m(f) := h^{-1}(\boldsymbol{\Delta}_m(f_*))$. By Theorem 4.10, h is close to the identity map and $\boldsymbol{\Delta}_m(f)$ approximates the Herman quasicircle $\mathbf{H}_*$ of f_*.

Fix a small neighborhood D of the critical value $c_1(f_*)$ of f_*. For large $n \in \mathbb{N}$ and $t \in \{\mathbf{a}_n, \mathbf{b}_n\}$, $c_{1+t}(f_*)$ is sufficiently close to $c_1(f_*)$, and so it is also contained in D. As s is picked to be large,

$$t \leq \max\{\mathbf{a}_n, \mathbf{b}_n\} - 1 < \min\{\mathbf{a}_m, \mathbf{b}_m\}.$$

Therefore, $\{c_j(f_*)\}_{j=1,2,\ldots t+1}$ never visits the tiles $\Delta_m(-\mathbf{p}_m, f_*)$ and $\Delta_m(-\mathbf{p}_{m+1}, f_*)$. For $j \in \{1, \ldots, t+1\}$, $c_j(f) = h(c_j(f_*))$. Since h is close to the identity, it follows that $c_{1+t}(f)$ also lies in D.

Let $D_0, D_1, \ldots, D_t := D$ denote the pullback of D along the orbit $c_1(f), c_2(f), \ldots, c_{1+t}(f)$. The goal is to show that for $i \in \{0, 1, \ldots, t-1\}$, the disk D_i does not intersect $\partial_{\mathrm{F}} U_f$ so that $f : D_i \to D_{i+1}$ is a branched covering.

An *interval* I in $\mathbb{Z}/\mathbf{q}_m\mathbb{Z}$ is a sequence of consecutive elements in $\mathbb{Z}/\mathbf{q}_m\mathbb{Z}$ of cardinality $< \mathbf{p}_m$. For any interval I in $\mathbb{Z}/\mathbf{q}_m\mathbb{Z}$, we write $\Delta_m(I) := \bigcup_{i\in I} \Delta_m(i)$ and

$$f^{-1}(I) = \begin{cases} I - \mathbf{p}_m & \text{if } I \cap \{-\mathbf{p}_m, -\mathbf{p}_m + 1, 0, 1\} = \varnothing, \\ (I - \mathbf{p}_m) \cup \{0, 1\} & \text{if } I \cap \{0, 1\} \neq \varnothing, \\ (I - \mathbf{p}_m) \cap \{-\mathbf{p}_m, -\mathbf{p}_m + 1\} & \text{if } I \cap \{-\mathbf{p}_m, -\mathbf{p}_m + 1\} \neq \varnothing. \end{cases}$$

Claim 1. For any interval I in $\mathbb{Z}/\mathbf{q}_m\mathbb{Z}$, the preimage of $\Delta_m(I)$ under $f|_{\boldsymbol{\Delta}_m}$ is contained in $\Delta_m(f^{-1}I)$.

Consider the dynamical plane of $f_m = \mathcal{R}^m f : U_m \to V$. For $i \in \{0, 1\}$, let $\Lambda_0(i, f_m)$ denote the closure of the connected component of $f_m^{-1}(U_m) \backslash \gamma_0(f_m) \cup \gamma_1$ contained in $\Delta_m(i)$. By spreading around, this produces a tiling $\boldsymbol{\Delta}_0(f_m)$, which is a skinnier version of $\Delta_0(f_m)$. We then embed it via Φ_m to the dynamical plane of f and spread it around to obtain the tiling $\boldsymbol{\Lambda}_m(f)$.

Claim 2. For any interval I in $\mathbb{Z}/\mathbf{q}_m\mathbb{Z}$, $\Lambda_m(I) = \boldsymbol{\Lambda}_m \cap \Lambda_m(I)$ and the preimage of $\Lambda_m(I)$ under $f|_{\boldsymbol{\Lambda}_m}$ is contained in $\Lambda_m(f^{-1}I)$.

The problem with the tiling $\mathbf{\Delta}_m$ is that even when the intersection $D_i \cap \mathbf{\Delta}_m$ is contained in $\Delta_m(I)$ for some interval I, it is possible that $D_{i-1} \cap \mathbf{\Delta}_m$ is not contained in $\Delta_m(f^{-1}(I))$. However, this issue does not occur for the tiling $\mathbf{\Lambda}_m$.

Claim 3. For any interval I in $\mathbb{Z}/\mathbf{q}_m\mathbb{Z}$, positive integer $i < \min\{\mathbf{a}_m, \mathbf{b}_m\}$, and any subset $T \subset V$,

$$T \cap \mathbf{\Delta}_m \subset \Delta_m(I) \Longrightarrow f^{-i}(T) \cap \mathbf{\Lambda}_m \subset \Lambda_m(f^{-i}(I)).$$

Consider the smallest interval $I_t \subset \mathbb{Z}/\mathbf{q}_m\mathbb{Z}$ containing $\{0,1\}$ such that $\Delta_m(I_t)$ contains the intersection $D_t \cap \mathbf{\Delta}_m(f)$. For $j \in \{0,1,\dots t-1\}$, let $I_j := f^{-(t-j)}(I_t)$. Then, the previous claim implies that

$$D_j \cap \mathbf{\Lambda}_m \subset \Lambda_m(I_j) \quad \text{ for all } j = 0,1,\dots t-1.$$

Let us fix some integer $\eta > 1$.

Claim 4. For any $j \in \{0,1,\dots t\}$,

(i) $|I_j|/\mathbf{q}_m$ is small and $\Delta_m(I_j, f_*)/\cap \mathbf{H}_*$ has a small combinatorial length;
(ii) if $j \le t-3-\eta$, the intervals I_j, I_{j+1}, ..., $I_{j+\eta+3}$ are pairwise disjoint;
(iii) if $j \ge t-1-\eta$, then I_j is disjoint from $\{-\mathbf{p}_m, -\mathbf{p}_m+1\}$.

Let us inductively define enlargements $\mathcal{D}_j$ and $\mathcal{D}'_j$ of each D_j as follows. We set $\mathcal{D}_t = \mathcal{D}'_t := D_t$. For $j < t$, we set $\mathcal{D}'_j$ to be the connected component of $f^{-1}(\mathcal{D}_{j+1}$ containing D_j. Then, set $\mathcal{D}_j$ to be the smallest topological disk containing $\mathcal{D}'_j$ and the interior of $(\Lambda_m(I_j))$.

Claim 5. For all j, $\mathcal{D}_j \cap \mathbf{\Lambda}_m$ is connected and its closure is $\Delta_m(I_j)$.

Recall that the preimage of $f_*^{-1}(\gamma_1)\backslash\gamma_0$ consists of arcs

$$\gamma_1^0, \dots, \gamma_{2(d_0-1)}^0 \subset \partial^0 U_{f_*}, \quad \gamma_1^\infty, \dots, \gamma_{2(d_\infty-1)}^\infty \subset \partial^\infty U_{f_*}.$$

The critical point $c_0(f_*)$ is the landing point of d_∞ external rays from $\partial^\infty_{\mathrm{L}} U_{f_*}$ and d_0 external rays from $\partial^0_{\mathrm{L}} U_{f_*}$. These external rays cut out the dynamical plane of f_* and form wakes W_1^0, ..., $W_{d_0-1}^0$, W_1^∞, ..., $W_{d_\infty-1}^\infty$, where each $W_i^\bullet$ contains $\sigma_{2i-1}^\bullet \cup \sigma_{2i}^\bullet$.

Set $k' := \lceil \frac{k}{2} \rceil$. For each $\gamma_k^\bullet$, let us pick an outer/inner bubble $B_k^\bullet$ attached to $\sigma_k^\bullet$ such that $B_k^\bullet$ is close to $\gamma_k^\bullet$ and that there is some $\eta_k^\bullet \in \mathbb{N}_{\ge 1}$ such that $f_*^{\eta_k^\bullet}$ univalently lifts the wake $W_{k'}^\bullet$ to a proper subset $\tilde{W}_k^\bullet$ containing $B_k^\bullet$. By Schwarz lemma, the map $f_*^{\eta_k^\bullet} : \tilde{W}_k^\bullet \to W_{k'}^\bullet$ has a unique fixed point $x_k^\bullet$ together with a unique $f_*^{\eta_k^\bullet}$-invariant bubble chain $\mathcal{Z}_k^\bullet = (Z_1 = B_k^\bullet, Z_2, Z_3, \dots)$. Denote by $\hat{\mathcal{Z}}_k^\bullet$ the bubble chain $(Z_2, Z_3, \dots)$, and by $R_k^\bullet$ the unique periodic external ray landing at $x_k^\bullet$; both are $\eta_k^\bullet$-periodic.

Let us denote by $\hat{\eta}_i^\bullet$ the least common period of $\hat{\mathcal{Z}}_{2i-1}^\bullet$ and $\hat{\mathcal{Z}}_{2i}^\bullet$.

As f is close to f_*, periodic rays $R_k^\bullet(f)$ exist in the dynamical plane of f and are close to the rays $R_k^\bullet(f_*)$ corresponding to f_*.

Let us set $\mathbf{\Lambda}_k^\bullet$ to be the closure of the connected component of $f^{-1}(\mathbf{\Delta}_m)\backslash\mathbf{\Delta}_m$ that intersects with $\sigma_m^\bullet$. Each of them is connected and

$$\mathbf{\Lambda}_k^\bullet \cap \mathbf{\Lambda} \subset \Lambda_m(\{-\mathbf{p}_m, -\mathbf{p}_{m+1}\}), \quad f(\mathbf{\Lambda}_k^\bullet) \subset \mathbf{\Lambda}_m.$$

Denote the union by

$$\mathbf{\Lambda}'_m := \mathbf{\Lambda}_1^0 \cup \dots \cup \mathbf{\Lambda}_{2(d_0-1)}^0 \cup \mathbf{\Lambda}_1^\infty \cup \dots \cup \mathbf{\Lambda}_{2(d_\infty-1)}^\infty.$$

For f, we define a *pseudo-bubble* of generation $g+1$ to be a lift of $\mathbf{\Lambda}'_m$ under f^g. We say that $\mathbf{\Lambda}'_m$ is attached to $\Lambda_m(\{-\mathbf{p}_m, -\mathbf{p}_{m+1}\})$.

Claim 6. Let us fix a large integer $M \gg 1$. Every bubble B_* of f_* of generation up to M is approximated by a pseudo-bubble B_f of f such that

(1) B_f is close to B_* and $f|_{B_f}$ is close to $f_*|_{B_*}$,
(2) if B_* is attached to another bubble B'_*, then B_f is attached to the pseudo-bubble corresponding to B'_*;
(3) if B_* is attached to $\mathbf{H}_*$, then B_f is attached to $\Lambda_m(I)$ for some interval I disjoint from $\{0,1\}$.

We can approximate the bubble chain $\mathcal{Z}_k^\bullet$ up to index M by pseudo-bubbles $B_0 = \mathbf{\Lambda}'_m(f)$, B_1, ..., B_M of f. This approximation can be extended infinitely by taking B_j for $j > M$ to be the pullback under $f^{\eta_k^\bullet}$ near $x_k^\bullet$ of B_{j-1}. This yields a pseudo-bubble chain $\mathcal{B}_k^\bullet(f)$ landing at the corresponding fixed point $x_k^\bullet(f)$.

We assume that D_t is small enough so that it is disjoint from $f^i(R_k^\bullet)$ for $i \in \{0,\dots,t\}$, $\bullet \in \{0,\infty\}$, and $k \in \{1,2,\dots,2(d_\bullet - 1)\}$.

Claim 7. The disks $\mathcal{D}_0$, ..., $\mathcal{D}_t$ are all disjoint from rays of the form $f^i(R_k^\bullet)$.

At last, to show that $f^t : D_0 \to D_t$ is a branched covering, we will prove by induction the following statements for $j = 0, 1, \dots t$.

(a) $\mathcal{D}_j$ intersects $\mathbf{\Lambda}'_m$ if and only if I_j contains $\{-\mathbf{p}_m, -\mathbf{p}_{m+1}\}$;
(b) If $\mathcal{D}_j$ intersects $\mathbf{\Lambda}'_m$, then the intersection is in a small neighborhood of c_0;
(c) If $\mathcal{D}_j$ intersects $\mathbf{\Lambda}'_m$ for $j < t-1$, then $j < t-1-\eta$ and $\mathcal{D}_{j+1}, \dots \mathcal{D}_{j+\eta+1}$ are all disjoint from $\mathbf{\Lambda}'_m$;
(d) If $\mathcal{D}_j$ intersects some pseudo-bubble chain $\mathcal{B}_k^\bullet(f)$, then the intersection is within $\mathbf{\Lambda}'_m$;
(e) $\mathcal{D}_j$ is an open disk disjoint from the forbidden boundary $\partial_{\mathrm{F}} U_f$.

Suppose (a)–(e) hold for $j+1$, $j+2$, $\dots t$. We will show that they also hold for j.

Suppose I_j contains $\{-\mathbf{p}_m, -\mathbf{p}_{m+1}\}$. Then, $\mathcal{D}_{j+1}$ contains $\Lambda_m(\{0,1,i\})$ where $i \in \{-1,2\}$, and so the lift $\mathcal{D}'_j$ of $\mathcal{D}_j$ contains $c_0(f)$ and intersects $\mathbf{\Lambda}'_m$.

Suppose I_j is disjoint from $\{-\mathbf{p}_m, -\mathbf{p}_{m+1}\}$. Then, $\mathcal{D}_{j+1}$ does not contain the critical value $c_1(f)$ and every point in $\mathcal{D}_{j+1}$ has at most one preimage under f in $\mathcal{D}'_j$. Since $\mathcal{D}_{j+1} \cap \mathbf{\Lambda}_m$ is connected, its preimage under $f|_{\mathcal{D}'_j}$ must be contained in $\mathbf{\Lambda}_m$. It follows that $\mathcal{D}'_j$ is disjoint from $\mathbf{\Lambda}'_m$. Since $\mathcal{D}'_j \cup \Lambda_m(I_j)$ does not surround $\mathbf{\Lambda}'_m$, then $\mathcal{D}_j$ is also disjoint from $\mathbf{\Lambda}'_m$.

We just proved (a). Then, (b) follows from Claim 5 and the fact that $\Lambda_m(I_{j+1})$ is a small neighborhood of $c_1(f)$, whereas (c) then follows from Claim 4 (ii).

By continuity, we can assume without loss of generality that (d) always holds for $j \geq t-\eta$ for every $\eta = \eta_k^\bullet$. Let us assume that $j < t-\eta$ and suppose for a contradiction that (d) fails, that is, there is some pseudo-bubble chain

$$\mathcal{B} = \mathcal{B}_k^\bullet = (Z_0 = \mathbf{\Lambda}'_m, Z_1, Z_2, \dots)$$

such that $\mathcal{D}_j$ intersects $\mathcal{B}^\bullet \backslash \mathbf{\Lambda}'_m$.

There is some minimal $i \geq 1$ such that $\mathcal{D}_j$ intersects Z_i. Since $\mathcal{D}'_j \cap \Lambda_m(I_j)$ is disjoint from the ray $R_k^\bullet$, then the subchain $\mathcal{B}^{(i)} = (Z_i, Z_{i+1}, \dots)$ intersects $\mathcal{D}'_j$ and its image $f(\mathcal{B}^{(i)})$ intersects $\mathcal{D}_{j+1}$. We claim that $i = 1$. Indeed, if $i > 1$, then by periodicity of the pseudo-bubble chain, $\mathcal{B}^{i-1}$ intersects $\mathcal{D}_{j+\eta}$, which is a contradiction to (4) for index $j+\eta$.

By Claim 6 (iii), each of the pseudo-bubbles $f(Z_1)$, $f^2(Z_1)$, $\ldots$, $f^\eta(Z_1)$ is attached to $\mathbf{\Lambda}_m \backslash \Lambda_m(\{0,1\})$. By a similar inductive argument, $\mathcal{B}$ must intersect $\mathcal{D}_{j+\eta}$. By (4) for index $j+\eta$, the disk $\mathcal{D}_{j+\eta}$ is disjoint from $\mathcal{B}^{(1)}$, so then $\mathcal{D}_{j+\eta}$ intersects $\mathbf{\Lambda}'_m$. By (1) for index $j+\eta$, the interval $I_{j+\eta}$ contains $\{-\mathbf{p}_m, -\mathbf{p}_{m+1}\}$, so for $l \in \{1, 2, \ldots, \eta\}$, $f^l(Z_1)$ is attached to $\Lambda_m(I_{j+l})$. Moreover, since c_1 isn't contained in $\mathcal{D}_{j+l} \cap \mathbf{\Lambda}_m$, every point in $\mathcal{D}_{j+l}$ has at most one preimage under $f|_{\mathcal{D}'_{j+l-1}}$.

Consider the lift Z'_1 of $f(Z_1)$ under f that is attached to $\Lambda_m(I_j)$. Since c_1 is not contained nor surrounded by $\mathcal{D}_{j+1} \cap f(\mathbb{Z}_1)$, the lift E of $f(\mathcal{D}_j \cap Z_1)$ under $f|_{\mathcal{D}'_j}$ agrees with the lift under $f|_{Z'_1}$. Therefore, E would be contained in Z'_1, not Z_1. This is a contradiction. Therefore, (d) holds.

We claim that (e) follows from (b) and (d). Indeed, if $\mathcal{D}_k$ were to intersect $\partial_{\mathrm{F}} U_f$, then it must intersect some $\mathcal{B}^\bullet_j(f)$ and so its intersection is contained in $\mathbf{\Lambda}'_m$. In particular, $\mathcal{D}_k$ can only intersect $\mathbf{\Lambda}'_m$ in a small neighborhood of c_0, which implies that $\mathcal{D}_k$ cannot intersect $\partial_{\mathrm{F}} U_f$. □

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