Symmetries in Quantum Mechanics

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1 Symmetries in Quantum Mechanics

Consider a unitary operator, $\mathcal{S}$ (called a symmetry operation), which is associated with some operation, such as translation or rotation. We have learned that for symmetry operations that differ infinitesimally from the identity operator, we can write

$$\mathcal{S} = 1 - \frac{i\epsilon}{\hbar} G, \tag{1}$$

where G is the Hermitian operator that is the generator of the symmetry operator. For example, the momentum operator is the generator of translation and angular momentum is the generator of rotation. If the Hamiltonian is invariant under $\mathcal{S}$, then

$$\mathcal{S}^\dagger H \mathcal{S} = H \Rightarrow [G, H] = 0. \tag{2}$$

Then from the Heisenberg equations of motion, we have

$$\frac{dG}{dt} = 0. \tag{3}$$

From this, we can see that if the Hamiltonian is invariant under a symmetry operator (e.g. position) then the generator of that operation (e.g. momentum) is a constant of motion.

2 Degeneracies

Suppose the Hamiltonian is invariant under some symmetry operator,

$$[H, \mathcal{S}] = 0 \tag{4}$$

and $|n\rangle$ is an energy eigenket with E_n. Then

$$H(\mathcal{S}|n\rangle) = \mathcal{S}H|n\rangle = E_n(\mathcal{S}|n\rangle). \tag{5}$$

So $|n\rangle$ and $\mathcal{S}|n\rangle$ have the same energy, E_n. If they represent different states, then they are degenerate. If $\mathcal{S}$ is characterized by some continuous parameter λ, then all states of the form $\mathcal{S}(\lambda)|n\rangle$ have the same energy.

3 Discrete Symmetries – Parity

3.1 Properties of the Parity Operator

The parity operator is unitary and Hermitian:

$$\pi^{-1} = \pi^\dagger = \pi. \tag{6}$$

An eigenstate of the parity operator can only have eigenvalue ± 1.

3.2 Operators Under Parity

The position operator and the momentum operator are odd under parity:

$$\pi^\dagger \mathbf{x} \pi = -\mathbf{x} \Rightarrow \mathbf{x}\pi = -\pi\mathbf{x} \tag{7}$$

$$\pi^\dagger \mathbf{p} \pi = -\mathbf{p} \Rightarrow \mathbf{p}\pi = -\pi\mathbf{p}. \tag{8}$$

That is, they both anticommute with the parity operator.

The angular momentum and spin operator is even under parity:

$$\pi^\dagger \mathbf{J} \pi = \mathbf{J} \Rightarrow \mathbf{J}\pi = \pi\mathbf{J}, \tag{9}$$

$$\pi^\dagger \mathbf{S} \pi = \mathbf{S}. \tag{10}$$

That is, it commutes with the parity operator.

A pseudoscalar is an operator that behaves like

$$\pi^\dagger \mathbf{S} \cdot \mathbf{x} \pi = -\mathbf{S} \cdot \mathbf{x} \Rightarrow \mathbf{S} \cdot \mathbf{x}\pi = -\pi\mathbf{S} \cdot \mathbf{x}. \tag{11}$$

Whereas for ordinary scalars we have

$$\pi^\dagger \mathbf{L} \cdot \mathbf{S} \pi = \mathbf{L} \cdot \mathbf{S} \Rightarrow \mathbf{L} \cdot \mathbf{S}\pi = \pi\mathbf{L} \cdot \mathbf{S}. \tag{12}$$

3.3 Wave Functions Under Parity

The wave function is given by

$$\psi(\mathbf{x}') = \langle \mathbf{x}' | \alpha \rangle . \tag{13}$$

Then the space inverted state is given by

$$\langle \mathbf{x}' | \pi | \alpha \rangle . \tag{14}$$

We can apply the parity operator to either the bra or the ket, and if we suppose $|\alpha\rangle$ is an parity eigenstate, then

$$\langle \mathbf{x}' | \pi | \alpha \rangle = \langle -\mathbf{x}' | \alpha \rangle = \psi(-\mathbf{x}') = \pm \langle \mathbf{x}' | \alpha \rangle = \pm\psi(\mathbf{x}'). \tag{15}$$

So the state $|\alpha\rangle$ is either even or odd under parity depending on whether the corresponding wave function satisfies

$$\psi(-\mathbf{x}') = \pm\psi(\mathbf{x}') \begin{cases} even \\ odd \end{cases} . \tag{16}$$

For an eigenstate of angular momentum, we have

$$\pi |\alpha, l, m\rangle = (-1)^l |\alpha, l, m\rangle = \pm |\alpha, l, m\rangle \begin{cases} l = even \\ l = odd \end{cases} . \tag{17}$$

3.4 Parity Operator Theorem

Suppose the Hamiltonian is invariant under parity:

$$[H, \pi] = 0. \tag{18}$$

Also suppose $|n\rangle$ is a nondegenerate eigenket of H with eigenvalue E_n:

$$H |n\rangle = E_n |n\rangle . \tag{19}$$

Then $|n\rangle$ is also a parity eigenket.

3.5 Parity Selection Rule

Suppose $|a\rangle$ and $|b\rangle$ are both eigenstates of the parity operator:

$$\pi |a\rangle = \epsilon_a |a\rangle \tag{20}$$

$$\pi |b\rangle = \epsilon_b |b\rangle , \tag{21}$$

where ϵ_a and ϵ_b are either ± 1. Then

$$\langle b|\mathbf{x}|a\rangle = \langle b|\pi^(-1)\pi\mathbf{x}\pi\pi^(-1)|a\rangle = \epsilon_a\epsilon_b \langle b|\pi\mathbf{x}\pi^(-1)|a\rangle = -\epsilon_a\epsilon_b \langle b|\mathbf{x}|a\rangle . \tag{22}$$

For the above equation to be true, either $\langle b|\mathbf{x}|a\rangle = 0$ or $\epsilon_a = \epsilon_b$. So we can see that the parity-odd operator $\mathbf{x}$ connects states of opposite parity. Similarly, we see that a parity-even operator connects states of the same parity.

4 Discrete Symmetries – Time-Reversal

4.1 Properties of the Time-Reversal Operator

The time-reversal operator is an anti-unitary operator. That is,

$$\langle\tilde{\beta}|\tilde{\alpha}\rangle = \langle\beta|\alpha\rangle^* \tag{23}$$

$$\Theta(c_1 |\alpha\rangle + c_2 |\beta\rangle) = c_1^*\Theta |\alpha\rangle + c_2^*\Theta |\beta\rangle , \tag{24}$$

where $|\tilde{\alpha}\rangle = \Theta |\alpha\rangle$ and $|\tilde{\beta}\rangle = \Theta |\beta\rangle$.

4.2 Operators Under Time-Reversal

The position operator is even under time-reversal:

$$\Theta\mathbf{x}\Theta^{-1} = \mathbf{x}. \tag{25}$$

The momentum, angular momentum, and spin operators are odd under time-reversal:

$$\Theta\mathbf{p}\Theta^{-1} = -\mathbf{p}, \tag{26}$$

$$\Theta\mathbf{J}\Theta^{-1} = -\mathbf{J}, \tag{27}$$

$$\Theta\mathbf{S}\Theta^{-1} = -\mathbf{S}. \tag{28}$$

4.3 Wave Function Under Time-Reversal

The spherical harmonics transform as

$$\Theta |l, m\rangle = (-1)^m |l, -m\rangle . \tag{29}$$

4.4 Time-Reversal Theorem

Suppose the Hamiltonian is invariant under time reversal and the energy eigenket $|n\rangle$ is nondegenerate. Then the corresponding energy eigenfunction is real.

To prove this, we note that

$$H\Theta |n\rangle = \Theta H |n\rangle = E_n\Theta |n\rangle . \tag{30}$$

So the state $|n\rangle$ and $\Theta\,|n\rangle$ have the same energy. But since the states are nondegenerate, they must represent the same state. The wavefunctions in position space are then

$$\langle \mathbf{x}|n\rangle = \langle \mathbf{x}|\Theta|n\rangle = \langle \mathbf{x}|n\rangle^*\,. \tag{31}$$

So we see that the wavfunction must be real.