Electrostatics

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1 Maxwell's Equations for Electrostatics

Maxwell's equations for electrostatics are (in differential form):

$$\nabla \times \mathbf{E} = 0, \tag{1}$$

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0. \tag{2}$$

1.1 The Electric Potential

Eq. 1 tells us that the electric field can be derived from a scalar potential:

$$\mathbf{E} = -\nabla V. \tag{3}$$

Then Eq. 2 takes the form of Poisson's equation

$$\nabla^2 V = -\rho/\epsilon_0. \tag{4}$$

This has the solution

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r'. \tag{5}$$

Here, we integrate over a volume charge distribution. For a surface charge and line charge distribution, this equation becomes:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{A}} \frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} da', \tag{6}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{C}} \frac{\lambda(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dl'. \tag{7}$$

1.2 The Electric Field

Given the above section, the electric field for a volume charge distribution is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2} \widehat{|\mathbf{r}-\mathbf{r}'|} d^3r'. \tag{8}$$

1.3 The Point Charge Case

The electric potential and electric field for a point charge are given by

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}'|}, \tag{9}$$

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\widehat{|\mathbf{r}-\mathbf{r}'|} = \frac{q}{4\pi\epsilon_0}\frac{|\mathbf{r}-\mathbf{r}'|}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{10}$$

For a collection of point charges, we can use the idea of superposition to find the potential and electric field:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\mathbf{r}-\mathbf{r}'|_i}, \tag{11}$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_i q_i \frac{(\mathbf{r}-\mathbf{r}')_i}{|\mathbf{r}-\mathbf{r}'|_i^3}. \tag{12}$$

The force on a charge Q from a collection of point charges is given by

$$\mathbf{F} = Q\mathbf{E} = Q\frac{1}{4\pi\epsilon_0}\sum_i q_i \frac{(\mathbf{r}-\mathbf{r}')_i}{|\mathbf{r}-\mathbf{r}'|_i^3}. \tag{13}$$

1.4 Boundary Conditions for the Potential and Electric Field

The boundary conditions for the electric potential are:

$$V_{above} = V_{below}, \tag{14}$$

$$\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\epsilon_0}, \tag{15}$$

where $\frac{\partial V}{\partial n} = \nabla V \cdot \hat{\mathbf{n}}$ denotes the normal derivative of V with respect to some surface. The first boundary condition tells us that the potential is continuous across a boundary.

The boundary conditions for the electric field are:

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0}, \tag{16}$$

$$\mathbf{E}^{\parallel}_{above} = \mathbf{E}^{\parallel}_{below}. \tag{17}$$

The superscripts here correspond to the part of the electric field that is perpendicular ($\perp$) or parallel ($\parallel$) to the surface. These two equations can be combined into a single equation:

$$\mathbf{E}_{above} - \mathbf{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}, \tag{18}$$

where $\hat{\mathbf{n}}$ is the normal vector perpendicular to the surface pointing from "below" to "above".

2 Gauss's Law

If we integrate Eq. 2 and apply Stoke's Theorem, then we arrive at Gauss's Law, which tells us that the flux of the electric field through some Gaussian surface that we choose is equal to the charge enclosed within that surface:

$$\int_{\mathcal{A}} \mathbf{E}\cdot d\mathbf{a} = \frac{Q_{enclosed}}{\epsilon_0}. \tag{19}$$

Using Gauss's Law requires the problem to have a high degree of symmetry. I'll go through the three classic examples of applying Gauss's Law.

2.1 Spherical Symmetry

For problems with spherical symmetry, we choose our Gaussian surface to be a concentric sphere. The radius of this sphere can be whatever we need to determine the field in whatever region we're interested in. In this case, $\mathbf{E}$ points radially outwards, so Gauss's Law becomes

$$\int_{\mathcal{S}} \mathbf{E}\cdot d\mathbf{a} = \int_{\mathcal{S}} |\mathbf{E}|da = |\mathbf{E}|\int_{\mathcal{S}} da = |\mathbf{E}|4\pi r^2. \tag{20}$$

Then the electric field is given by

$$|\mathbf{E}|4\pi r^2 = \frac{Q_{enclosed}}{\epsilon_0} \Rightarrow \mathbf{E} = \frac{Q_{enclosed}}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}. \tag{21}$$

2.2 Cylindrical Symmetry

For problems with cylindrical symmetry, we choose our Gaussian surface to be a coaxial cylinder of length l and radius s. Then Gauss's Law becomes

$$\int_{\mathcal{S}} \mathbf{E} \cdot \mathbf{da} = \int_{\mathcal{S}} |\mathbf{E}| da = |\mathbf{E}| \int_{\mathcal{S}} da = |\mathbf{E}| 2\pi s l. \tag{22}$$

Then the electric field is given by

$$\mathbf{E} = \frac{Q_{enclosed}}{2\pi\epsilon_0 s l}\hat{\mathbf{s}}. \tag{23}$$

2.3 Planar Symmetry

For problems with planar symmetry (an infinite plane with surface charge σ), we choose our Gaussian surface to be a "pillbox" of area A extending equal distances above and below the surface. In this case, $Q_{enclosed} = \sigma A$. By symmetry, the electric field points away from the surface, passing through both ends of the box. Thus,

$$\int_{\mathcal{S}} \mathbf{E} \cdot \mathbf{da} = 2A|\mathbf{E}|. \tag{24}$$

The electric field is then given by

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}, \tag{25}$$

where $\hat{\mathbf{n}}$ is the normal vector pointing away from the surface.

3 Work in Electrostatics

The work needed to assemble a configuration of point charges is given by

$$W = \frac{1}{2}\sum_i q_i V(\mathbf{r}_i) = \frac{1}{2}\sum_i q_i \sum_{j \neq i} \frac{q_j}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}'|_j}. \tag{26}$$

4 Conductors and Capacitors

4.1 Conductors

The properties of an ideal conductor are:

1. $\mathbf{E} = 0$ inside a conductor.
2. $\rho = 0$ inside a conductor.
3. Any net charge resides on the surface of the conductor.
4. A conductor is an equipotential. That is, any two points on or within the conductor have the same potential.

4.2 Capacitors

A capacitor is made from two conductors with opposite charge ($\pm Q$). The capacitance is defined as

$$C \equiv \frac{Q}{V}. \tag{27}$$

The energy stored in a capacitor is given by

$$W_{cap} = \frac{1}{2} C V^2. \tag{28}$$

This is also the work required to "charge up" the capacitor.

5 Multipole Expansion in Electrostatics

5.1 The Electric Dipole

For a collection of point charges, the dipole moment is given by

$$\mathbf{p} = \sum_i q_i \mathbf{r'}_i, \tag{29}$$

where $\mathbf{r'}$ is the vector pointing from the origin to the charge q_i.

In the limit $r >> R$, the potential and field for an electric dipole is given by

$$V_{dip}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}, \tag{30}$$

$$\mathbf{E}_{dip}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{p}) - \mathbf{p}}{r^3}. \tag{31}$$

In the case where $\mathbf{p} = p\hat{\mathbf{z}}$, we can write the field in polar coordinates as

$$\mathbf{E}_{dip}(r, \theta) = \frac{p}{4\pi\epsilon_0} \left[2\cos\theta \hat{\mathbf{r}} + \sin\theta \hat{\theta} \right]. \tag{32}$$

The torque on a dipole in an electric field is given by

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}. \tag{33}$$

The force on the dipole is

$$\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}. \tag{34}$$

5.2 The Quadrupole Moment

In the limit $r >> R$, the potential and field for an electric quadrupole is given by

$$V_{quad}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} Q_{ij} \frac{3r_i r_j - \delta_{ij} r^2}{r^5}, \tag{35}$$

where the components of the electric quadrupole tensor are given by

$$Q_{ij} = \frac{1}{2} \int d^3 r \rho(\mathbf{r}) r_i r_j. \tag{36}$$

The quadrupole tensor is a symmetric 3×3 tensor with six independent values:

$$\mathbf{Q} = \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{xy} & Q_{yy} & Q_{yz} \\ Q_{xz} & Q_{yz} & Q_{zz} \end{bmatrix}. \tag{37}$$

6 Electric Fields in Matter

6.1 Bound Charges

Suppose we have some polarized material with dipole moment per unit volume $\mathbf{P}$. The polarization will produce a bound surface and volume charge, given by

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}, \tag{38}$$
$$\rho_b = -\nabla \cdot \mathbf{P}. \tag{39}$$

The potential produced by these bound charges is given by

$$V_{pol}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\oint_{\mathcal{S}} \frac{\sigma_b}{|\mathbf{r}-\mathbf{r}'|} da' + \int_{\mathcal{V}} \frac{\rho_b}{|\mathbf{r}-\mathbf{r}'|} d^3r'\right] \tag{40}$$

6.2 Maxwell's Equation in Dielectrics

Within a dielectric, we can write the total charge density in terms of the free and bound charge density:

$$\rho = \rho_f + \rho_b. \tag{41}$$

Then we can write the differential form of Gauss's law as

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_f + \rho_b = -\nabla \cdot \mathbf{P} = \rho_f. \tag{42}$$

Combining the divergence terms gives

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f \Rightarrow \nabla \cdot \mathbf{D} = \rho_f, \tag{43}$$

where we have defined the electric displacement,

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}. \tag{44}$$

Unlike the electric field, the curl of the electric displacement is not zero:

$$\nabla \times \mathbf{D} = \nabla \times \mathbf{P}. \tag{45}$$

6.3 Gauss's Law in Dielectrics

The integral form of Gauss's Law in dielectrics is

$$\int_{\mathcal{A}} \mathbf{D} \cdot \mathbf{da} = Q_{f,enclosed}. \tag{46}$$

6.4 Linear Dielectrics

For a linear dielectric, the polarization is proportional to the electric field

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \tag{47}$$

where χ_e is the electric susceptibility of the medium. In linear media, the displacement field is given by

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \mathbf{D} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon_0 (1 + \chi_e) \mathbf{E} = \epsilon \mathbf{E}, \tag{48}$$

where we have defined the permittivity of the material

$$\epsilon \equiv \epsilon_0 (1 + \chi_e). \tag{49}$$

6.5 Boundary Conditions in Dielectrics

In the presence of dielectrics, the boundary conditions for the potential are

$$V_{above} = V_{below}, \tag{50}$$

$$\epsilon_{above}\frac{\partial V_{above}}{\partial n} - \epsilon_{below}\frac{\partial V_{below}}{\partial n} = -\sigma_f. \tag{51}$$

The boundary condition for the electric field in the presence of dielectrics is given by

$$\epsilon_{above}E^{\perp}_{above} - \epsilon_{below}E^{\perp}_{below} = \sigma_f.$$