Problem 9.1 *worth 7 points*
Compartmental system. A *compartmental system* is a model used to describe the movement of some material over time among a set of n compartments of a system, and the outside world. It is widely used in *pharmaco-kinetics*, the study of how the concentration of a drug varies over time in the body. In this application, the material is a drug, and the compartments are the bloodstream, lungs, heart, liver, kidneys, and so on. Compartmental systems are special cases of linear dynamical systems.

In this problem we will consider a very simple compartmental system with 3 compartments. We let $(x_t)_i$ denote the amount of material (say, a drug) in compartment i at time period t. Between period t and period $t+1$, the material moves as follows.

- 10% of the material in compartment 1 moves to compartment 2. (This decreases the amount in compartment 1 and increases the amount in compartment 2).
- 5% of the material in compartment 2 moves to compartment 3.
- 5% of the material in compartment 3 moves to compartment 1.
- 5% of the material in compartment 3 is eliminated.

Express this compartmental system as a linear dynamical system, $x_{t+1} = Ax_t$. (Give the matrix A.) Be sure to account for all the material entering and leaving each compartment.

Solution: Breaking down each piece allows us to see how each is affected by time. Starting with compartment 1 $[(x_0)_1]$ we see it loses 10% to compartment 2 and gains 5% from compartment 3. This means we should see $(x_{t+1})_1 = .9(x_t)_1 + .05(x_t)_3$. Using knowledge of linear equations, we know that our matrix A is going to be the coefficients needed to make the next iteration:

$$\begin{bmatrix}(x_{t+1})_1\\(x_{t+1})_2\\(x_{t+1})_3\end{bmatrix} = A\begin{bmatrix}(x_t)_1\\(x_t)_2\\(x_t)_3\end{bmatrix}$$
$$\begin{bmatrix}(x_{t+1})_1\\(x_{t+1})_2\\(x_{t+1})_3\end{bmatrix} = \begin{bmatrix}0.9 & 0 & 0.05\\ .1 & .95 & 0\\ 0 & .05 & .95\end{bmatrix}\begin{bmatrix}(x_t)_1\\(x_t)_2\\(x_t)_3\end{bmatrix}$$
$$\begin{bmatrix}(x_{t+1})_1\\(x_{t+1})_2\\(x_{t+1})_3\end{bmatrix} = \begin{bmatrix}0.9(x_t)_1 + 0(x_t)_2 + 0.05(x_t)_3\\ .1(x_t)_1 + .95(x_t)_2 + 0(x_t)_3\\ 0(x_t)_1 + .05(x_t)_2 + .95(x_t)_3\end{bmatrix}$$
$$\begin{bmatrix}(x_{t+1})_1\\(x_{t+1})_2\\(x_{t+1})_3\end{bmatrix} = \begin{bmatrix}0.9(x_t)_1 + 0.05(x_t)_3\\ .1(x_t)_1 + .95(x_t)_2\\ .05(x_t)_2 + .9(x_t)_3\end{bmatrix}$$

We can verify the other two quickly.

$$(x_{t+1})_2 = .1(x_t)_1 + .95(x_t)_2$$

Because compartment 2 gains 10% from compartment 1 and loses 5% from itself, meaning it has 95% left.

$$(x_{t+1})_3 = .05(x_t)_2 + .9(x_t)_3$$

Because compartment 3 gains 5% of compartment 2 and loses 5% of itself, but also gives 5% of its material to compartment 2, totaling 10% lost.

Problem 9.3 *worth 6 points*
Equilibrium point for linear dynamical system. Consider a time-invariant linear dynamical system with offset, $x_{t+1} = Ax_t + c$, where x_t is the state n-vector. We say that a vector z is an *equilibrium point* of the linear dynamical system if $x_1 = z$ implies $x_2 = z$, $x_3 = z$,... (In words: If the system starts in state z, it stays in state z.)

Find a matrix F and vector g for which the set of linear equations $Fz = g$ characterizes equilibrium points. (This means: If z is an equilibrium point, then $Fz = g$; conversely if $Fz = g$, then z is an equilibrium point.) Express F and g in terms of A, c, any standard matrices or vectors (e.g., I, $\mathbf{1}$, or 0), and matrix and vector operations.

Remark. Equilibrium points often have interesting interpretations. For example, if the linear dynamical system describes the population dynamics of a country, with the vector c denoting immigration (emigration when entries of c are negative), an equilibrium point is a population distribution that does not change, year to year. In other words, immigration exactly cancels the changes in population distribution caused by aging, births, and deaths.

Solution: We will first plug in two identical vectors and see what we need A and c_n to be.

$$Fz = g$$

$$A \begin{bmatrix} z \\ z \\ z \end{bmatrix} + c_n = \begin{bmatrix} z \\ z \\ z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ z \\ z \end{bmatrix} + c_n = \begin{bmatrix} z \\ z \\ z \end{bmatrix}$$

$$\begin{bmatrix} z \\ z \\ z \end{bmatrix} + c_n = \begin{bmatrix} z \\ z \\ z \end{bmatrix}$$

The only thing c_n should be is 0_n, and $A = I_{n \times n}$

$$\begin{bmatrix} z \\ z \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} z \\ z \\ z \end{bmatrix}$$

And so now we have $z = z$, meaning we have equilibrium.

Problem 9.5 modified *worth 7 points*
Suppose we were to represent the $(n+1)^{th}, n^{th}$ Fibonacci numbers as a linear dynamical system, we would go about it as follows, we first define the initial values to be $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, representing the second and first Fibonacci numbers respectively (0 and 1). That is we consider $x_1 = \begin{bmatrix} fib(2) \\ fib(1) \end{bmatrix}$. Where $fib(n)$ denotes the n^{th} Fibonacci number for your convenience. Your task is to find the matrix A such that $x_{t+1} = Ax_t$, that is matrix A such that on multiplying it to the vector containing the $(n)^{th}$ and $(n-1)^{th}$ Fibonacci numbers would give us the $(n+1)^{th}$ and $(n)^{th}$ Fibonacci numbers. That is find A such that

$$\begin{bmatrix} fib(n+1) \\ fib(n) \end{bmatrix} = A \begin{bmatrix} fib(n) \\ fib(n-1) \end{bmatrix}$$

Solution: since the Fibonacci sequence is defined as $fib(n+1) = fib(n) + fib(n-1)$

$$\begin{bmatrix} fib(n+1) \\ fib(n) \end{bmatrix} = A \begin{bmatrix} fib(n) \\ fib(n-1) \end{bmatrix}$$
$$\begin{bmatrix} fib(n+1) \\ fib(n) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} fib(n) \\ fib(n-1) \end{bmatrix}$$
$$\begin{bmatrix} fib(n+1) \\ fib(n) \end{bmatrix} = \begin{bmatrix} fib(n)(1) & +fib(n-1)(1) \\ fib(n)(1) & +0 \end{bmatrix}$$
$$\begin{bmatrix} fib(n+1) \\ fib(n) \end{bmatrix} = \begin{bmatrix} fib(n) + fib(n-1) \\ fib(n) \end{bmatrix}$$

so we see $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ satisfies the sequence conditions because $fib(n+1) = fib(n) + fib(n-1)$

Problem 9.4 Modified *worth 10 extra credit points*
Express the K-Markov model as a linear dynamical system with state $z_t = (x_t, \ldots, x_{t-K+1})$, (As in $z_{t+1} = Bz_t$, find B) where $x_{t+1} = A_1x_t + A_2x_{t-1} + A_3x_{t-2} + \ldots + A_Kx_{t-K+1}$ (Hint: Use Block Matrices)

Solution: To begin I started with the original problem to find the pattern occuring. It came down to multiplying the original vector by a matrix that would create the a new vector from x_2 and x_3 to x_3 and x_4. This was because the $n+1th$ element is a recurrence relation and needs the two preceding vectors to make itself.

$$\begin{bmatrix} x_{t+1} \\ x_t \\ x_{t-1} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} x_t \\ x_{t-1} \\ x_{t-2} \end{bmatrix}$$
$$\begin{bmatrix} x_{t+1} \\ x_t \\ x_{t-1} \end{bmatrix} = \begin{bmatrix} A_1x_t + A_2x_{t-1} + A_3x_{t-2} \\ Ix_t + 0 + 0 \\ 0 + Ix_{t-1} + 0 \end{bmatrix}$$
$$\begin{bmatrix} x_{t+1} \\ x_t \\ x_{t-1} \end{bmatrix} = \begin{bmatrix} A_1x_t + A_2x_{t-1} + A_3x_{t-2} \\ x_t \\ x_{t-1} \end{bmatrix}$$

To make this go to $t - K + 1$:

$$B = \begin{bmatrix} A_1 & A_2 & A_3 & \cdots & A_K \\ I & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix}$$