Complex Integration

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1 Singularities

A function is called *analytic* (or regular, or holomorphic) in a region of the complex plane if it is both single valued and differentiable everywhere in that region. We can also have functions that are analytic except at certain points called singularities. The most important kind of singularity is probably poles. If the complex function $f(z)$ has the form

$$f(z) = \frac{g(z)}{(z - z_0)^n}$$

with $n > 0$ and $g(z)$ analytic and non-zero, then $f(z)$ has a pole of order n at $z = z_0$. If there is a pole at $z = z_0$, then

$$\lim_{z \to z_0} [(z - z_0)^n f(z)] \neq 0,$$

while it would be equal to zero if there were not a pole. Poles of order $n = 1$ are called *simple poles.*

2 Branch Points and Cuts

2.1 Square Roots

For multivalued complex functions, we get the emergence of branch points. For example,

$$f(z) = \sqrt{z} = \sqrt{re^{i\theta}} = \sqrt{r}(\cos(\theta/2) + i\sin(\theta/2))$$

has a branch point at $z = 0$ and infinity. The important thing to see in this example is that as you go around the complex plane by an angle 2π, you do not get the same value of $f(z)$ as you started with! To make $\sqrt{z}$ single-valued, we need to consider multiple complex planes, called Riemann sheets. In the example given, we would need two Riemann sheets, since when we go around by 4π, we return to our original value. Note that for the function $f(z) = \sqrt{z - z_0}$, there are branch points at $z = z_0$ and infinity. Also, we can write

$$f(z) = \sqrt{z - z_0} = |z - z_0|^{\frac{1}{2}} e^{i\theta/2}.$$

2.2 Complex Logarithms

Another example of a multi-valued complex function is the logarithm:

$$log(z) = log(re^{i\theta}) = log(r) + i\theta.$$

We see that if we go around the complex plane by an angle 2π n times, then

$$log(z) = log(r) + i\theta + i2\pi n.$$

So it looks like we need an infinite number of Riemann sheets, since each rotation by 2π takes you to a new sheet. Here, the branch point is at $z = 0$, so we have two logical choices for the branch cut:

1. Put the branch cut along the positive real axis, so that your principle branch corresponds to the angles $0 < \theta < 2\pi$.

2. Put the branch cut along the negative real axis, so that your principle branch corresponds to the angles $-\pi < \theta < \pi$.

3 Complex Integration

When integrating in the complex plane, you need to specify a curve $\mathbb{C}$ that you want to integrate along:

$$\int_{\mathbb{C}} f(z)dz.$$

One way to evaluate this is to parameterize the curve $z = x + iy$. However, the value of the integral is path-dependent. So let's look at some other ways to evaluate complex integrals.

3.1 Cauchy's Theorem

If $f(z)$ is analytic everywhere within and on a contour $\mathbb{C}$ (no singularities), then

$$\oint_{\mathbb{C}} f(z)dz = 0.$$

One consequence of this theorem is that we can deform the contour $\mathbb{C}$ into any shape we want given that $f(z)$ remains analytic in and on $\mathbb{C}$.

3.2 Cauchy's Integral Formula

If $f(z)$ is analytic within and on a closed contour $\mathbb{C}$, and z_0 is a point within $\mathbb{C}$, then

$$f(z_0) = \frac{1}{2\pi i} \oint_{\mathbb{C}} \frac{f(z)}{z - z_0} dz.$$

More generally, we can define the derivative of a function as

$$f^n(z) = \frac{n!}{2\pi i} \oint_{\mathbb{C}} \frac{f(w)}{(w - z)^{n+1}} dz.$$

Note that all derivatives exist for analytic functions.

3.3 Residue Theorem

Consider integrating a function $f(z)$ around a closed contour $\mathbb{C}$, when $f(z)$ has a singularity at $z = z_0$ inside of $\mathbb{C}$. Then

$$\oint_{\mathbb{C}} f(z)dz = 2\pi i \sum_k (res)_k,$$

where the residue is defined as

$$res(z = z_0) = \lim_{z \to z_0} \left[\frac{1}{(p-1)!} \frac{d^{p-1}}{dx^{p-1}} \left((z - z_0)^p f(z) \right) \right],$$

where p is the order of the pole at $z = z_0$. When evaluating real integrals using contour integration, a typical choice of $\mathbb{C}$ is a circle or a semicircle, where the radius is taken to infinity such that the value of the integral along that curve goes to zero.

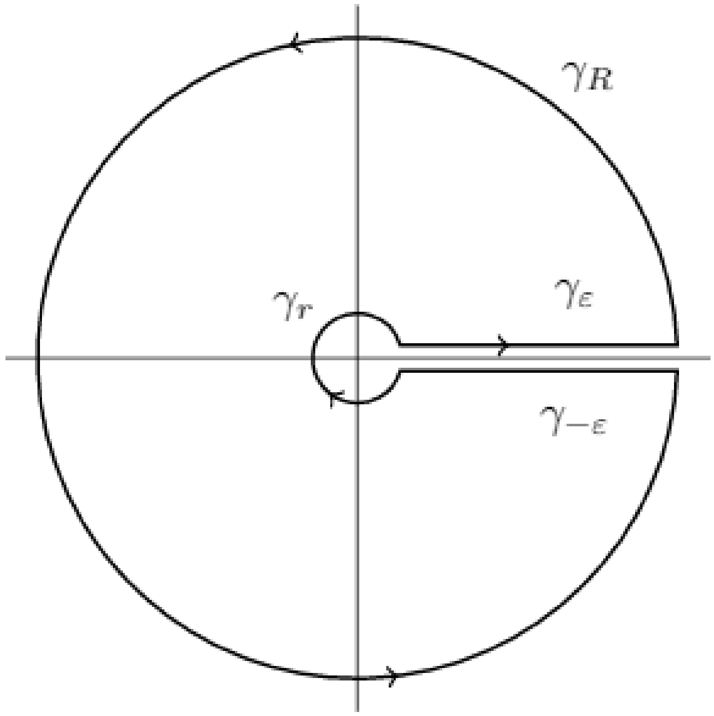


Figure 1: Example of a keyhole contour with branch point at $z = 0$ and the branch cut along the positive real axis.

3.4 Integrals with a Branch Cut

For integrals with a branch point at $z = z_0$ and a branch cut along the positive real axis (just for example), choose a keyhole contour, as shown in Figure 1.

Remember, since we're dealing with a multi-valued function, you need to say which Riemann sheet you're working on when evaluating the integral! If the contour is chosen correctly, when you take the radius of the large circle $R \to \infty$ and the radius of the small circle $r \to 0$, then the integral along those parts of the contour should go to zero.

3.5 Principle Value of Integral

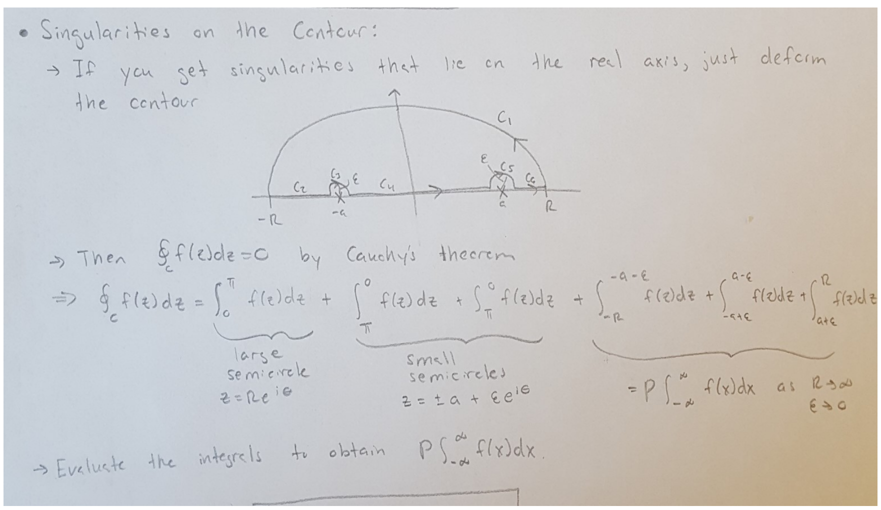
Just look at Figure 2...

3.6 Wedge Contour (or Pi Slice)

Here's a nice explanation:

https://math.stackexchange.com/questions/242514/using-contour-integration/242533

The basic idea is that you choose your contour to be a wedge and choose the angle of that wedge such that some nice cancellation occurs.



- Singularities on the Contour:
  - → If you get singularities that lie on the real axis, just deform the contour

→ Then $\oint_C f(z)dz = 0$ by Cauchy's theorem

$$\Rightarrow \oint_C f(z)\,dz = \underbrace{\int_0^\pi f(z)dz}_{\substack{\text{large}\\ \text{semicircle}\\ z=Re^{i\theta}}} + \underbrace{\int_\pi^0 f(z)\,dz + \int_\pi^0 f(z)dz}_{\substack{\text{small}\\ \text{semicircles}\\ z=\pm a+\varepsilon e^{i\theta}}} + \underbrace{\int_{-R}^{-a-\varepsilon} f(z)dz + \int_{-a+\varepsilon}^{a-\varepsilon} f(z)dz + \int_{a+\varepsilon}^{R} f(z)dz}_{=P\int_{-\infty}^{\infty} f(x)dx \text{ as } R\to\infty,\ \varepsilon\to 0}$$

→ Evaluate the integrals to obtain $P\int_{-\infty}^{\infty} f(x)dx$.

Figure 2: How to obtain the principle value of an integral.