Problem 11.4 *worth 4 points*
Transpose of orthogonal matrix. Let U be an orthogonal $n \times n$ matrix. Show that its transpose U^T is also orthogonal.

Solution: If U is orthogonal, $U^TU = I$ and $U^{-1} = U^T \longrightarrow U^{-1}U = UU^{-1} = I$.
Orthogonal matrices imply orthonormal columns:

$$U_iU_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases},$$

where i and j denote column number. Let $A = U^T$ and suppose the rows of A are orthonormal since they're the columns of U. If we let our rows of A represent $1 \times n$, a, orthonormal vectors and our columns of U as $n \times 1$, u, orthonormal vectors, when we transpose the vector of block vectors we find:

$$\begin{aligned} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} &= \\ \begin{bmatrix} a_1u_1 & a_1u_2 & a_1u_3 \\ a_2u_1 & a_2u_2 & a_2u_3 \\ a_3u_1 & a_3u_2 & a_3u_3 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= I \end{aligned}$$

This satisfies our condition $U^{-1}U = UU^{-1} = I$ because columns become the rows of transposed vectors, so the only time we'll get a one are when the row number matches the column number.

Problem 11.6 *worth 6 points*
Inverse of a block upper triangular matrix. Let B and D be invertible matrices of sizes $m \times m$ and $n \times n$, respectively, and let C be any $m \times n$ matrix. Find the inverse of

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

in terms of B^{-1}, C, and D^{-1}. (The matrix A is called *block upper triangular*.)
Hints. First get an idea of what the solution should look like by considering the case when B, C, and D are scalars. For the matrix case, your goal is to find matrices W, X, Y, Z (in terms of B^{-1}, C, and D^{-1}) that satisfy

$$A \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = I.$$

Use block matrix multiplication to express this as a set of four matrix equations that you can then solve. The method you will find is sometimes called *block back substitution*.

Solution:

$$A \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} B & C \\ 0 & D \end{bmatrix} \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

(first index will be I) $$\begin{bmatrix} B \\ C \end{bmatrix} \begin{bmatrix} W \\ Y \end{bmatrix} = I$$

$$(BW + CY) = I$$

(second index will be 0) $$\begin{bmatrix} B \\ C \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} = 0$$

$$(BX + CZ) = 0$$

(third index will be 0) $$\begin{bmatrix} 0 \\ D \end{bmatrix} \begin{bmatrix} W \\ Y \end{bmatrix} = 0$$

this implies Y = 0 $$DY = 0$$

(fourth index will be I) $$\begin{bmatrix} 0 \\ D \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} = I$$

this implies $Z = D^{-1}$ $$DZ = I$$

Since we've figured out $Y = 0$ we plug this into the equation for the first index $(BW + CY) = I$ and we see $BW = I$; from our identities we can deduce $W = B^{-1}$. To find X we will use $(BX + CZ) = 0$.

$$\begin{aligned} (BX + CZ) &= 0 \\ BX &= -CZ \\ (B^{-1})BX &= -(B^{-1})CZ \\ IX &= -B^{-1}CZ \\ X &= -B^{-1}CD^{-1} \end{aligned}$$

So this means our answer is

$$\begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} B^{-1} & -B^{-1}CD^{-1} \\ 0 & D^{-1} \end{bmatrix}.$$

Proven:

$$\begin{bmatrix} B & C \\ 0 & D \end{bmatrix} \begin{bmatrix} B^{-1} & -B^{-1}CD^{-1} \\ 0 & D^{-1} \end{bmatrix} = I$$

$$\text{(first entry)} \begin{bmatrix} B \\ C \end{bmatrix} \begin{bmatrix} B^{-1} \\ 0 \end{bmatrix} = BB^{-1} + C0 = I$$

$$\begin{bmatrix} B \\ C \end{bmatrix} \begin{bmatrix} -B^{-1}CD^{-1} \\ D^{-1} \end{bmatrix} = -BB^{-1}CD^{-1} + CD^{-1}$$

$$\text{(second entry)} - ICD^{-1} + CD^{-1} = CD^{-1} - CD^{-1} = 0$$

$$\text{(third entry)} \begin{bmatrix} 0 \\ D \end{bmatrix} \begin{bmatrix} B^{-1} \\ 0 \end{bmatrix} = B^{-1}(0) + D(0) = 0$$

$$\text{(fourth entry)} \begin{bmatrix} 0 \\ D \end{bmatrix} \begin{bmatrix} -B^{-1}CD^{-1} \\ D^{-1} \end{bmatrix} = -B^{-1}CD^{-1}(0) + DD^{-1} = I$$

Problem 11.8 *worth 4 points*
If a matrix is small, its inverse is large. If a number a is small, its inverse $\frac{1}{a}$ (assuming $a \neq 0$) is large. In this exercise you will explore a matrix analog of this idea. Suppose the $n \times n$ matrix A is invertible. Show that $||A^{-1}|| \geq \frac{\sqrt{n}}{||A||}$. This implies that if a matrix is small, its inverse is large. *Hint.* You can use the inequality $||AB|| \leq ||A||||B||$ which holds for any matrices for which the product makes sense. (See exercise 10.12).

Solution: Let $B = A^{-1}$. Since the norm produces a scalar, we will treat it as such:

$$\begin{aligned} ||AB|| &\leq ||A||||B|| \\ ||AA^{-1}|| &\leq ||A||||B|| \\ ||I|| &\leq ||A||||B|| \\ \sqrt{n} &\leq ||A||||B|| \\ \frac{\sqrt{n}}{||A||} &\leq ||B|| \\ ||A^{-1}|| &\geq \frac{\sqrt{n}}{||A||} \end{aligned}$$

Problem 11.9 *worth 6 points*
Push-through identity. Suppose A is $m \times n$, B is $n \times m$, and the $m \times m$ matrix $I + AB$ is invertible.

(a) Show that the $n \times n$ matrix $I + AB$ is invertible. *Hint.* Show that $(I + BA)x = 0$ implies $(I + AB)y = 0$, where $y = Ax$.

(b) Establish the identity
$$B(I + AB)^{-1} = (I + BA)^{-1}B.$$
This is sometimes called th e*push-through identity* since the matrix B appearing on the left 'moves' into the inverse, and 'pushes' the B in the inverse out to the right side.*Hint.* Start with the identity
$$B(I + AB) = (I + BA)B,$$
and multiply on the right by $(I + AB)^{-1}$, and on the left by $(I + BA)^{-1}$.

Solution:

(a) Set $Ix + BAx = Iy + ABy$, and plug Ax in for y:

$$\begin{aligned}(I + BA)x &= Ix + BAx = 0\\ (I + AB)y &= Iy + ABy = 0\end{aligned}$$

$$\begin{aligned}Ix + BAx &= Iy + ABy\\ x + BAx &= y + ABy\\ y &= Ax\\ x + BAx &= Ax + ABAx\\ x + BAx &= A(Ix + BAx)\\ x + BAx &= A(0)\end{aligned}$$

(b) It gave the solution in the hint:

$$\begin{aligned}B(I + AB) &= (I + BA)B\\ (I + BA)^{-1}B(I + AB)(I + AB)^{-1} &= (I + BA)^{-1}(I + BA)B(I + AB)^{-1}\\ (I + BA)^{-1}BI &= IB(I + AB)^{-1}\\ (I + BA)^{-1}B &= B(I + AB)^{-1}\\ B(I + AB)^{-1} &= (I + BA)^{-1}B\end{aligned}$$