

Solutions 1

1. The Cartesian and polar forms are as follows.

$$\begin{array}{ll} (a) \ i, e^{i\pi/2}, & (b) \ 1 + i, \sqrt{2}e^{i\pi/4} \\ (c) \ -16\sqrt{3} + 16i, 32e^{5\pi i/6}, & (d) \ -2, 2e^{\pi i}. \end{array}$$

2. It's sufficient to show $|z| - |w| \leq |z - w|$ and $|w| - |z| \leq |z - w|$. Both come from triangle inequality.

3. Since $\langle z, w \rangle = z\bar{w} = (x + iy)(u - iv) = (ux + vy) + i(uy - vx)$,

$$\begin{aligned} \operatorname{Re}\langle z, w \rangle &= ux + vy = (x, y) \cdot (u, v), \\ \overline{\langle w, z \rangle} &= \overline{wz} = \bar{w}z = \langle z, w \rangle, \\ \langle z, z \rangle &= z\bar{z} = |z|^2 = x^2 + y^2 \geq 0. \end{aligned}$$

Equality on the last line holds if and only if x and y are 0.

4. We can use the identity $|z|^2 = z\bar{z}$. For every $z, w \in \mathbb{C}$,

$$\begin{aligned} |z \pm w|^2 &= (z \pm w)(\bar{z} \pm \bar{w}) = z\bar{z} + w\bar{w} \pm z\bar{w} \pm w\bar{z} \\ &= |z|^2 + |w|^2 \pm (z\bar{w} + \overline{z\bar{w}}) = |z|^2 + |w|^2 \pm 2\operatorname{Re}(z\bar{w}). \end{aligned}$$

Then,

$$\begin{aligned} |z + w|^2 - |z - w|^2 &= (|z|^2 + |w|^2 + 2\operatorname{Re}(z\bar{w})) - (|z|^2 + |w|^2 - 2\operatorname{Re}(z\bar{w})) \\ &= 4\operatorname{Re}(z\bar{w}). \end{aligned}$$

5. Since $w \neq 1$ and $w^n - 1 = 0$,

$$1 + w + \dots + w^{n-1} = \frac{w^n - 1}{w - 1} = 0.$$

Take the real value of the equation above to get:

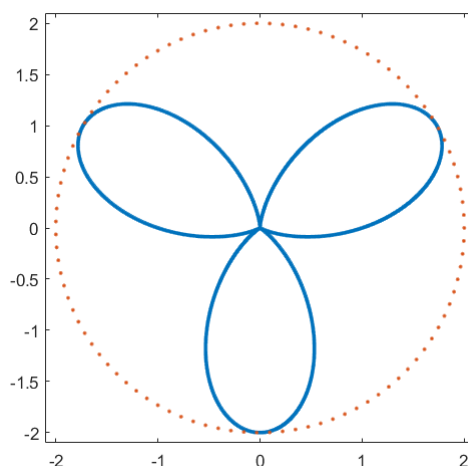
$$\cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{4\pi}{n}\right) + \dots + \cos\left(\frac{2(n-1)\pi}{n}\right) = 0.$$

6. Since $|-8 + 8i\sqrt{3}| = 16$ and $\operatorname{Arg}(-8 + 8i\sqrt{3}) = \frac{2\pi}{3}$, then $z^4 = 2^4 e^{2\pi i(3k+1)/3}$ for any integer k . Then,

$$z = 2e^{\pi i(3k+1)/6}, \text{ for } k \in \{0, 1, 2, 3\}.$$

Simplifying the expression, the roots are $\pm(\sqrt{3} + i)$ and $\pm(-1 + i\sqrt{3})$.

7. Let $\alpha = \cos(\frac{2\pi}{5})$ and $w = e^{2\pi i/5}$.
- (a) $\alpha = \operatorname{Re}(w) = \frac{w+\bar{w}}{2} = \frac{w+w^4}{2}$ and $\alpha^2 = \frac{w^2+w^3+2}{4}$.
 - (b) This is 0 from exercise 5.
 - (c) From part (b), we can pick $p = 4$, $q = 2$, and $r = -1$.
 - (d) By quadratic formula, $\alpha = \frac{-1 \pm \sqrt{5}}{4}$. We pick the $+$ sign since $\alpha > 0$.
8. I will only sketch (a); the rest should be fairly easy to illustrate.
- (a) It's the boundary of a 'flower' with three petals of maximum radius 2 centered at 0. See below.



- (b) When $z = x + iy$, the equation can be rewritten as $x^2 - y^2 = 1$, a hyperbola.
- (c) When $z = x + iy$, multiplying both top and bottom with the complex conjugate $\bar{z} - i$ gives you:

$$\frac{z - i}{z + i} = \frac{x^2 + y^2 - 1 - 2ix}{x^2 + (y + 1)^2}.$$

The denominator is always positive unless $z = -i$, on which the fraction is undefined. The real value is negative exactly when $x^2 + y^2 - 1 < 0$. This gives us the unit disk $\mathbb{D} = \{|z| < 1\}$.

- (d) The imaginary part of the fraction above is 0 when $-2ix = 0$. This gives us the set of purely imaginary numbers $\{iy \mid y \in \mathbb{R} \setminus \{-1\}\}$. We exclude $-i$ since the fractional expression is not defined at that point.

- (e) When $z = x + iy$, $\operatorname{Im} z^2 < 0$ exactly when $xy < 0$ and $\operatorname{Im}(z + 1 + i)^2 < 0$ exactly when $(x + 1)(y + 1) < 0$. This is the set $\{x + iy \mid x < -1, y > 0\} \cup \{x + iy \mid x > 0, y < -1\}$.
9. For each of the five sets in Exercise 9 above, determine whether or not they are open, closed, bounded, connected, simply connected or multiply connected.
- not open, compact, connected, multiply connected.
 - not open, closed, unbounded and disconnected.
 - open, not closed, bounded, simply connected.
 - not open, not closed, unbounded, disconnected.
 - open, not closed, unbounded, disconnected.
10. Refer to the definition of convergence of complex numbers.
11. No. Let $r_n = \frac{1}{n}$, $r = 0$, $\theta_n = (-1)^n \frac{\pi}{2}$, and $\theta = 0$. Then, $r_n e^{i\theta_n} = \frac{(-1)^n i}{n}$ converges to $re^{i\theta} = 0$. Even though $r_n \rightarrow r$, unfortunately $\theta_n \not\rightarrow \theta$.
12. It is easier when f is rewritten as $f(z) = z^2$. Then, for any $a \in \mathbb{C}$, the derivative always exists:

$$f'(a) = \lim_{z \rightarrow a} \frac{z^2 - a^2}{z - a} = \lim_{z \rightarrow a} z + a = 2a.$$

Alternatively, you may show that Cauchy Riemann equations hold throughout \mathbb{C} .

13. Upon computing the derivative at an arbitrary point $a \in \mathbb{C}$,

$$\lim_{z \rightarrow 0} \frac{|a + z|^2 - |a|^2}{z} = \lim_{z \rightarrow 0} \frac{z\bar{z} + \bar{a}z + a\bar{z}}{z} = \lim_{z \rightarrow 0} \bar{z} + \bar{a} + a\frac{\bar{z}}{z} = \bar{a} + a \lim_{z \rightarrow 0} \frac{\bar{z}}{z}.$$

When $a = 0$, it is clear that the limit above exists and is equal to 0. However, when $a \neq 0$, the limit does not exist since $\lim_{z \rightarrow 0} \frac{\bar{z}}{z}$ does not exist. Since $|z|^2$ is only complex differentiable at one point, it is not holomorphic on any domain.

Solutions 2

1. If $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$, then $\overline{f(\bar{z})} = p(x, y) + iq(x, y)$ where $p(x, y) = u(x, -y)$ and $q(x, y) = -v(x, -y)$. It remains to show that Cauchy Riemann equations still hold for the pair p and q on the domain $\bar{U} := \{\bar{z} \mid z \in U\}$, which is the reflection of U in the real axis. (Note that the correct domain for $\overline{f(\bar{z})}$ is \bar{U} , not U .)

2. This is merely an exercise in multivariable calculus. Use the chain rules:

$$\frac{\partial f}{\partial x} = \frac{x}{r} \frac{\partial f}{\partial r} - \frac{y}{r^2} \frac{\partial f}{\partial \theta}, \quad \frac{\partial f}{\partial y} = \frac{y}{r} \frac{\partial f}{\partial r} + \frac{x}{r^2} \frac{\partial f}{\partial \theta}.$$

Log is holomorphic because

$$\frac{d}{d\bar{z}} \text{Log} z = \frac{1}{2\bar{z}} \left(r \frac{\partial}{\partial r} + i \frac{\partial}{\partial \theta} \right) (\ln r + i\theta) = \frac{1}{2\bar{z}} (1 - 1) = 0.$$

Its derivative is

$$\frac{d}{dz} \text{Log} z = \frac{1}{2z} \left(r \frac{\partial}{\partial r} - i \frac{\partial}{\partial \theta} \right) (\ln r + i\theta) = \frac{1}{2z} (1 + 1) = \frac{1}{z}.$$

3. Let $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$. If $f(z) = \overline{f(z)}$, then $v(x, y) \equiv 0$. By Cauchy Riemann equations, $u_x = v_y \equiv 0$ and $u_y = -v_x \equiv 0$, so then $u(x, y) = c$ for some constant $c \in \mathbb{R}$. Therefore, $f(z) \equiv c$ on U .
4. Let $z = x + iy$. When $x \in \mathbb{R}$ and $|y| < \pi$, $e^{x+iy} = e^x e^{iy}$. The function is surjective because the image is

$$\{e^x e^{iy} \mid x \in \mathbb{R}, |y| < \pi\} = \{r e^{iy} \mid r > 0, |y| < \pi\} = \mathbb{C} \setminus (-\infty, 0].$$

Since e^{iy} is 2π -periodic with respect to y and since the height of the strip is at most 2π , e^z is injective. The inverse of e^z is $\text{Log}(z)$ which is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$ with derivative z^{-1} .

5. The preimage is

$$\{z \mid 1 - z^{-1} \in (-\infty, 0]\} = \{(1 - x)^{-1} \mid x \in (-\infty, 0]\} = [-1, 0).$$

This can be taken as the branch cut because its image under $1 - z^{-1}$ is $(-\infty, 0]$, the usual branch cut for $\log(z)$.

6. Let $\tanh^{-1}(z) = w$, then $z = \frac{e^w - e^{-w}}{e^w + e^{-w}}$. This can be rewritten as

$$e^{2w} = \frac{1+z}{1-z}$$

Using logarithm, the expression becomes

$$w = \frac{1}{2} \log \frac{1+z}{1-z}.$$

7. Here, k represents any integer.

- (a) $\frac{1}{2} \ln 2 + i\frac{\pi}{4}(8k-3)$,
- (b) $e^{i \ln \pi}$,
- (c) $\frac{\pi}{2}(1+4k) - i \ln(1+\sqrt{2})$,
- (d) $e^{-\frac{\pi}{8}(1+4k)(\sqrt{3}+i)}$.

8. All are smooth and closed. The only simple ones are $n = 1, 2$.

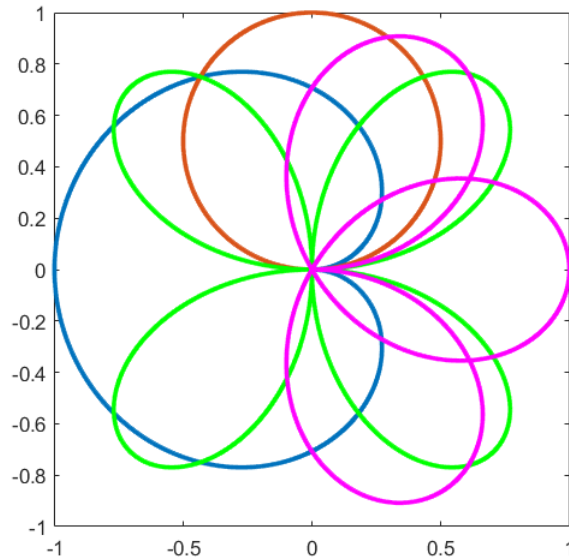


Figure 1: $n = 1$ in blue, $n = 2$ in red, $n = 3$ in pink and $n = 4$ in green

9. The integrals are as follows:

- (a) Use $\gamma(t) = (3+4i)t$ for $0 \leq t \leq 1$. Since $\gamma'(t) = 5$,

$$\int_{\gamma} \operatorname{Im} z dz = \int_0^1 4t \cdot |\gamma'(t)| dt = \int_0^1 20t dt = 10.$$

(b) Since $\gamma'(t) = 2ie^{it}$,

$$\int_{\gamma} i\bar{z} + iz^2 dz = \int_{\pi/2}^{\pi} (2ie^{-it} + 8e^{3it}) \cdot 2ie^{it} dt = \int_{\pi/2}^{\pi} -4 + 16ie^{4it} dt = -2\pi.$$

(c) Since $\gamma'(t) = ie^{it}$,

$$\begin{aligned} \int_{\gamma} \text{pv } z^i dz &= \int_{-\pi/2}^{\pi/2} e^{i\text{Log}(e^{it})} \cdot ie^{it} dt = \int_{-\pi/2}^{\pi/2} ie^{t(-1+i)} dt \\ &= \frac{i}{-1+i} (e^{\frac{\pi}{2}(-1+i)} - e^{\frac{\pi}{2}(1-i)}) = \frac{1-i}{2} (ie^{-\pi/2} + ie^{\pi/2}) \\ &= (1+i) \cosh(\pi/2). \end{aligned}$$

10. Since $\gamma'(t) = (-1+i)e^{(-1+i)t}$, the length of γ is

$$L(\gamma) = \int_0^{2\pi} |-1+i| dt = 2\pi\sqrt{2}.$$

11. The distance between the line segment γ and the point 1 is $2^{-1/2}$, so then

$$\max_{z \in \gamma} |(z-1)^{-3}| = (\min_{z \in \gamma} |z-1|)^{-3} = (2^{-1/2})^{-3} = 2\sqrt{2}.$$

Since $L(\gamma) = |2i-2| = 2\sqrt{2}$, then by ML inequality,

$$\left| \int_{\gamma} \frac{1}{(z-1)^3} dz \right| \leq 2\sqrt{2} \cdot L(\gamma) = 8.$$

12. Let $z = x + iy \in \gamma$, then $|e^{\bar{z}}| = e^x \leq e^2$ because $0 \leq x \leq 2$. Therefore,

$$\left| \int_{\gamma} e^{\bar{z}} dz \right| \leq L(\gamma) \cdot \max_{z \in \gamma} |e^{\bar{z}}| = 8e^2.$$

13. One primitive is $\frac{z^{i+1}}{i+1}$ because by chain rule, on $\mathbb{C} \setminus (-\infty, 0]$,

$$\frac{dz^{i+1}}{dz} = \frac{de^{(i+1)\text{Log}z}}{dz} = \frac{i+1}{z} \cdot e^{(i+1)\text{Log}z} = (i+1)z^i.$$

The curve γ lies in the domain $\mathbb{C} \setminus (-\infty, 0]$ and it travels from $-i$ to i . Then,

$$\begin{aligned} \int_{\gamma} \text{pv } z^i dz &= \frac{i^{i+1}}{i+1} - \frac{(-i)^{i+1}}{i+1} = \frac{1}{1+i} (ie^{i\text{Log}i} - (-i)e^{i\text{Log}(-i)}) \\ &= \frac{1-i}{2} (ie^{-\pi/2} + ie^{\pi/2}) = (1+i) \cosh(\pi/2). \end{aligned}$$

14. Both integrands are entire functions. As such, the integrals are independent of the choice of the contour.

(a) The integrand has primitive $iz + z^3/3$. Then,

$$\int_0^i z^2 + idz = iz + z^3/3 \Big|_0^i = -1 - i/3.$$

(b) The integrand has primitive $i \cosh z$. Then,

$$\int_{-\pi}^{\pi} \sin(iz) = i \cosh z \Big|_{-\pi}^{\pi} = 0.$$

15. The integrand can be rewritten as $\frac{2}{5} \left(\frac{1}{z-3/2} - \frac{1}{z+1} \right)$. Since -1 is outside of the pentagon γ but $3/2$ is enclosed by γ , we apply Cauchy-Goursat so that the integral is reduced to

$$\frac{2}{5} \int_{\gamma} \frac{1}{z - 3/2} dz.$$

By deformation theorem, we can replace γ with any small circle centered at $3/2$. The integral is then reduced to $4\pi i/5$.

Solutions 3

1. By partial fractions, the integral can be rewritten as

$$\frac{i}{12} \oint_{\gamma} \frac{dz}{z + 1.5i} - \frac{i}{12} \oint_{\gamma} \frac{dz}{z - 1.5i}.$$

The singular points we need to keep our eye on are $\pm 1.5i$.

- (a) The rectangle does not enclose $\pm 1.5i$. Both integrands are holomorphic along and inside γ . By Cauchy-Goursat, the integral is 0.
- (b) The circle only encloses $1.5i$, but not $-1.5i$. The first integral is 0 by Cauchy-Goursat. The second becomes $-\frac{i}{12} \cdot 2\pi i = \frac{\pi}{6}$. In total, the integral is πi .
- (c) Check that γ is a negatively oriented circle centered at 0 of radius π , enclosing both $\pm 1.5i$. Therefore, the integral evaluates to

$$\frac{i}{12} \cdot 2\pi i - \frac{i}{12} \cdot 2\pi i = 0.$$

2. The following functions g are holomorphic along and inside the domain enclosed by $C(0, 2)$.

- (a) Apply Cauchy's formula to $g(z) = \frac{z+2}{z-3}$ at the point $z_0 = 1$. The integral is

$$\oint_{C(0,2)} \frac{g(z)}{z-1} dz = 2\pi i g(1) = -3\pi i.$$

- (b) Apply Cauchy's formula to $g(z) = e^{e^z}$ at the point $z_0 = i\pi/2$. The integral is

$$\oint_{C(0,2)} \frac{g(z)}{z-i} dz = 2\pi i g(i) = 2\pi i e^{e^{i\pi/2}} = 2\pi i e^i.$$

- (c) Apply Cauchy's differentiation formula to get the 3rd derivative of $g(z) = \sinh(\pi z)$ at the point $z_0 = 0$. The integral is

$$\oint_{C(0,2)} \frac{g(z)}{z^4} dz = \frac{2\pi i}{3!} \frac{d^3}{dz^3} (\sinh(\pi z)) \Big|_{z=0} = \frac{\pi^4 i}{3}.$$

3. For any point $z_0 \in \mathbb{C}$, radius $r > 0$ and point w on the circle $C(z_0, r)$, we can apply triangle inequality to get $|w| \leq |w - z_0| + |z_0| = r + |z_0|$ and consequently $|f(w)| \leq \pi(r + |z_0|)$. By Cauchy's inequality,

$$|f''(z_0)| \leq \frac{2\pi(r + |z_0|)}{r^2}.$$

Taking the limit as $r \rightarrow \infty$, the right hand side goes to 0. Since $|f''(z_0)|$ is independent of r , $f''(z_0) = 0$ for all z_0 . The primitive f' must be some constant a and the primitive f of f' must be of the form $az + b$. However, since $|f(0)| \leq \pi \cdot 0 = 0$, b must be 0.

4. Since $f^{(6)}$ is bounded and entire, it is a constant function of some value a where $|a| > 0$. By taking primitive 6 times, f must be a polynomial of degree 6 because it has a leading term $\frac{a}{6!}z^6$.
5. The inequality implies that $f(z) \neq 0$ for all z , so $1/f(z)$ is a well-defined entire function. Since $|1/f(z)| \leq 1$, it is bounded and therefore constant. f is then constant too.
6. There is some constant $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. By ML inequality,

$$\begin{aligned} \left| \oint_{C(0,R)} \frac{f(z)}{(z - z_0)(z - z_1)} dz \right| &\leq 2\pi R \max_{|z|=R} \left| \frac{f(z)}{(z - z_0)(z - z_1)} \right| \\ &= 2\pi R \frac{M}{\min_{|z|=R} |(z - z_0)(z - z_1)|} \\ &\leq \frac{2\pi MR}{(R - |z_0|)(R - |z_1|)}. \end{aligned}$$

where the final inequality comes from triangle inequality. By taking the limit as $R \rightarrow \infty$, this upper bound clearly goes to 0, so then

$$\lim_{R \rightarrow \infty} \oint_{C(0,R)} \frac{f(z)}{(z - z_0)(z - z_1)} dz = 0.$$

This integral can be separated by partial fractions and evaluated by Cauchy's integral formula.

$$\begin{aligned} \oint_{C(0,R)} \frac{f(z)}{(z - z_0)(z - z_1)} dz &= \frac{1}{z_0 - z_1} \left[\oint_{C(0,R)} \frac{f(z)}{z - z_0} - \oint_{C(0,R)} \frac{f(z)}{z - z_1} dz \right] \\ &= \frac{f(z_0) - f(z_1)}{2\pi i(z_0 - z_1)}. \end{aligned}$$

This expression is independent of R , so then it must be 0. Therefore $f(z_0) = f(z_1)$.

7. It is holomorphic with derivative $2z$ on \mathbb{D} and 0 on the annulus $\{2 < |z| < 3\}$. It attains maximum on the annulus with $|f(z)| = 2$. The set U is disconnected and therefore the maximum modulus principle does not apply.
8. As f is entire, by maximum modulus principle, it is sufficient to see the behavior of f on the circle $\{|z| = 2\}$ to find maximum points. When $z = 2e^{it}$ where $t \in \mathbb{R}$,

$$\begin{aligned} |z^3 + i| &= |8e^{3it} + 1| = |(8 \cos 3t + 1) + i8 \sin 3t| \\ &= [64 \cos^2 3t + 16 \cos 3t + 1 + 64 \sin^2 3t]^{1/2} = [65 + 16 \cos 3t]^{1/2} \end{aligned}$$

The real function $\cos 3t$ attains its maximum value 1 at $t = 0, \pm \frac{2\pi}{3}$. At any of these values, we have $|z^3 + 1| = 9$, and this is attained by $z = 2, -1 \pm i\sqrt{3}$.

9. The function $e^{(1+i)z}$ is entire. By the maximum modulus principle, to find the maximum value of $e^{(1+i)z}$ on the closed square $\{x + iy \mid 1 < x, y < \pi\}$, it is sufficient to look at the the function along the boundary of the square. Let $z = x + iy$.

$$|e^{(1+i)z}| = |e^{(x-y)+i(x+y)}| = e^{x-y}.$$

The maximum of $x - y$ is attained on the boundary of the square when $x = \pi$ and $y = 1$. Therefore, the smallest radius is $r = e^{\pi-1}$.

10. Part (a) follows from applying the minimum modulus principle on $\mathbb{D}(z_0, \epsilon)$. If the lemma weren't true, it would in the most direct way contradict the minimum modulus principle. Part (b) follows from triangle inequality:

$$\begin{aligned} |f(z)| &\geq |a_d z^d| - \sum_{n=0}^{d-1} |a_n z^n| \geq |a_d| |z|^d - \sum_{n=0}^{d-1} |a_n| |z|^{d-1} \\ &\geq |z|^{d-1} \left(|a_d| |z| - \sum_{n=0}^{d-1} |a_n| \right) \geq |z|^{d-1} \geq R^{d-1}. \end{aligned}$$

For part (c), $|f|$ must attain minimum on the compact disk $\overline{\mathbb{D}(0, R)}$ where R is from part (b). Let z_0 be a minimum point in this compact disk. If $f(z_0) \neq 0$, then it will contradict part (a). Therefore, $f(z_0) = 0$.