Rigid Body Kinematics and Motion

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1 Rigid Body Kinematics

1.1 Euler's Theorem and Chasle's Theorem

- Euler's Theorem: *The general displacement of a rigid body with one point fixed is a rotation about some axis.*
- Chasle's Theorem: *The most general displacement of a rigid body is a translation plus a rotation.*

1.2 The Rate of Change of a Vector

The time rate of change of some vector $\mathbf{G}$ is

$$\left(\frac{d\mathbf{G}}{dt}\right)_{space} = \left(\frac{d\mathbf{G}}{dt}\right)_{body} + (\boldsymbol{\omega} \times \mathbf{G})$$

where $\boldsymbol{\omega}$ is the angular velocity vector which lies along the axis of infinitesimal rotation.

1.3 Coriolis and Centrifugal Forces

If we apply the rate of change equation to the radius vector $\mathbf{r}$ twice, then we can obtain an expression for the acceleration in the space set of axes:

$$\left(\frac{d\mathbf{r}}{dt}\right)_{space} = \mathbf{v}_s = \mathbf{v}_r + (\boldsymbol{\omega} \times \mathbf{r}).$$

Now plugging in $\mathbf{v}_s$ into the rate of change equation yields

$$\mathbf{a}_s = \mathbf{a}_r + 2(\boldsymbol{\omega} \times \mathbf{v}_r) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

We can then use Newton's second law to obtain the equations of motion:

$$\mathbf{F} = m\mathbf{a}_s \Rightarrow \mathbf{F} - 2m(\boldsymbol{\omega} \times \mathbf{v}_r) - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = m\mathbf{a}_r \Rightarrow \mathbf{F}_{eff} = m\mathbf{a}_r.$$

So to an observer in the rotating system, it appears as if the particle, denoted by the vector $\mathbf{r}$, is moving under the influence of an effective force, $\mathbf{F}_{eff}$. We define the Coriolis and Centrifugal forces as:

$$\mathbf{F}_{Coriolis} = -2m(\boldsymbol{\omega} \times \mathbf{v}_r),$$
$$\mathbf{F}_{Centrifugal} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

2 Rigid Body Motion

2.1 Angular Momentum

For rigid body motion with one point stationary, the angular momentum about that point is

$$\mathbf{L} = m_i(\mathbf{r}_i \times \mathbf{v}_i),$$

where the sum over i is implied. If we note that $\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i$, then

$$\begin{aligned}\mathbf{L} &= m_i \left[\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)\right] \\ &= m_i \left[\boldsymbol{\omega} r_i^2 - \mathbf{r}_i(\mathbf{r}_i \cdot \boldsymbol{\omega})\right].\end{aligned}$$

So then

$$\begin{aligned}L_x &= \omega_x m_i(r_i^2 - x_i^2) - \omega_y m_i x_i y_i - \omega_z m_i x_i z_i \\ &= \omega_x I_{xx} + \omega_y I_{xy} + \omega_z I_{xz},\end{aligned}$$

and similarly for L_y and L_z. We can construct a matrix $\mathbb{I}$ containing the I_{jk} components, where

$$I_{jk} = \int_{\mathcal{V}} \rho(\mathbf{r})(r^2\delta_{jk} - r_j r_k) d\mathcal{V},$$

where $\rho(\mathbf{r})$ is the mass density. Then,

$$\mathbf{L} = \mathbb{I}\boldsymbol{\omega}.$$

2.2 Kinetic Energy

The kinetic energy of motion about a fixed point is given by

$$T = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I} \cdot \boldsymbol{\omega}.$$

If $\boldsymbol{\omega} = \omega \hat{n}$, then

$$T_{rot} = \frac{1}{2} I \omega^2,$$

where

$$I = \hat{n} \cdot \mathbb{I} \cdot \hat{n} = m_i \left[r_i^2 - (\mathbf{r}_i \cdot \hat{n})^2\right]$$

is the moment of inertia of the body.

2.3 Parallel Axis Theorem

The moment of inertia about an axis that is parallel to one that goes through the center of mass of a body is given by

$$I_a = I_b + MR^2 \sin^2\theta.$$

Figure 1 depicts this theorem.

2.4 Principle Moments of Inertia

There exists a set of coordinates in which the inertia tensor $\mathbb{I}$ is diagonal with three principle values. We can diagonalize $\mathbb{I}$ through the transformation

$$\mathbb{I}_D = \mathbb{R}\mathbb{I}\mathbb{R}^T,$$

where $\mathbb{R}$ is the rotation matrix whose columns define the principle axes. The principle moments of inertia are found by solving for the eigenvalues of $\mathbb{I}$. The corresponding principle axes are found by solving for the eigenvectors of $\mathbb{I}$.

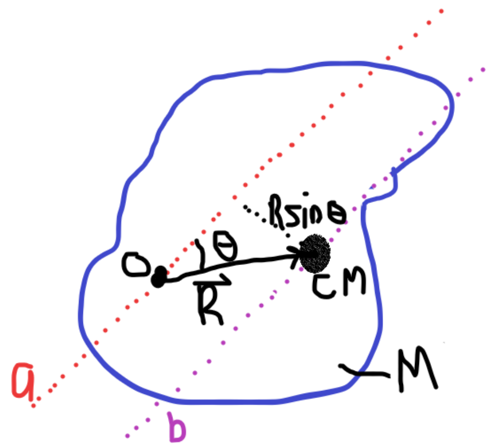

Figure 1: Parallel axis theorem.

2.5 The Euler Equations of Motion

Consider an inertial reference frame whose origin is a fixed point of a rigid boy which could also be its center of mass. The torque on the rigid body is given by

$$\mathbf{N} = \left(\frac{d\mathbf{L}}{dt}\right)_{space}.$$

But we also know that the rate of change of a vector is given by

$$\left(\frac{d\mathbf{L}}{dt}\right)_{space} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + (\boldsymbol{\omega} \times \mathbf{L}).$$

So then

$$\mathbf{N} = \left(\frac{d\mathbf{L}}{dt}\right) + (\boldsymbol{\omega} \times \mathbf{L}) \Rightarrow N_i = \frac{dL_i}{dt} + \epsilon_{ijk}\omega_j L_k,$$

where we use the Levi-Cevita symbol to rewrite the vector equation in component form. If the body set of axes are taken to be the principle axes, then $L_i = I\omega_i$, and

$$I\frac{d\omega_i}{dt} + \epsilon_{ijk}\omega_j\omega_k I_k = N_i.$$

Note that there is no sum over i in the first term on the RHS, but there is a sum over j and k in the second term on the RHS. If we look at each component ($i = 1, 2, 3$), then we obtain Euler's equations of motion:

$$\begin{aligned}
I_1\dot{\omega}_1 + \omega_2\omega_3(I_2 - I_3) &= N_1 \\
I_2\dot{\omega}_2 + \omega_3\omega_1(I_3 - I_1) &= N_2 \\
I_3\dot{\omega}_3 + \omega_1\omega_2(I_1 - I_2) &= N_3.
\end{aligned}$$

In the case that one of the angular velocities is much greater than the other two ($\omega_1 >> \omega_2, \omega_3$ for example) then the equation that has the two smaller angular velocities multiplied together (in the example, this would be the first equation) becomes negligible compared to the other two and can be ignored.

2.6 Torque-Free Motion

When no net torque is applied to the body, $\mathbf{N} = 0$, and the Euler equations become:

$$
\begin{aligned}
I_1\dot{\omega_1} + \omega_2\omega_3(I_2 - I_3) &= 0 \\
I_2\dot{\omega_2} + \omega_3\omega_1(I_3 - I_1) &= 0 \\
I_3\dot{\omega_3} + \omega_1\omega_2(I_1 - I_2) &= 0.
\end{aligned}
$$

There are two constants of motion when there is no net torque:

1. The total angular momentum is constant.

$$
\begin{aligned}
L^2 &= L_1^2 + L_2^2 + L_3^2 \\
&= I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2.
\end{aligned}
$$

2. The rotational kinetic energy is constant.

$$
\begin{aligned}
T &= \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2 \\
&= \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3}.
\end{aligned}
$$

2.7 Lagrangian Formalism for Rigid Bodies

Recall Chasle's theorem, which states that the most general motion of a rigid body is a translation plus a rotation. So when applying the Lagrangian formalism to problems involving rigid bodies, split up the kinetic energy into two parts: the kinetic energy of the center of mass of the rigid body (translational) and the kinetic energy about the center of mass of the rigid body (rotational).

$$
T = T_{translational} + T_{rotational} \Rightarrow L = T - V = T_{translational} + T_{rotational} - V.
$$

Then just apply Lagrange's equations as usual to find the equations of motion.