Solving Problems on Orbits

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1 Kepler's Laws

1. The motion of a planet is an ellipse with the Sun at one of the two foci.
2. The radius vector sweeps out equal areas in equal times. If we look at the Lagrangian of an object in a central potential,

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r),$$

then we see that L does not explicitly depend on the coordinate θ. Therefore the corresponding conjugate momentum

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = l,$$

is conserved, where l is the angular momentum. So then

$$\dot{p_\theta} = \frac{d}{dt}(mr^2\dot{\theta}) = 0 = \frac{d}{dt}(\frac{1}{2}r^2\dot{\theta}),$$

where we divided by $2m$ in the last step. Now we define the areal velocity (the area swept out by the radius vector per unit time)

$$dA = \frac{1}{2}r(rd\theta).$$

Then

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}.$$

So we can see that the conservation of angular momentum is equivalent to saying that the areal velocity is constant.

3. The square of the orbital period is proportional to the cube of their major axis:

$$\tau = \frac{2\mu}{l}\pi a^(3/2)\sqrt{\frac{l^2}{mk}} = 2\pi a^(3/2)\sqrt{\frac{\mu}{k}},$$

where μ is the reduced mass, defined as

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{1}{m_1} + \frac{1}{m_2}.$$

2 Boundedness of Orbits

The total energy in spherical coordinates for a particle in a plane subject to a conservative central potential V(r) is

$$H = T + V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = \frac{1}{2}m\dot{r}^2 + V_{eff},$$

where the effective potential is defined as

$$V_{eff} = \frac{1}{2}mr^2\dot{\theta}^2 + V(r) = \frac{l^2}{2mr^2} + V(r).$$

We can use the effective potential to determine the boudedness of the orbits:

1. To find the extrema of V_{eff}, look at $\frac{\partial V_{eff}}{\partial r} = 0$ and solve for the values of r that satisfy the equation. We'll denote those points as r_0.
2. Look at
$$\frac{\partial^2 V_{eff}}{\partial r^2}|_{r=r_0}$$
for each extrema r_0. There are two cases to look for:
   - $\frac{\partial^2 V_{eff}}{\partial r^2}|_{r=r_0} > 0$. If this is true then the orbit is bounded.
   - $\frac{\partial^2 V_{eff}}{\partial r^2}|_{r=r_0} < 0$. If this is true then the orbit is unbounded.

Consider the following plot of effective potential from Goldstein pg.78: For energies equal to

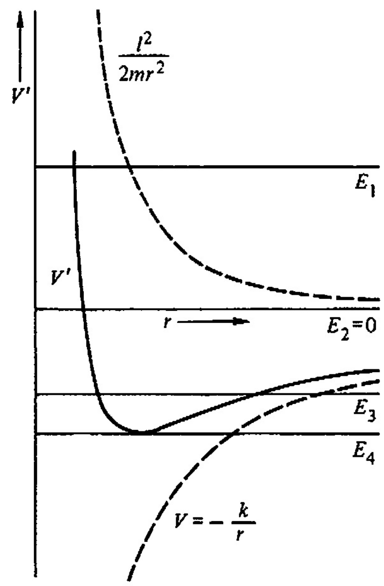

FIGURE 3.3 The equivalent one-dimensional potential for attractive inverse-square law of force.

Figure 1:

and above $E_2 = 0$, we have unbounded orbits. At energy E_3 we have a bounded orbit with two turning points, as indicated by the intersection of V' and the E_3 line. At energy E_4, we get $\dot{r} = 0$, corresponding to a circular orbit. This occurs when $r = r_0$, the minimum of the V' graph.

2.1 Condition for Closed Orbits: Bertrand's Theorem

The only central forces that result in closed orbits for all bound particles are the inverse-square law ($1/r^2$) and Hooke's Law (r^2).