Full name.:	William Mardick-Kanter
Student id.:	109388896

1. Question:

Determine which of the following statements are **true** or **false**. Put a cross on the corresponding check box. Note that each correct answer is 1 point, each wrong answer is -1 point, and each unanswered one is 0 point. The total minimum number of points for this question is zero.

17 points

true	false	
☐	☒	Function $f : \mathbb{R}^n \to \mathbb{R}$ defined as $f(x) = \|x\|$ is a linear function.
☒	☐	If the angle between two n-vectors a and b is $\pi/6$, then they are linearly independent.
☐	☒	If we apply Gram–Schmidt algorithm to k vectors and it terminates sooner than k iterations, then those vectors are linearly independent.
☒	☐	The transpose of an upper triangular matrix is a lower triangular matrix.
☒	☐	If $A = \mathbf{diag}(\lambda_1, \dots, \lambda_n)$, then $\|A\| = \|(\lambda_1, \dots, \lambda_n)\|$.
☐	☒	Equation $Ax = b$ doe not have any solution for x if columns of A are linearly dependent.
☒	☐	Consider square matrices A, B. $AB = BA$ if A and B are diagonal.
☐	☒	If $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear and $g : \mathbb{R}^p \to \mathbb{R}^n$ is affine, then $f(g(\cdot)) : \mathbb{R}^p \to \mathbb{R}^m$ is linear.
☒	☐	If matrix A is orthogonal, then $\|Ax\| = \|x\|$ for any x.
☒	☐	If A is a tall matrix, then it is not right-invertible.
☐	☒	Consider $A \in \mathbb{R}^{n \times m}$. If $\text{rank}(A) = m$, then map $f(x) = Ax$ is one-to-one.
☐	☒	If matrix A is orthogonal, then $\det(A) = 1$.
☒	☐	Consider $A \in \mathbb{R}^{n \times m}$. If A is right-invertible, then map $f(x) = Ax$ is onto.
☒	☐	If $\det(A) = 0$, then at least one of the eigenvalues of A is zero.
☒	☐	The characteristic polynomial of a matrix A is $(\lambda + 3)(\lambda + 2)(\lambda + 1)$. Then A is diagonalizable.
☐	☐	Square matrices A and $Q^T AQ$ have the same eigenvalues, where Q is a nonsingular matrix.
☐	☒	Given any $A \in \mathbb{R}^{n \times m}$, $A^T A$ is always real symmetric.

Problem 2 *worth 10 points*
i) Find the QR factorization of the matrix

$$A = \begin{bmatrix} 0 & -3 & 2 \\ 1 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

ii) Given

$$A^{-1} = \begin{bmatrix} 0 & 1 & -4 \\ -\frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix},$$

compute R^{-1}. *(Hint: one knows that $A^{-1} = R^{-1}Q^T$).*

Solution: First we apply Gram-Schmidt to A:

$$\begin{aligned}
\tilde{q_1} &= a_1 \\
\tilde{q_1} &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
q_1 &= \frac{\tilde{q_1}}{||\tilde{q_1}||} \\
||\tilde{q_1}|| &= \sqrt{0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0} \\
||\tilde{q_1}|| &= 1 \\
q_1 &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\tilde{q_2} &= a_2 - (q_1^T a_2) q_1 \\
\tilde{q_2} &= \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
\tilde{q_2} &= \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} - (0) \\
q_2 &= \frac{\tilde{q_2}}{||\tilde{q_2}||} \\
||\tilde{q_2}|| &= \sqrt{-3 \cdot -3 + 0 \cdot 0 + 0 \cdot 0} \\
||\tilde{q_2}|| &= 3 \\
q_2 &= \begin{bmatrix} \frac{-3}{3} \\ \frac{0}{3} \\ \frac{0}{3} \end{bmatrix} \\
q_2 &= \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

$$\tilde{q_3} = a_3 - (q_1^T a_3) q_1 - (q_2^T a_3) q_2$$

$$\tilde{q_3} = \begin{bmatrix}2\\4\\1\end{bmatrix} - \left(\begin{bmatrix}0\\1\\0\end{bmatrix}^T \begin{bmatrix}2\\4\\1\end{bmatrix}\right)\begin{bmatrix}0\\1\\0\end{bmatrix} - \left(\begin{bmatrix}-1\\0\\0\end{bmatrix}^T \begin{bmatrix}2\\4\\1\end{bmatrix}\right)\begin{bmatrix}-1\\0\\0\end{bmatrix}$$

$$\tilde{q_3} = \begin{bmatrix}2\\4\\1\end{bmatrix} - (4)\begin{bmatrix}0\\1\\0\end{bmatrix} - (-2)\begin{bmatrix}-1\\0\\0\end{bmatrix}$$

$$\tilde{q_3} = \begin{bmatrix}2\\4\\1\end{bmatrix} - \begin{bmatrix}0\\4\\0\end{bmatrix} - \begin{bmatrix}2\\0\\0\end{bmatrix}$$

$$\tilde{q_3} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$q_3 = \frac{\tilde{q_3}}{||\tilde{q_3}||}$$

$$||\tilde{q_3}|| = \sqrt{0\cdot 0 + 0\cdot 0 + 1\cdot 1}$$

$$||\tilde{q_3}|| = 1$$

$$q_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$Q = \begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

So now we have Q since the columns of Q are the orthogonalized columns of A. Since we know Q and R are invertible we can solve for R^{-1} from $A = QR$

$$A = QR$$

$$AQ^{-1} = R$$

$$A^{-1}Q = R^{-1}$$

$$\begin{bmatrix}0 & 1 & -4\\-\frac{1}{3} & 0 & \frac{2}{3}\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix} = R^{-1}$$

$$\begin{bmatrix}0\cdot 0 + 1\cdot 1 + -4\cdot 0 & 0\cdot -1 + 1\cdot 0 + -4\cdot 0 & 0\cdot 0 + 1\cdot 0 + -4\cdot 1\\-\frac{1}{3}\cdot 0 + 0\cdot 1 + \frac{2}{3}\cdot 0 & -\frac{1}{3}\cdot -1 + 0\cdot 0 + \frac{2}{3}\cdot 0 & -\frac{1}{3}\cdot 0 + 0\cdot 0 + \frac{2}{3}\cdot 1\\0\cdot 0 + 0\cdot 1 + 1\cdot 0 & 0\cdot -1 + 0\cdot 0 + 1\cdot 0 & 0\cdot 0 + 0\cdot 0 + 1\cdot 1\end{bmatrix} = R^{-1}$$

$$\begin{bmatrix}1 & 0 & -4\\0 & \frac{1}{3} & \frac{2}{3}\\0 & 0 & 1\end{bmatrix} = R^{-1}$$

Problem 3 *worth 5 points*
A real symmetric matrix $B \in \mathbb{R}^{n\times n}$ (i.e. $B^T = B$) is said to be positive definite if all of its eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are positive. (Recall that λ is an eigenvalue of B if and only if there exits a nonzero vector t such that $Bt = \lambda t$). Show that B^{-1} is also positive definite. That is, you need to show that all the eigenvalues of B^{-1} are also positive. (Hint: consider equation $Bt_i = \lambda_i t_i$ for all $i \in 1, \ldots, n$. What happens when we multiply B^-1 from left to both sides of this equality?)

Solution: If B is positive definite and invertible then we know that all of the eigenvalues of B are greater than 0. Since it is invertible we also know $B^{-1}B = I$ which leads to

$$I = B(B^{-1})^T$$

$$I = BB^{-1}$$

Since this is the case then it follows that all the eigenvalues of B^{-1} are of the form $\frac{1}{\lambda_i}$ where λ_i is an eigenvalue of B. Since all of B's eigenvalues are positive, inverting them will retain their positive value, meaning all the eigenvalues of B^{-1} are positive, which means B^{-1} is positive definite.

Problem 4 *worth 5 points*
Show that $A = AA^{\dagger}A$ and $A^{\dagger} = A^{\dagger}AA^{\dagger}$ where $A^{\dagger} = (A^TA)^{-1}A^T$

Solution:

$$A = AA^{\dagger}A$$
$$A = A((A^TA)^{-1}A^T)A$$
$$A = A(A^TA)^{-1}(A^TA)^1$$
$$A = A(A^TA)^{-1+1}$$
$$A = A(A^TA)^0$$
$$A = A(1)$$
$$A = A$$

So we can now see that $A^{\dagger}A = 1$

$$A^{\dagger} = A^{\dagger}AA^{\dagger}$$
$$A^{\dagger} = (1)A^{\dagger}$$
$$A^{\dagger} = A^{\dagger}$$

Problem 5 *worth 5 points*
Show that matrix

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is orthogonal and then find

$$RR^T R \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Solution: To prove a matrix is orthogonal we multiply the matrix by its transpose, if we get the identity matrix, we have an orthogonal matrix

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad R^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{aligned} RR^T &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \cos\theta\sin\theta \\ \cos\theta\sin\theta - \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} \end{aligned}$$

From Calculus we know $\cos^2\theta + \sin^2\theta = 1$. So

$$RR^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which proves R is orthogonal. Next we will solve for the equation:

$$\begin{aligned} RR^T R \begin{bmatrix} 3 \\ 4 \end{bmatrix} &= IR \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 3\cos\theta & -4\sin\theta \\ 3\sin\theta & 4\cos\theta \end{bmatrix} \end{aligned}$$

Problem 6 *worth 10 points*
Suppose A is an $m \times n$ matrix with linearly independent columns and QR factorization $A = QR$, and b is an m-vector. The vector $A\hat{x}$ is the linear combination of the columns of A that is closest to the vector b, i.e., it is the projection of b onto the set of linear combinations of the columns of A. Show that $\|A\hat{x} - b\|^2 = \|b\|^2 - \left\|Q^T b\right\|^2$ (This is the square of the distance between b and the closest linear combination of the columns of A) (Hint: vector $\hat{x}$ is the least squares approximate solution of $Ax = b$.)

Solution: We know that $A\hat{x} = QQ^T b$ (proven in Homework 10), so we can start with expanding the left hand side. We also know that $||A\hat{x} - b||^2 = ||A\hat{x}||^2 + ||b||^2 - 2A\hat{x}^T b$.

$$\begin{aligned}
\|A\hat{x} - b\|^2 &= \|A\hat{x}\|^2 + \|b\|^2 - 2A\hat{x}^T b \\
\|A\hat{x} - b\|^2 &= \|A\hat{x}\|^2 + \|b\|^2 - 2\|A\hat{x}\|\|b\| \cos\theta \\
\|A\hat{x} - b\|^2 &= \|A\hat{x}\|^2 + \|b\|^2 - 2\|A\hat{x}\|\|b\|(0)
\end{aligned}$$

We know that the angle between the m-vector $A\hat{x}$ and the m-vector b is perpendicular to any linear combination of the vectors in the space of A (given by the Orthogonality Principle, textbook Equation 12.9) so the dot product will be zero. Next, we see we're left with $\|A\hat{x}\|^2 + \|b\|^2$ but we know the right hand side of the equation is going to be equal to 0 because the solution to the least squares problem is to minimize that equation.

$$\begin{aligned}
0 &= \|A\hat{x}\|^2 + \|b\|^2 \\
0 &= \|QQ^T b\|^2 + \|b\|^2
\end{aligned}$$

The only way this solution will work is if Q is the **-1** diagonal matrix. If this were the case then $QQ^T = I$ and $Q^T b = -b$, but since we're taking the square of the norm it won't matter. We can pull out the first Q as a negative sign so we're left with

$$\begin{aligned}
\|A\hat{x} - b\|^2 &= \|QQ^T b\|^2 + \|b\|^2 \\
\|A\hat{x} - b\|^2 &= \mathbf{-1}\|Q^T b\|^2 + \|b\|^2 \\
\|A\hat{x} - b\|^2 &= \|b\|^2 - \|Q^T b\|^2 \\
\|A\hat{x} - b\|^2 &= \|b\|^2 - \|b\|^2 \\
0 &= 0
\end{aligned}$$