

Special Relativity

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1 Basic Postulates of Special Relativity

There are two basic postulates of special relativity:

- The laws of physics are the same to all inertial observers.
- The speed of light is the same to all inertial observers.

The formulation of physics that incorporates these two postulates is called covariant.

2 The Spacetime Interval

If we define ds to be the distance between two points (or events) in 4D Minkowski space, then we see that

$$(ds)^2 = (cdt)^2 - [(dx)^2 + (dy)^2 + (dz)^2].$$

We can note that the quantity $(ds)^2$ is invariant regardless of reference frame. That is,

$$(ds)^2 = (ds')^2.$$

3 Time Dilation and Length Contraction

3.1 Time Dilation

Let's define the proper time τ to be the time measured by a clock at rest in frame S' , which is moving with velocity $\mathbf{v} = v\hat{x}$ relative to a stationary frame S . In the S' frame,

$$(ds')^2 = (cd\tau)^2,$$

since there is no motion in the rest frame of the clock. In the stationary S frame, there is no motion of the clock in the y or z direction, so $dy = dz = 0$. There is, however, motion in the x direction. If we note that $dx = vdt$, then

$$(ds)^2 = (cdt)^2 - (vdt)^2.$$

Since the quantity $(ds)^2$ is an invariant quantity,

$$\begin{aligned}(ds')^2 &= (ds)^2 \\ (cd\tau)^2 &= (cdt)^2 - (vdt)^2 \\ dt &= \gamma d\tau,\end{aligned}$$

where

$$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}} > 1.$$

So we see that $d\tau < dt$, which means that time is measured slower in a moving reference frame relative to a stationary reference frame.

3.2 Length Contraction

Suppose there is a rod of length l_0 stationary in frame S , and a clock at rest in frame S' , which moves at a speed $\mathbf{v} = v\hat{x}$ relative to a stationary frame S . If the clock starts recording time when the origins of the two frames overlap, then the length of the rod as measured in each reference frame is given by

$$l_0 = vt \quad (S), \quad l' = v\tau \quad (S'),$$

where l' is the length of the rod as measured in frame S' . Remembering that $t = \gamma\tau$, we can see that

$$v = \frac{l_0}{t} = \frac{l'}{\tau} = \frac{\gamma l'}{t} \Rightarrow l' = \frac{l_0}{\gamma} < l_0.$$

So we see that $l' < l_0$, which tells us that an observer in the moving frame will measure the rod to be smaller than it is in the stationary frame.

4 Lorentz Transformations

A Lorentz transformation is one which preserves the invariance of the interval $(ds)^2$. This gives us a way of relating the sets of coordinates in the frames S and S' . If S' moves with a velocity $\mathbf{v} = v\hat{x}$ relative to a stationary frame S , then the coordinates are related by

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix},$$

where $\beta = v/c$.

5 Four Vectors and Energy

- 4-vector $\Rightarrow x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$.
- 4-velocity $\Rightarrow u^\mu = (\gamma c, \gamma v_x, \gamma v_y, \gamma v_z)$.
- 4-momentum $\Rightarrow p = mu$.

We can look at the quantity

$$p \cdot p = m^2 c^2 = \gamma^2 m^2 c^2 - \gamma^2 m^2 v^2.$$

Noting that $E = \gamma mc^2$ and $\mathbf{p} = \gamma m\mathbf{v}$, we can rewrite the above equation as

$$m^2 c^2 = \frac{E^2}{c^2} - |\mathbf{p}|^2 \Rightarrow E^2 = m^2 c^4 + p^2 c^2.$$

The relativistic kinetic energy is defined as

$$T = E - mc^2 = \sqrt{m^2 c^4 + p^2 c^2} - mc^2.$$

6 Velocity Addition

If we want to do a Lorentz transformation from stationary frame S_1 to frame S_3 , where

- S_3 moves with velocity $\mathbf{v}' = v'\hat{x}$ with respect to frame S_2
- S_2 moves with velocity $\mathbf{v} = v\hat{x}$ with respect to frame S_1

then we just do successive Lorentz transforms. The transformation matrix is given by

$$\begin{aligned}
 L_{1 \rightarrow 3} &= \begin{bmatrix} \gamma' & -\beta'\gamma' & 0 & 0 \\ -\beta'\gamma' & \gamma' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} \gamma\gamma'(1 + \beta\beta') & -\gamma\gamma'(\beta + \beta') & 0 & 0 \\ -\gamma\gamma'(\beta + \beta') & \gamma\gamma'(1 + \beta\beta') & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} \gamma'' & -\beta''\gamma'' & 0 & 0 \\ -\beta''\gamma'' & \gamma'' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
 \end{aligned}$$

where

$$\gamma'' = \gamma\gamma'(1 + \beta\beta'), \quad \beta'' = \frac{\beta + \beta'}{1 + \beta\beta'}.$$