Magnetostatics

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1 Maxwell's Equations in Magnetostatics

Maxwell's equations in magnetostatics are

$$\nabla \cdot \mathbf{B} = 0, \tag{1}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad (Ampere's \quad Law). \tag{2}$$

We also have the continuity equation

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}, \tag{3}$$

where ρ is the charge density. In magnetostatics, we are dealing with steady currents, so

$$\nabla \cdot \mathbf{J} = 0. \tag{4}$$

1.1 The Magnetic Vector Potential

The first equation tells us that the magnetic field can be derived by a vector potential $\mathbf{A}$ as

$$\nabla \times \mathbf{A} = \mathbf{B}. \tag{5}$$

It also follows that

$$\nabla \cdot \mathbf{A} = 0. \tag{6}$$

Using the vector potential, we can rewrite Ampere's Law as

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}. \tag{7}$$

This is just Poisson's equation, which has the solution (for different current distributions)

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 r' \quad (volume), \tag{8}$$
$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{S}} \frac{\mathbf{K}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} da' \quad (surface), \tag{9}$$
$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{C}} \frac{\mathbf{I}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dl' \quad (line). \tag{10}$$

1.2 The Magnetic Field – The Biot-Savart Law

To calculate the magnetic field of a steady current, we can use the Biot-Savart Law:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r' \quad (volume), \tag{11}$$
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{S}} \frac{\mathbf{K}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} da' \quad (surface), \tag{12}$$
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{C}} \frac{\mathbf{I}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dl' = \frac{\mu_0 I}{4\pi} \int_{\mathcal{C}} \frac{d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \quad (line). \tag{13}$$

1.3 Boundary Conditions for B and A

The boundary conditions for the magnetic field are

$$B^{\perp}_{above} = B^{\perp}_{below}, \tag{14}$$

$$B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K, \tag{15}$$

where K is the surface current. These can be combined into a single equation:

$$\mathbf{B}_{above} - \mathbf{B}_{above} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}}), \tag{16}$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the surface pointing from "below" to "above".

The boundary conditions for the magnetic vector potential are

$$\mathbf{A}_{above} = \mathbf{A}_{below}, \tag{17}$$

$$\frac{\partial \mathbf{A}_{above}}{\partial n} - \frac{\partial \mathbf{A}_{below}}{\partial n} = -\mu_0 \mathbf{K}. \tag{18}$$

2 Ampere's Law

Using Stoke's theorem, we can rewrite the differential form of Ampere's Law in integral form as

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a} = \mu_0 I_{enclosed}. \tag{19}$$

Here, we integrate around some chosen "Amperian loop" placed in our region of interest. Just like Gauss's Law, our problem needs a good deal of symmetry in order to apply Ampere's Law. Let's go through a few classic applications of Ampere's Law.

2.1 Long Straight Wire

To find the magnetic field a distance s from a long straight wire with current I along the $+z$ direction, we choose our Amperian loop to be a concentric circle centered on the wire with radius s. By symmetry, the magnitude of the magnetic field is constant around the loop, so

$$\oint \mathbf{B} \cdot d\mathbf{l} = B \oint dl = B 2\pi s = \mu_0 I_{enclosed} = \mu_0 I. \tag{20}$$

So the magnetic field is given by

$$\mathbf{B} = \frac{\mu 0 I}{2\pi s} \hat{\boldsymbol{\phi}}, \tag{21}$$

where we determined the direction of the field using the right hand rule.

2.2 Infinite Surface

Consider an infinite plane with surface current $\mathbf{K} = K\hat{\mathbf{x}}$ in the xy-plane. To fine the magnetic field, we choose our Amperian loop to be a rectangle of length l parallel to the yz-plane extending equal distances above and below the surface. Then

$$\oint \mathbf{B} \cdot d\mathbf{l} = 2Bl = \mu_0 I_{enclosed} = \mu_0 K l, \tag{22}$$

where the factor of 2 comes from the contribution of the rectangle above and below the surface. Then the magnetic field is given by

$$\mathbf{B} = \begin{cases} +(\mu_0/2)K\hat{\mathbf{y}}, & z < 0 \\ -(\mu_0/2)K\hat{\mathbf{y}}, & z > 0 \end{cases}, \tag{23}$$

where the direction of the field is determined by the right hand rule, and noting that the field can only have a $\hat{\mathbf{y}}$ component.

2.3 The Long Solenoid

Consider a very long solenoid consisting of n closely wound turns per unit length on a cylinder of radius R and carrying a steady current I. To find the field outside of the solenoid, we choose an Amperian loop of length L with sides a distance a and b away from the edge of the solenoid. Then

$$\oint \mathbf{B} \cdot d\mathbf{l} = [B(a) - B(b)]L = \mu_0 I_{enclosed} = 0 \Rightarrow B(a) = B(b). \tag{24}$$

So the field outside the solenoid does not depend on the distance from the axis. Since the field must go to zero as $s \to \infty$, then $\mathbf{B} = 0$ outside the solenoid. To find the field inside the solenoid, we choose an Amperian loop to be a rectangle of length L with equal parts inside and outside the solenoid. Then

$$\oint \mathbf{B} \cdot d\mathbf{l} = BL = \mu_0 I_{enclosed} = \mu_0 nIL. \tag{25}$$

Then

$$\mathbf{B} = \begin{cases} \mu_0 nI\hat{\mathbf{z}}, & inside \\ 0, & outside \end{cases}. \tag{26}$$

3 Magnetic Forces

The magnetic force on a charge q moving with velocity $\mathbf{v}$ in a field $\mathbf{B}$ is given by the Lorentz force law:

$$\mathbf{F}_{mag} = q(\mathbf{v} \times \mathbf{B}). \tag{27}$$

Combining this with the force from an electric field,

$$\mathbf{F} = q[(\mathbf{E} + (\mathbf{v} \times \mathbf{B})]. \tag{28}$$

One thing to note is that **magnetic forces do no work**.

3.1 Force From a Current Carrying Wire

The force from a current carrying wire is given by

$$\mathbf{F}_{mag} = \int (\mathbf{I} \times \mathbf{B}) dl. \tag{29}$$

If $\mathbf{I}$ and $d\mathbf{l}$ point in the same direction,

$$\mathbf{F}_{mag} = \int I(d\mathbf{l} \times \mathbf{B}). \tag{30}$$

4 Multipole Expansion

4.1 The Dipole Vector Potential

The dipole term in the expansion of the magnetic vector potential is

$$\mathbf{A}_{dipole}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3}, \tag{31}$$

where the magnetic dipole moment is given by

$$\mathbf{m} = \frac{1}{2}\int(\mathbf{r}\times\mathbf{J})d^3r. \tag{32}$$

For a loop of current, the dipole moment is given by

$$\mathbf{m} = I\int d\mathbf{a} = IA\hat{\mathbf{n}}, \tag{33}$$

where A is the area of the surface enclosed by the loop and $\hat{\mathbf{n}}$ is the unit vector perpendicular to the surface enclosed by the loop. The direction of this normal unit vector is determined by the right hand rule.

If we place our dipole at the origin and choose the coordinate system such that the dipole moment points in the z-direction, then we can express the dipole vector potential in spherical coordinates:

$$\mathbf{A}_{dipole}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}}. \tag{34}$$

4.2 The Dipole Magnetic Field

The magnetic field of a dipole is given by

$$\mathbf{B}_{dipole}(\mathbf{r}) = \nabla\times\mathbf{A}_{dipole}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}]. \tag{35}$$

If we place our dipole at the origin and choose the coordinate system such that the dipole moment points in the z-direction, then we can express the dipole magnetic field in spherical coordinates:

$$\mathbf{B}_{dipole}(\mathbf{r}) = \nabla\times\mathbf{A}_{dipole}(\mathbf{r}) = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\boldsymbol{\theta}}). \tag{36}$$

5 Magnetic Fields in Matter

5.1 Force and Torque on Dipole

The force on a dipole placed in a magnetic field is given by

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B}). \tag{37}$$

The torque on a dipole placed in a magnetic field is

$$\mathbf{N} = \mathbf{m}\times\mathbf{B}. \tag{38}$$

The dipole will experience this torque and will want to align it's dipole moment with the direction of the field.

5.2 Bound Currents

Consider an object with magnetic dipole moment per unit volume, $\mathbf{M}$. The potential of this magnetized object is the same as one produced by a volume current

$$\mathbf{J}_b = \nabla \times \mathbf{M} \tag{39}$$

and a surface current

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}. \tag{40}$$

The potential produced by these bound currents is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\int_{\mathcal{V}} \frac{\mathbf{J}_b(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r' + \oint_{\mathcal{S}} \frac{\mathbf{K}_b(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} da'\right]. \tag{41}$$

5.3 Maxwell's Equations for Magnetostatics in Matter

We can write the total current as

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f. \tag{42}$$

Then we can rewrite Ampere's Law in differential form as

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J} = \mathbf{J}_b + \mathbf{J}_f = \mathbf{J}_f + (\nabla \times \mathbf{M}). \tag{43}$$

Collecting the two curls terms together we get

$$\nabla \times \left(\frac{1}{\mu_0}\mathbf{B} - \mathbf{M}\right) = \nabla \times \mathbf{H} = \mathbf{J}_f, \tag{44}$$

where we've defined the auxiliary field as

$$\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}. \tag{45}$$

In integral form, Ampere's Law is

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{f,encl}. \tag{46}$$

5.4 Boundary Conditions in Matter

The boundary conditions for the auxiliary field are

$$H^{\perp}_{above} - H^{\perp}_{below} = -(M^{\perp}_{above} - M^{\perp}_{below}) \tag{47}$$

$$\mathbf{H}^{\parallel}_{above} - \mathbf{H}^{\parallel}_{below} = \mathbf{K}_f \times \hat{\mathbf{n}}. \tag{48}$$

5.5 Linear Media

Linear materials obey the equation

$$\mathbf{M} = \chi_m \mathbf{H}, \tag{49}$$

where χ_m is the magnetic susceptibility. Then from Eq. 45,

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1+\chi_m)\mathbf{H} = \mu\mathbf{H}, \tag{50}$$

where we've defined the permeability

$$\mu \equiv \mu_0(1+\chi_m). \tag{51}$$

We can also relate the volume bound current density to the free current density:

$$\mathbf{J}_b = \nabla \times \mathbf{M} = \mathbf{J}_b = \nabla \times (\chi_m \mathbf{H}) = \chi_m \mathbf{J}_f. \tag{52}$$