# Solving Laplace's Equation Using Separation of Variables

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This document will only go over separation of variables in Cartesian and spherical polar coordinates, as those seem to be the most applicable to the qualifying exam. It is probably important to memorize the main results from these derivations, namely the general solutions for the potential as well as some useful identities. However, it is also nice to go through the derivation a couple of times just in case you forget the final result.

## 1 Laplace's Equation

Poisson's equation is

$$\nabla^2 V = \rho/\epsilon_0.$$

In the case where we are dealing with a region with no charge ($\rho = 0$), then this reduces to Laplace's equation:

$$\nabla^2 V = 0.$$

When solving problems, we often want to choose a convenient coordinate system to exploit the geometry of the system. We will assume a separable solution ($V$ is given by the product of functions that each depend on only one of the independent coordinates). Once we find a solution using this procedure, we can construct the most general solution by summing over all possible solutions.

### 1.1 Cartesian Coordinates

Consider a box with side lengths $(x, y, z) = (a, b, c)$ and one point located at the origin. Let one side of the box be held at the potential $V(x, y, z = c) = \phi(x, y)$ and all other sides grounded ($V = 0$). We assume the potential has the form

$$V(x, y, z) = X(x)Y(y)Z(z).$$

In Cartesian coordinates, the Laplacian is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Then Laplace's equation yields:

$$\nabla^2 V = Y(y)Z(z)\frac{\partial^2 X(x)}{\partial x^2} + X(x)Z(z)\frac{\partial^2 Y(y)}{\partial y^2} + X(x)Y(y)\frac{\partial^2 Z(z)}{\partial z^2} = 0.$$

Diving by $V(x, y, z) = X(x)Y(y)Z(z)$ gives

$$\frac{1}{X}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y(y)}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z(z)}{\partial z^2} = 0.$$

For this equation to hold, each term must equal a constant. So then

$$\frac{1}{X}\frac{\partial^2 X(x)}{\partial x^2} = -\alpha^2,$$
$$\frac{1}{Y}\frac{\partial^2 Y(y)}{\partial y^2} = -\beta^2,$$
$$\frac{1}{Z}\frac{\partial^2 Z(z)}{\partial z^2} = \gamma^2,$$

where $\gamma^2 = \alpha^2 + \beta^2$. We have cleverly chosen the constants such that we get an exponential solution in the coordinate that has the potential. In this case, it is the z-coordinate. The solutions to the above equations are

$$\begin{aligned}
X(x) &= A\sin(\alpha x) + B\cos(\alpha x), \\
Y(y) &= C\sin(\beta y) + D\cos(\beta y), \\
Z(z) &= F\sinh(\gamma z) + G\cosh(\gamma z).
\end{aligned}$$

Now we can apply the boundary conditions, given by the zero potential on five of the sides of the box:

$$\begin{aligned}
X(0) &= X(a) = 0, \\
Y(0) &= Y(b) = 0, \\
Z(0) &= 0.
\end{aligned}$$

These boundary conditions tell us that

$$\begin{aligned}
&B = D = G = 0, \\
&\alpha = \frac{n\pi}{a}, \quad (n = 1, 2, ...) \\
&\beta = \frac{m\pi}{b}, \quad (m = 1, 2, ...) \\
&\gamma^2 = \alpha^2 + \beta^2 \Rightarrow \gamma_{nm} = \sqrt{\alpha^2 + \beta^2} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}.
\end{aligned}$$

The potential is then

$$V(x, y, z) = X(x)Y(y)Z(z) = A_{nm}\sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})\sinh(\gamma_{nm}z).$$

The most general form is

$$V(x, y, z) = \sum_{m,n} A_{nm}\sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})\sinh(\gamma_{nm}z).$$

To find the constant $A_{nm}$, we can use the final boundary condition

$$V(x, y, z = c) = \phi(x, y) = \sum_{m,n} A_{nm}\sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})\sinh(\gamma_{nm}c).$$

Then using Fourier's trick, we exploit the orthogonality of the sine functions,

$$\int_0^a \sin(\frac{n\pi x}{a})\sin(\frac{n'\pi x}{a})dx = \frac{a}{2}\delta_{n,n'}$$

(and likewise for the y term), and multiply each side of the equation by $\sin(\frac{n'\pi x}{a})\sin(\frac{m'\pi y}{b})$ and integrate from $0 \rightarrow a$ and $0 \rightarrow b$. The delta functions with select out only one term from the sum, and we get that

$$A_{nm} = \frac{4\int_0^a\int_0^b \sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})\phi(x, y)dxdy}{ab\sinh(\gamma_{nm}c)}.$$

We now have the most general solution, so we just need to evaluate the above integral for a given $\phi(x, y)$ to solve the problem. One important thing to note is that we can use the idea of superposition to solve more complex problems. For instance, consider the case in which two sides of the box are held at different potentials (with the other sides grounded). This may seem difficult to solve at first, but we can simply treat the problem as two boxes with one side at a given potential (as done above), and then just add the two solutions together!

## 1.2 Spherical Polar Coordinates

In spherical coordinates, Laplace's equation is

$$\nabla^2 V = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rV) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0.$$

Let's assume a separable solution

$$V(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) = \frac{U(r)}{r}\Theta(\theta)\Phi(\phi),$$

where we have written $R(r) = U(r)/r$ for convenience. Plugging this solution into Laplace's equation yields

$$\frac{1}{Ur}\Theta\Phi\frac{\partial^2 U}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)\frac{U}{r}\Phi + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}\frac{U}{r}\Theta = 0.$$

Dividing by $V(r,\theta,\phi)$ and rearranging yields

$$r^2\sin^2\theta\left[\frac{1}{U}\frac{\partial^2 U}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{1}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)\right] + \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = 0.$$

For this equation to be true, we need each term to equal a constant. That is,

$$r^2\sin^2\theta\left[\frac{1}{U}\frac{\partial^2 U}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{1}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)\right] = m^2,$$
$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = -m^2.$$

We can see from our choice of constant that the $\Phi$ equation has the solution

$$\Phi(\phi) \propto e^{\pm im\phi}. \tag{1}$$

### 1.2.1 The Radial Part

Looking at the first equation, we can divide it into a radial part that only depends on $r$ and an angular part that depends on $\theta$:

$$\frac{r^2}{U}\frac{\partial^2 U}{\partial r^2} + \left[\frac{1}{\sin\theta}\frac{1}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta}\right] = 0. \tag{2}$$

Again, for this equation to hold, we need each part to equal a constant. In this case, we choose the angular part to be equal to $-l(l+1)$ such that

$$\frac{r^2}{U}\frac{\partial^2 U}{\partial r^2} = l(l+1) \Rightarrow \frac{\partial^2 U}{\partial r^2} - \frac{l(l+1)U}{r^2} = 0.$$

To solve the above equation, we assume a solution of the form $U = r^\alpha$ and find the solution to be

$$U(r) = Ar^{l+1} + Br^{-l} \Rightarrow R(r) = Ar + \frac{B}{r^{l+1}}.$$

We can see that in the limit $r \to 0$, we need $B = 0$ to maintain a regular solution that doesn't blow up. Likewise, for $r \to \infty$, we need $A = 0$.

### 1.2.2 The Angular Part (Azimuthal Symmetry)

In the case of azimuthal symmetry (that is, if you rotate your system about the $\phi$ direction the problem doesn't change), then our potential is independent of the coordinate $\phi$:

$$V = V(r,\theta) = R(r)\Theta(\theta).$$

This is equivalent to setting $m = 0$ in our previous work. Doing this, the angular part of Laplace's equation becomes

$$\frac{1}{\sin\theta}\frac{1}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) = -l(l+1).$$

The solutions to the above differential equation are the Legendre polynomials, so

$$\Theta(\theta) = P_l(cos\theta).$$

The first few Legendre polynomials are given by:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= (3x^2-1)/2 \\ P_3(x) &= (5x^3-3x)/2 \end{aligned}$$

The general solution for the potential is then given by

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r + \frac{B_l}{r^{l+1}}\right)P_l(cos\theta).$$

The coefficients $A_l$ and $B_l$ are then found by applying boundary conditions. A common problem deals with a spherical shell held at some potential $V(R,\theta) = \xi(\theta)$. The first thing to do would be to express $\xi(\theta)$ in terms of Legendre polynomials if possible, and note that you only need a few of the $P_l(cos\theta)$ terms in the sum to be non-zero in order to satisfy the boundary condition at $r = R$. Otherwise you can solve for the coefficients by using Fourier's trick and exploiting the orthogonality of the Legendre polynomials:

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn}.$$

Another important property of the Legendre polynomials is that

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta), \tag{3}$$

where $r_< = min(|\mathbf{r}|,|\mathbf{r}'|)$ and $r_> = max(|\mathbf{r}|,|\mathbf{r}'|)$. This is a good relation to know for cases where you want to move off-axis.

### 1.2.3 The Angular Part (Beyond Azimuthal Symmetry)

In the case where we don't have azimuthal symmetry ($m \neq 0$), the solution to Eq. 2 is the associated Legendre polynomials $P_l^m(cos\theta)$. If we combine these solutions with the solution to the $\phi$ equation (Eq. 1), we can obtain the spherical harmonics, given by

$$Y_l^m(\theta,\phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(cos\theta)e^{im\phi}.$$

The spherical harmonics form a complete orthonormal set, so

$$\int Y_l^m(\theta,\phi) Y_{l'}^{m'}(\theta,\phi) d\Omega = \delta_{l,l'}\delta_{m,m'}.$$

Then we can write the potential in terms of the spherical harmonics:

$$V(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}r + \frac{B_{lm}}{r^{l+1}}\right) Y_l^m(\theta,\phi).$$

The coefficients can then be solved using either Fourier's trick to exploit the orthogonality or by writing the boundary condition in terms of spherical harmonics to truncate the sum.

An important theorem to know is the addition theorem for the spherical harmonics:

$$P_l(cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}^*(\theta'\phi') Y_l^m(\theta,\phi).$$

As usual, the primed coordinates characterize the vector to the charge distribution, while the unprimed coordinates characterize the vector to the point you want to know the potential at. The angle $\gamma$ gives the angle between those two vectors.

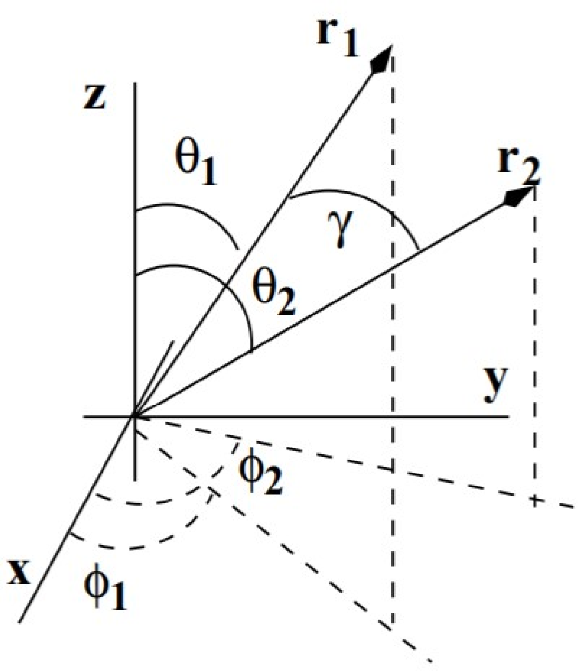


Figure 1: The addition theorem for spherical harmonics.

Using the addition theorem, we can write Eq. 3 as

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}^*(\theta'\phi') Y_l^m(\theta,\phi).$$