> **Problem 11.4** *worth 4 points*
> *Transpose of orthogonal matrix.* Let U be an orthogonal $n \times n$ matrix. Show that its transpose U^T is also orthogonal.

Solution: If U is orthogonal, $U^T U = I$ and $U^{-1} = U^T \longrightarrow U^{-1}U = UU^{-1} = I$.
Orthogonal matrices imply orthonormal rows:

$$U_i U_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases},$$

where i and j denote row number. Let $A = U^T$ and suppose the columns of A are orthonormal. If we let our rows of U represent $1 \times n$ orthonormal vectors and our columns of A as $n \times 1$ orthonormal vectors, when we transpose the vector of block vectors we find:

$$\begin{bmatrix} U_{R1}A_{C1} & U_{R1}A_{C2} & U_{R1}A_{C3} \\ U_{R2}A_{C1} & U_{R2}A_{C2} & U_{R2}A_{C3} \\ U_{R3}A_{C1} & U_{R3}A_{C2} & U_{R3}A_{C3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= I$$

This satisfies our condition $U^{-1}U = UU^{-1} = I$ because rows become the columns of transposed vectors, so the only time we'll get a one are when the row number matches the column number.

> **Problem 11.6** *worth 6 points*
> *Inverse of a block upper triangular matrix.* Let B and D be invertible matrices of sizes $m \times m$ and $n \times n$, respectively, and let C be any $m \times n$ matrix. Find the inverse of
> $$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$
> in terms of B^{-1}, C, and D^{-1}. (The matrix A is called *block upper triangular.*)
> *Hints.* First get an idea of what the solution should look like by considering the case when B, C, and D are scalars. For the matrix case, your goal is to find matrices W, X, Y, Z (in terms of B^{-1}, C, and D^{-1}) that satisfy
> $$A \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = I.$$
> Use block matrix multiplication to express this as a set of four matrix equations that you can then solve. The method you will find is sometimes called *block back substitution.*

Solution:

$$A \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} B & C \\ 0 & D \end{bmatrix} \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

(first index will be I)
$$\begin{bmatrix} B \\ C \end{bmatrix} \begin{bmatrix} W \\ Y \end{bmatrix} = I$$
$$(BW + CY) = I$$

(second index will be 0)
$$\begin{bmatrix} B \\ C \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} = 0$$
$$(BX + CZ) = 0$$

(third index will be 0)
$$\begin{bmatrix} 0 \\ D \end{bmatrix} \begin{bmatrix} W \\ Y \end{bmatrix} = 0$$

this implies Y = 0
$$DY = 0$$

(fourth index will be I)
$$\begin{bmatrix} 0 \\ D \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} = I$$

this implies $Z = D^{-1}$
$$DZ = I$$

Since we've figured out $Y = 0$ we plug this into the equation for the first index $(BW + CY) = I$ and we see $BW = I$; from our identities we can deduce $W = B^{-1}$. To find X we will use $(BX + CZ) = 0$.

$$(BX + CZ) = 0$$
$$BX = -CZ$$
$$(B^{-1})BX = -(B^{-1})CZ$$
$$IX = -B^{-1}CZ$$
$$X = -B^{-1}CD^{-1}$$

So this means our answer is
$$\begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} B^{-1} & -B^{-1}CD^{-1} \\ 0 & D^{-1} \end{bmatrix}.$$

Proven:

$$\begin{bmatrix} B & C \\ 0 & D \end{bmatrix} \begin{bmatrix} B^{-1} & -B^{-1}CD^{-1} \\ 0 & D^{-1} \end{bmatrix} = I$$

$$\text{(first entry) } \begin{bmatrix} B \\ C \end{bmatrix} \begin{bmatrix} B^{-1} \\ 0 \end{bmatrix} = BB^{-1} + C0 = I$$

$$\begin{bmatrix} B \\ C \end{bmatrix} \begin{bmatrix} -B^{-1}CD^{-1} \\ D^{-1} \end{bmatrix} = -BB^{-1}CD^{-1} + CD^{-1}$$

$$\text{(second entry) } -ICD^{-1} + CD^{-1} = CD^{-1} - CD^{-1} = 0$$

$$\text{(third entry) } \begin{bmatrix} 0 \\ D \end{bmatrix} \begin{bmatrix} B^{-1} \\ 0 \end{bmatrix} = B^{-1}(0) + D(0) = 0$$

$$\text{(fourth entry) } \begin{bmatrix} 0 \\ D \end{bmatrix} \begin{bmatrix} -B^{-1}CD^{-1} \\ D^{-1} \end{bmatrix} = -B^{-1}CD^{-1}(0) + DD^{-1} = I$$

Problem 11.8 *worth 4 points*
If a matrix is small, its inverse is large. If a number a is small, its inverse $\frac{1}{a}$ (assuming $a \neq 0$) is large. In this exercise you will explore a matrix analog of this idea. Suppose the $n \times n$ matrix A is invertible. Show that $||A^{-1}|| \geq \frac{\sqrt{n}}{||A||}$. This implies that if a matrix is small, its inverse is large. *Hint.* You can use the inequality $||AB|| \leq ||A||||B||$ which holds for any matrices for which the product makes sense. (See exercise 10.12).

Solution: Let $B = A^{-1}$. Since the norm produces a scalar, we will treat it as such:

$$||AB|| \leq ||A||||B||$$
$$||AA^{-1}|| \leq ||A||||B||$$
$$||I|| \leq ||A||||B||$$
$$\sqrt{n} \leq ||A||||B||$$
$$\frac{\sqrt{n}}{||A||} \leq ||B||$$
$$||A^{-1}|| \geq \frac{\sqrt{n}}{||A||}$$

Problem 11.9 *worth 6 points*
Push-through identity. Suppose A is $m \times n$, B is $n \times m$, and the $m \times m$ matrix $I + AB$ is invertible.

(a) Show that the $n \times n$ matrix $I + AB$ is invertible. *Hint.* Show that $(I + BA)x = 0$ implies $(I + AB)y = 0$, where $y = Ax$.

(b) Establish the identity
$$B(I + AB)^{-1} = (I + BA)^{-1}B.$$

This is sometimes called th e*push-through identity* since the matrix B appearing on the left 'moves' into the inverse, and 'pushes' the B in the inverse out to the right side.*Hint.* Start with the identity

$$B(I + AB) = (I + BA)B,$$

and multiply on the right by $(I + AB)^{-1}$, and on the left by $(I + BA)^{-1}$.

Solution:

(a) Set $Ix + BAx = Iy + ABy$, and plug Ax in for y:

$$(I + BA)x = Ix + BAx = 0$$
$$(I + AB)y = Iy + ABy = 0$$

$$Ix + BAx = Iy + ABy$$
$$x + BAx = y + ABy$$
$$y = Ax$$
$$x + BAx = Ax + ABAx$$
$$x + BAx = A(Ix + BAx)$$
$$x + BAx = A(0)$$

(b) It gave the solution in the hint:

$$B(I + AB) = (I + BA)B$$
$$(I + BA)^{-1}B(I + AB)(I + AB)^{-1} = (I + BA)^{-1}(I + BA)B(I + AB)^{-1}$$
$$(I + BA)^{-1}BI = IB(I + AB)^{-1}$$
$$(I + BA)^{-1}B = B(I + AB)^{-1}$$
$$B(I + AB)^{-1} = (I + BA)^{-1}B$$