Solving Problems using Lagrangian and Hamiltonian Mechanics

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1 Lagrangian Mechanics

To solve mechanics problems using the Lagrangian formalism, just follow these easy steps:

1. Read the problem and draw the relevant system.

2. Determine the generalized coordinates that will be used to characterize the system. To do this, use the relationship
$$\#coordinates = DN - k,$$
where D is the number of dimensions of the system, N is the number of objects in the problem, and k is the number of constraints in the problems. Examples of constraints are the length of a pendulum being constant, or an object being constraint to move on a plane, etc.

3. Write out the positions of the objects in Cartesian coordinates (x,y,z) in terms of the generalized coordinates.

4. Write the kinetic (T) and potential (V) energy of each object in terms of their Cartesian coordinates. The kinetic energy is given by
$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2).$$

5. Construct the Lagrangian, defined as
$$L(q_i, \dot{q_i}, t) = T - V.$$

6. Obtain the equations of motion for each generalized coordinate q_i using Lagrange's equations of motion:
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_i}}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{1}$$

1.1 Constants of Motion

If the Lagrangian doesn't depend explicitly on one of the generalized coordinates q_i, then we can see from Lagrange's equations that
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_i}}\right) = 0.$$
We can see from the above equation that the quantity $\frac{\partial L}{\partial \dot{q_i}}$ is constant in time. We then define the canonical momentum as
$$p_i = \frac{\partial L}{\partial \dot{q_i}}.$$
The canonical momentum is then a constant of motion if the Lagrangian doesn't depend on the corresponding generalized coordinate, which we call cyclic in this case.

1.2 Systems with Constraints and Lagrange Multipliers

The advantage of the Lagrangian formalism is that we are able to bypass the forces of constraint. But what if we actually want to know what these forces are. In this case we need to employ the use of Lagrange multipliers and make a few modifications to our previous method.

1. Follow steps 1-5 of the previous method.

2. Define the equation of constraint

$$f_j(q_1, q_2, ..., q_n, t) = 0.$$

   Examples of equations of constraint are the length of a pendulum being constant or the equation of rolling constraint:

$$rd\theta = dx.$$

   Note that there may be more than one equation of constraint for a given problem.

3. For each equation of constraint, assign a Lagrange multiplier λ_j.

4. Lagrange's equations are modified to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial f_j}{\partial q_i} = 0.$$

   Here, j=1,2,...,m, where m is the number of constraints in the problem. We can note that

$$-\sum_{j=1}^{m} \lambda_j \frac{\partial f_j}{\partial q_i} = Q_i,$$

   where Q_i are the generalized forces that produce the individual constraints.

2 Hamiltonian Mechanics

To solve a mechanics problem using the Hamiltonian formalism, follow these steps:

1. Follow the appropriate steps to construct the Lagrangian that defines the system (L=T-V).

2. Construct the Hamiltonian, defined as

$$H(q_i, p_i, t) = \sum_i \dot{q}_i p_i - L(q_i, \dot{q_i}, t).$$

3. Since we want to get rid of the $\dot{q}_i$, use the equation for the canonical momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

   to write the $\dot{q}_i$ in terms of the p_i, then use those to eliminate the $\dot{q}_i$ in H.

4. Use the Hamilton equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

5. Something to keep in mind with the Hamiltonian. If the Lagrangian does not explicitly depend on time, then the Hamiltonian is a constant of motion (conserved quantity). Another note is that the Hamiltonian is not necessarily the total energy of the system. However, if the kinetic energy is quadratic in the $\dot{q}_i$ terms, and the potential energy depends only on the coordinates q_i, the the Hamiltonian is the total energy of the system. That is

$$H = T + V.$$