Chapter 2 Convex sets

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three groups of concepts

affine combination	convex combination	conic combination
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Convex combination

convex combination of $x_1, \cdots, x_k \in \mathbb{R}^n$: points of the form

$$\theta_1 x_1 + \cdots + \theta_k x_k, \quad \text{where} \quad \theta_1, \cdots, \theta_k \geq 0 \quad \text{and} \quad \theta_1 + \cdots + \theta_k = 1$$

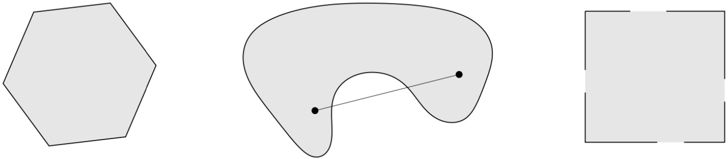
line segment between x_1 and x_2: the set of all convex combinations of x_1 and x_2

$$\{x = \theta x_1 + (1-\theta) x_2 \mid 0 \leq \theta \leq 1\}$$

Convex set

convex set: $C \subseteq \mathbb{R}^n$ is convex if contains line segment between any pair of points in C

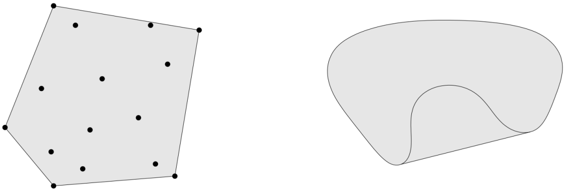
examples (one convex, two nonconvex)



Convex hull

convex hull of $C \subseteq \mathbb{R}^n$: the set of all convex combinations of points in C

$$\mathbf{conv}\, C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_1, \cdots, x_k \in C;\ \theta_1, \cdots, \theta_k \geq 0;\ \theta_1 + \cdots + \theta_k = 1\}$$



facts

- the convex hull of C is the smallest convex set containing C
- if C is a convex set, then $\mathbf{conv}\, C = C$

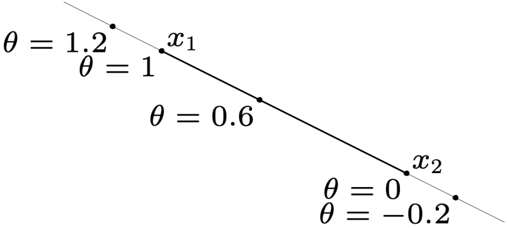
Affine combination

affine combination of $x_1, \cdots, x_k \in \mathbb{R}^n$: points of the form

$$\theta_1 x_1 + \cdots + \theta_k x_k, \quad \text{where} \quad \theta_1 + \cdots + \theta_k = 1$$

line through x_1 and x_2: the set of all affine combinations of x_1 and x_2

$$\{x = \theta x_1 + (1 - \theta) x_2 \mid \theta \in \mathbb{R}\}$$



Affine set

affine set: $C \subseteq \mathbb{R}^n$ is affine if it contains the line through any pair of points in C

example

- the solution set of linear equations $\{x \mid Ax = b\}$ is an affine set
- conversely, every affine set can be expressed as the solution set of a system of linear equations

Affine hull

affine hull of $C \subseteq \mathbb{R}^n$: the set of all affine combinations of points in C

$$\mathbf{aff}\, C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_1, \cdots, x_k \in C,\ \theta_1 + \cdots + \theta_k = 1\}$$

facts

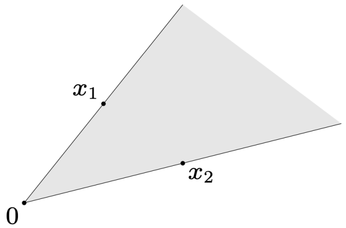
- the affine hull of C is the smallest affine set containing C
- if C is an affine set, then $\mathbf{aff}\, C = C$

Conic combination

cone: $C \subseteq \mathbb{R}^n$ is a cone if $\theta x \in C$ for every $x \in C$ and $\theta \geq 0$.

conic combination of $x_1, \cdots, x_k \in \mathbb{R}^n$: points of the form

$$\theta_1 x_1 + \cdots + \theta_k x_k, \quad \text{where} \quad \theta_1, \cdots, \theta_k \geq 0$$



Convex cone

convex cone: $C \subseteq \mathbb{R}^n$ is a convex cone if it is convex and a cone

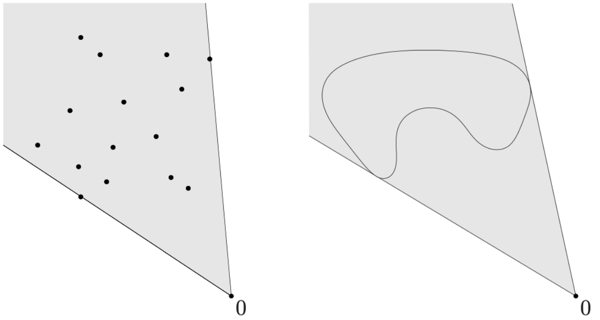
fact

C is a convex cone $\iff$ C contains all conic combinations of points in itself

Conic hull

conic hull of $C \subseteq \mathbb{R}^n$: the set of all conic combinations of points in C

$$\{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_1, \cdots, x_k \in C;\ \theta_1, \cdots, \theta_k \geq 0\}$$



facts

- the conic hull of C is the smallest convex cone containing C
- if C is a convex cone, then its conic hull is itself

Exercises

- Study the following concepts from text:

  affine dimension, **relative interior**.

- Suppose that $C \subseteq \mathbb{R}^n$ is convex, then $\mathbf{int}\, C$ and $\mathbf{cl}\, C$ are also convex.
- Suppose that $C \subseteq \mathbb{R}^n$ is convex, then
$$\mathbf{int}\, C = \varnothing \qquad \Longleftrightarrow \qquad C \text{ is contained in a hyperplane.}$$
- Suppose that $C \subseteq \mathbb{R}^n$ is convex, and $\mathbf{int}\, C \neq \varnothing$, then
$$\mathbf{cl}(\mathbf{int}\, C) = \mathbf{cl}\, C.$$

Look at $d + 1$, the largest number of affinely independent points from C. Let $x_0, \ldots, x_d$ one such affinely independent subset of largest size. Note that every other point is an affine combination of the points x_k, so lies in the affine subspace generated by them, which is of dimension d.

If $d < n$ then this subspace is contained in an affine hyperplane.

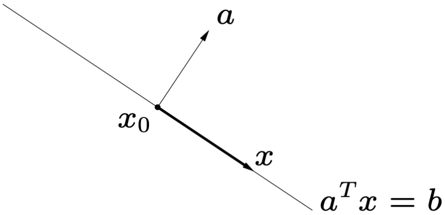
If $d = n$, then C contains $d + 1$ affinely independent points. Since C is convex, it will also contain the convex hull of those $n + 1$ points. Now, in an n-dimensional space the convex hull of $n + 1$ affinely independent points has non-empty interior. So the interior of C is non-empty.

four types of elementary examples

- LP type: hyperplanes, halfspaces, polyhedra
- ball type: Euclidean balls, ellipsoids, norm balls
- cone type: second-order cone, norm cones
- matrix type: positive semidefinite cone

Hyperplanes

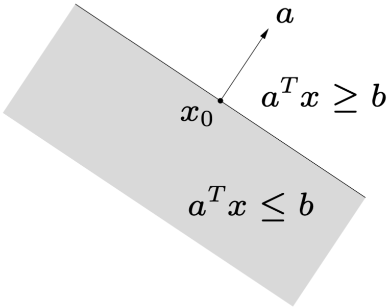
hyperplane: set of the form $\{x \mid a^T x = b\}$ $(a \in \mathbb{R}^n, a \neq 0, b \in \mathbb{R})$



fact: hyperplanes are affine and convex

Halfspaces

halfspace: set of the form $\{x \mid a^T x \leq b\}$ $(a \in \mathbb{R}^n, a \neq 0, b \in \mathbb{R})$



fact: halfspaces are convex

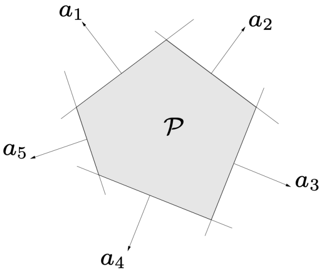
Polyhedra

polyhedron: solution set of finitely many linear inequalities and equalities

$$Ax \preceq b, \quad Cx = d$$

where $A \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{p \times n}$, $\preceq$ is componentwise inequality

i.e. polyhedra are intersections of finite number of halfspaces and hyperplanes;



fact: polyhedra are convex (page 31)

Euclidean balls

Euclidean ball with center x_c and radius r: two equivalent representations

- set of the form
$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \le r\}$$
- set of the form
$$B(x_c, r) = \{x_c + ru \mid \|u\|_2 \le 1\}$$

fact: Euclidean balls are convex

Ellipsoids

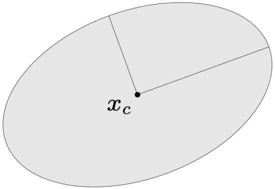
ellipsoid: two equivalent representations

- set of the form

$$\{x \mid (x - x_c)^T P^{-1}(x - x_c) \leq 1\} \quad \text{with} \quad P \in \mathbb{S}^n_{++}$$

- set of the form

$$\{x_c + Au \mid \|u\|_2 \leq 1\} \quad \text{with } A \text{ square and nonsingular}$$



fact: ellipsoids are convex

Norm balls

norm ball with center x_c and radius r: set of the form

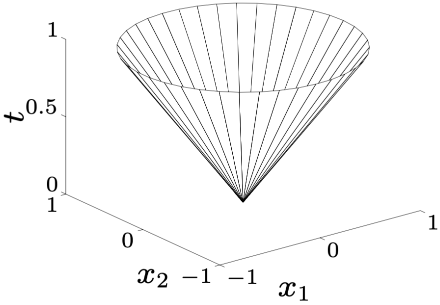
$$\{x \mid \|x - x_c\| \leq r\}$$

fact: norm balls are convex

Norm cones

second-order cone:

$$\{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\|_2 \le t\}$$



norm cone: for any norm $\|\cdot\|$ on $\mathbb{R}^n$

$$\{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\| \le t\}$$

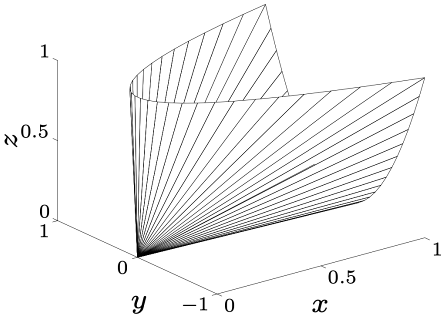
fact: norm cones are convex cones

Positive semidefinite cone

positive semidefinite cone

$$\mathbb{S}^n_+ = \{X \in \mathbb{S}^n \mid X \succeq 0\}$$

fact: positive semidefinite cone $\mathbb{S}^n_+$ is a convex cone



- Affine and convex sets
- Elementary examples
- **Operations preserving convexity**
- Separating and supporting hyperplanes
- Generalized inequalities
- Dual cones and generalized inequalities

Establishing convexity

Practical methods for establishing convexity of a set C

1. apply definition

$$x_1, x_2 \in C, 0 \leq \theta \leq 1 \qquad \Longrightarrow \qquad \theta x_1 + (1-\theta) x_2 \in C$$

2. reconstruct C from known convex sets by operations preserving convexity:

   - intersection
   - affine functions
   - perspective function
   - linear-fractional functions

Intersection

an arbitrary intersection of convex sets is convex

example

the positive semidefinite cone $\mathbb{S}_{+}^{n}$ is convex

Affine function

affine function $f : \mathbb{R}^n \to \mathbb{R}^m$ is of the form

$$f(x) = Ax + b \qquad \text{with } A \in \mathbb{R}^{m \times n} \text{ and } b \in \mathbb{R}^m$$

- the image of a convex set under f is convex

$$S \subseteq \mathbb{R}^n \text{ convex} \qquad \Longrightarrow \qquad f(S) = \{f(x) \mid x \in S\} \text{ convex}$$

- the inverse image $f^{-1}(C)$ of a convex set under f is convex

$$C \subseteq \mathbb{R}^m \text{ convex} \qquad \Longrightarrow \qquad f^{-1}(C) = \{x \in \mathbb{R}^n \mid f(x) \in C\} \text{ convex}$$

examples

- scaling and translation: if $S \subseteq \mathbb{R}^n$ is convex, $\alpha \in \mathbb{R}$ and $a \in \mathbb{R}^n$, then

$$\alpha S = \{\alpha x \mid x \in S\} \qquad \text{and} \qquad S + a = \{x + a \mid x \in S\}$$

  are convex

- projection: if $S \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is convex, then

$$T = \{x_1 \in \mathbb{R}^m \mid (x_1, x_2) \in S \text{ for some } x_2 \in \mathbb{R}^n\}$$

  is convex

- sum: if $S_1, S_2 \subseteq \mathbb{R}^n$ are both convex, then

$$S_1 + S_2 = \{x + y \mid x \in S_1,\ y \in S_2\}$$

  is convex

- solution set of linear matrix inequality

$$\{x \in \mathbb{R}^n \mid x_1 A_1 + \cdots + x_n A_n \preceq B\}$$

where $A_i, B \in \mathbb{S}^m$, is convex

proof
inverse image of the positive semidefinite cone under the affine function

$$f: \mathbb{R}^n \to \mathbb{S}^m, \qquad f(x) = B - (x_1 A_1 + \cdots + x_n A_n)$$

- hyperbolic cone

$$\left\{x \in \mathbb{R}^n \,\middle|\, x^T P x \leq \left(c^T x\right)^2, c^T x \geq 0\right\}$$

where $P \in \mathbb{S}^n_+$ and $c \in \mathbb{R}^n$, is convex

proof

inverse image of the second-order cone

$$\{(z,t) \mid z^T z \leq t^2, t \geq 0\}$$

under the affine function $f\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ given by $f(x) = (P^{1/2}x, c^T x)$

Perspective function

perspective function $P\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ given by

$$P(x,t) = x/t, \qquad \mathbf{dom}\, P = \mathbb{R}^n \times \mathbb{R}_{++} = \{(x,t) \mid t > 0\}$$

- images of convex sets under perspective function are convex
- inverse images of convex sets under perspective function are convex
- prove it

Linear-fractional functions

linear-fractional function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ given by

$$f(x) = \frac{Ax+b}{c^T x + d}, \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$$

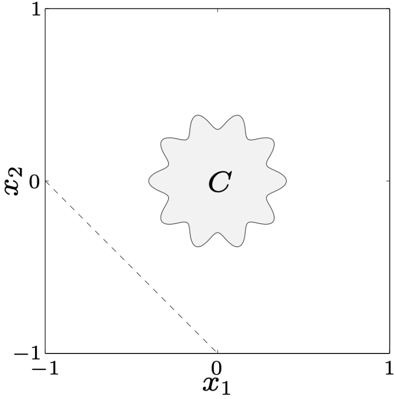
it is the composition of an affine function g and the perspective function P, where

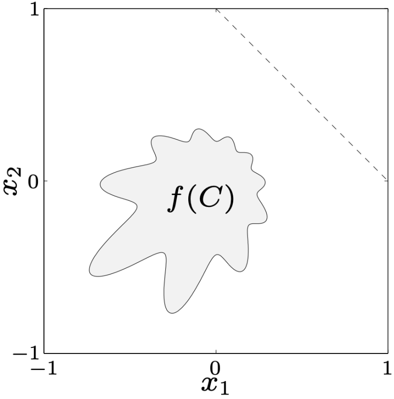
$$g(x) = \begin{bmatrix} A \\ c^T \end{bmatrix} x + \begin{bmatrix} b \\ d \end{bmatrix}$$

- images of convex sets under linear-fractional functions are convex
- inverse images of convex sets under linear-fractional functions are convex

example

$$f(x) = \frac{x}{x_1 + x_2 + 1}, \qquad \mathbf{dom}\, f = \{(x_1, x_2) \mid x_1 + x_2 + 1 > 0\}$$





two fundamental properties of convex sets

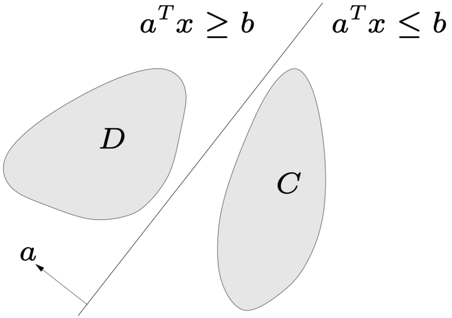
- separating hyperplane theorem
- supporting hyperplane theorem

Separating hyperplane theorem

Theorem

If C and D are nonempty disjoint convex sets, then there exist $a \neq 0$ and b such that

$$a^T x \leq b \text{ for all } x \in C, \qquad a^T x \geq b \text{ for all } x \in D.$$



Remark $\{x \mid a^T x = b\}$ is called a **separating hyperplane**

Sketch of proof

Step 1. Theorem holds if there exists $c \in C$ and $d \in D$ such that

$$\|c - d\|_2 \leq \|u - v\|_2$$

for all $u \in C$ and $v \in D$.

Idea: take the perpendicular bisector of the line segment connecting c and d.

Step 2. If $C = \{0\}$, then there exists $a \neq 0$ such that $a^T x \geq 0$ for all $x \in D$.

Idea: if $0 \notin \mathbf{cl}\, D$, apply Step 1 to $\{0\}$ and $\mathbf{cl}\, D$; if $\mathbf{int}\, D = \varnothing$, D is contained in a hyperplane; if $0 \in \mathbf{cl}\, D$ and $\mathbf{int}\, D \neq \varnothing$, shrink D by ε and let $\varepsilon \to 0$.

Step 3. Prove the general case.

Idea: apply Step 2 to $\{0\}$ and $S = \{y - x \mid x \in C,\ y \in D\}$.

Step 1. Define

$$f(x) = (d-c)^T \left(x - \frac{d+c}{2} \right) = (d-c)^T x - \frac{1}{2} \|d-c\|_2^2.$$

For any $v \in D$, we have $tv + (1-t)d \in D$ for $t \in [0,1]$, hence the function

$$g(t) = \|tv + (1-t)d - c\|_2^2$$

satisfies $g(t) \geq g(0)$ for $t \in [0,1]$, which implies $g'(0) \geq 0$. Since

$$g'(t) = 2(tv + (1-t)d - c)^T (v-d)$$

we obtain $g'(0) = 2(d-c)^T (v-d) \geq 0$, therefore

$$\begin{aligned} f(v) = (d-c)^T \left(v - \frac{d+c}{2} \right) &= (d-c)^T \left(v - d + \frac{d-c}{2} \right) \\ &= (d-c)^T (v-d) + \frac{1}{2} \|d-c\|_2^2 \geq 0. \end{aligned}$$

Similarly we can show $f(u) \leq 0$ for any $u \in C$.

Step 2. We first check two simple cases, then prove the remaining cases.

Case 1. Assume $0 \notin \mathbf{cl}\, D$.

Note that $\mathbf{cl}\, D$ is convex, and there exists some $d \in \mathbf{cl}\, D$ such that $\|d - 0\|_2 \leq \|y - 0\|_2$ for all $y \in \mathbf{cl}\, D$ (why?). Applying Step 1 to $\{0\}$ and $\mathbf{cl}\, D$, there exists $a \neq 0$ and b such that $a^T y \geq b \geq a^T 0 = 0$ for all $y \in \mathbf{cl}\, D$.

Case 2. Assume $\mathbf{int}\, D = \varnothing$.

In such a case D is contained in a hyperplane $\{z \mid a^T z = b\}$ for some $a \neq 0$ and b. Assume wlog $b \geq 0$, then $a^T y = b \geq 0$ for all $y \in D$.

Case 3. Consider the remaining cases.

For each sufficiently small $\varepsilon > 0$, the set $D_{-\varepsilon} = \{z \mid B(z, \varepsilon) \subseteq D\}$ is nonempty and convex, and $0 \notin \mathbf{cl}\, D_{-\varepsilon}$ (why?). By Case 1 there exists $a_\varepsilon \neq 0$ such that $a_\varepsilon^T z \geq 0$ for all $z \in D_{-\varepsilon}$. Assume wlog $\|a_\varepsilon\|_2 = 1$. Choose any positive sequence $\{\varepsilon_i\}$ converging to 0, a subsequence of $\{a_{\varepsilon_i}\}$ converges to some $\bar{a}$ with $\|\bar{a}\|_2 = 1$. Then $\bar{a}^T z \geq 0$ for all $z \in \mathbf{int}\, D$ (why?) hence for all $z \in D$ (why?).

Step 3. The nonempty set

$$S = \{y - x \mid x \in C,\ y \in D\}$$

is convex (why?) and disjoint from $\{0\}$. By Step 2, there exists $a \neq 0$ such that

$$a^T(y - x) \geq 0 \qquad \Longleftrightarrow \qquad a^T x \leq a^T y$$

for all $x \in C$ and $y \in D$. It follows that

$$\mathbf{sup}\{a^T x \mid x \in C\} \leq \mathbf{inf}\{a^T y \mid y \in D\}$$

whose both sides are finite. Then any b in between satisfies

$$a^T x \leq b \leq a^T y$$

for all $x \in C$ and $y \in D$.

Remarks

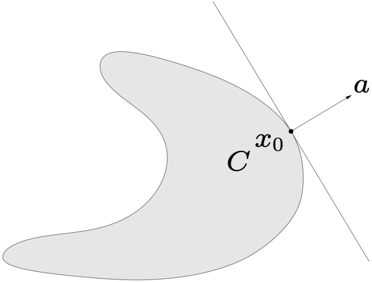
- strict separation requires additional assumptions (e.g. point and closed convex set)
- converse separating theorem requires additional assumptions (e.g. one set is open)

Supporting hyperplane theorem

A **supporting hyperplane** to a set C at a boundary point x_0 is a hyperplane

$$\{x \mid a^T x = a^T x_0\}$$

where $a \neq 0$, such that $a^T x \leqslant a^T x_0$ for all $x \in C$



Theorem

If C is convex, then supporting hyperplane exists at every boundary point of C.

Sketch of proof

Step 1. Assume $\mathbf{int}\, C \neq \varnothing$. (Idea: use separating hyperplane for $\{x_0\}$ and $\mathbf{int}\, C$.)

Since $\mathbf{int}\, C$ is also convex (why?), we apply separating hyperplane theorem to $\mathbf{int}\, C$ and $\{x_0\}$ to conclude the existence of $a \neq 0$ and b such that $a^T x \leq b \leq a^T x_0$ for all $x \in \mathbf{int}\, C$. Since $\mathbf{cl}\, C = \mathbf{cl}(\mathbf{int}\, C)$ (why?) it follows that $a^T x \leq b$ for all $x \in \mathbf{cl}\, C$. In particular $a^T x \leq b = a^T x_0$ for all $x \in C$ and the given $x_0 \in \mathbf{bd}\, C$.

Step 2. Assume $\mathbf{int}\, C = \varnothing$. (Idea: C is contained in a hyperplane.)

Then C is contained in a hyperplane $\{x \mid a^T x = b\}$ for some $a \neq 0$ and b (why?). It follows $a^T x = b = a^T x_0$ for all $x \in C$.

Tasks Study the following extensions of both theorems from the text

- strict separation theorem
- converse of separating hyperplane theorem
- converse of supporting hyperplane theorem

Proper cones

proper cone: a cone $K \subseteq \mathbb{R}^n$ satisfying

- K is convex
- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line, or equivalently, $\pm x \in K \implies x = 0$)

examples

- nonnegative orthant $K = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_i \geq 0, i = 1, \cdots, n\}$
- positive semidefinite cone $K = \mathbb{S}^n_+$

Generalized inequalities

generalized inequality on $\mathbb{R}^n$ defined by a proper cone $K \subseteq \mathbb{R}^n$

$$x \preceq_K y \quad \Longleftrightarrow \quad y - x \in K$$
$$x \prec_K y \quad \Longleftrightarrow \quad y - x \in \textbf{int}\, K$$

examples (similar for $\prec$, $\succeq$, $\succ$)

- nonnegative orthant and componentwise inequality ($K = \mathbb{R}^n_+$)

$$x \preceq_{\mathbb{R}^n_+} y \quad \Longleftrightarrow \quad x_i \le y_i, \quad i = 1, \cdots, n$$

- positive semidefinite cone and symmetric matrix inequality ($K = \mathbb{S}^n_+$)

$$X \preceq_{\mathbb{S}^n_+} Y \quad \Longleftrightarrow \quad Y - X \text{ positive semidefinite}$$

these two types are so common that we drop the subscript in $\preceq_K$

properties (same for $\prec$, $\succeq$, $\succ$)

- many properties of $\preceq_K$ are similar to $\leq$ on $\mathbb{R}$; for example:

$$x \preceq_K y, \quad u \preceq_K v \qquad \Longrightarrow \qquad x + u \preceq_K y + v$$

- partial ordering (not always a total/linear ordering)

  it could happen that $x \not\preceq_K y$ and $y \not\preceq_K x$

- pay attention to other subtleties

  for example: $x \preceq_K y$ does not imply $x \prec_K y$ or $x = y$

- if $x \prec_K y$, then for u and v small enough, $x + u \prec_K y + v$

Minimum and minimal elements

$x \in S$ is the **minimum element** of S with respect to $\preceq_K$ if for all

$$y \in S \qquad \Longrightarrow \qquad x \preceq_K y$$

$x \in S$ is a **minimal element** of S with respect to $\preceq_K$ if

$$y \in S, \quad y \preceq_K x \qquad \Longrightarrow \qquad y = x$$

Remark Minimum element, if exists, must be unique.

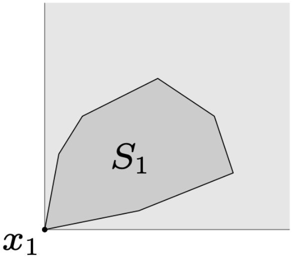
property A point $x \in S$ is the minimum element of S if and only if

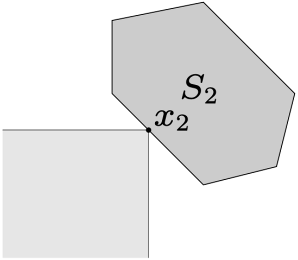
$$S \subseteq x + K.$$

property A point $x \in S$ is the minimal element of S if and only if

$$(x - K) \cap S = \{x\}.$$

example for $K = \mathbb{R}^2_+$





x_1 is the minimum element of S_1 $\qquad$ x_2 is a minimal element of S_2

coming up $\qquad$ criteria for determining minimum/minimal elements

Affine and convex sets

Elementary examples

Operations preserving convexity

Separating and supporting hyperplanes

Generalized inequalities

Dual cones and generalized inequalities

Dual cones

dual cone of a cone $K \subseteq \mathbb{R}^n$

$$K^* = \{y \in \mathbb{R}^n \mid y^T x \geq 0 \text{ for all } x \in K\}$$

examples

$$\begin{aligned}
K = \mathbb{R}^n_+ \quad &\Longrightarrow \quad K^* = \mathbb{R}^n_+ \\
K = \mathbb{S}^n_+ \quad &\Longrightarrow \quad K^* = \mathbb{S}^n_+ \text{ (example 2.24)} \\
K = \{(x,t) \mid \|x\|_2 \leq t\} \quad &\Longrightarrow \quad K^* = \{(x,t) \mid \|x\|_2 \leq t\} \\
K = \{(x,t) \mid \|x\|_1 \leq t\} \quad &\Longrightarrow \quad K^* = \{(x,t) \mid \|x\|_\infty \leq t\}
\end{aligned}$$

first three examples are **self-dual** cones

Properties of dual cones

- K^* is a cone, and is always convex, even when the original cone K is not convex
- K^* is closed and convex
- $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$
- If K has nonempty interior, then K^* is pointed
- If the closure of K is pointed then K^* has nonempty interior
- K^{**} is the closure of the convex hull of K. (Hence if K is convex and closed, $K^{**} = K$)
- If K is a proper cone, then so is its dual K^*, and moreover, that $K^{**} = K$

Dual generalized inequalities

- assume K is a proper cone, then K^* is also a proper cone, and $K^{**} = K$
- in such a case K^* also defines generalized inequalities
- generalized inequalities with respect to K^* can usually be interpreted via K

examples

$$y \succeq_{K^*} 0 \quad \Longleftrightarrow \quad y \in K^* \quad \Longleftrightarrow \quad y^T x \geq 0 \text{ for all } x \succeq_K 0$$

$$y \succ_{K^*} 0 \quad \Longleftrightarrow \quad y \in \textbf{int}\, K^* \quad \Longleftrightarrow \quad y^T x > 0 \text{ for all } x \succeq_K 0 \text{ and } x \neq 0$$

Dual characterization of the minimum element

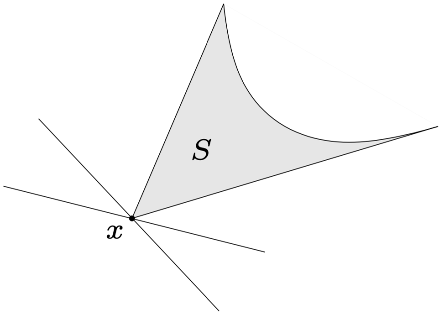
x is the minimum element of S with respect to $\preceq_K$	$\Longleftrightarrow$	x is the unique minimizer of $\lambda^T z$ over $z \in S$ for each $\lambda \succ_{K^*} 0$

Geometrically, this means that for any $\lambda \succ_{K^*} 0$, the hyperplane

$$\{z | \lambda^T (z - x) = 0\}$$

is a strict supporting hyperplane to S at x.

Note that convexity of S is **not required**



Sketch of proof

($\Longrightarrow$) for each $z \in S$ and $z \neq x$

- x is minimum $\Longrightarrow$ $z - x \succeq_K 0$ and $z - x \neq 0$
- $\lambda^T(z - x) > 0$ for each $\lambda \succ_{K^*} 0$ $\Longrightarrow$ x is the unique minimizer of $\lambda^T z$

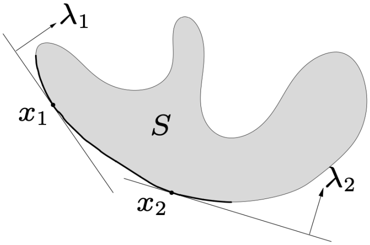
($\Longleftarrow$) for each $z \in S$ and $z \neq x$

- x is the unique minimizer of $\lambda^T z$ $\Longrightarrow$ $\lambda^T(z - x) > 0$ for each $\lambda \succ_{K^*} 0$
- continuity $\Longrightarrow$ $\lambda^T(z - x) \geq 0$ for each $\lambda \succeq_{K^*} 0$ $\Longrightarrow$ $z - x \succeq_K 0$

Dual characterization of a minimal element

There is a gap between the necessary and sufficient conditions. Compared to the minimum, there is no uniqueness in the condition.

x is a minimal element of S with respect to $\preceq_K$	$\Longleftarrow$	x minimizes $\lambda^T z$ over $z \in S$ for some $\lambda \succ_{K^*} 0$



x is a minimal element of S with respect to $\preceq_K$ and S is **convex**	$\Longrightarrow$	x minimizes $\lambda^T z$ over $z \in S$ for some nonzero $\lambda \succeq_{K^*} 0$

Sketch of proof

x is a minimal element of S with respect to $\preceq_K$	$\Longleftarrow$	x minimizes $\lambda^T z$ over $z \in S$ for some $\lambda \succ_{K^*} 0$

($\Longleftarrow$)

- suppose that x is **not** minimal, i.e., there exists a $z \in S, z \neq x$, and $z \prec_K x$ (by def. 2.1 on page 45)
- Then $\lambda^T(x - z) > 0$ which is obtained from the second property relating a generalized inequality and its dual on page 53
- However, this contradicts our condition that x is the minimizer of $\lambda^T z$ over S

A point x can be minimal in S, but not a minimizer of $\lambda^T z$ over $z \in S$, for any λ (see Fig. 2.25 on page 56). This suggests that convexity plays an important role in the converse, which is correct.

Sketch of proof

x is a minimal element of S with respect to $\preceq_K$ and S is **convex**	$\Longrightarrow$	x minimizes $\lambda^T z$ over $z \in S$ for some nonzero $\lambda \succeq_{K^*} 0$

($\Longrightarrow$)

- x is minimal $\quad\Longrightarrow\quad ((x-K)\backslash\{x\}) \cap S = \emptyset$
- Applying the separating hyperplane theorem to the disjoint convex sets $((x-K)\backslash\{x\})$ and S, we conclude that there is a $\lambda \neq 0$ and μ such that $\lambda^T(x-y) \leq \mu$ for all $y \in K$, and $\lambda^T z \geq \mu$ for all $z \in S$
- From the first inequality, we have $\lambda^T y \geq \lambda^T x - \mu \geq 0$ since we take $z = x$ in the second inequality $\lambda^T z \geq \mu$ for all $z \in S$. We conclude $\lambda \succeq_{K^*} 0$ based on page 53.
- Since $x \in S$ and $x \in x - K$, we have $\lambda^T x = \mu$ on the separating hyperplane, so the second inequality implies that μ is the minimum value of $\lambda^T z$ over S.
- x is a minimizer of $\lambda^T z$ over S, where $\lambda \neq 0$, $\lambda \succeq_{K^*} 0$