ADVANCED TOPICS SECTION

The Advanced Topics Section is meant for articles that deal with physics more advanced than is typical of regular articles in AJP. Though advanced, these articles have been judged to give clear presentations of material useful to a segment of AJP readers.

Dipoles in quantum field theory

William Miyahira and David C. Latimer
Department of Physics, University of Puget Sound, Tacoma, Washington 98416-1031

(Received 4 August 2018; accepted 15 December 2018)

In quantum field theory, a fundamental spin-$\frac{1}{2}$ particle can have intrinsic magnetic and electric dipole moments. Herein, we examine the way in which a charged fermion or photon is scattered by a massive neutral fermion via its dipole moment. In the low-energy limit, the field theoretic scattering amplitude can be related to a semi-classical electromagnetic interaction Hamiltonian. We find, in this limit, that the dipole field of the fundamental particle is that of a classical pure dipole, including the correct contact field. Additionally, we examine the optical properties of a medium consisting of magnetic dipoles. By computing the Compton scattering amplitude for a magnetic dipole, we find that a medium of polarized dipoles is circularly birefringent; that is, the index of refraction depends upon the polarization of the light. Our presentation is geared toward advanced undergraduate or beginning graduate students.
https://doi.org/10.1119/1.5086009

I. INTRODUCTION

In classical electromagnetism, electric and magnetic dipoles are of primary importance because electric dipoles are the simplest non-trivial neutral charge distribution, and magnetic dipoles represent the simplest source of magnetic fields. In fact, the leading order behavior of the electric or magnetic field of a bulk (neutral) material is determined by summing up the fields from the microscopic dipole moments of infinitesimal volume elements. Curiously, the field due to a microscopic dipole has the same structure for both magnetic and electric dipoles far from the source. However, on the microscopic level, electric and magnetic dipole fields do differ. We will elaborate on this point below.

Let us first consider a physical electric dipole located at the origin, consisting of two opposite charges, $\pm q$, separated by some distance d. Its dipole moment is $\mathbf{p}_{\text{phys}} = q\mathbf{d}$, pointing to the positive charge. From this physical dipole, we can posit the abstract point-like pure dipole by shrinking d to zero while keeping the dipole moment $\mathbf{p}$ fixed. The pure dipole's electric field is[1]

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{r^3} - \frac{1}{3\epsilon_0}\mathbf{p}\,\delta^{(3)}(\mathbf{r}). \qquad (1)$$

The first term in this expression is the familiar dipole field for $r > 0$, and the second term is called a contact term because it is only nonzero at the field source, located at the origin. Qualitatively, the contact term is reasonable for a physical dipole (finite d) in the sense that the electric field at the origin (*between* the dipole's two charges) points in the direction opposite to the dipole moment. Quantitatively, the term is necessary to satisfy the theorem that states that the average electric field over a sphere of radius R is proportional to the dipole moment (with respect to the sphere's center) of the charges contained in the sphere; namely, $\mathbf{E}_{\text{av}} = -(1/4\pi\epsilon_0)(\mathbf{p}/R^3)$. Reference 2 contains an accessible discussion of this point.

Similarly, the field of a pure magnetic dipole is the limiting case of the field of a physical current loop with dipole moment $\boldsymbol{\mu}_{\text{phys}} = I\mathbf{a}$ held constant while the area of the current loop a is taken to zero. The pure dipole field is[3]

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{3(\boldsymbol{\mu}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \boldsymbol{\mu}}{r^3} + \frac{2\mu_0}{3}\boldsymbol{\mu}\,\delta^{(3)}(\mathbf{r}). \qquad (2)$$

For $r > 0$, the structure of the field is the same as that of an electric dipole, but the contact term differs. In particular, the sign on the contact term is opposite to that of the electric dipoles. This is because the magnetic field at the center of a physical current loop points in the same direction as the area vector $\mathbf{a}$. Quantitatively, this particular contact term is required for reasons similar to those for the electric dipole.[2]

The difference between the two dipole contact terms can be traced to the difference between the field sources.[4] Electric fields are sourced by point charges while magnetic fields are sourced by currents, i.e., *moving* charges. In fact, if magnetic monopoles were to exist, then a magnetic dipole created from two opposite magnetic charges would have a contact field akin to that of the electric field in Eq. (1). Furthermore, an electric dipole created by a monopole current would have a contact field akin to that of the magnetic field in Eq. (2). If we exclude magnetic monopoles from classical electromagnetism, the dipole contact fields are unambiguously known, but one might wonder if this fact holds in the realm of quantum theory. Electrons, for instance, possess an intrinsic magnetic dipole moment, and it is natural to question whether the contact magnetic field is similar to that produced by a current, Eq. (2), or that produced by two oppositely charged magnetic monopoles. The resolution of the question lies in experiment, and as we see below, intrinsic magnetic dipoles have contact fields like those produced by classical currents.

Though a pure dipole is an idealization in classical physics, it has relevance in the realm of particle physics where particles (composite or fundamental) can approach the limit of point-like electromagnetic objects. At the quantum level, the contact magnetic field of a point particle can have measurable consequences. Perhaps the most well-known example of this can be found in the hyperfine structure of atomic hydrogen.[2] The proton has a magnetic dipole moment given by $\boldsymbol{\mu}_{\rm p} = \gamma_{\rm p}\mathbf{S}$ with a gyromagnetic ratio of $\gamma_{\rm p} = g_{\rm p}\frac{e}{2m_{\rm p}}$ and a g-factor of $g_{\rm p} = 5.58$.[5,6] For an electron in the ground state of hydrogen, the electron's wave function overlaps sufficiently with the proton such that the interaction between the electron's magnetic dipole moment and the proton's contact magnetic field produces an energy difference between the two spin states of the electron. This energy difference is independent of the far-field behavior of the proton's magnetic field and thus serves as an exclusive probe of its contact field. The system's energy is slightly greater when the electron and proton spins are aligned than when they are anti-aligned, and the energy difference corresponds to a photon wavelength of 21 cm, consistent with a proton contact magnetic field produced by a classical current, Eq. (2), rather than one produced by oppositely charged magnetic monopoles, Eq. (1). Observations of this 21 cm line have proven to be a key signal in mapping clouds of neutral hydrogen in our galaxy.[7]

The hyperfine structure of hydrogen was first computed by Fermi in 1930.[8] Subsequently, the topic has been discussed in many quantum mechanics texts[9–12] and the pedagogical literature.[2,13–16] In these works, the electronic state is treated quantum mechanically (as a solution of either the Dirac or Schrödinger equation), but the proton's magnetic field is treated classically in one of two ways. In some treatments, the electron is coupled to the proton's magnetic field, Eq. (2), by an interaction term in the Hamiltonian given by $-\mathbf{m}\cdot\mathbf{B}$, where $\mathbf{m}$ is the electron's magnetic dipole moment. Other treatments incorporate electron and magnetic field interactions through the vector potential in the minimal coupling procedure, $-i\hbar\boldsymbol{\nabla}\mapsto -i\hbar\boldsymbol{\nabla} - e\mathbf{A}$ with $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. Either procedure results in the correct hyperfine correction because the contact magnetic field truly is that of a classical pure dipole.

These semiclassical treatments are effective in making the physics more accessible to students in an introductory quantum mechanics course; however, more advanced students might find the arguments lacking, particularly in light of the differences between the contact terms for the classical electric and magnetic dipole fields. If we could effectively model the proton as a spinning spherical shell of charge, then it is reasonable to suppose that the contact magnetic field of a proton is that produced by a classical dipole, Eq. (2). In fact, using the Dirac equation, one *can* interpret the fermion's spin and magnetic dipole moment as a circular flow of energy and charge in the particle's matter wave,[17] but these arguments are likely not well known to students. Instead, students are more familiar with a common exercise in electrodynamics[18] that makes them skeptical of the "spinning sphere" model of a point-like quantum particle. In this exercise, students find that the classical spinning-sphere model of the electron requires surface speeds that exceed the speed of light.

In this paper, we study point-like electric and magnetic dipole moments in the framework of low-energy quantum field theory. In particular, within the context of a scattering experiment, we examine how a charged fermion and a neutral fermion interact by virtue of the neutral fermion's (electric or magnetic) dipole moment. Our calculations treat both (charged and neutral) fermions on the same footing as quantum mechanical objects, and we recover the classical behavior of the point-like dipole moment in the low-energy limit. This is to be expected because we are computing tree-level, non-relativistic processes. An alternate approach would be to impose the non-relativistic limits at the Lagrangian level of the field theory;[19] in this case, it is clear that the quantum field theory resembles its classical counterpart.

In what follows, we will first compute the amplitude for a charged fermion projectile scattered by a very massive neutral fermion that interacts only via its magnetic dipole moment. Because we work with states of definite momentum, we must take the inverse Fourier transform (IFT) of the scattering amplitude in the low-energy limit in order to recover an expression for the interaction Hamiltonian in coordinate space. We discuss how to carefully evaluate the IFT using results from distribution theory in order to correctly capture the contact magnetic field. In the end, we find that the magnetic dipole interacts with a charged fermion via a direct dipole-dipole mode and through a Lorentz force on the moving charge.

We then discuss the interaction between a charged fermion and a particle with an electric dipole moment. In the low energy limit, the scattering amplitude has the expected term due to the electrostatic interaction between the light fermion's electric charge and the dipole potential of the heavy particle. But it also has another term, due to the interaction between the electric dipole moment that the light fermion has in the rest frame of the heavy particle: in this frame, the fact that the light fermion has an intrinsic magnetic dipole moment means that it also has an electric dipole moment by virtue of its motion, and this electric dipole moment also couples to the dipole field of the static, heavy particle.[20]

Finally, we discuss one example dealing with the interaction of photons and magnetic dipoles. From a calculation of the Compton scattering amplitude for the magnetic dipole, we can understand some optical properties for a medium containing such dipoles. In particular, we find that if the dipoles are oriented along the direction of light propagation, then the medium is circularly birefringent.

II. CHARGED FERMION AND MAGNETIC DIPOLE INTERACTION

Let us first consider the interaction between a charged fermion, ψ_f, with charge e and mass m, and a neutral fermion, ψ_F, with magnetic dipole moment μ and mass M. The charged fermion has initial four-momentum k, while the neutral fermion's initial four-momentum is p; the momenta of outgoing states carry primes. The leading-order contribution to the scattering amplitude can be computed from the tree-level Feynman diagram in Fig. 1. Our conventions will follow those used in Ref. 21. In particular, when working with field theory amplitudes, we will use natural units, $\hbar = 1$ and $c = 1$. As a result, the expressions for electric and magnetic fields that follow will have $\epsilon_0 = 1$ and $\mu_0 = 1$.

To evaluate the diagram, we implement the usual Feynman rules, along with the magnetic-dipole vertex factor $-i\Gamma^\mu$. This factor is given by $\Gamma^\mu := -if_2\gamma\sigma^{\mu\nu}q_\nu$ where $\sigma^{\mu\nu} := (i/2)[\gamma^\mu, \gamma^\nu]$, f_2 is the magnetic form factor, γ the gyromagnetic ratio, and $q := p - p'$ is the four-momentum transfer.[21] A pedagogical derivation of this operator can be found in Ref. 22. By coupling the electromagnetic fermion current to a classical electromagnetic four-potential A^μ in the non-relativistic limit, one finds that the magnetic dipole moment of ψ_F is proportional to its spin, $\mathbf{S}$, according to $\boldsymbol{\mu} = 2f_2\gamma\mathbf{S}$.[23] The amplitude is given by

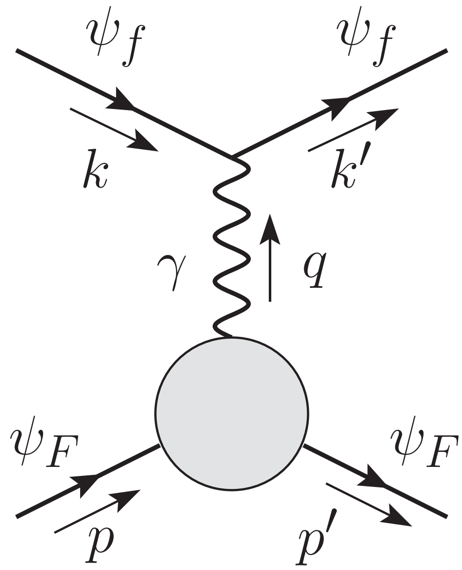


Fig. 1. Feynman diagram used to compute the scattering amplitude between fermions ψ_f and ψ_F via the exchange of a photon γ. The neutral fermion, ψ_F, couples to the photon via its magnetic dipole, indicated by the shaded circle. Fermion flow is indicated by the arrows on the fermion lines, whereas external arrows label the four momenta of the particles.

$$\mathcal{M} = -ief_2\gamma\, \bar{u}^{s'}(k')\gamma_\mu u^s(k)\, \frac{q_\nu}{q^2}\, \bar{u}^{S'}(p')\sigma^{\mu\nu}u^S(p). \tag{3}$$

The Dirac spinor associated with incoming fermion, ψ_F, is

$$u^S(p) = \frac{1}{\sqrt{2(p^0+M)}}\begin{pmatrix} \left[(p^0+M)1 - \mathbf{p}\cdot\boldsymbol{\sigma}\right]\xi^S \\ \left[(p^0+M)1 + \mathbf{p}\cdot\boldsymbol{\sigma}\right]\xi^S \end{pmatrix}, \tag{4}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices and ξ^S is a two-component Pauli spinor that encodes the spin state of the fermion. The other states are described by similar Dirac spinors; for the charged fermion ψ_f, we represent its spin state with the two-component spinor η^s.

In order to gain some understanding of the magnetic-dipole moment interaction in quantum field theory, it is easiest to make some simplifying assumptions. First, we work in the rest frame of the neutral fermion; that is, we set $p = (M, 0)$. Second, we assume that the neutral fermion remains non-relativistic after interacting with the charged fermion. In particular, we require $p' := (M - q^0, -\mathbf{q}) \approx (M, -\mathbf{q}) + \mathcal{O}(|\mathbf{q}|^2/M)$, which can be achieved if M is sufficiently large. In essence, we focus on a scattering scenario in which a charged particle interacts elastically with a static magnetic dipole that undergoes minimal recoil in the collision. Finally, we assume that the spin states of the fermions are unchanged in the collision;[24] that is, we set $S' = S$ and $s' = s$ and define the spin vectors to be $\mathbf{S} := (1/2)\xi^{S\dagger}\boldsymbol{\sigma}\xi^S$ and $\mathbf{s} := (1/2)\eta^{s\dagger}\boldsymbol{\sigma}\eta^s$. Implementing these additional constraints, we find that the scattering amplitude simplifies to

$$\mathcal{M} \approx 4eM\left\{-i\mathbf{k}\cdot\frac{(\boldsymbol{\mu}\times\mathbf{q})}{|\mathbf{q}|^2} + \left[\mathbf{s}\cdot\boldsymbol{\mu} - \frac{1}{|\mathbf{q}|^2}(\mathbf{s}\cdot\mathbf{q})(\boldsymbol{\mu}\cdot\mathbf{q})\right]\right\}, \tag{5}$$

dropping terms that are $\mathcal{O}(|\mathbf{q}|^1)$ and $\mathcal{O}(M^0)$.

In the non-relativistic limit, we expect this scattering amplitude to be proportional to the matrix element for the quantum-mechanical interaction Hamiltonian mediating the transition from initial to final states, $H_{\mathrm{fi}} \sim -\mathcal{M}$.[25] To construct the interaction Hamiltonian, we note that a charged fermion can interact with an external magnetic field through two channels. Moving charge results in a current, $\mathbf{J} = (e/m)\mathbf{k}$, which can be scattered by the field by virtue of the Lorentz force. Additionally, the charged fermion's intrinsic magnetic moment, $\mathbf{m} = g(e/2m)\mathbf{s}$, can interact with the field regardless of the fermions' relative motion. In terms of energetics, these two interactions would enter the Hamiltonian as $H_{\mathrm{fi}} = -\mathbf{J}\cdot\mathbf{A} - \mathbf{m}\cdot\mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are the external magnetic vector potential and field, respectively, due to the static neutral fermion. This is exactly the structure of the scattering amplitude in Eq. (5) if we identify the vector potential and field as

$$\widehat{\mathbf{A}}(\mathbf{q}) = -i\frac{(\boldsymbol{\mu}\times\mathbf{q})}{|\mathbf{q}|^2}, \quad \widehat{\mathbf{B}}(\mathbf{q}) = \boldsymbol{\mu} - \frac{(\boldsymbol{\mu}\cdot\mathbf{q})\mathbf{q}}{|\mathbf{q}|^2}. \tag{6}$$

The "hat" over these fields indicates that they are expressed in terms of the momentum transfer to the charged fermion.

As per the Born approximation, we can relate the momentum representation of each field to its spatial representation through an inverse Fourier transform. We begin with the vector potential; its spatial representation is

$$\mathbf{A}(\mathbf{r}) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\widehat{\mathbf{A}}(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}}\,\mathrm{d}^3q. \tag{7}$$

It is perhaps easiest to evaluate the integral using spherical coordinates, choosing a coordinate system in which the vector $\mathbf{r}$ is aligned with the z-axis. With this choice, only one component of the integral survives. In the end, we find

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi}\frac{\boldsymbol{\mu}\times\widehat{\mathbf{r}}}{r^2}. \tag{8}$$

This has the structure of the vector potential of a pure magnetic dipole. To find the field, we take the curl of the potential, being mindful of the Dirac delta function that arises from differentiation of $\widehat{\mathbf{r}}/r^2$; namely, we note from Ref. 26 the derivative[27]

$$\partial_j\partial_k\frac{1}{|\mathbf{r}|} = \frac{3x_jx_k - |\mathbf{r}|^2\delta_{jk}}{|\mathbf{r}|^5} - \frac{4\pi}{3}\delta_{jk}\delta^{(3)}(\mathbf{r}). \tag{9}$$

From the curl, we find

$$\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi}\frac{3(\boldsymbol{\mu}\cdot\widehat{\mathbf{r}})\widehat{\mathbf{r}} - \boldsymbol{\mu}}{|\mathbf{r}|^3} + \frac{2}{3}\boldsymbol{\mu}\delta^{(3)}(\mathbf{r}). \tag{10}$$

This turns out to be the correct expression for the dipole magnetic field of the fermion, ψ_F, including the correct contact field at the origin. But, we were a bit too cavalier in our computation of the inverse Fourier transform of the vector potential, and the same approach would fail if we were to attempt to compute the transform of the magnetic field in Eq. (6) because we would miss a delta function.[28] Strictly speaking, Fourier transforms are defined for functions which are absolutely integrable. For a component of $\widehat{\mathbf{A}}(\mathbf{q})$, we see that this is not the case because $\int_{\mathbb{R}^3}(|q_j|/|\mathbf{q}|^2)\,\mathrm{d}^3q$ clearly diverges. But all is not lost because we can formally compute the Fourier transform of the vector potential if we view it as a tempered distribution.

A readable account of tempered distributions can be found in Ref. 29. For our purposes, we only need two facts. The first is the following Fourier transform of the distribution $|\mathbf{r}|^{-1}$

$$\widehat{|\mathbf{r}|^{-1}} = 4\pi|\mathbf{q}|^{-2}; \tag{11}$$

or, alternatively, we have the inverse Fourier transform $\widetilde{|\mathbf{q}|^{-2}} = (1/4\pi)|\mathbf{r}|^{-1}$. The second fact is that the nice features that apply to the Fourier transform of functions map over to tempered distributions. In particular, differentiation becomes pointwise multiplication under the Fourier transform, and vice versa. That is, we have $(\widehat{\partial f/\partial x_j})(\mathbf{q}) = iq_j\widehat{f}(\mathbf{q})$ and $\widehat{x_j f}(\mathbf{q}) = i(\partial/\partial q_j)\widehat{f}(\mathbf{q})$.

To compute the position representation of the vector potential, Eq. (6), we need the IFT of $\mathbf{q}/|\mathbf{q}|^2$. Considering a single component of this field, we find

$$\widetilde{\left(\frac{q_j}{|\mathbf{q}|^2}\right)} = -i\frac{\partial}{\partial x_j}\widetilde{|\mathbf{q}|^{-2}} = -\frac{i}{4\pi}\frac{\partial}{\partial x_j}|\mathbf{r}|^{-1} = \frac{i}{4\pi}\frac{x_j}{|\mathbf{r}|^3}, \tag{12}$$

using Eq. (11). This yields the same vector potential as before, Eq. (8). Furthermore, we can compute the position representation of the magnetic field, Eq. (6), mediating the dipole-dipole interaction. For the second term in the magnetic field, we evaluate the IFT of $q_j q_k/|\mathbf{q}|^2$ as above

$$\widetilde{\left(\frac{q_j q_k}{|\mathbf{q}|^2}\right)} = -\frac{\partial^2}{\partial x_k \partial x_j}\widetilde{|\mathbf{q}|^{-2}} = -\frac{1}{4\pi}\partial_k\partial_j|\mathbf{r}|^{-1}. \tag{13}$$

We use Eq. (9) to evaluate these last two derivatives. The first term in the magnetic field in Eq. (6) is independent of $\mathbf{q}$, and thus, its IFT is a delta function. Putting these terms together, we find that the neutral fermion's field mediating the dipole-dipole interaction is consistent with the previous result from the vector potential

$$\mathbf{B} = \frac{1}{4\pi}\frac{3(\boldsymbol{\mu}\cdot\widehat{\mathbf{r}})\widehat{\mathbf{r}} - \boldsymbol{\mu}}{|\mathbf{r}|^3} + \frac{2}{3}\boldsymbol{\mu}\delta^{(3)}(\mathbf{r}). \tag{14}$$

III. CHARGED FERMION AND ELECTRIC DIPOLE INTERACTION

We now examine how electric dipoles interact with charged fermions in the low-energy limit. To compute the leading order contribution to the scattering amplitude, we use the same Feynman diagram as in Fig. 1 but consider ψ_F to be a neutral fermion with an electric-dipole moment. The appropriate vertex factor is then $-i\Gamma^\mu := -if_e\sigma^{\mu\nu}\gamma^5 q_\nu$ with $q = p - p'$,[22,30,31] and the particle's electric dipole moment is given by $\mathbf{p} = 2f_e\mathbf{S}$. We refer readers to Ref. 22 for a pedagogical discussion of the operator encoding a fermion's electric dipole moment.

The presence of the γ^5 matrix signals that the electric-dipole moment is odd under a parity transformation. Furthermore, the term is also odd under a time reversal transformation. These two features can be understood by imagining a classical physical dipole with moment $\mathbf{p}_{\text{phys}} = q\mathbf{d}$, which is collinear with the particle's spin. Under a parity transformation, the vector $\mathbf{p}_{\text{phys}}$ changes sign, but the spin pseudovector is unchanged. Under time reversal, the quantities exhibit the opposite behavior; that is, the vector $\mathbf{p}_{\text{phys}}$ is unchanged, but the spin pseudovector changes sign. Taken together, these two transformation properties require asymmetry under a joint charge and parity transformation.[32] Parity alone is maximally violated in the weak interactions,[33] but charge and parity violation occur at a small level in the standard model of particle physics.[34] As such, the electric dipole moment of fundamental particles in the standard model is expected to be extremely small. However, some molecules (such as water) can have permanent electric dipole moments, so our calculations could be relevant for these objects, as a first approximation.

The amplitude for the scattering process in Fig. 1 is given by

$$\mathcal{M} = ef_e\,\bar{u}^{s'}(k')\gamma_\mu u^s(k)\,\frac{q_\nu}{q^2}\,\bar{u}^{S'}(p')\sigma^{\mu\nu}\gamma^5 u^S(p). \tag{15}$$

We again make several simplifying assumptions. As before, we work in the rest frame of the neutral particle, ψ_F, and assume it to be non-relativistic; that is, $p = (M, \mathbf{0})$ and $p' \approx (M, -\mathbf{q})$. We will also assume that the charged fermion, ψ_f, is non-relativistic so that its energy is given by its rest mass $k^0 \approx k'^0 \approx m$. Implementing these limits, we only consider terms that are at most $\mathcal{O}(|\mathbf{q}|^0)$.

The leading order term is $\mathcal{O}(|\mathbf{q}|^{-1})$

$$\mathcal{M}_{\text{lot}} = i4emM\frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{q}|^2}. \tag{16}$$

The interaction Hamiltonian between a charged particle and an electric field is $H_{\text{fi}} = eV$, and so, we identify the neutral fermion's electric potential with

$$\widehat{V}(\mathbf{q}) = -i\frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{q}|^2}. \tag{17}$$

Taking the inverse Fourier transform, Eq. (12), the position representation of the potential is that of a classical electric dipole

$$V(\mathbf{r}) = \frac{1}{4\pi}\frac{\mathbf{p}\cdot\mathbf{r}}{|\mathbf{r}|^3}. \tag{18}$$

Beyond the leading order term, there are corrections to the interaction, dependent upon the incoming momentum of ψ_f (but independent of its spin), that come in at $\mathcal{O}(|\mathbf{q}|^0)$. We leave the exploration of these terms as an exercise for the reader.

There is another mode by which the fermion, ψ_f, can interact with the electric field of the target particle. In the amplitude, this term is $\mathcal{O}(|\mathbf{q}|^0)$ and depends on ψ_f's spin, $\mathbf{s} := \frac{1}{2}\eta'^\dagger\boldsymbol{\sigma}\eta$; this spin-spin contribution is given by

$$\mathcal{M}_{ss} = -2M\frac{(\mathbf{k}\times\mathbf{m})\cdot\mathbf{q}\,(\mathbf{q}\cdot\mathbf{p})}{|\mathbf{q}|^2}, \tag{19}$$

where the magnetic dipole moment of the charged fermion, ψ_f, is $\mathbf{m} = (e/m)\mathbf{s}$. This term has the structure of an electric dipole moment interacting with the neutral particle's (dipole) electric field, $H_{\text{fi}} = -\mathbf{p}_f\cdot\mathbf{E}$. Evidently, the moving charged fermion, ψ_f, acquires an electric dipole moment in the rest frame of the target particle (by virtue of its magnetic dipole moment). Indeed, in classical electrodynamics, a magnetic dipole moving with velocity $\mathbf{v}$ does acquire an electric dipole moment given by, in the non-relativistic limit, $\mathbf{p}_f = \mathbf{v} \times \mathbf{m} = 1/m(\mathbf{k} \times \mathbf{m})$ (with $c = 1$).[20] Thus, the neutral particle's electric field (in momentum space) appears as

$$\widehat{\mathbf{E}}(\mathbf{q}) = -\frac{\mathbf{q}\,(\mathbf{q}\cdot\mathbf{p})}{|\mathbf{q}|^2}, \tag{20}$$

consistent with the scalar potential in Eq. (17). We recall in electrostatics that the field is given by the gradient of the scalar potential, $\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$. Taking the Fourier transform of one component of this equation yields $\widehat{E_j}(\mathbf{q}) = -\widehat{\frac{\partial}{\partial x_j} V}(\mathbf{q}) = -iq_j\widehat{V}(\mathbf{q})$; thus, we have $\widehat{\mathbf{E}}(\mathbf{q}) = -i\mathbf{q}\widehat{V}(\mathbf{q})$. As such, to find the particle's electric field, we can take the gradient of Eq. (18). Alternatively, we can evaluate the inverse Fourier transform of Eq. (20) as before (using Eq. (13))

$$\mathbf{E}(\mathbf{r}) = \frac{3(\mathbf{p}\cdot\widehat{\mathbf{r}})\widehat{\mathbf{r}} - \mathbf{p}}{4\pi|\mathbf{r}|^3} - \frac{1}{3}\mathbf{p}\delta^{(3)}(\mathbf{r}). \tag{21}$$

We note that the contact term for the dipole field is consistent with the point-like limit of a classical electric dipole, Eq. (1).

IV. PHOTON AND MAGNETIC DIPOLE INTERACTION

Aside from the interaction between charged particles and neutral particles with dipole moments, one can also investigate the interaction between light (real photons) and dipoles. In Fig. 2, we include the leading order Feynman diagrams that contribute to the Compton amplitude. Here, a photon with four-momentum k scatters off a particle (with dipole moment) to a photon with final four-momentum k'. Because an electron has a magnetic dipole moment, the photon's interaction with the dipole moment results in higher order corrections to the usual Compton amplitude.[35–38] Experimentally, magnetic Compton scattering can be used to probe the magnetization of a material.[39] Additionally, at low energies, the Compton amplitude informs us of the optical features of a collection of such fermions. In this section, we will confine ourselves to an exploration of the index of refraction of a dilute gas of neutral magnetic dipoles.

In classical electromagnetism, the index of refraction of a dilute gas is due to the interference between scattered spherical waves and the incident plane wave. The deviation of the index from unity is proportional to the number density of scatterers and the forward scattering amplitude, assuming a random array of scatterers.[40–42] Fermi generalized this result to the realm of particle physics so that the notion of an index of refraction can be mapped over to neutrons[43] and even neutrinos.[44] For photons of energy (or angular frequency) ω, the index of refraction for light is

$$n = 1 + \frac{N}{4M\omega^2}\mathcal{M}(k, p \to k, p), \tag{22}$$

where N is the number density of scatterers and $\mathcal{M}(k, p \to k, p)$ is the forward Compton amplitude.[42] As with the classical electromagnetic derivation, only the forward Compton amplitude contributes to the index because scattering in other directions interferes destructively.

Rather generally,[45] the forward Compton amplitude has the following structure:

$$\mathcal{M} = g(\omega)\boldsymbol{\epsilon}'^{*}\cdot\boldsymbol{\epsilon} + ih(\omega)\mathbf{S}\cdot(\boldsymbol{\epsilon}'^{*}\times\boldsymbol{\epsilon}), \tag{23}$$

where g and h are real analytic functions (up to some threshold value for ω) and the $\boldsymbol{\epsilon}$ vectors represent the photon polarization for the initial (unprimed) and final (primed) states.[35–37] The photons are transverse so that $\widehat{\mathbf{k}}\cdot\boldsymbol{\epsilon} = 0 = \widehat{\mathbf{k}}'\cdot\boldsymbol{\epsilon}'$. Suppose we consider photons propagating in the $\widehat{\mathbf{k}} = \widehat{\mathbf{z}}$ direction; then, the photon polarization vectors could be in the $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ directions. In particular, we consider states of definite circular polarization: $\boldsymbol{\epsilon}_{\pm} := (1/\sqrt{2})(\widehat{\mathbf{x}}\pm i\widehat{\mathbf{y}})$. Inserting these polarization vectors into the general expression for the amplitude, Eq. (23), and then into the expression for the refractive index, we discover

$$n_{\pm} = 1 + \frac{N}{4M\omega^2}\left[g(\omega)\mp h(\omega)\langle S_z\rangle\right], \tag{24}$$

where the subscript corresponds to initial and final polarization states $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}' = \boldsymbol{\epsilon}_{\pm}$ and $\langle S_z\rangle$ is the average z-component of spin in the medium. We observe that the term proportional to $h(\omega)$ in the amplitude, Eq. (23), results in a birefringent medium; that is, the medium's index of refraction depends upon the polarization of the light.

Returning to our specific system at hand, we use the Feynman diagrams in Fig. 2 to compute the forward Compton amplitude assuming that ψ_F is a neutral fermion which interacts via a magnetic dipole moment. The result is

$$\mathcal{M}(k, p \to k, p) = -i8M\omega(f_2\gamma)^2\mathbf{S}\cdot(\boldsymbol{\epsilon}'^{*}\times\boldsymbol{\epsilon}). \tag{25}$$

As a consequence, we find a refractive index of

$$n_{\pm} = 1 \pm \frac{2N(f_2\gamma)^2}{\omega}\langle S_z\rangle. \tag{26}$$

If, perhaps, there is some net alignment of spin along the z-axis (perhaps due to some external magnetic field), we find that the medium is in fact birefringent. Such a medium would demonstrate the magnetic Faraday effect. Its birefringence would rotate the plane of polarization for a linearly polarized wave. Such an effect is important in mapping magnetic fields in our galaxy,[46] and it has even been suggested as a means by which one might observationally constrain a potential magnetic dipole moment of the universe's dark matter.[47]

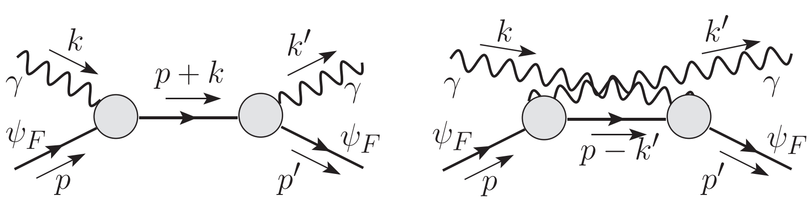


Fig. 2. Feynman diagrams used to compute the Compton scattering amplitude for ψ_F, which couples to the photon, γ, via a magnetic dipole moment. Fermion flow is indicated by the arrows on the fermion lines while four momenta are represented by external arrows.

V. CONCLUSION

In this paper, we have focused on low-energy interactions between electromagnetic dipoles and charged fermions or photons within the context of quantum field theory. In this limit, the dipole fields of the neutral fermions reproduce those of classical electromagnetism, including the correct contact fields for point-like pure dipoles. For students just beginning their studies of particle physics or quantum field theory, the syntax and manipulations needed to compute scattering amplitudes in field theory are often opaque. We hope that the techniques in this paper will equip students with some tools that can help them connect the physics of quantum field theory to more familiar classical systems.

In considering dipole interactions, we have focused on a few examples, keeping only leading order contributions in the non-relativistic limit. This work could easily be extended to include higher energy corrections, or one could even explore the physics in the ultra-relativistic regime. Additionally, one could consider other non-relativistic applications. For example, the scattering amplitude between two neutral electric dipoles can be used to explore the interaction between polar molecules in a dilute gas. If the dipole spins are randomly oriented, then the computation will need to be carried out to higher order in perturbation theory. That is, the process will involve a two-photon exchange between the particles, resulting in a one-loop Feynman diagram. The phenomena can be compared to the van der Waals interaction in which neutral molecules interact via induced electric dipole moments.[48]

ACKNOWLEDGMENTS

D.C.L. thanks Eli Gandour-Rood for locating an annotated collection of Fermi's original papers. W.M. was funded, in part, by a Sherman Fairchild Research Award from the University of Puget Sound.

[1] John David Jackson, *Classical Electrodynamics*, 3rd ed. (John Wiley and Sons, Inc., New York, NY, 1999), p. 149.
[2] David J. Griffiths, "Hyperfine splitting in the ground state of hydrogen," Am. J. Phys. **50**, 698–703 (1982).
[3] Reference 1, p. 188.
[4] David J. Griffiths, "Dipoles at rest," Am. J. Phys. **60**, 979–987 (1992).
[5] M. Tanabashi *et al.* (Particle Data Group), Phys. Rev. D **98**, 030001 (2018).
[6] We assume that the elementary charge constant e is positive so that the proton's charge is e and the electron's charge is $-e$.
[7] Gart Westerhout, "The radio galaxy," Sci. Am. **201**, 44–51 (1959).
[8] E. Fermi, "Über die magnetischen Momente der Atomkerne," Z. Phys. **60**, 320–333 (1930).
[9] Claude Cohen-Tannoudji, Bernard Diu, and Franck Laloë, *Quantum Mechanics* (John Wiley and Sons, Inc., New York, NY, 1977), Vol. 2, pp. 1249–1256.
[10] Stephen Gasiorowicz, *Quantum Physics*, 2nd ed. (John Wiley and Sons, Inc., New York, NY, 1996), pp. 287–289.
[11] David J. Griffiths, *Introduction to Quantum Mechanics*, 2nd ed. (Pearson Prentice Hall, Upper Saddle River, NJ, 2005), pp. 283–285.
[12] Albert Messiah, *Quantum Mechanics* (Dover, Mineola, NY, 1999), pp. 938–940.
[13] Richard A. Ferrell, "Electron-nucleus hyperfine interaction in atoms," Am. J. Phys. **28**, 484–486 (1960).
[14] S. M. Blinder, "The hyperfine interaction Hamiltonian," J. Mol. Spectrosc. **5**, 17–23 (1961).
[15] George T. Rado, "Simple derivation of the electron-nucleus contact hyperfine interaction," Am. J. Phys. **30**, 716–718 (1962).
[16] Manfred Bucher, "The electron inside the nucleus: An almost classical derivation of the isotropic hyperfine interaction," Eur. J. Phys. **21**, 19–22 (2000).
[17] Hans C. Ohanian, "What is spin?" Am. J. Phys. **54**, 500–505 (1986).
[18] David J. Griffiths, *Introduction to Electrodynamics*, 4th ed. (Pearson, Boston, MA, 2013), p. 379.
[19] W. E. Caswell and G. P. Lepage, "Effective Lagrangians for bound state problems in QED, QCD, and other field theories," Phys. Lett. **167B**, 437–442 (1986).
[20] Wolfgang K. H. Panofsky and Melba Phillips, *Classical Electricity and Magnetism*, 2nd ed. (Addison-Wesley, Reading, MA, 1962), pp. 332–334.
[21] Michael E. Peskin and Daniel V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading, MA, 1995).
[22] M. Nowakowski, E. A. Paschos, and J. M. Rodriguez, "All electromagnetic form-factors," Eur. J. Phys. **26**, 545–560 (2005).
[23] Walter Greiner and Joachim Reinhardt, *Quantum Electrodynamics*, 3rd ed. (Springer, Berlin, 2003), pp. 121–129.
[24] The amplitude for the spin-flip process vanishes in the leading order limit of low-momentum transfer.
[25] An accessible discussion of the relationship between the interaction Hamiltonian matrix element and the field theoretic invariant amplitude can be found in J. J. Sakurai, *Advanced Quantum Mechanics* (Addison Wesley, Reading, MA, 1967), Sec. 4.2. We note that the minus sign that occurs in our expression (relative to the Sakurai text) is a consequence of the conventions that we adopt from Ref. 21.
[26] Charles P. Frahm, "Some novel delta-function identities," Am. J. Phys. **51**, 826–829 (1983).
[27] For a discussion of generalizations of the derivatives in Ref. 26, see Jerrold Franklin, "Comment on 'Some novel delta-function identities' by Charles P. Frahm [Am. J. Phys. **51**, 826 (1983)]," Am. J. Phys. **78**, 1225–1226 (2010); V. Hnizdo, "Generalized second-order partial derivatives of $1/r$," Eur. J. Phys. **32**, 287–297 (2011).
[28] One of us, in fact, missed such a contact term in a previous paper: K. M. Whitcomb and D. C. Latimer, "Scattering from a quantum anapole at low energies," Am. J. Phys. **85**, 932–936 (2017).
[29] Robert S. Strichartz, *A Guide to Distribution Theory and Fourier Transforms* (World Scientific, Singapore, 2003).
[30] Boris Kayser, "Majorana neutrinos and their electromagnetic properties," Phys. Rev. D **26**, 1662–1670 (1982).
[31] Jose F. Nieves, "Electromagnetic properties of Majorana neutrinos," Phys. Rev. D **26**, 3152–3158 (1982).
[32] L. Landau, "On the conservation laws for weak interactions," Nucl. Phys. **3**, 127–131 (1957).
[33] C. S. Wu, E. Ambler, R. W. Hayward, D. D. Hoppes, and R. P. Hudson, "Experimental test of parity conservation in beta decay," Phys. Rev. **105**, 1413–1415 (1957).
[34] J. H. Christenson, J. W. Cronin, V. L. Fitch, and R. Turlay, "Evidence for the 2π decay of the ${K_2}^0$ meson," Phys. Rev. Lett. **13**, 138–140 (1964).
[35] M. Gell-Mann, M. L. Goldberger, and W. E. Thirring, "Use of causality conditions in quantum theory," Phys. Rev. **95**, 1612–1627 (1954).
[36] F. E. Low, "Scattering of light of very low frequency by systems of spin 1/2," Phys. Rev. **96**, 1428–1432 (1954).
[37] M. Gell-Mann and M. L. Goldberger, "Scattering of low-energy photons by particles of spin 1/2," Phys. Rev. **96**, 1433–1438 (1954).
[38] A. B. Kukanov and T'hai K'uang, "Theory of Compton scattering from a charged Dirac particle having intrinsic dipole moments," Sov. Phys. J. **15**, 92–94 (1972).
[39] M. J. Cooper, C. Shenton-Taylor, J. A. Duffy, C. A. Steer, and L. V. Blaauw, "A short history of magnetic Compton scattering," Nucl. Instrum. Methods Phys. Res., A **580**, 1–7 (2007).
[40] Reference 1, pp. 500–506.
[41] Richard P. Feynman, Robert B. Leighton, and Matthew L. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1963), Vol. 1, Secs. 31-1 and 31-2.
[42] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley and Sons, Inc., New York, NY, 1964), p. 559.
[43] E. Fermi, *Nuclear Physics*, revised ed. (University of Chicago Press, Chicago, 1950), pp. 201–202.

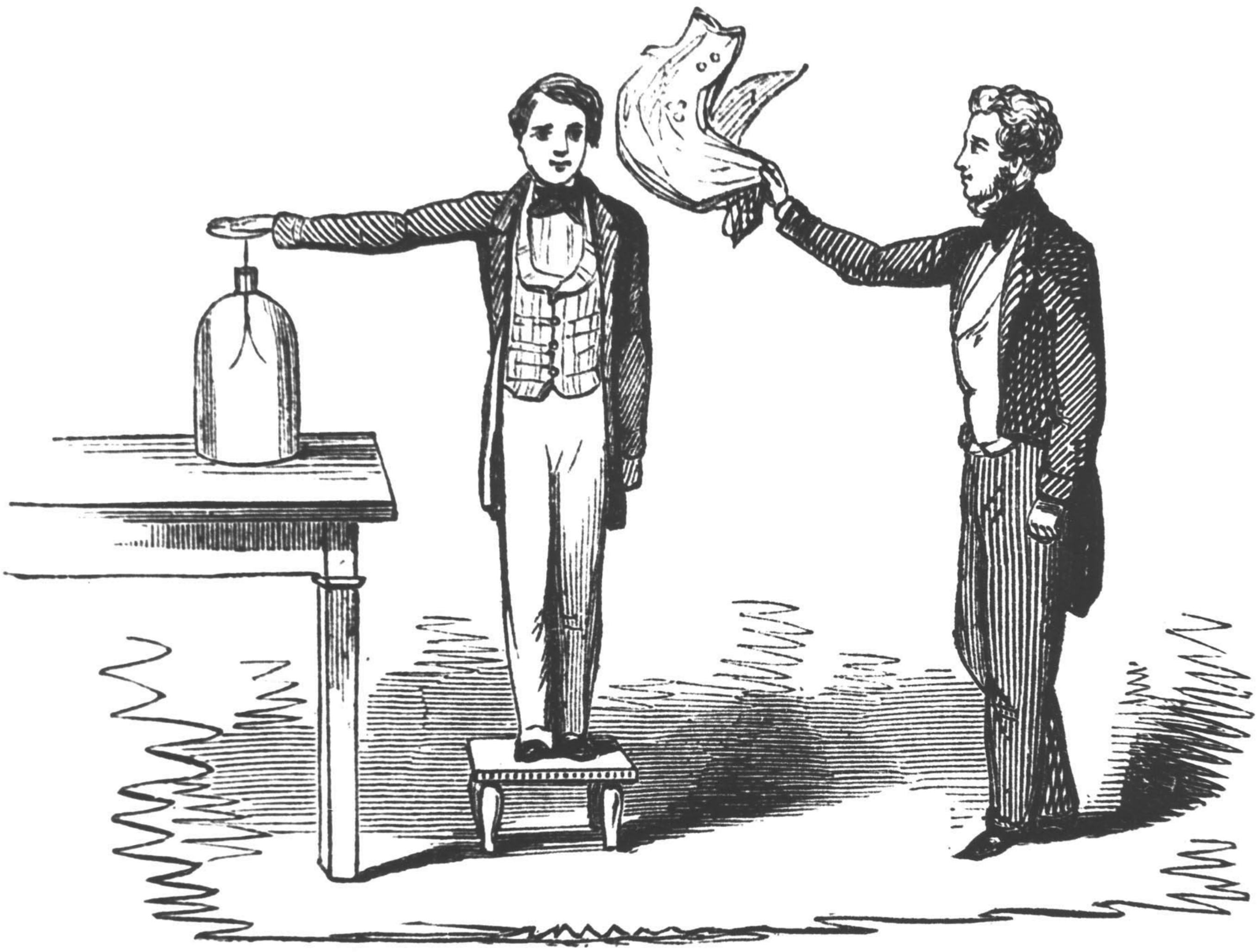
[44]J. Liu, "Neutrino coherent forward scattering and its index of refraction," Phys. Rev. D **45**, 1428–1431 (1992).

[45]We assume that the amplitude is Lorentz invariant and respects charge conjugation, parity, and time reversal symmetries. If these discrete symmetries are not conserved, then other terms can arise in the forward Compton amplitude; see M. Gorchtein, "CP-violation in Compton scattering," Phys. Rev. C **77**, 065501 (2008); P. F. Bedaque and M. J. Savage, "Parity violation in gamma polarized-p Compton scattering," Phys. Rev. C **62**, 018501 (2000); J.-W. Chen, T. D. Cohen, and C. W. Kao, "Parity violation in polarized-gamma proton Compton scattering," Phys. Rev. C **64**, 055206 (2001).

[46]N. Oppermann *et al*., "An improved map of the galactic Faraday sky," Astron. Astrophys. **542**, A93 (2012).

[47]S. Gardner, "Observing dark matter via the gyromagnetic Faraday effect," Phys. Rev. Lett. **100**, 041303 (2008).

[48]Barry R. Holstein, "The van der Waals interaction," Am. J. Phys. **69**, 441–449 (2001).



A Novel Way of Electrifying a Person

Many of us electrify a person, preferably with long, thin hair, by using a Wimshurst machine. Here is a very different way of doing the demonstration that saves the cost of the machine. "Assistant standing on the insulating stool and touching the disc of the electroscope while being struck with a dry handkerchief", from John Henry Pepper, *The Boy's Playbook of Science* (George Routledge and Sons, London. ca. 1873), pg 180. (Picture and Text by Thomas B. Greenslade, Jr., Kenyon College.)