Solving Problems on Small Oscillations

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The steps to solving a problem involving small oscillations are as follows:

1. Find the Lagrangian of the system, defined as $L = T - V$.

2. Invoke small oscillations to the system. These include, for instance, having an angle be small or making some displacement small. In your Lagrangian

   - Use the small angle approximation, keeping terms up to order θ in your Lagrangian.
   $$\sin\theta \approx \theta, \quad \cos\theta \approx 1 - \frac{1}{2}\theta^2.$$
   - Make the relevant displacements small. This may involve doing Taylor expansions, keeping terms up to first order in the coordinate in your Lagrangian.
   - Use the small displacements, denoted as η_i, the generalized coordinates of your system.

3. Apply Lagrange's equations using the η_i as the generalized coordinates, giving you i equations:
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_i}\right) - \frac{\partial L}{\partial \eta_i} = 0.$$
Doing this will give you factors of $\ddot{\eta}_i$ in your equations of motion.

4. Write the generalized coordinates as
$$\eta_i = a_i e^{i\omega t},$$
and substitute into the equations of motion found in step 3. The time derivatives are
$$\dot{\eta}_i = i\omega a_i e^{i\omega t}, \quad \ddot{\eta}_i = -\omega^2 a_i e^{i\omega t}.$$
After doing this, the $e^{i\omega t}$ terms should cancel out.

5. Write the system of equations (for i generalized coordinates you should have a system of i equations) in the form
$$(\mathbb{V} - \omega^2\mathbb{T})\mathbf{a} = 0.$$
In the above equation,

   - $\mathbb{V}$ and $\mathbb{T}$ are the potential energy and kinetic energy matrices, respectively. To form the kinetic energy matrix, group together all of the terms in your system of equations that contain an ω^2. Those terms, with the ω^2 factored out, form the $\mathbb{T}$ matrix. The other terms form the $\mathbb{V}$ matrix.
   - $\mathbf{a}$ is a vector that contains the a_i's.
   - The ω^2's are the normal mode frequencies.

6. Solve the eigenvalue problem from step 5.

   - First find the eigenfrequencies (normal mode frequencies) ω_k^2 by looking at
   $$det(\mathbb{V} - \omega^2\mathbb{T}) = 0, \tag{1}$$
   and solving for the ω_k^2's.

- Next, find the eigenvectors $\mathbf{a}_k$ for each normal mode frequency from

$$(\mathbb{V} - \omega_k^2 \mathbb{T})\mathbf{a}_k = 0.$$

Solve for the corresponding $\mathbf{a}_k$'s and be sure to normalize them using the condition

$$\mathbf{a}^{(k)T} \mathbb{T} \mathbf{a}^{(k)} = 1,$$

where the superscript T corresponds to transposing the vector. Our normalized eigenvectors (with normalization constant a_k) have the form

$$\mathbf{a}^{(k)} = a_k \mathbf{a}_k.$$

Note that the superscript k corresponds to the eigenvector being normalized, having the form of the above equation.

7. Construct the modal matrix $\mathbb{A}$. This is done by using the normalized eigenvectors $\mathbf{a}^{(k)}$ as the columns of $\mathbb{A}$.

8. Find the normal coordinates, denoted by ξ_i. The relationship between the ξ_i and the η_i is

$$\xi = \mathbb{A}^T \mathbb{T} \eta,$$

where

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \end{bmatrix}, \quad \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \end{bmatrix}.$$

These normal coordinates diagonalize the Lagrangian, and decouple the equations of motion.

9. Write the Lagrangian in terms of the normal coordinates ξ_i:

   - Write the generalized coordinates and velocities η_i and $\dot{\eta}_i$ in terms of the normal coordinates and velocities ξ_i and $\dot{\xi}_i$.
   - Apply Lagrange's equations, this time using the normal coordinates ξ_i as your new generalized coordinates:
   $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_i}\right) - \frac{\partial L}{\partial \xi_i} = 0.$$
   - The result should be that you obtain i uncoupled differential equations corresponding to simple harmonic oscillators with frequency ω_k (the normal mode frequencies found in step 6).
   - Note that each normal coordinate corresponds to a vibration of the system with only one frequency.