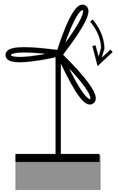
Melbourne School of Engineering
MCEN90018 Advanced Fluid Dynamics

Lecture BL01: Overview

1 March 2016

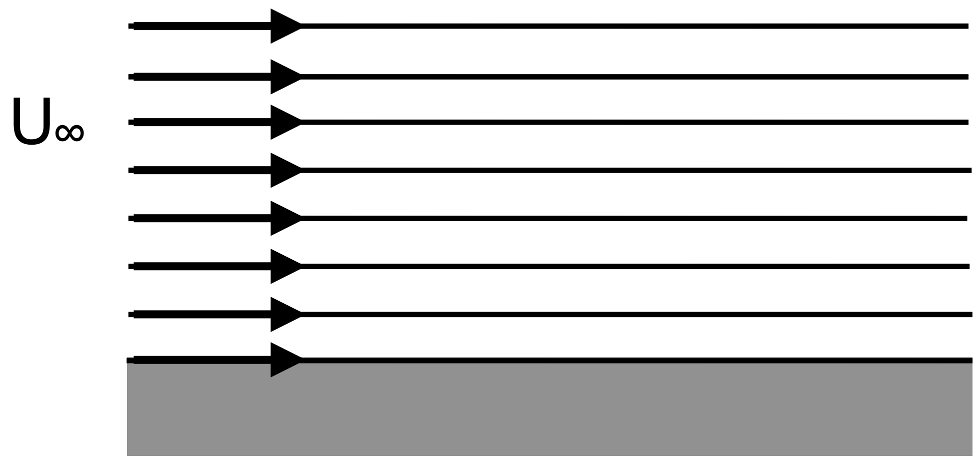
Advanced Fluid Dynamics. Why?



(Gharib 2009)

Advanced Fluid Dynamics. Why?

Potential flow $\psi = U_\infty y$



Advanced Fluid Dynamics. Why?

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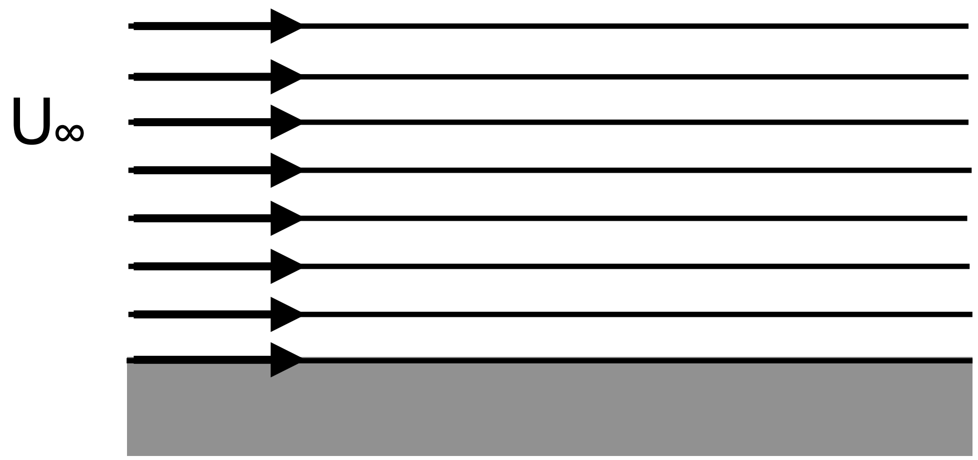
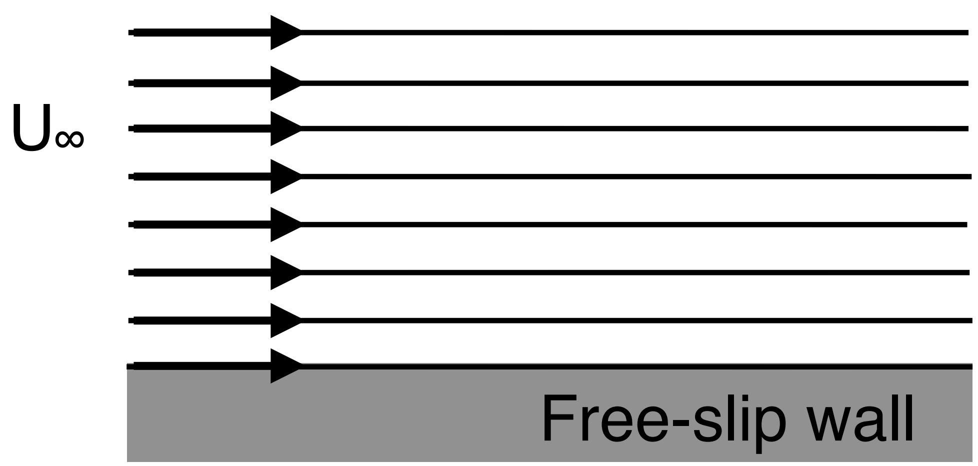


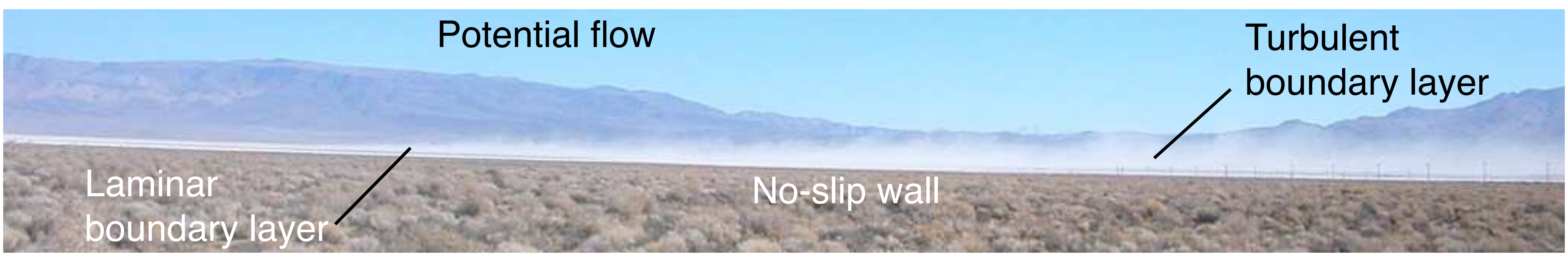
Real flow

Advanced Fluid Dynamics. Why?

Potential flow $\psi = U_\infty y$



Real flow



Advanced Fluid Dynamics. Why?

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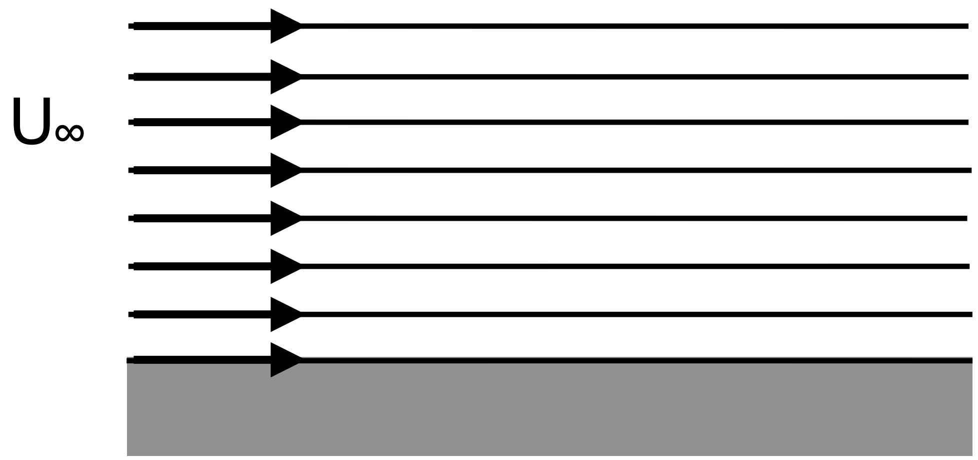
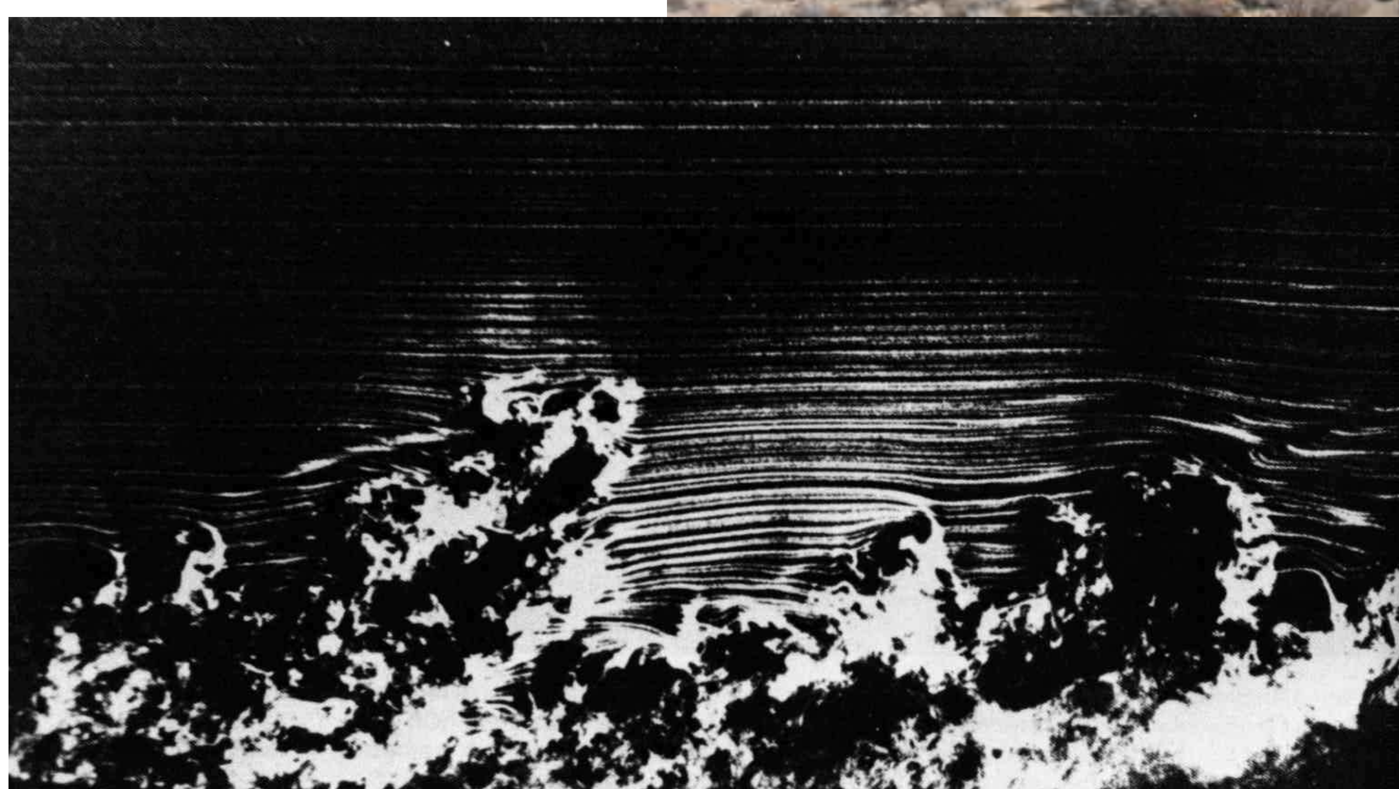


Real flow



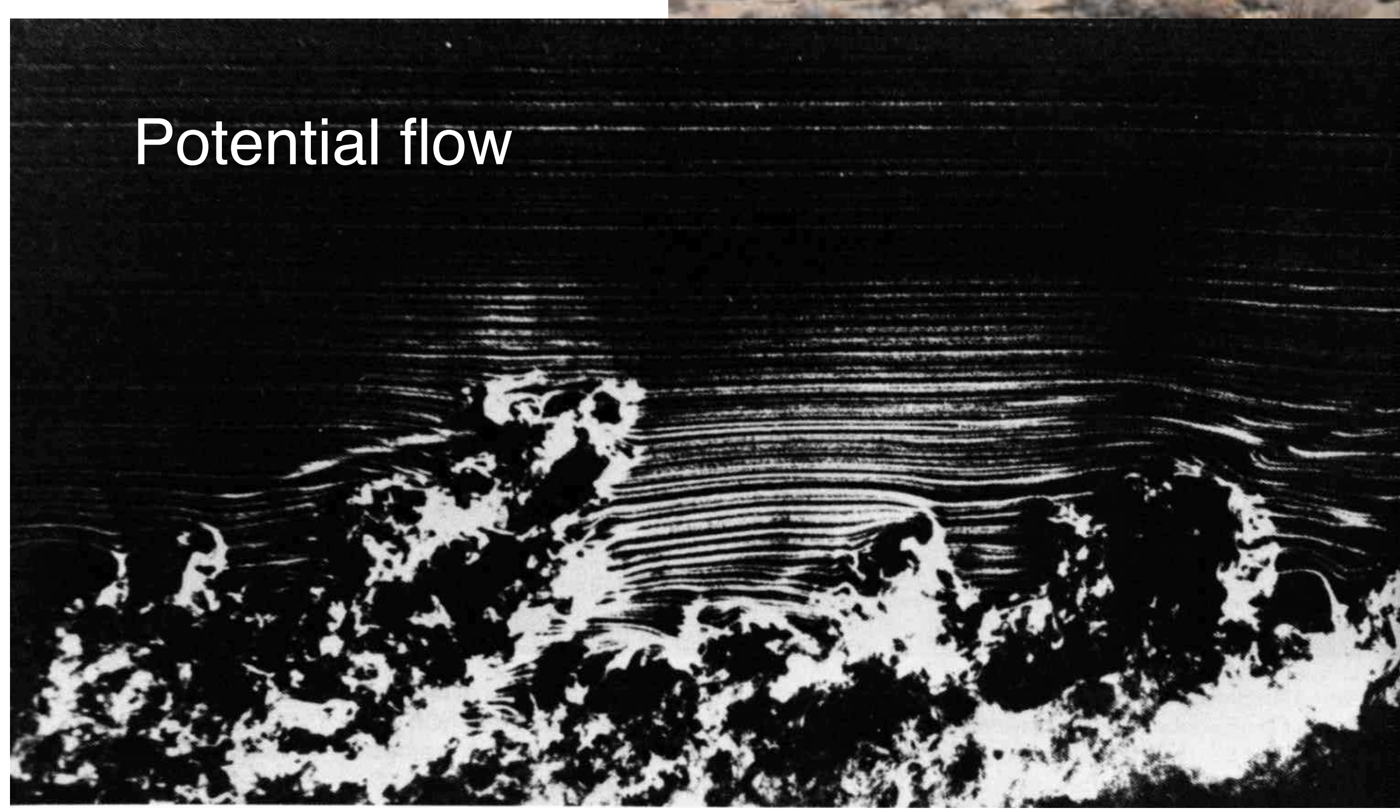
(Van Dyke 1982)

Advanced Fluid Dynamics. Why?

Potential flow $\psi = U_\infty y$

U_∞

Real flow

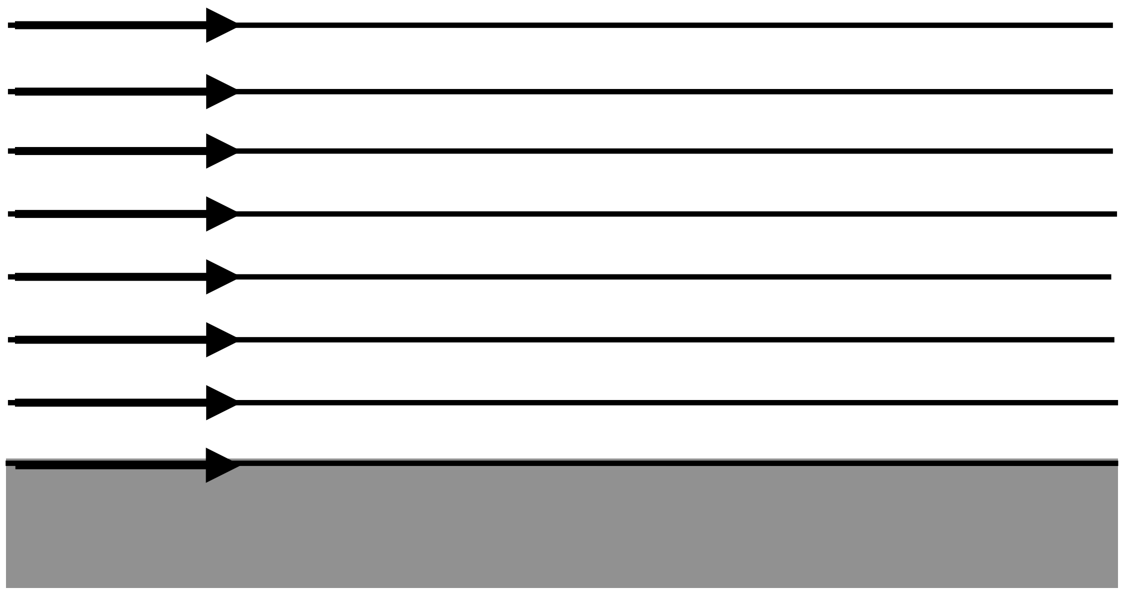


Turbulent boundary layer

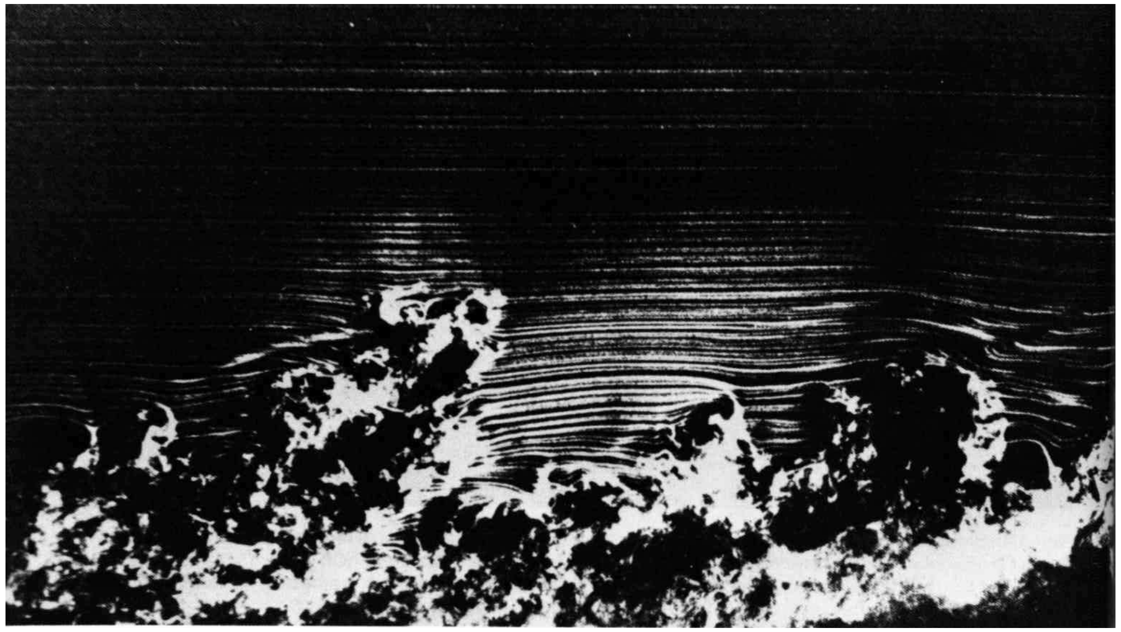
No-slip wall

Advanced Fluid Dynamics. Why?

Potential flow



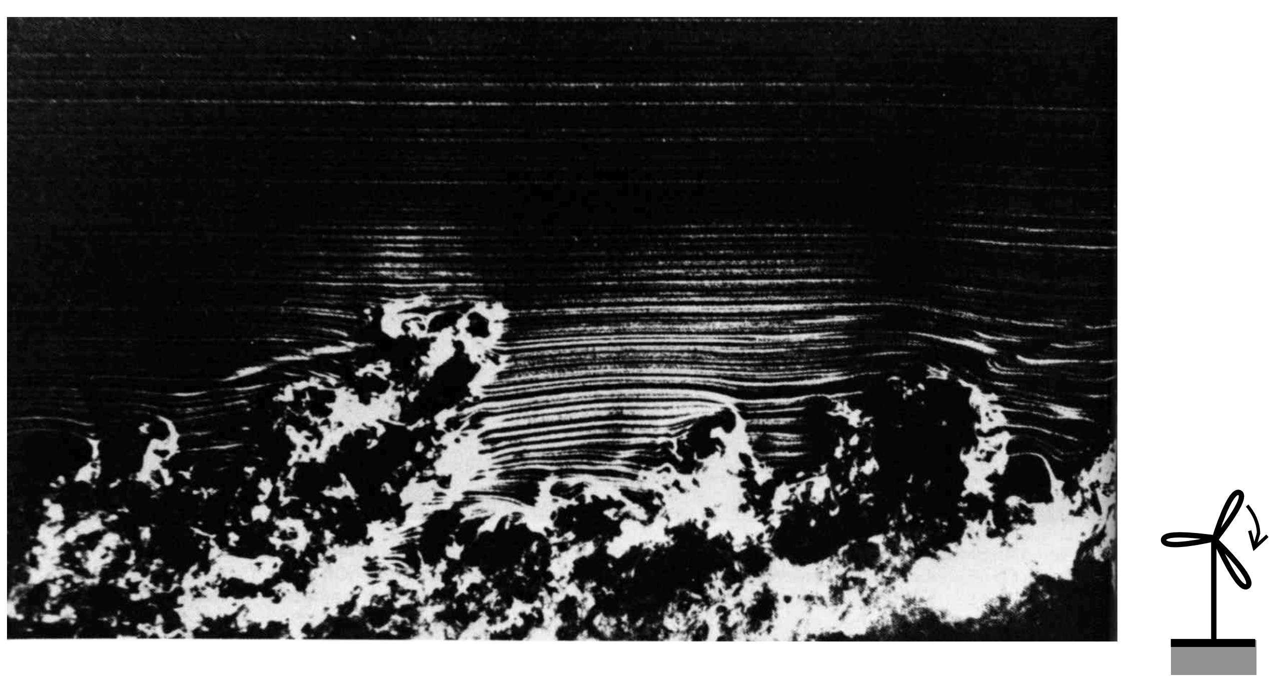
Real flow



Advanced Fluid Dynamics. Why?

Potential flow

Real flow

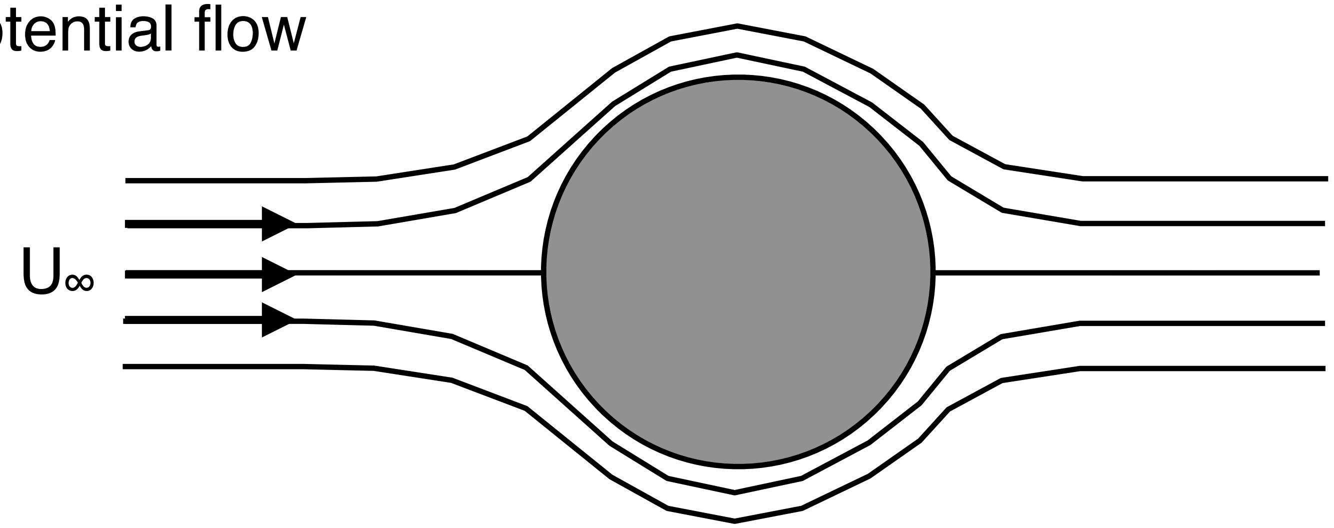


Compare design considerations for the two situations.

- Wind engineering (A Webb)

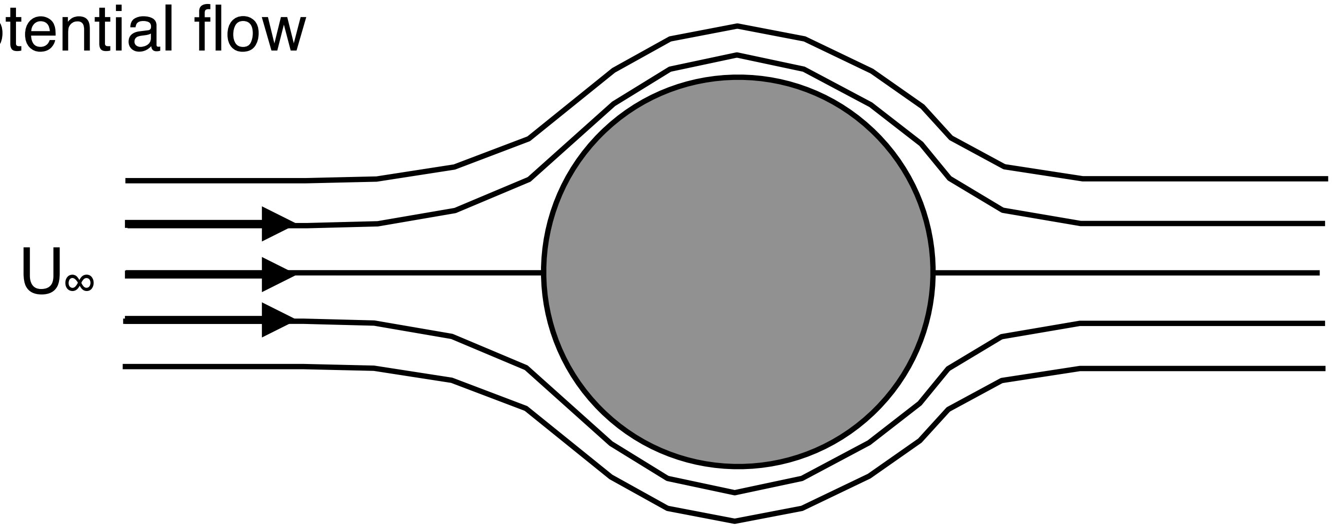
Advanced Fluid Dynamics. Why?

Potential flow



Advanced Fluid Dynamics. Why?

Potential flow



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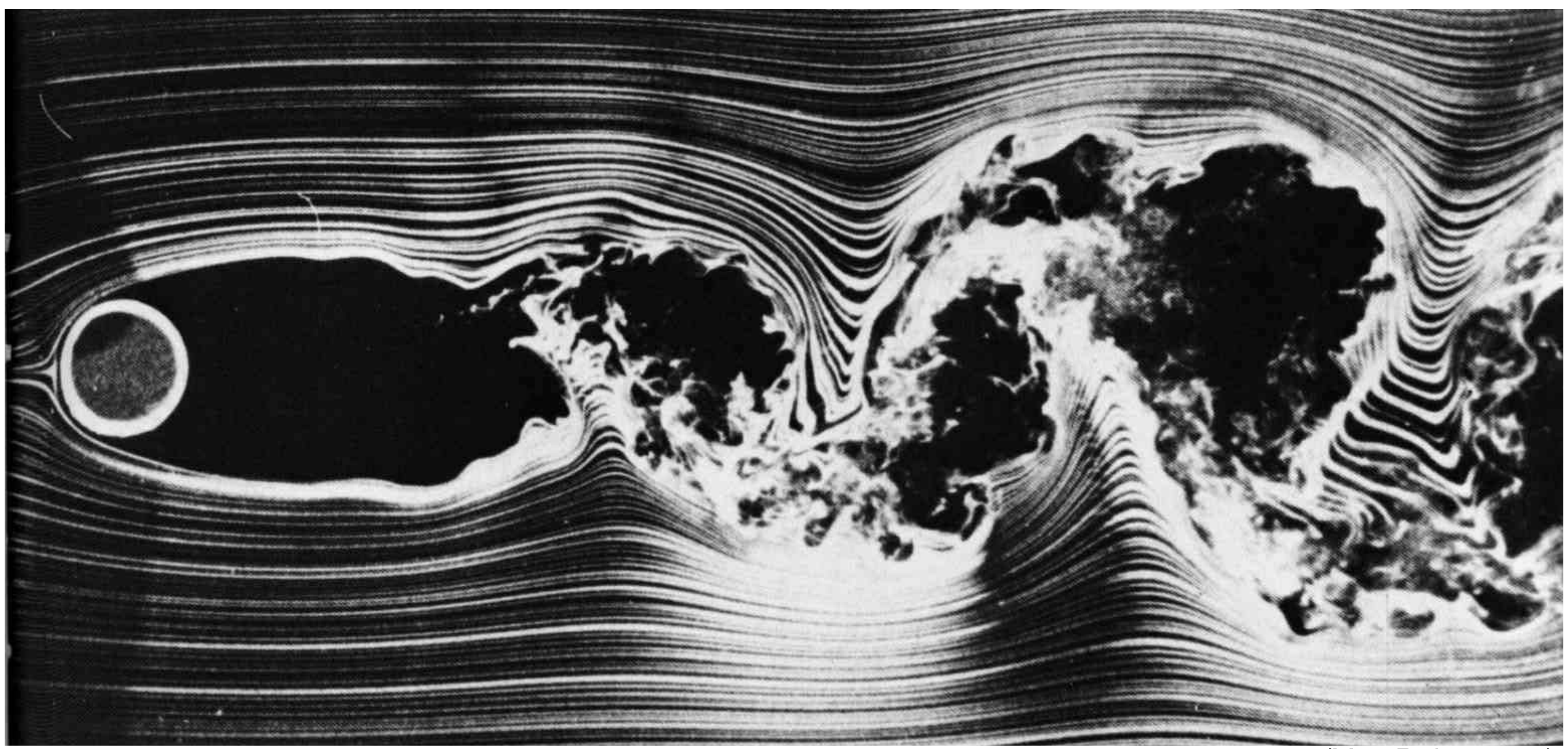
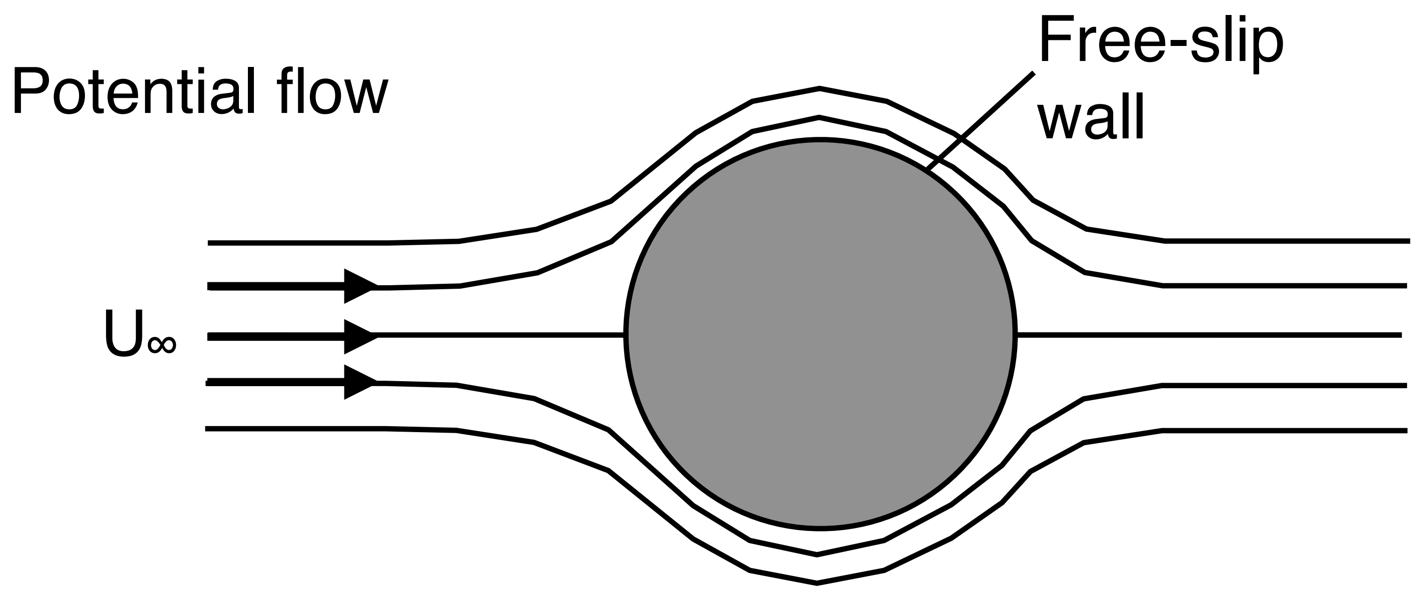
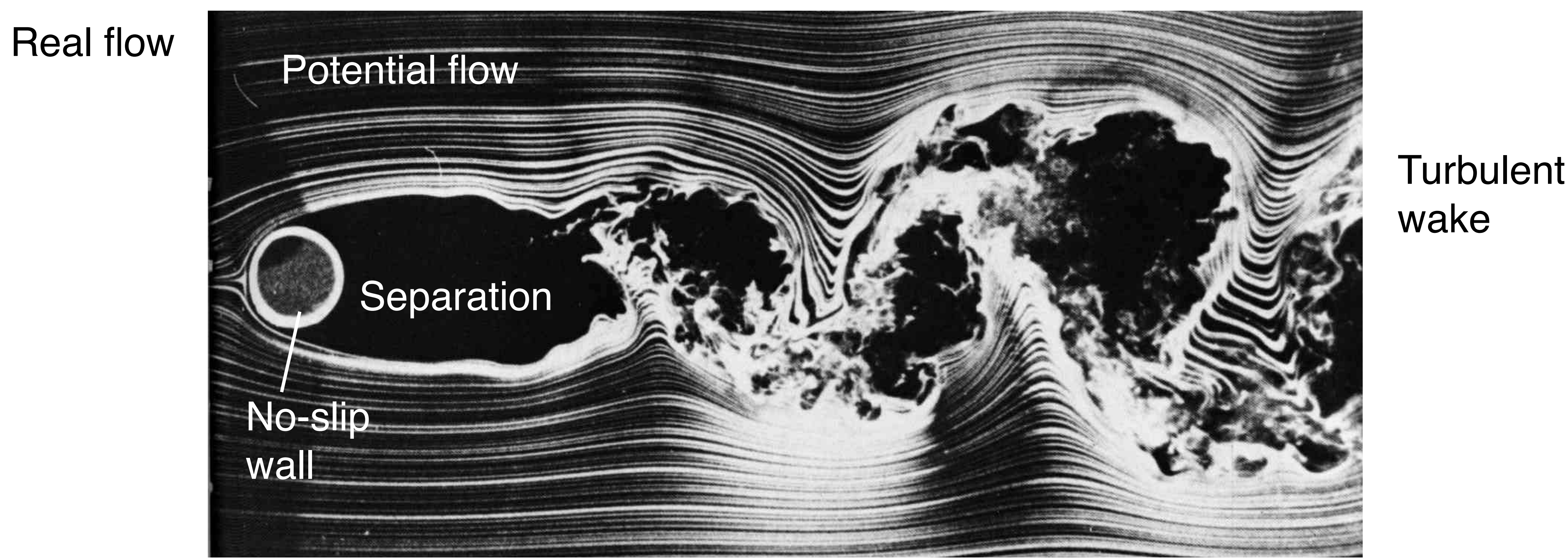


(Van Dyke 1982)

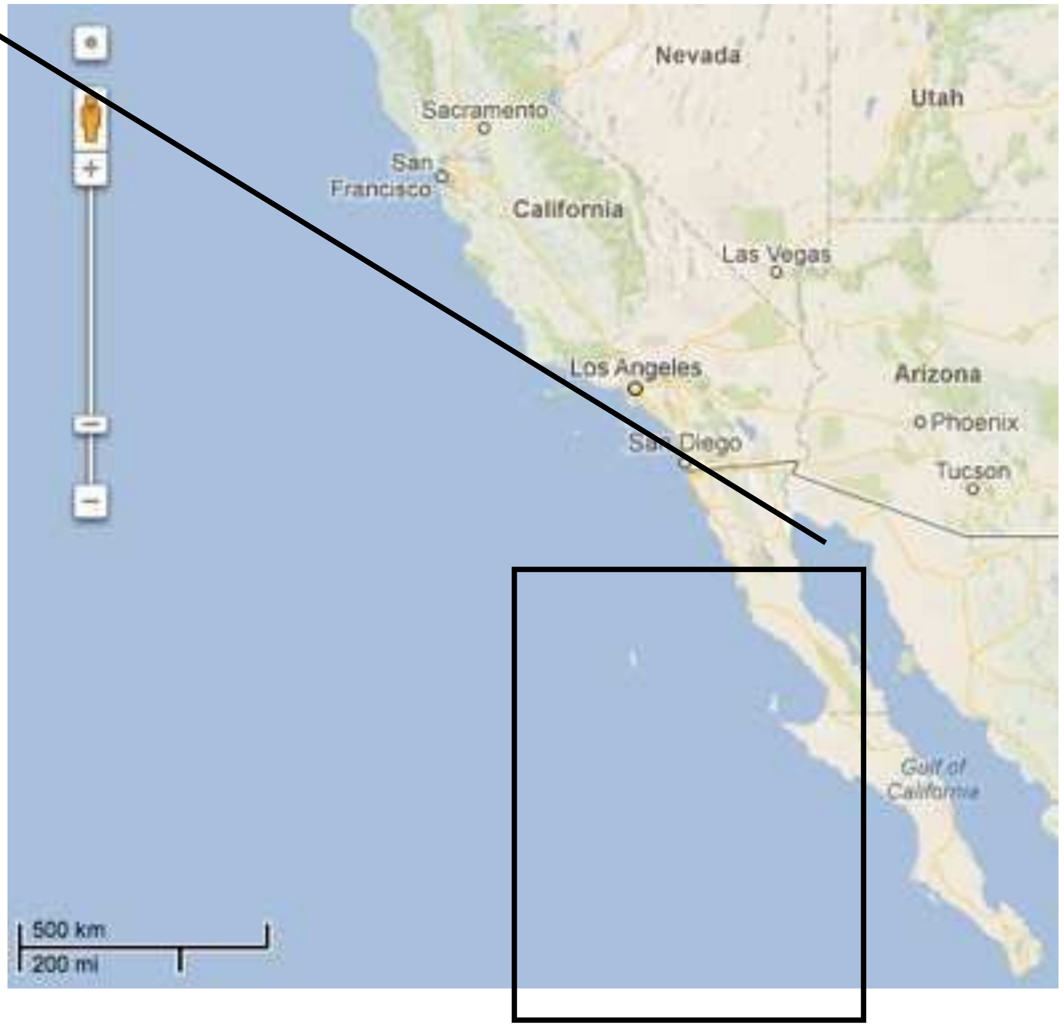
Advanced Fluid Dynamics. Why?





Advanced Fluid Dynamics. Why?

(http://visibleearth.nasa.gov/view.php?id=59600)



3 June 2002,
Guadalupe Island,
Mexico

Advanced Fluid Dynamics. Why?

Wait, don’t throw the baby out with the bath water!

- Potential flow is very good except when it’s not and must therefore be used with care.

- Design can be exploited to follow potential flow.

- Panel methods (N Hutchins)

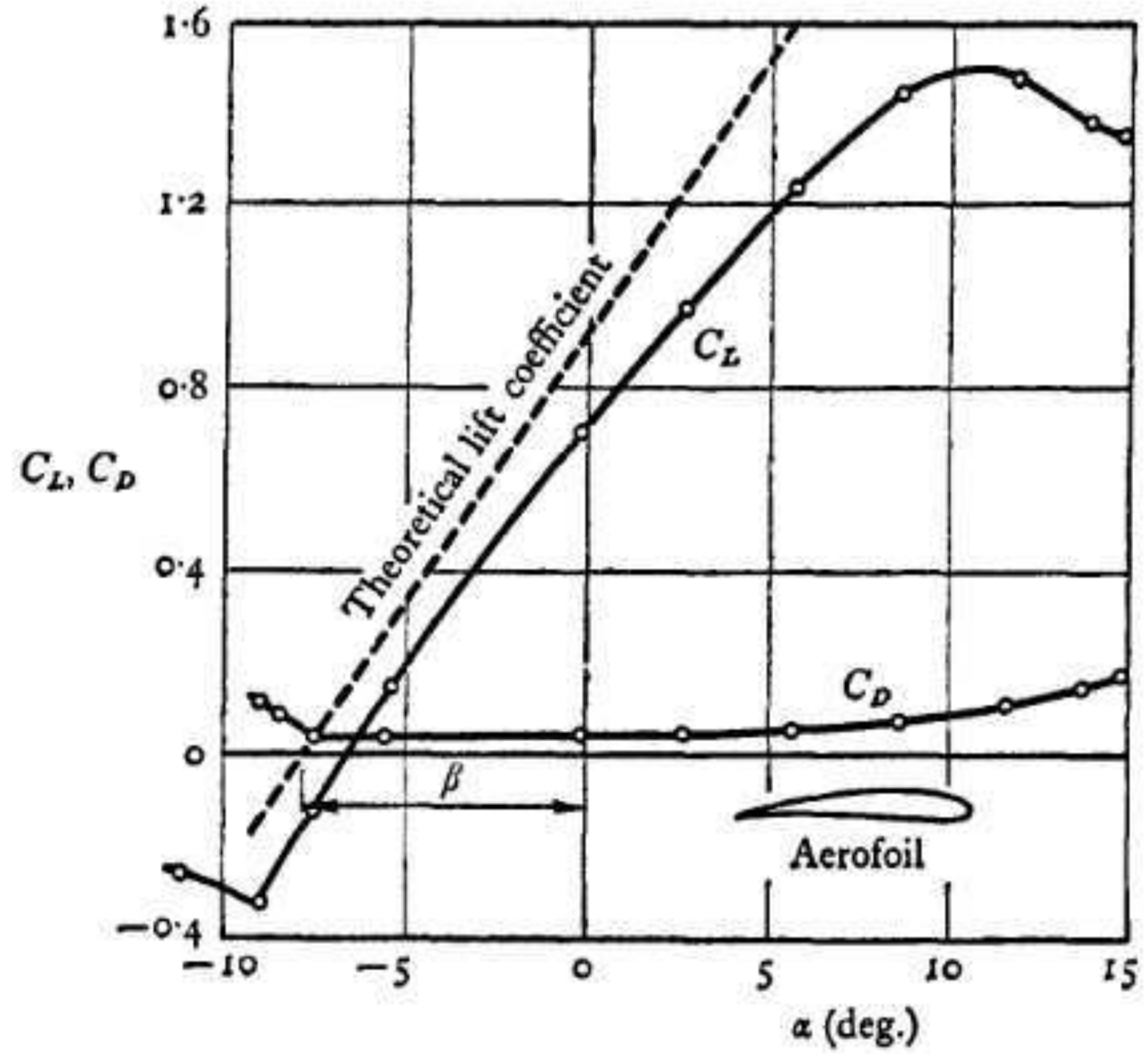
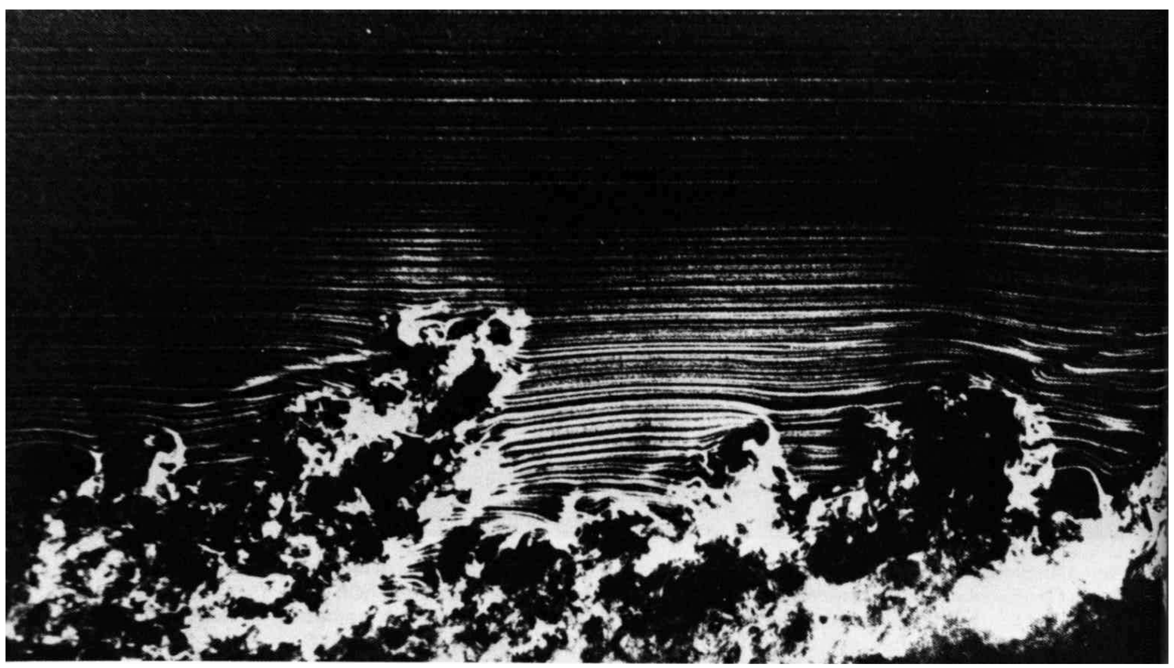


Figure 6.7.10. Observed values of the lift and drag on a cambered Joukowski aerofoil as a function of angle of incidence. (From Betz 1915.)

(Batchelor 1967)

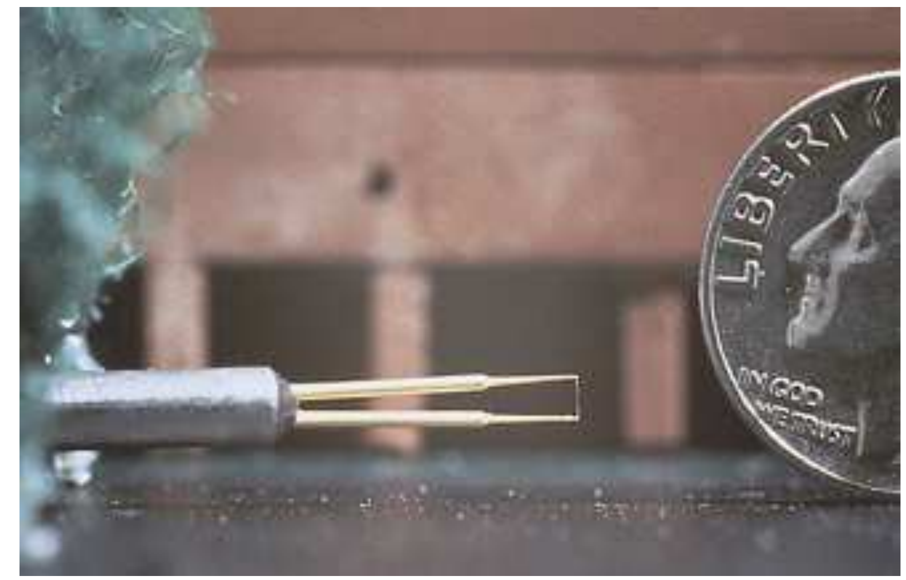
Advanced Fluid Dynamics. Why?

Understanding a real flow
requires measurement skills.



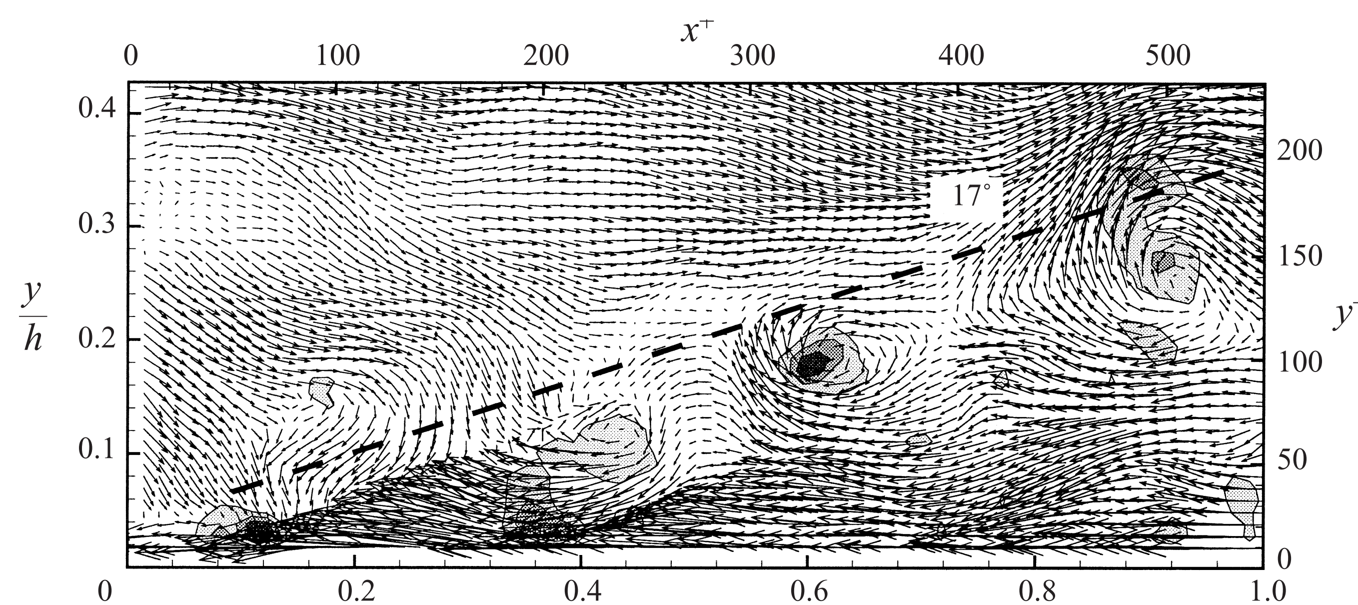
- Experimental methods (N Hutchins)

Hot-wire anemometry



(http://www.aawe.org/images/gallery/
Hotwire(with%20Dime).jpg)

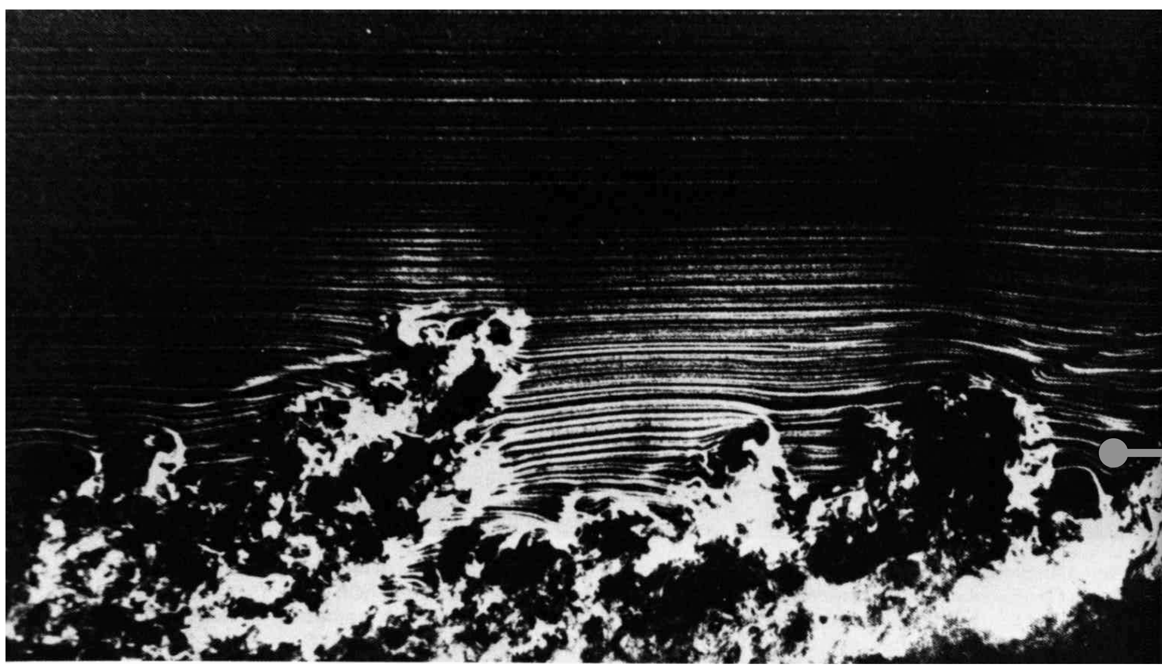
Particle image velocimetry



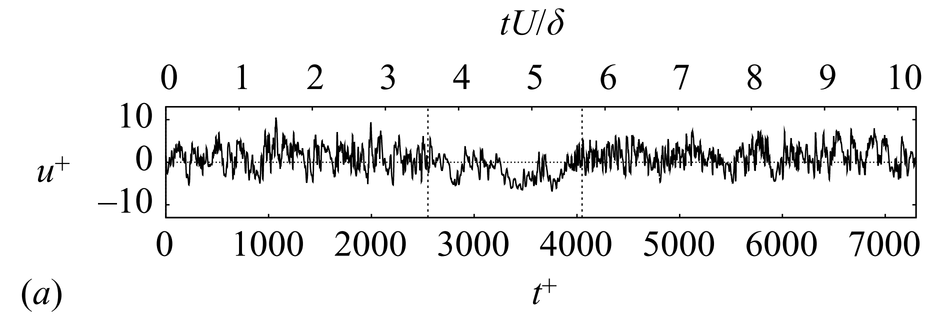
(Christensen & Adrian 2001)

Advanced Fluid Dynamics. Why?

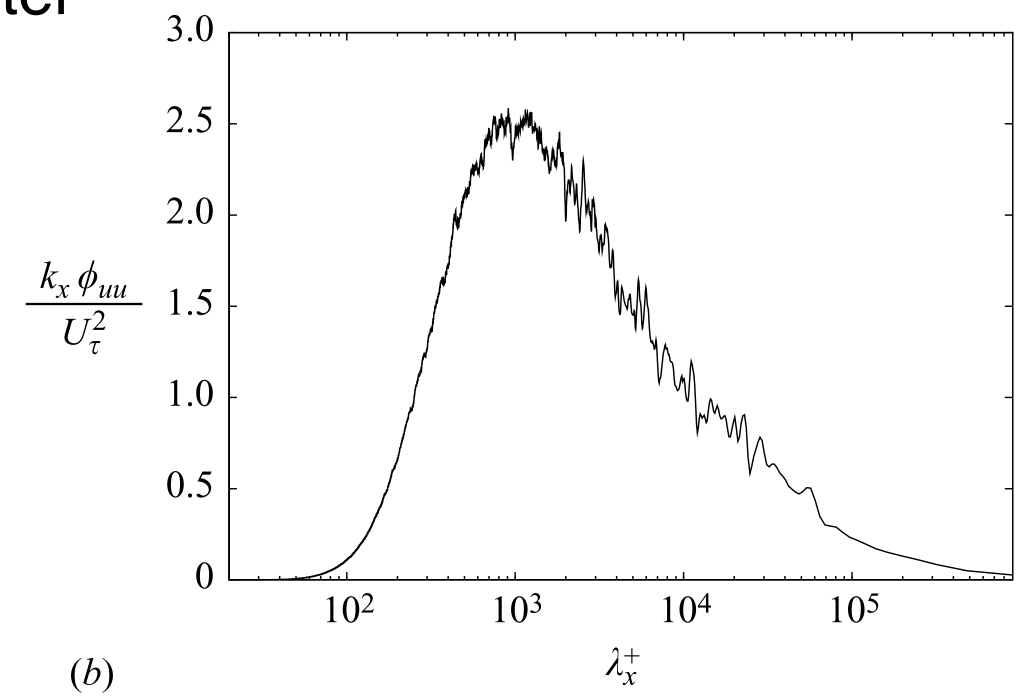
Understanding a real flow requires measurement skills.



Hot-wire anemometer



Fourier analysis

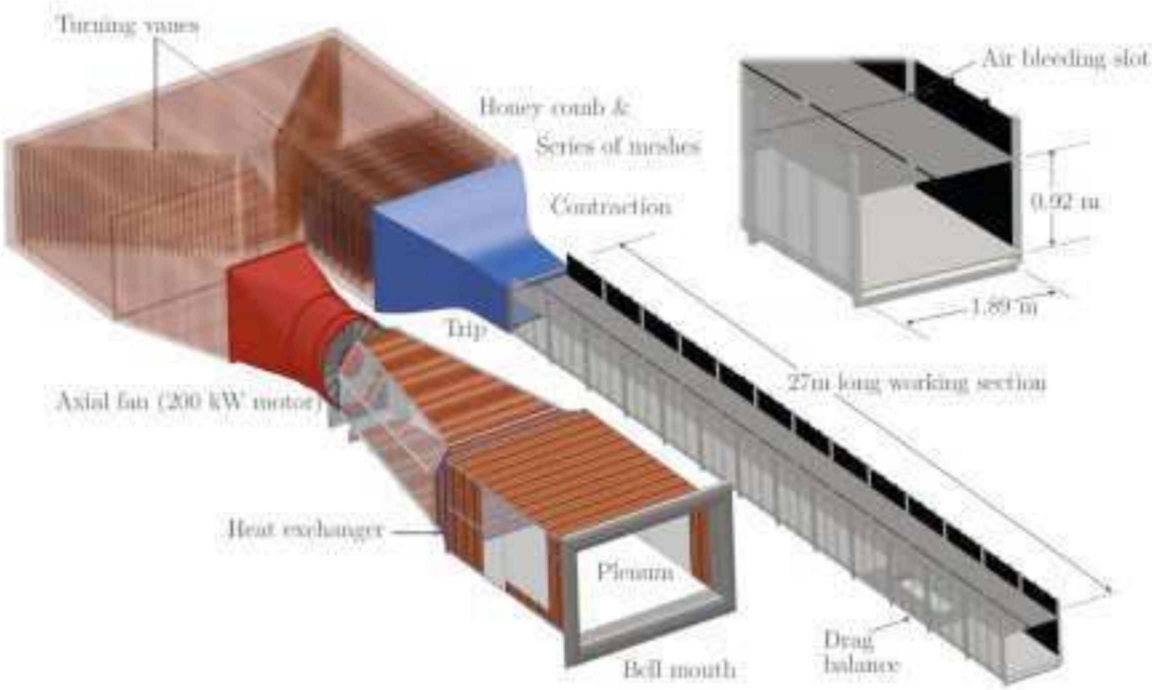


(Mathis *et al.* 2009)

Advanced Fluid Dynamics. Why?

Melbourne University: the “tastemaker” in turbulent boundary layer research

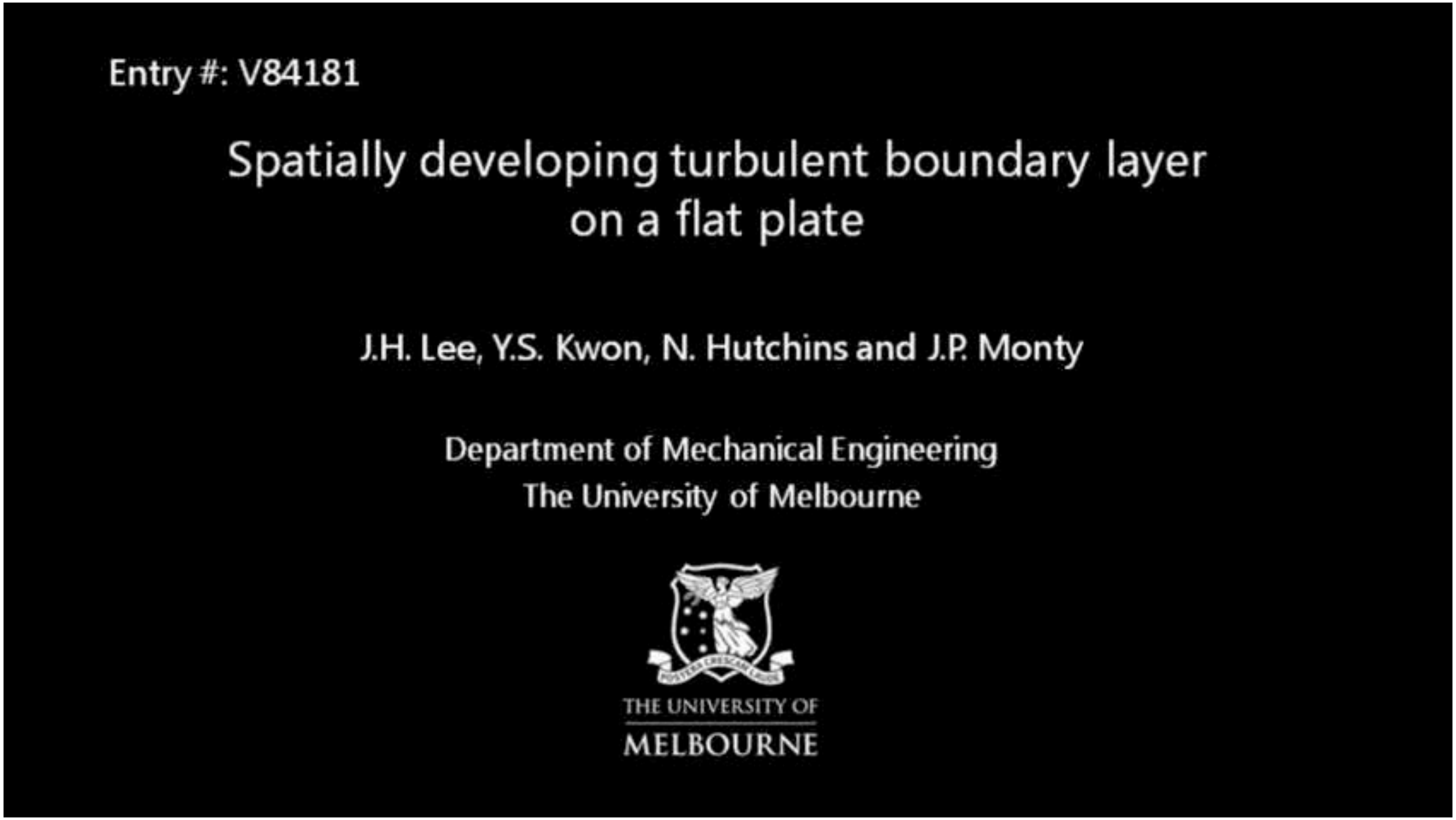
University of Melbourne’s
High Reynolds Number Boundary Layer
Wind Tunnel, 27m working section
(1st floor block D)



University of New Hampshire’s Flow
Physics Facility, 72m working section
(Prof Joe Klewicki, also at Melbourne)

Advanced Fluid Dynamics. Why?

Melbourne University: the “tastemaker” in turbulent boundary layer research



University of Melbourne’s Michell lab tow tank, 60m (2nd floor Block C)

(http://arxiv.org/abs/1210.3881)

Advanced Fluid Dynamics. Why?

Melbourne University: the “tastemaker” in turbulent boundary layer research

(Kozul 2015)

Navier–Stokes solved on high-performance computers by Melbourne University researchers

(http://nci.org.au/nci-systems/national-facility/peak-system/raijin/)

Advanced Fluid Dynamics

Unit 1. Boundary layers and turbulence (50%, assessed in exam)

Aim is to learn how, why and when the boundary layer is important in practical applications and how to do some ‘simple’ calculations taking it into account.

Unit 2. Experimental methods (50%, assessed in assignments/ lab reports)

Aim is to learn key concepts in modern experimental and numerical techniques centred around particle-image velocimetry and hot-wire anemometry.

Advanced Fluid Dynamics: The Rules

Lectures
Assessment
Previous feedback
Labs and assignments
Tutorials
Lecture notes
Textbooks
Schedule

- Topics: see schedule
- Written notes for BL lectures will be recorded on the document camera
- BL lectures (D Chung) assessed in exam
- EX lectures (N Hutchins) assessed in assignments/lab reports
- Guest lectures

Advanced Fluid Dynamics: The Rules

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50% two-hour exam (boundary layers and turbulence)
10% lab (particle image velocimetry PIV)
10% lab (hot wire anemometry HWA)
10% assignment (image analysis)
10% assignment (signal analysis)
10% assignment (panel methods)

How to do well in the exam?

- Do all past exam questions	P
- Do all tutorial questions	H3
- Understand derivations in lectures	H2
- Extra reading	H1

Advanced Fluid Dynamics: The Rules

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- We like labs
- We like guest lecturers (see me if you want to attend the Fluids Research Seminars)
- We like tutors
- We like partial solutions
- We like hand-written BL lectures
- Questions in assignments are unrelated to the exam (Yes, because experimental methods is half of the subject and assignments/lab reports are how you are assessed.)
- Teamwork is hard (Yes, because you should learn to manage it as part of your accredited education. Choose your teammates wisely)
- ☠ "...coding, coding, coding. I despise MATLAB. Nightmare..." (Drop the class. 5 MATLAB help sessions, 3 in the week when assignments are due. Bring your laptop.)
- Assignments are difficult (Yes, because you should spend 13−15 hours per 10% assignment/lab report according to accreditation. 2 people = 26−30 hours)
- Want assignments/lab reports back sooner (OK)

Advanced Fluid Dynamics: The Rules

Lectures
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Schedule

- 50% of assessment
- Lab class in groups of 5 or 6
- Sign up to lab class
- Attendance at lab class is compulsory
- 2 reports per lab session (i.e. groups of 2 or 3)
- Assignments in groups of 2
- No plagiarism or copying but free to discuss results
- Lab coordinators: Dougie Squire, Kevin Kevin, Dileep Chandran

Advanced Fluid Dynamics: The Rules

Lectures
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Previous feedback
Labs and assignments
Tutorials
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Textbooks
Schedule

- 6 tutorial sheets
- Attend 6 sessions
- Tutors: Dougie Squire, Kevin Kevin
- 3 repeat sessions (about 25 students each)
- Excellent exam preparation
- No filming please

Advanced Fluid Dynamics: The Rules

Lectures
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Textbooks
Schedule

- Prof Chong's lecture notes are on the LMS
- This year's slides will be on the LMS after the lecture
- Previous year's slides will be on the LMS prior to lecture

Advanced Fluid Dynamics: The Rules

Lectures
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Schedule

- No textbook to buy, but the list given in Prof Chong's lecture notes is a good place to start for further reading
- Bedtime reading will be recommended from time to time and it is fun to check those out
- Google 'FYFD'

Advanced Fluid Dynamics: The Rules

Lectures
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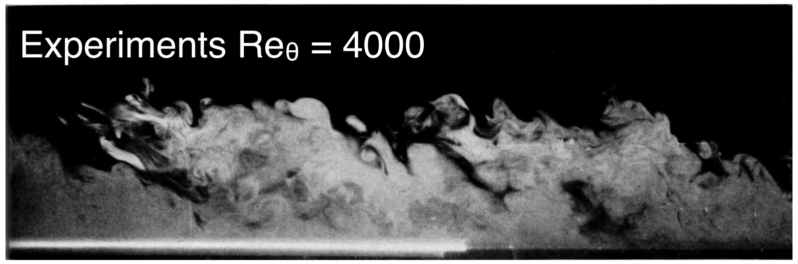
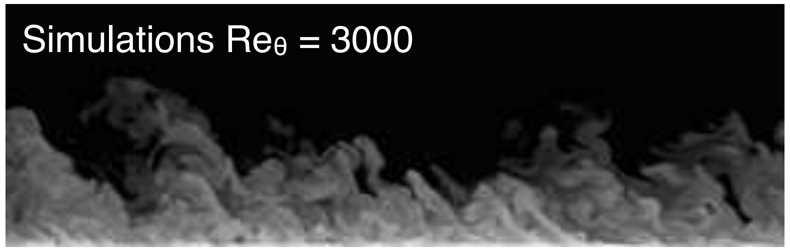
Schedule

- On the LMS under ‘Subject Information’
- Subject to changes

Navier–Stokes equations for incompressible flow



(Van Dyke 1982)



(Kozul 2015)

$\rho = \text{const.}$

Conservation of mass:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

Rate of change of x-momentum = sum of x-forces:

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu)}{\partial y} + \frac{\partial(\rho wu)}{\partial z} = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

Rate of change of y-momentum = sum of y-forces:

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho vv)}{\partial y} + \frac{\partial(\rho wv)}{\partial z} = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

Rate of change of z-momentum = sum of z-forces:

$$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho uw)}{\partial x} + \frac{\partial(\rho vw)}{\partial y} + \frac{\partial(\rho ww)}{\partial z} = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Impermeable, no-slip boundary conditions at wall:

$u = v = w = 0$

Navier–Stokes equations for incompressible flow: 2 components (of velocity) and 2 dimensions

$\rho = \text{const.}$

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Rate of change of x-momentum = sum of x-forces:

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u u)}{\partial x} + \frac{\partial(\rho v u)}{\partial y} = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

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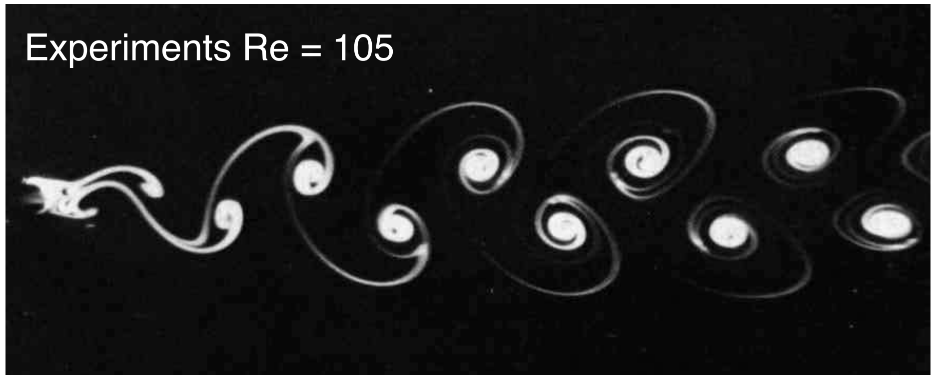
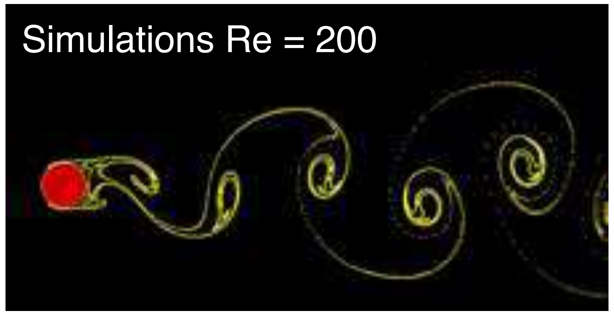


(Van Dyke 1982)



(http://www.lstm.uni-erlangen.de/projekt/breuer/vortex_street.gif)

Navier–Stokes equations for incompressible flow: 2 components (of velocity) and 2 dimensions

$\rho = \text{const.}$

Conservation of mass:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

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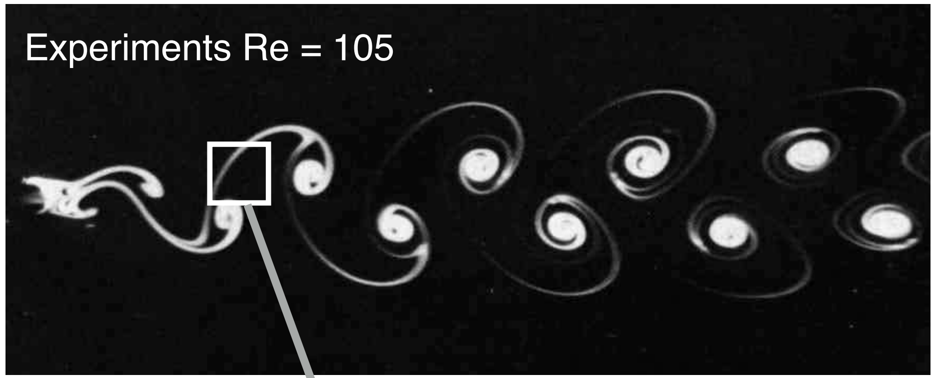
Rate of change of y-momentum = sum of y-forces:

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho u v)}{\partial x} + \frac{\partial(\rho v v)}{\partial y} = -\frac{\partial p}{\partial y} + \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2})$$

Impermeable, no-slip boundary conditions at wall:

$u = v = 0$



(Van Dyke 1982)

Stationary control volume

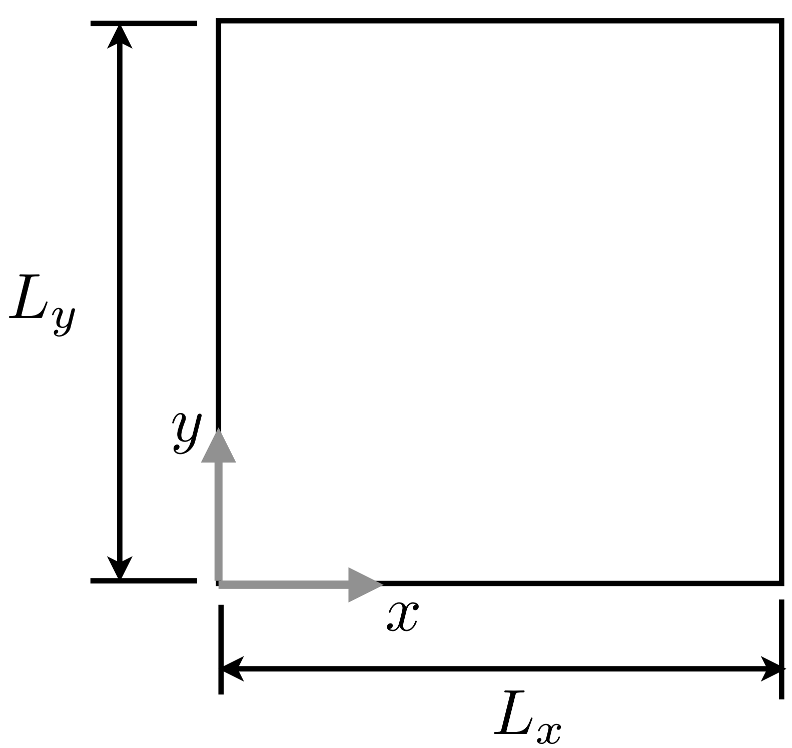
To better understand these equations, consider a stationary control volume.

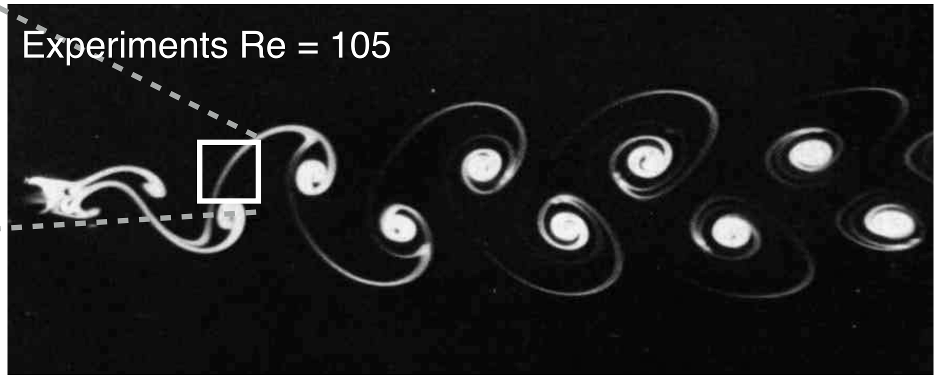
Mass equation

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

Integrate over an arbitrary rectilinear stationary control volume:

$$\int_0^{L_y}\int_0^{L_x} \frac{\partial(\rho u)}{\partial x}\,\mathrm{d}x\,\mathrm{d}y + \int_0^{L_y}\int_0^{L_x} \frac{\partial(\rho v)}{\partial y}\,\mathrm{d}x\,\mathrm{d}y = 0$$



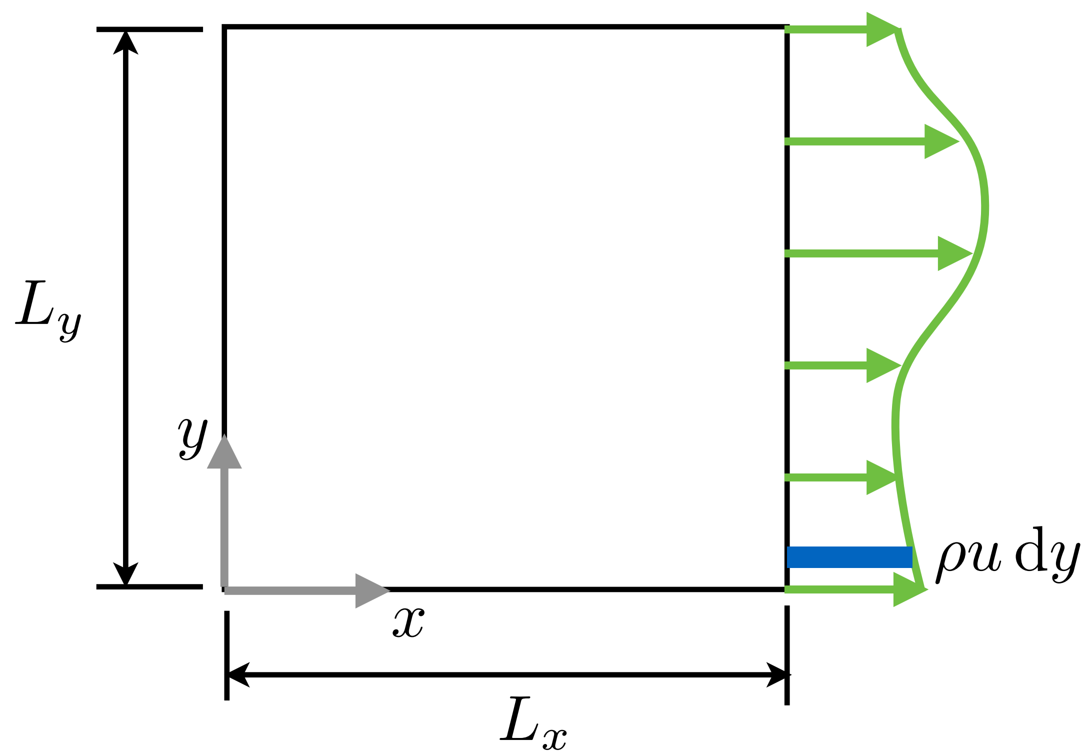


(Van Dyke 1982)

Mass equation

Right outflow of mass

$$\boxed{\left[\int_0^{L_y} \rho u \,\mathrm{d}y\right]_{x=L_x}} - \left[\int_0^{L_y} \rho u \,\mathrm{d}y\right]_{x=0} + \left[\int_0^{L_x} \rho v \,\mathrm{d}x\right]_{y=L_y} - \left[\int_0^{L_x} \rho v \,\mathrm{d}x\right]_{y=0} = 0$$



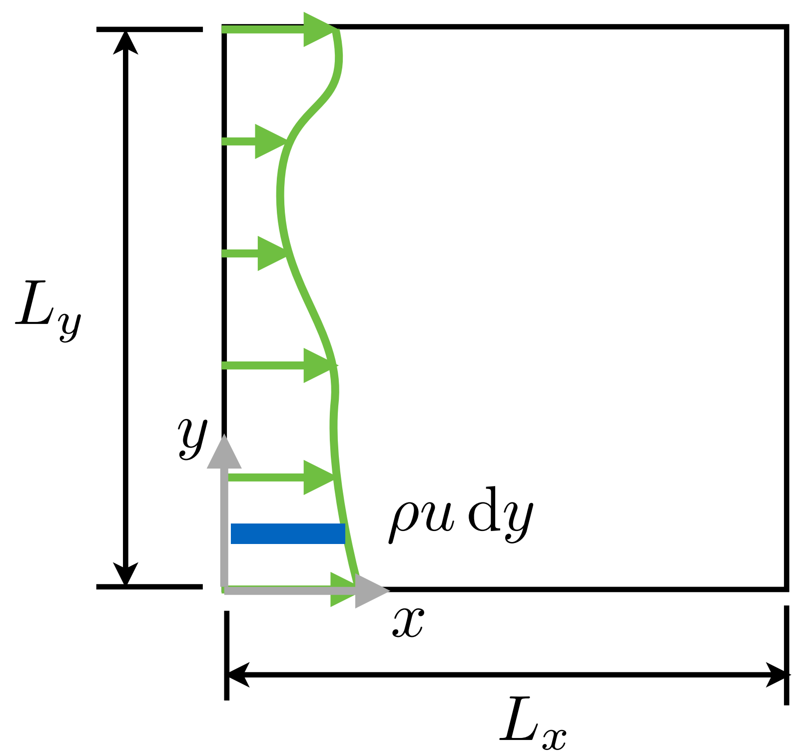
ρudy has the SI units,
$(\mathrm{kg/m^3})(\mathrm{m/s})(\mathrm{m}) = (\mathrm{kg/s})/\mathrm{m}$,
i.e. mass flow rate per unit depth.

So the integral from $y = 0$ to L_y represents the mass flow rate across the edge of the control volume.

Mass equation

Left inflow of mass

$$\left[\int_0^{L_y} \rho u \, \mathrm{d}y\right]_{x=L_x} \boxed{- \left[\int_0^{L_y} \rho u \, \mathrm{d}y\right]_{x=0}} + \left[\int_0^{L_x} \rho v \, \mathrm{d}x\right]_{y=L_y} - \left[\int_0^{L_x} \rho v \, \mathrm{d}x\right]_{y=0} = 0$$



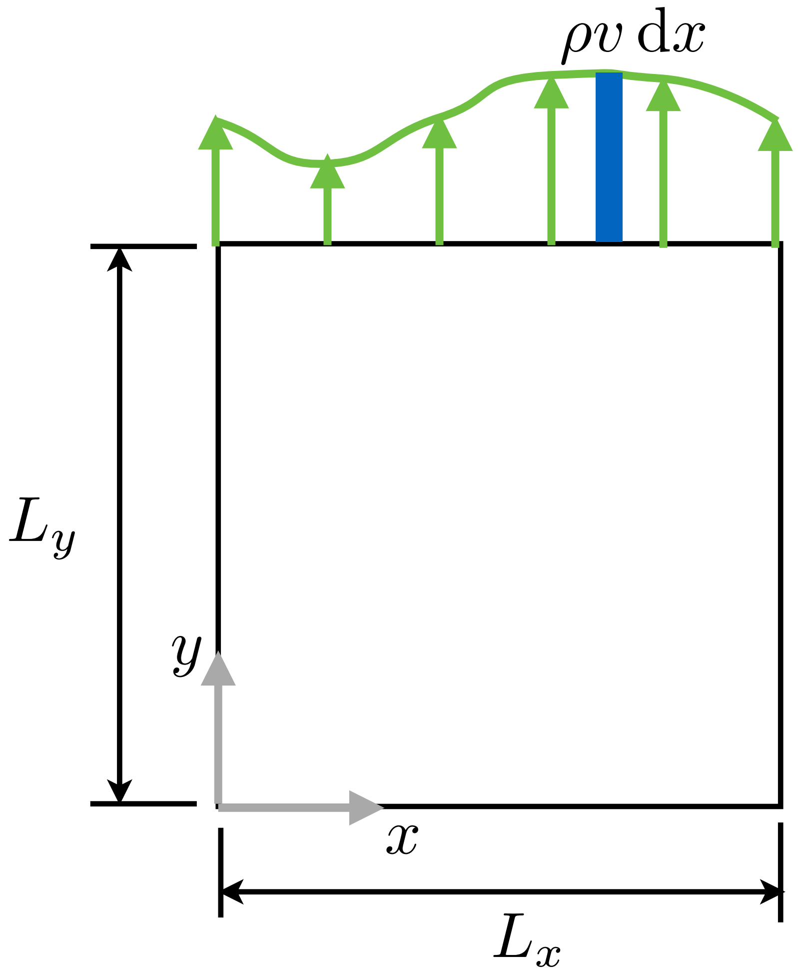
ρudy has the SI units,
$(kg/m^3)(m/s)(m) = (kg/s)/m$,
i.e. mass flow rate per unit depth.

So the integral from $y = 0$ to L_y represents the mass flow rate across the edge of the control volume.

Mass equation

Top outflow of mass

$$\left[\int_0^{L_y} \rho u \,\mathrm{d}y\right]_{x=L_x} - \left[\int_0^{L_y} \rho u \,\mathrm{d}y\right]_{x=0} + \boxed{\left[\int_0^{L_x} \rho v \,\mathrm{d}x\right]_{y=L_y}} - \left[\int_0^{L_x} \rho v \,\mathrm{d}x\right]_{y=0} = 0$$



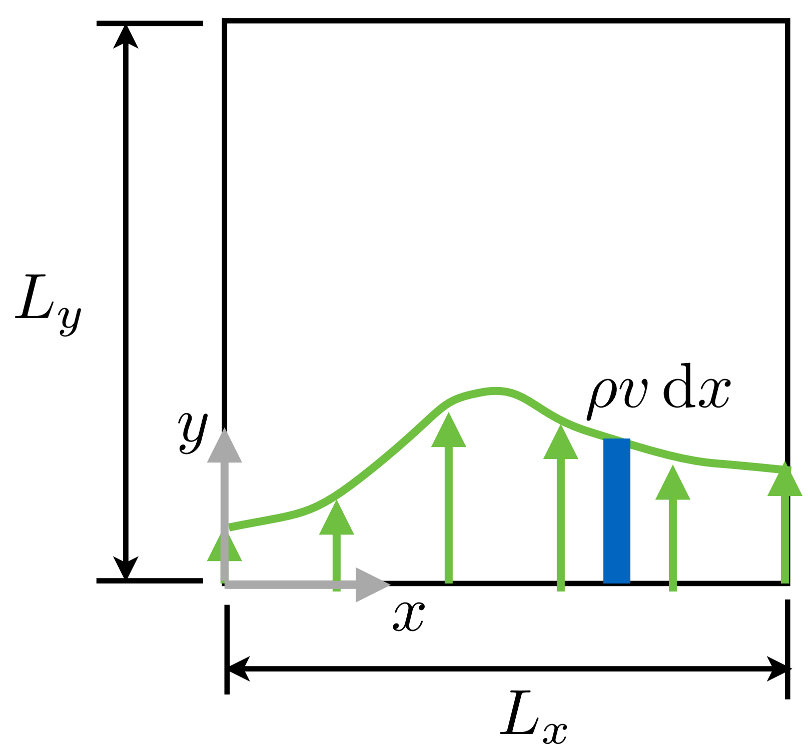
ρvdx has the SI units,
$(\mathrm{kg/m^3})(\mathrm{m/s})(\mathrm{m}) = (\mathrm{kg/s})/\mathrm{m}$,
i.e. mass flow rate per unit depth.

So the integral from $x = 0$ to L_x represents the mass flow rate across the edge of the control volume.

Mass equation

Bottom inflow of mass

$$\left[\int_0^{L_y} \rho u \,\mathrm{d}y\right]_{x=L_x} - \left[\int_0^{L_y} \rho u \,\mathrm{d}y\right]_{x=0} + \left[\int_0^{L_x} \rho v \,\mathrm{d}x\right]_{y=L_y} \boxed{- \left[\int_0^{L_x} \rho v \,\mathrm{d}x\right]_{y=0}} = 0$$



ρvdx has the SI units,
(kg/m³)(m/s)(m) = (kg/s)/m,
i.e. mass flow rate per unit depth.

So the integral from x = 0 to L_x represents the mass flow rate across the edge of the control volume.

Mass equation

Hopefully you are now convinced that the equation below is equivalent to the conservation of mass, i.e. mass flow rate in = mass flow rate out.

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad \rho = \text{const.}$$

or

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

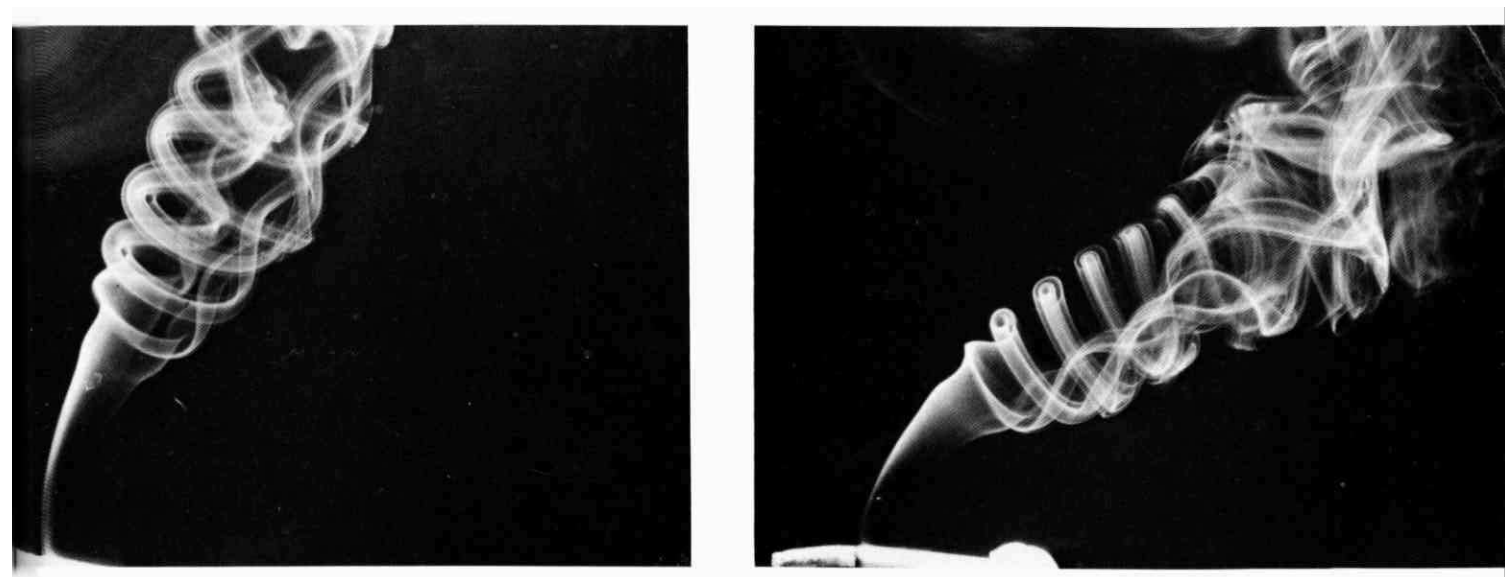
Melbourne School of Engineering
MCEN90018 Advanced Fluid Dynamics

Lecture BL02: Navier–Stokes equations
3 March 2016

Can you explain the messy/smooth behaviour of rising smoke from cigarette?

(http://askphysics.com/wp-content/uploads/2012/01/cigarette.gif)

From Melbourne Uni:



107. Instability of smoke from a cigarette. These two views of a familiar phenomenon, photographed in a slight draft, show that the initially rectilinear plume becomes unstable and forms two trailing vortices interconnected by a ladder-like network of vortex loops. *Perry & Lim 1978*

(Van Dyke 1982)

Navier–Stokes equations for incompressible flow: 2 components (of velocity) and 2 dimensions

$\rho = \text{const.}$

Conservation of mass:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

Rate of change of x-momentum = sum of x-forces:

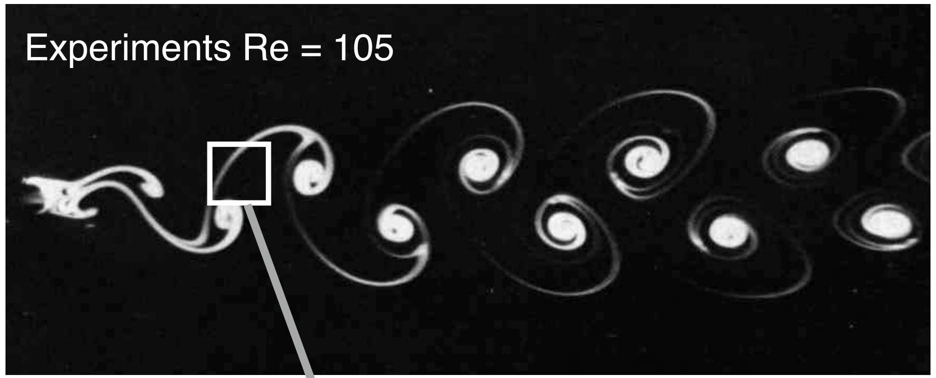
$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu)}{\partial y} = -\frac{\partial p}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$$

Rate of change of y-momentum = sum of y-forces:

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho vv)}{\partial y} = -\frac{\partial p}{\partial y} + \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2})$$

Impermeable, no-slip boundary conditions at wall:

$u = v = 0$



(Van Dyke 1982)

Stationary control volume

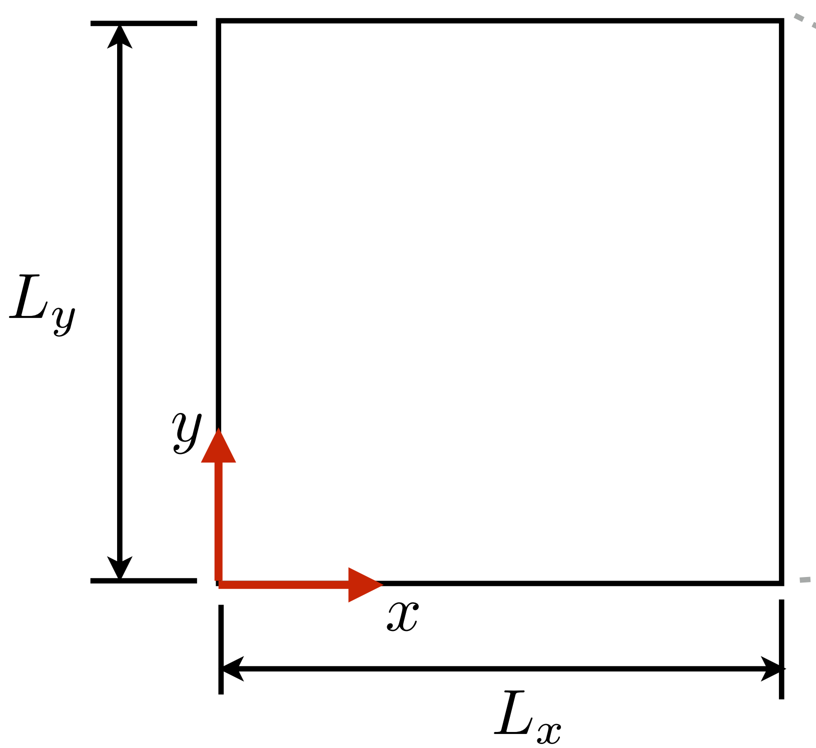
To better understand these equations, consider a stationary control volume.

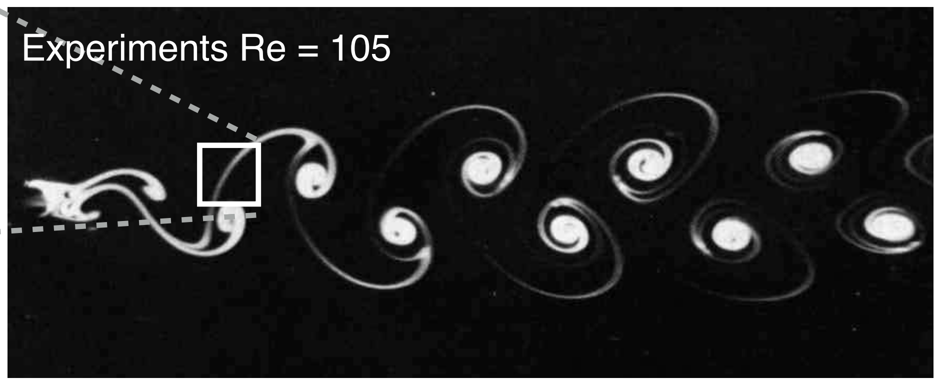
x-momentum equation: LHS

$$\boxed{\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u u)}{\partial x}+\frac{\partial(\rho v u)}{\partial y}}=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Integrate left-hand side over an arbitrary stationary control volume:

$$\int_0^{L_y}\int_0^{L_x}\frac{\partial(\rho u)}{\partial t}\,\mathrm{d}x\,\mathrm{d}y+\int_0^{L_y}\int_0^{L_x}\frac{\partial(\rho u u)}{\partial x}\,\mathrm{d}x\,\mathrm{d}y+\int_0^{L_y}\int_0^{L_x}\frac{\partial(\rho v u)}{\partial y}\,\mathrm{d}x\,\mathrm{d}y$$



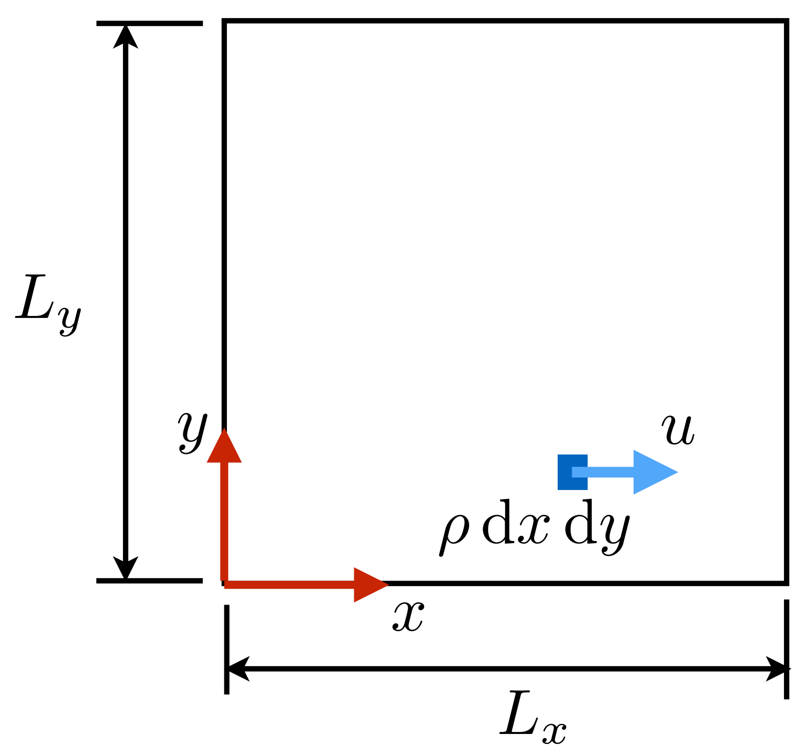


(Van Dyke 1982)

x-momentum equation: LHS

Rate of change of x-momentum in control volume

$$\boxed{\frac{\partial}{\partial t}\int_0^{L_y}\int_0^{L_x}\rho u\,\mathrm{d}x\,\mathrm{d}y} + \left[\int_0^{L_y}\rho u u\,\mathrm{d}y\right]_{x=L_x} - \left[\int_0^{L_y}\rho u u\,\mathrm{d}y\right]_{x=0} + \left[\int_0^{L_x}\rho v u\,\mathrm{d}x\right]_{y=L_y} - \left[\int_0^{L_x}\rho v u\,\mathrm{d}x\right]_{y=0}$$

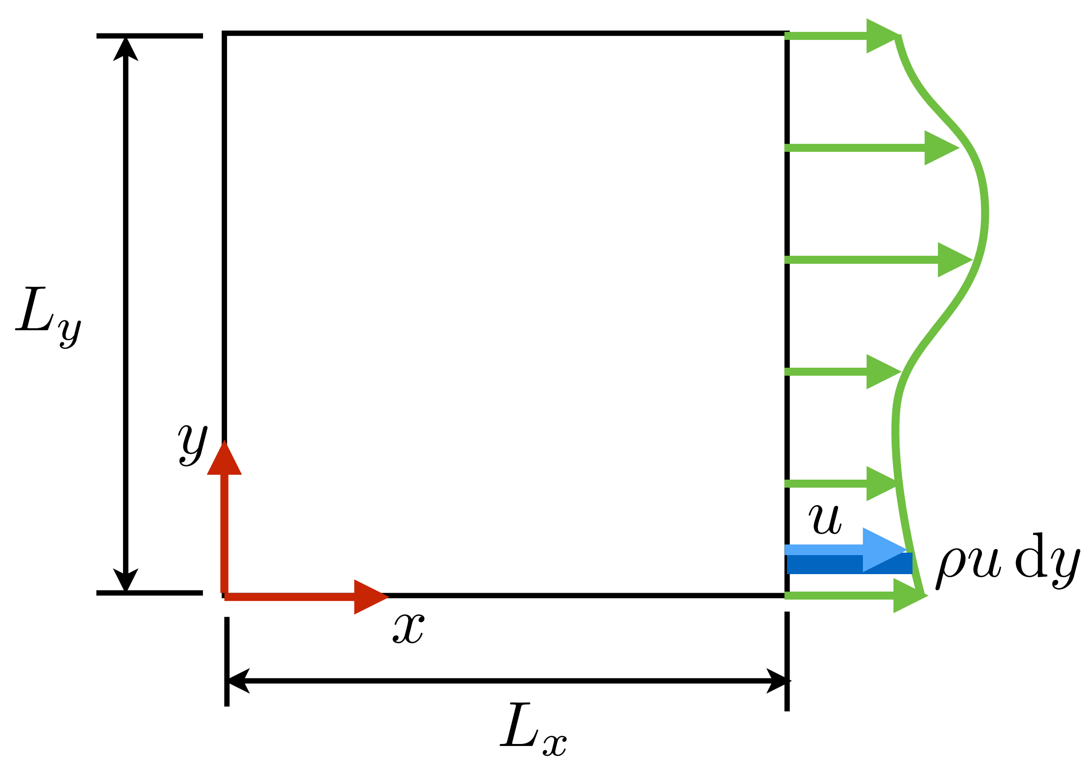


ρdxdy has the SI units,
(kg/m³)(m)(m) = kg/m,
i.e. mass per unit depth.

x-momentum equation: LHS

Right outflow of x-momentum

$$\frac{\partial}{\partial t}\int_0^{L_y}\int_0^{L_x}\rho u\,\mathrm{d}x\,\mathrm{d}y \boxed{+\left[\int_0^{L_y}\rho u u\,\mathrm{d}y\right]_{x=L_x}} - \left[\int_0^{L_y}\rho u u\,\mathrm{d}y\right]_{x=0} + \left[\int_0^{L_x}\rho v u\,\mathrm{d}x\right]_{y=L_y} - \left[\int_0^{L_x}\rho v u\,\mathrm{d}x\right]_{y=0}$$

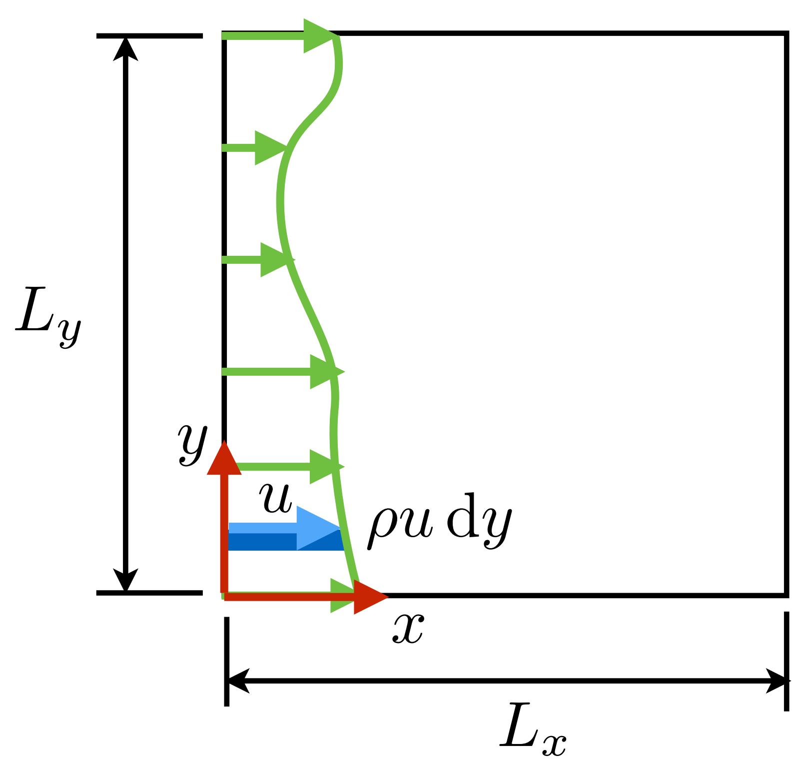


ρudy has the SI units,
(kg/m³)(m/s)(m) = (kg/s)/m,
i.e. mass flow rate per unit depth.

x-momentum equation: LHS

Left inflow of x-momentum

$$\frac{\partial}{\partial t}\int_0^{L_y}\int_0^{L_x}\rho u \,\mathrm{d}x\,\mathrm{d}y + \left[\int_0^{L_y}\rho u u\,\mathrm{d}y\right]_{x=L_x} \boxed{-\left[\int_0^{L_y}\rho u u\,\mathrm{d}y\right]_{x=0}} + \left[\int_0^{L_x}\rho v u\,\mathrm{d}x\right]_{y=L_y} - \left[\int_0^{L_x}\rho v u\,\mathrm{d}x\right]_{y=0}$$

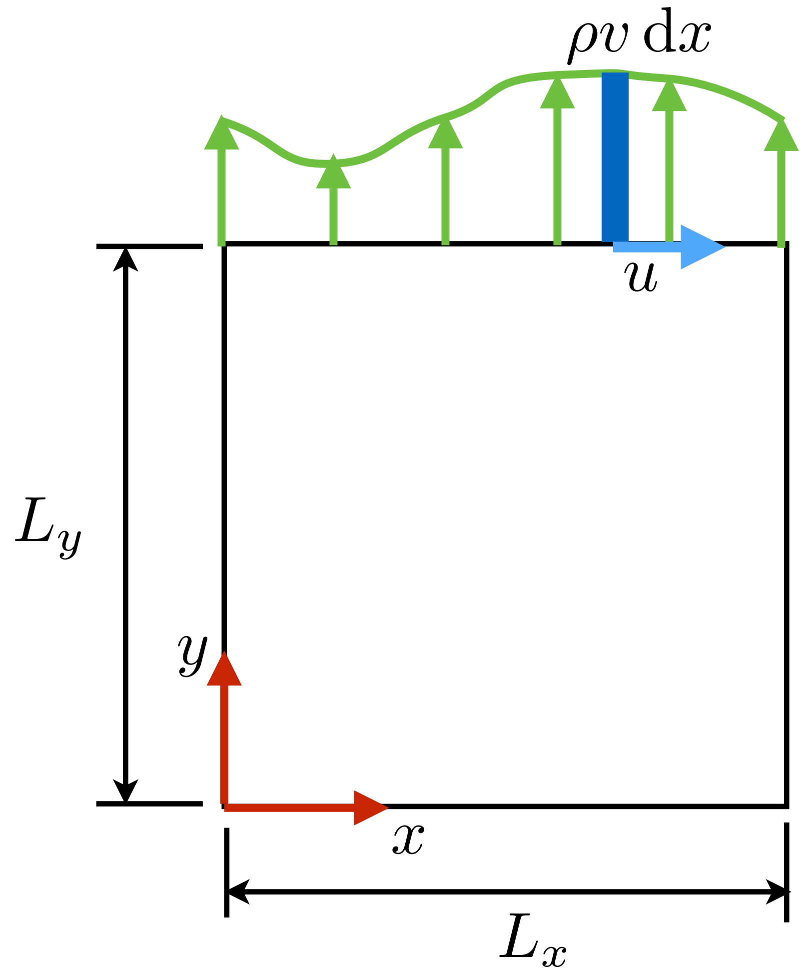


ρudy has the SI units,
(kg/m³)(m/s)(m) = (kg/s)/m,
i.e. mass flow rate per unit depth.

x-momentum equation: LHS

Top outflow of x-momentum

$$\frac{\partial}{\partial t}\int_0^{L_y}\int_0^{L_x}\rho u \, \mathrm{d}x \, \mathrm{d}y + \left[\int_0^{L_y}\rho u u \, \mathrm{d}y\right]_{x=L_x} - \left[\int_0^{L_y}\rho u u \, \mathrm{d}y\right]_{x=0} \boxed{+ \left[\int_0^{L_x}\rho v u \, \mathrm{d}x\right]_{y=L_y}} - \left[\int_0^{L_x}\rho v u \, \mathrm{d}x\right]_{y=0}$$

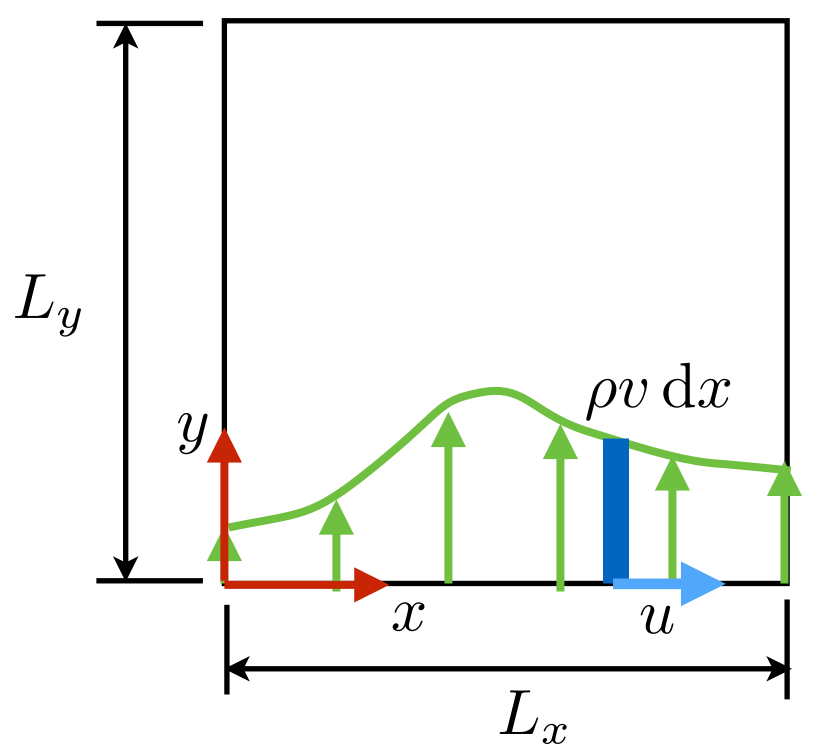


ρvdx has the SI units,
(kg/m³)(m/s)(m) = (kg/s)/m,
i.e. mass flow rate per unit depth.

x-momentum equation: LHS

Bottom inflow of x-momentum

$$\frac{\partial}{\partial t}\int_0^{L_y}\int_0^{L_x}\rho u\,\mathrm{d}x\,\mathrm{d}y+\left[\int_0^{L_y}\rho uu\,\mathrm{d}y\right]_{x=L_x}-\left[\int_0^{L_y}\rho uu\,\mathrm{d}y\right]_{x=0}+\left[\int_0^{L_x}\rho vu\,\mathrm{d}x\right]_{y=L_y}\boxed{-\left[\int_0^{L_x}\rho vu\,\mathrm{d}x\right]_{y=0}}$$



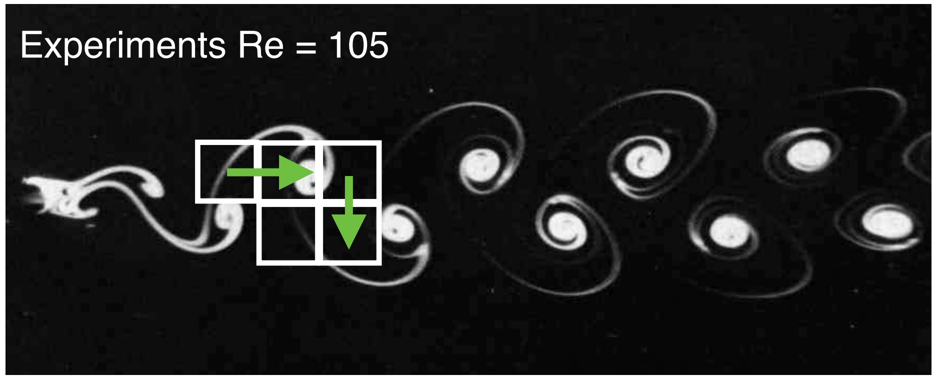
ρvdx has the SI units,
(kg/m³)(m/s)(m) = (kg/s)/m,
i.e. mass flow rate per unit depth.

x-momentum equation: LHS

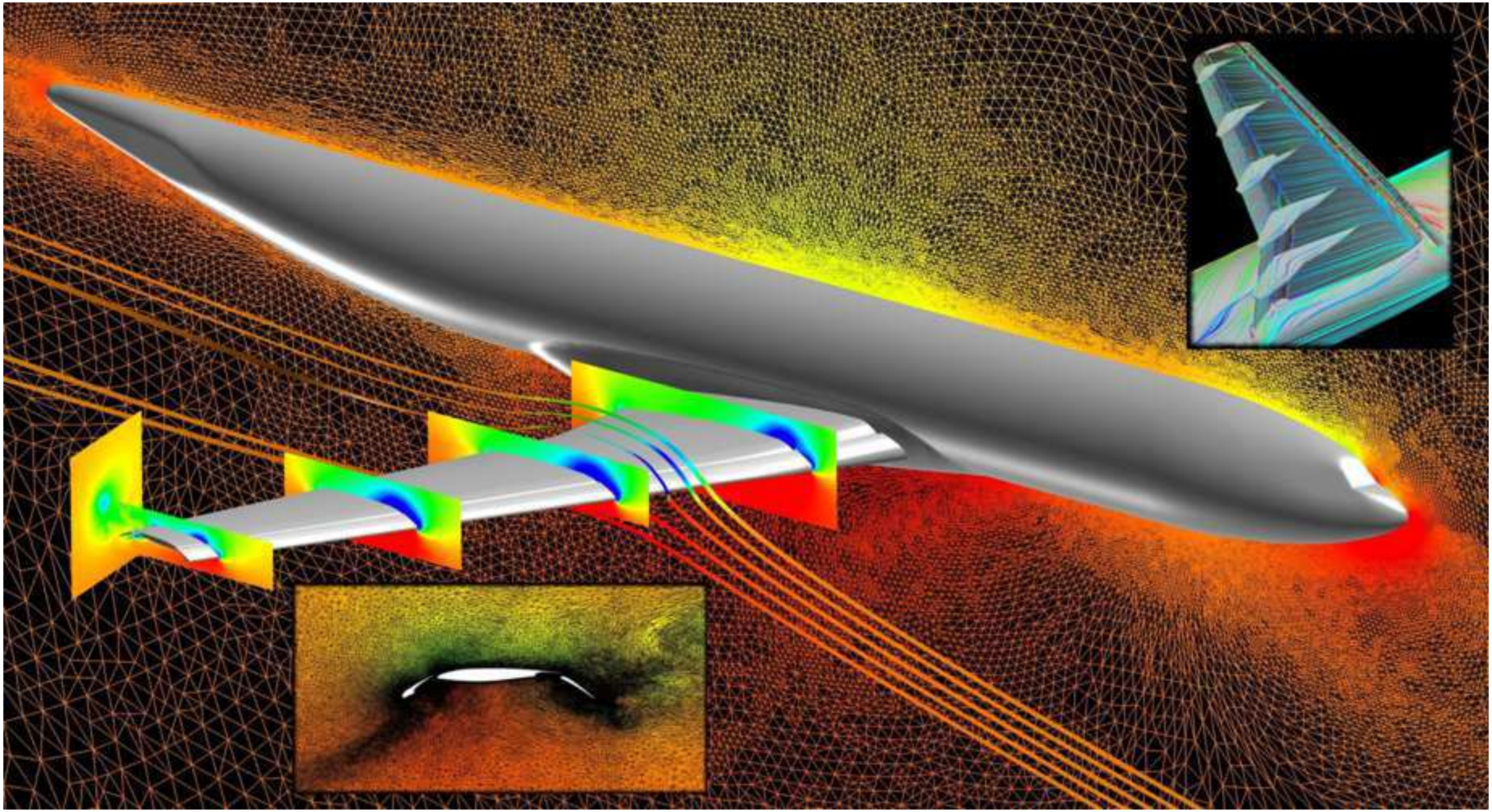
$$\frac{\partial}{\partial t}\int_0^{L_y}\int_0^{L_x}\rho u\,\mathrm{d}x\,\mathrm{d}y+\left[\int_0^{L_y}\rho uu\,\mathrm{d}y\right]_{x=L_x}-\left[\int_0^{L_y}\rho uu\,\mathrm{d}y\right]_{x=0}+\left[\int_0^{L_x}\rho vu\,\mathrm{d}x\right]_{y=L_y}-\left[\int_0^{L_x}\rho vu\,\mathrm{d}x\right]_{y=0}$$

So if left-hand side of x-momentum equation sums to zero (no external forces acting on the control volume), the total x-momentum in the control volume can only change by mass flow carrying x-momentum in or out of the control volume.

cf. computational fluid dynamics: consider an arrangement of control volumes, each shuffling momenta with its neighbour.



(Van Dyke 1982)



http://www.nasa.gov/sites/default/files/hi_lift_mesh_transport_sim.jpg

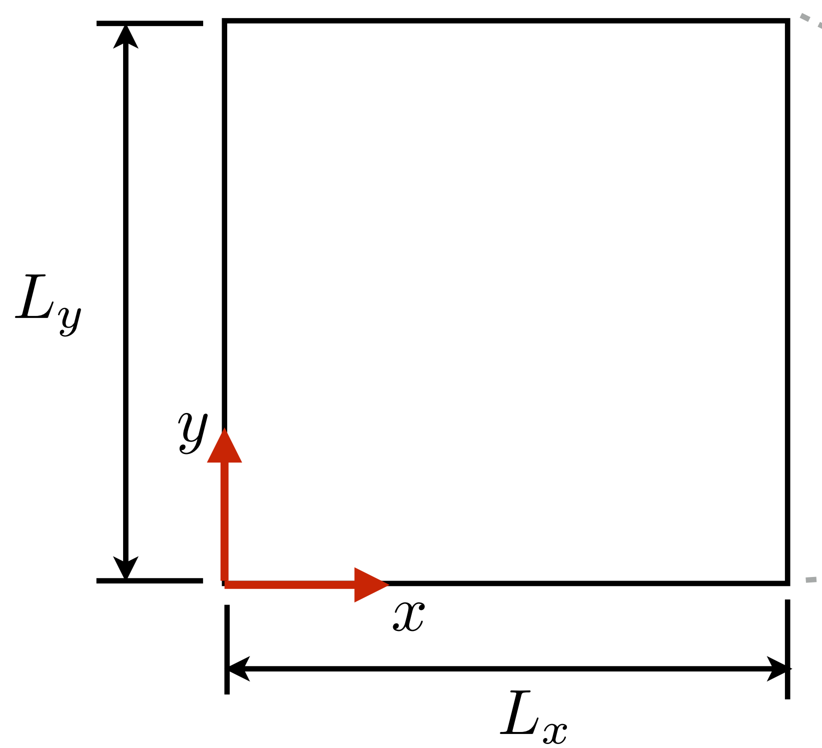
cf. computational fluid dynamics: consider an arrangement of control volumes, each shuffling momenta with its neighbour.

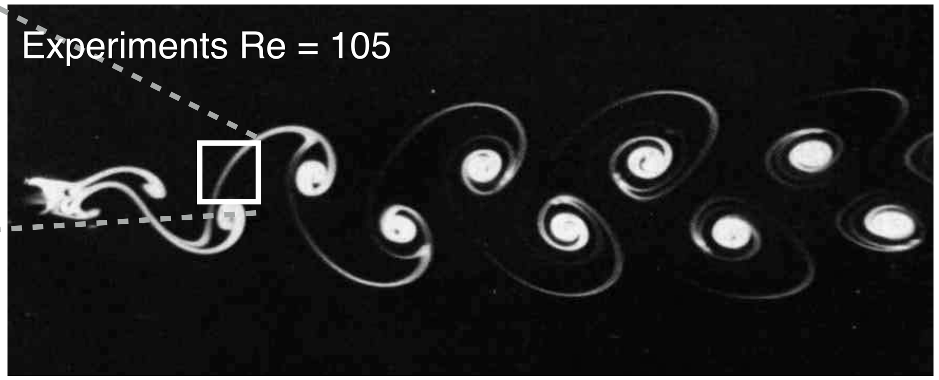
x-momentum equation: RHS

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u u)}{\partial x} + \frac{\partial(\rho v u)}{\partial y} = \boxed{-\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}$$

Integrate right-hand side over an arbitrary stationary control volume:

$$\int_0^{L_y}\int_0^{L_x} -\frac{\partial p}{\partial x}\,\mathrm{d}x\,\mathrm{d}y + \int_0^{L_y}\int_0^{L_x} \mu\frac{\partial^2 u}{\partial x^2}\,\mathrm{d}x\,\mathrm{d}y + \int_0^{L_y}\int_0^{L_x} \mu\frac{\partial^2 u}{\partial y^2}\,\mathrm{d}x\,\mathrm{d}y$$



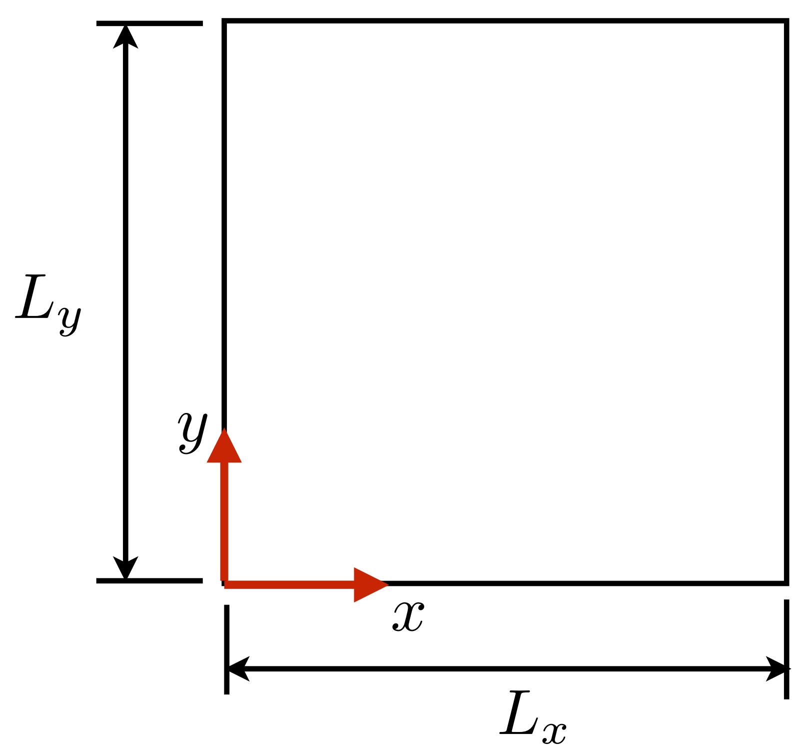


(Van Dyke 1982)

x-momentum equation: RHS

$$\left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=0} - \left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=L_x}$$

$$+\left[\int_0^{L_y} \mu\frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=L_x} - \left[\int_0^{L_y} \mu\frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=0} + \left[\int_0^{L_x} \mu\frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=L_y} - \left[\int_0^{L_x} \mu\frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=0}$$



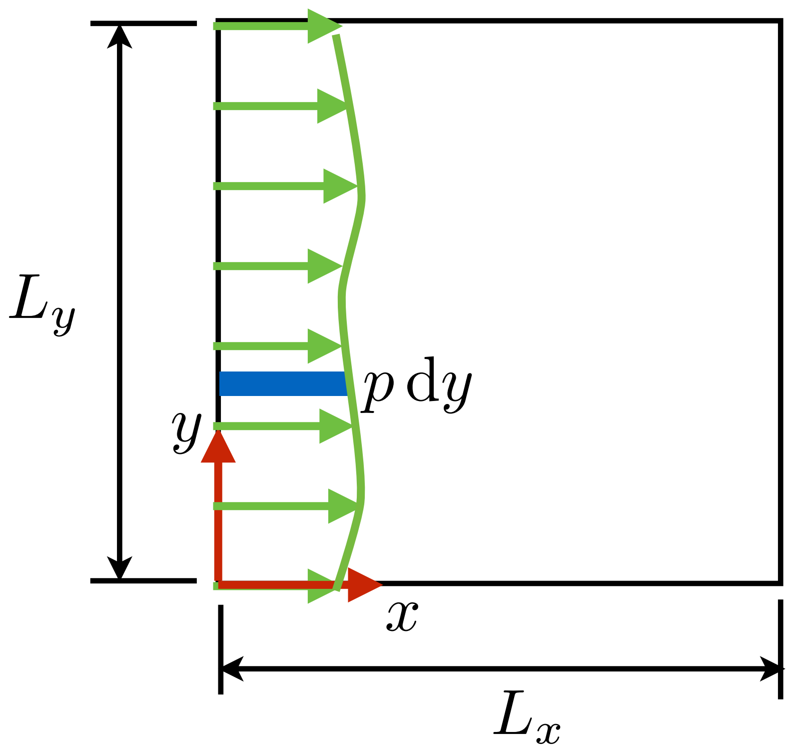
Here we assume that there are no body forces such as gravity so that all the remaining forces are surface forces, i.e. they only act at the boundaries of the control volume.

x-momentum equation: RHS

Pressure force acting on left edge

$$\boxed{\left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=0}} - \left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=L_x}$$

$$+\left[\int_0^{L_y} \mu\frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=L_x} - \left[\int_0^{L_y} \mu\frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=0} + \left[\int_0^{L_x} \mu\frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=L_y} - \left[\int_0^{L_x} \mu\frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=0}$$



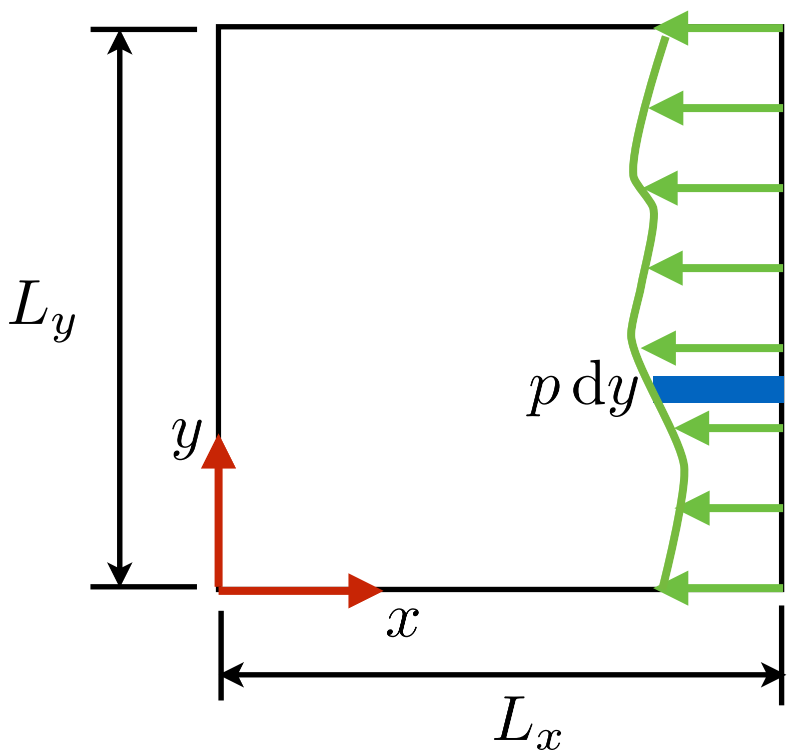
pdy has the SI units, (N/m²)(m) = N/m, i.e. force per unit depth.

x-momentum equation: RHS

Pressure force acting on right edge

$$\left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=0} \boxed{- \left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=L_x}}$$

$$+ \left[\int_0^{L_y} \mu \frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=L_x} - \left[\int_0^{L_y} \mu \frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=0} + \left[\int_0^{L_x} \mu \frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=L_y} - \left[\int_0^{L_x} \mu \frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=0}$$

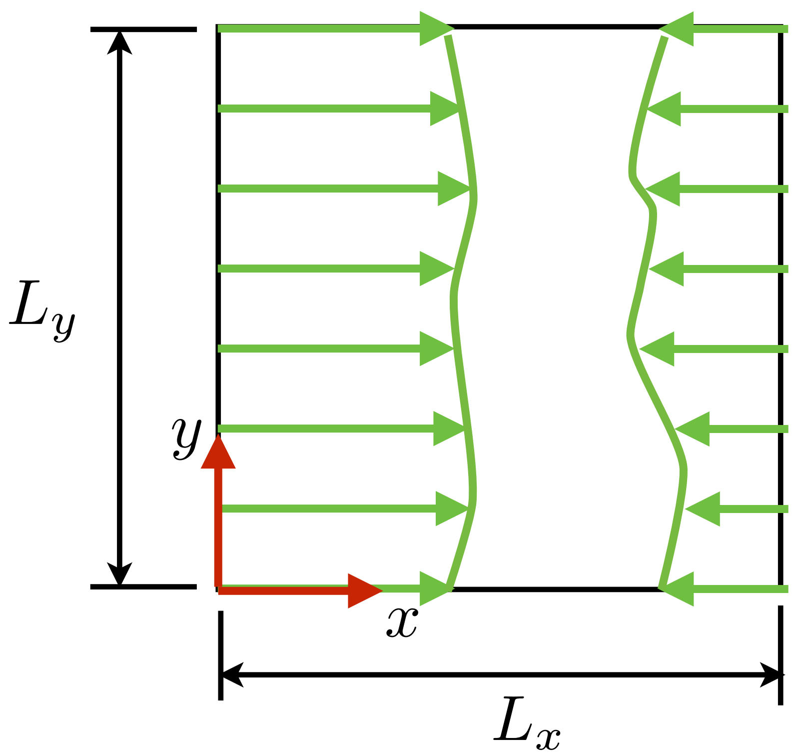


pdy has the SI units, (N/m^2)(m) = N/m, i.e. force per unit depth.

x-momentum equation: RHS

$$\boxed{\left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=0} - \left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=L_x}} > 0 \text{ (increases x-momentum)}$$

$$+\left[\int_0^{L_y} \mu\frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=L_x} - \left[\int_0^{L_y} \mu\frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=0} + \left[\int_0^{L_x} \mu\frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=L_y} - \left[\int_0^{L_x} \mu\frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=0}$$

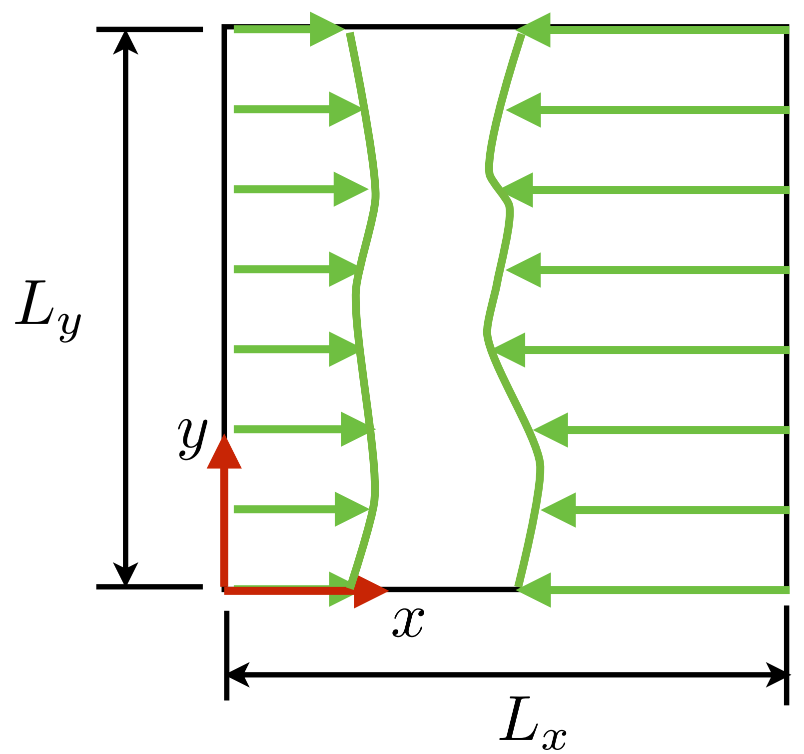


It is the pressure difference that drives the flow (tends to increase or decrease x-momentum). In particular, the absolute pressure is immaterial in an incompressible flow.

x-momentum equation: RHS

$$\boxed{\left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=0} - \left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=L_x}} < 0 \text{ (decreases x-momentum)}$$

$$+\left[\int_0^{L_y} \mu\frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=L_x} - \left[\int_0^{L_y} \mu\frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=0} + \left[\int_0^{L_x} \mu\frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=L_y} - \left[\int_0^{L_x} \mu\frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=0}$$

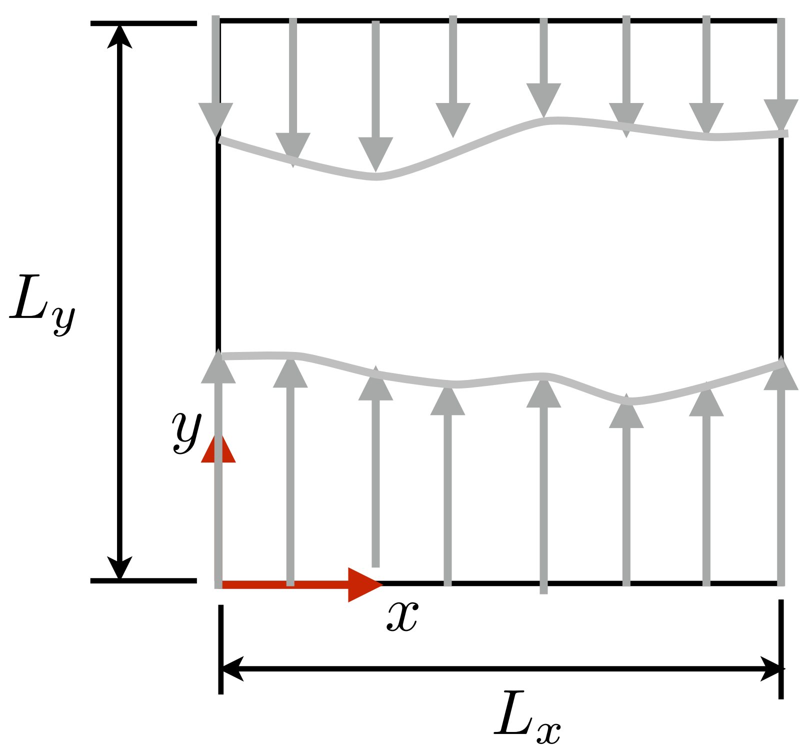


It is the pressure difference that drives the flow (tends to increase or decrease x-momentum). In particular, the absolute pressure is immaterial in an incompressible flow.

x-momentum equation: RHS

$$\left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=0} - \left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=L_x}$$

$$+\left[\int_0^{L_y} \mu\frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=L_x} - \left[\int_0^{L_y} \mu\frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=0} + \left[\int_0^{L_x} \mu\frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=L_y} - \left[\int_0^{L_x} \mu\frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=0}$$

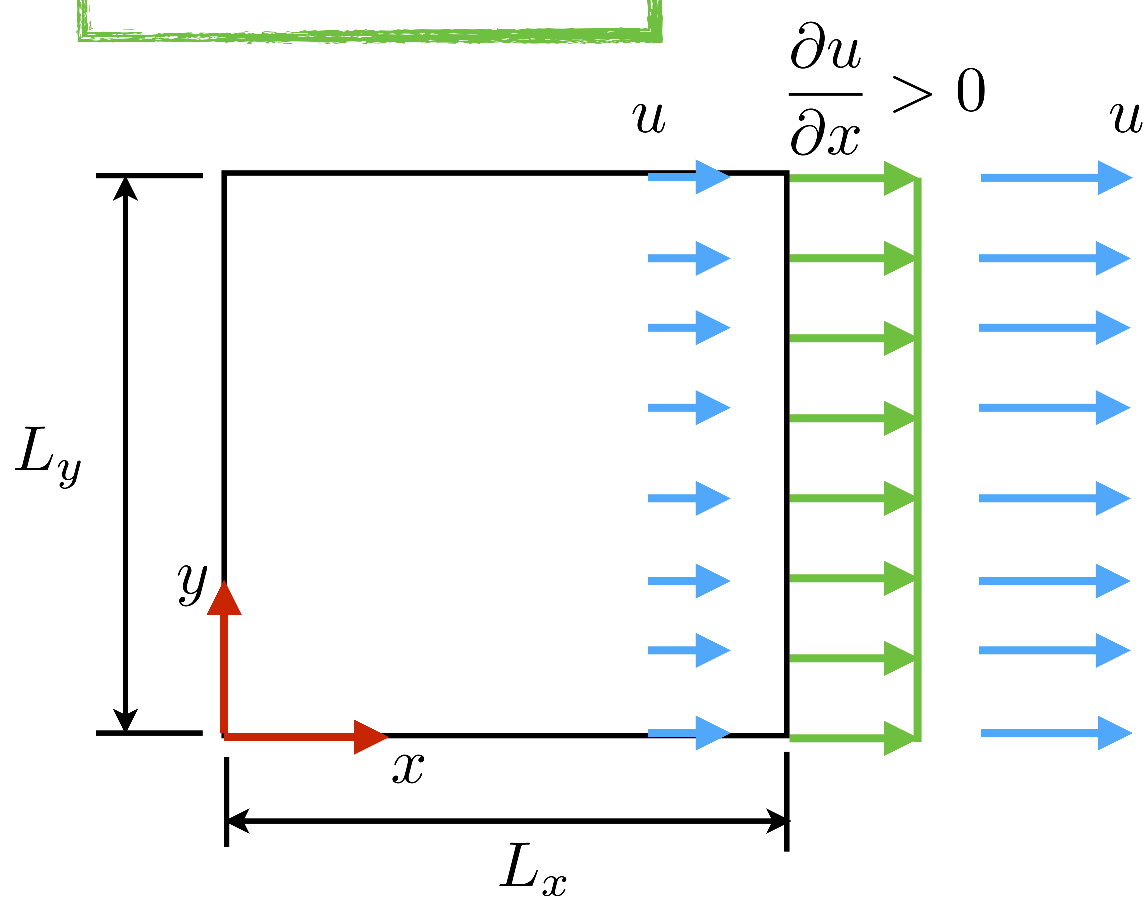


Top and bottom pressures act on the control volume as well, but do not contribute to the x-momentum equation.

x-momentum equation: RHS

$$\left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=0} - \left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=L_x}$$

$$\boxed{+\left[\int_0^{L_y} \mu\frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=L_x}} - \left[\int_0^{L_y} \mu\frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=0} + \left[\int_0^{L_x} \mu\frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=L_y} - \left[\int_0^{L_x} \mu\frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=0}$$



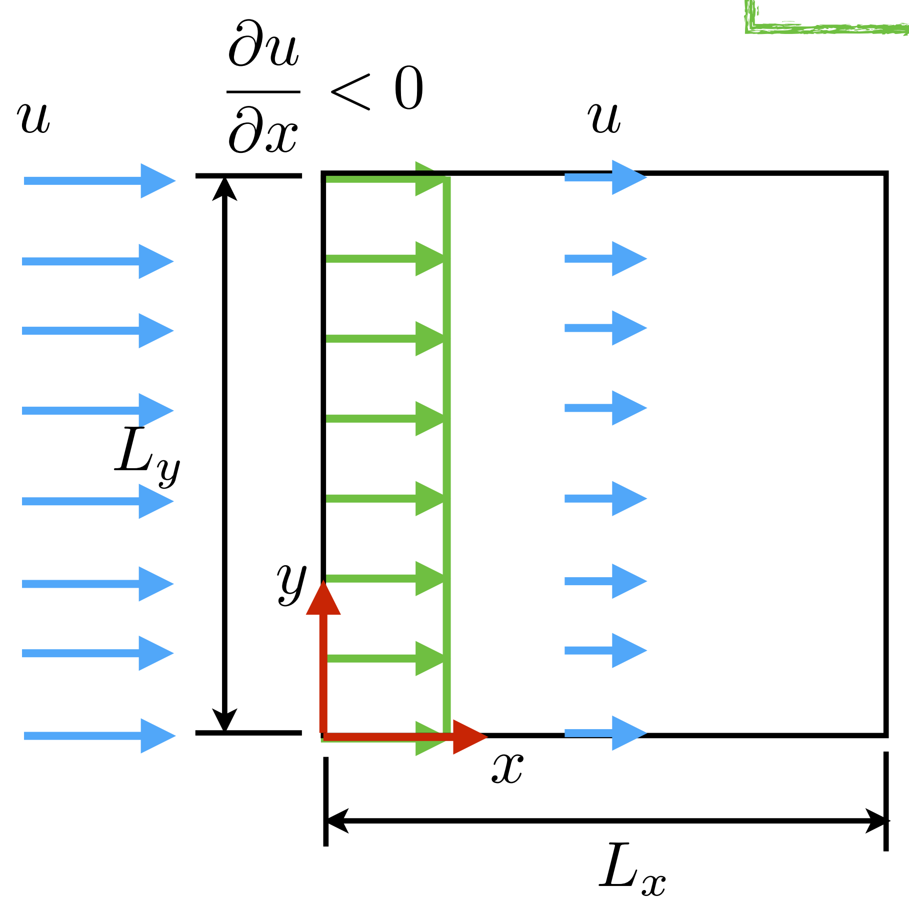
μ(du/dx)dy has the SI units, (Ns/m²)(1/s)(m) = N/m, i.e. force per unit depth.

Viscous forces smooth the velocity field. e.g. if there is a higher velocity outside the control volume, the viscous force increases the velocity inside the control volume, stopping only when the flow field is uniform.

x-momentum equation: RHS

$$\left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=0} - \left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=L_x}$$

$$+\left[\int_0^{L_y} \mu\frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=L_x} \boxed{-\left[\int_0^{L_y} \mu\frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=0}} + \left[\int_0^{L_x} \mu\frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=L_y} - \left[\int_0^{L_x} \mu\frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=0}$$



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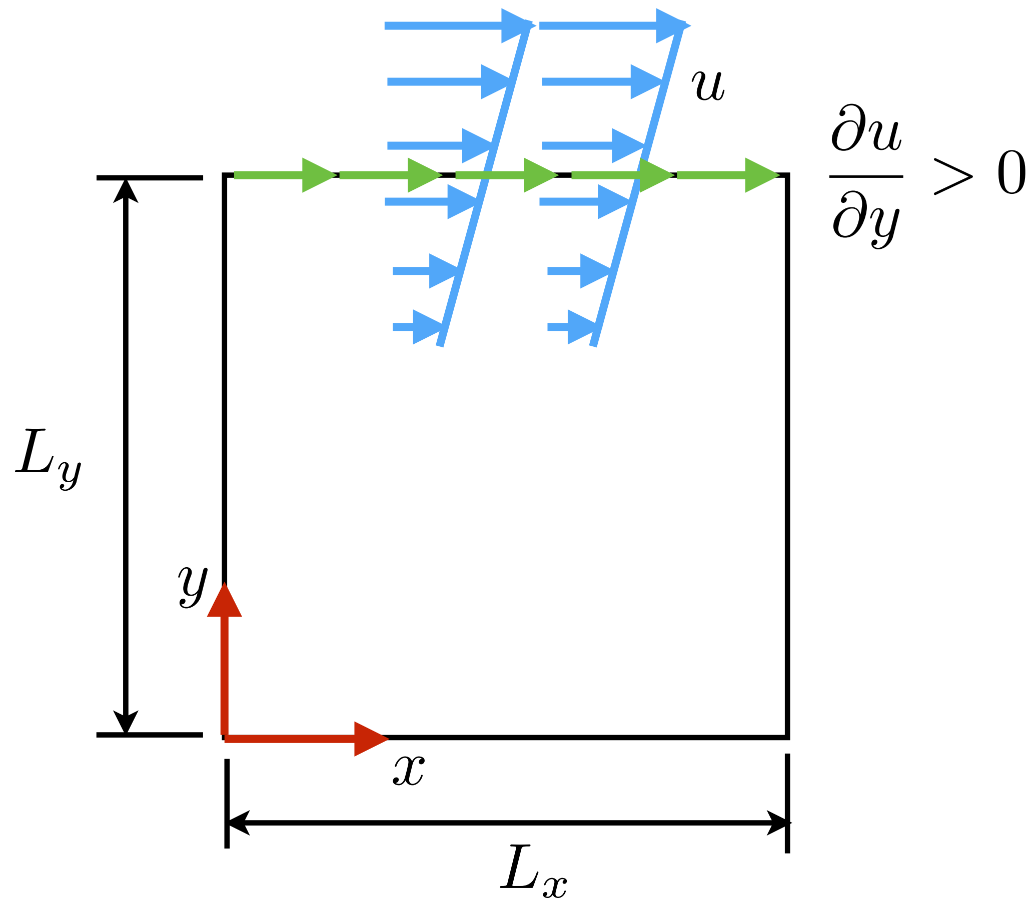
x-momentum equation: RHS

$$\left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=0} - \left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=L_x}$$

$$+\left[\int_0^{L_y} \mu\frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=L_x} - \left[\int_0^{L_y} \mu\frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=0} \boxed{+\left[\int_0^{L_x} \mu\frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=L_y}} - \left[\int_0^{L_x} \mu\frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=0}$$



μ(du/dx)dy has the SI units, (Ns/m²)(1/s)(m) = N/m, i.e. force per unit depth.

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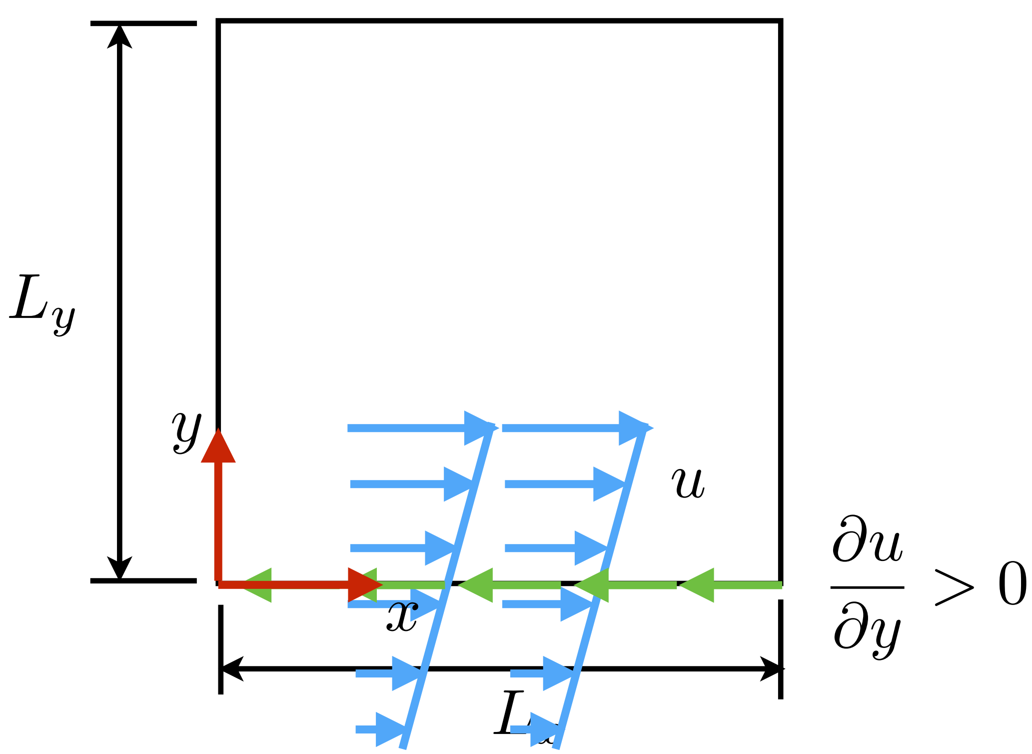
x-momentum equation: RHS

$$\left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=0} - \left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=L_x}$$

$$+\left[\int_0^{L_y} \mu\frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=L_x} - \left[\int_0^{L_y} \mu\frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=0} + \left[\int_0^{L_x} \mu\frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=L_y} \boxed{- \left[\int_0^{L_x} \mu\frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=0}}$$



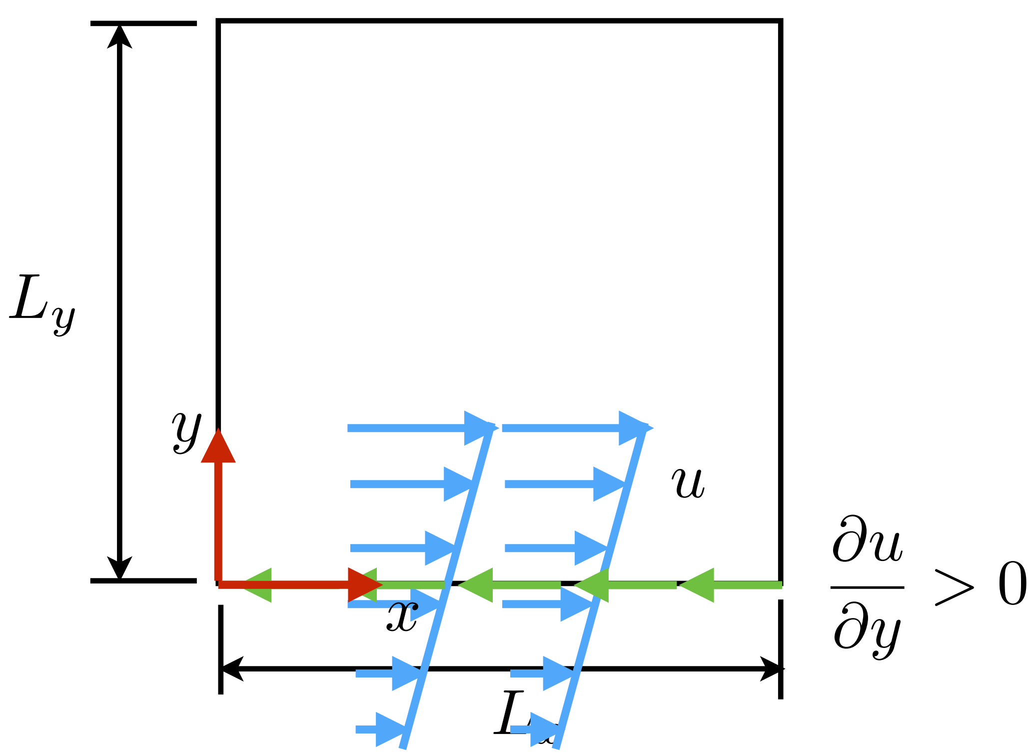
μ(du/dx)dy has the SI units, (Ns/m²)(1/s)(m) = N/m, i.e. force per unit depth.

Viscous forces smooth the velocity field. e.g. if there is a higher velocity outside the control volume, the viscous force increases the velocity inside the control volume, stopping only when the flow field is uniform.

x-momentum equation: RHS

$$\left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=0} - \left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=L_x}$$

$$+\left[\int_0^{L_y} \mu\frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=L_x} - \left[\int_0^{L_y} \mu\frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=0} + \left[\int_0^{L_x} \mu\frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=L_y} \boxed{- \left[\int_0^{L_x} \mu\frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=0}}$$

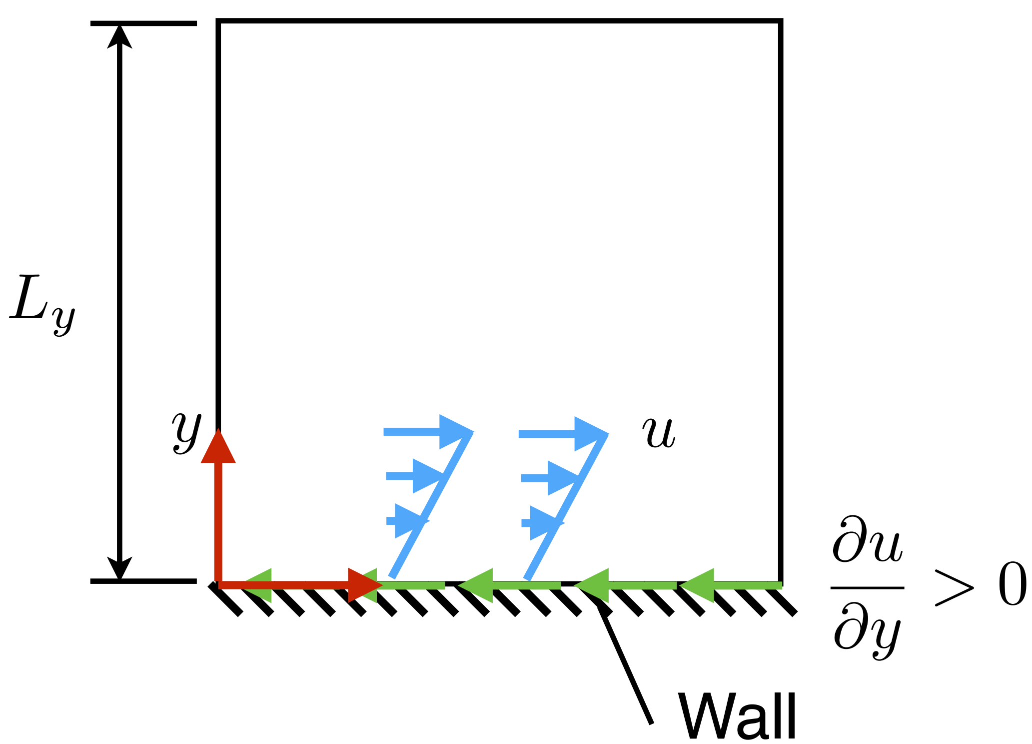


The higher the velocity gradient, the greater the viscous force.

x-momentum equation: RHS

$$\left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=0} - \left[\int_0^{L_y} p\,\mathrm{d}y\right]_{x=L_x}$$

$$+\left[\int_0^{L_y} \mu\frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=L_x} - \left[\int_0^{L_y} \mu\frac{\partial u}{\partial x}\,\mathrm{d}y\right]_{x=0} + \left[\int_0^{L_x} \mu\frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=L_y} \boxed{-\left[\int_0^{L_x} \mu\frac{\partial u}{\partial y}\,\mathrm{d}x\right]_{y=0}}$$



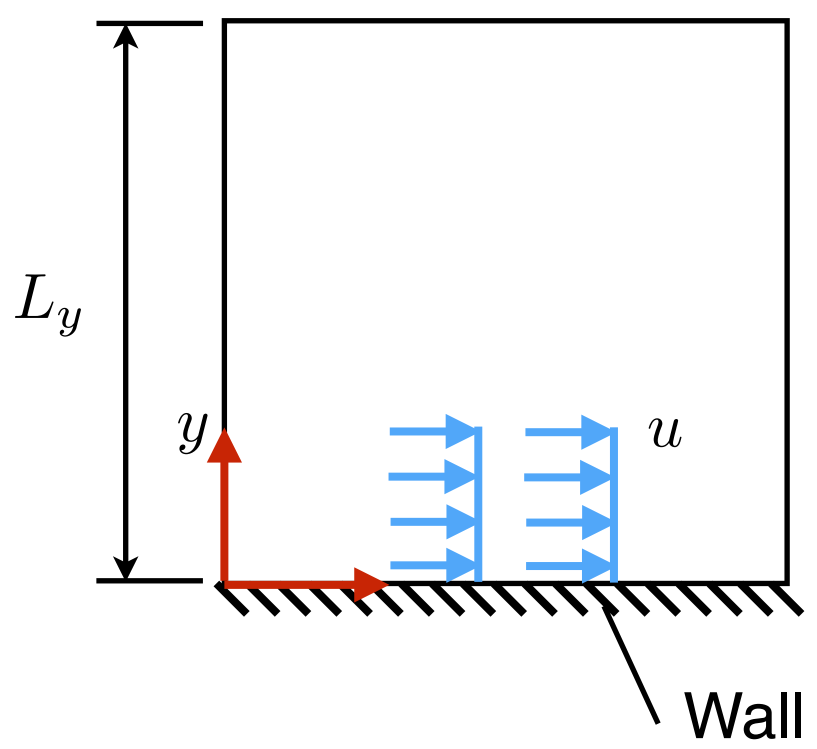
If the bottom edge were an impermeable (v = 0), no-slip (u = 0) wall, then only the viscous force can act at the bottom edge; no x-momentum change in control volume can occur from flow through the bottom edge (v = 0) or from pressure, which acts normal to boundary.

μ(du/dy) at the wall is called the (viscous) wall shear stress (wall-tangential force per unit area) and gives rise to skin-friction drag. This is how the no-slip wall announces its presence to the flow.

x-momentum equation: RHS

$$\left[\int_0^{L_y} p \,\mathrm{d}y\right]_{x=0} - \left[\int_0^{L_y} p \,\mathrm{d}y\right]_{x=L_x}$$

$$+ \left[\int_0^{L_y} \mu \frac{\partial u}{\partial x} \,\mathrm{d}y\right]_{x=L_x} - \left[\int_0^{L_y} \mu \frac{\partial u}{\partial x} \,\mathrm{d}y\right]_{x=0} + \left[\int_0^{L_x} \mu \frac{\partial u}{\partial y} \,\mathrm{d}x\right]_{y=L_y} \boxed{- \left[\int_0^{L_x} \mu \frac{\partial u}{\partial y} \,\mathrm{d}x\right]_{y=0}}$$



If viscosity were zero (ideal fluid), then there is no way for the wall to apply any wall-tangential force on the control volume, and so the flow must slip along the wall. In other words, the presence of a no-slip wall cannot be communicated and the skin-friction drag is zero (c.f. potential flow).

Viscous force is needed for imposing no slip!

x-momentum equation

Hopefully you are now convinced that the equation below is equivalent to Newton's second law, i.e. rate of change of x-momentum = sum of x-forces.

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u u)}{\partial x} + \frac{\partial(\rho v u)}{\partial y} = -\frac{\partial p}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \qquad \rho = \text{const.}$$

or

$$\frac{\partial u}{\partial t} + \frac{\partial u u}{\partial x} + \frac{\partial v u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \nu \equiv \mu/\rho$$

Relative size of forces in momentum equation

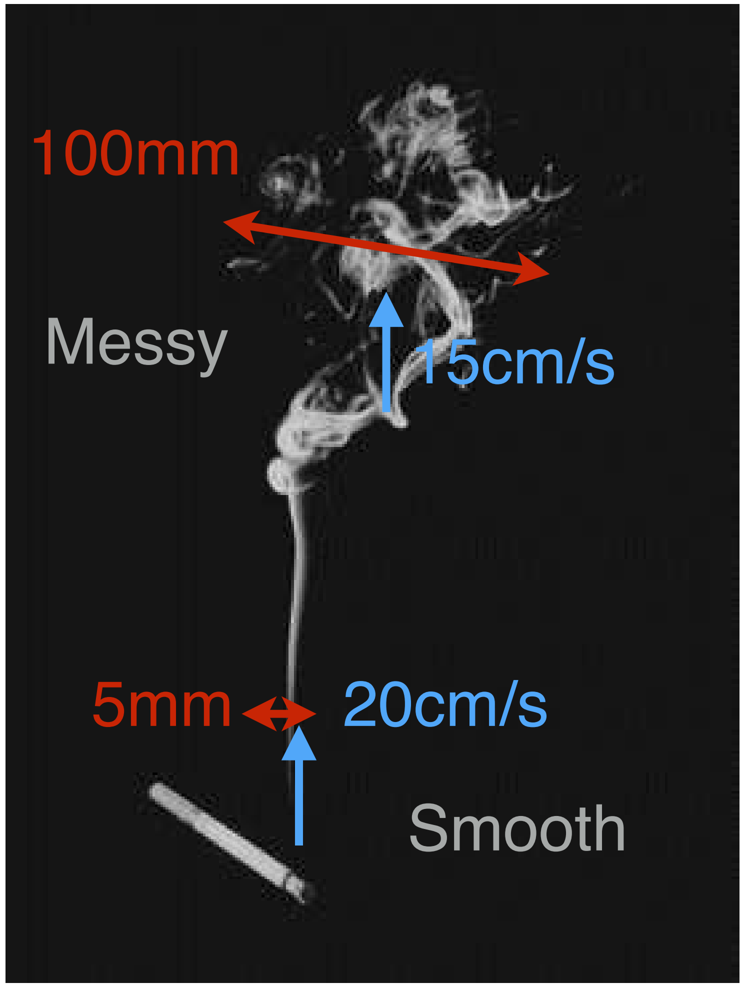
$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu)}{\partial y} = -\frac{\partial p}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \qquad \rho = \text{const.}$$

Inertia force:
shuffles x-momentum,
scales with $\rho U^2/L$

Viscous force:
smooths x-momentum,
scales with $\mu U/L^2$

So: $\dfrac{\text{Shuffling action}}{\text{Smoothing action}} = \dfrac{\text{Inertia force}}{\text{Viscous force}} = \dfrac{\rho U^2/L}{\mu U/L^2} = \dfrac{UL}{\nu} = Re$

Reynolds number in rising smoke from cigarette



Re = 1000

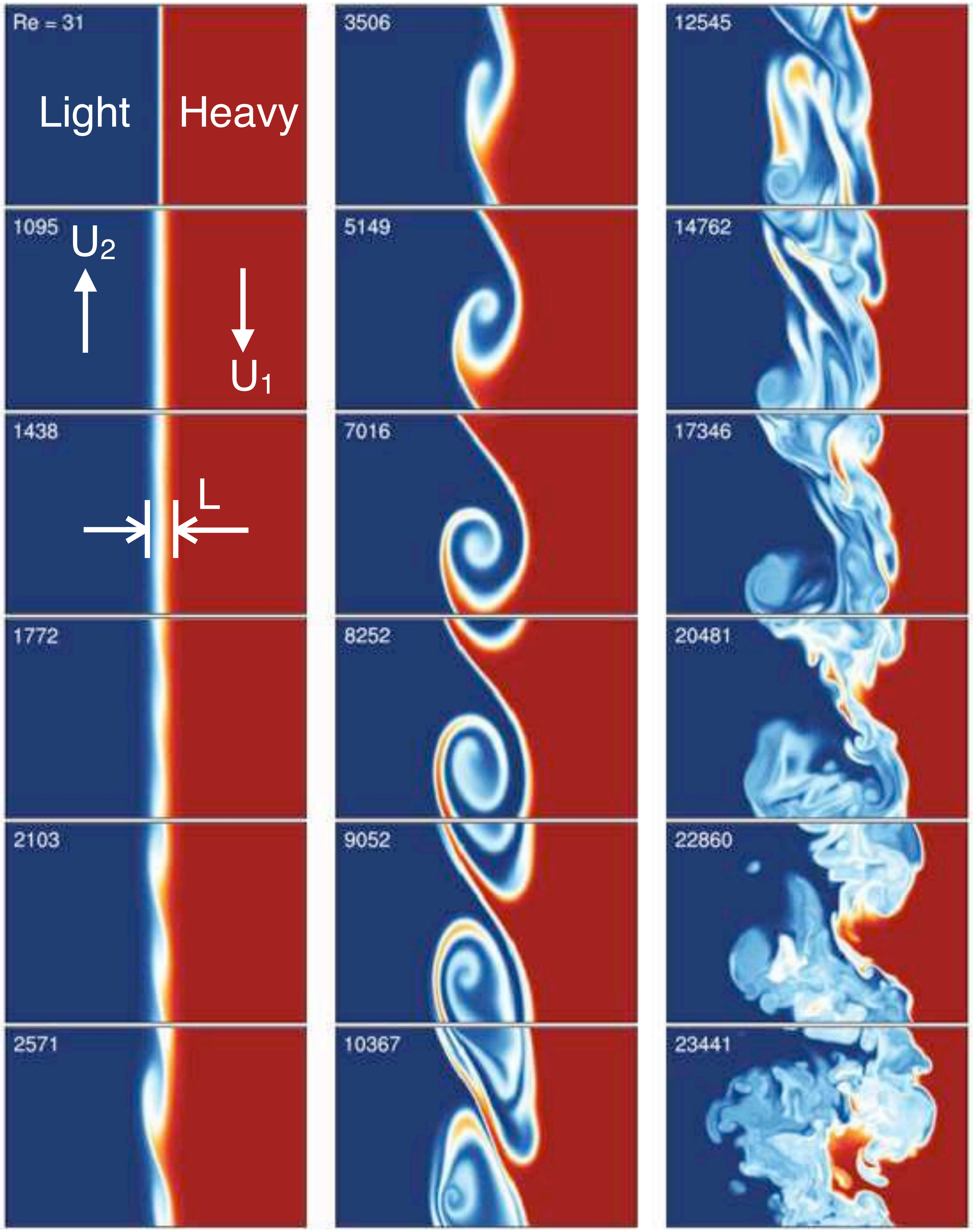
Re = 70

(http://askphysics.com/wp-content/uploads/2012/01/cigarette.gif)

(Dynamic viscosity of air: 1.5×10^{-5} m^2/s)

Observe how the flow visually looks at the different Reynolds numbers, i.e. ratio of inertia force on viscous force, or ratio of shuffling action on smoothing action.

$$Re = \frac{(U_2 - U_1)L}{\nu}$$



Reynolds number

Sample Reynolds Numbers

Re	animal	*Re*	aircraft
62,000	seagull	2,000,000,000	boeing 747
50,000	large fish	110,000,000	typical commercial jet
3,900	butterfly	6,300,000	cessna
1,000	honeybee	4,700,000	light plane
300	african frog tadpole	1,600,000	glider
120	housefly	250,000	model airplane
15	chalcid wasp	47,000	paper airplane
0.2	paramecium		
0.025	dinoflagellate		
0.0035	spermatozoa, sea urchin		
0.000,01	bacterium		

http://physics.info/turbulence/

(http://anandgreenwell.com/wp-content/uploads/2008/07/contrails317140916_std1.jpg)

Very high Reynolds number

(Roshko 2007)

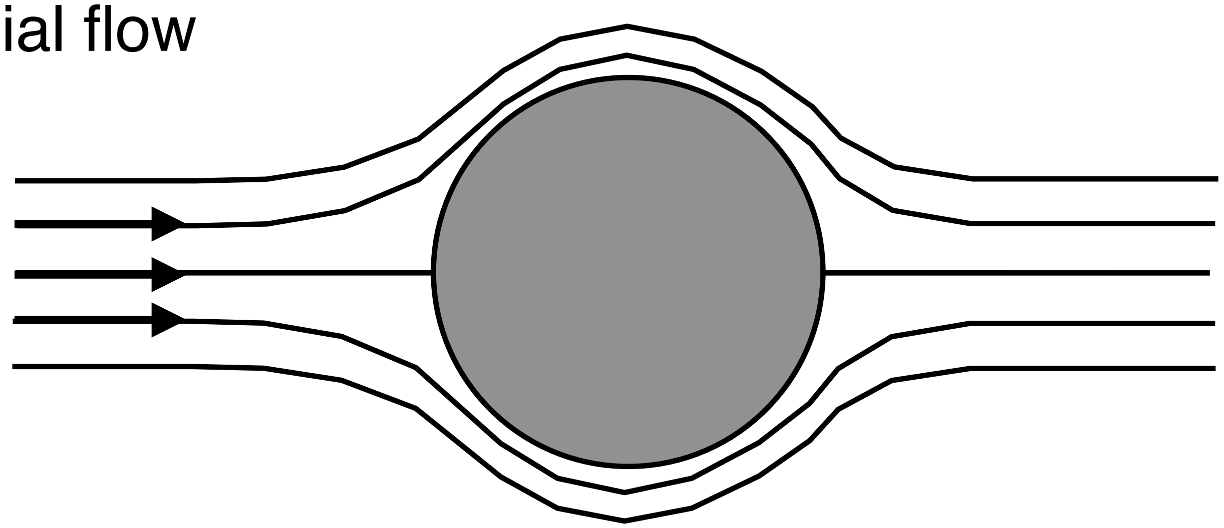
Rocket testing near
Los Angeles in 1969.

Melbourne School of Engineering
MCEN90018 Advanced Fluid Dynamics

Lecture BL03: d'Alembert's paradox
4 March 2016

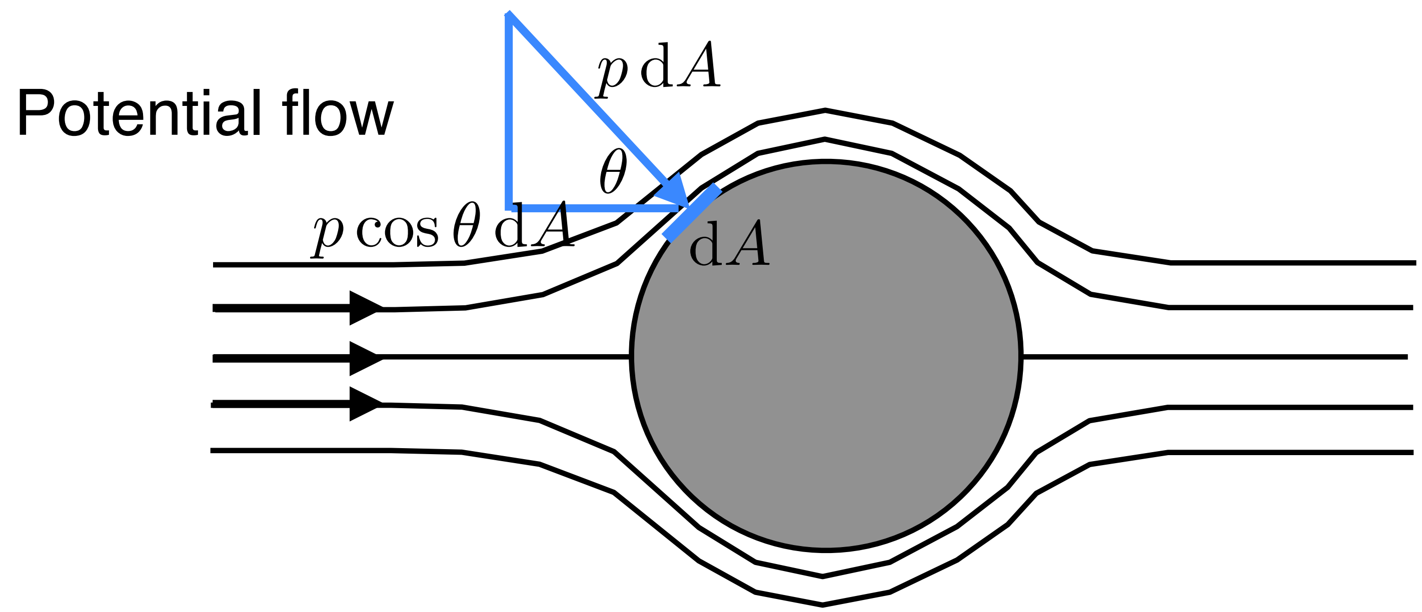
d'Alembert's (1717 – 1783) paradox

Potential flow



https://en.wikipedia.org/wiki/D%27Alembert%27s_paradox

d'Alembert's (1717 – 1783) paradox



$$D = \int p\cos\theta\,\mathrm{d}A = 0$$

d'Alembert's paradox:

no pressure/form drag for the case of potential flow!
True for any object in general, not just for a cylinder.

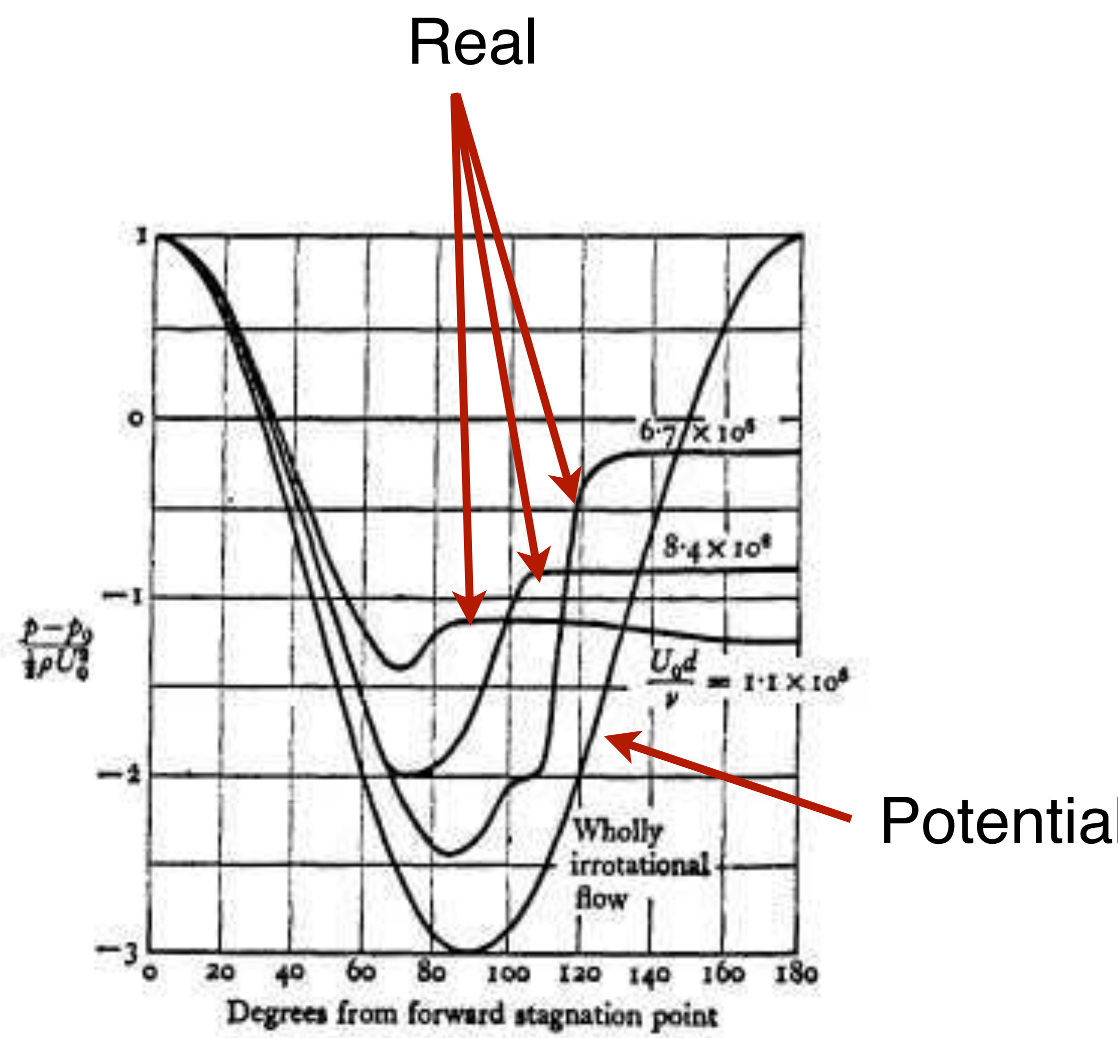
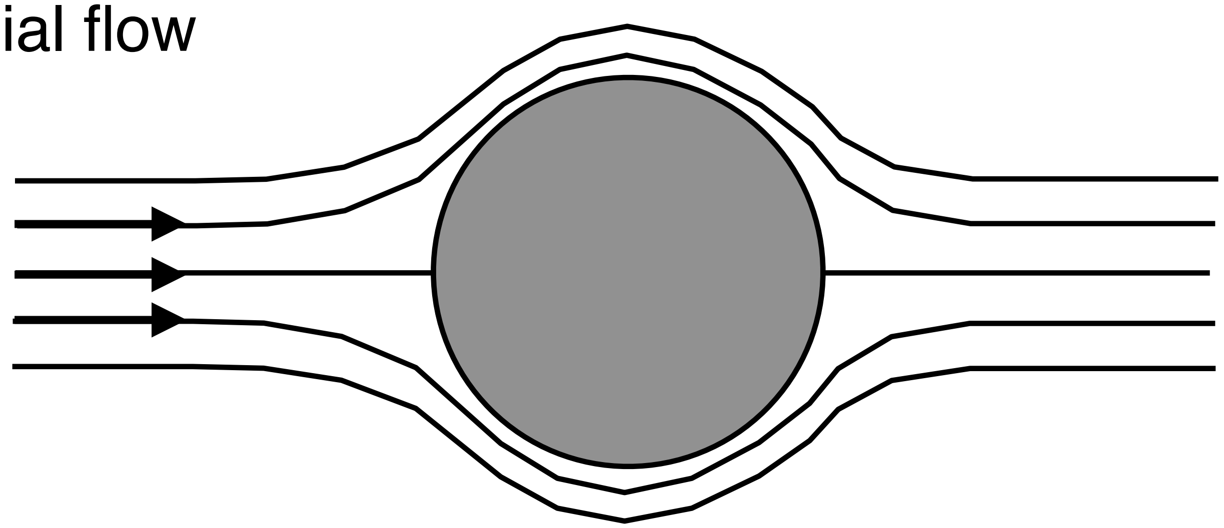


Figure 5.11.5. The measured pressure distribution at the surface of a circular cylinder in a stream of speed U_0, at different Reynolds numbers; p_0 = pressure at infinity.

(Batchelor 1967)

d’Alembert’s (1717 – 1783) paradox

Potential flow



https://en.wikipedia.org/wiki/D%27Alembert%27s_paradox

“Fluid mechanics was thus discredited by engineers from the start, which resulted in an unfortunate split – between the field of hydraulics, observing phenomena which could not be explained, and theoretical fluid mechanics explaining phenomena which could not be observed – in the words of the Chemistry Nobel Laureate Sir Cyril Hinshelwood.”

https://en.wikipedia.org/wiki/D%27Alembert%27s_paradox

Potential flow

What is potential flow?
Answer: irrotational, incompressible flow.

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{0} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

In 2D:

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Potential flow

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Answer: irrotational, incompressible flow.

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In 2D:

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Satisfied by streamfunction, ψ, or velocity potential, ϕ,

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

such that

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Potential flow

Why use potential flow?

High Re -> set viscosity to zero

Navier–Stokes

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu)}{\partial y} = -\frac{\partial p}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho vv)}{\partial y} = -\frac{\partial p}{\partial y} + \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2})$$

$$u_n|_w = u_t|_w = 0$$

Potential flow

Why use potential flow?

High Re -> set viscosity to zero -> Navier–Stokes becomes Euler

Navier–Stokes

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu)}{\partial y} = -\frac{\partial p}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho vv)}{\partial y} = -\frac{\partial p}{\partial y} + \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2})$$

$$u_n|_w = u_t|_w = 0$$

Euler

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \quad (3)$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu)}{\partial y} = -\frac{\partial p}{\partial x} \quad (1)$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho vv)}{\partial y} = -\frac{\partial p}{\partial y} \quad (2)$$

$$u_n|_w = 0 \qquad u_t|_w \neq 0$$

Potential flow

Why use potential flow?

High Re -> set viscosity to zero -> Navier–Stokes becomes Euler -> Solution: potential flow

Navier–Stokes

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u u)}{\partial x} + \frac{\partial(\rho v u)}{\partial y} = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho u v)}{\partial x} + \frac{\partial(\rho v v)}{\partial y} = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$u_n|_w = u_t|_w = 0$$

Euler

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \quad (3)$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u u)}{\partial x} + \frac{\partial(\rho v u)}{\partial y} = -\frac{\partial p}{\partial x} \quad (1)$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho u v)}{\partial x} + \frac{\partial(\rho v v)}{\partial y} = -\frac{\partial p}{\partial y} \quad (2)$$

$$u_n|_w = 0 \qquad u_t|_w \neq 0$$

Obtain the vorticity equation:

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Need to use (3) to simplify to these terms

$$\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1) \quad \Rightarrow \quad \frac{\partial \omega}{\partial t} + \frac{\partial u\omega}{\partial x} + \frac{\partial v\omega}{\partial y} = 0$$

1) See that $\omega = 0$ is a non-trivial, i.e. $(u, v) \neq 0$, solution of Euler.

2) Can find potential flow solution that satisfies the free-slip BC.

Potential flow

Can show that potential flow also satisfy the Navier–Stokes equations:

Navier–Stokes

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \quad (3)$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu)}{\partial y} = -\frac{\partial p}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \quad (1)$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho vv)}{\partial y} = -\frac{\partial p}{\partial y} + \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) \quad (2)$$

$$u_n|_w = u_t|_w = 0$$

Obtain the vorticity equation:

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Need to use (3) to simplify to these terms

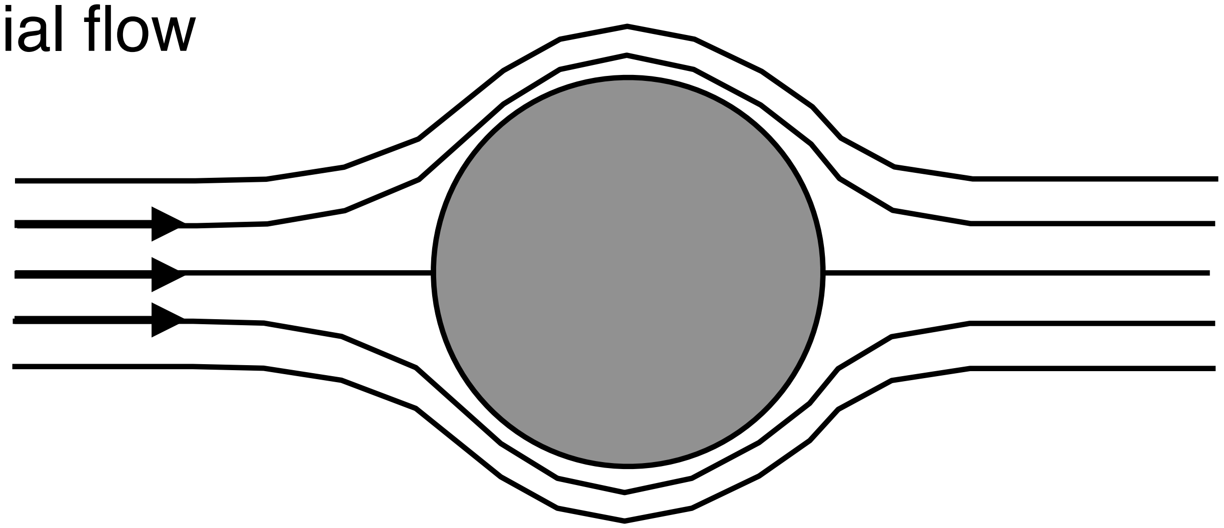
$$\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1) \quad \Rightarrow \quad \frac{\partial \omega}{\partial t} + \frac{\partial u\omega}{\partial x} + \frac{\partial v\omega}{\partial y} = \nu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right)$$

1) See that $\omega = 0$ is a non-trivial, i.e. $(u, v) \neq 0$, solution of Navier–Stokes.

2) Cannot find potential flow solution that satisfies both free-slip and no-slip BCs!

Resolution of d'Alembert's paradox

Potential flow



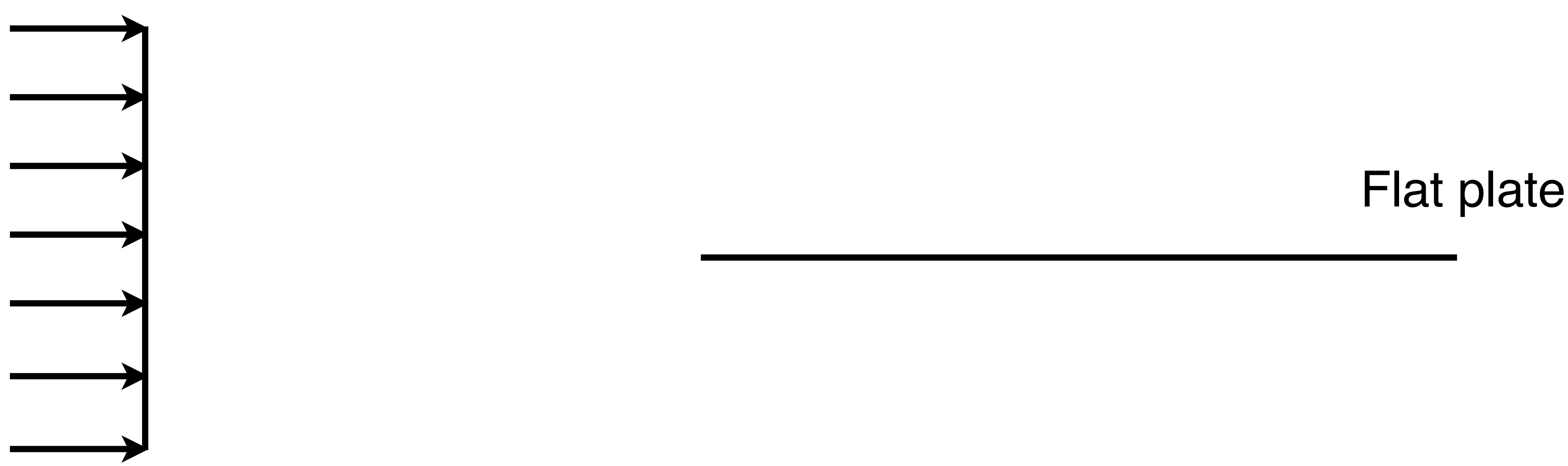
- So what went wrong with potential flow? Potential flow (irrotational, incompressible flow) does not satisfy the no-slip boundary condition, although it satisfies the impermeable boundary condition.

- So what gives? Incompressibility is still satisfied, but the flow can no longer be irrotational in order to satisfy the no-slip boundary condition. However, the rotational region is confined to a small region adjacent to the boundary called the boundary layer.

- Viscosity allows the no-slip boundary condition, which 1) gives rise to skin-friction (viscous) drag (direct effect) and 2) gives rise to the boundary layer, which, if separated (see later), significantly affects the form (pressure) drag (indirect effect).

Potential flow (irrotational, incompressible flow) and boundary layers

How does a flow that is irrotational become rotational (gain vorticity)?



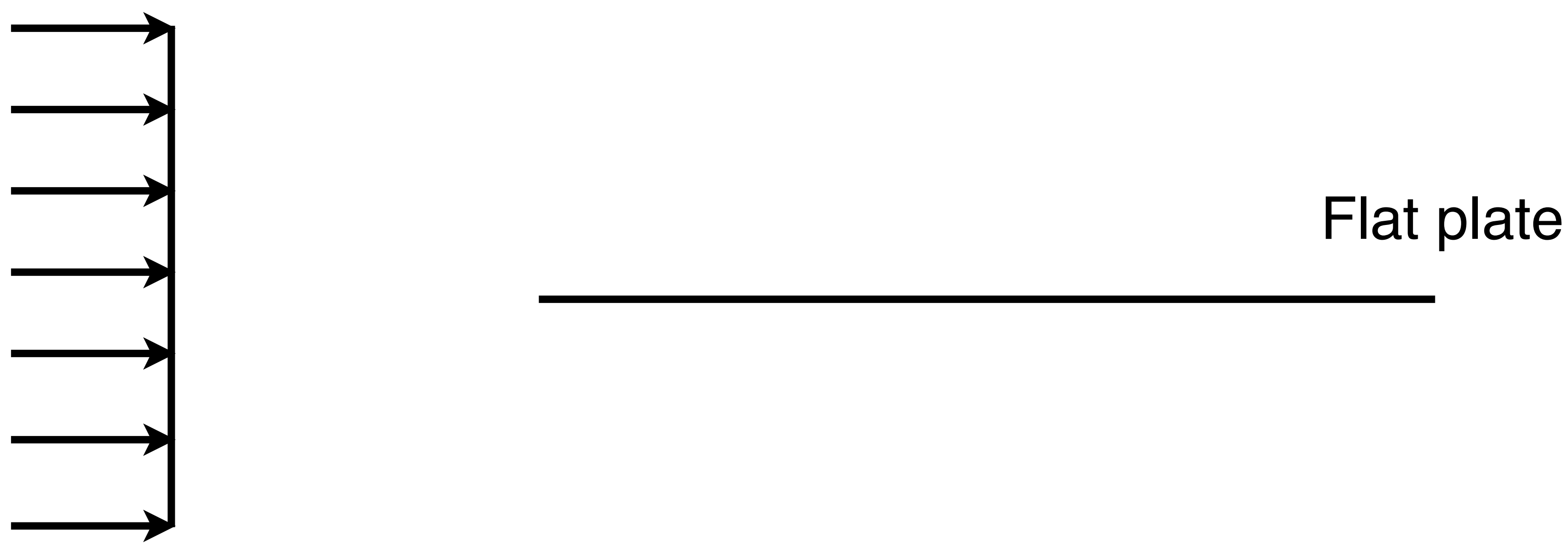
$$u = U_\infty$$

$$v = 0$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

Potential flow (irrotational, incompressible flow) and boundary layers

How does a flow that is irrotational become rotational (gain vorticity)?



Potential flow (irrotational, incompressible flow) and boundary layers

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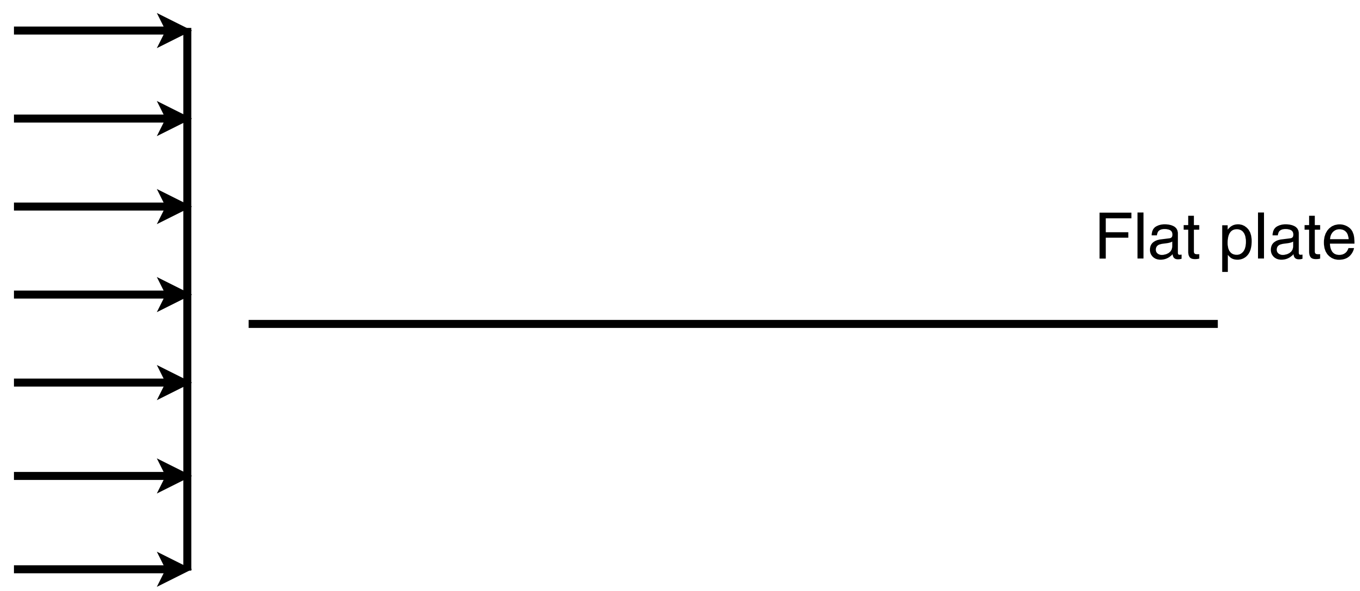
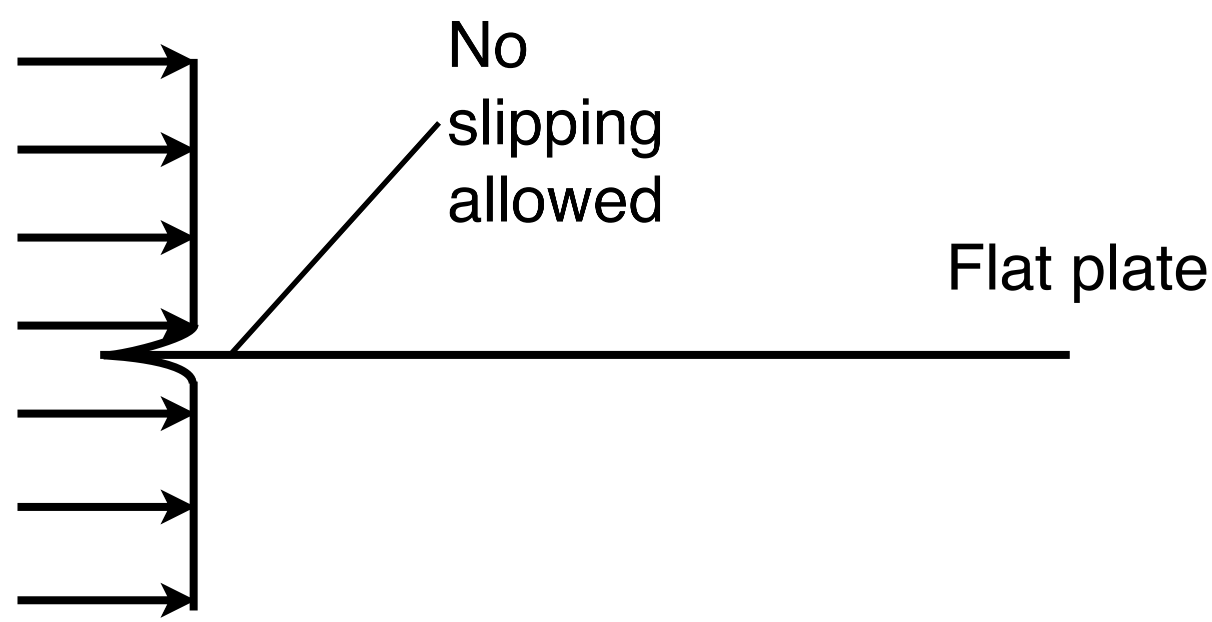


Potential flow (irrotational, incompressible flow) and boundary layers

How does a flow that is irrotational become rotational (gain vorticity)?



Potential flow (irrotational, incompressible flow) and boundary layers

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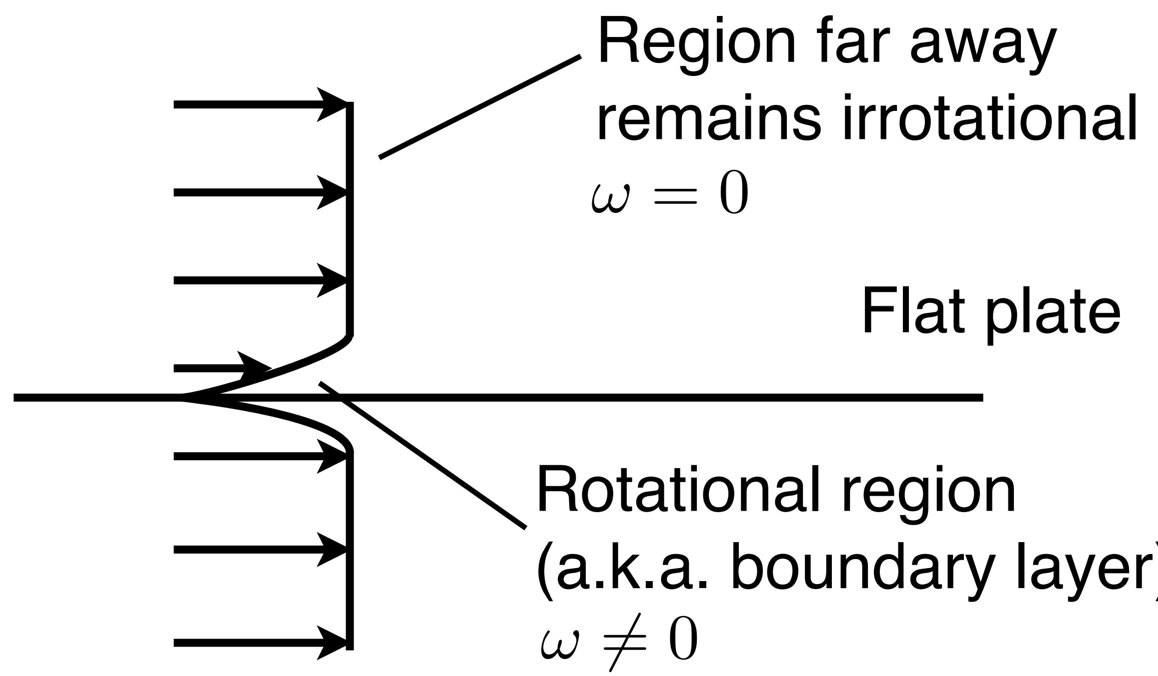
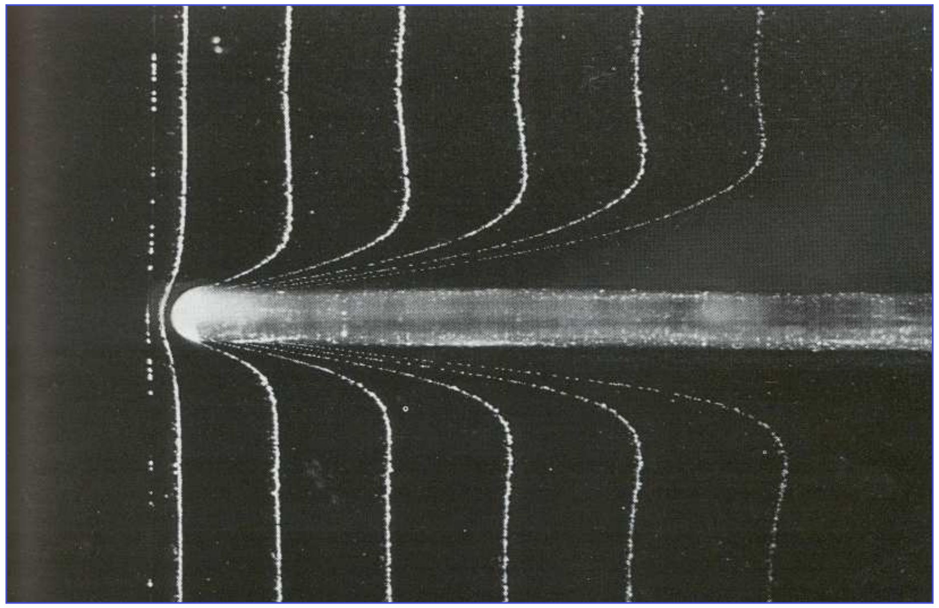


Don’t believe it? Watch this video:

http://www.youtube.com/watch?v=loCLkcYEWD4&t=8m35s

Potential flow (irrotational, incompressible flow) and boundary layers

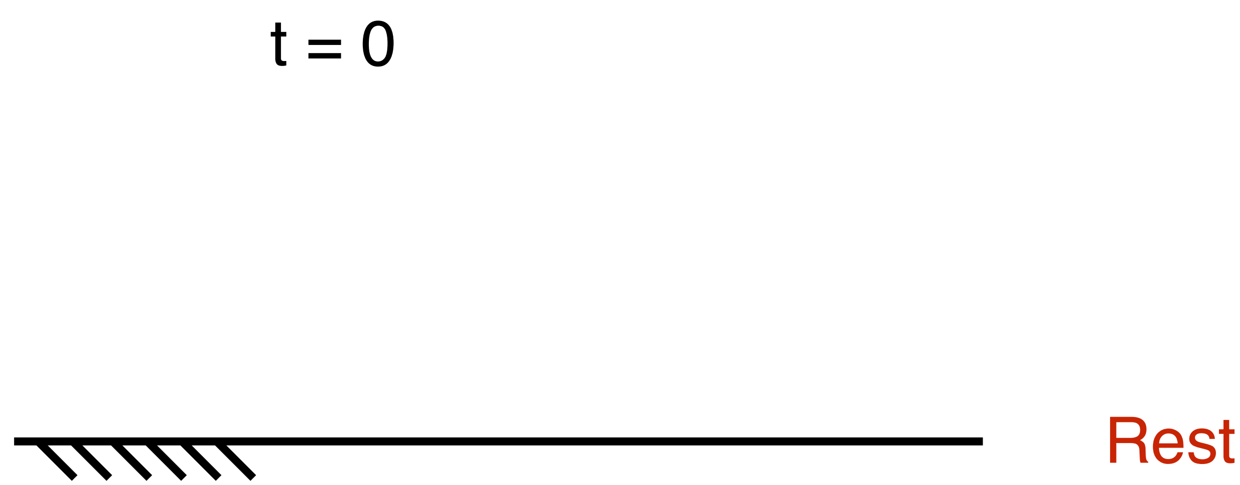
How does a flow that is irrotational become rotational (gain vorticity)?



(http://maeresearch.ucsd.edu/~sarkar/jim_rohr/mae101b-SumII2006/Ch9a.ppt)

Stokes' first problem/Rayleigh problem

Recall example of impulsively accelerated wall (see Schlichting).



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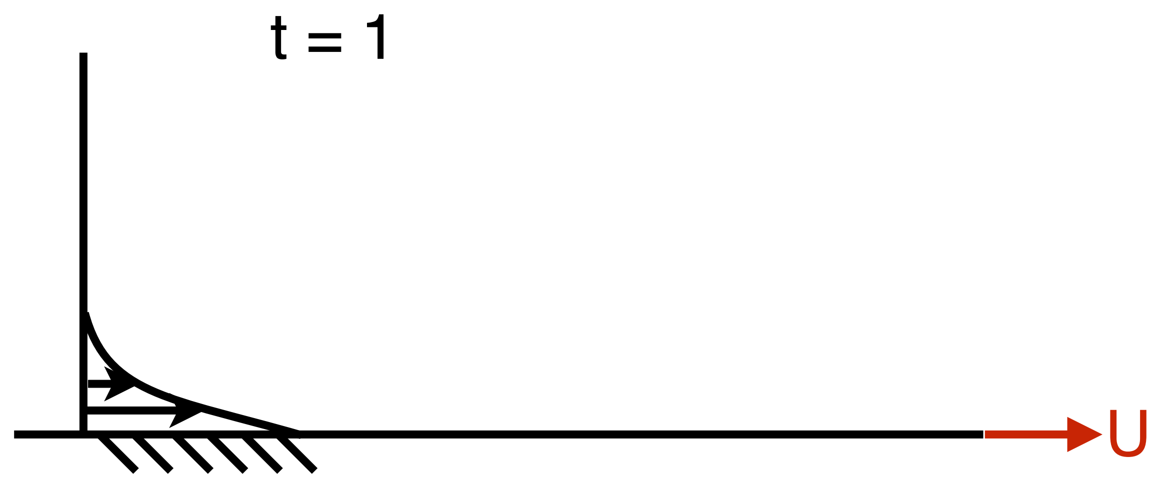

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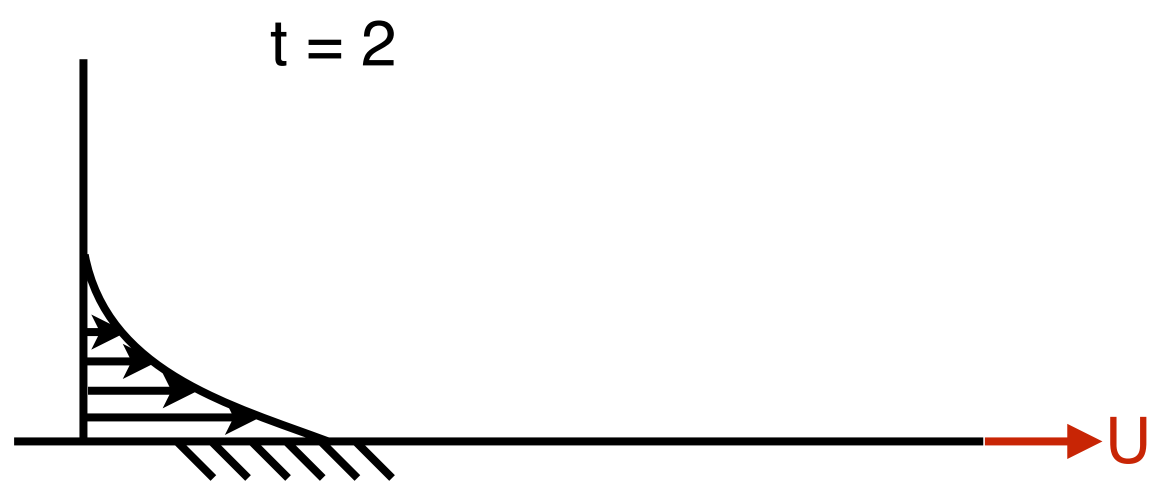

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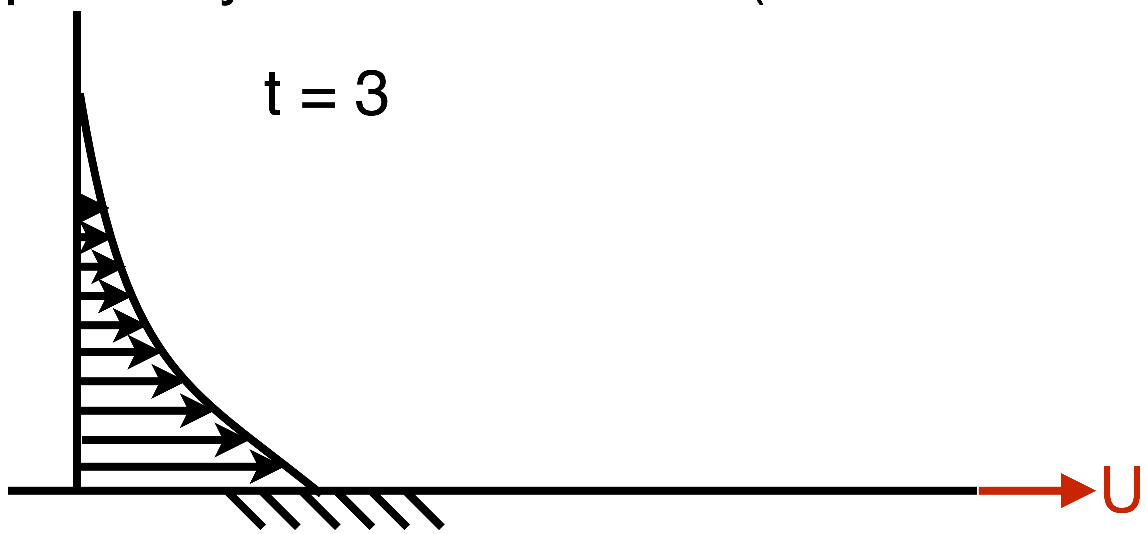

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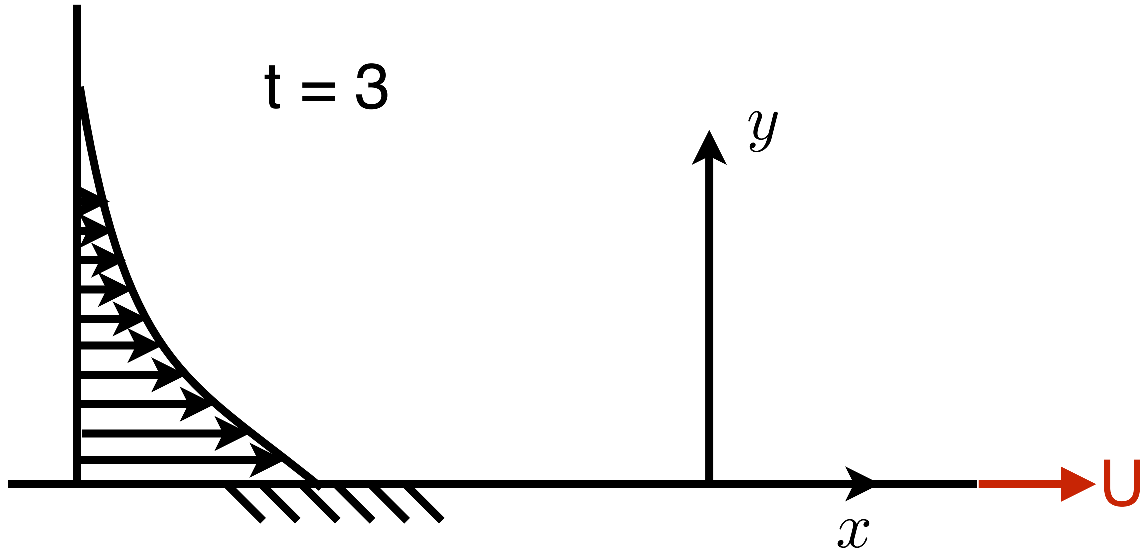


Governing equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Boundary conditions:

$$u(y=0) = U \qquad u(y\to\infty) = 0$$

$$v(y=0) = 0 \qquad v(y\to\infty) = 0$$

Infinite plate: $\dfrac{\partial}{\partial x}(\cdot) = 0$

Can show from this that:

$$v = 0, \quad p = \text{const.},$$

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}$$

Stokes' first problem/Rayleigh problem

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

Pro tip: similarity solution typically arises when there is no natural length scale (no pipe diameter, no channel width)

Guess similarity solution: $u(y,t) = Uf(\eta), \quad \eta = y/\delta(t)$

Plug into x-momentum equation to find: $f'' + \dfrac{\delta\delta'}{\nu}\eta f' = 0$

For similarity solution to exist, we must arrive at an ODE in terms of f and its derivatives and η.

In particular, no dependence on y and t.

This can only be achieved if $\delta\delta'/\nu = \text{const.} = C$

That is, $\dfrac{\mathrm{d}\delta^2}{\mathrm{d}t} = 2\nu C$ and $f'' + C\eta f' = 0$

Melbourne School of Engineering
MCEN90018 Advanced Fluid Dynamics

Lecture BL04: Stokes' first problem/Rayleigh problem

15 March 2016

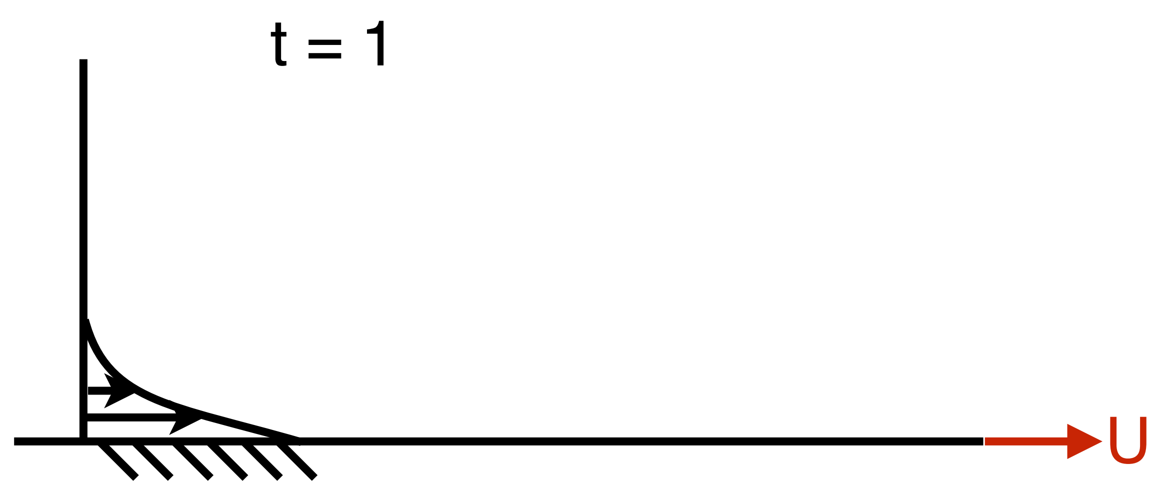
Stokes' first problem/Rayleigh problem

Recall example of impulsively accelerated wall (see Schlichting).

$t = 0$

Stokes' first problem/Rayleigh problem

Recall example of impulsively accelerated wall (see Schlichting).



Stokes' first problem/Rayleigh problem

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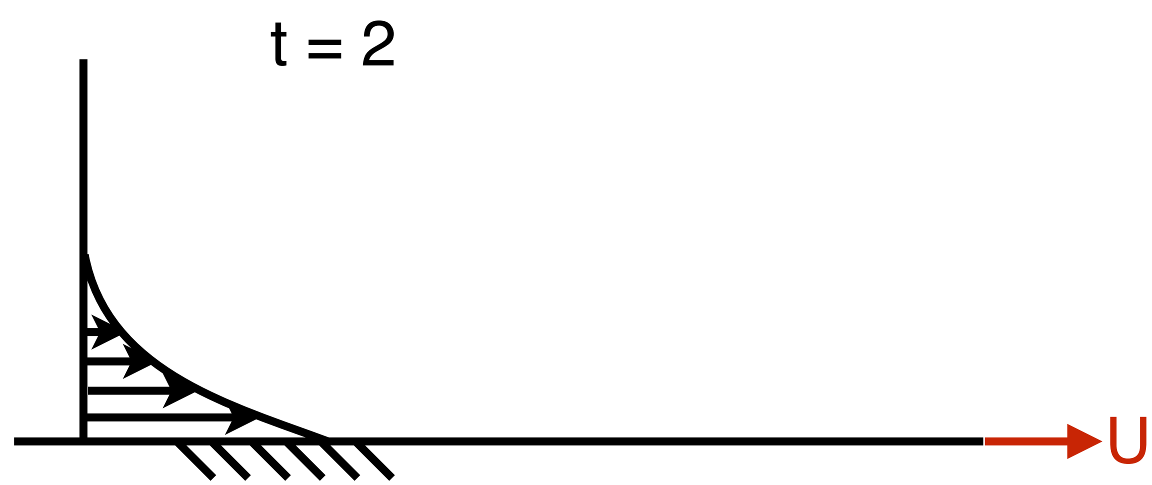

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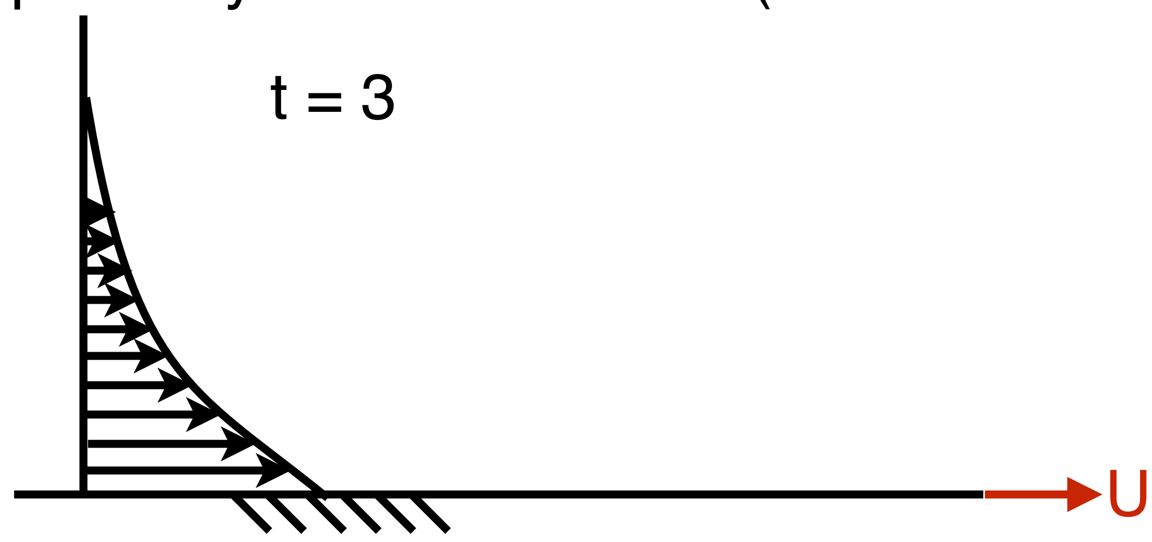

Stokes' first problem/Rayleigh problem

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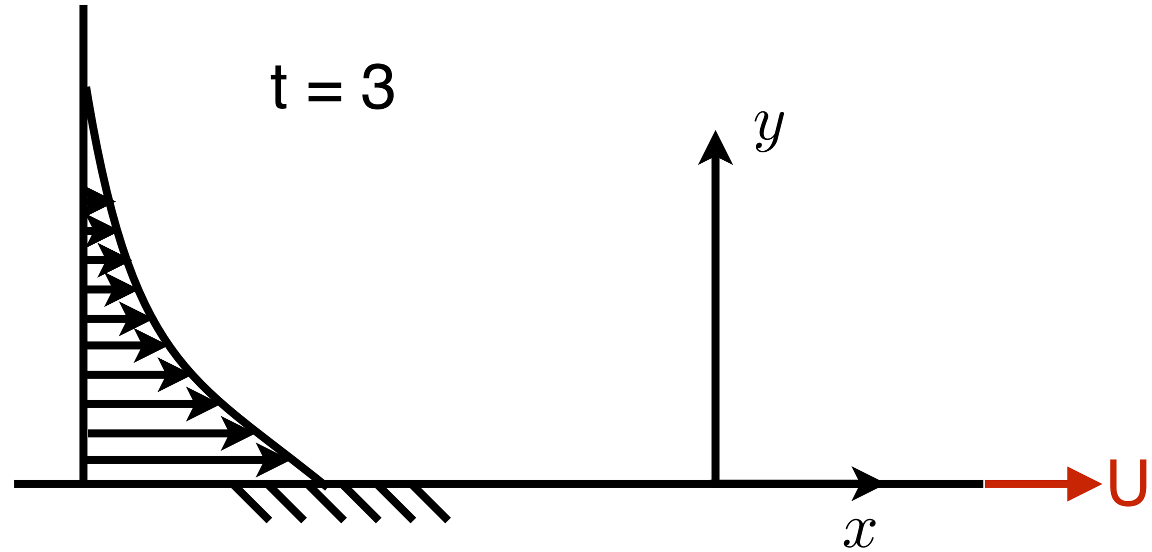

Governing equations:

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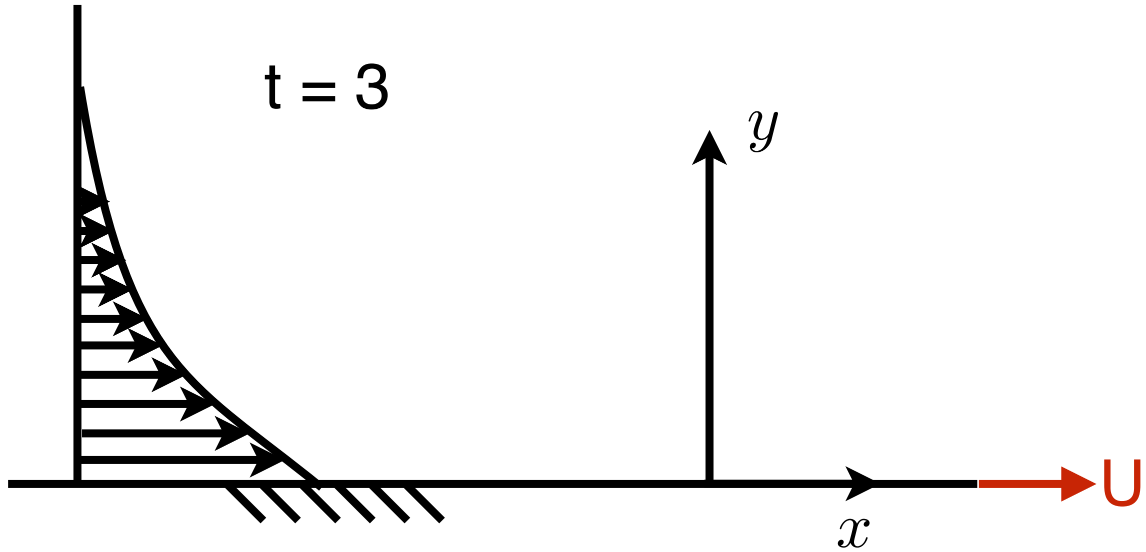
$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Stokes' first problem/Rayleigh problem

Recall example of impulsively accelerated wall (see Schlichting).



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Stokes' first problem/Rayleigh problem

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

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Guess similarity solution: $u(y,t) = Uf(\eta), \quad \eta = y/\delta(t)$

Plug into x-momentum equation to find: $f'' + \frac{\delta\delta'}{\nu}\eta f' = 0$

For similarity solution to exist, we must arrive at an ODE in terms of f and its derivatives and η.

In particular, no dependence on y and t.

This can only be achieved if $\delta\delta'/\nu = \text{const.} = C$

That is, $\frac{\mathrm{d}\delta^2}{\mathrm{d}t} = 2\nu C$ and $f'' + C\eta f' = 0$

Stokes' first problem/Rayleigh problem

$$u(y,t) = Uf(\eta), \quad \eta = y/\delta(t)$$

Boundary/initial conditions:

$$u = 0 \quad \text{at} \quad t = 0, y \geq 0 \quad \Rightarrow \quad f(\frac{y}{\delta(0)}) = 0$$

$$u = U \quad \text{at} \quad t > 0, y = 0 \quad \Rightarrow \quad f(0) = 1$$

$$u = 0 \quad \text{at} \quad t > 0, y = \infty \quad \Rightarrow \quad f(\infty) = 0$$

Boundary conditions must collapse so $\delta(0) = 0$

Summary:

$$\frac{\mathrm{d}\delta^2}{\mathrm{d}t} = 2\nu C \qquad \delta(0) = 0$$

and

$$f'' + C\eta f' = 0 \qquad f(0) = 1, \quad f(\infty) = 0$$

Stokes' first problem/Rayleigh problem

Solution: $u = U \operatorname{erfc}(\eta)$ $\qquad \eta \equiv \dfrac{y}{2\sqrt{\nu t}}$ $\qquad \operatorname{erfc}(\eta) \equiv 1 - \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^{\eta} \mathrm{e}^{-s^2}\, \mathrm{d}s$

Stokes' first problem/Rayleigh problem

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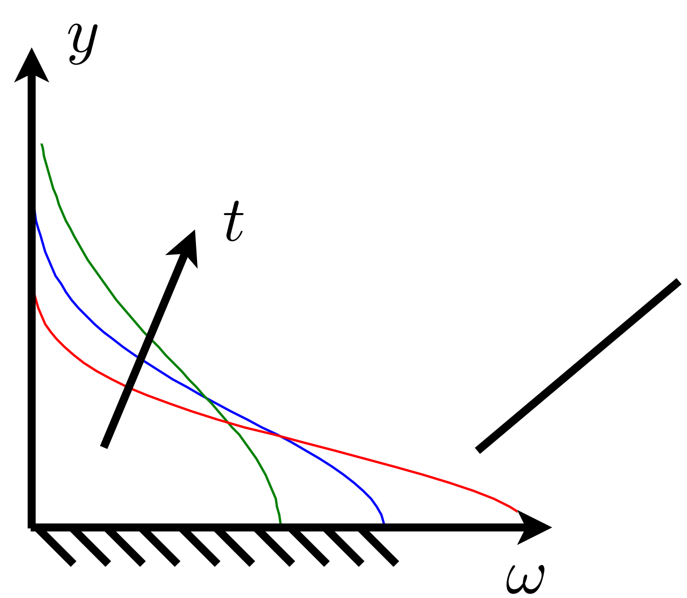
Vorticity: $\omega = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$

$$\omega = \frac{U}{\sqrt{\pi\nu t}}\mathrm{e}^{-[y/(2\sqrt{\nu t})]^2}$$



Vorticity confined near wall, and diffuses away from wall with time.

So potential flow (irrotational, incompressible flow) is not applicable near wall.

But note that the thickness of the layer is small if viscosity is small (for fixed U and t).

Stokes' first problem/Rayleigh problem

Solution: $u = U\operatorname{erfc}(\eta)$ $\quad \eta \equiv \dfrac{y}{2\sqrt{\nu t}}$ $\quad \operatorname{erfc}(\eta) \equiv 1 - \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^{\eta} \mathrm{e}^{-s^2}\,\mathrm{d}s$

Vorticity: $\omega = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$

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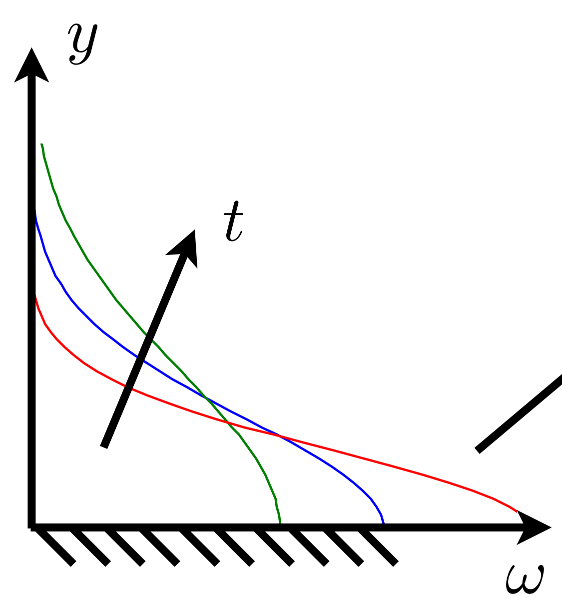


Vorticity confined near wall, and diffuses away from wall with time.

Skin friction: $\tau_0 = \mu \dfrac{\partial u}{\partial y}\Big|_{y=0} = -\mu\omega\big|_{y=0}$

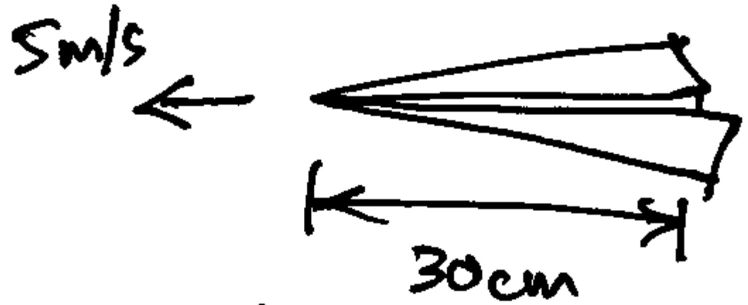
$$\frac{\tau_0}{\rho} = -\frac{\sqrt{\nu}U}{\sqrt{\pi t}}$$

Melbourne School of Engineering
MCEN90018 Advanced Fluid Dynamics

Lecture BL05: Prandtl's boundary layer approximation
17 March 2016

Estimate thickness of boundary layer on paper aeroplane

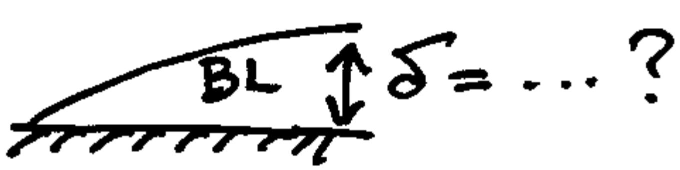
Paper aeroplane



Air: $\nu = 1.5 \times 10^{-5}\ m^2/s$

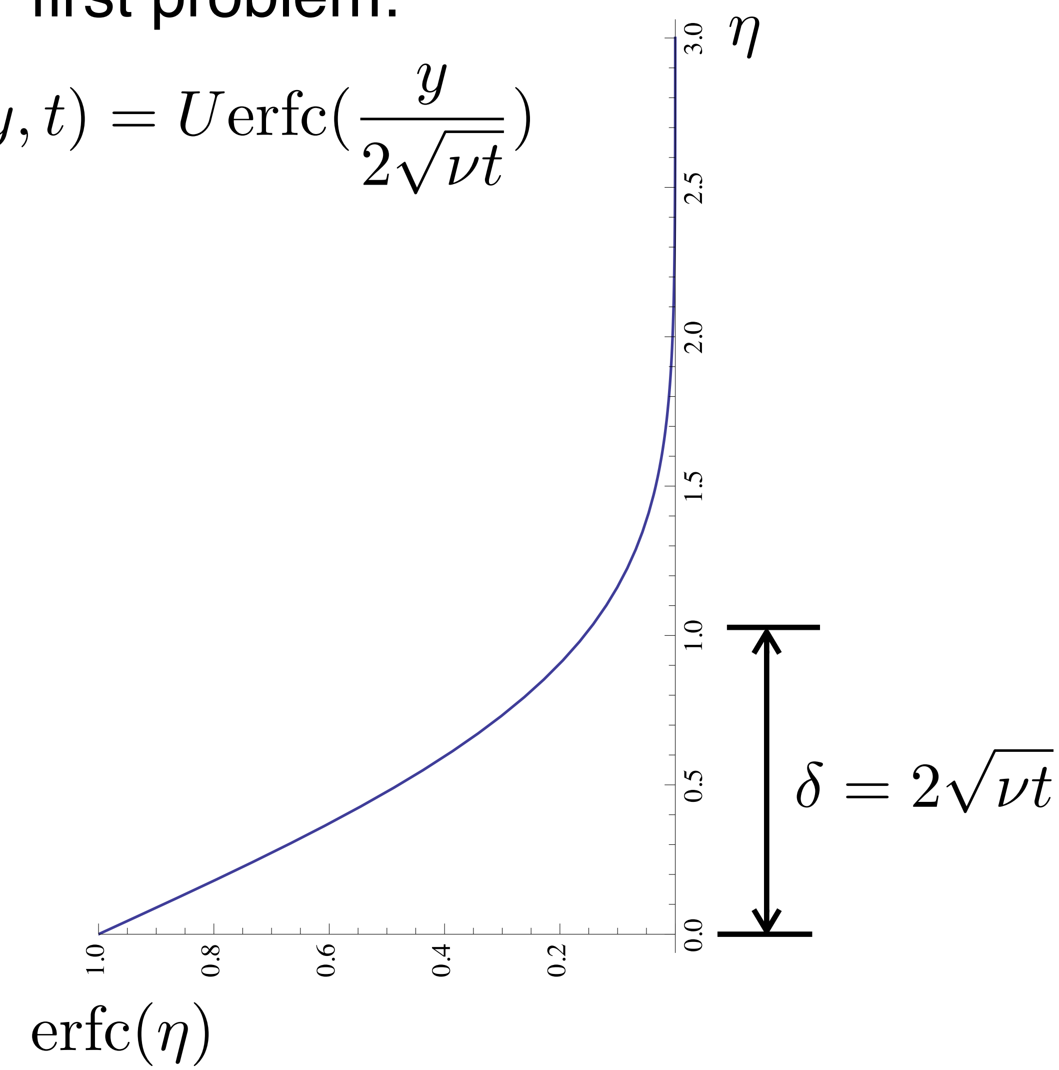
$Re = 100\ 000$

Model with infinite flat plate

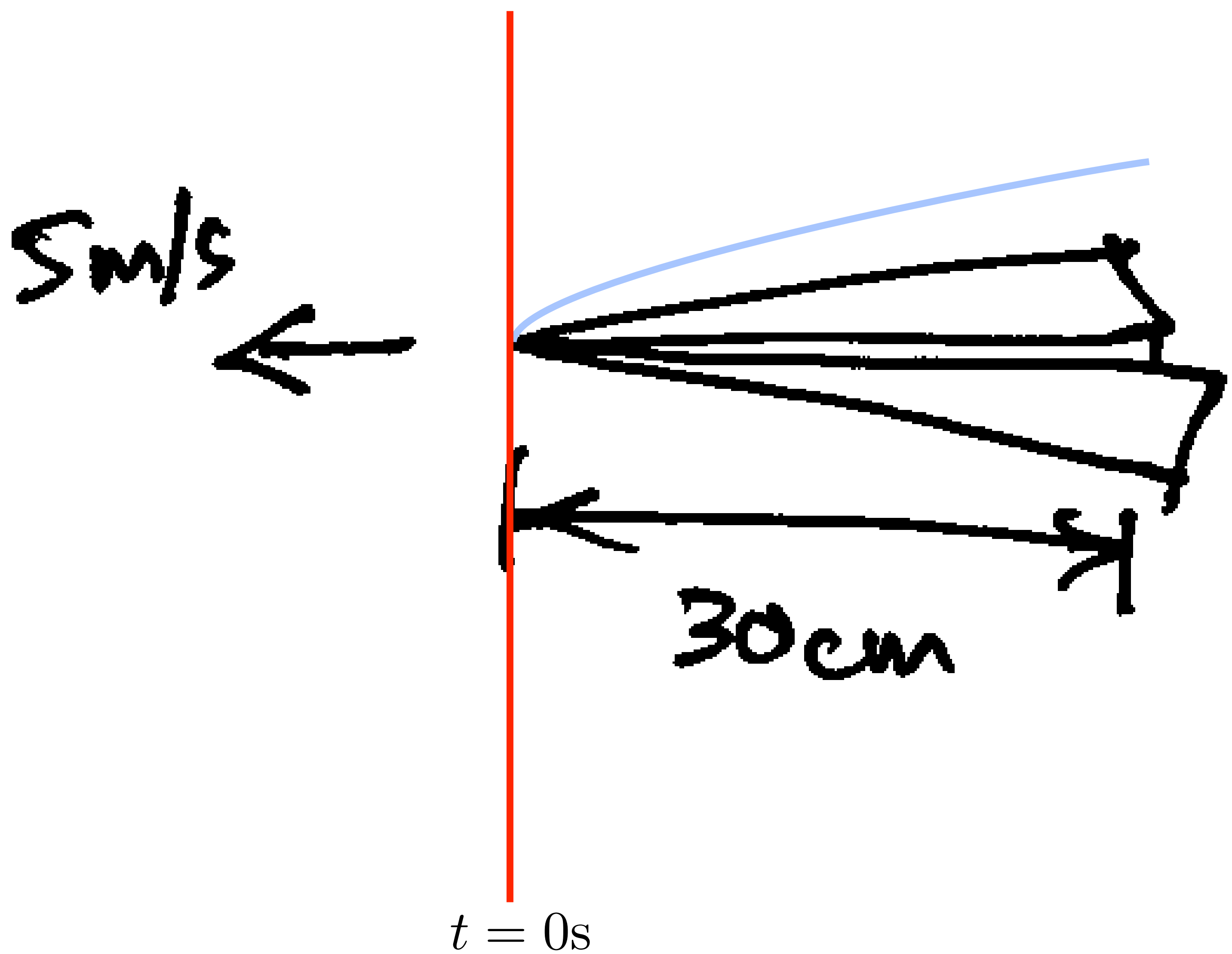


Stokes' first problem:

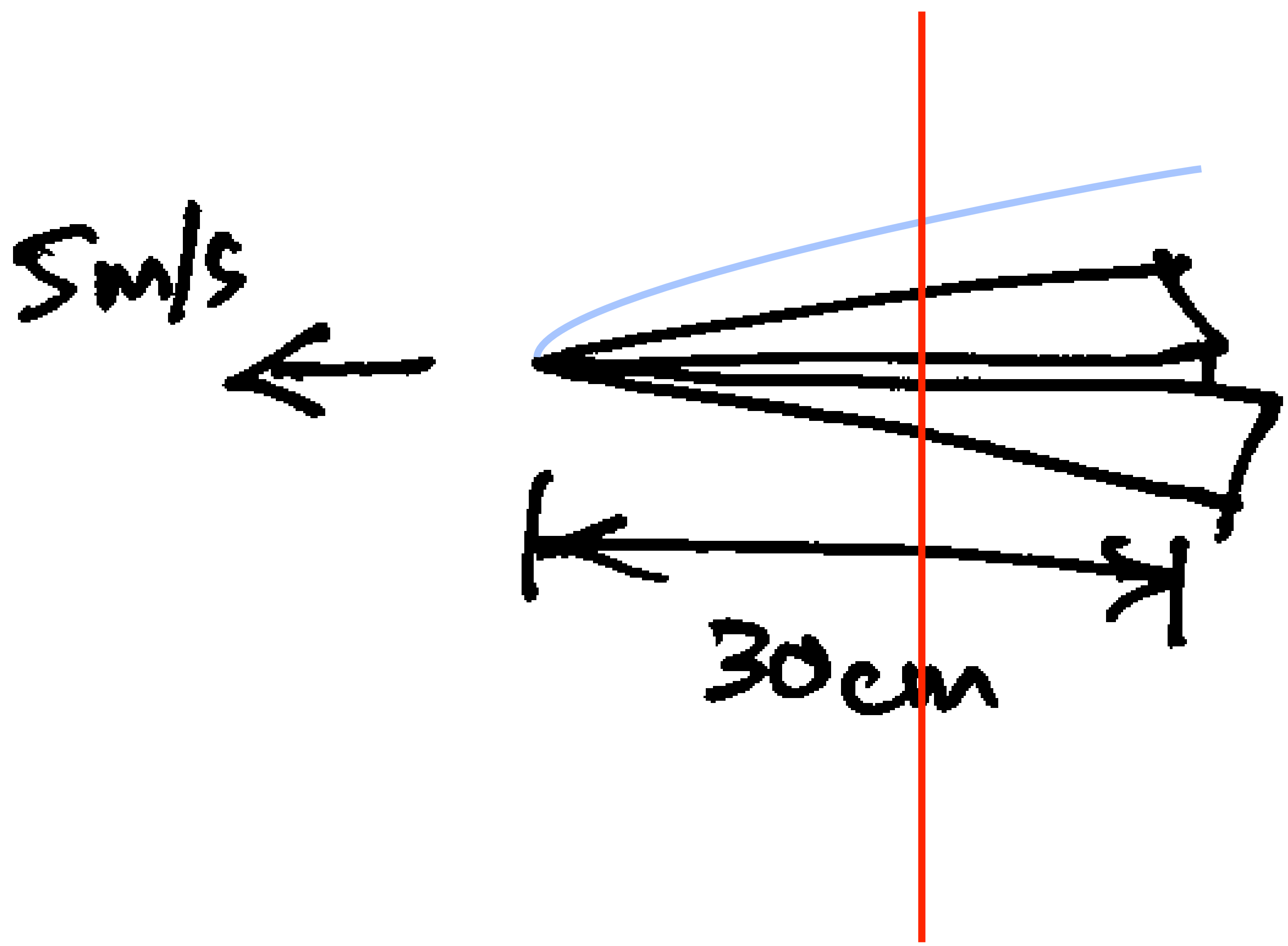
$$u(y,t) = U \mathrm{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right)$$



Estimate thickness of boundary layer on paper aeroplane

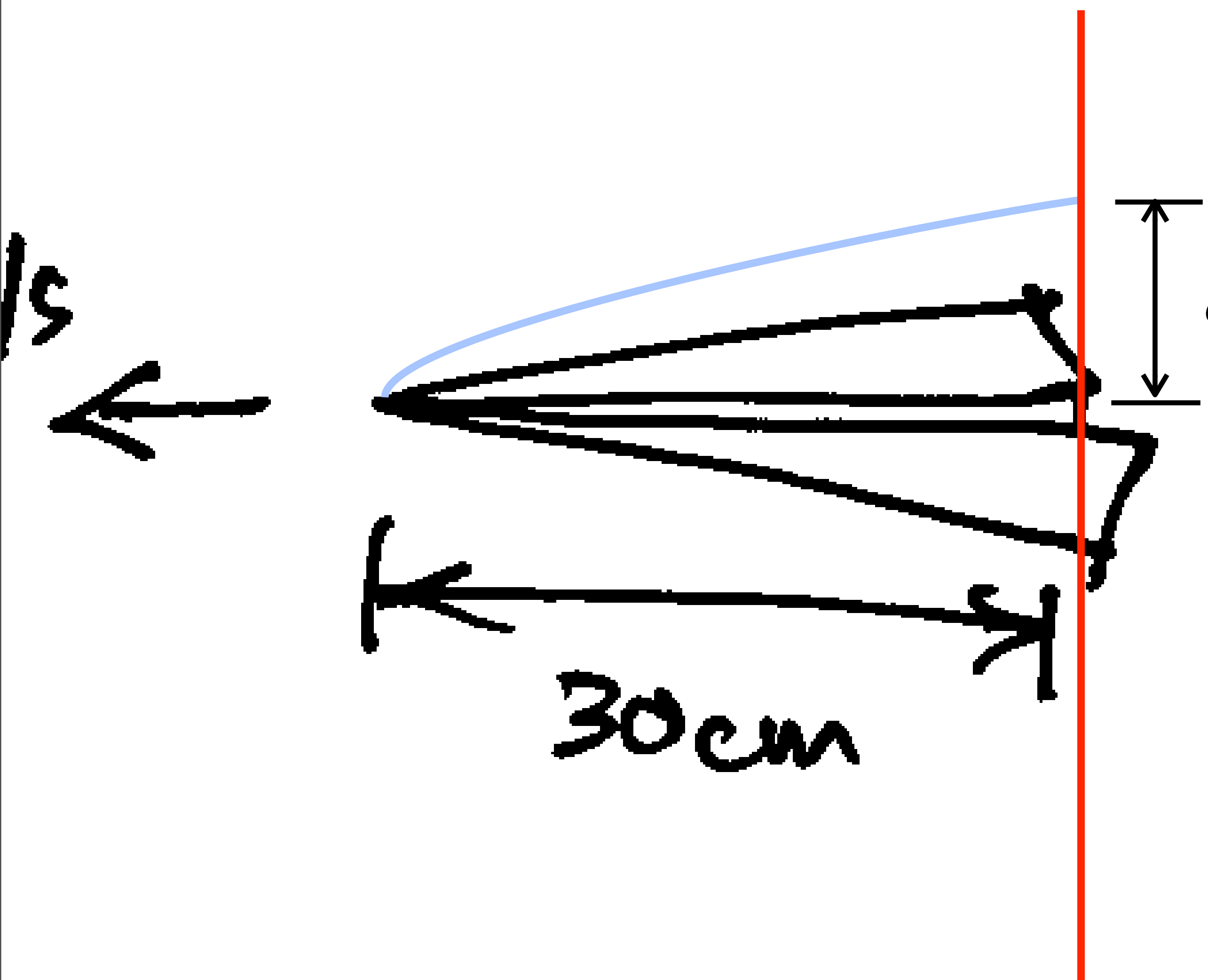


Estimate thickness of boundary layer on paper aeroplane



$t = 0.03\mathrm{s}$

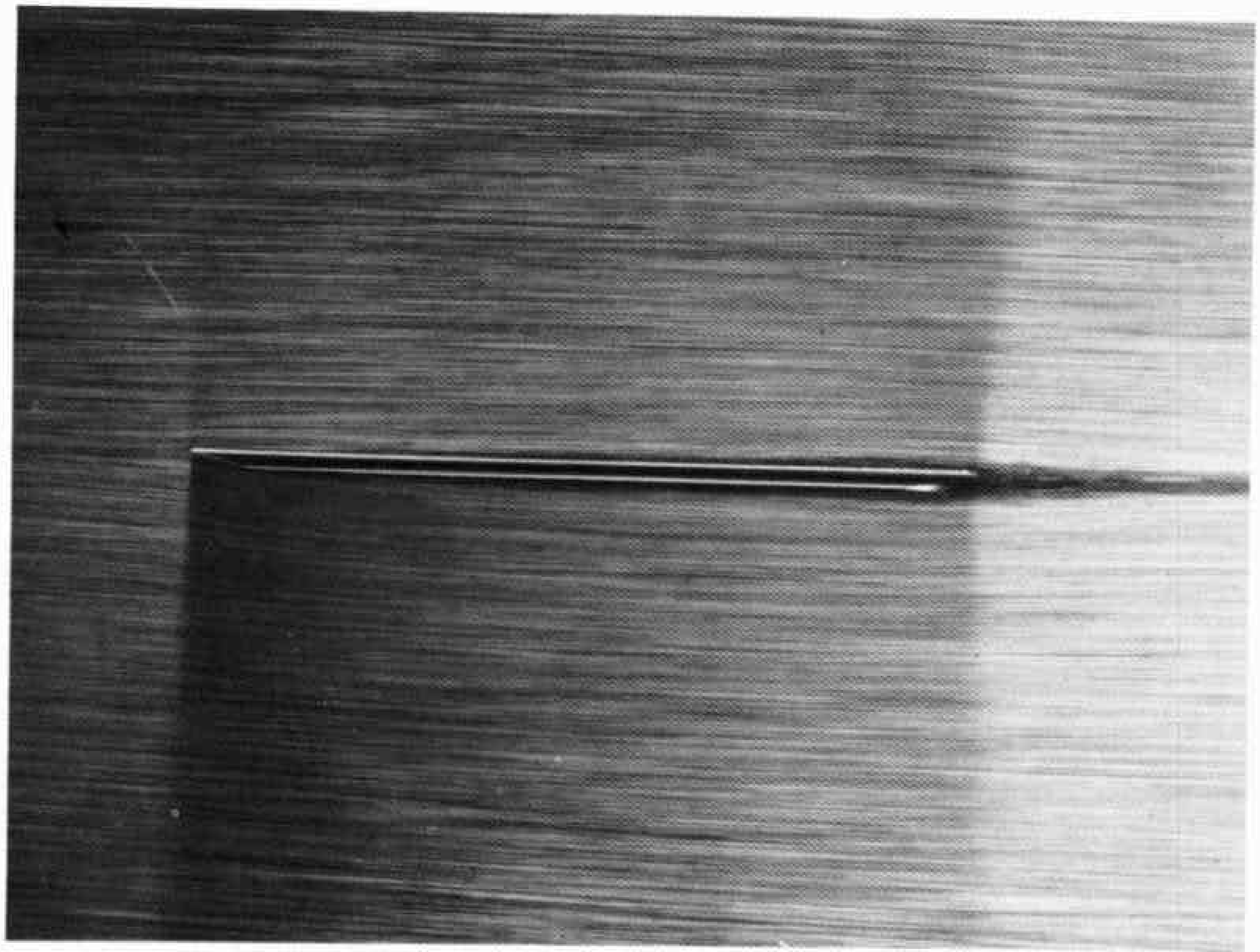
Estimate thickness of boundary layer on paper aeroplane



$$\delta = 2\sqrt{\nu t} = 2\sqrt{(1.5 \times 10^{-5}\,\mathrm{m}^2/\mathrm{s})(0.06\,\mathrm{s})}$$
$$= 1.9\,\mathrm{mm}$$

$t = 0.06\mathrm{s}$

Prandtl's idea

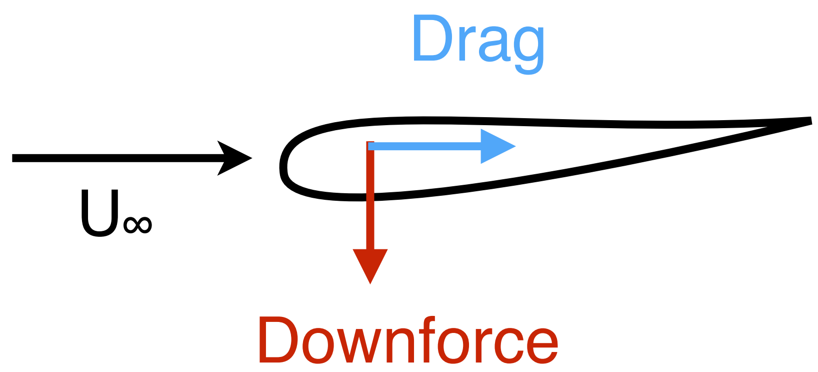


Flat plate (Re = 10 000, $\alpha = 0$) (Van Dyke)

Motivating problem

Say you are tasked with the aerodynamic design of a front wing on an F1 car.

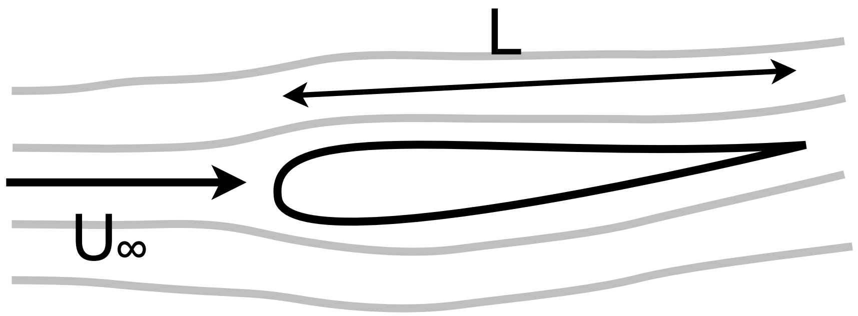
http://www.grandprix.com.au/fan-zone/galleries/2016-testing-2-day-3



Navier–Stokes equations for incompressible, constant-density, steady, two-dimensional flow

Conservation of mass:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$



x-momentum:

$$\frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

y-momentum:

$$\frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

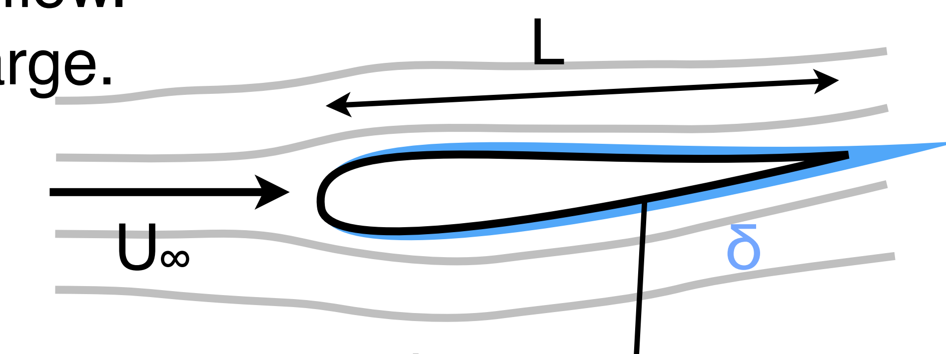
Boundary conditions at wall:

$$u = v = 0$$

Equations are quite complicated and do not necessarily help in understanding the various aspects of the flow needed for design.

Prandtl’s idea

Viscous forces unimportant over most of flow. $U_\infty L/\nu$ large.

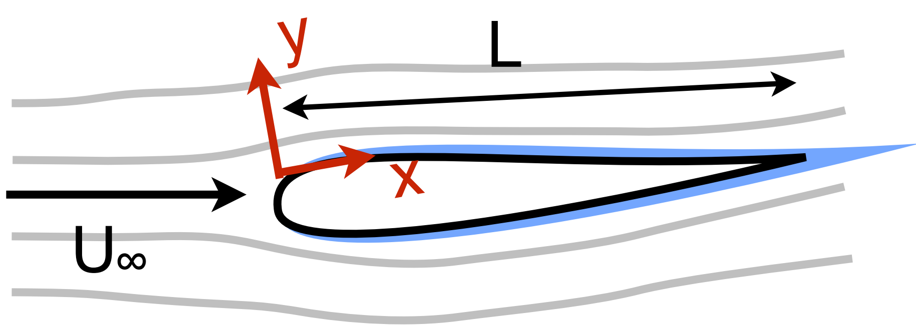


Thin layer, viscous forces important, no-slip BC. $U_\infty \delta/\nu$ small!

“The boundary-layer hypothesis helps to reconcile the intuitive expectation that the effects of viscosity on the flow are unimportant, at any rate over most of the flow field, when the viscosity is small, with the fact that the no-slip condition must be satisfied at a solid boundary however small the viscosity may be; indeed this reconciliation was Prandtl’s main objective and was a land-mark in the development of fluid mechanics.” (Batchelor 1967)

Now the flow has a simpler structure and a framework for understanding the flow. But we need to turn words and ideas into precise mathematical statements.

Prandtl's idea



We want to focus on the action. What we need is the right scales/rulers to observe the various flow features of interest (boundary layer/far field).

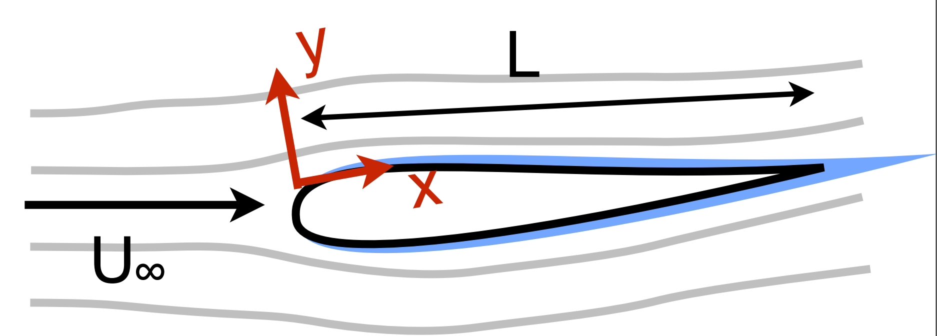
-You don't bring a knife to a gun fight.
-You don't use a microscope to look at stars.
-You don't use a kitchen scale to weigh yourself.
-Neither should you use the wing's chord length as ruler to observe the boundary layer.

Navier–Stokes equations describe fluid flow across the full range of scales. So let's apply the right scales for each of the flow features of interest.

Here, we can see that X = L is a good choice for the wall-parallel length scale. Also U = U_∞ is a good choice for wall-parallel velocity scale.

Prandtl's idea

Scale quantities with appropriate ruler so that variations in tilde quantities can be represented by "nice" numbers (as Re gets large), e.g. 0.2, 1, 5, but probably not 0.001 or 1000.



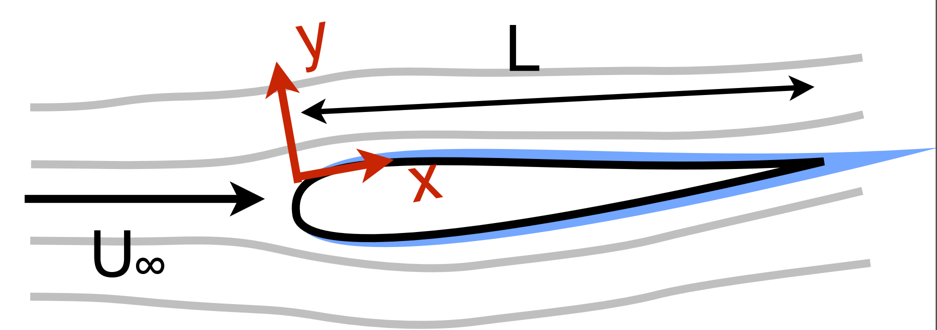
$$\delta x = X \delta \widetilde{x} \qquad \delta u = U \delta \widetilde{u}$$

$$\delta y = Y \delta \widetilde{y} \qquad \delta v = V \delta \widetilde{v}$$

U, V, X and Y are the scales/rulers to be determined for each flow problem.

We are interested in scaling the change in these quantities, not their absolute values, because the absolute values are inconsequential in the equations of motion. The equations only contain d(something)/d(something) terms, i.e. only changes in quantities.

Prandtl's idea



$$\delta x = X\delta\widetilde{x} \qquad \delta u = U\delta\widetilde{u}$$

$$\delta y = Y\delta\widetilde{y} \qquad \delta v = V\delta\widetilde{v}$$

Explicit change of variables:
$$y(\tilde{y}) = y_0 + Y(\tilde{y} - \tilde{y}_0)$$
$$x(\tilde{x}) = x_0 + X(\tilde{x} - \tilde{x}_0)$$
$$u(x, y) = u_0 + U\tilde{u}(\tilde{x}(x), \tilde{y}(y))$$
$$v(x, y) = v_0 + V\tilde{v}(\tilde{x}(x), \tilde{y}(y))$$

So: $$\frac{\partial u}{\partial x} = U\frac{\partial \tilde{u}}{\partial \tilde{x}}\frac{\mathrm{d}\tilde{x}}{\mathrm{d}x} = U\frac{\partial \tilde{u}}{\partial \tilde{x}}\frac{1}{X} = \frac{U}{X}\frac{\partial \tilde{u}}{\partial \tilde{x}}$$ etc.

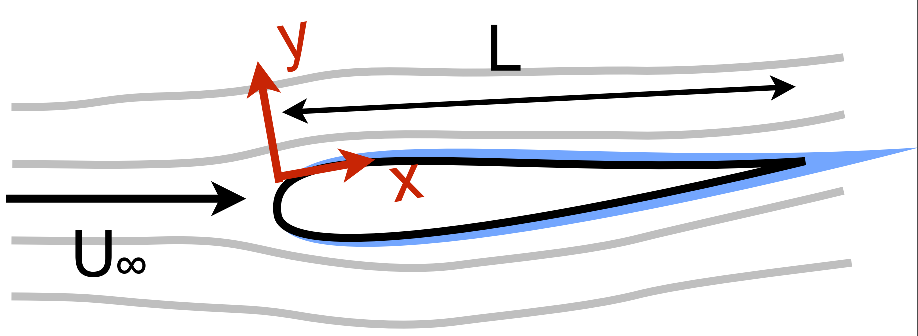
Conservation of mass then becomes: $$\frac{U}{X}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{V}{Y}\frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

Prandtl's idea

Use rulers in conservation of mass:

$$\frac{U}{X}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{V}{Y}\frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

O(1) O(1)



This constrains the wall-normal velocity to

$$V = U\frac{Y}{X}$$

$$\boxed{\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0} \quad (1)$$

Prandtl's idea

Use rulers in x-momentum:

$$\frac{U^2}{X}\frac{\partial \tilde{u}\tilde{u}}{\partial \tilde{x}} + \frac{VU}{Y}\frac{\partial \tilde{v}\tilde{u}}{\partial \tilde{y}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu U}{X^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\nu U}{Y^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

Use constraint from continuity V = U(Y/X):

$$\frac{U^2}{X}\frac{\partial \tilde{u}\tilde{u}}{\partial \tilde{x}} + \frac{U^2}{X}\frac{\partial \tilde{v}\tilde{u}}{\partial \tilde{y}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu U}{X^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\nu U}{Y^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

× X/U²:

$$\boxed{\underbrace{\frac{\partial \tilde{u}\tilde{u}}{\partial \tilde{x}}}_{O(1)} + \underbrace{\frac{\partial \tilde{v}\tilde{u}}{\partial \tilde{y}}}_{O(1)} = -\frac{\partial[p/(\rho U^2)]}{\partial(x/X)} + \frac{\nu}{UX}\underbrace{\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}}_{O(1)} + \frac{\nu}{UX}\frac{X^2}{Y^2}\underbrace{\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}}_{O(1)}} \qquad (2)$$

Prandtl's idea

Use rulers in y-momentum:

$$\frac{UV}{X}\frac{\partial \tilde{u}\tilde{v}}{\partial \tilde{x}} + \frac{V^2}{Y}\frac{\partial \tilde{v}\tilde{v}}{\partial \tilde{y}} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\nu V}{X^2}\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{\nu V}{Y^2}\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}$$

Use constraint from continuity V = U(Y/X):

$$\frac{U^2 Y}{X^2}\frac{\partial \tilde{u}\tilde{v}}{\partial \tilde{x}} + \frac{U^2 Y}{X^2}\frac{\partial \tilde{v}\tilde{v}}{\partial \tilde{y}} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\nu U Y}{X^3}\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{\nu U}{XY}\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}$$

× Y/U^2:

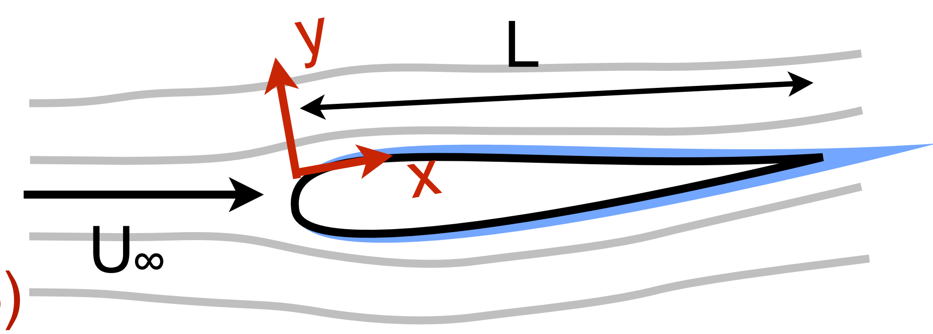
$$\boxed{\frac{Y^2}{X^2}\underbrace{\frac{\partial \tilde{u}\tilde{v}}{\partial \tilde{x}}}_{O(1)} + \frac{Y^2}{X^2}\underbrace{\frac{\partial \tilde{v}\tilde{v}}{\partial \tilde{y}}}_{O(1)} = -\frac{\partial[p/(\rho U^2)]}{\partial(y/Y)} + \frac{\nu}{UX}\frac{Y^2}{X^2}\underbrace{\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2}}_{O(1)} + \frac{\nu}{UX}\underbrace{\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}}_{O(1)}} \qquad (3)$$

Prandtl's idea

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0 \quad (1)$$

$$\frac{\partial \tilde{u}\tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}\tilde{u}}{\partial \tilde{y}} = -\frac{\partial[p/(\rho U^2)]}{\partial(x/X)} + \frac{\nu}{UX}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\nu}{UX}\frac{X^2}{Y^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} \quad (2)$$

$$\frac{Y^2}{X^2}\frac{\partial \tilde{u}\tilde{v}}{\partial \tilde{x}} + \frac{Y^2}{X^2}\frac{\partial \tilde{v}\tilde{v}}{\partial \tilde{y}} = -\frac{\partial[p/(\rho U^2)]}{\partial(y/Y)} + \frac{\nu}{UX}\frac{Y^2}{X^2}\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{\nu}{UX}\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2} \quad (3)$$



Typically, viscosity is fixed (air or water), X = L is fixed (given object), but the Reynolds number, Re = $U_\infty L/v$ changes through U = U_∞ (high speed).

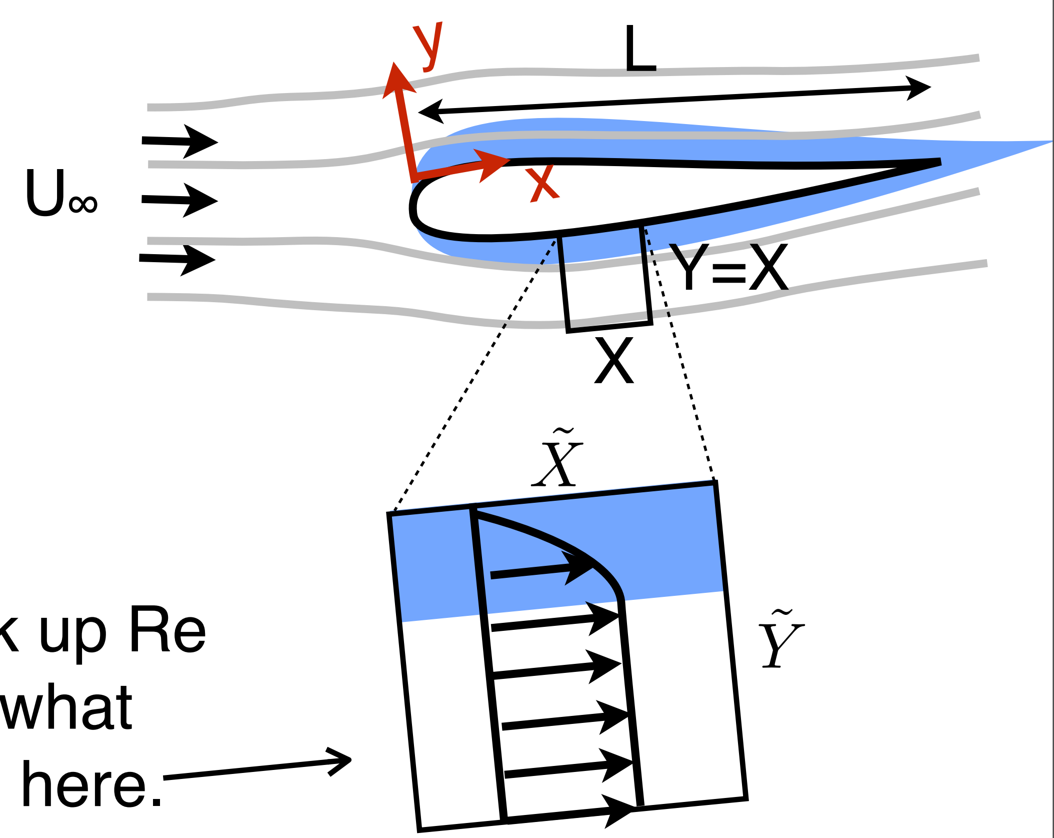
We're down to the choice of Y/X = function(Re) as Re becomes large. Why are we obsessed with Re large? Answer: 1) typical of many practical problems, and 2) terms drop out of the equations of motion.

Case 1: Y/X = 1 leads to Euler's equation satisfied by potential flow in the outer region

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

$$\frac{\partial \tilde{u}\tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}\tilde{u}}{\partial \tilde{y}} = -\frac{\partial[p/(\rho U^2)]}{\partial(x/X)} + \frac{1}{Re}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{1}{Re}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

$$\frac{\partial \tilde{u}\tilde{v}}{\partial \tilde{x}} + \frac{\partial \tilde{v}\tilde{v}}{\partial \tilde{y}} = -\frac{\partial[p/(\rho U^2)]}{\partial(y/Y)} + \frac{1}{Re}\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{1}{Re}\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}$$



Now, Crank up Re and watch what happens in here.

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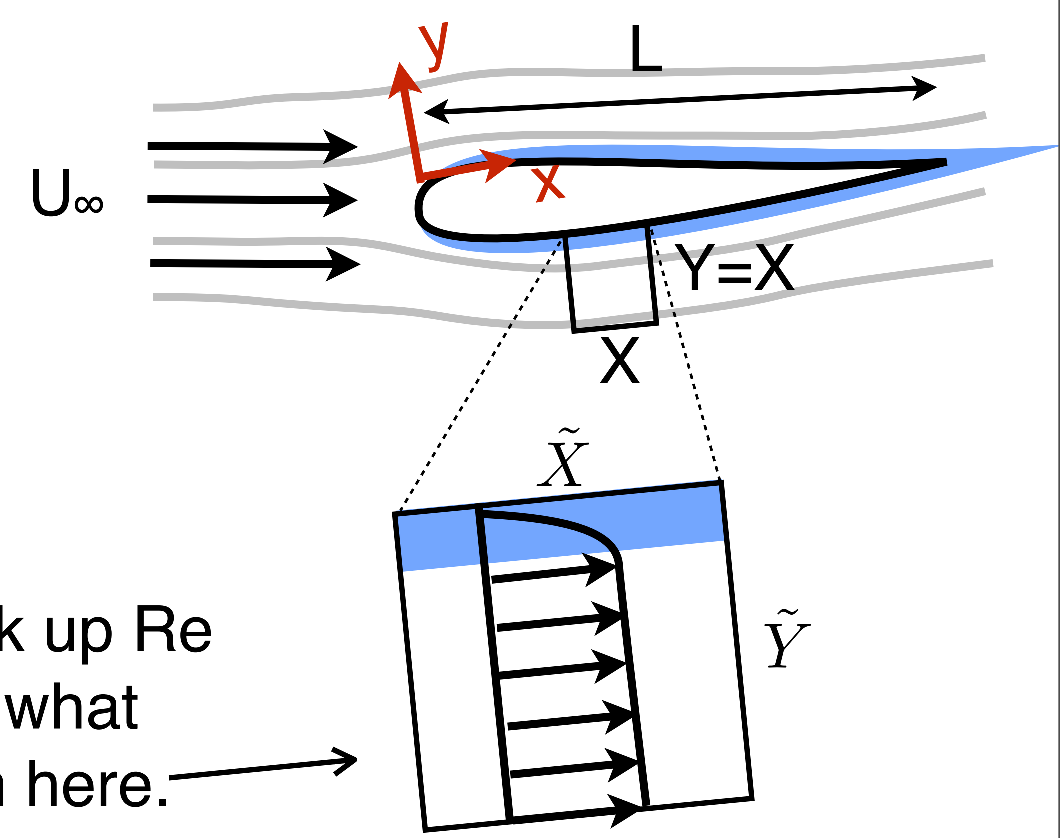

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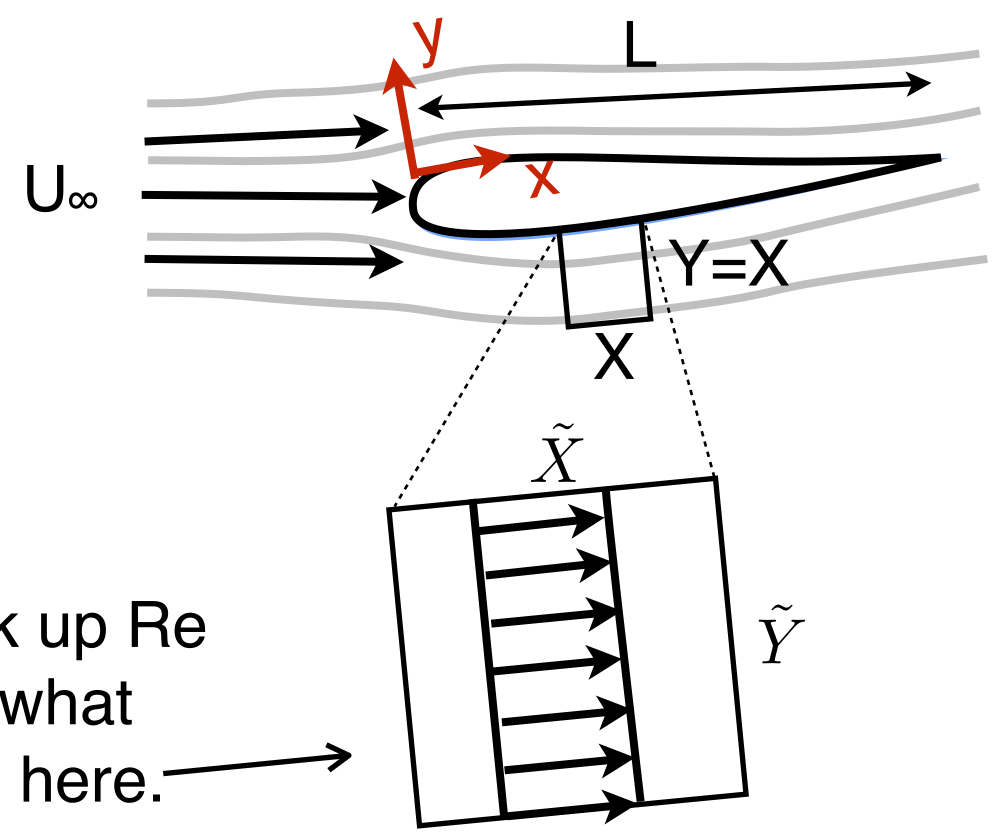
$$\frac{\partial \tilde{u}\tilde{v}}{\partial \tilde{x}} + \frac{\partial \tilde{v}\tilde{v}}{\partial \tilde{y}} = -\frac{\partial [p/(\rho U^2)]}{\partial (y/Y)}$$



Now, Crank up Re and watch what happens in here.

Potential flow!

No-slip BC cannot be satisfied. Equations of motion reduced to first-order PDE (impermeable BC still satisfied).

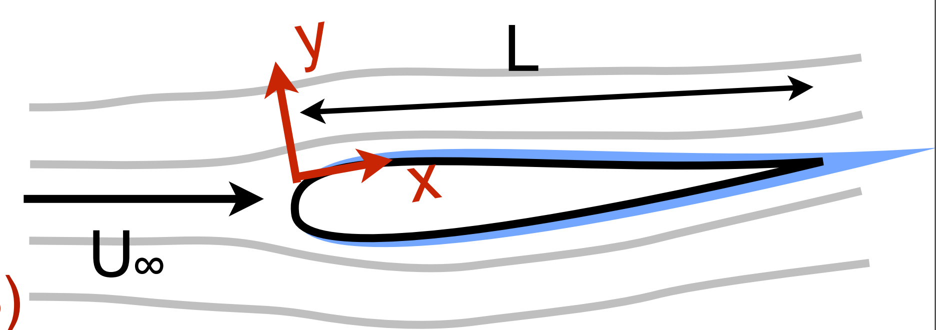
Case 2: Y/X = 1/Re$^{1/2}$ leads to boundary layer approximation in inner region

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0 \quad (1)$$

No-slip BC needs this term

$$\frac{\partial \tilde{u}\tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}\tilde{u}}{\partial \tilde{y}} = -\frac{\partial[p/(\rho U^2)]}{\partial(x/X)} + \frac{\nu}{UX}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \boxed{\frac{\nu}{UX}\frac{X^2}{Y^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}} \quad (2)$$

$$\frac{Y^2}{X^2}\frac{\partial \tilde{u}\tilde{v}}{\partial \tilde{x}} + \frac{Y^2}{X^2}\frac{\partial \tilde{v}\tilde{v}}{\partial \tilde{y}} = -\frac{\partial[p/(\rho U^2)]}{\partial(y/Y)} + \frac{\nu}{UX}\frac{Y^2}{X^2}\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{\nu}{UX}\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2} \quad (3)$$



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$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

$$\frac{\partial \tilde{u}\tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}\tilde{u}}{\partial \tilde{y}} = -\frac{\partial [p/(\rho U^2)]}{\partial (x/X)} + \frac{1}{Re}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

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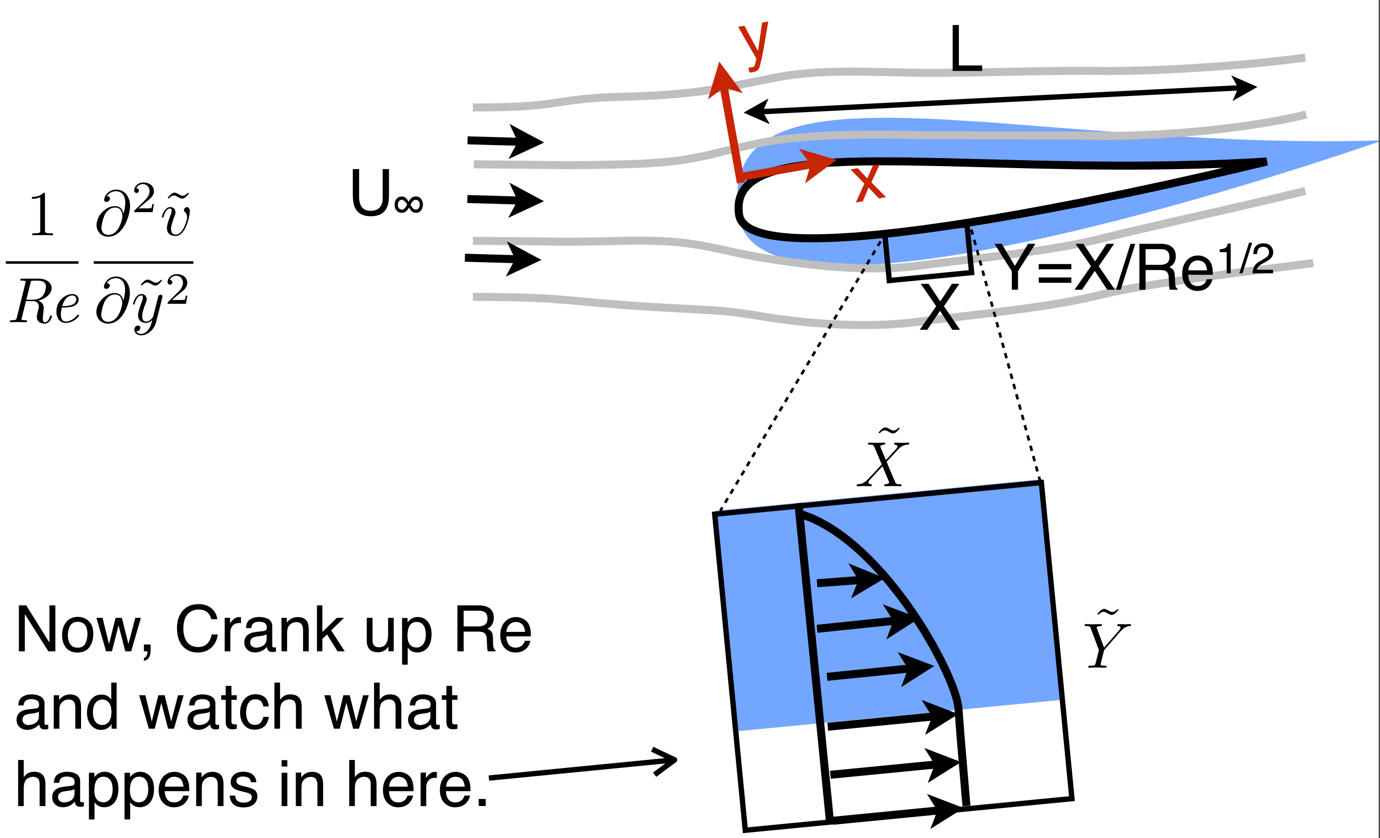

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$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

$$\frac{\partial \tilde{u}\tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}\tilde{u}}{\partial \tilde{y}} = -\frac{\partial [p/(\rho U^2)]}{\partial (x/X)} + \frac{1}{Re}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

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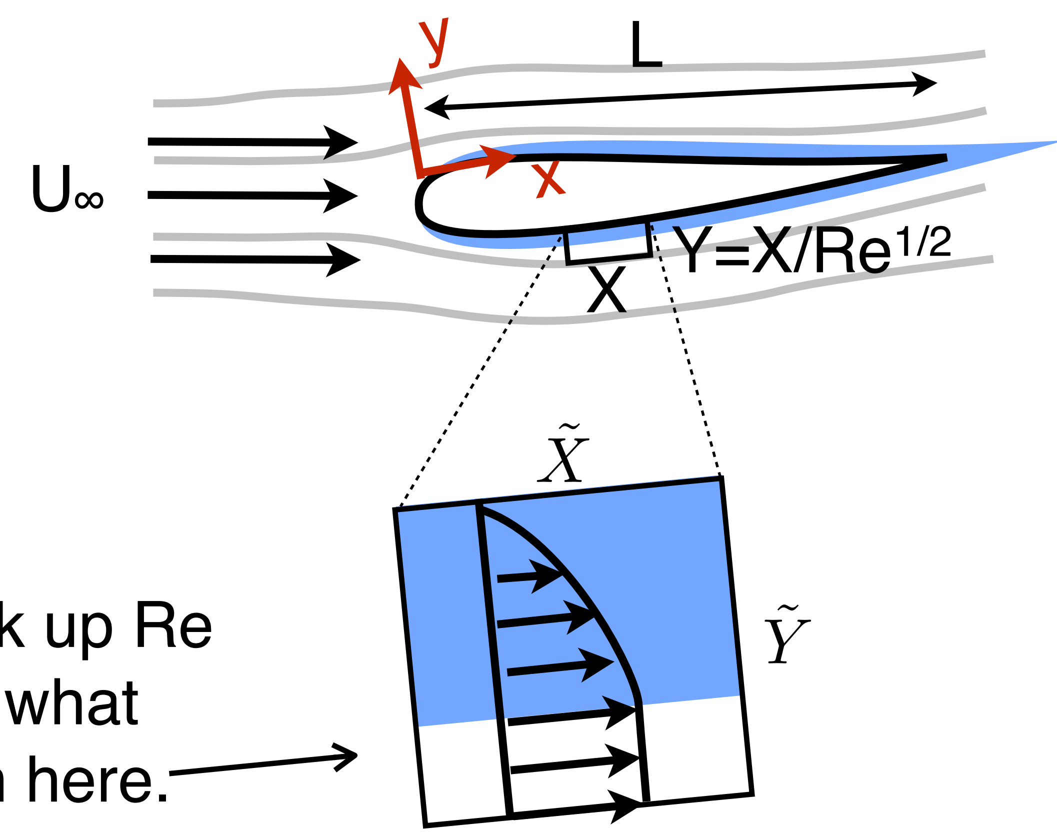


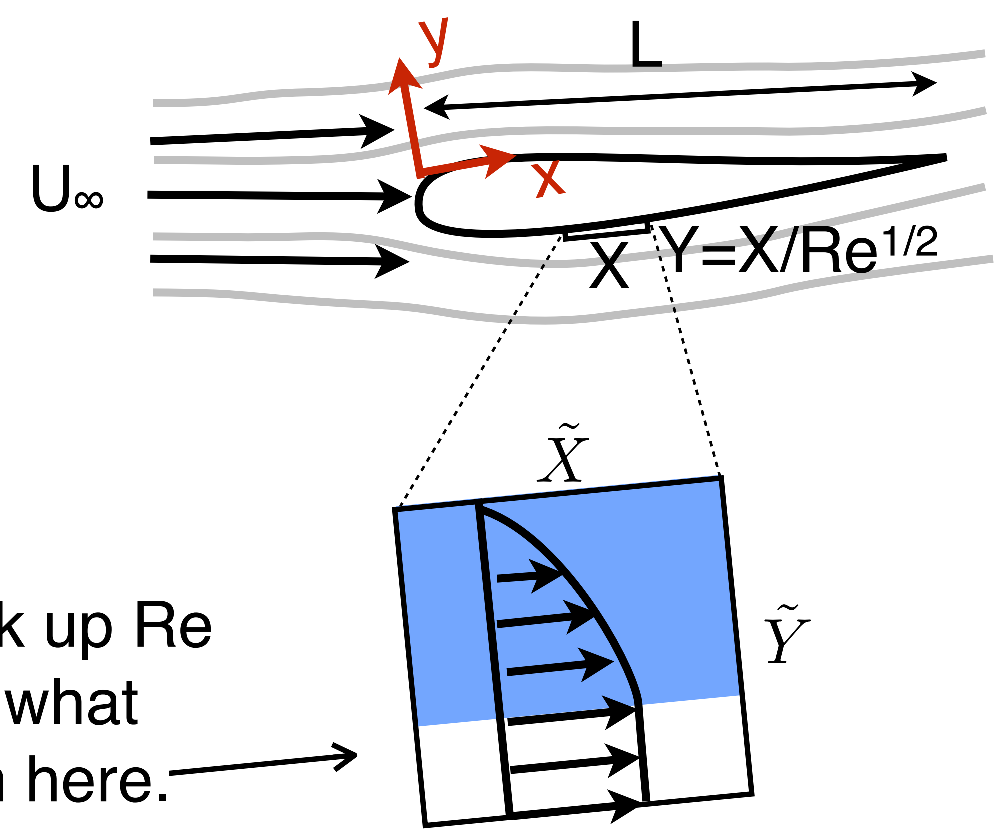
Now, Crank up Re and watch what happens in here.

Case 2: Y/X = 1/Re$^{1/2}$ leads to boundary layer approximation in inner region

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

$$\frac{\partial \tilde{u}\tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}\tilde{u}}{\partial \tilde{y}} = -\frac{\partial [p/(\rho U^2)]}{\partial (x/X)} \qquad + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

$$= -\frac{\partial [p/(\rho U^2)]}{\partial (y/Y)}$$



Now, Crank up Re and watch what happens in here.

Boundary layer approximation!

Can satisfy no-slip BC because equations of motion remain second-order PDE.

Summary for inner and outer regions

Inner region: boundary layer approximation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

No-slip and impermeable BC at y = 0

Outer region: Euler's equation (solution: potential flow)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Impermeable BC at y = 0

Summary for inner and outer regions

Inner region: boundary layer approximation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Straight d's

$$\frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad \Rightarrow \quad p = p(x)$$

No-slip and impermeable BC at y = 0

Outer region: Euler's equation (solution: potential flow)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Impermeable BC at y = 0

Summary for inner and outer regions

Inner region: boundary layer approximation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} + \nu\frac{\partial^2 u}{\partial y^2}$$

No-slip and impermeable BC at y = 0

Outer region: Euler's equation (solution: potential flow)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Impermeable BC at y = 0

Melbourne School of Engineering
MCEN90018 Advanced Fluid Dynamics

Lecture BL06: Similarity solution

18 March 2016

Summary for inner and outer regions

Inner region: boundary layer approximation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} + \nu\frac{\partial^2 u}{\partial y^2}$$

No-slip and impermeable BC at y = 0

Outer region: Euler's equation (solution: potential flow)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Impermeable BC at y = 0

Summary for inner and outer regions

Inner region: boundary layer approximation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} + \nu\frac{\partial^2 u}{\partial y^2}$$

No-slip and impermeable BC at y = 0

Outer region: Euler's equation (solution: potential flow)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Impermeable BC at y = 0

Overlap region:

$$\frac{L}{Re^{1/2}} \ll y \ll L$$

$$\tilde{y}_i \to \infty, \quad \tilde{y}_o \to 0$$

Match inner flow at infinity
to outer flow at wall.

Outer slip velocity at wall: $U_1(x)$

Inner flow x-momentum at infinity (u is not function of y, otherwise still in boundary layer, v is not zero):

$$u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x}$$

Match: $u = U_1$

$$U_1\frac{\mathrm{d}U_1}{\mathrm{d}x} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x}$$

Conundrum: but v is not matched (see later).

Summary for inner and outer regions

Inner region: boundary layer approximation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} = U_1 \frac{\mathrm{d}U_1}{\mathrm{d}x} + \nu \frac{\partial^2 u}{\partial y^2}$$

No-slip and impermeable BC at y = 0

Outer region: Euler's equation (solution: potential flow)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x}$$

$$\frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y}$$

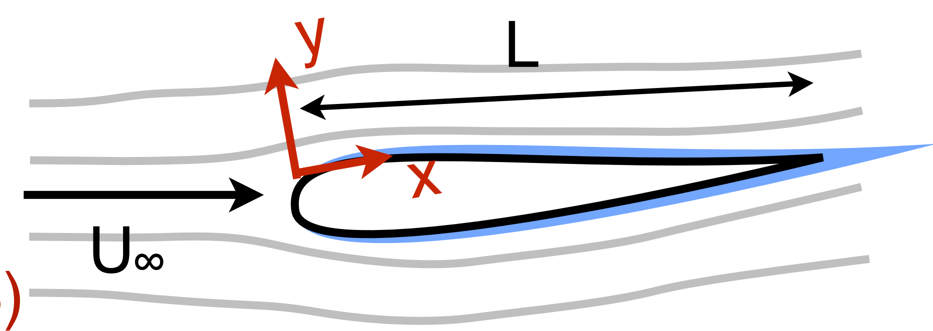
Impermeable BC at y = 0

Case 3 (bonus): Y/X = 1, Re → 0 leads to Stokes' (viscous/creeping) flow

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0 \quad (1)$$

$$\frac{\partial \tilde{u}\tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}\tilde{u}}{\partial \tilde{y}} = -\frac{\partial [p/(\rho U^2)]}{\partial (x/X)} + \frac{\nu}{UX}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\nu}{UX}\frac{X^2}{Y^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} \quad (2)$$

$$\frac{Y^2}{X^2}\frac{\partial \tilde{u}\tilde{v}}{\partial \tilde{x}} + \frac{Y^2}{X^2}\frac{\partial \tilde{v}\tilde{v}}{\partial \tilde{y}} = -\frac{\partial [p/(\rho U^2)]}{\partial (y/Y)} + \frac{\nu}{UX}\frac{Y^2}{X^2}\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{\nu}{UX}\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2} \quad (3)$$



First multiply (2) and (3) by UX/v.

$$\frac{UX}{\nu}\frac{\partial \tilde{u}\tilde{u}}{\partial \tilde{x}} + \frac{UX}{\nu}\frac{\partial \tilde{v}\tilde{u}}{\partial \tilde{y}} = -\frac{\partial [p/(\rho U\nu/X)]}{\partial (x/X)} + \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{X^2}{Y^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

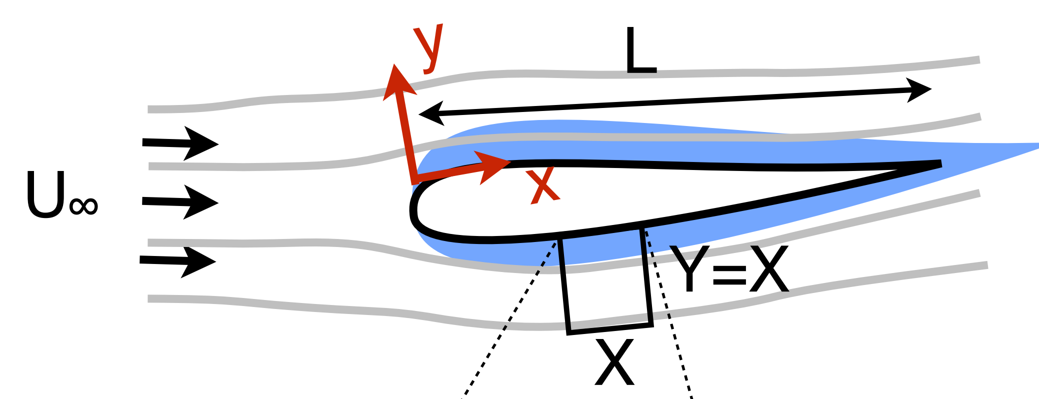
$$\frac{UX}{\nu}\frac{Y^2}{X^2}\frac{\partial \tilde{u}\tilde{v}}{\partial \tilde{x}} + \frac{UX}{\nu}\frac{Y^2}{X^2}\frac{\partial \tilde{v}\tilde{v}}{\partial \tilde{y}} = -\frac{\partial [p/(\rho U\nu/X)]}{\partial (y/Y)} + \frac{Y^2}{X^2}\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}$$

Case 3 (bonus): Y/X = 1, Re → 0 leads to Stokes' (viscous/creeping) flow

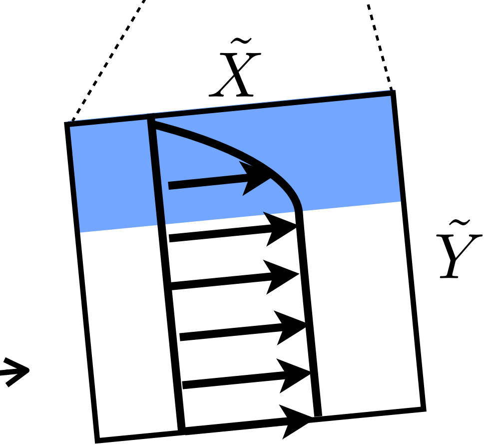
$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

$$Re\frac{\partial \tilde{u}\tilde{u}}{\partial \tilde{x}} + Re\frac{\partial \tilde{v}\tilde{u}}{\partial \tilde{y}} = -\frac{\partial[p/(\rho U\nu/X)]}{\partial(x/X)} + \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

$$Re\frac{\partial \tilde{u}\tilde{v}}{\partial \tilde{x}} + Re\frac{\partial \tilde{v}\tilde{v}}{\partial \tilde{y}} = -\frac{\partial[p/(\rho U\nu/X)]}{\partial(y/Y)} + \frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}$$



Now, wind down Re
and watch what
happens in here.

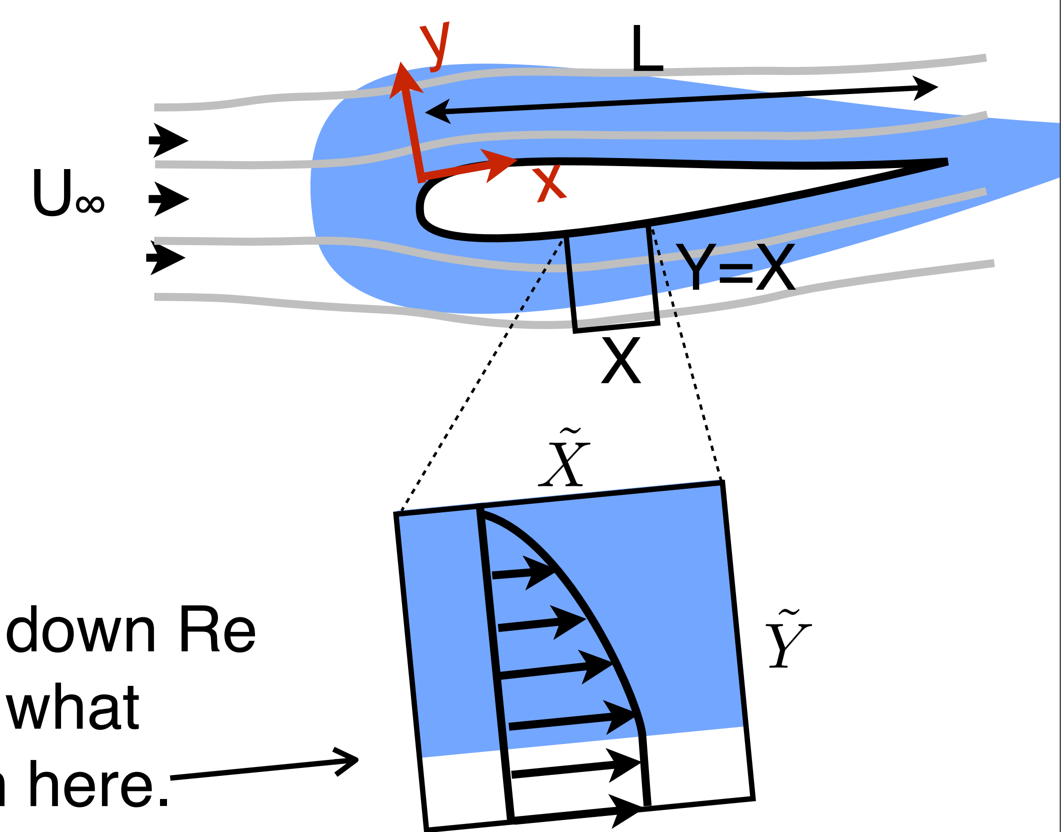


Case 3 (bonus): Y/X = 1, Re → 0 leads to Stokes' (viscous/creeping) flow

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

$$Re\frac{\partial \tilde{u}\tilde{u}}{\partial \tilde{x}} + Re\frac{\partial \tilde{v}\tilde{u}}{\partial \tilde{y}} = -\frac{\partial[p/(\rho U\nu/X)]}{\partial(x/X)} + \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

$$Re\frac{\partial \tilde{u}\tilde{v}}{\partial \tilde{x}} + Re\frac{\partial \tilde{v}\tilde{v}}{\partial \tilde{y}} = -\frac{\partial[p/(\rho U\nu/X)]}{\partial(y/Y)} + \frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}$$



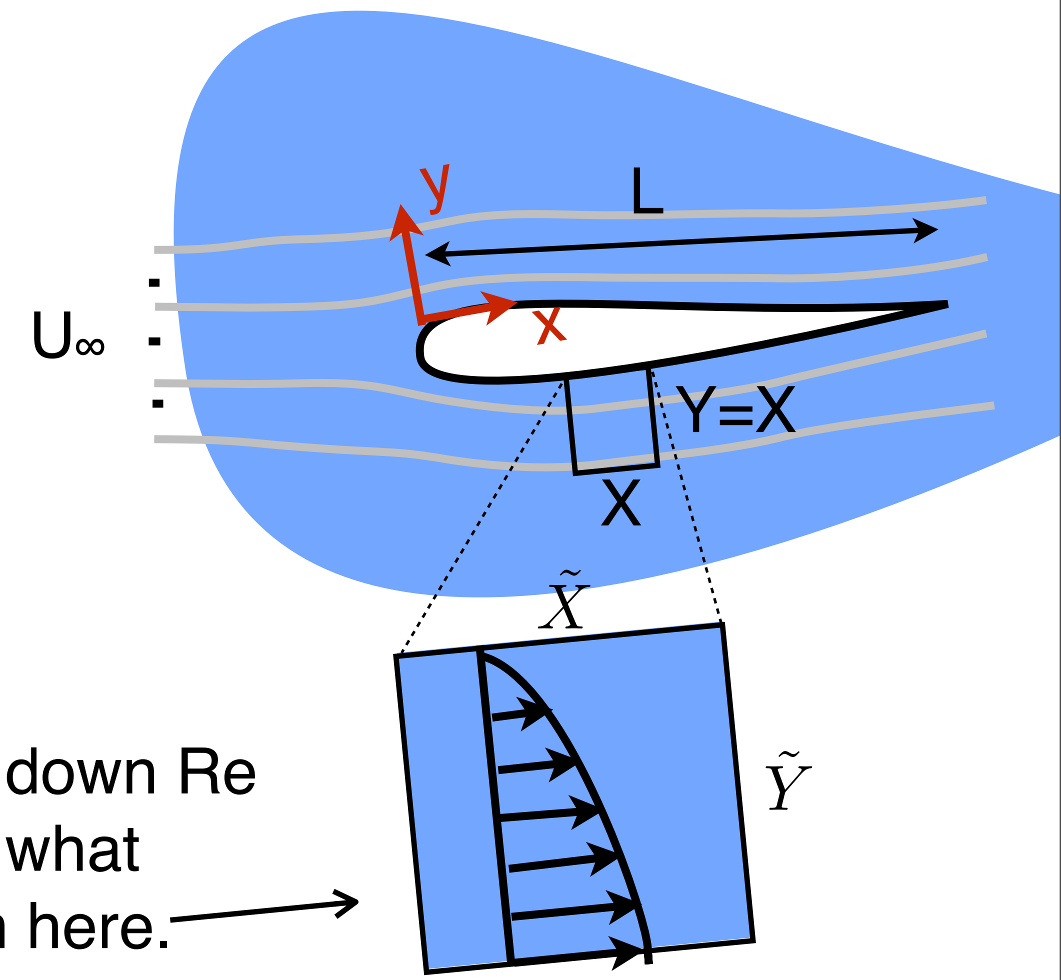
Now, wind down Re and watch what happens in here.

Case 3 (bonus): Y/X = 1, Re → 0 leads to Stokes' (viscous/creeping) flow

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

$$= -\frac{\partial[p/(\rho U \nu / X)]}{\partial(x/X)} + \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

$$= -\frac{\partial[p/(\rho U \nu / X)]}{\partial(y/Y)} + \frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}$$



Now, wind down Re and watch what happens in here.

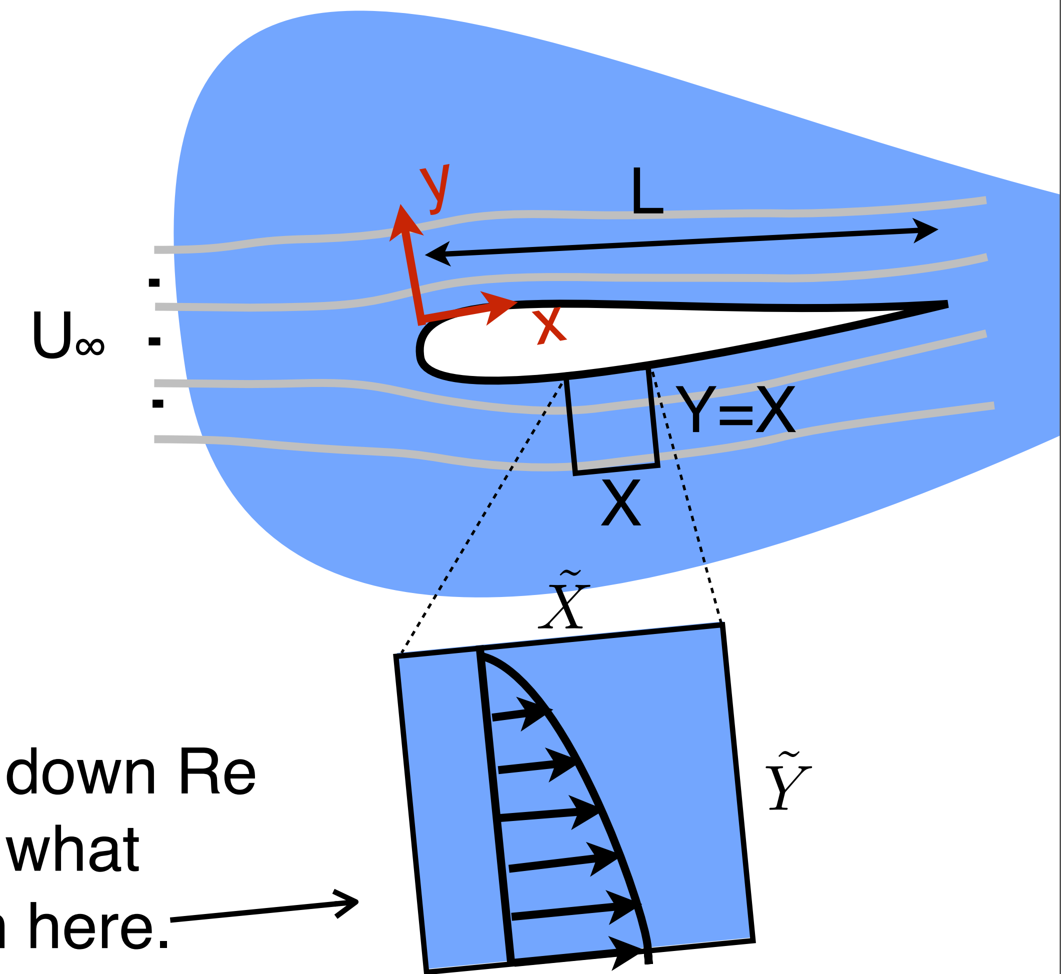
Stokes flow!

Case 3 (bonus): Y/X = 1, Re → 0 leads to Stokes' (viscous/creeping) flow

i.e. $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$0=-\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$0=-\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$



Equations of motion is linear.

Now, wind down Re and watch what happens in here.

Application in microfluidics: medicine (lab on a chip), lubrication, PIV, sperm swimming.

Fun fact: flows are reversible in time.

Summary for inner and outer regions

Inner region: boundary layer approximation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} = U_1 \frac{\mathrm{d}U_1}{\mathrm{d}x} + \nu \frac{\partial^2 u}{\partial y^2}$$

No-slip and impermeable BC at y = 0

Outer region: Euler's equation (solution: potential flow)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

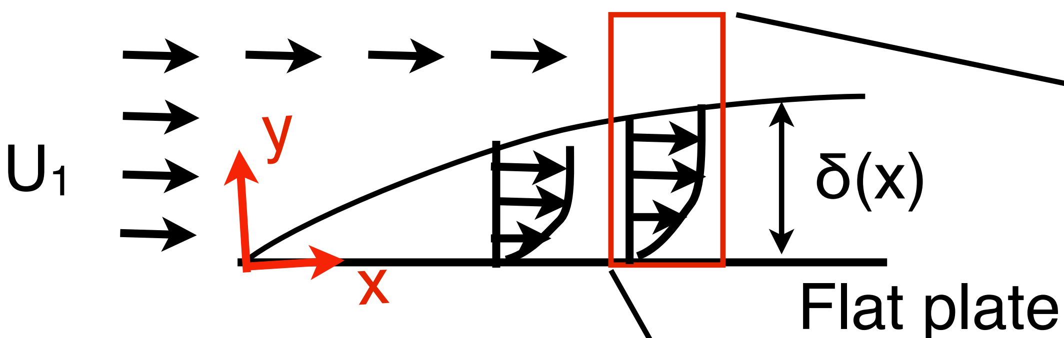
$$\frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

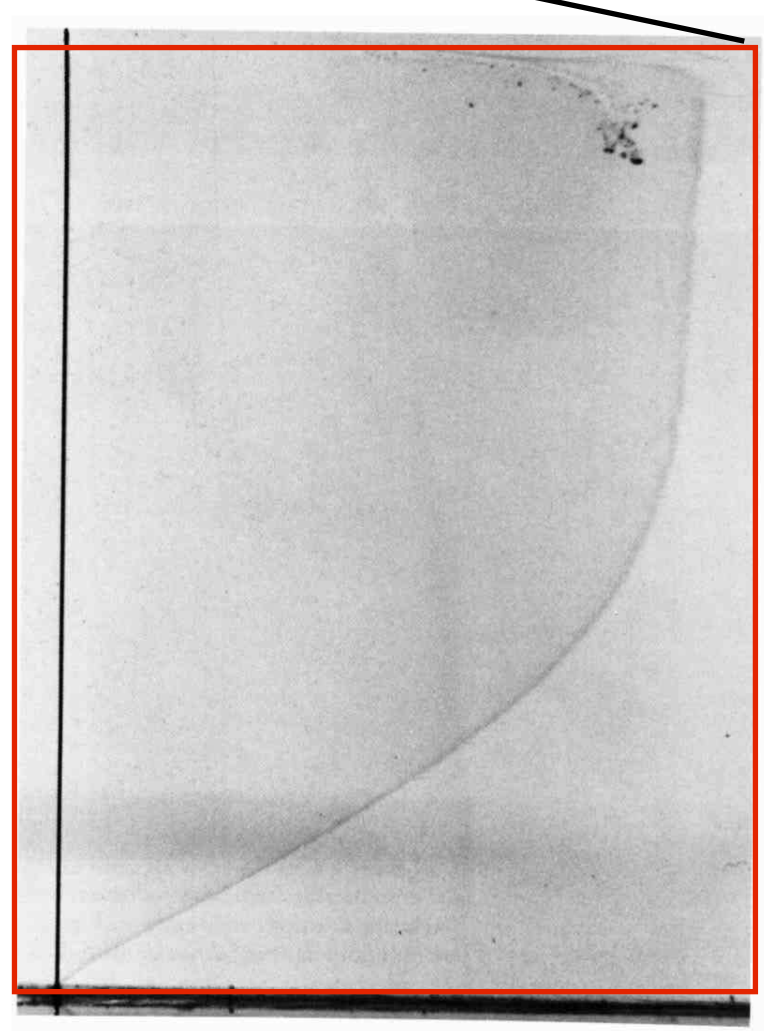
$$\frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Impermeable BC at y = 0

Blasius boundary layer

Zero pressure gradient ⇔ U_1 does not vary with x

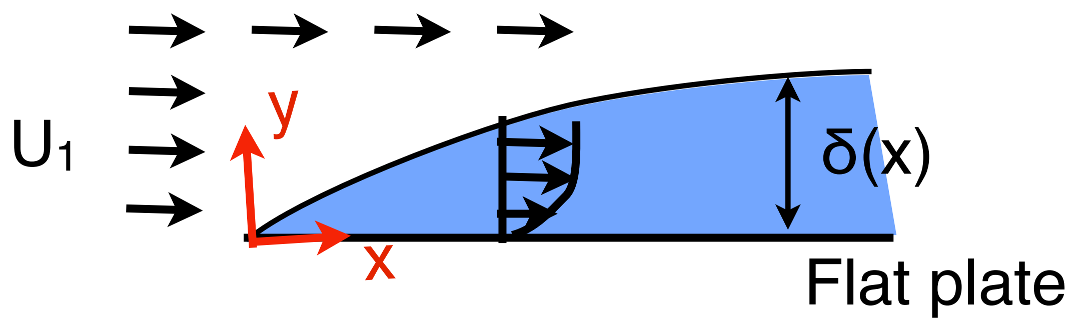




30. Blasius boundary-layer profile on a flat plate. The tangential velocity profile in the laminar boundary layer on a flat plate, discovered by Prandtl and calculated accurately by Blasius, is made visible by tellurium. Water is flowing at 9 cm/s. The Reynolds number is 500 based on distance from the leading edge, and the displacement thickness is about 5 mm. A fine tellurium wire perpendicular to the plate at the left is subjected to an electrical impulse of a few milliseconds duration. A chemical reaction produces a slender colloidal cloud, which drifts with the stream and is photographed a moment later to define the velocity profile. *Photograph by F. X. Wortmann*

(Van Dyke 1982)

Blasius boundary layer: inner BL flow



Equations of motion incorporating zero-pressure-gradient information:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

Solve for dependent variables:

$$u, v$$

(Two unknowns)

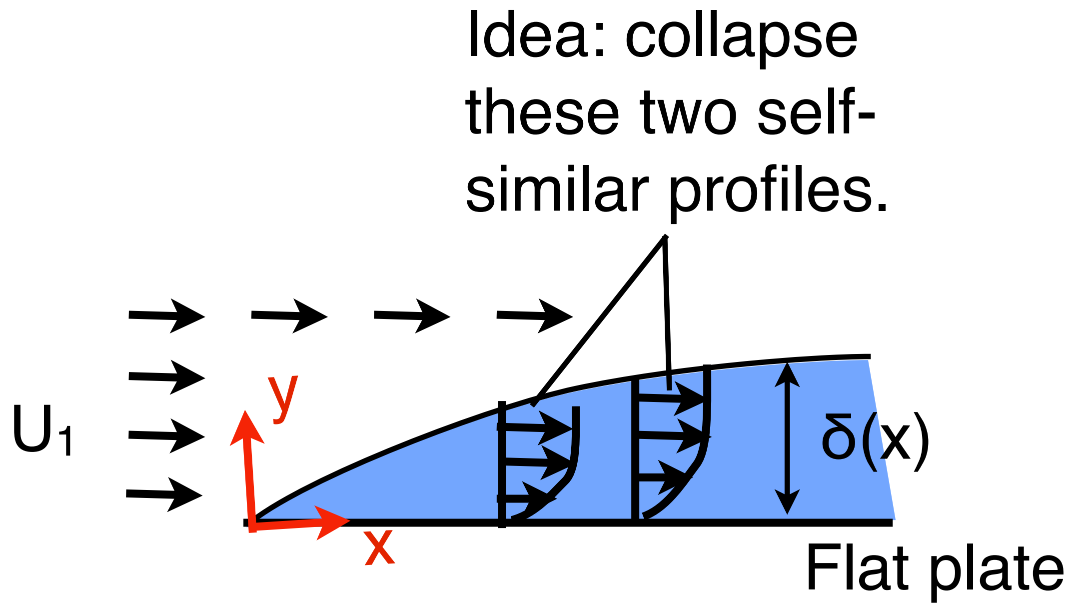
Boundary conditions:

$$u = U_1$$

at y → ∞ (overlap) and x = 0 (freestream)

$u = 0, \quad v = 0$ at y = 0

Blasius boundary layer: similarity solution for inner BL flow



Similarity solution:

$$u(x,y) = U_1 F(\eta(x,y))$$

$$\eta(x,y) = \frac{y}{\delta(x)}$$

Calculate terms occurring in equations of motion:

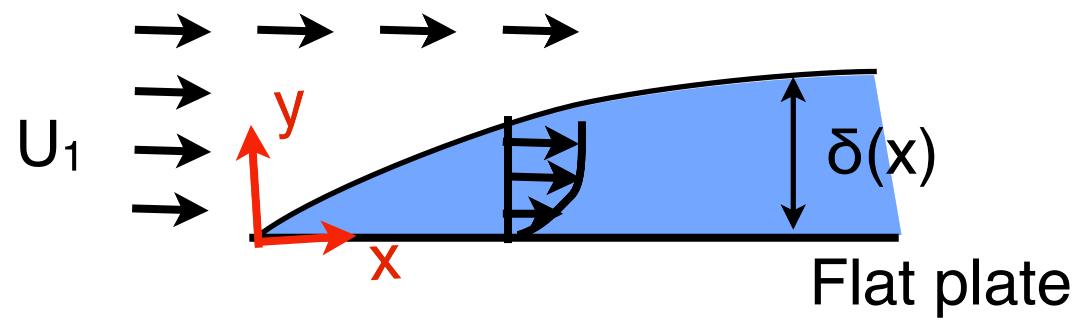
$$\frac{\partial u}{\partial x} = -\frac{U_1 \delta'}{\delta}\eta F'$$

$$\frac{\partial u}{\partial y} = \frac{U_1}{\delta}F'$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_1}{\delta^2}F''$$

$$v(x,y) = \int_0^y -\frac{\partial u}{\partial x}(x,y_1)\,\mathrm{d}y_1 = U_1\delta'\big(\eta F - \int_0^\eta F(\eta_1)\,\mathrm{d}\eta_1\big)$$

Blasius boundary layer: similarity solution for inner BL flow



Similarity solution:

$$u(x,y) = U_1 F(\eta(x,y))$$
$$\eta(x,y) = \frac{y}{\delta(x)}$$

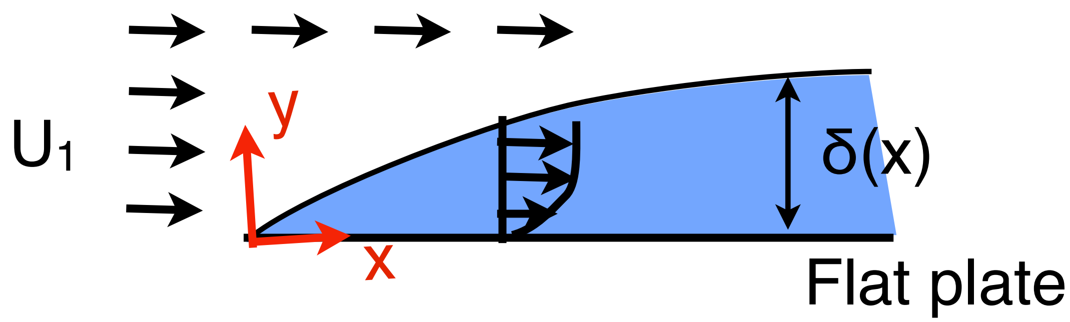
Calculate terms occurring in equations of motion:

$$\frac{\partial u}{\partial x} = -\frac{U_1\delta'}{\delta}\eta F' = -\frac{U_1\delta'}{\delta}\eta f''$$

Define: $f' \equiv F \quad \Leftrightarrow \quad f = \int_0^\eta F(\eta_1)\,\mathrm{d}\eta_1$

$$\frac{\partial u}{\partial y} = \frac{U_1}{\delta}F' = \frac{U_1}{\delta}f''$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{U_1}{\delta^2}F'' = \frac{U_1}{\delta^2}f'''$$
$$v(x,y) = \int_0^y -\frac{\partial u}{\partial x}(x,y_1)\,\mathrm{d}y_1 = U_1\delta'\big(\eta F - \int_0^\eta F(\eta_1)\,\mathrm{d}\eta_1\big) = U_1\delta'\big(\eta f' - f\big)$$

Blasius boundary layer: similarity solution for inner BL flow



Similarity solution:

$$u(x,y) = U_1 F(\eta(x,y))$$

$$\eta(x,y) = \frac{y}{\delta(x)}$$

Substitute in x-momentum equation:

$$(U_1 f')(-\frac{U_1\delta'}{\delta}\eta f'') + U_1\delta'(\eta f' - f)\frac{U_1}{\delta}f'' = \nu\frac{U_1}{\delta^2}f'''$$

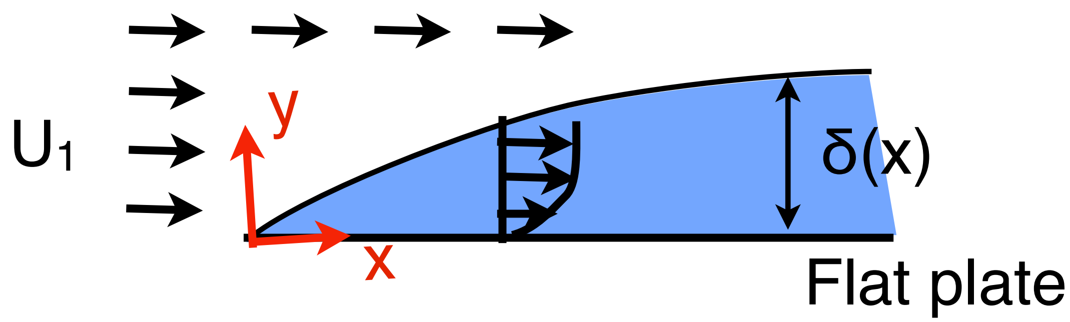
$$f''' + \frac{U_1\delta\delta'}{\nu}f''f = 0$$

For similarity solution to exist, we must arrive at an ODE in terms of f and its derivatives and η.

In particular, no dependence on y and x.

This can only be achieved if $U_1\delta\delta'/\nu = \mathrm{const.} = 1/2$, say.

Blasius boundary layer: similarity solution for inner BL flow



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$$f''' + \frac{U_1\delta\delta'}{\nu}f''f = 0$$

For similarity solution to exist, we must arrive at an ODE in terms of f and its derivatives and η.

In particular, no dependence on y and x.

This can only be achieved if $U_1\delta\delta'/\nu = \text{const.} = 1/2$, say.

$\Rightarrow$

$$2f''' + f''f = 0$$

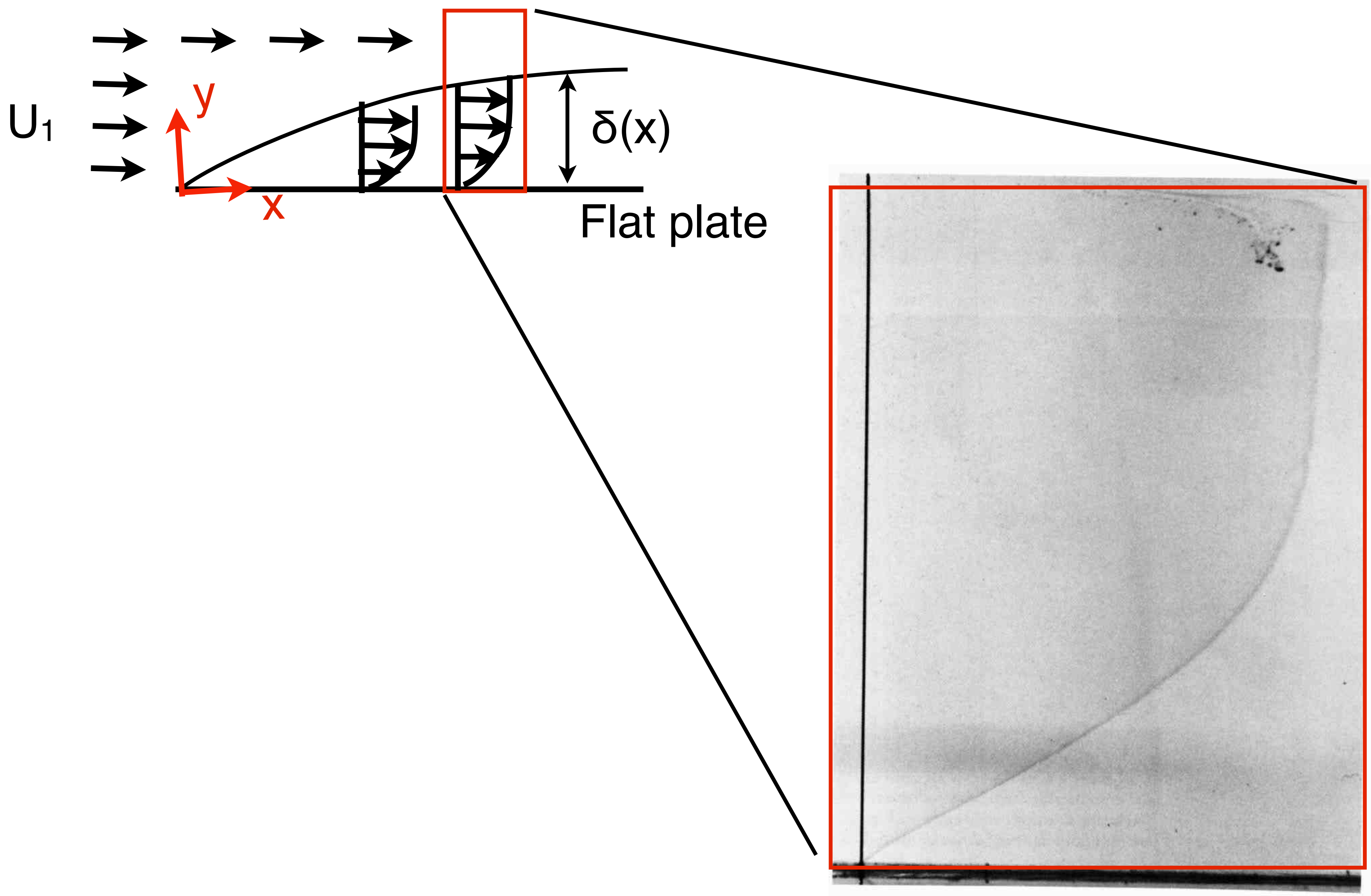
$$\frac{\mathrm{d}(\delta^2)}{\mathrm{d}x} = \frac{\nu}{U_1}$$

Melbourne School of Engineering
MCEN90018 Advanced Fluid Dynamics

Lecture BL07: Blasius' boundary layer
22 March 2016

Blasius’ boundary layer

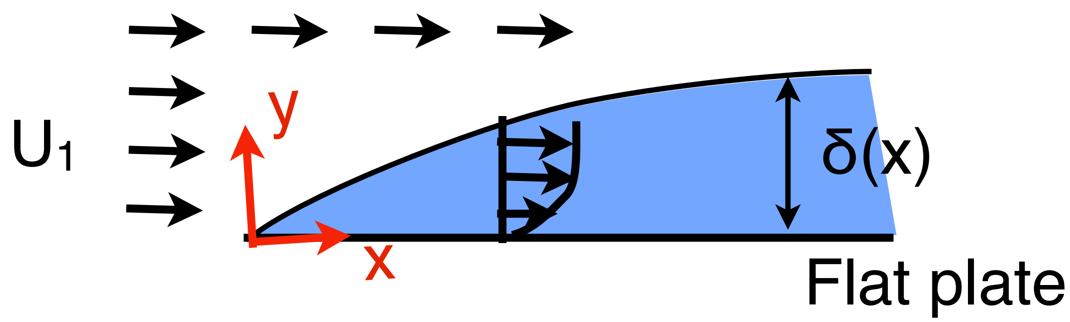
Zero pressure gradient ⇔ U_1 does not vary with x



30. Blasius boundary-layer profile on a flat plate. The tangential velocity profile in the laminar boundary layer on a flat plate, discovered by Prandtl and calculated accurately by Blasius, is made visible by tellurium. Water is flowing at 9 cm/s. The Reynolds number is 500 based on distance from the leading edge, and the displacement thickness is about 5 mm. A fine tellurium wire perpendicular to the plate at the left is subjected to an electrical impulse of a few milliseconds duration. A chemical reaction produces a slender colloidal cloud, which drifts with the stream and is photographed a moment later to define the velocity profile. *Photograph by F. X. Wortmann*

(Van Dyke 1982)

Blasius' boundary layer



Equations of motion incorporating zero-pressure-gradient information:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

(Two 2nd-order equations)

Solve for dependent variables:

$$u, v$$

(Two unknowns)

Boundary conditions:

$u = U_1$

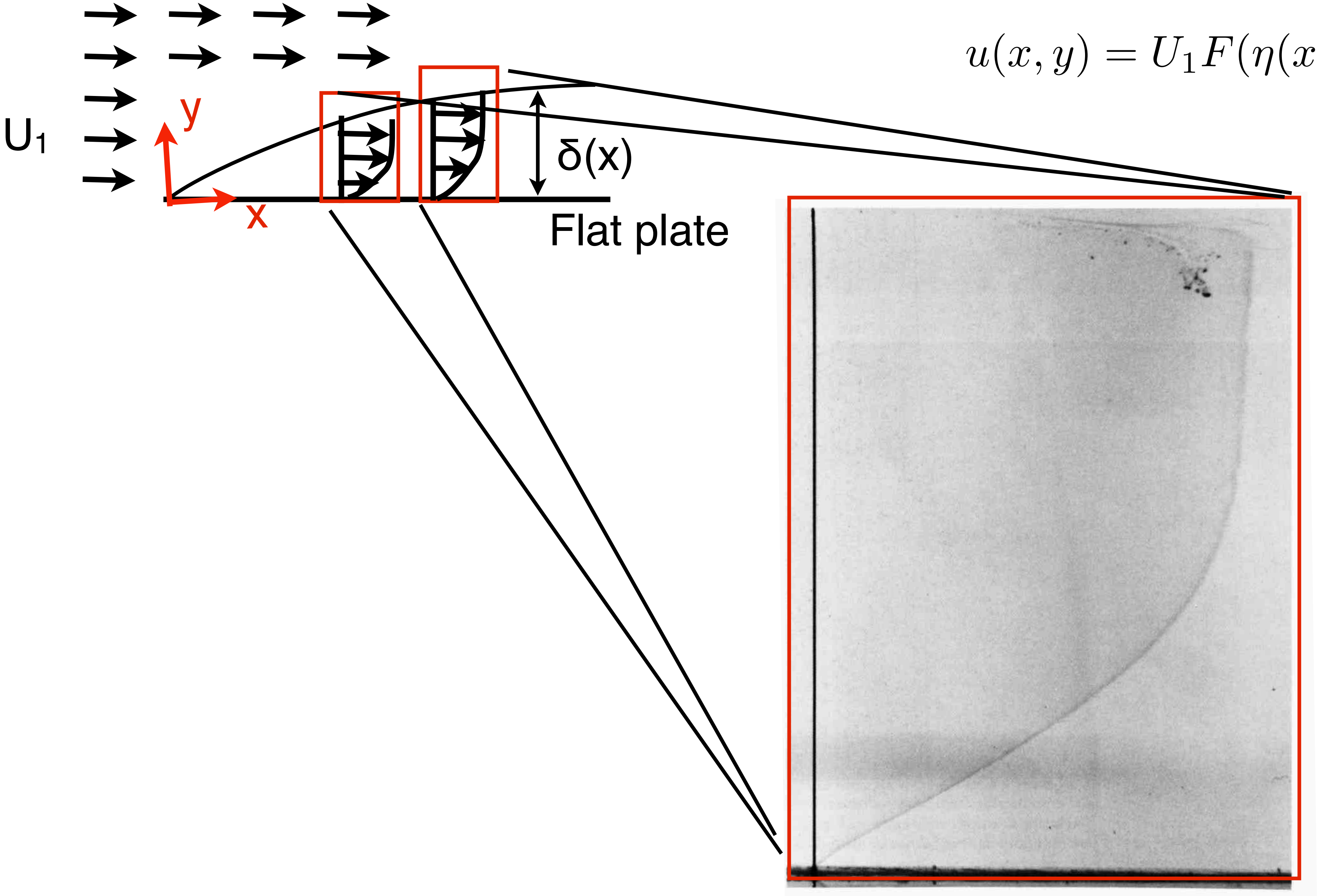
at y → ∞ (overlap) and x = 0 (freestream)

$u = 0, \quad v = 0$ at y = 0

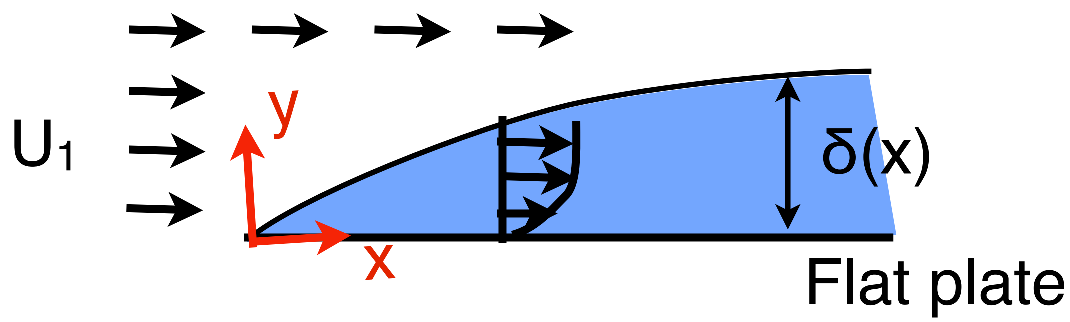
Blasius' boundary layer

Similarity solution:

$$u(x,y) = U_1 F(\eta(x,y)) \qquad \eta(x,y) = \frac{y}{\delta(x)}$$



Blasius' boundary layer



Similarity solution:

$$u(x,y) = U_1 F(\eta(x,y)) \qquad \eta(x,y) = \frac{y}{\delta(x)}$$

So we have:

$$2f''' + f''f = 0$$

$$\frac{\mathrm{d}(\delta^2)}{\mathrm{d}x} = \frac{\nu}{U_1}$$

$$f = \int_0^\eta F(\eta_1)\,\mathrm{d}\eta_1$$

Boundary conditions in similarity form:

$$f(0) = 0$$

$$u = U_1$$

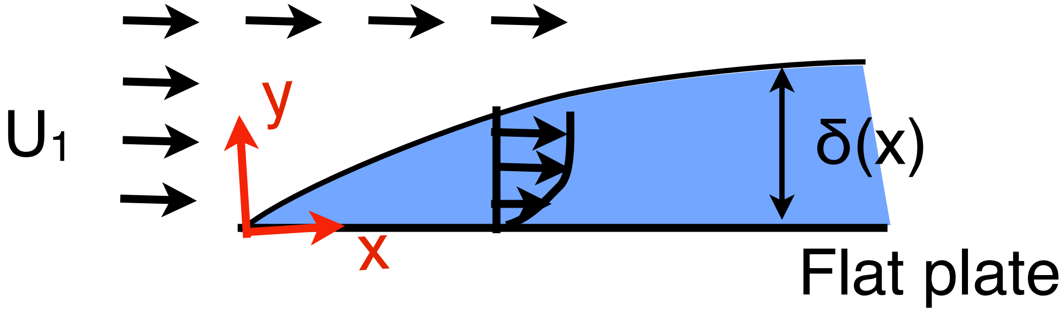
at y → ∞ (overlap) and
x = 0 (freestream)

$$f'(\infty) = F(\infty) = 1, \quad \delta(0) = 0$$

$$f'(0) = F(0) = 0$$

$u = 0, \quad v = 0$ at y = 0

Blasius' boundary layer



Similarity solution:

$$u(x,y) = U_1 F(\eta(x,y)) \qquad \eta(x,y) = \frac{y}{\delta(x)}$$

So we have:

$$2f''' + f''f = 0$$

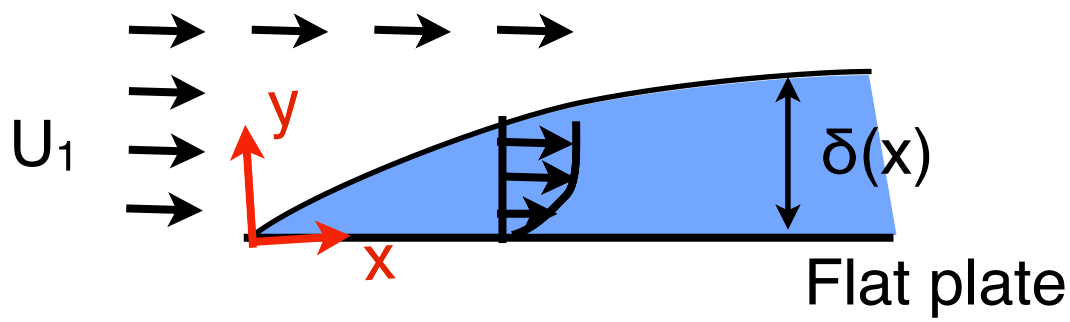
$$\frac{\mathrm{d}(\delta^2)}{\mathrm{d}x} = \frac{\nu}{U_1}$$

Boundary conditions:

$$f(0) = 0 \qquad \delta(0) = 0$$

$$f'(\infty) = 1 \qquad f'(0) = 0$$

Blasius' boundary layer



Similarity solution:

$$u(x,y) = U_1 F(\eta(x,y)) \qquad \eta(x,y) = \frac{y}{\delta(x)}$$

So we have:

$$2f''' + f''f = 0$$

$$\frac{\mathrm{d}(\delta^2)}{\mathrm{d}x} = \frac{\nu}{U_1}$$

Boundary conditions:

$$f(0) = 0 \qquad \delta(0) = 0$$

$$f'(\infty) = 1 \qquad f'(0) = 0$$

How does the boundary-layer thickness grow with x?

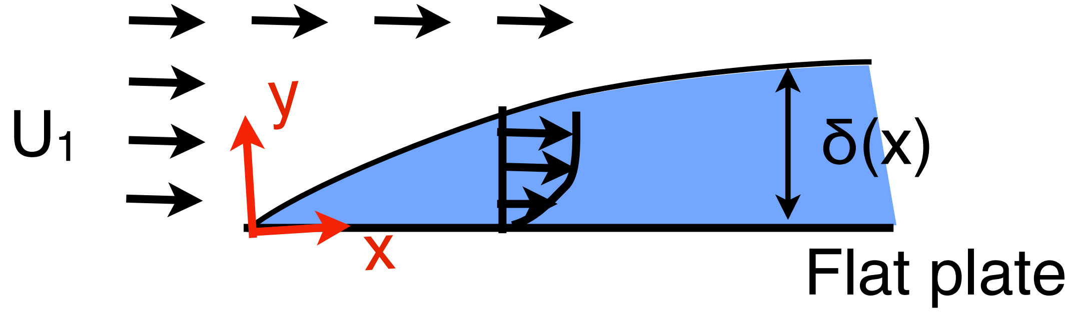
$$\delta(x) = \sqrt{\frac{\nu}{U_1} x}$$

cf. Stokes' first problem:

$$\delta(t) \propto \sqrt{\nu t}$$

i.e. $$\frac{\delta}{x} = \sqrt{\frac{\nu}{U_1 x}} = Re_x^{-1/2}$$

Blasius' boundary layer



Similarity solution:

$$u(x,y) = U_1 F(\eta(x,y)) \qquad \eta(x,y) = \frac{y}{\delta(x)}$$

How to solve

$$2f''' + f''f = 0$$

subject to BCs:

$$f(0) = 0$$

$$f'(\infty) = 1$$

$$f'(0) = 0$$

??

Numerically solve initial value problem (Runge–Kutta, etc.)

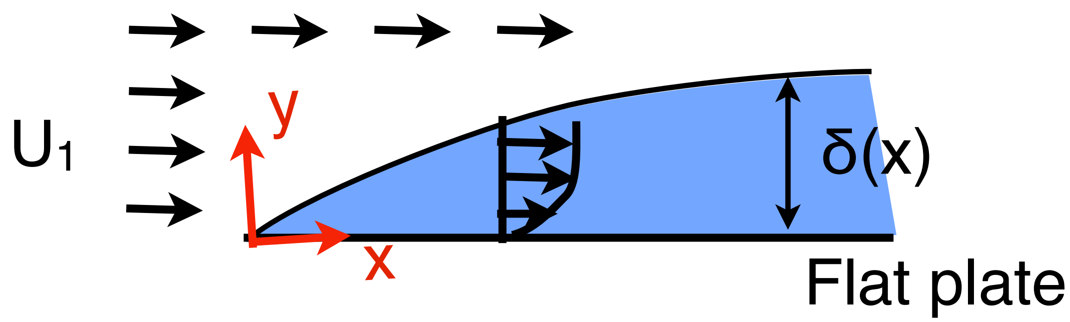
$$\frac{\mathrm{d}}{\mathrm{d}\eta}\begin{bmatrix} f'' \\ f' \\ f \end{bmatrix} = \begin{bmatrix} -f''f/2 \\ f'' \\ f' \end{bmatrix}$$

Initial condition:

$$\begin{bmatrix} f'' \\ f' \\ f \end{bmatrix}_{\eta=0} = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}$$

Tinker with c until f'(big number) = 1.
This is called 'shooting'.

Blasius' boundary layer



Numerically solve initial value problem (Runge–Kutta, etc.)

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\begin{bmatrix} f'' \\ f' \\ f \end{bmatrix} = \begin{bmatrix} -f''f/2 \\ f'' \\ f' \end{bmatrix}$$

Initial condition:

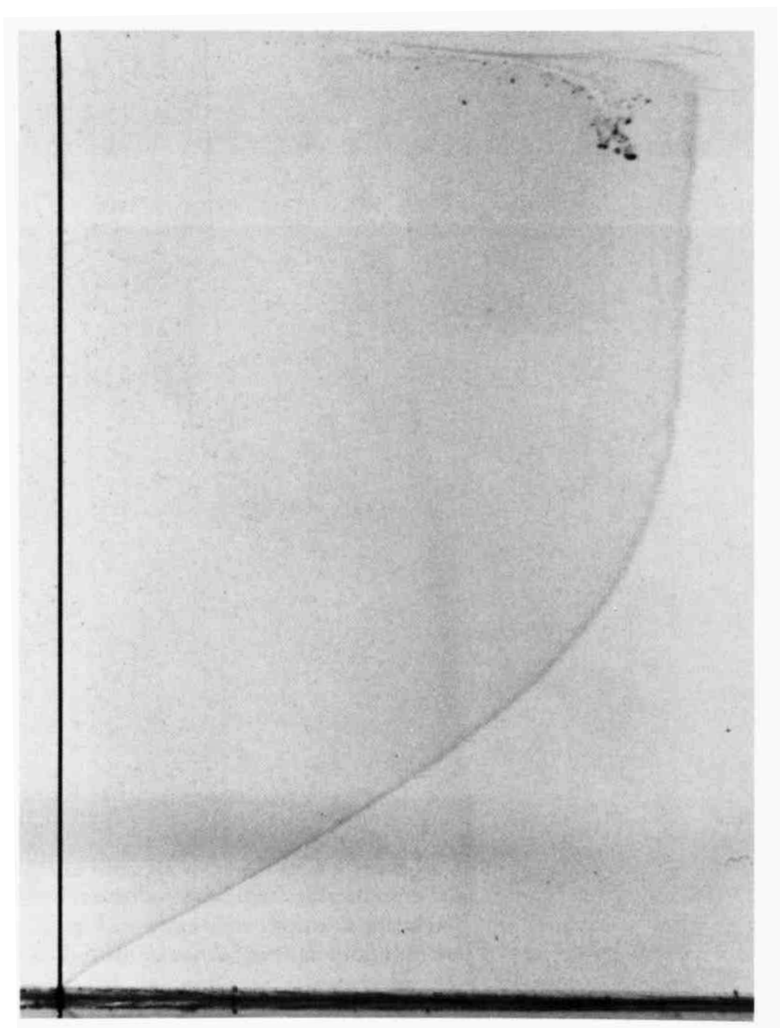
$$\begin{bmatrix} f'' \\ f' \\ f \end{bmatrix}_{\eta=0} = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}$$

Tinker with c until f'(big number) = 1. This is called 'shooting'.

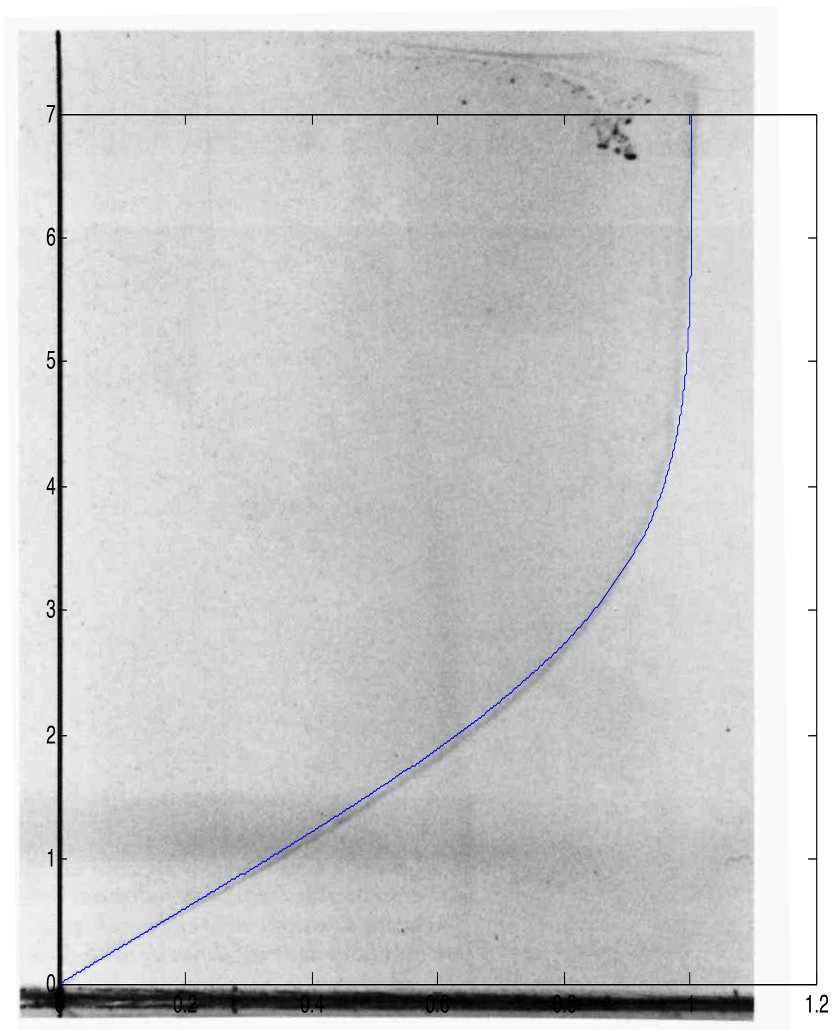
```
clear all;

n  = 5001;
dh = 0.004;

h  = zeros(1,n);
f0 = zeros(1,n);
f1 = zeros(1,n);
f2 = zeros(1,n);

% Initial conditions
h(1)  = 0.0;
f0(1) = 0.0;
f1(1) = 0.0;
f2(1) = 0.332;

% Euler integration
for i=1:n-1
   h(i+1)  = h(i)  + dh;
   f0(i+1) = f0(i) + dh * f1(i);
   f1(i+1) = f1(i) + dh * f2(i);
   f2(i+1) = f2(i) + dh * (-f2(i)*f0(i)/2.0);
end

f1(n)
```

Find c = 0.332 (here, we have used big number = 20).

Blasius' boundary layer: comparison with experiment

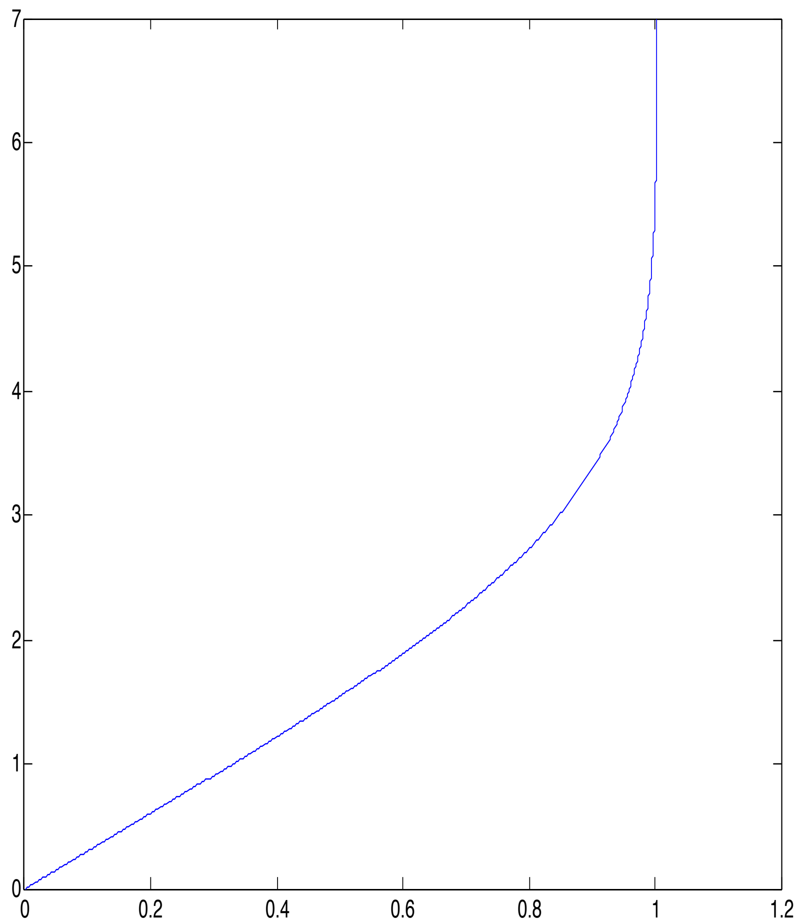


Blasius' boundary layer: comparison with experiment



Blasius' boundary layer: comparison with experiment

$$\eta = \frac{y}{\delta} = \frac{y}{\sqrt{\nu x/U_1}}$$

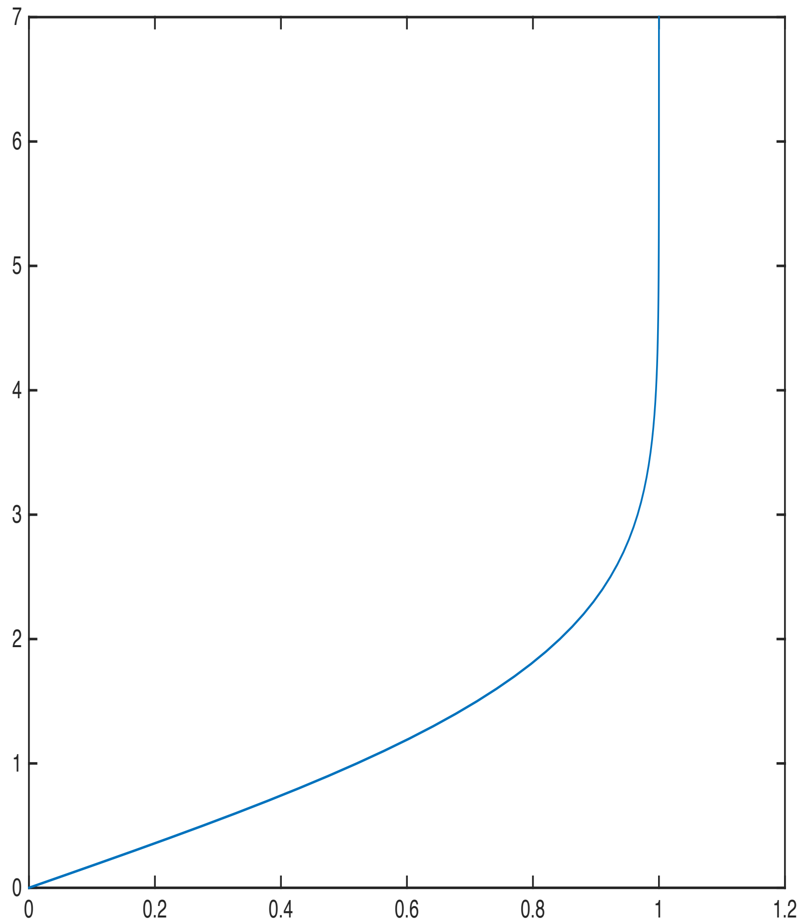


$$\frac{u}{U_1}$$

Blasius' boundary layer: comparison with Stokes' first problem x = $U_1 t$

$$\eta = \frac{y}{\delta}$$

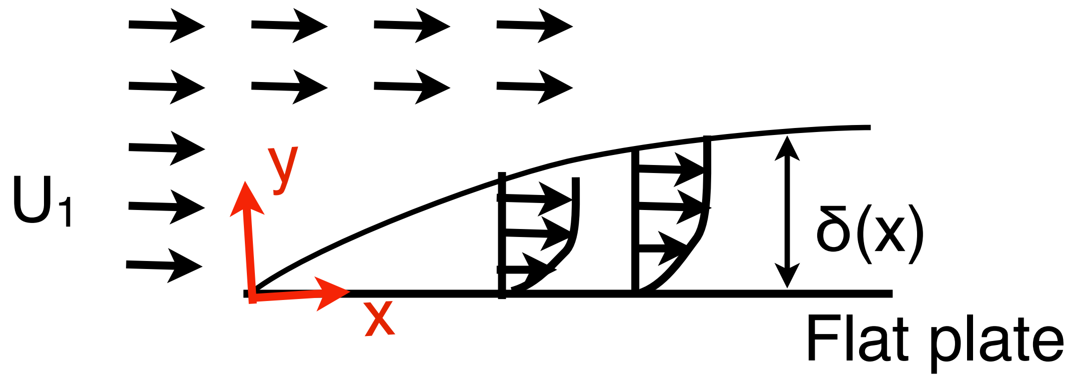
$$= \frac{y}{\sqrt{\nu t}}$$



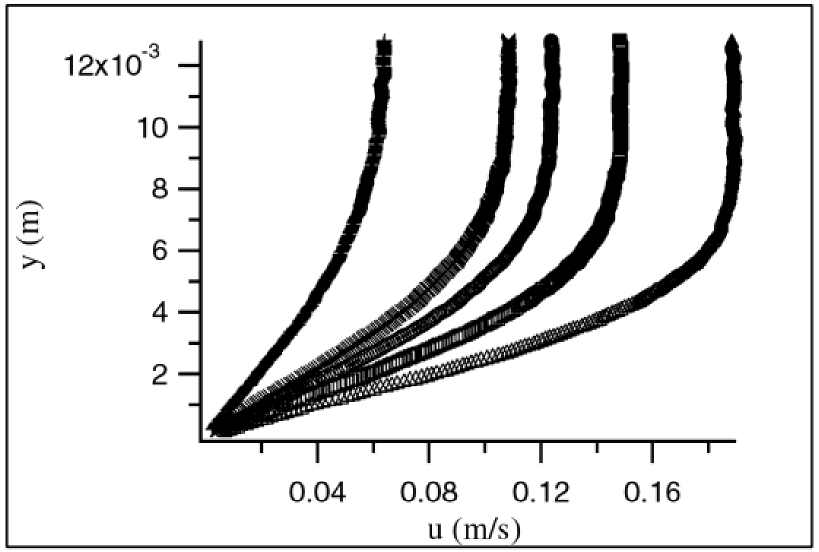
$$1 - \frac{u}{U} = 1 - \mathrm{erfc}(\frac{1}{2}\frac{y}{\sqrt{\nu t}})$$

Blasius' boundary layer: comparison with experiment

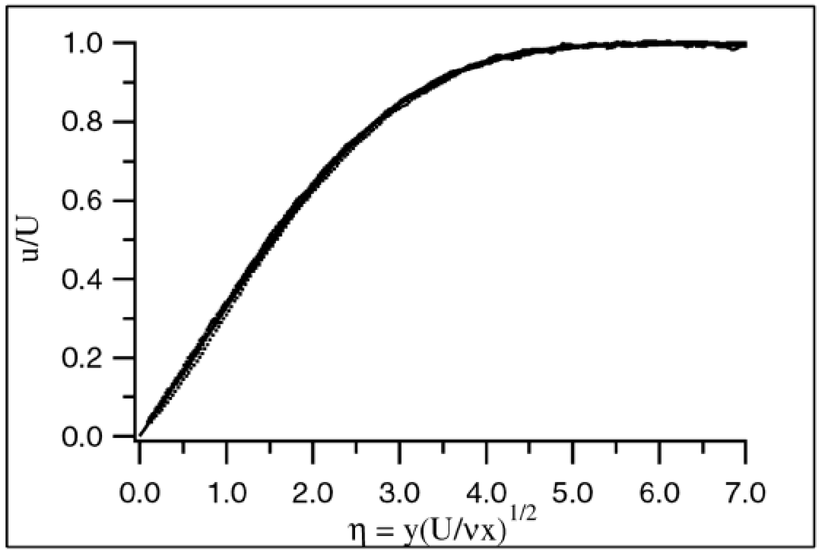
Check self-similarity and boundary-layer approximation.



Experimental data (molecular tagging velocimetry):



(a)

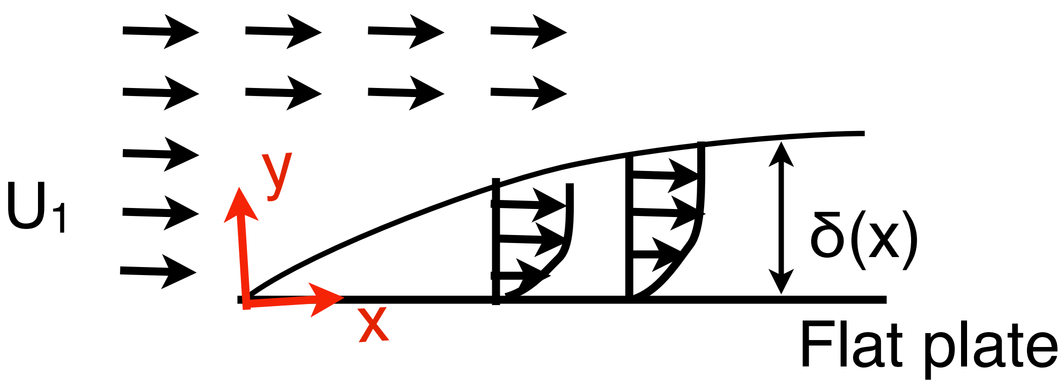


(b)

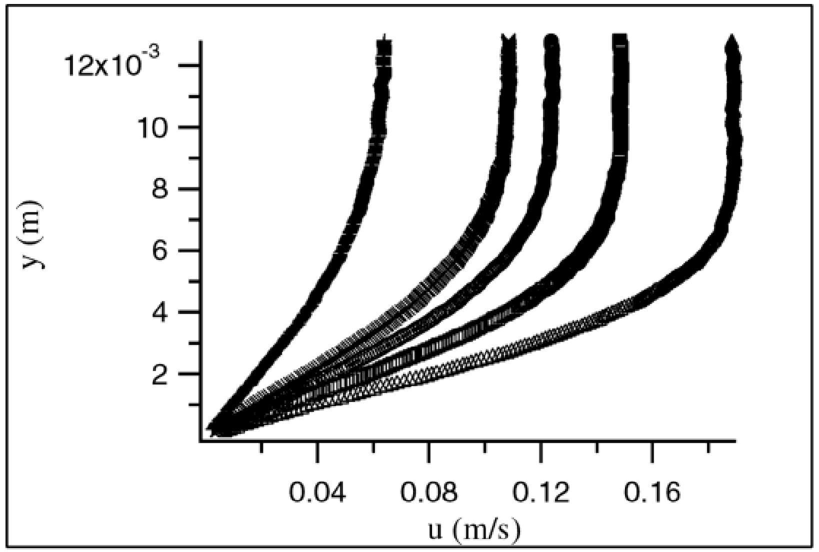
(Klewicki & Hill 2003)

Blasius' boundary layer: comparison with experiment

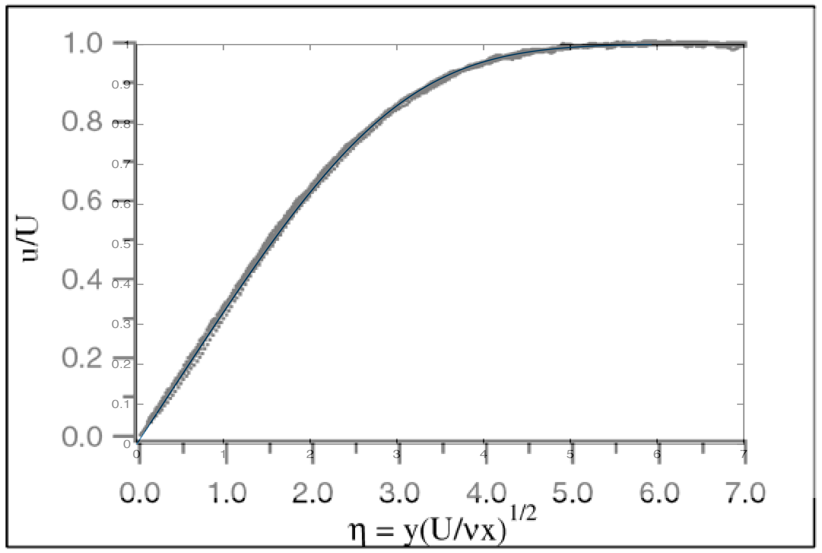
Check self-similarity and boundary-layer approximation.



Experimental data (molecular tagging velocimetry):



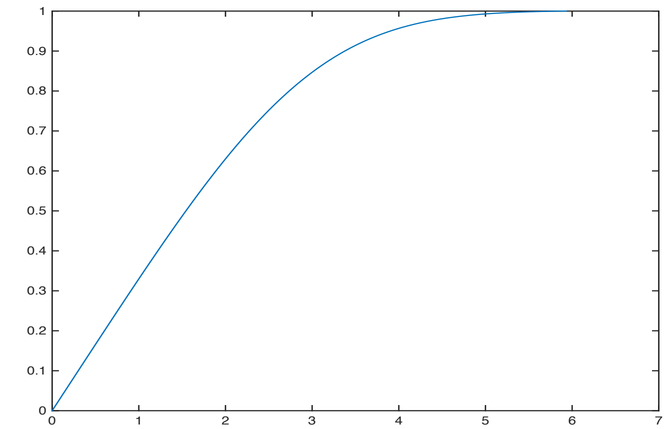
(a)



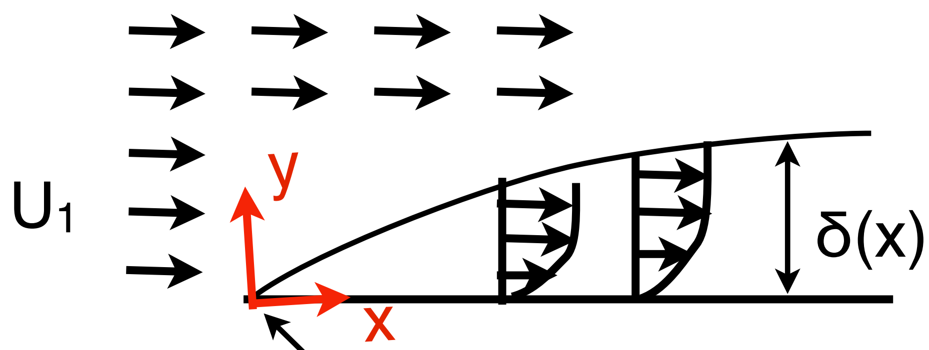
(b)

(Klewicki & Hill 2003)

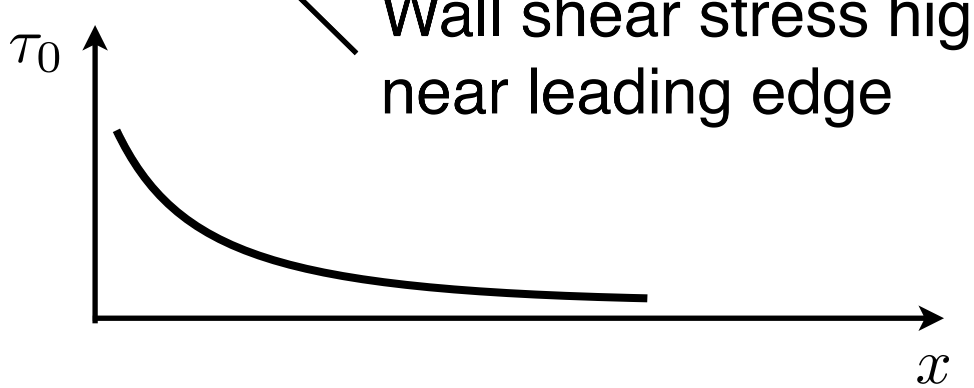
Blasius' boundary layer: comparison with experiment



Blasius' boundary layer: calculate wall shear stress



Wall shear stress highest near leading edge



$$\tau_0 = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0}$$

cf. Stokes' first problem: $\tau_0 \propto \mu U / \delta$

$$\tau_0 = \mu \frac{U_1}{\delta} f''(0) = \mu \frac{U_1}{\delta}(0.332)$$

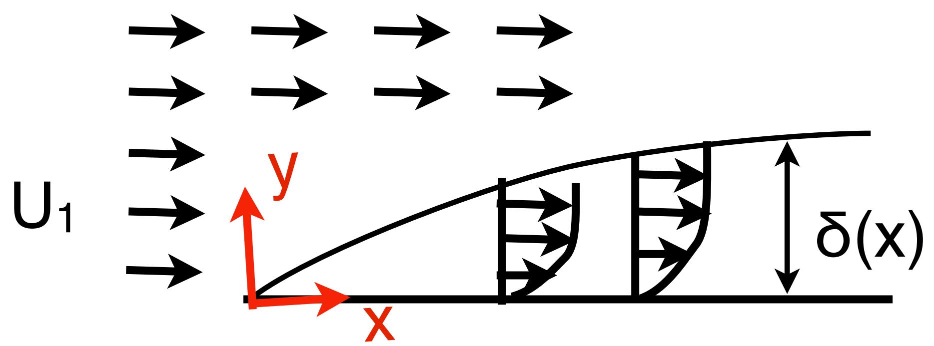
Recall: $\delta(x) = \sqrt{\frac{\nu}{U_1} x}$

So: $\tau_0 = \rho U_1^2 (U_1 x / \nu)^{-1/2} (0.332)$

Wall shear stress coefficient:

$$c_f \equiv \frac{\tau_0}{\frac{1}{2}\rho U_1^2} = 2 Re_x^{-1/2} (0.332)$$

Blasius' boundary layer: calculate skin-friction drag



Depth of flat plate = b

Length of flat plate = L

Recall τ_0 is force per unit area.

$$\tau_0 = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0}$$

$$\tau_0 = \rho U_1^2 (U_1 x/\nu)^{-1/2} (0.332)$$

$$D = \int_0^L \tau_0 b \, \mathrm{d}x$$

$$D = \rho U_1^2 (U_1/\nu)^{-1/2} (0.332) b \int_0^L x^{-1/2} \, \mathrm{d}x$$

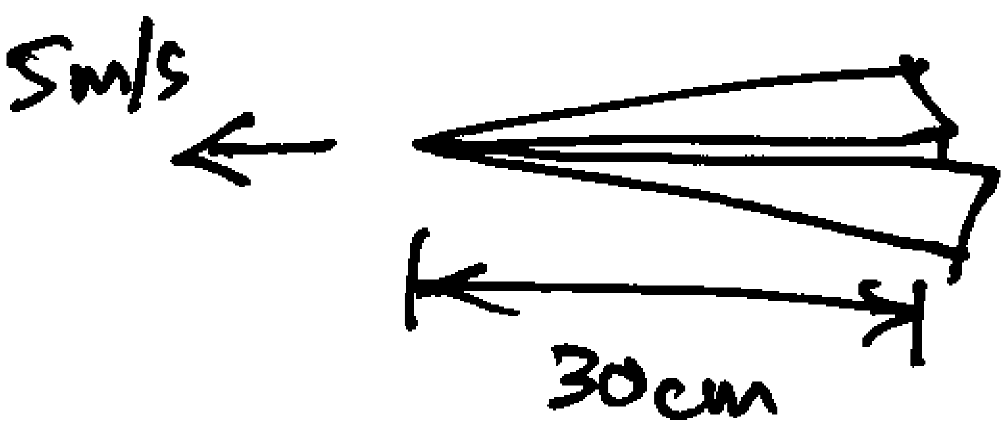
$$= \rho U_1^2 (U_1/\nu)^{-1/2} (0.332) b (2L^{1/2})$$

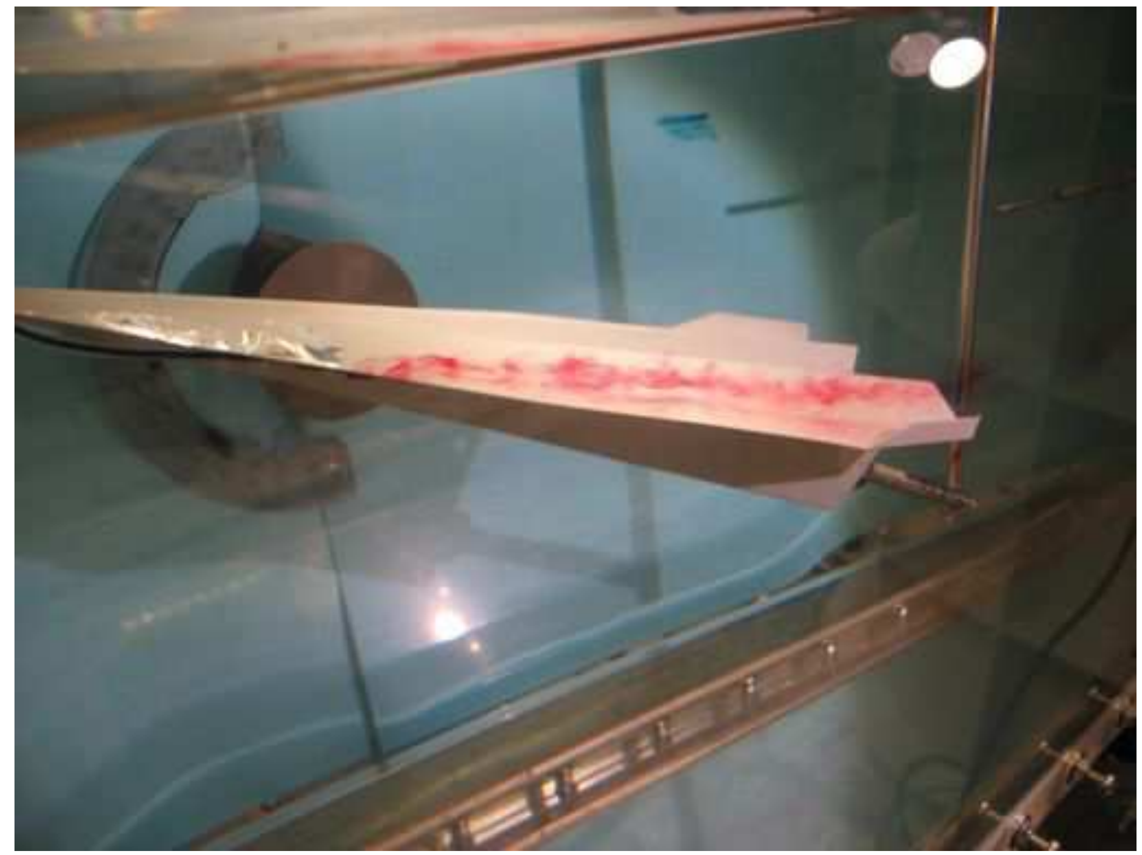
Skin-friction drag coefficient:

$$C_D \equiv \frac{D}{\frac{1}{2}\rho U_1^2 (bL)}$$

$$= (U_1 L/\nu)^{-1/2} 4(0.332) = 4(0.332)\, Re^{-1/2}$$

Exercise: estimate drag on paper aeroplane



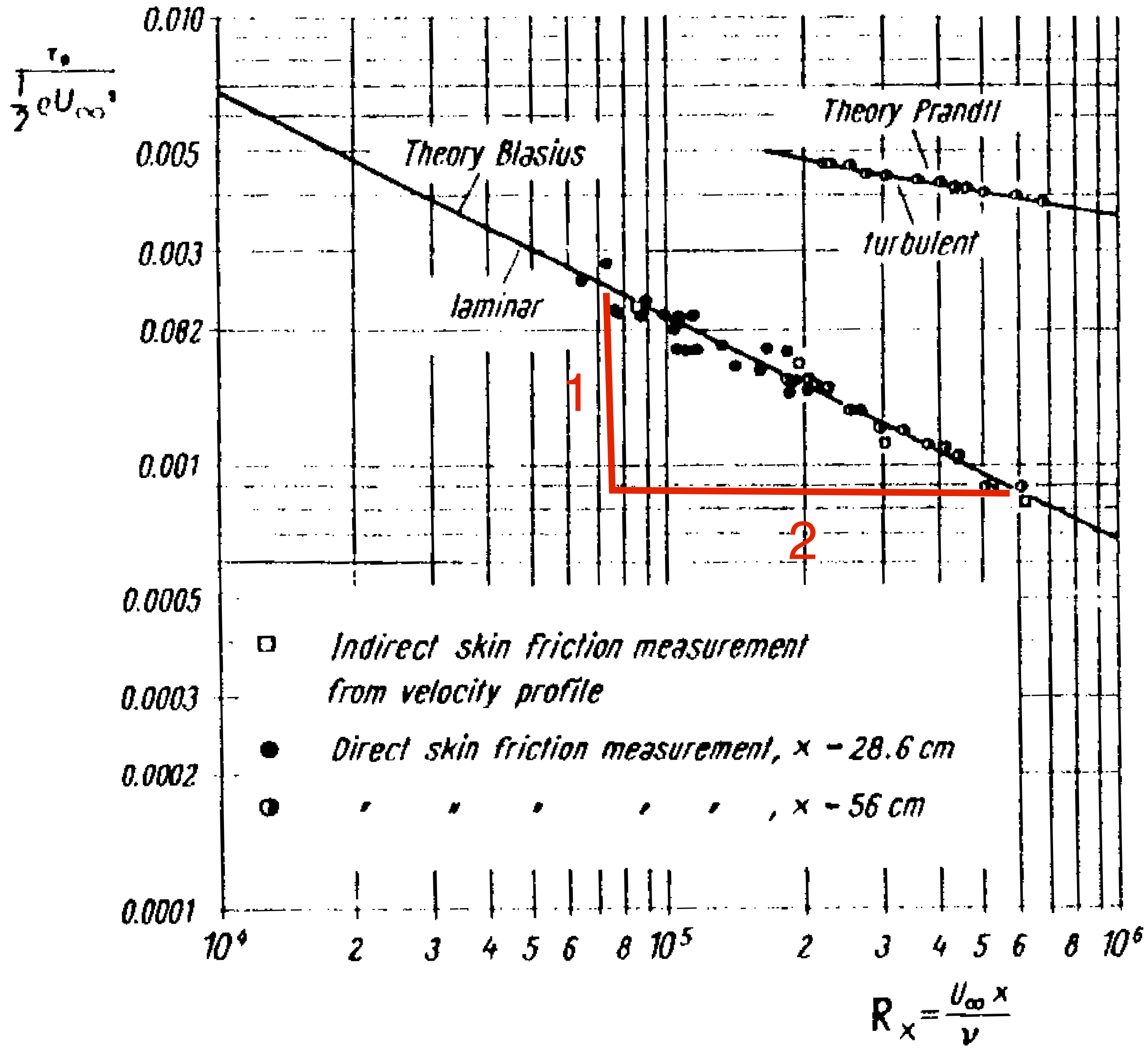


Compare your drag estimate
with data and discuss.

Figure 5: Wing vortex at high angles of attack (Re=55,000).

(Ng *et al.* AIAA 2009-3958)

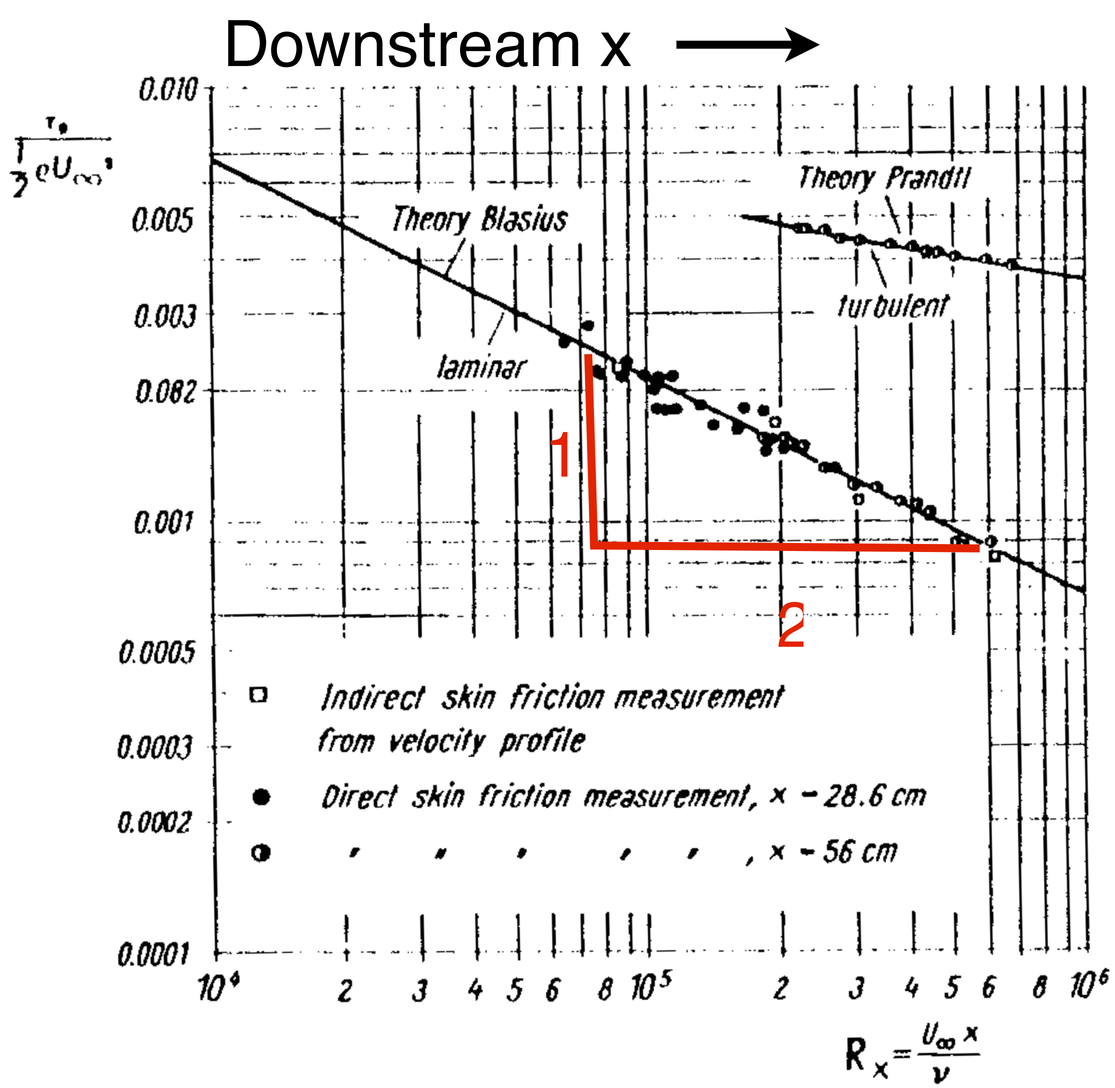
Blasius' boundary layer: calculate wall shear stress



Wall shear stress coefficient:

$$c_f \equiv \frac{\tau_0}{\frac{1}{2}\rho U_1^2} = 2Re_x^{-1/2}(0.332)$$

Blasius' boundary layer: calculate wall shear stress

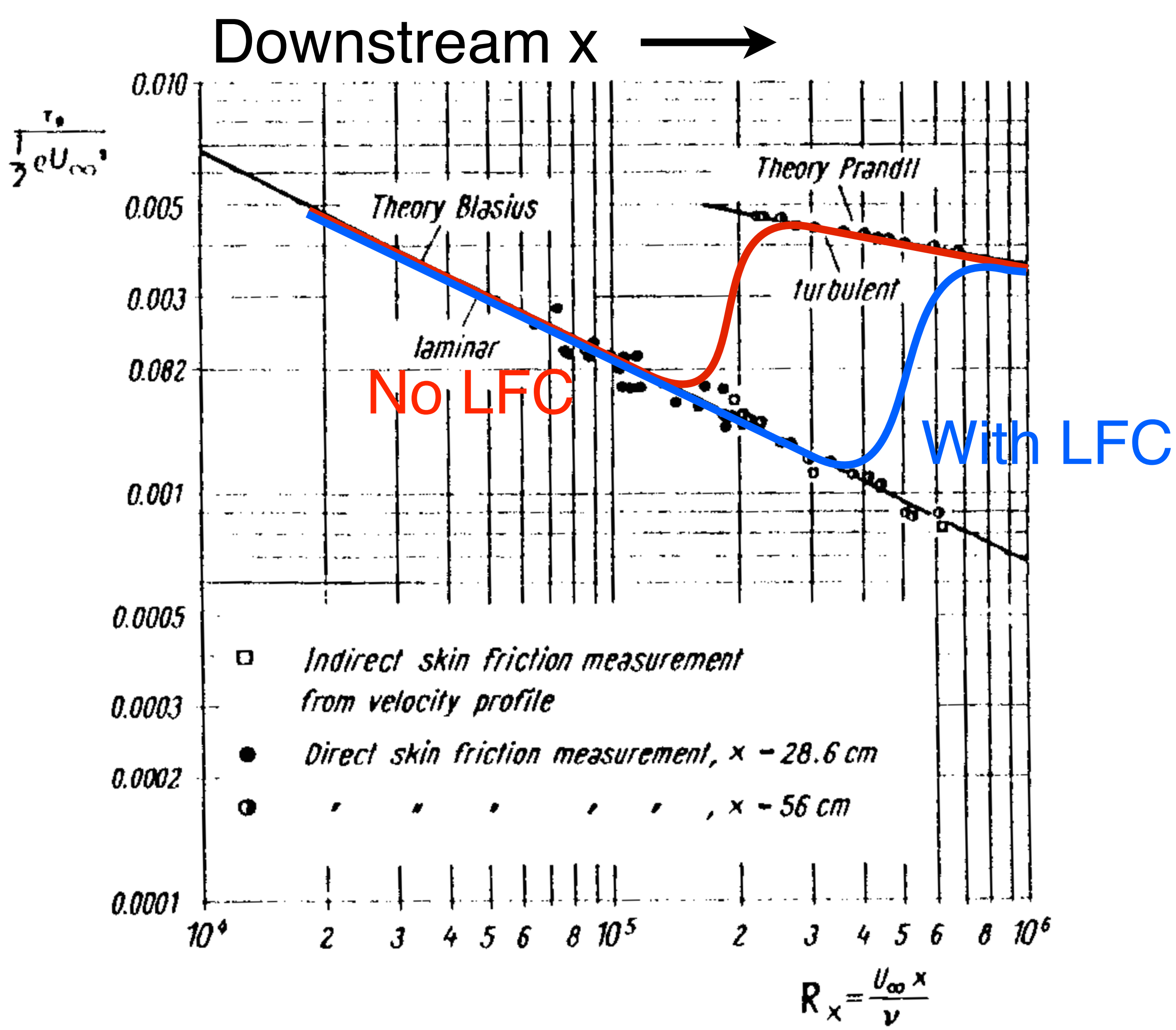


Wall shear stress coefficient:

$$c_f \equiv \frac{\tau_0}{\frac{1}{2}\rho U_1^2} = 2Re_x^{-1/2}(0.332)$$

Recall:

Application: laminar flow control on Boeing 787



http://www.newairplane.com/787/#/design-highlights/visionary-design/propulsion/

Melbourne School of Engineering
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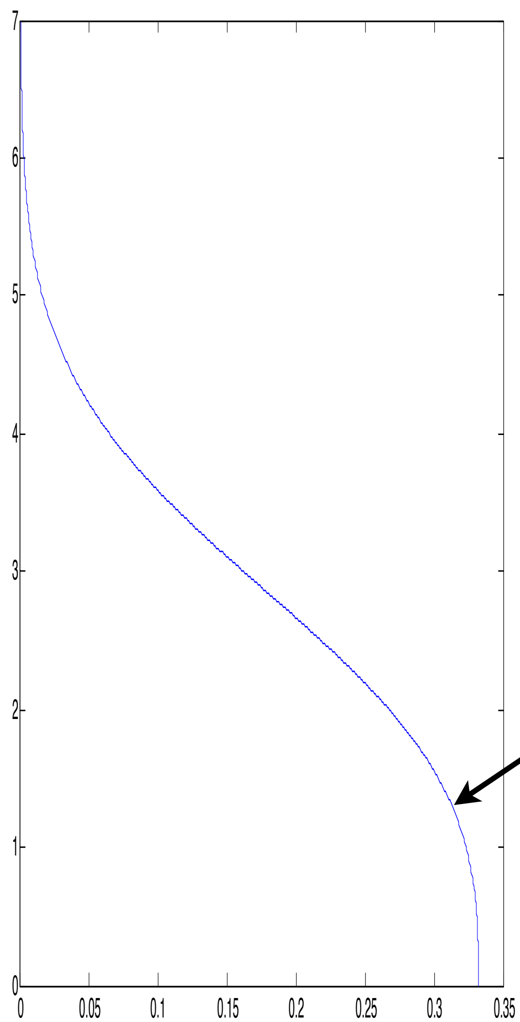
Lecture BL08: Displacement and momentum thicknesses

24 March 2016

Blasius' boundary layer: vorticity

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{V}{X}\frac{\partial \tilde{v}}{\partial \tilde{x}} - \frac{U}{Y}\frac{\partial \tilde{u}}{\partial \tilde{y}} = \frac{U}{Y}\left[\frac{Y^2}{X^2}\frac{\partial \tilde{v}}{\partial \tilde{x}} - \frac{\partial \tilde{u}}{\partial \tilde{y}}\right] = \frac{U}{Y}\left[\frac{1}{Re}\frac{\partial \tilde{v}}{\partial \tilde{x}} - \frac{\partial \tilde{u}}{\partial \tilde{y}}\right] \approx -\frac{\partial u}{\partial y} = \frac{U_1}{\delta} f''$$

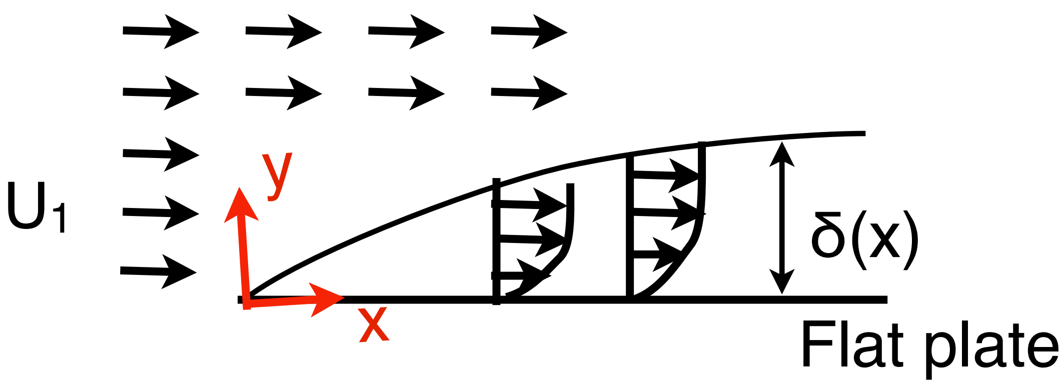
$$\eta = \frac{y}{\delta}$$



Vorticity concentrated near wall

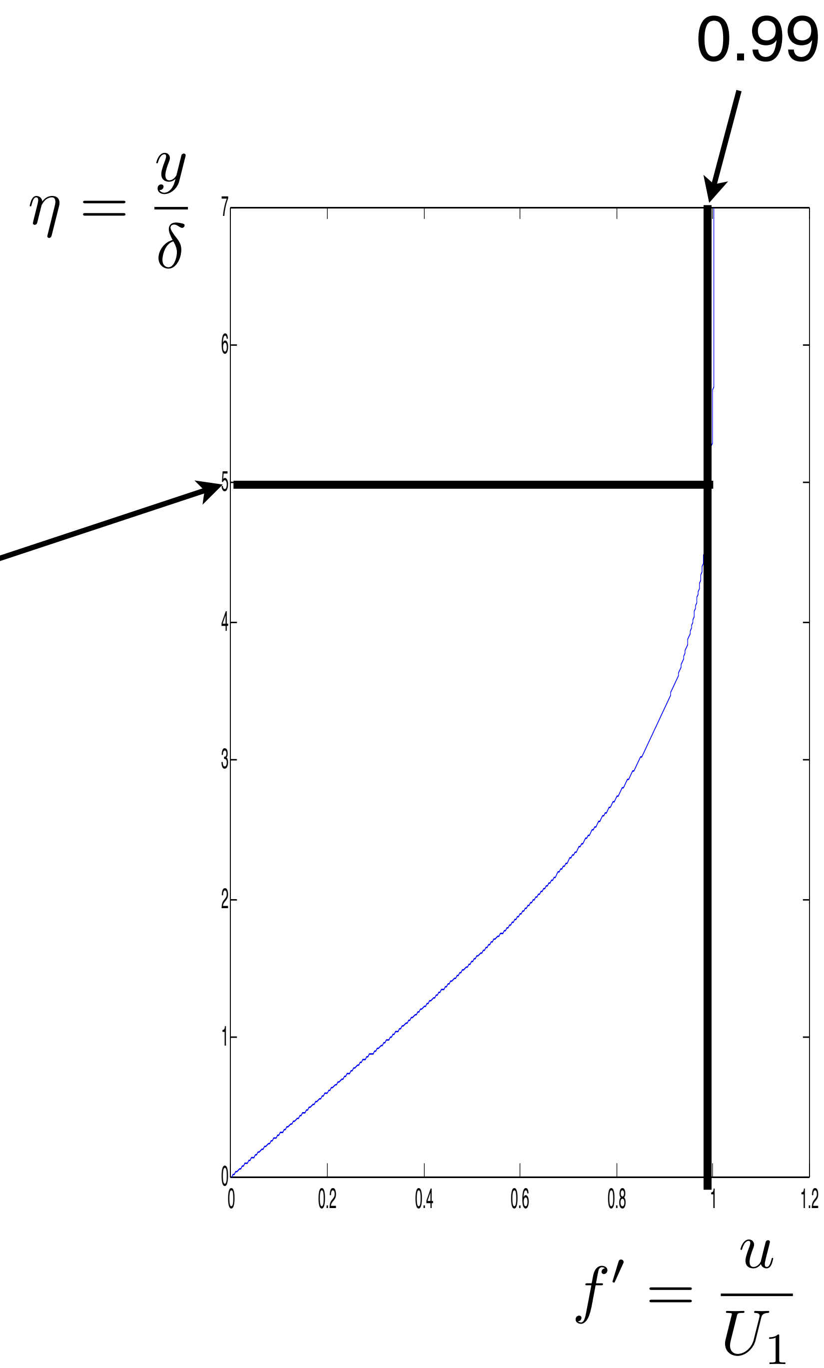
$$\omega \frac{\sqrt{\nu x / U_1}}{U_1} = f''$$

Blasius' boundary layer: 99% thickness



$$\frac{\delta_1}{\delta} = \frac{\delta_1}{\sqrt{\nu x/U_1}} \approx 5.0$$

$$\delta_1 \approx 5.0\sqrt{\nu x/U_1}$$

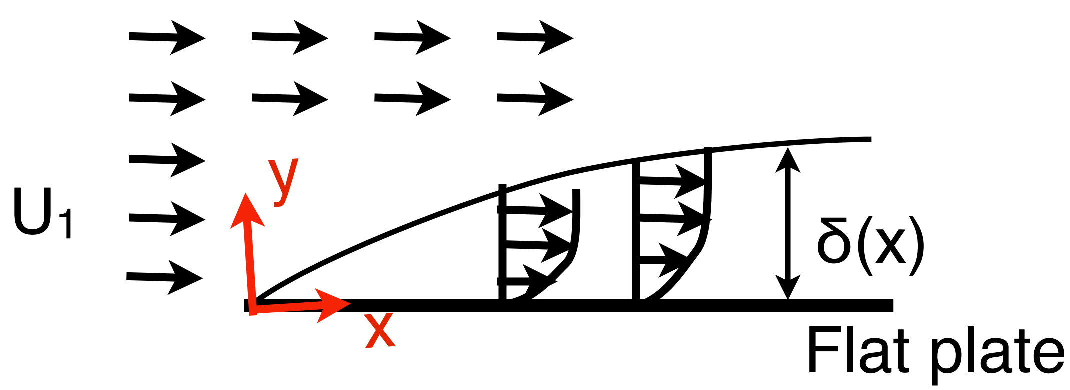


Not always reliable, because one needs to have u(y) with 1% maximum error.

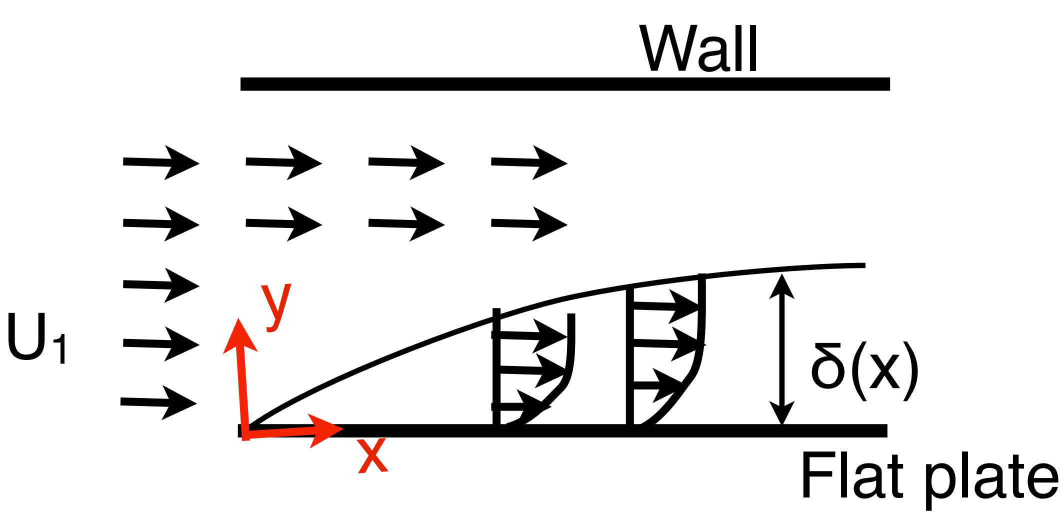
Question:

Is

No wall



the same as



?

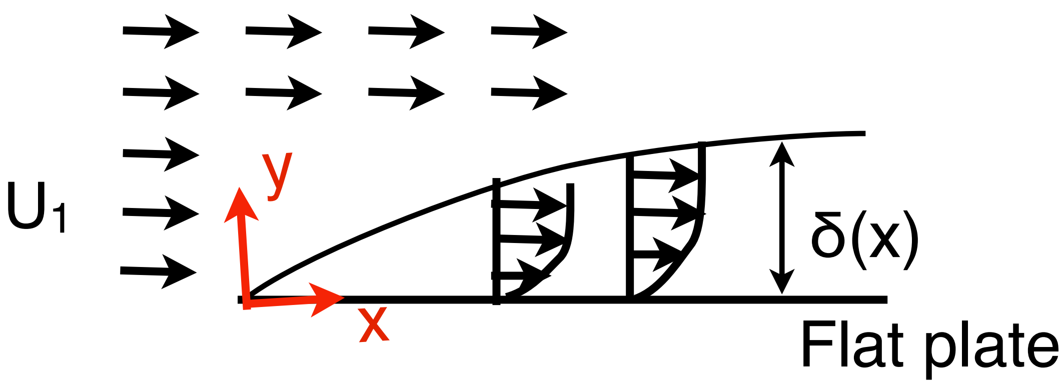
Wing above ground

Melbourne High Reynolds Number Boundary Layer Wind Tunnel

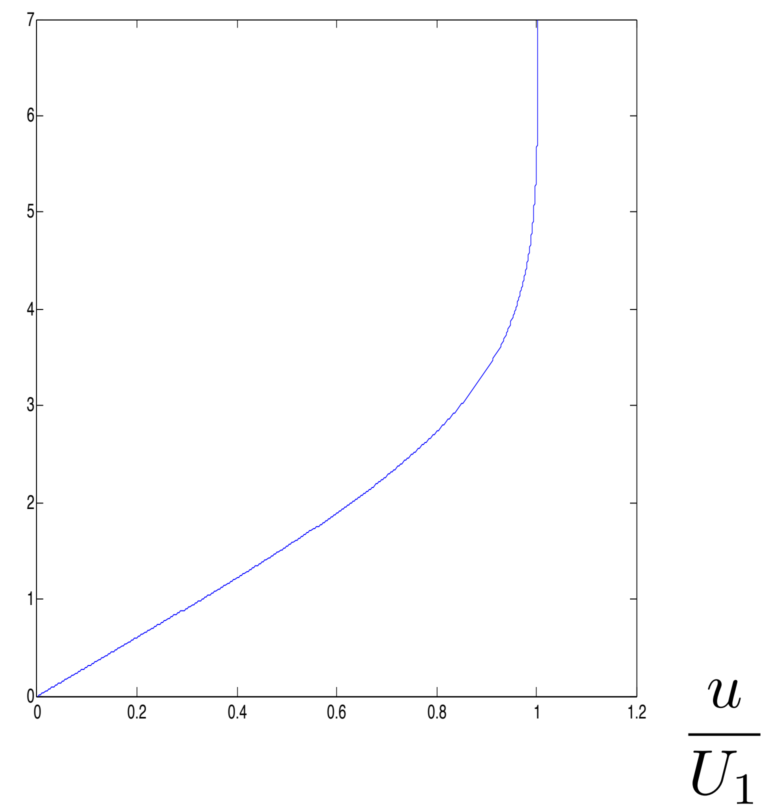
Question: why are there gaps on the roof of the wind tunnel?

http://www.abc.net.au/catalyst/stories/3285559.htm

Blasius’ boundary layer



$$\eta = \frac{y}{\delta} = \frac{y}{\sqrt{\nu x / U_1}}$$



- Near leading edge, the boundary-layer approximation does not hold. (Reynolds number, Re_x, is not large enough.)

- Downstream when Re_x is large enough, the boundary layer behaves as if there is an effective origin at x = 0, which may not coincide with the leading edge. So we also need a better way to track boundary-layer ‘age’ other than x, e.g. 99% thickness, displacement thickness, momentum thickness.

Blasius' boundary layer: wall-normal velocity

$$v = \int_0^y -\frac{\partial u}{\partial x}\,\mathrm{d}y_1 = U_1 \frac{\mathrm{d}\delta}{\mathrm{d}x}(\eta f' - f)$$

$$\delta(x) = \sqrt{\frac{\nu}{U_1}x}$$

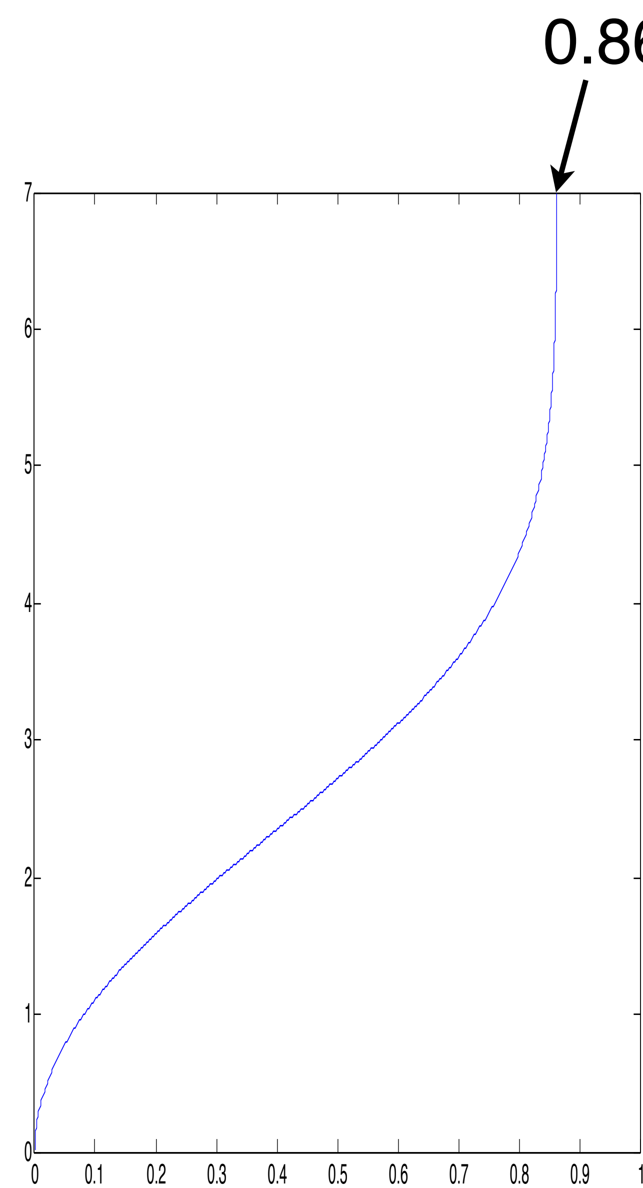
$$\frac{v}{U_1} = (U_1 x/\nu)^{-1/2}\frac{1}{2}(\eta f' - f)$$

Note that $\displaystyle \frac{v}{U_1} = 0.86(U_1 x/\nu)^{-1/2}$

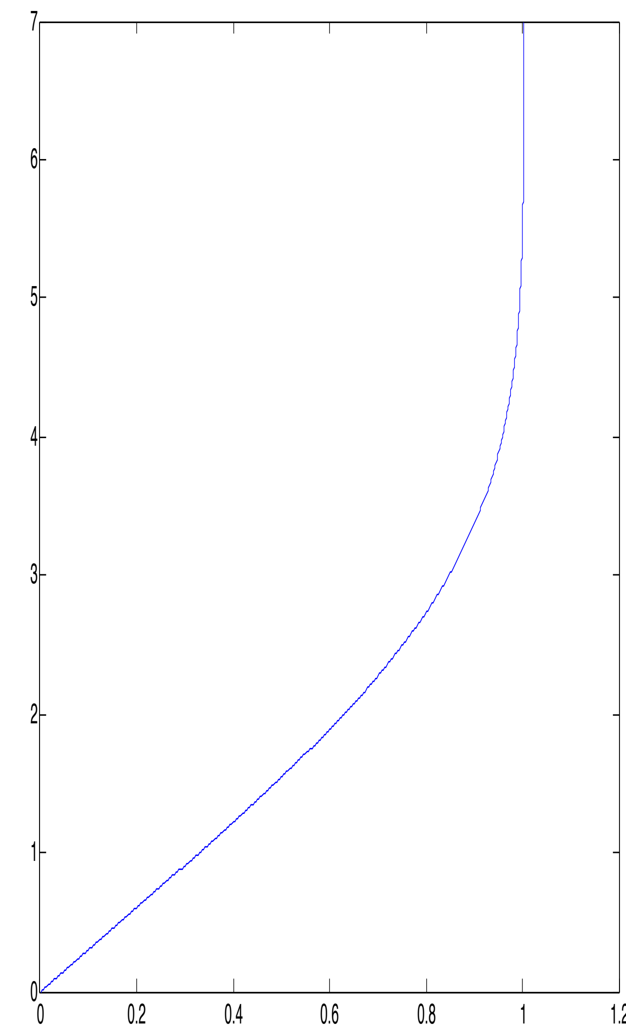
or $\displaystyle \frac{v}{u} = 0.86(U_1 x/\nu)^{-1/2}$

far away from the boundary.
i.e. streamlines are not straight
as we first expected, but
becomes straighter as *Re*$_x$
becomes large.

$$\eta = \frac{y}{\delta}$$

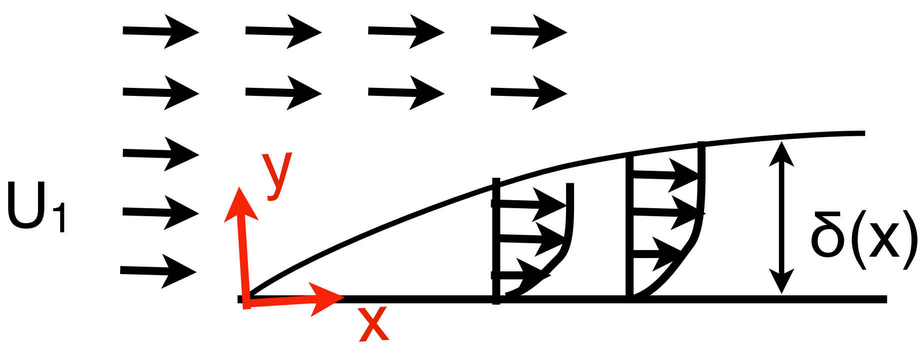


$$\frac{v}{U_1}(U_1 x/\nu)^{1/2} = \frac{1}{2}(\eta f' - f)$$

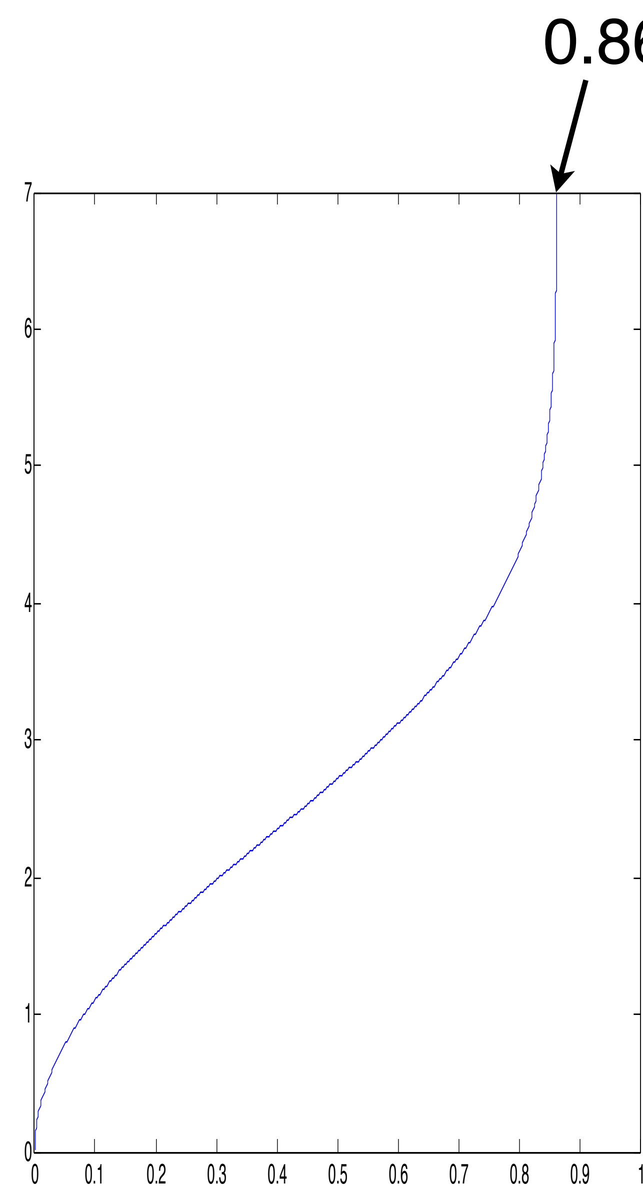


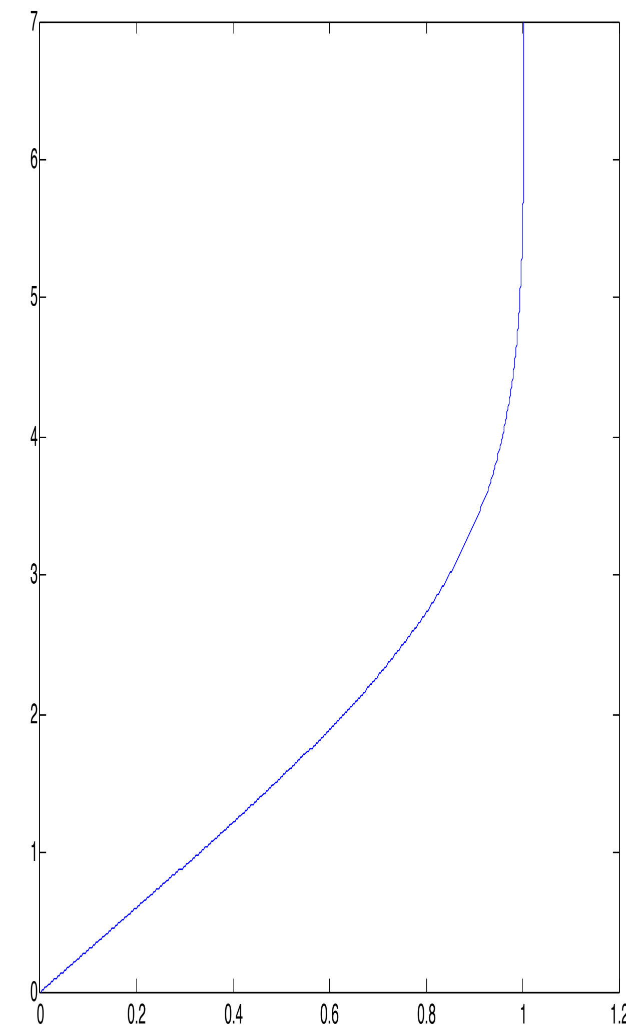
$$f' = \frac{u}{U_1}$$

Blasius' boundary layer: wall-normal velocity



$$\eta = \frac{y}{\delta}$$





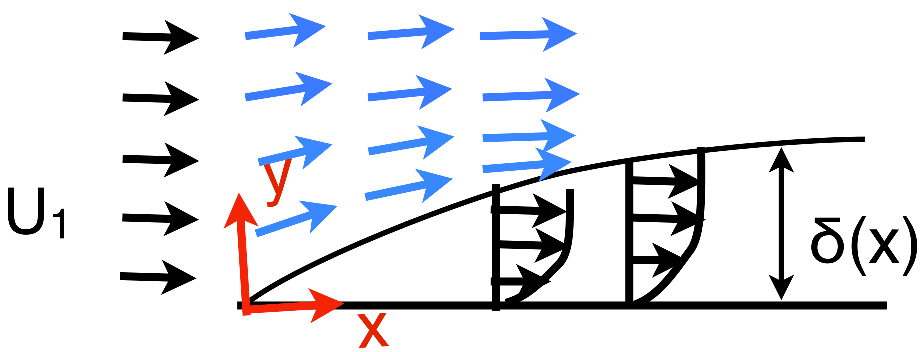
$$\frac{v}{U_1}(U_1 x/\nu)^{1/2} = \frac{1}{2}(\eta f' - f) \qquad f' = \frac{u}{U_1}$$

Note that $\dfrac{v}{U_1} = 0.86(U_1 x/\nu)^{-1/2}$

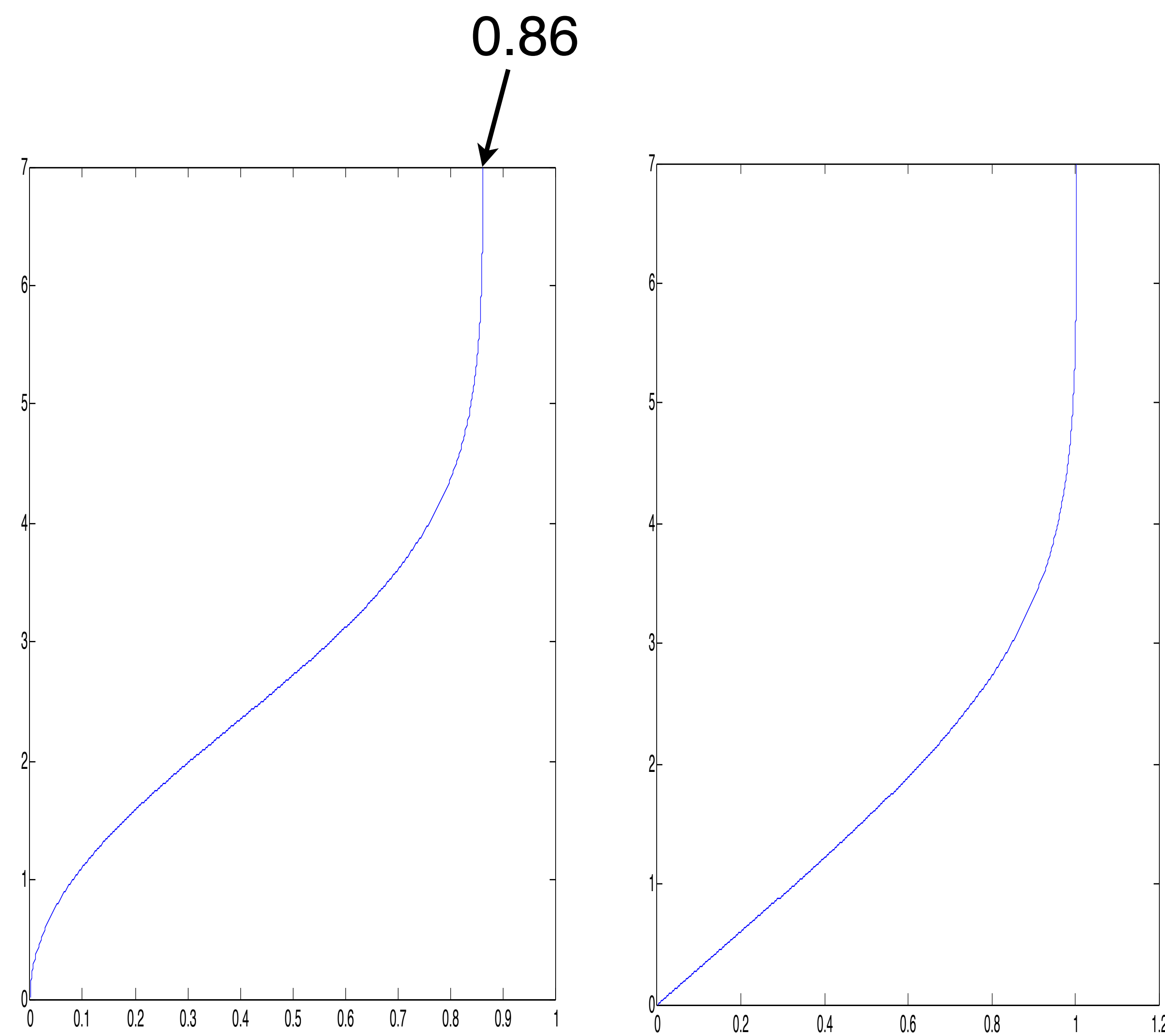
or $\dfrac{v}{u} = 0.86(U_1 x/\nu)^{-1/2}$

far away from the boundary.
i.e. streamlines are not straight as we first expected, but becomes straighter as Re_x becomes large.

Blasius' boundary layer: wall-normal velocity



$$\eta = \frac{y}{\delta}$$



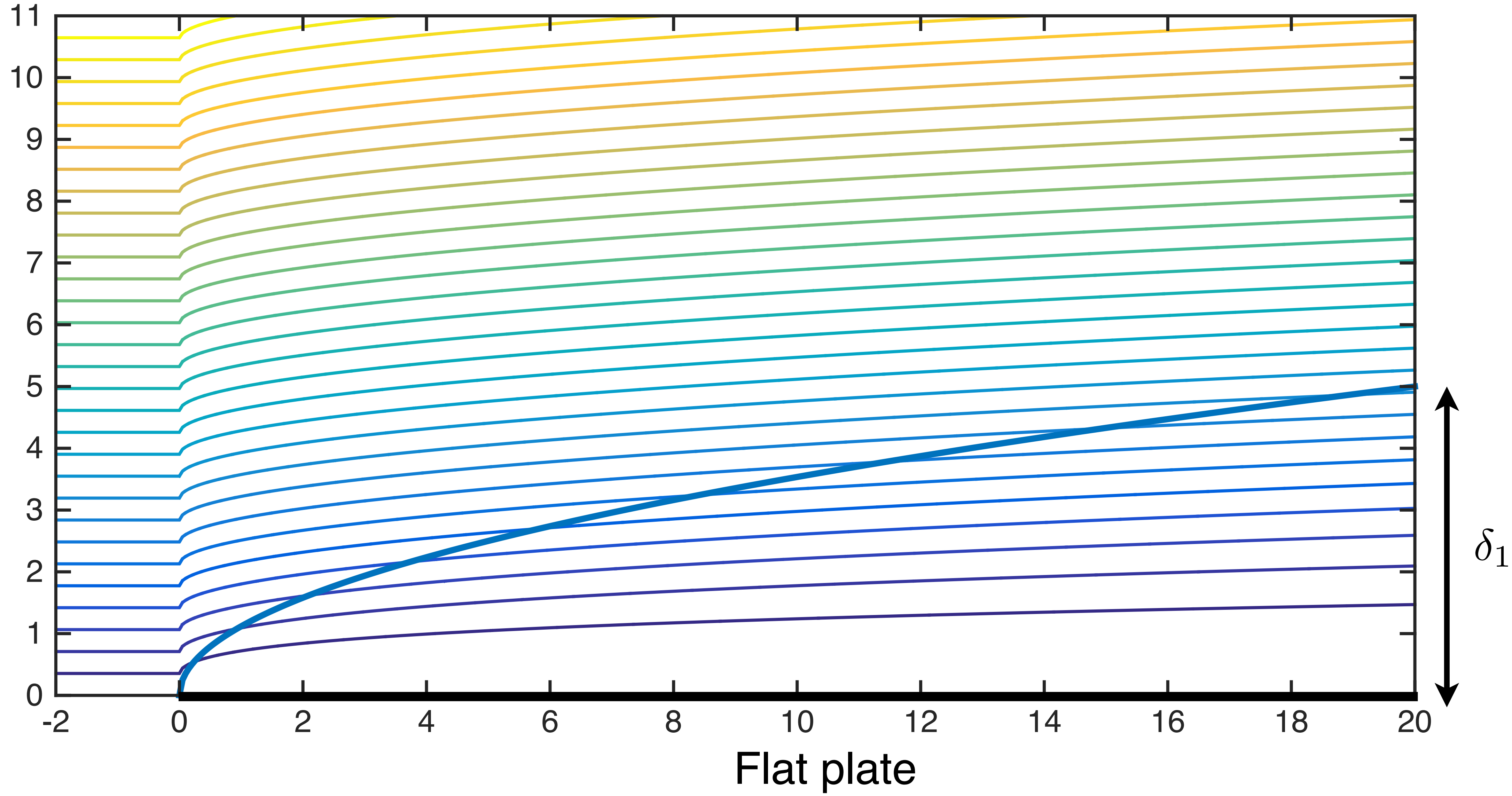
$$\frac{v}{U_1}(U_1 x/\nu)^{1/2} = \frac{1}{2}(\eta f' - f) \qquad f' = \frac{u}{U_1}$$

Note that $$\frac{v}{U_1} = 0.86(U_1 x/\nu)^{-1/2}$$

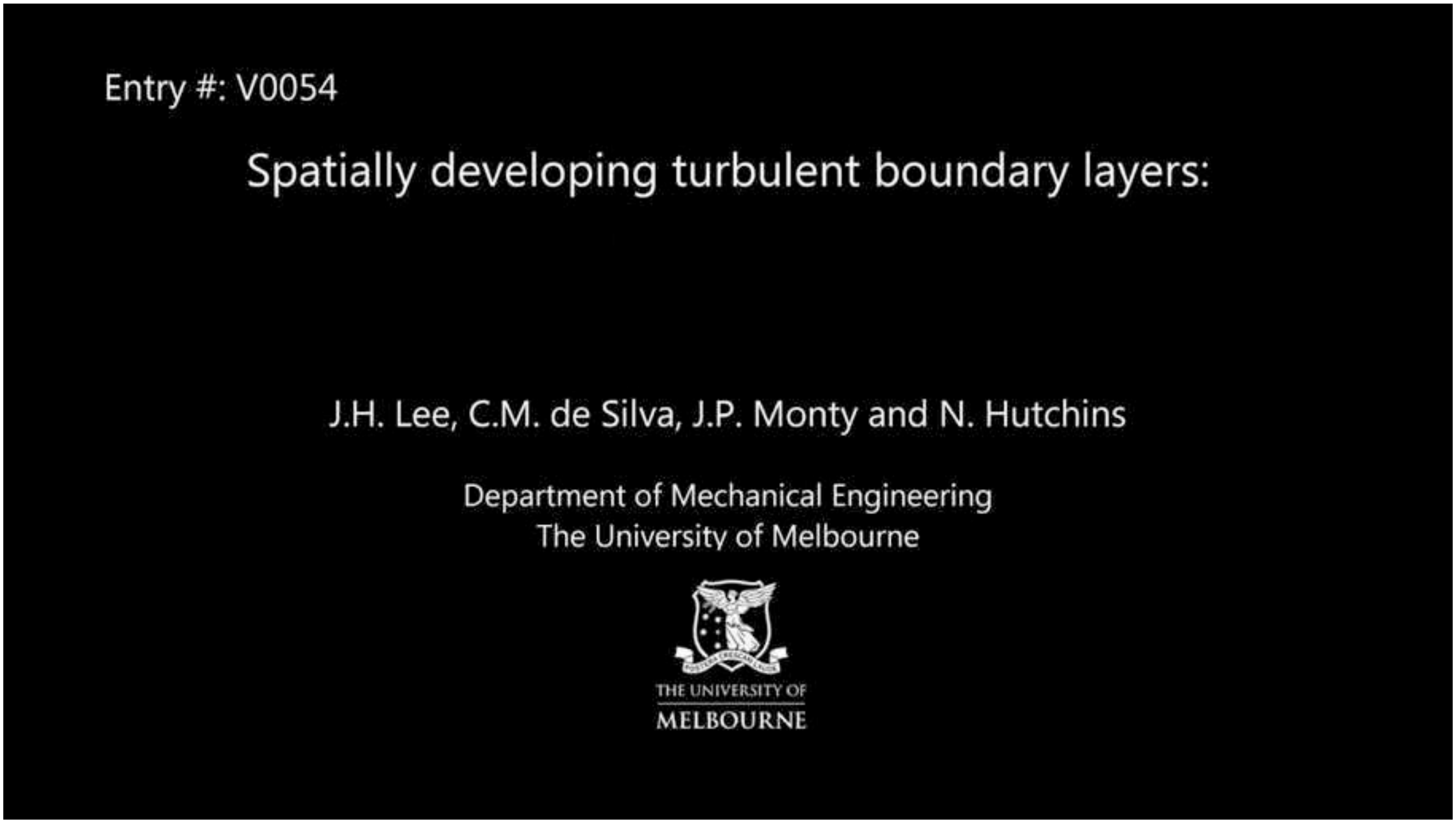
or $$\frac{v}{u} = 0.86(U_1 x/\nu)^{-1/2}$$

far away from the boundary.
i.e. streamlines are not straight as we first expected, but becomes straighter as *Re_x* becomes large.

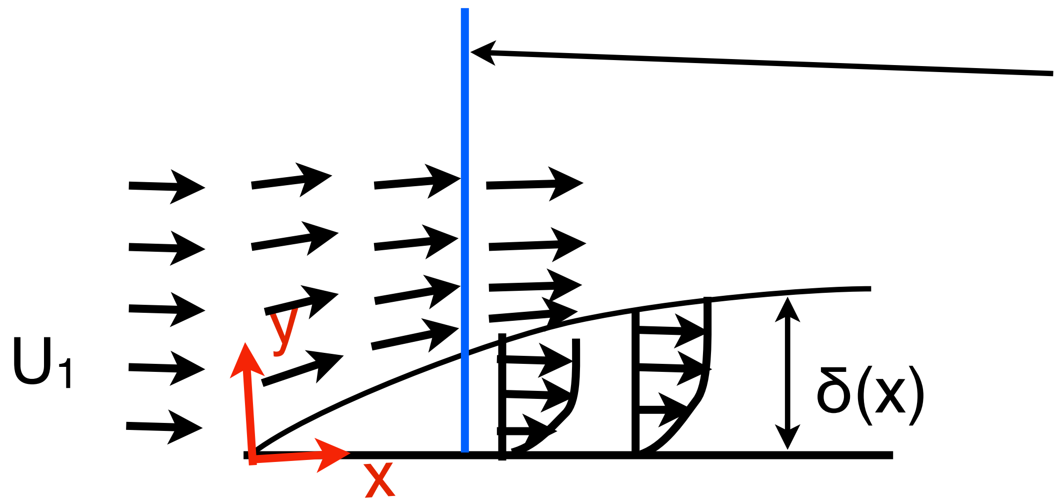
Blasius' boundary layer: streamlines



Blasius' boundary layer: effect of boundary-layer displacement

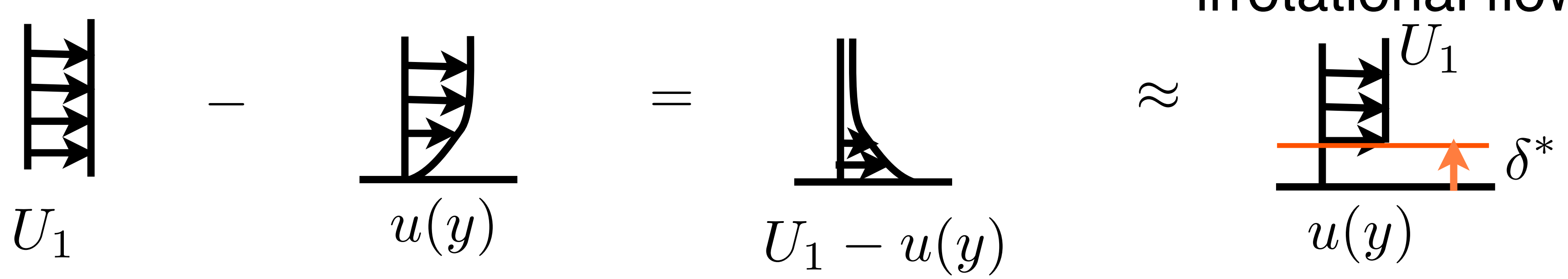


Blasius' boundary layer: displacement thickness



Question: What is the volume flux per unit depth loss due to flat plate?

Without plate | With plate | Effect of plate | Effective displacement of irrotational flow

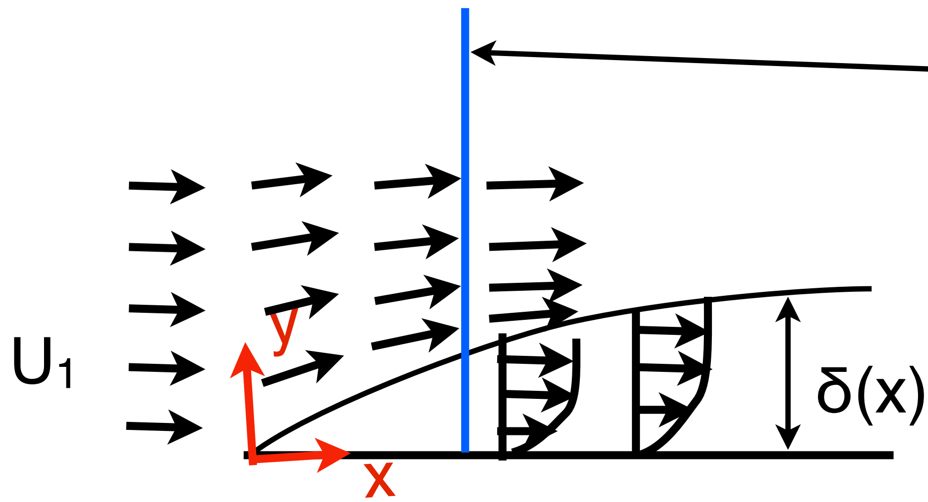


Volume flux loss:

$$\int_0^\infty (U_1 - u(y))\,\mathrm{d}y = U_1 \boxed{\int_0^\infty (1 - \frac{u(y)}{U_1})\,\mathrm{d}y} = U_1\delta^*$$

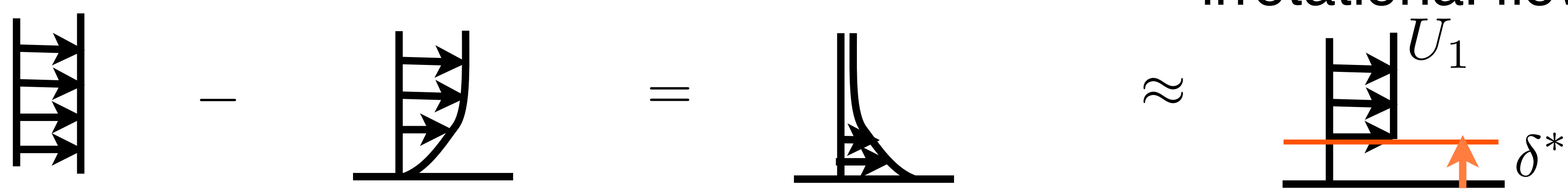
Displacement thickness

Blasius' boundary layer: displacement thickness



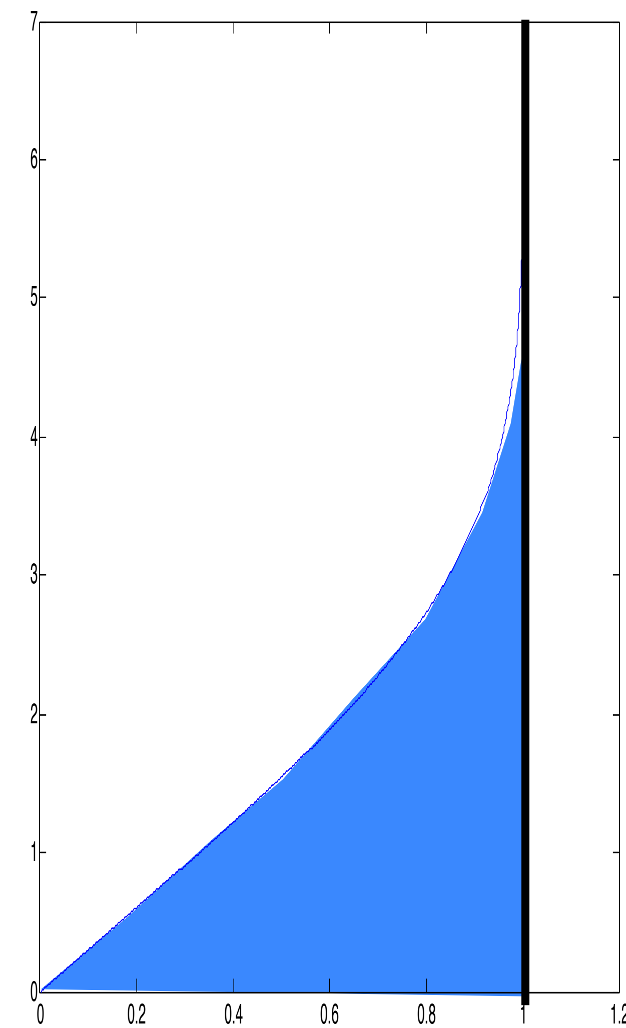
Question: What is the volume flux per unit depth loss due to flat plate?

Without plate | With plate | Effect of plate | Effective displacement of irrotational flow



$U_1 \quad - \quad u(y) \quad = \quad U_1 - u(y) \quad \approx \quad u(y)$

$$\eta = \frac{y}{\delta}$$



$$f' = \frac{u}{U_1}$$

Displacement thickness:

$$\delta^* \equiv \int_0^\infty (1 - \frac{u(y)}{U_1})\,\mathrm{d}y = \delta \int_0^\infty [1 - f'(\eta)]\,\mathrm{d}\eta \approx (1.7)\delta$$

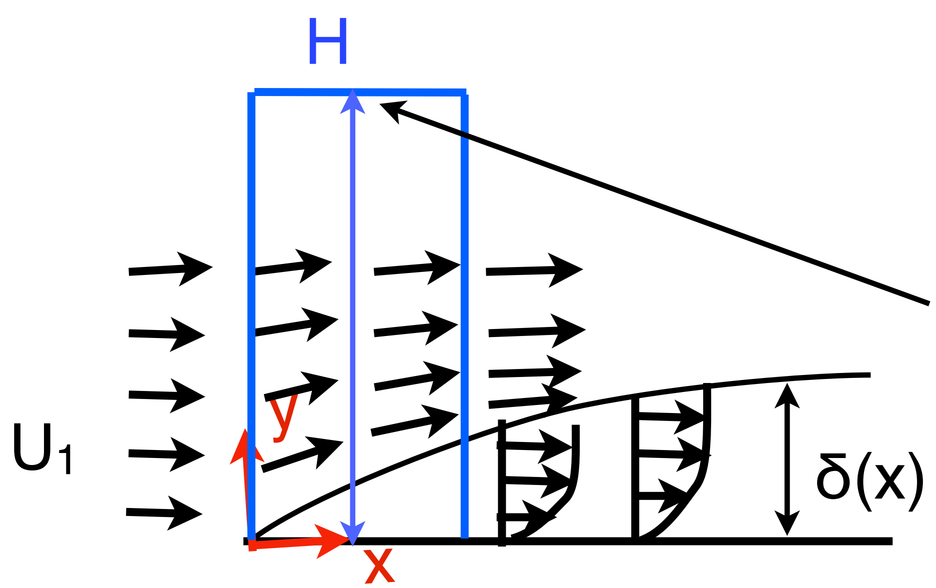
$$\delta \approx \frac{\delta_1}{5.0} \approx \frac{\delta^*}{1.7} \qquad\qquad \delta^* \approx 0.34\,\delta_1$$

Melbourne School of Engineering
MCEN90018 Advanced Fluid Dynamics

Lecture BL09: Separation

12 April 2016

Blasius' boundary layer: bleeding in wind-tunnel design



So boundary layers behave as an effective contraction and will accelerate the flow in a wind tunnel if not bled.

Amount of 'bleeding' through top of straight wind tunnel to maintain pressure gradient.

Mass conservation in blue control volume:

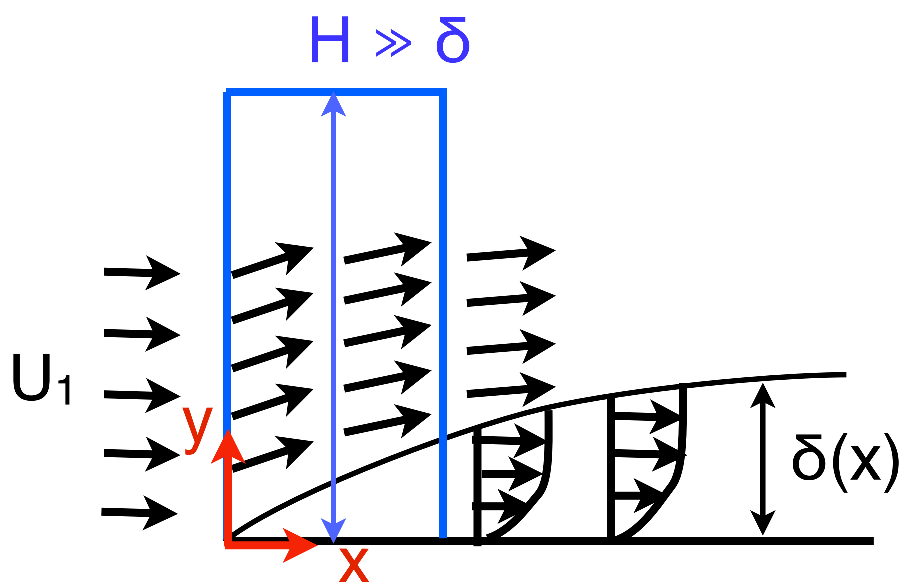
(Flow into from left) = (Bled flow from top) + (Reduced flow out of right)

$U_1 H$ $\quad\quad$ $\displaystyle\int_0^x v\,\mathrm{d}x$ $\quad\quad$ $(U_1 H - U_1 \delta^*)$

$$\int_0^x v\,\mathrm{d}x = U_1 \delta^*$$

$$v = \frac{\mathrm{d}}{\mathrm{d}x}(U_1 \delta^*)$$

Displacement thickness, δ*



Mass conservation of control volume:

(Flow in from left) = (Flow out of top) + (Flow out of right)

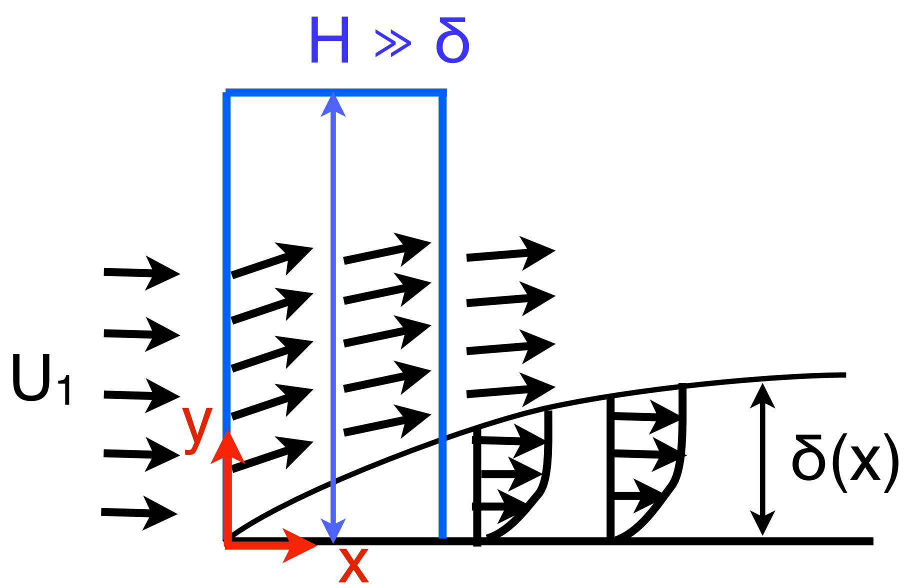
$$\int_0^H U_1 \, \mathrm{d}y \quad = \quad \int_0^x v \, \mathrm{d}x \quad + \quad \int_0^H u \, \mathrm{d}y$$

Flow per unit depth, actually

Because integrand vanishes for H ≫ δ

$$\Rightarrow \quad \int_0^x v \, \mathrm{d}x = \int_0^H (U_1 - u) \, \mathrm{d}y = U_1 \int_0^H (1 - \frac{u}{U_1}) \mathrm{d}y = U_1 \int_0^\infty (1 - \frac{u}{U_1}) \, \mathrm{d}y = U_1 \delta^*$$

Momentum thickness, θ



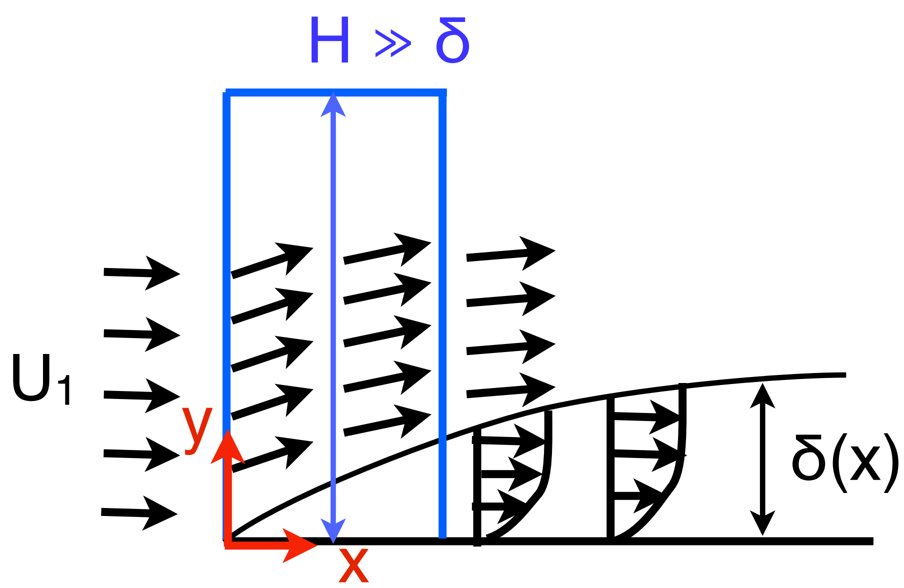
x-momentum of control volume (per unit depth):

0 = (x-mom. in from left) – (x-mom. out of top) – (x-mom. out of right) – (Drag on plate)

$$0 = \int_0^H \rho U_1^2 \,\mathrm{d}y - \int_0^x \rho U_1 v \,\mathrm{d}x - \int_0^H \rho u^2 \,\mathrm{d}y - \frac{D}{b}$$

$$\rho U_1 \int_0^x v \,\mathrm{d}x = \rho U_1 \int_0^H (U_1 - u) \,\mathrm{d}y$$

Momentum thickness, θ



x-momentum of control volume (per unit depth):

$$\frac{D}{\rho b U_1^2} = \int_0^H \mathrm{d}y - \int_0^H (1 - \frac{u}{U_1})\,\mathrm{d}y - \int_0^H \frac{u^2}{U_1^2}\,\mathrm{d}y$$

$$= \int_0^H \frac{u}{U_1}(1 - \frac{u}{U_1})\,\mathrm{d}y$$

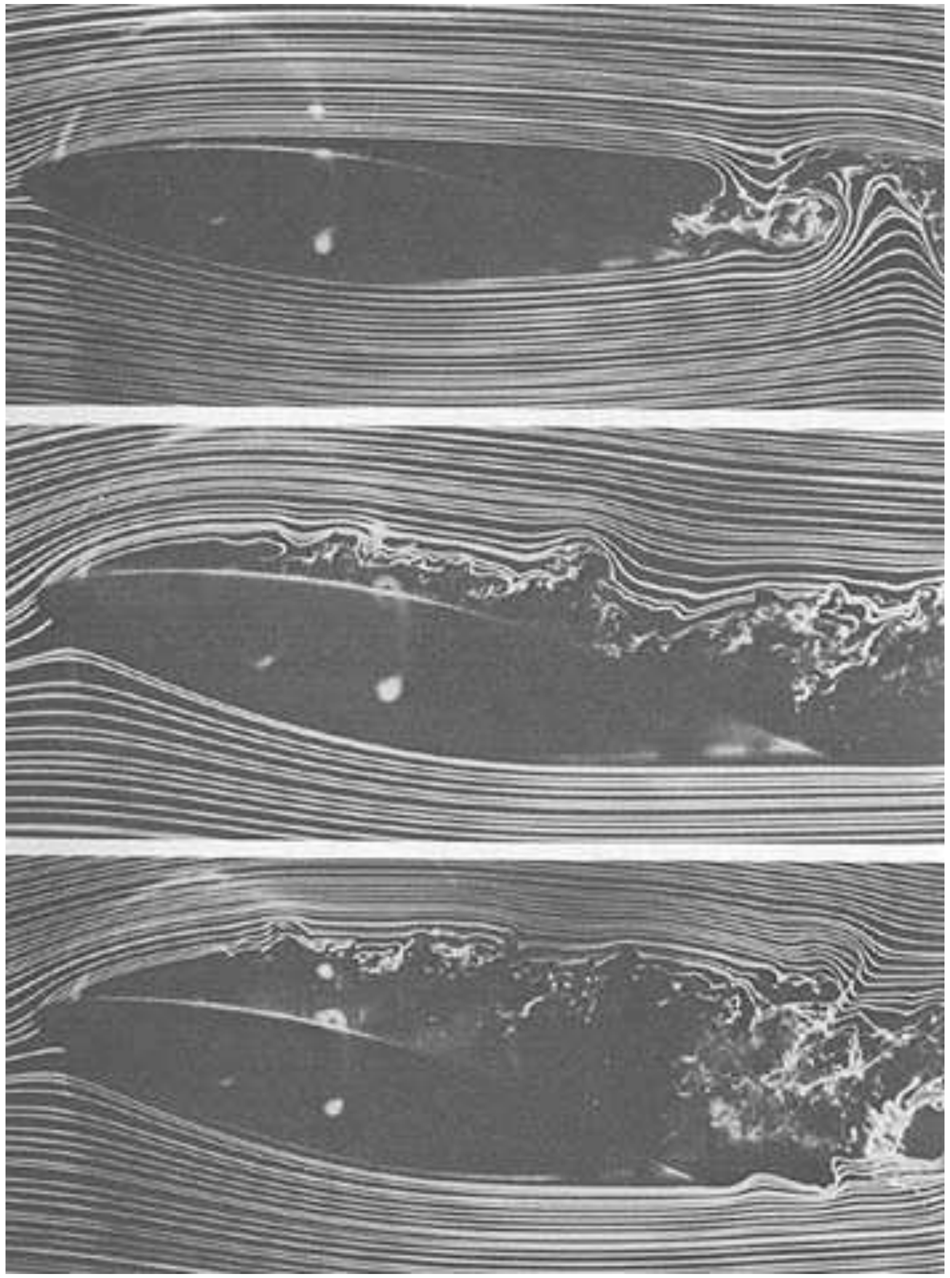
$$= \int_0^\infty \frac{u}{U_1}(1 - \frac{u}{U_1})\,\mathrm{d}y$$

$$= \theta(x)$$

$$C_D = \frac{D}{\frac{1}{2}\rho b x U_1^2} = \frac{\theta(x)}{\frac{1}{2}x}$$

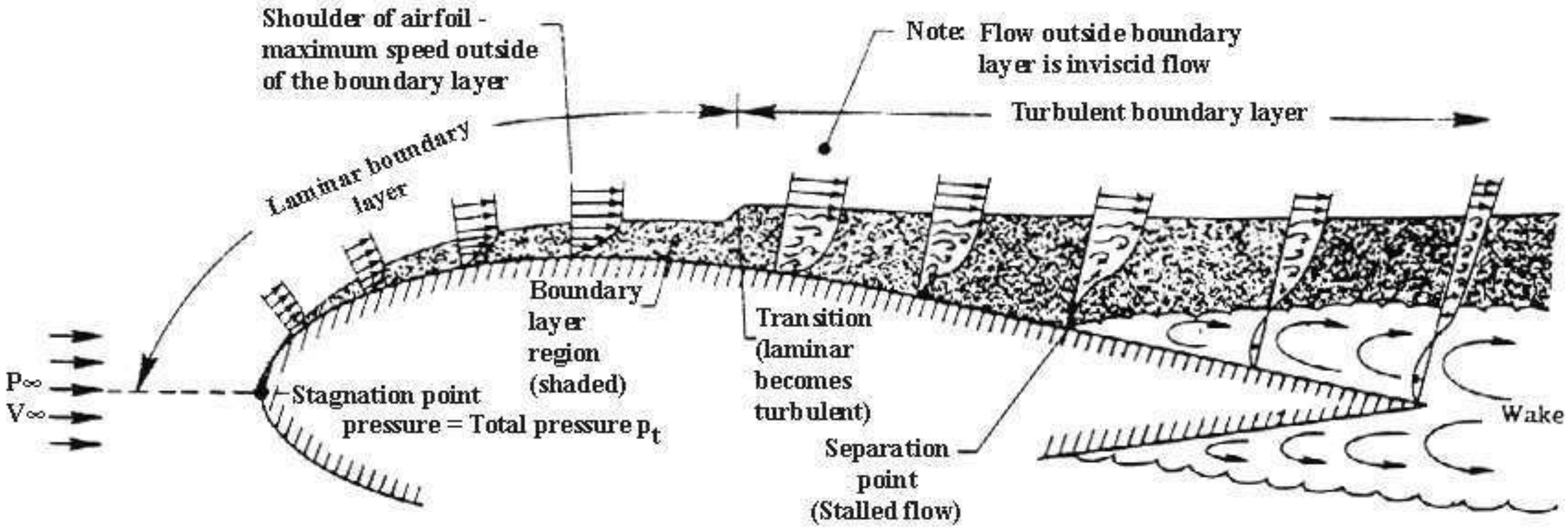
Pressure gradient effects in the boundary layer and separation

Prof Chong's notes pp. 25–29



http://history.nasa.gov/SP-4103/app-f.htm

Pressure gradient effects in the boundary layer and separation



http://www.centennialofflight.net/essay/Theories_of_Flight/Skin_Friction/TH11G3.htm

Fun bed-time reading:

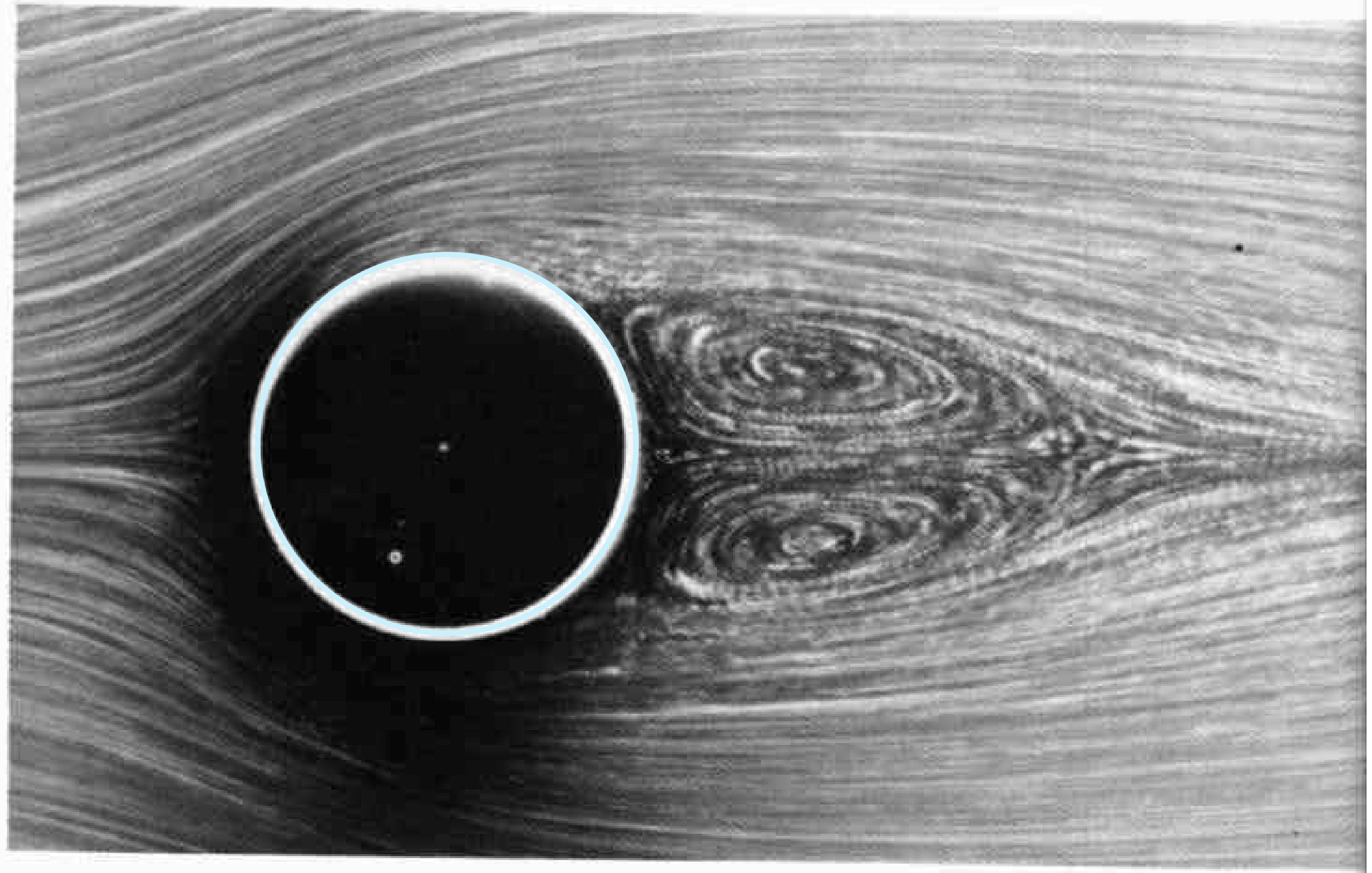
http://www.centennialofflight.net/essay/Theories_of_Flight/Prandtl/TH10.htm

http://www.centennialofflight.net/essay/Theories_of_Flight/Skin_Friction/TH11.htm

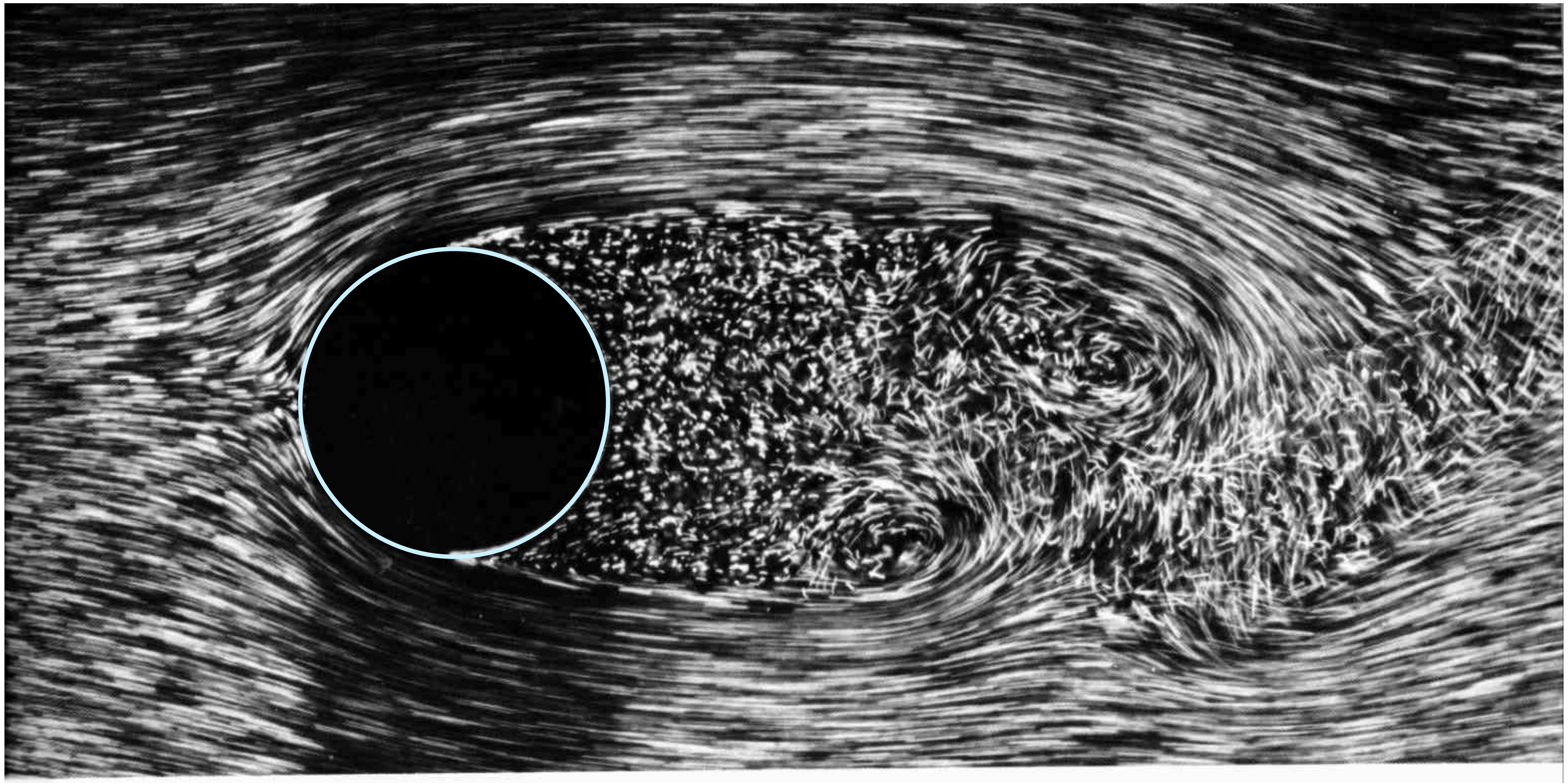
Melbourne School of Engineering
MCEN90018 Advanced Fluid Dynamics

Lecture BL10: Falker–Skan boundary layers

14 April 2016

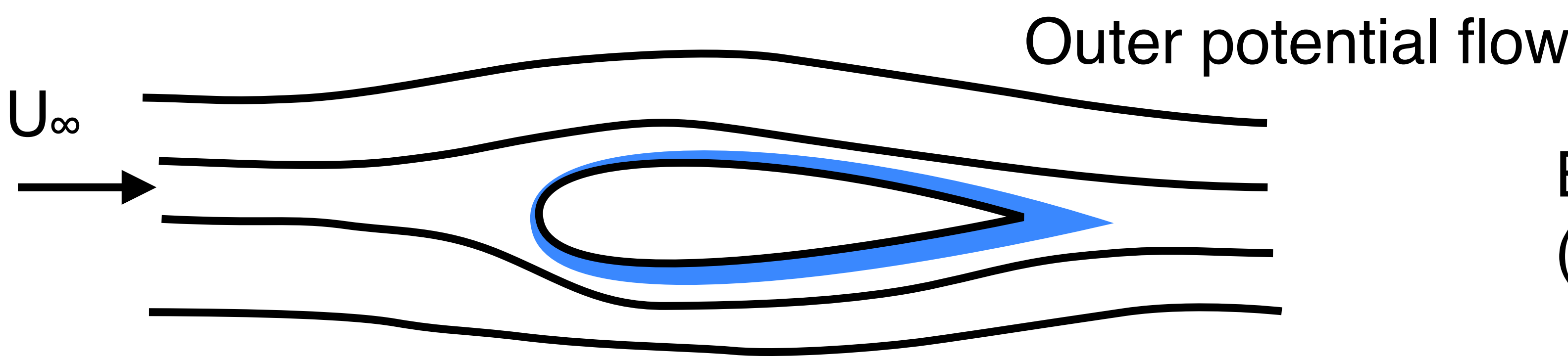


42. Circular cylinder at $R=26$. The downstream distance to the cores of the eddies also increases linearly with Reynolds number. However, the lateral distance between the cores appears to grow more nearly as the square root. *Photograph by Sadatoshi Taneda*



47. Circular cylinder at R**=2000.** At this Reynolds number one may properly speak of a boundary layer. It is laminar over the front, separates, and breaks up into a turbulent wake. The separation points, moving forward as the Reynolds number is increased, have now attained their upstream limit, ahead of maximum thickness. Visualization is by air bubbles in water. *ONERA photograph, Werlé & Gallon 1972*

Pressure-gradient effects on the boundary layer

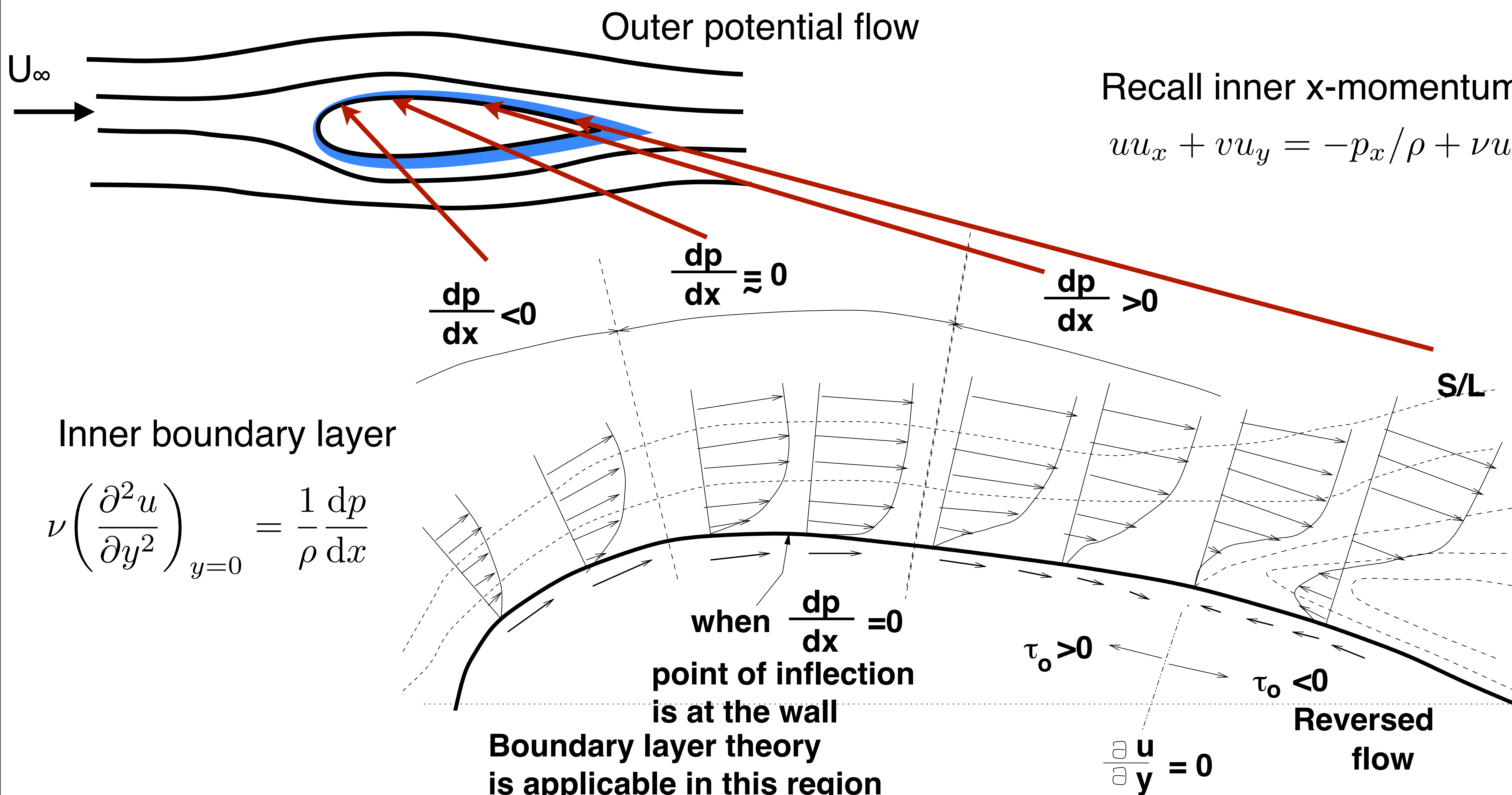


Evaluate Euler's equation on the airfoil
(x: along wall, y: normal to wall)

$$U_1 \frac{\mathrm{d}U_1}{\mathrm{d}x} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} \qquad \text{(cf. Bernoulli)}$$

U_1 is outer velocity evaluated at the 'wall', the velocity along the dividing streamline.

Pressure-gradient effects on the boundary layer



Recall inner x-momentum:

$$uu_x + vu_y = -p_x/\rho + \nu u_{yy}$$

Inner boundary layer

$$\nu\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = \frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x}$$

Boundary layer approximation is no longer applicable and we have to solve the full N/S eqns.

(Chong, p. 28)

Falkner–Skan

Recall Blasius:

$$u(x,y) = U_1 F(\eta(x,y))$$

$$\eta(x,y) = \frac{y}{\delta(x)}$$

Now Falkner–Skan:

$$u(x,y) = U_1(x) F(\eta(x,y))$$

$$\eta(x,y) = \frac{y}{\delta(x)}$$

Falkner–Skan

Recall Blasius:

$$u(x,y) = U_1 F(\eta(x,y))$$
$$\eta(x,y) = \frac{y}{\delta(x)}$$

$$\frac{\partial u}{\partial x} = -\frac{U_1\delta'}{\delta}\eta F'$$
$$\frac{\partial u}{\partial y} = \frac{U_1}{\delta}F'$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{U_1}{\delta^2}F''$$
$$v(x,y) = \int_0^y -\frac{\partial u}{\partial x}(x,y_1)\,\mathrm{d}y_1$$
$$= U_1\delta'\big(\eta F - \int_0^\eta F(\eta_1)\,\mathrm{d}\eta_1\big)$$

Now Falkner–Skan:

$$u(x,y) = U_1(x) F(\eta(x,y))$$
$$\eta(x,y) = \frac{y}{\delta(x)}$$

$$\frac{\partial u}{\partial x} = U_1' F - \frac{U_1\delta'}{\delta}\eta F'$$
$$\frac{\partial u}{\partial y} = \frac{U_1}{\delta}F'$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{U_1}{\delta^2}F''$$
$$v(x,y) = \int_0^y -\frac{\partial u}{\partial x}(x,y_1)\,\mathrm{d}y_1$$
$$= -U_1'\delta\int_0^\eta F(\eta_1)\,\mathrm{d}\eta_1$$
$$+ U_1\delta'\big(\eta F - \int_0^\eta F(\eta_1)\,\mathrm{d}\eta_1\big)$$

Falkner–Skan

Define:

$$f' \equiv F \quad \Leftrightarrow \quad f = \int_0^{\eta} F(\eta_1)\,\mathrm{d}\eta_1$$

$$f(0) = 0$$

Then:

$$u(x,y) = U_1(x) f'(\eta(x,y))$$

$$\eta(x,y) = \frac{y}{\delta(x)}$$

$$\frac{\partial u}{\partial x} = U_1' f' - \frac{U_1 \delta'}{\delta} \eta f''$$

$$\frac{\partial u}{\partial y} = \frac{U_1}{\delta} f''$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_1}{\delta^2} f'''$$

$$v = -U_1' \delta f + U_1 \delta' \big(\eta f' - f\big)$$

Falkner–Skan

Put:

$$\boxed{\begin{aligned} u &= U_1 f' \\ \frac{\partial u}{\partial x} &= U_1' f' - \frac{U_1 \delta'}{\delta}\eta f'' \\ \frac{\partial u}{\partial y} &= \frac{U_1}{\delta} f'' \\ \frac{\partial^2 u}{\partial y^2} &= \frac{U_1}{\delta^2} f''' \\ v &= -U_1'\delta f + U_1\delta'\big(\eta f' - f\big) \end{aligned}}$$

in x-momentum:

$$uu_x + vu_y = U_1 U_1' + \nu u_{yy}$$

and multiply by

$$\frac{\delta^2}{U_1 \nu}$$

Check: get Blasius if $U_1' = 0$

Get: $\frac{U_1'\delta^2}{\nu}(f')^2 - \big(\frac{U_1'\delta^2}{\nu} + \frac{U_1\delta'\delta}{\nu}\big) f f'' = \frac{U_1'\delta^2}{\nu} + f'''$ and set $\frac{U_1\delta'\delta}{\nu} = \frac{1}{2}$

For a similarity solution to exist, we need this equation to depend only on the similarity variable. This means all the x-dependent coefficients must be constants.

Falkner–Skan

$$\boxed{\frac{U_1'\delta^2}{\nu}}(f')^2 - \left(\boxed{\frac{U_1'\delta^2}{\nu}} + \boxed{\frac{U_1\delta'\delta}{\nu}}\right)ff'' = \boxed{\frac{U_1'\delta^2}{\nu}} + f'''$$

As with Blasius, it doesn't matter what the specific choices for the constants are, as long as they are consistent. But here we follow the convention of Batchelor (1967, "An Introduction to Fluid Dynamics", p. 316).

$$\frac{U_1'\delta^2}{\nu} = m \qquad\qquad \frac{U_1\delta'\delta}{\nu} = \frac{1}{2}(1-m)$$

This gives:

$$m(f')^2 - \frac{1}{2}(m+1)ff'' = m + f''' \quad \text{and} \quad \frac{U_1'\delta^2}{\nu} = m \quad \text{and} \quad \frac{U_1(\delta^2)'}{\nu} = 1 - m$$

Boundary conditions:

$$U_1(x_0) = U_0$$

$$u(0) = 0, \quad u(\infty) = U_1$$

$$\delta(x_0) = \delta_0 \equiv (\nu x_0/U_0)^{1/2}$$

i.e. $f'(0) = 0, \quad f'(\infty) = 1$ Also: $f(0) = 0$

Falkner–Skan

$$m(f')^2 - \frac{1}{2}(m+1)ff'' = m + f'''$$

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1$$

$$\frac{U_1'\delta^2}{\nu} = m \qquad \frac{U_1(\delta^2)'}{\nu} = 1 - m$$

$$U_1(x_0) = U_0 \qquad \delta(x_0) = (\nu x_0/U_0)^{1/2}$$

Solution:

$$\begin{aligned} U_1(x) &= U_0(x/x_0)^m \\ \delta(x) &= (\nu x/U_1)^{1/2} \\ &= (\nu x_0/U_0)^{1/2}(x/x_0)^{(1-m)/2} \end{aligned}$$

Falkner–Skan

$$U_1(x) = U_0(x/x_0)^m \propto x^m$$

$$\delta(x) = (\nu x_0/U_0)^{1/2}(x/x_0)^{(1-m)/2} \propto x^{(1-m)/2}$$

Note: as U_1 increases, δ grows slower (thinner) than the $x^{1/2}$ of Blasius and vice versa.

	m	$\delta(x)$	$U_1(x)$
Blasius	0	$\propto x^{1/2}$	$\propto 1$

Falkner–Skan

$$U_1(x) = U_0(x/x_0)^m \propto x^m$$

$$\delta(x) = (\nu x_0/U_0)^{1/2}(x/x_0)^{(1-m)/2} \propto x^{(1-m)/2}$$

	m	$\delta(x)$	$U_1(x)$
Blasius	0	$\propto x^{1/2}$	$\propto 1$
Stagnation	1	$\propto 1$	$\propto x$

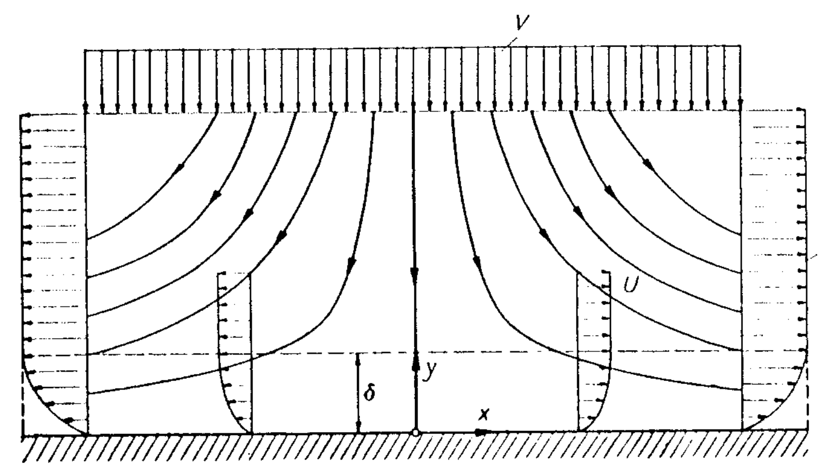


Fig. 5.10. Stagnation in plane flow

Falkner–Skan

$$U_1(x) = U_0(x/x_0)^m \propto x^m$$

$$\delta(x) = (\nu x_0/U_0)^{1/2}(x/x_0)^{(1-m)/2} \propto x^{(1-m)/2}$$

	m	$\delta(x)$	$U_1(x)$
Blasius	0	$\propto x^{1/2}$	$\propto 1$
Stagnation	1	$\propto 1$	$\propto x$
Sink/source (Use $U_0 < 0$ for sink)	–1	$\propto x$	$\propto x^{-1}$

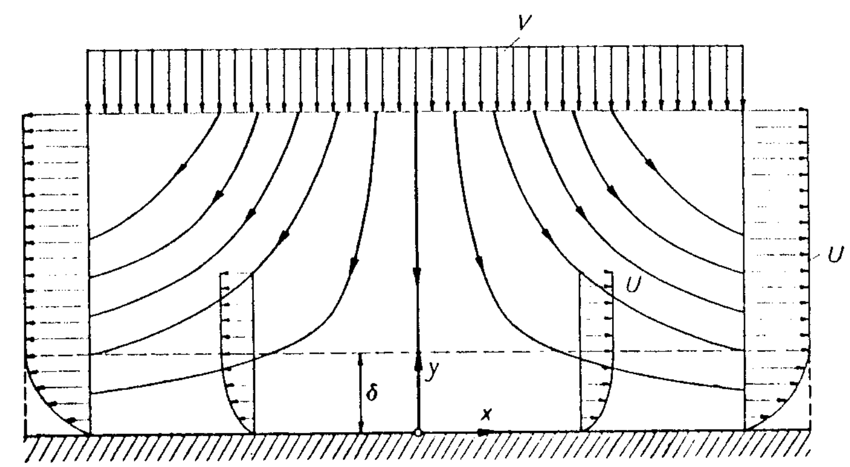
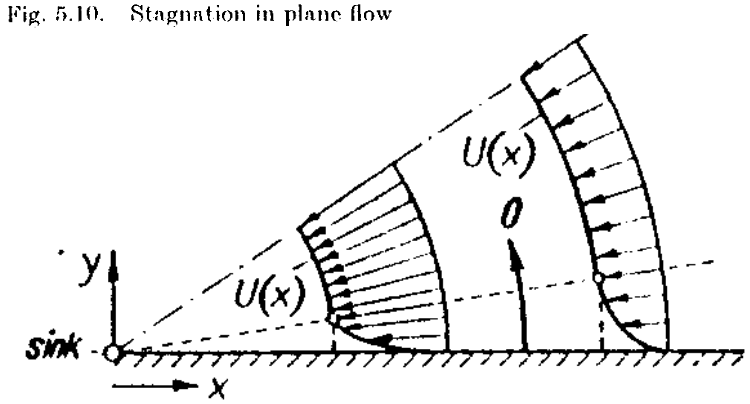


Fig. 5.10. Stagnation in plane flow



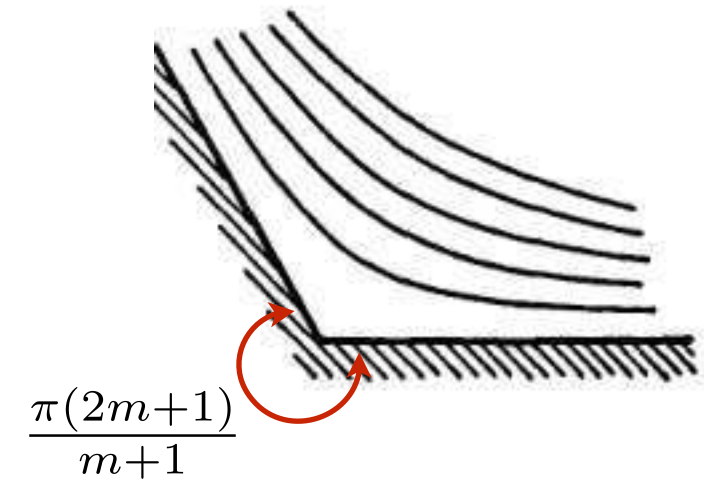
Falkner–Skan

cf. Batchelor, pp. 409–410

$$U_1(x) = U_0(x/x_0)^m \propto x^m$$

$$\delta(x) = (\nu x_0/U_0)^{1/2}(x/x_0)^{(1-m)/2} \propto x^{(1-m)/2}$$

	m	$\delta(x)$	$U_1(x)$
Blasius	0	$\propto x^{1/2}$	$\propto 1$
Wedge	–1/2 < m < ∞	$\propto x^{(1-m)/2}$	$\propto x^m$
	–1/2 < m < 0		Decreasing
	0 < m < ∞		Increasing
Stagnation	1	$\propto 1$	$\propto x$
Sink/source (Use $U_0 < 0$ for sink)	–1	$\propto x$	$\propto x^{-1}$



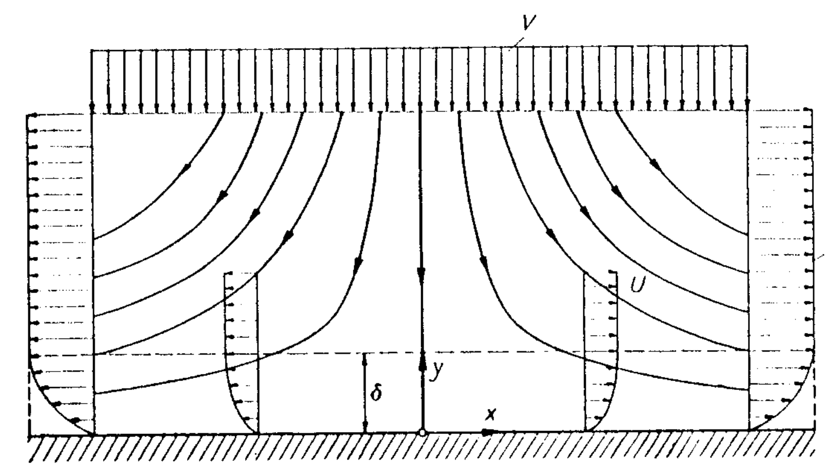
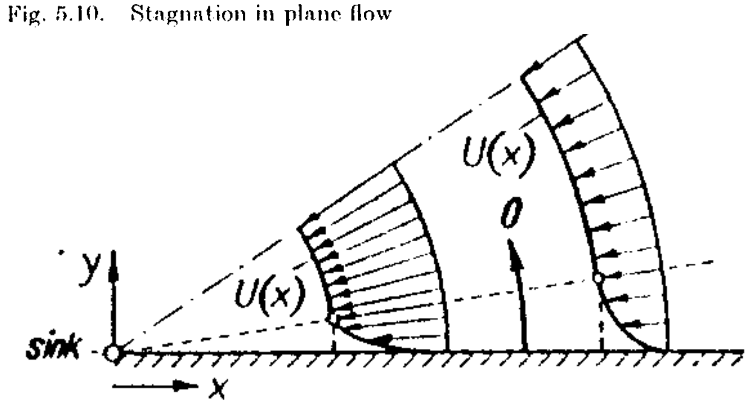


Fig. 5.10. Stagnation in plane flow



Falkner–Skan

$$U_1(x) = U_0(x/x_0)^m \propto x^m$$

$$\delta(x) = (\nu x_0/U_0)^{1/2}(x/x_0)^{(1-m)/2} \propto x^{(1-m)/2}$$

Pressure gradient:

$$\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} = -U_1\frac{\mathrm{d}U_1}{\mathrm{d}x} = -m\frac{U_0^2}{x_0}\left(\frac{x}{x_0}\right)^{2m-1}$$

	m	$\delta(x)$	$U_1(x)$		
Blasius	0	$\propto x^{1/2}$	$\propto 1$	0	
Wedge	–1/2 < m < ∞	$\propto x^{(1-m)/2}$	$\propto x^m$		
	–1/2 < m < 0		Decreasing	< 0	Adverse
	0 < m < ∞		Increasing	> 0	Favourable
Stagnation	1	$\propto 1$	$\propto x$	> 0	Favourable
Sink/source (Use $U_0 < 0$ for sink)	–1	$\propto x$	$\propto x^{-1}$	< 0	Adverse (Favourable for sink)

Falkner–Skan

$$U_1(x) = U_0(x/x_0)^m \propto x^m$$

$$\delta(x) = (\nu x_0/U_0)^{1/2}(x/x_0)^{(1-m)/2} \propto x^{(1-m)/2}$$

Just a number

Displacement thickness:

$$\delta^* = \int_0^\infty (1 - \frac{u}{U_1})\,\mathrm{d}y = \delta(x) \int_0^\infty (1 - f')\,\mathrm{d}\eta$$

$$\propto x^{(1-m)/2}$$

Just a number (but may be zero)

Wall shear stress:

$$\tau_0 = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu \frac{U_1(x)}{\delta(x)} f''(0)$$

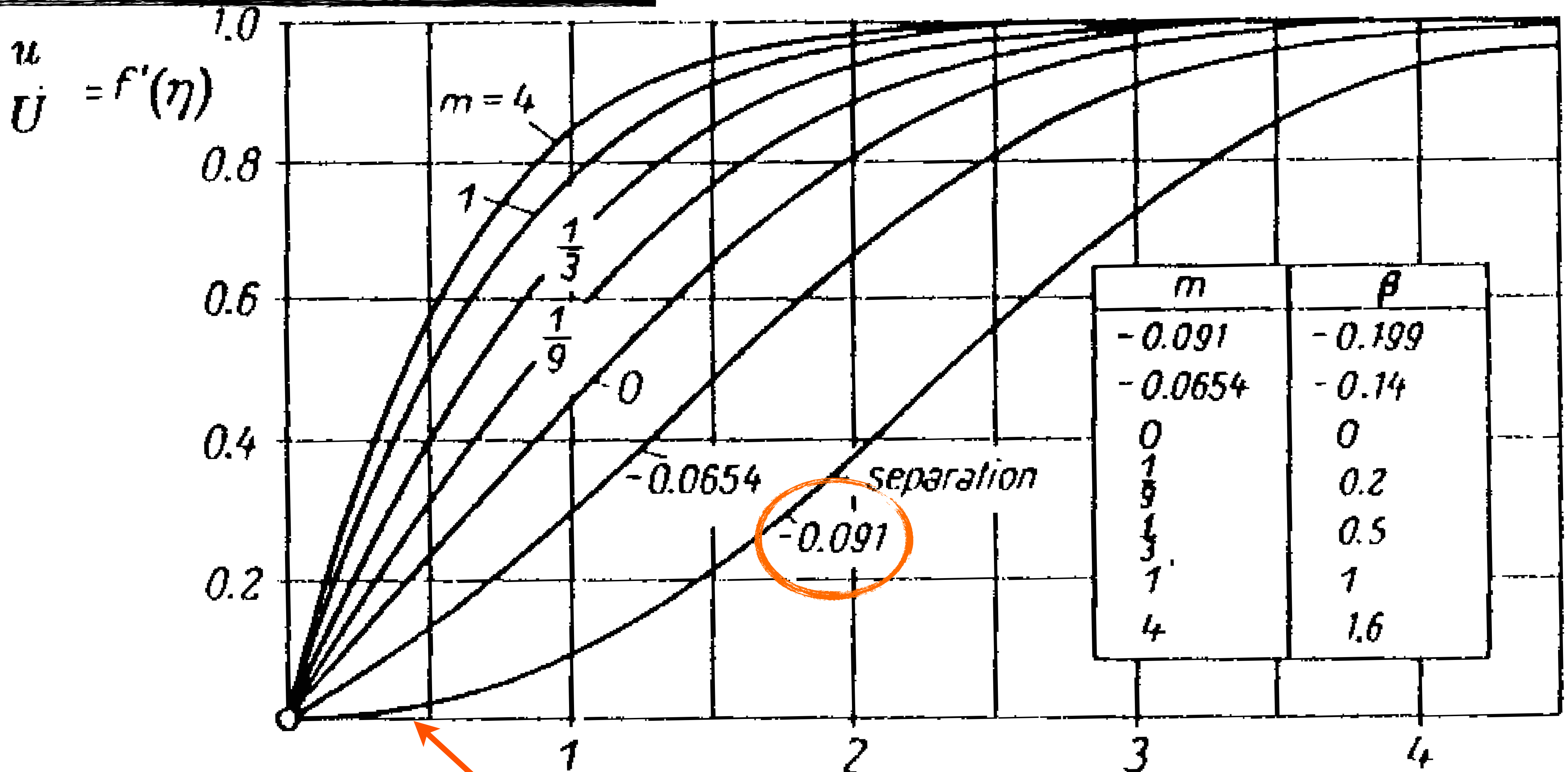
$$\propto \frac{x^m}{x^{(1-m)/2}} = x^{(3m-1)/2}$$

Falkner–Skan

Solve this numerically:

$$m(f')^2 - \frac{1}{2}(m+1)ff'' = m + f'''$$

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1$$



$$\beta = \frac{2m}{m+1}$$

Zero slope
(shear stress)

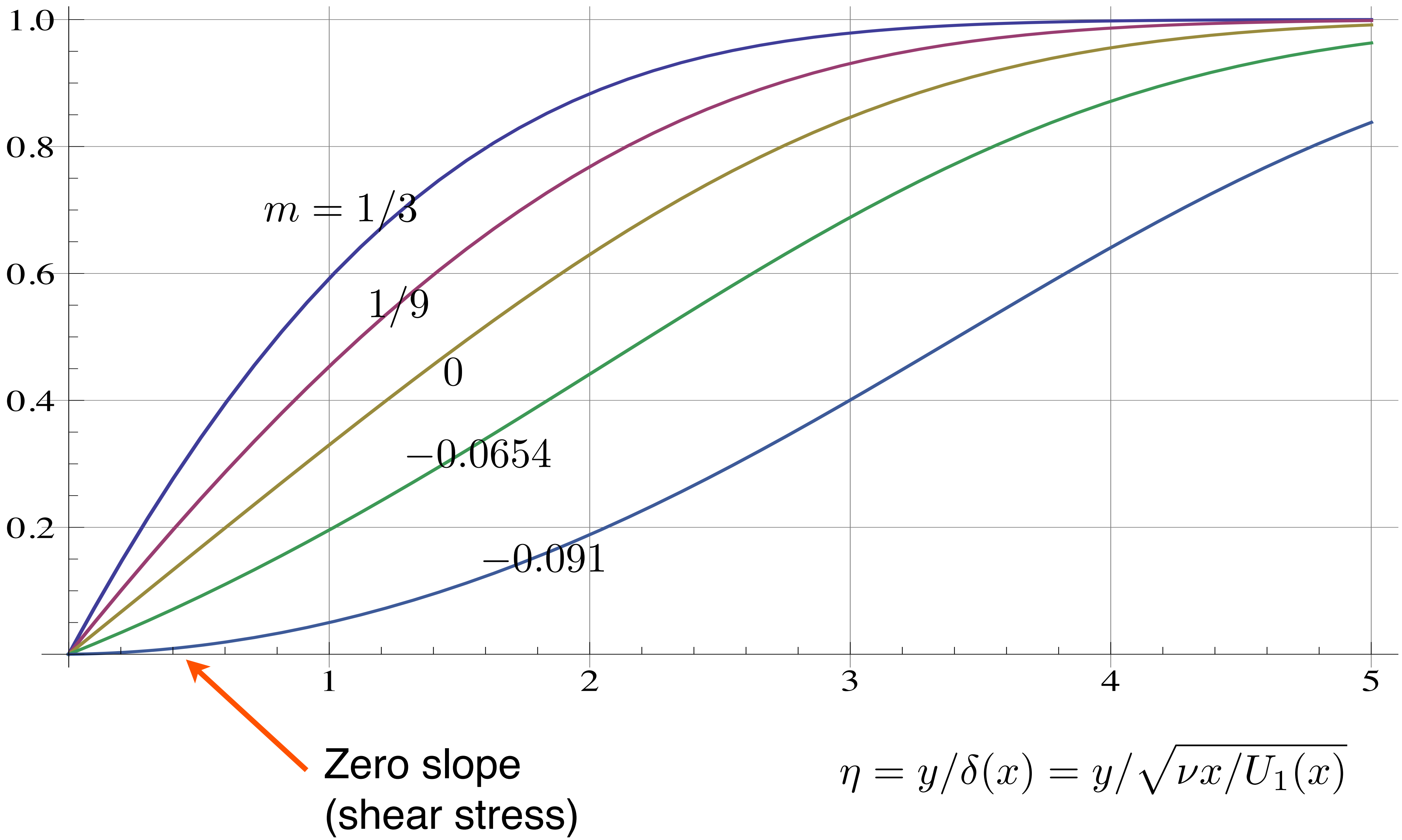
$$\sqrt{\frac{m+1}{2}}\sqrt{\frac{U}{\nu x}}\,y = \sqrt{\frac{m+1}{2}}\,\eta$$

Falkner–Skan

$$m(f')^2 - \frac{1}{2}(m+1)ff'' = m + f'''$$

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1$$

$$\frac{u}{U_1(x)} = f'(\eta)$$



$$\eta = y/\delta(x) = y/\sqrt{\nu x/U_1(x)}$$

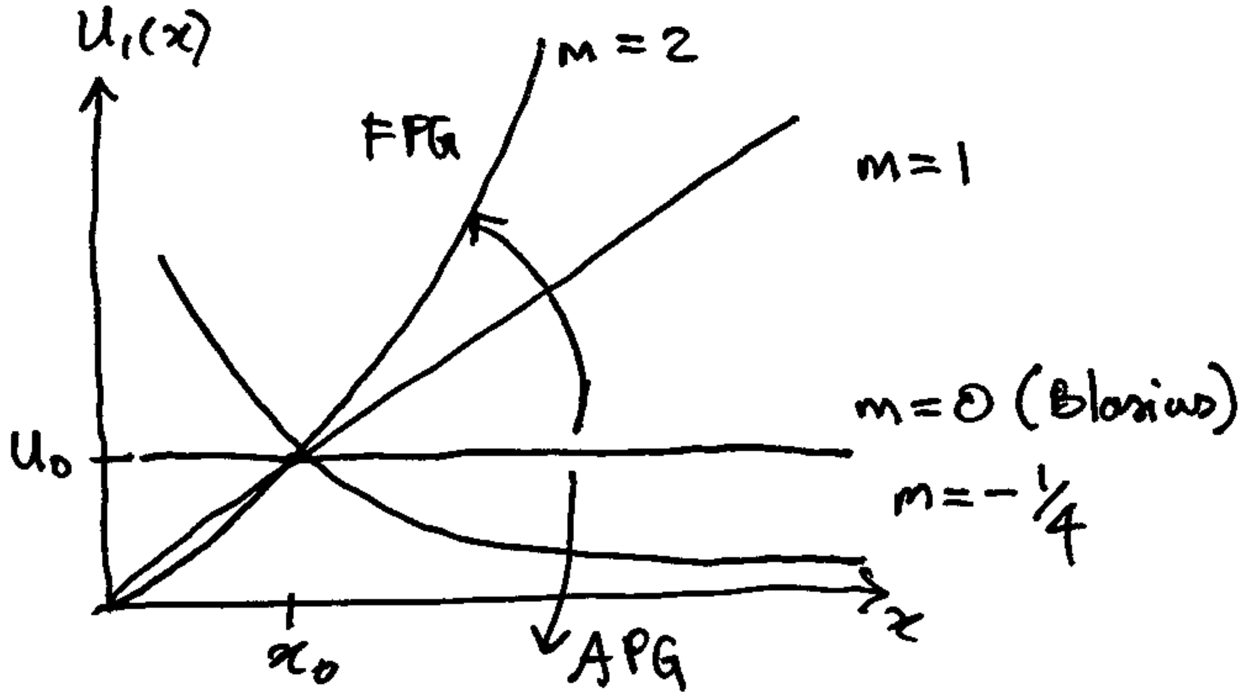
Melbourne School of Engineering
MCEN90018 Advanced Fluid Dynamics

Lecture BL11: von Karman momentum integral equation

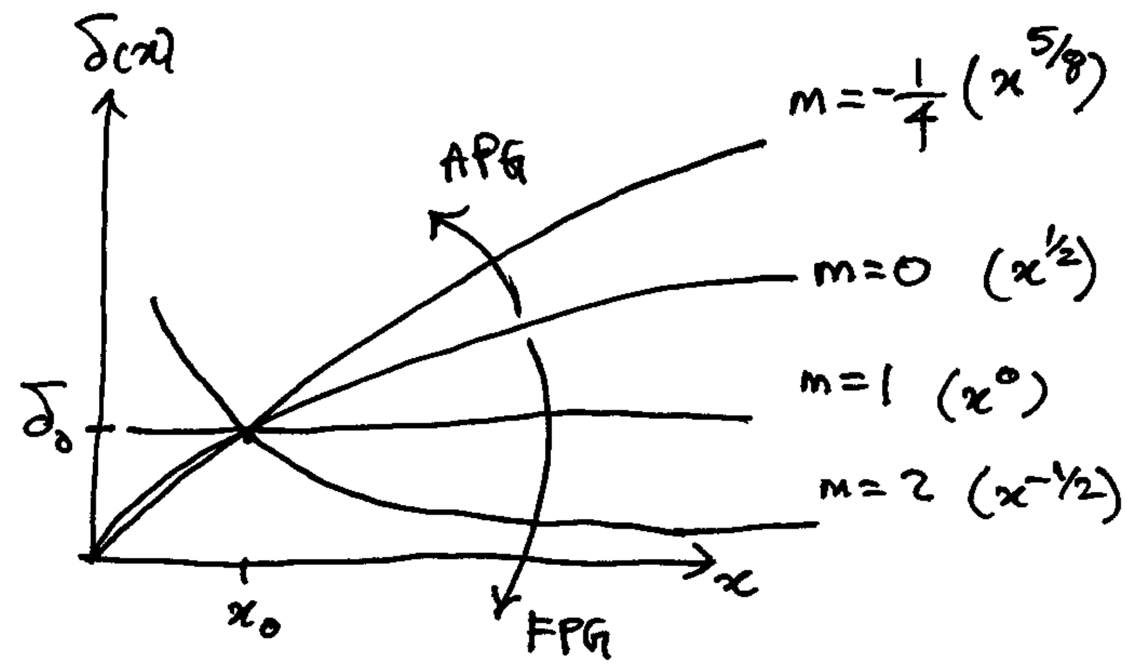
15 April 2016

Falkner–Skan

$U_1(x) \propto x^m$



$\delta(x) \propto x^{(1-m)/2}$

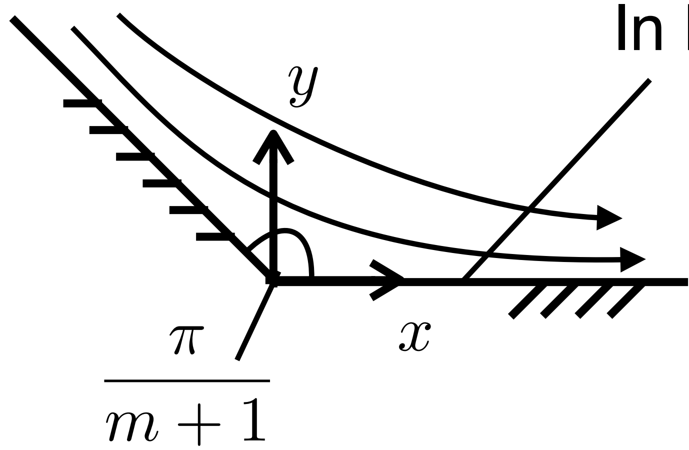


Falkner–Skan

What kind of potential flow has $U_1(x) \propto x^m$?

Try wedge flow: $w(z) = \phi + \mathrm{i}\,\psi = A(x + \mathrm{i}\,y)^{m+1}$

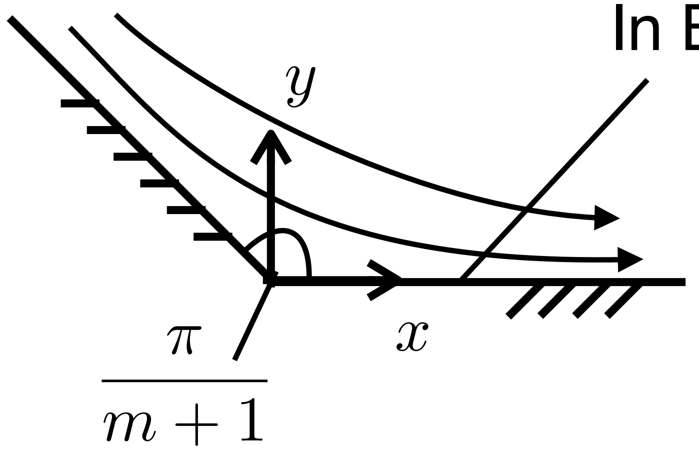
In BL: $u = \dfrac{\partial \psi}{\partial y} \propto x^m$



Falkner–Skan

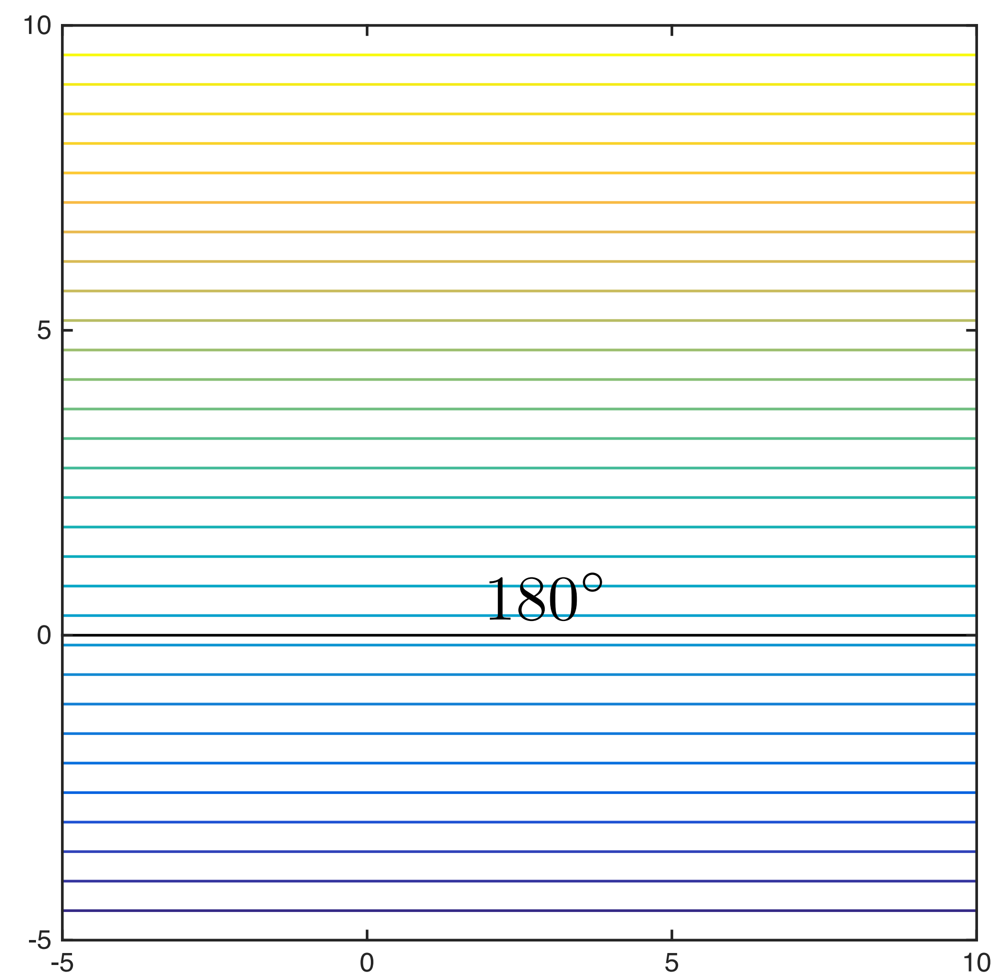
What kind of potential flow has $U_1(x) \propto x^m$?

Try wedge flow: $w(z) = \phi + \mathrm{i}\,\psi = A(x + \mathrm{i}\,y)^{m+1}$



In BL: $u = \dfrac{\partial \psi}{\partial y} \propto x^m$

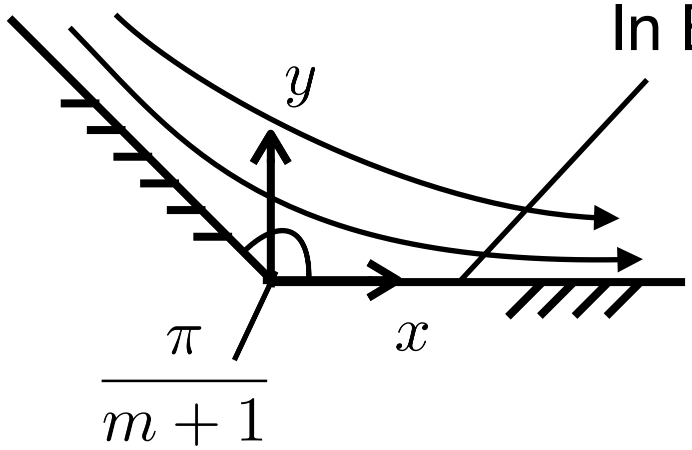
$m = 0$



Falkner–Skan

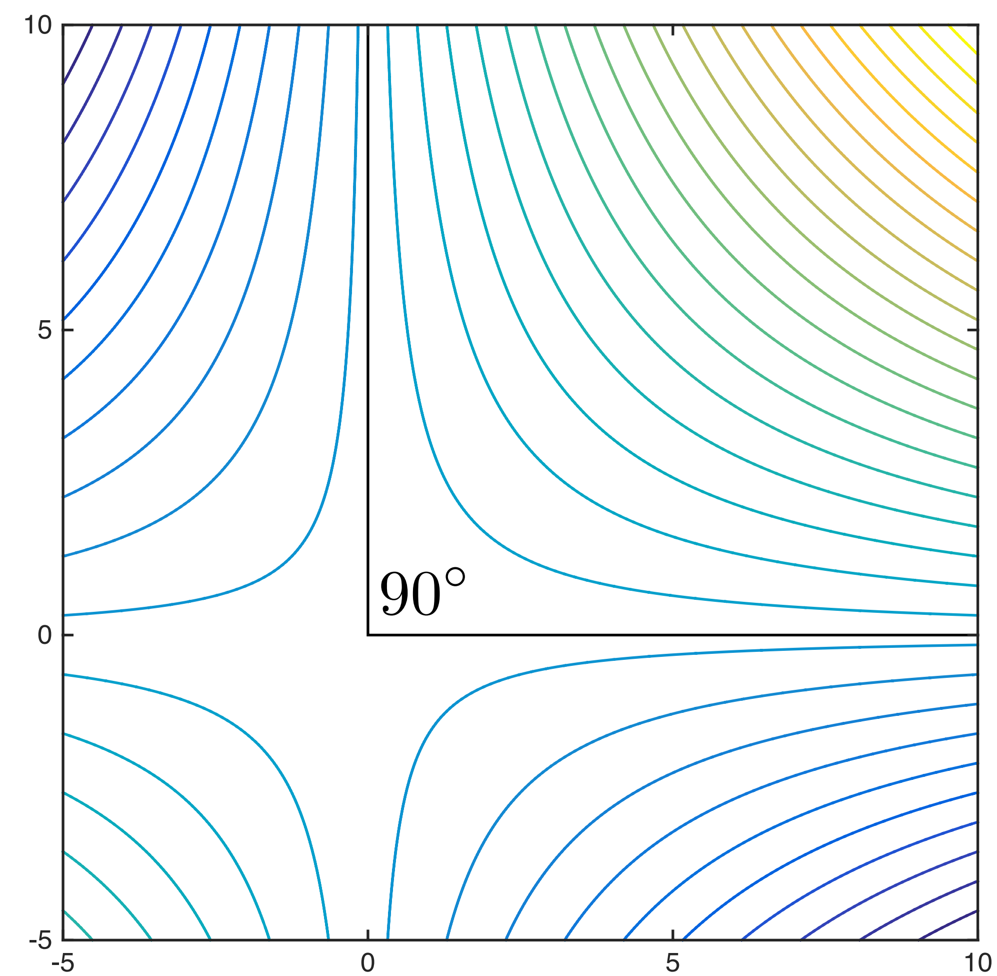
What kind of potential flow has $U_1(x) \propto x^m$?

Try wedge flow: $w(z) = \phi + \mathrm{i}\,\psi = A(x + \mathrm{i}\,y)^{m+1}$



In BL: $u = \dfrac{\partial \psi}{\partial y} \propto x^m$

$m = 1$

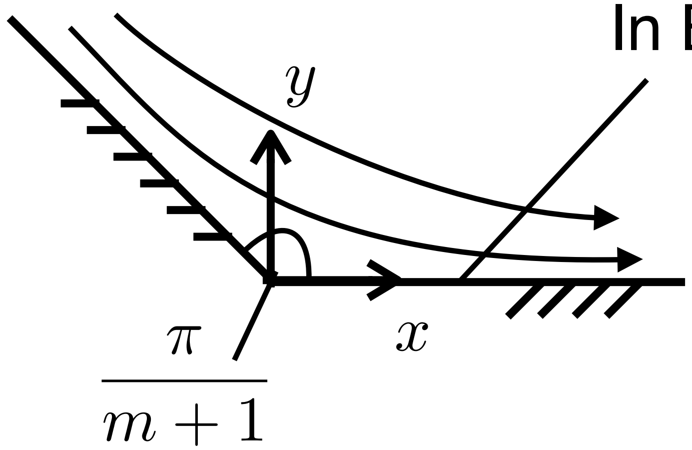


Falkner–Skan

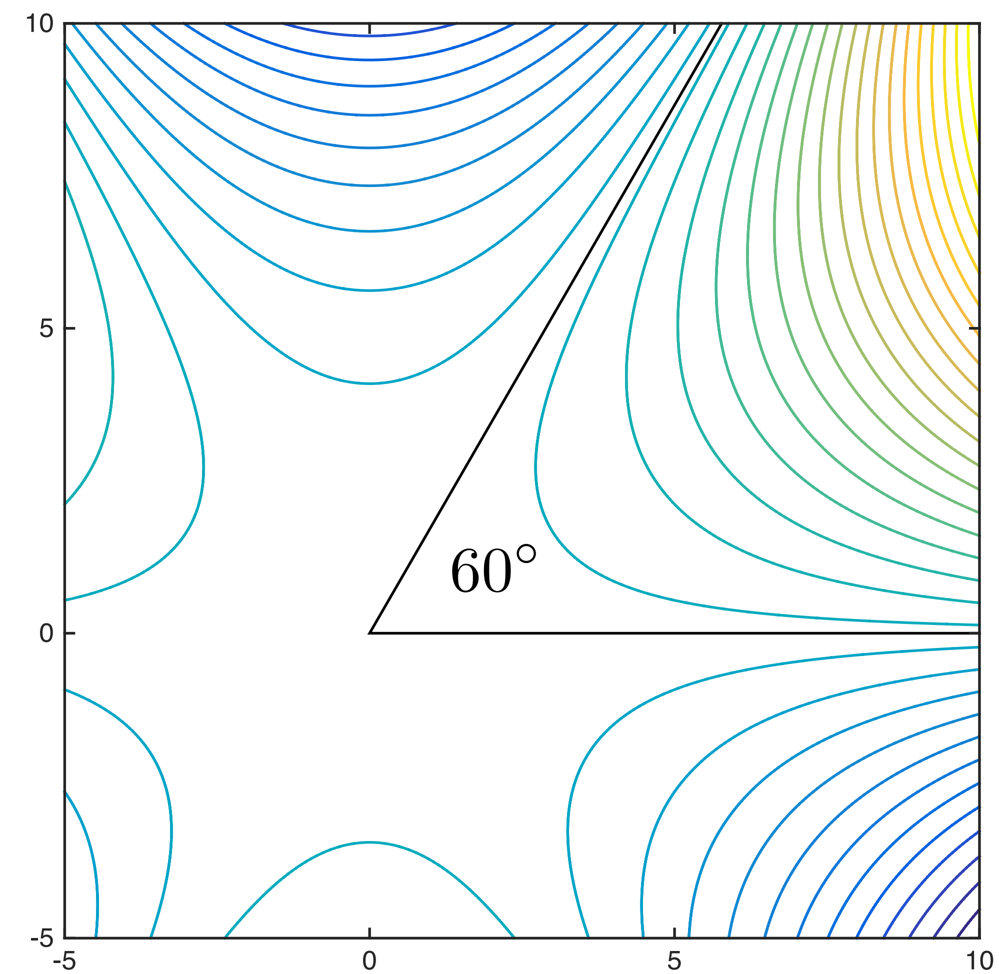
What kind of potential flow has $U_1(x) \propto x^m$?

Try wedge flow: $w(z) = \phi + \mathrm{i}\,\psi = A(x + \mathrm{i}\,y)^{m+1}$

In BL: $u = \dfrac{\partial \psi}{\partial y} \propto x^m$



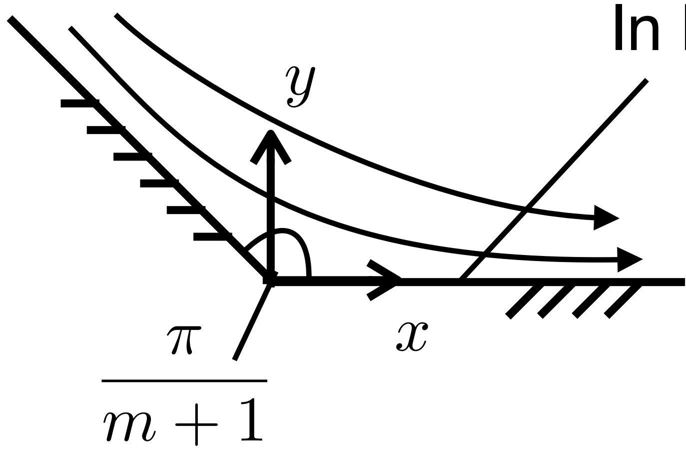
$m = 2$



Falkner–Skan

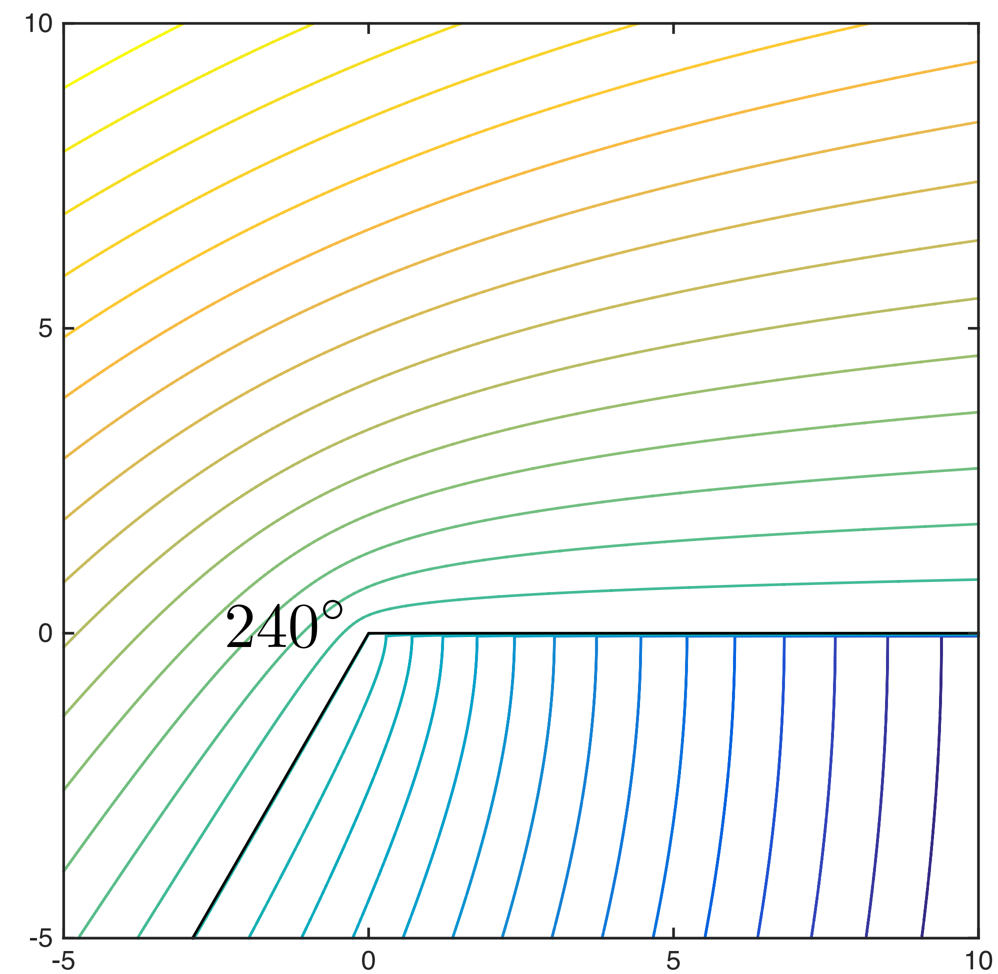
What kind of potential flow has $U_1(x) \propto x^m$?

Try wedge flow: $w(z) = \phi + \mathrm{i}\,\psi = A(x + \mathrm{i}\,y)^{m+1}$



In BL: $u = \dfrac{\partial \psi}{\partial y} \propto x^m$

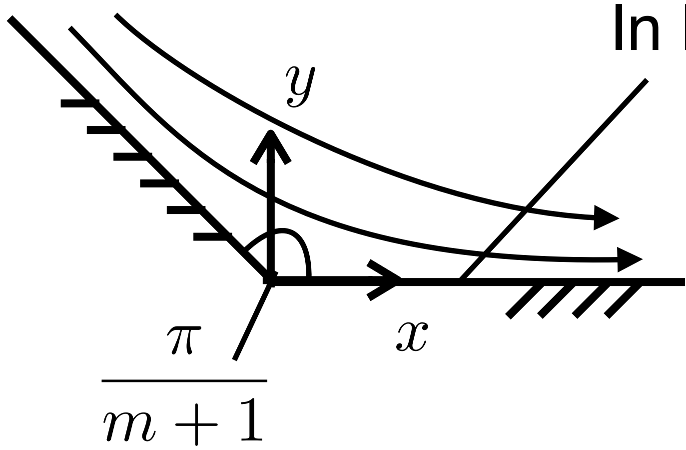
$m = -0.25$



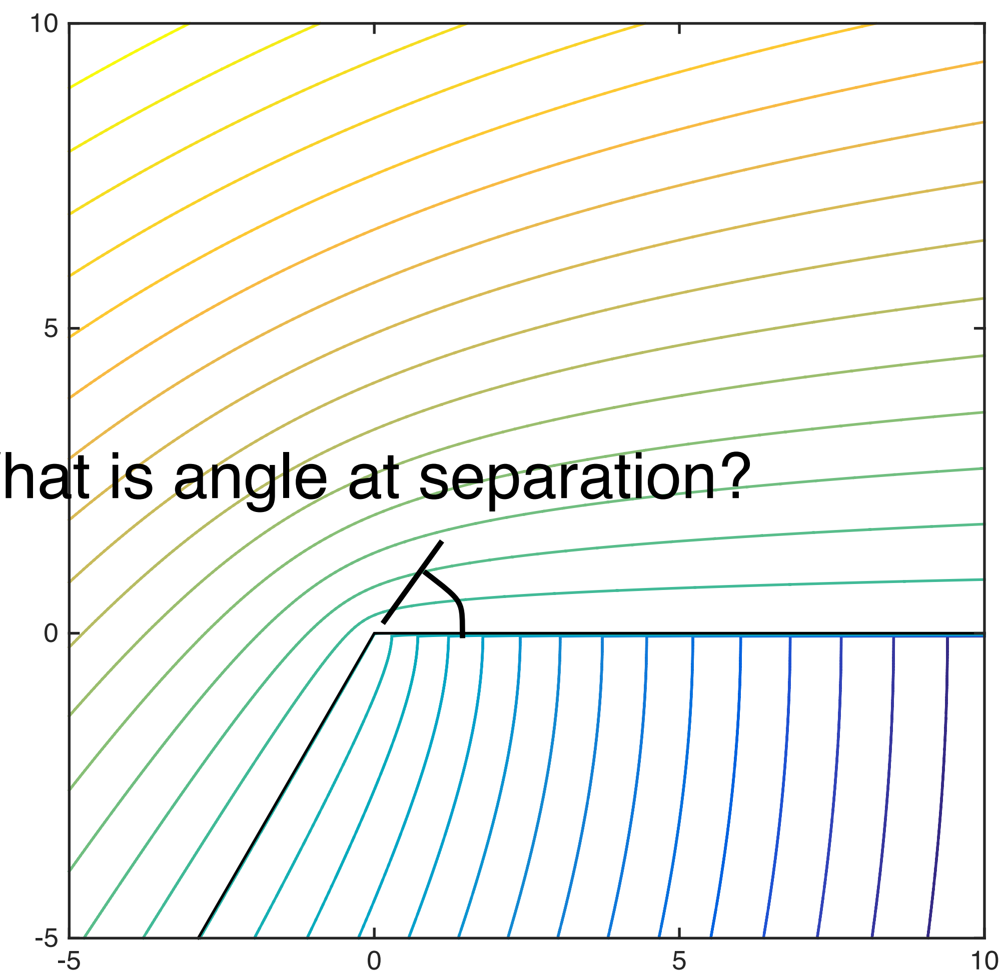
Falkner–Skan

What kind of potential flow has $U_1(x) \propto x^m$?

Try wedge flow: $w(z) = \phi + \mathrm{i}\,\psi = A(x + \mathrm{i}\,y)^{m+1}$



In BL: $u = \dfrac{\partial \psi}{\partial y} \propto x^m$



What is angle at separation?

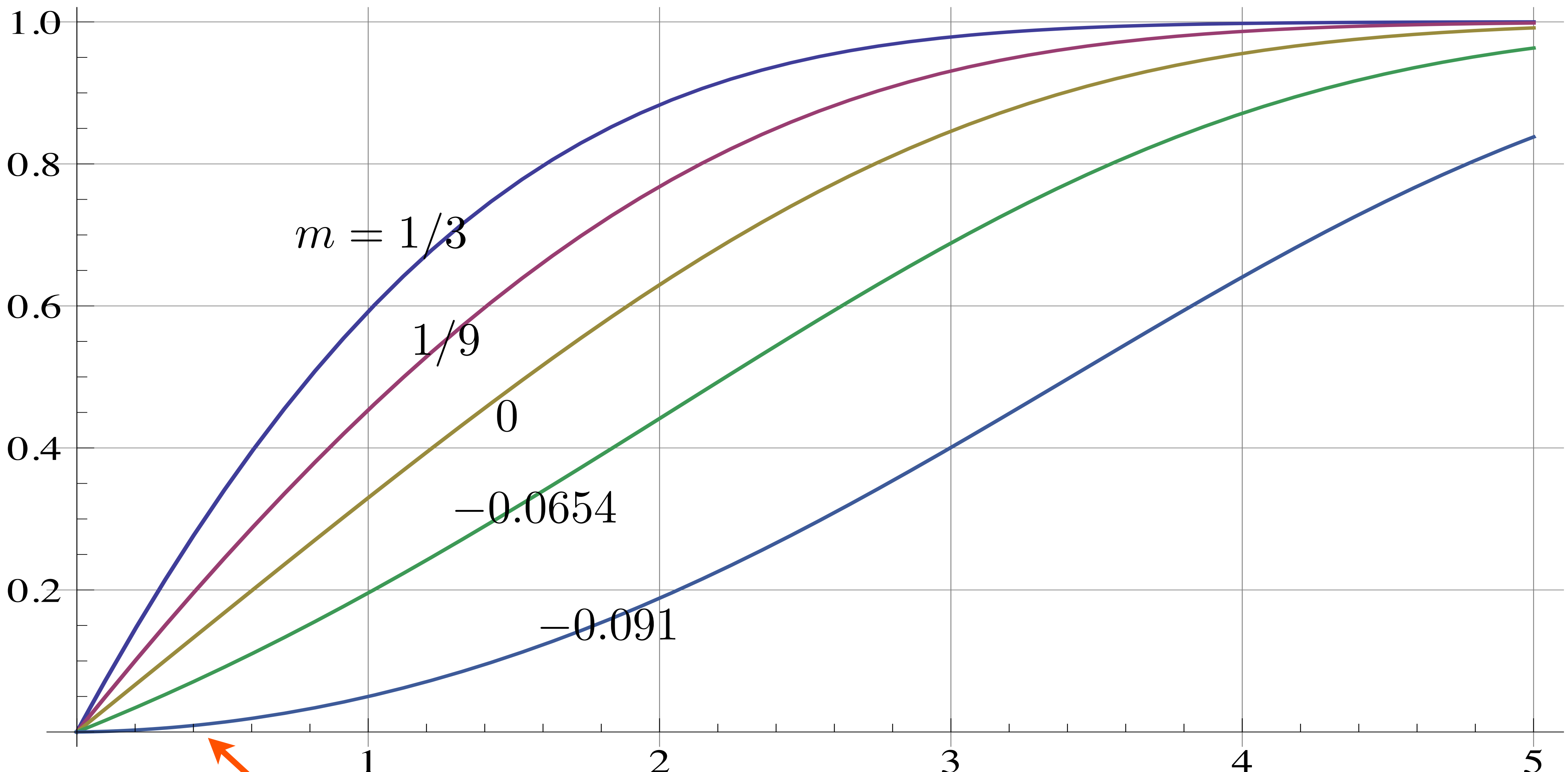
Falkner–Skan

$$m(f')^2 - \frac{1}{2}(m+1)ff'' = m + f'''$$

Solve for f and find m when f''(0) = 0.
m = -0.091 => angle is 18 degrees.

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1$$

$$\frac{u}{U_1(x)} = f'(\eta)$$



Zero slope
(shear stress)

$$\eta = y/\delta(x) = y/\sqrt{\nu x/U_1(x)}$$

Von Kármán momentum integral equation

$$\frac{c_f}{2} = \frac{\mathrm{d}\theta}{\mathrm{d}x} + \frac{\theta}{U_1}\frac{\mathrm{d}U_1}{\mathrm{d}x}\left(2 + \frac{\delta^*}{\theta}\right)$$

Read Chong pp. 40–45 and Schlichting (7th edition) pp.158–161

Von Kármán was above all an ingenious applied mathematician with the uncanny ability to find simple and direct solutions to all types of elusive problems. He reached the height of his productivity

Penner *et al.* 2009 Annu. Rev. Fluid Mech

http://usstampgallery.com/view.php?id=bf3d91a85a9fb744dee3094a7be132c76ee64047

Von Kármán momentum integral equation

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Von Kármán was above all an ingenious applied mathematician with the uncanny ability to find simple and direct solutions to all types of elusive problems. He reached the height of his productivity typically after midnight and after a leisurely banquet that generally included Hungarian Tokay wine, especially when accompanied by a young and enthusiastic researcher who often had to struggle staying awake long enough to grasp the great discovery being made without much of his help.

Penner *et al.* 2009 Annu. Rev. Fluid Mech

http://usstampgallery.com/view.php?id=bf3d91a85a9fb744dee3094a7be132c76ee64047

Melbourne School of Engineering
MCEN90018 Advanced Fluid Dynamics

Lecture BL12: Reynolds-averaged Navier−Stokes

19 April 2016

Turbulence

We've done laminar stuff, but most of
nature is turbulent.

Make sure you watch this classic video:

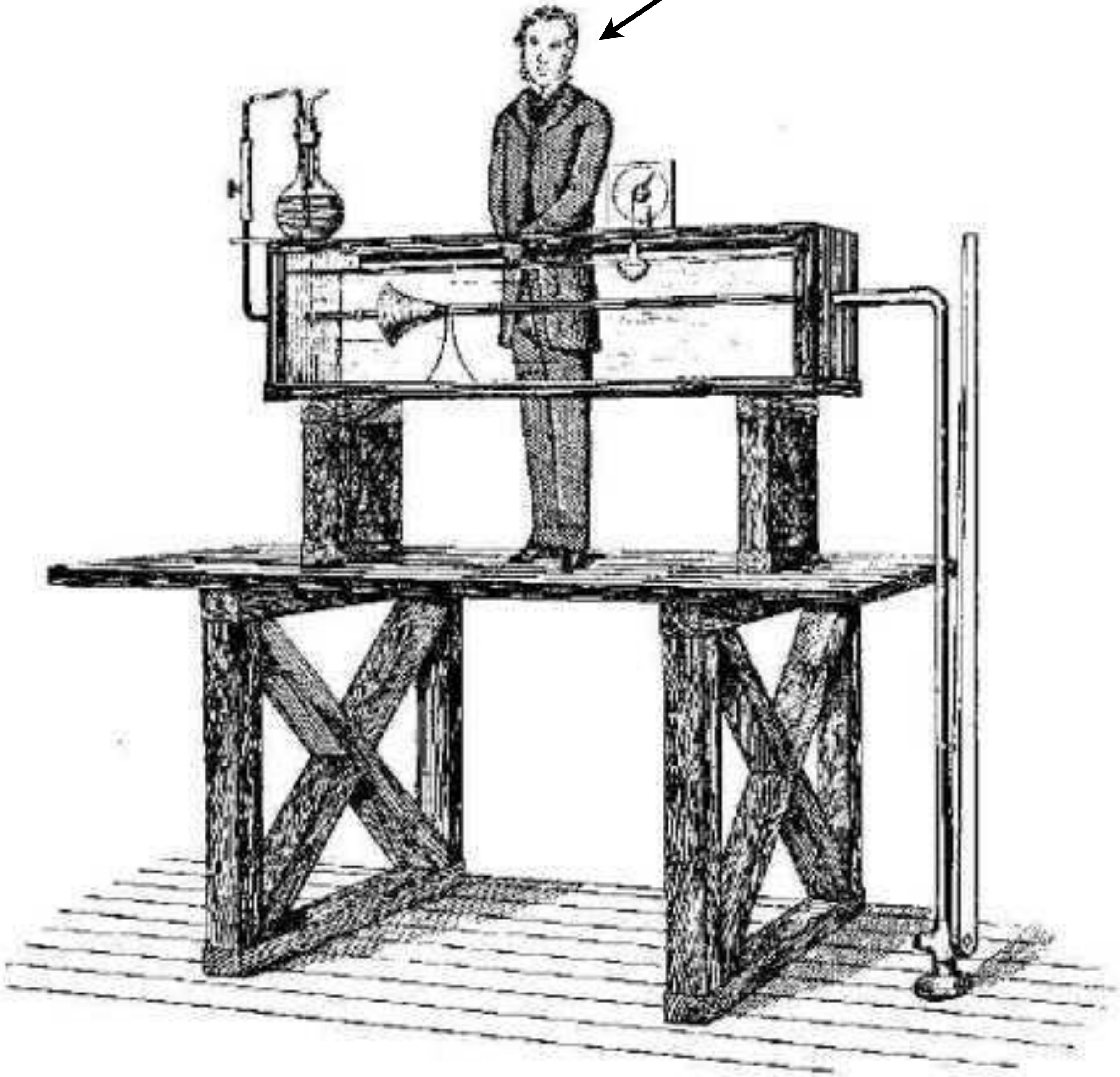
http://www.youtube.com/watch?v=1_oyqLOqwnI

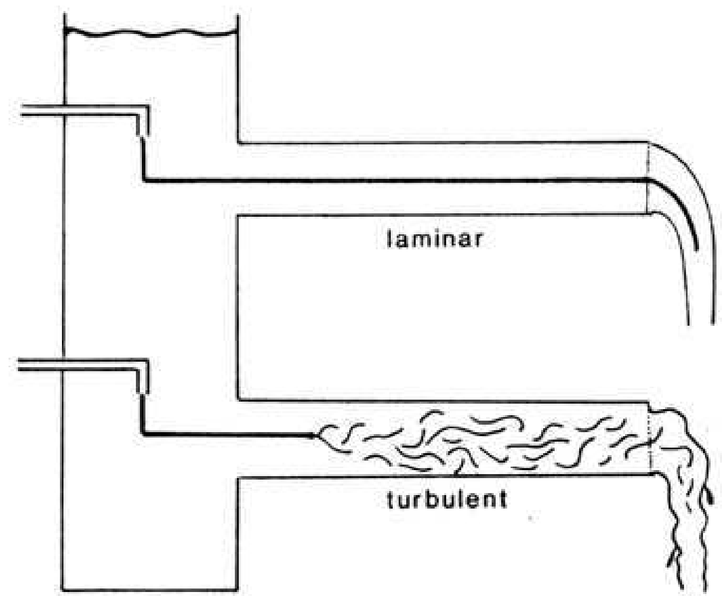
Turbulence:
1) Disorder (irreproducible in detail, but statistics are!)
2) Efficient mixing
3) Vorticity (irregularly distributed in 3D)

Really good reading:
https://www.princeton.edu/~asmits/Bicycle_web/transition.html

Transition to turbulence

Osborne Reynolds?





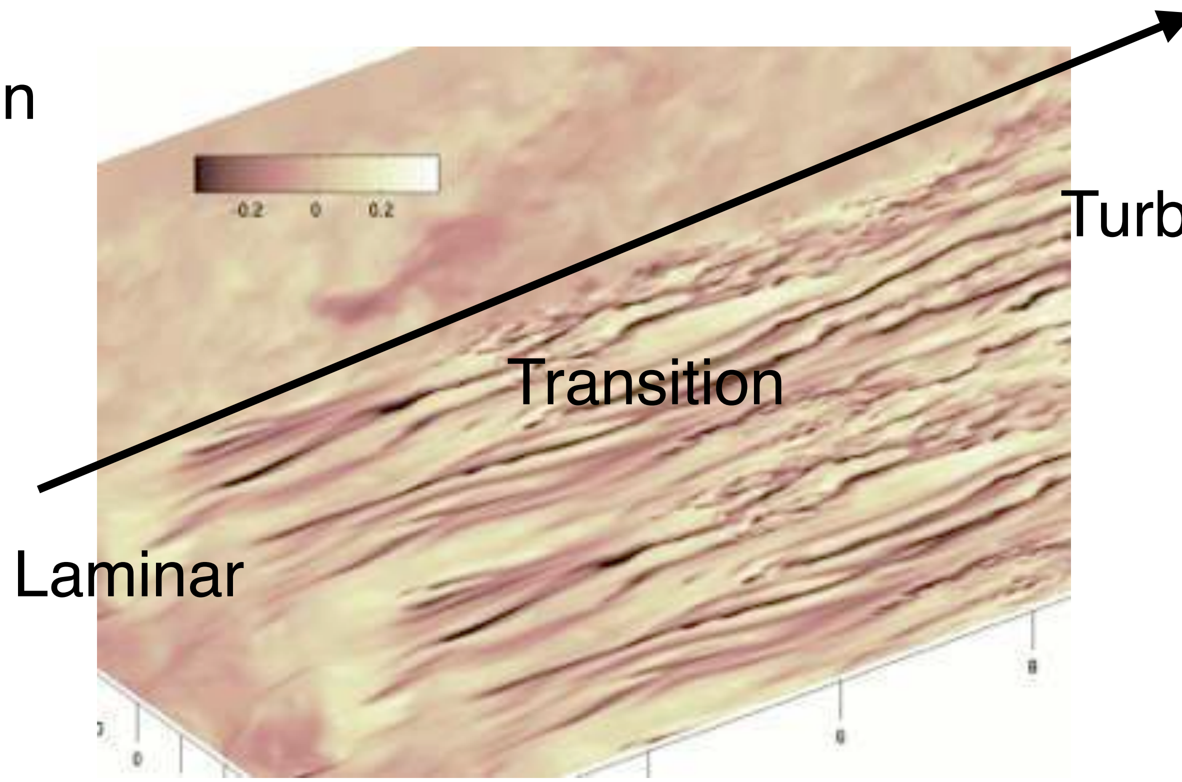
Re $\lesssim$ 500

Re $\gtrsim$ 500

Number depends on details of flow.

Transition to turbulence

Simulation

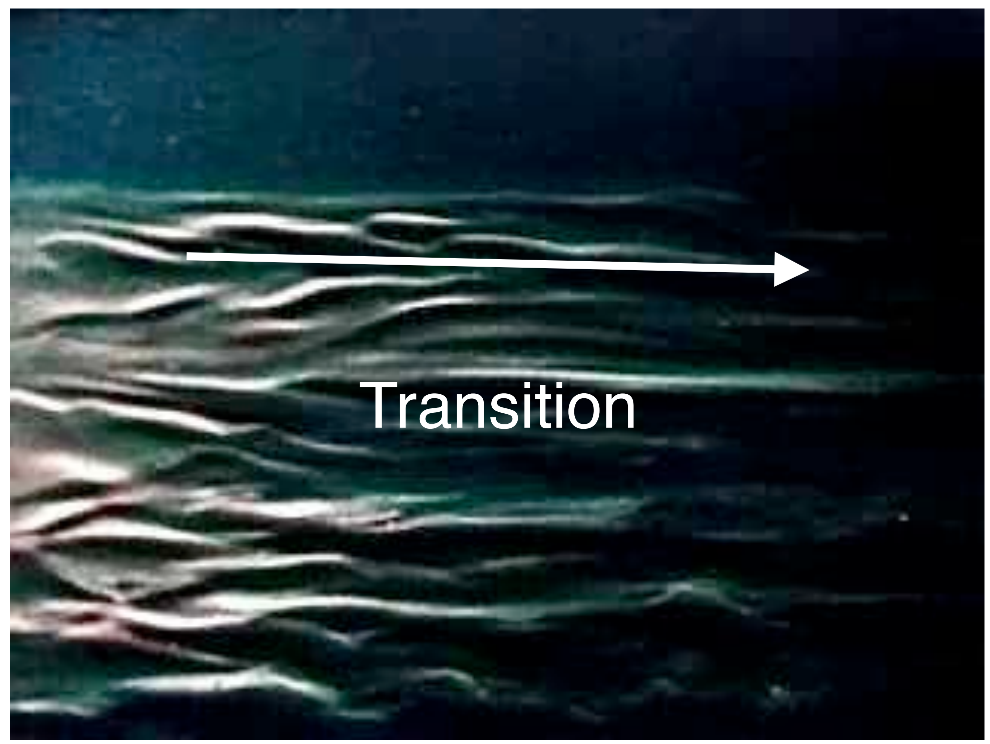


Isometric view

Transition to turbulence

Experiment

Laminar



Turbulence

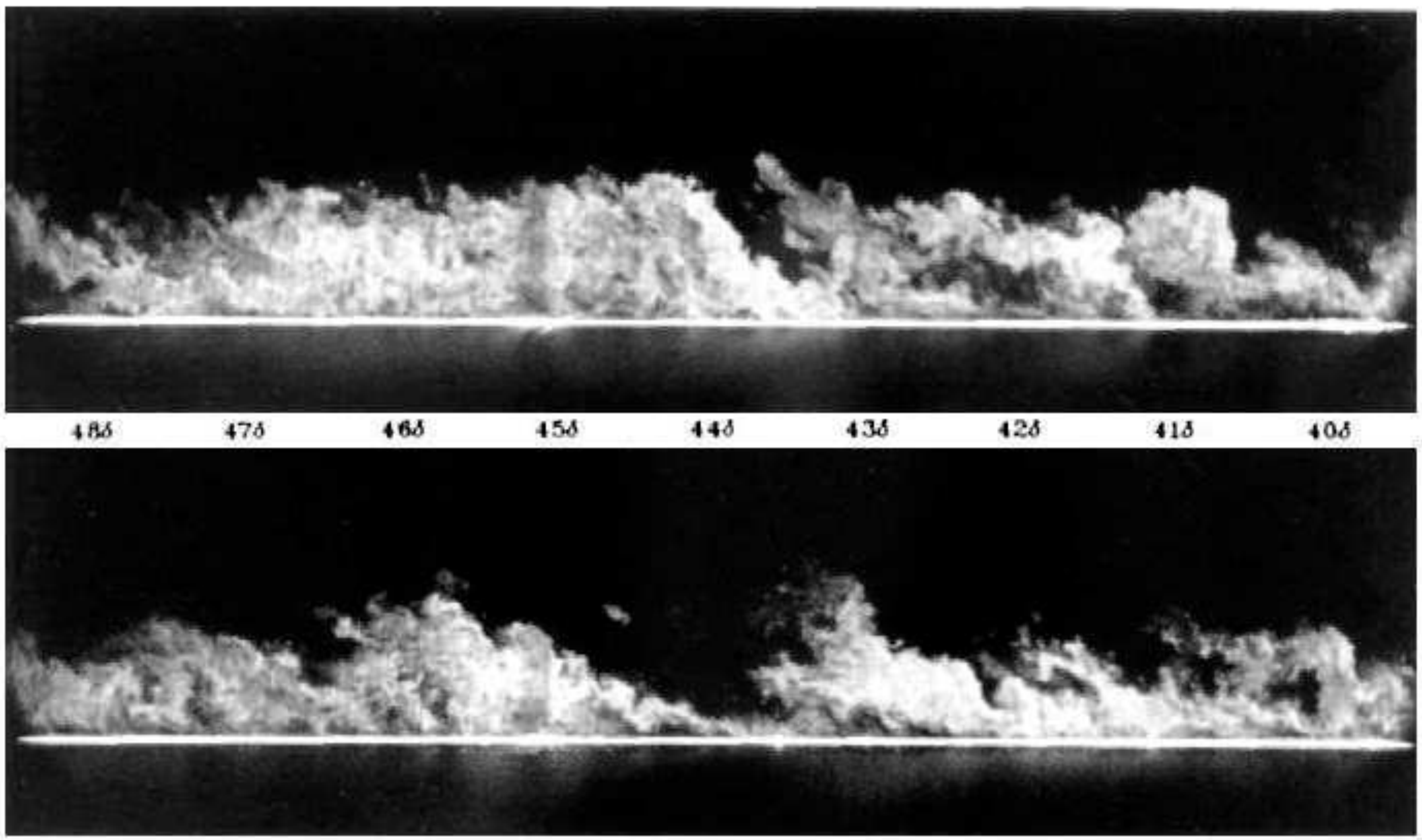
Top-down view

Transition to turbulence

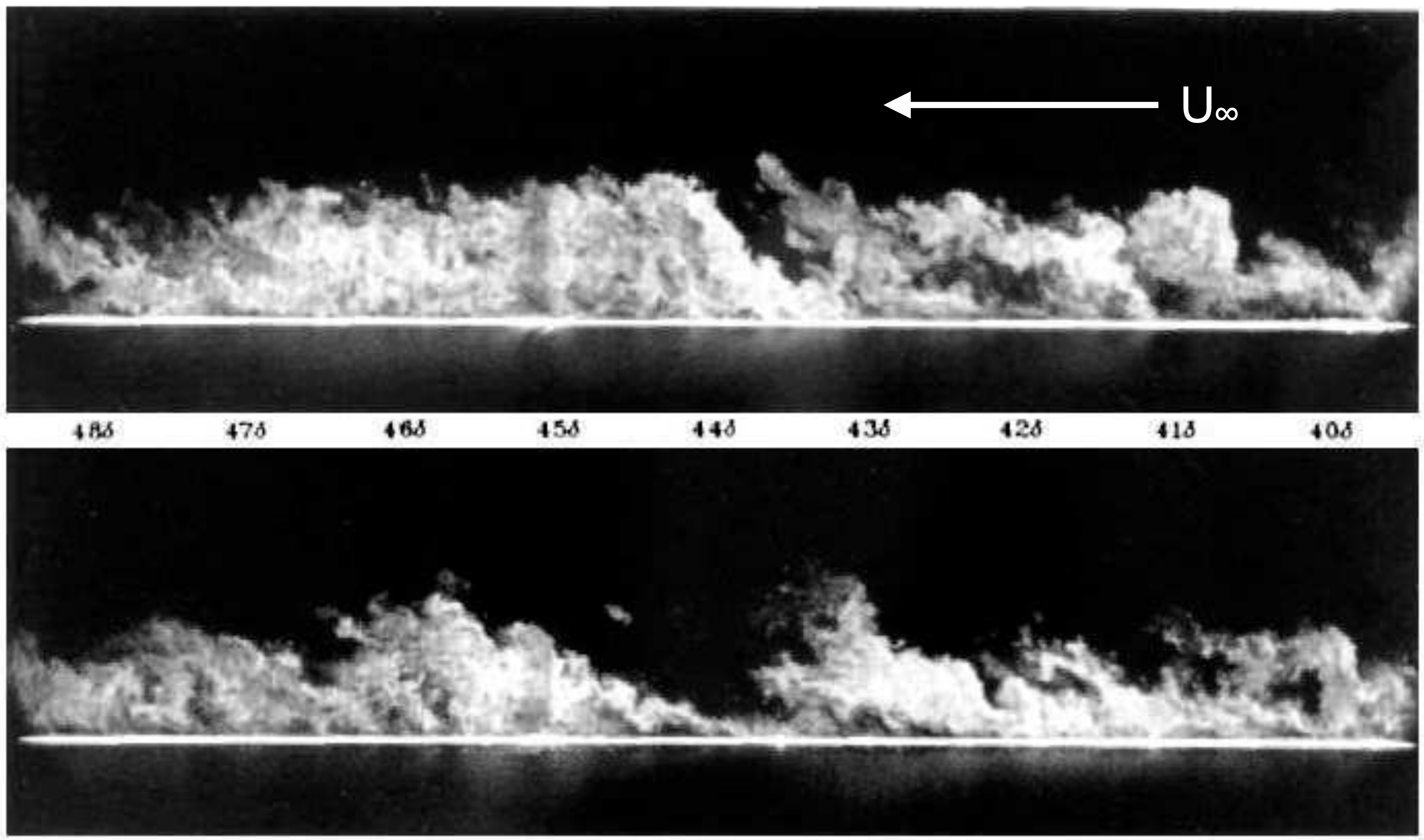
Schlatter *et al.* (2011):

https://www.youtube.com/watch?v=4KeaAhVoPIw

Turbulence



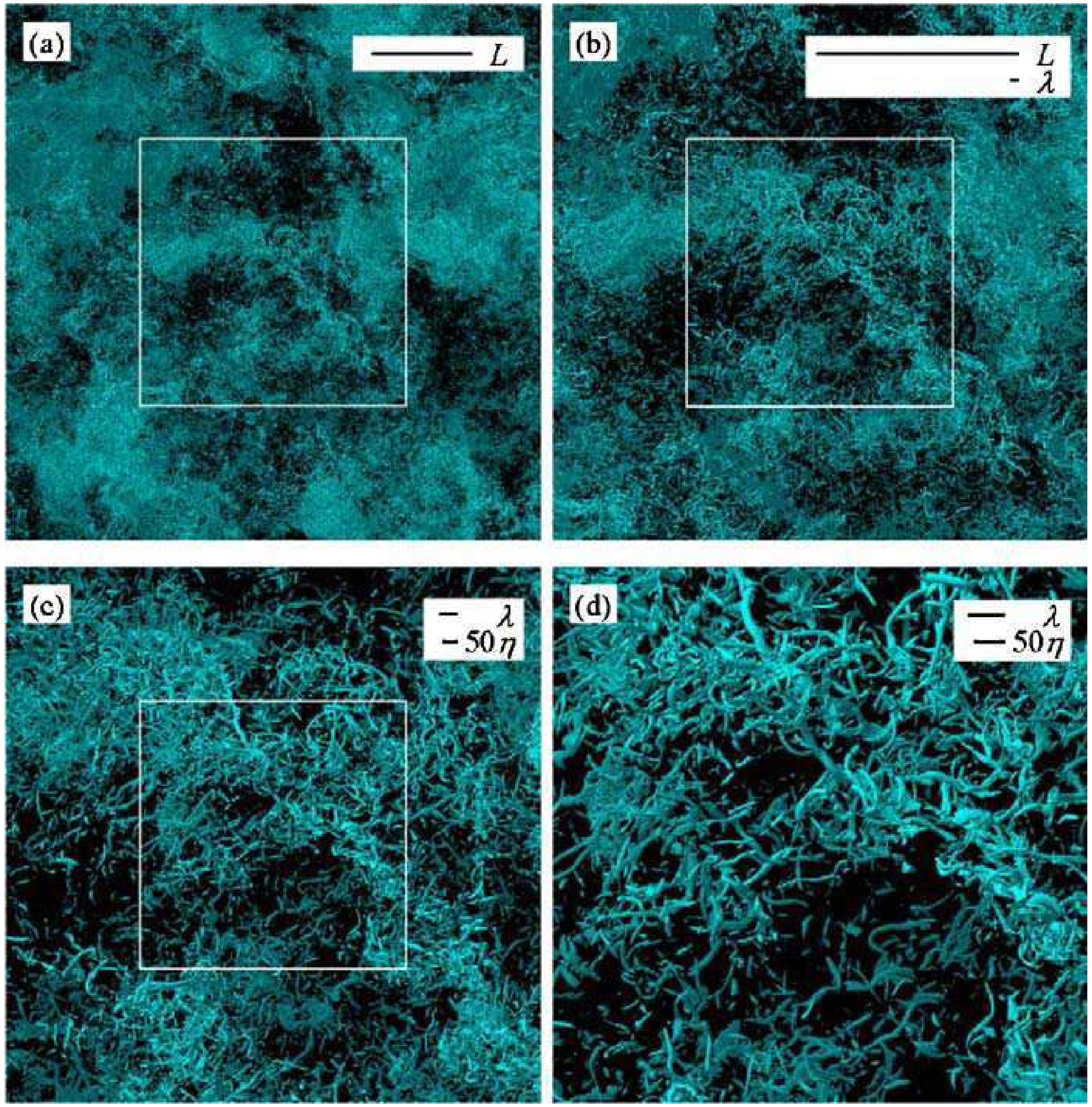
Turbulence



Turbulence

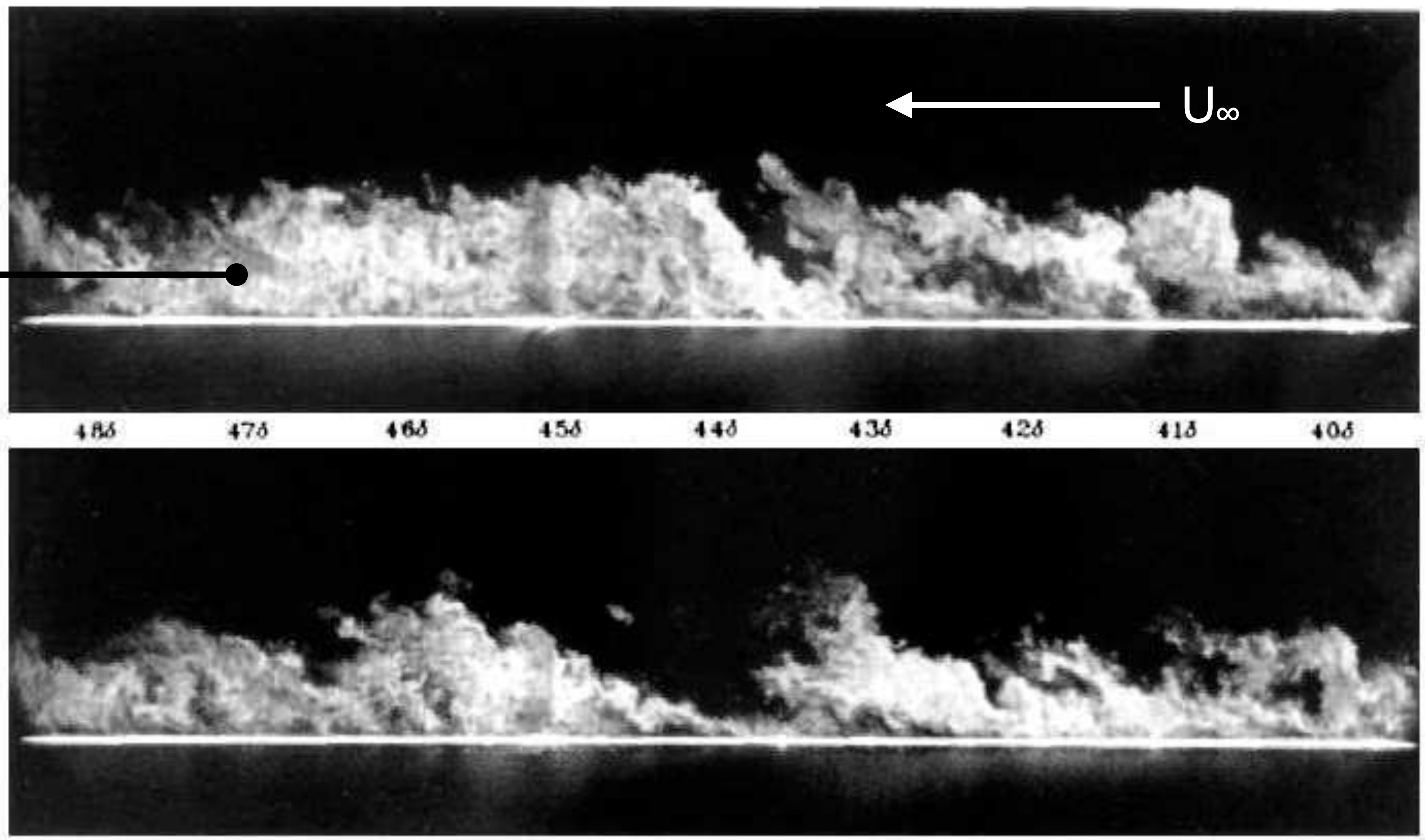
Homogeneous isotropic turbulence computed using the Earth Simulator. Note irregular distribution of vorticity.

http://en.wikipedia.org/wiki/Earth_Simulator



(Kaneda & Ishihara 2004)

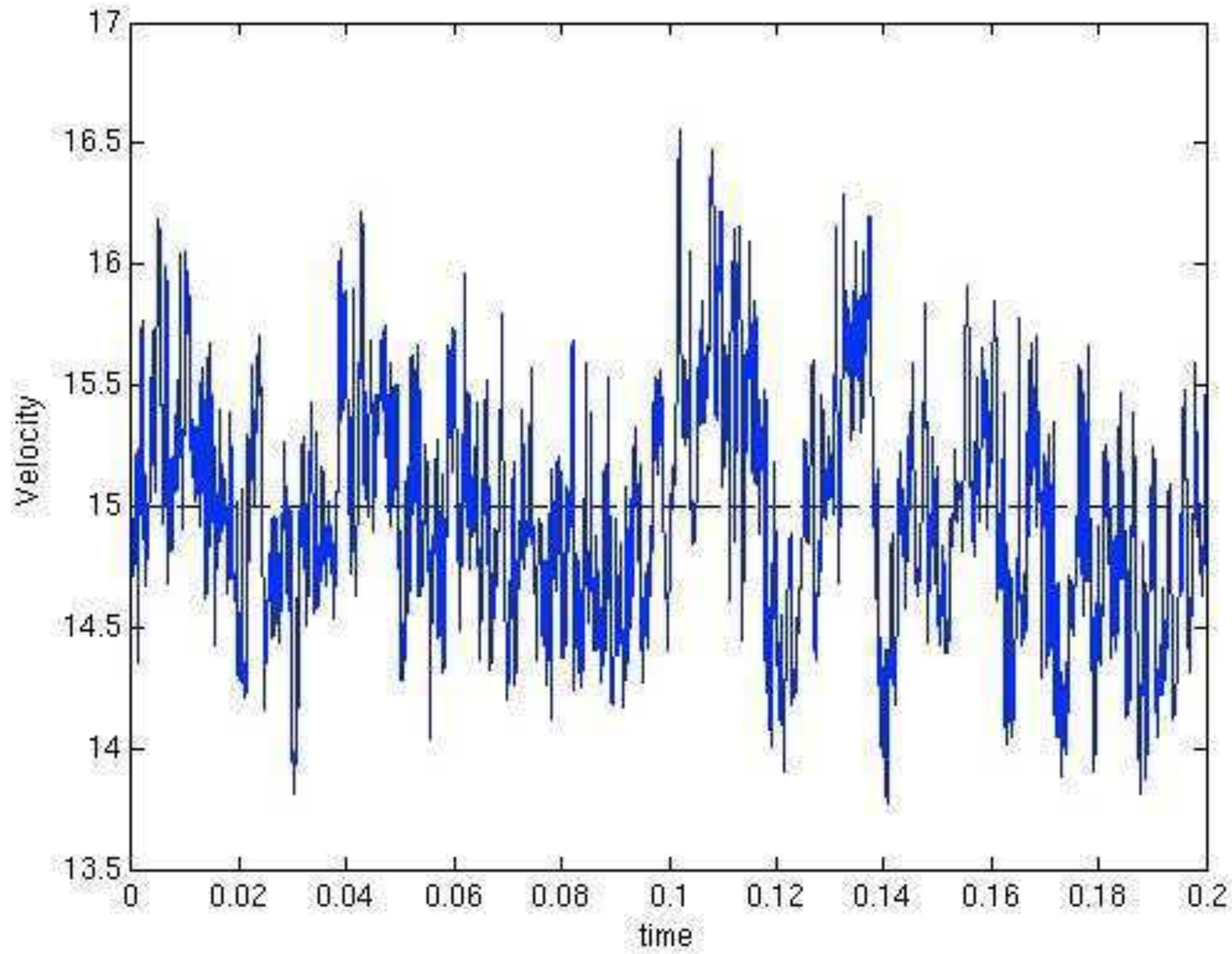
Turbulence



Stick a hot wire here.

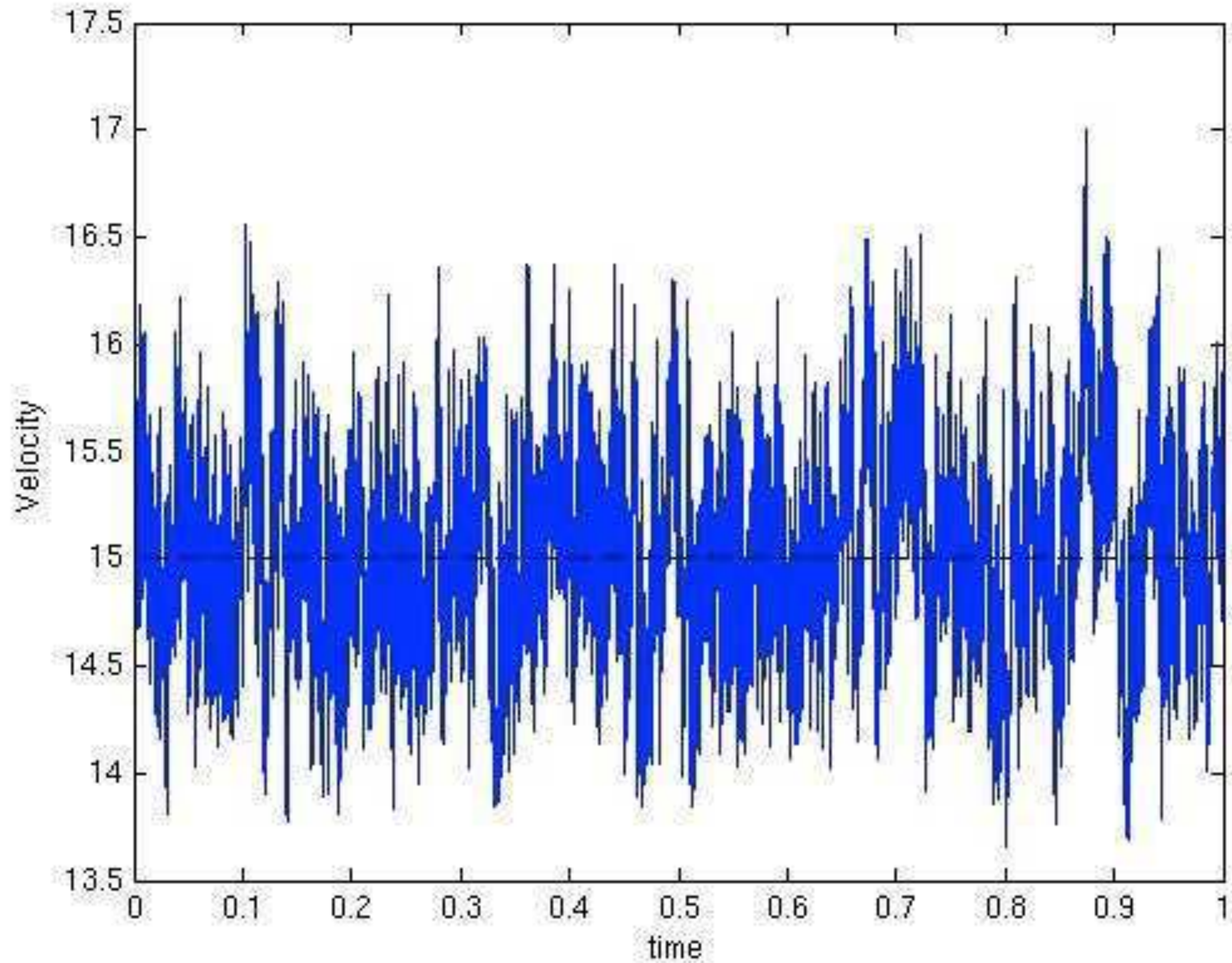
What do you get?

Turbulence



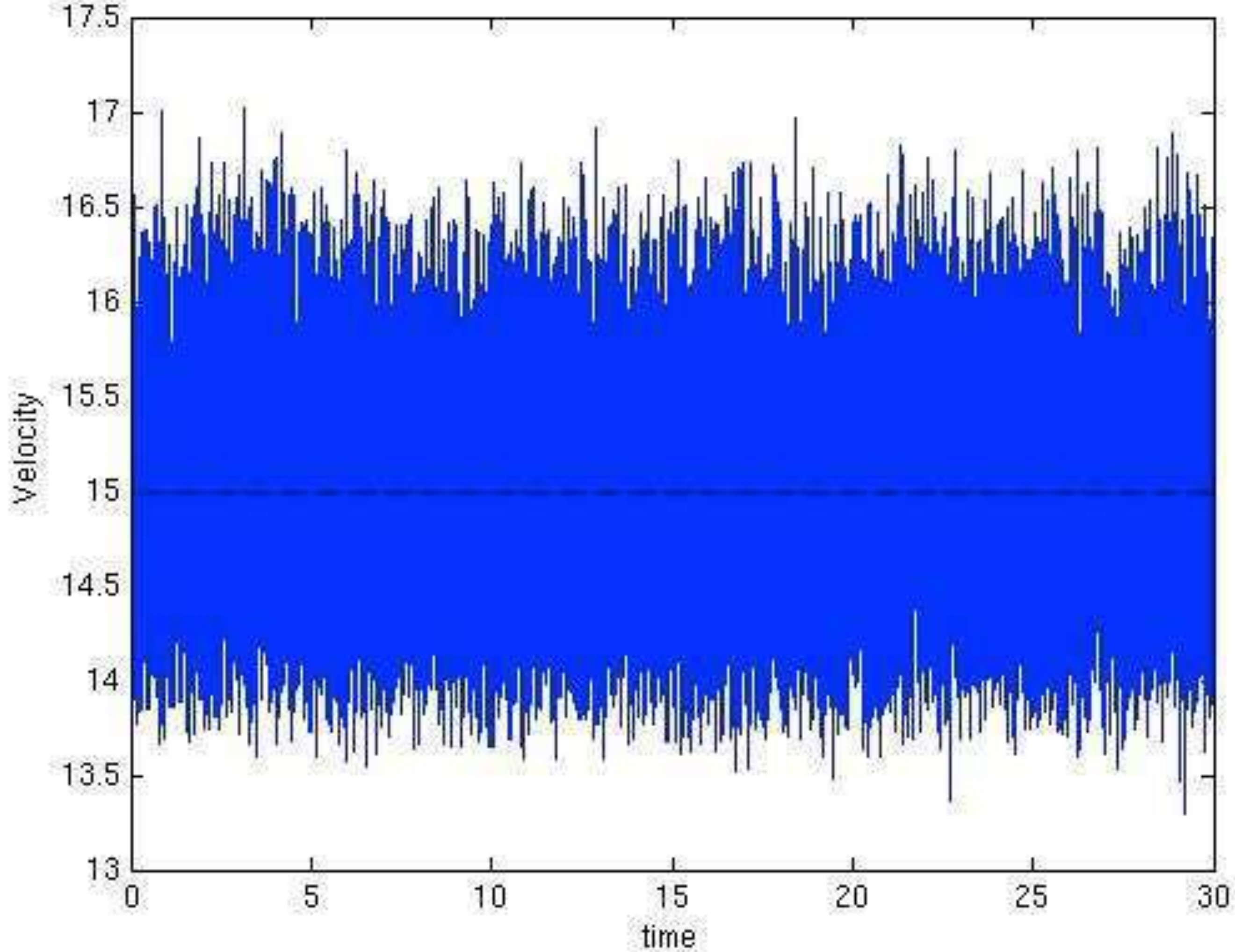
17
16.5
16
15.5
15
14.5
14
13.5
Velocity
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
time

Turbulence



17.5
17
16.5
16
15.5
15
14.5
14
13.5
Velocity
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
time

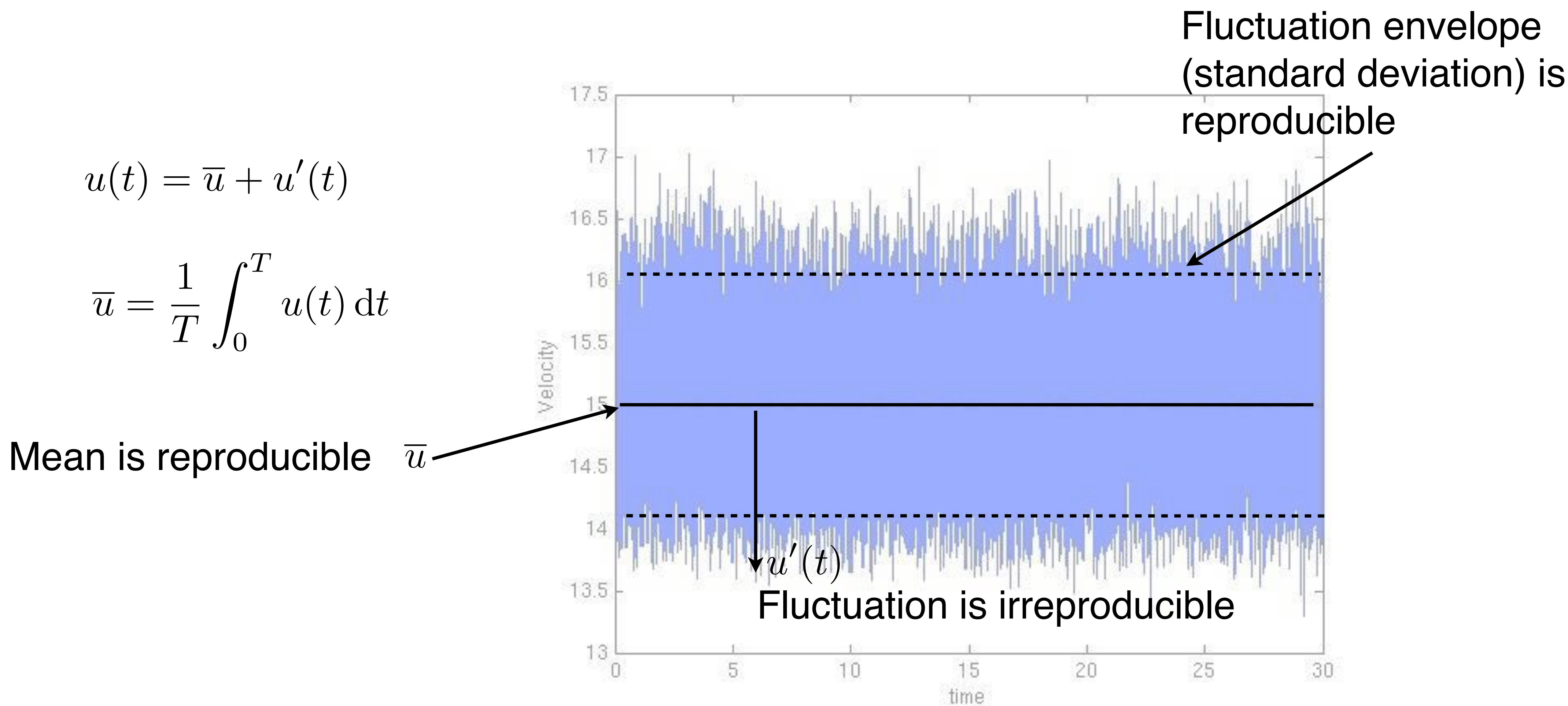
Turbulence



Turbulence

$$u(t) = \overline{u} + u'(t)$$

$$\overline{u} = \frac{1}{T}\int_0^T u(t)\,\mathrm{d}t$$



Reynolds averaging

Reynolds averaging (T large):

$$u(t) = \overline{u} + u'(t)$$

$$\overline{u} = \frac{1}{T}\int_0^T u(t)\,\mathrm{d}t$$

Navier–Stokes:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla}\cdot(\boldsymbol{u}\boldsymbol{u}) = -\frac{1}{\rho}\boldsymbol{\nabla}p + \nu\nabla^2\boldsymbol{u}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$$

Put decomposition into Navier–Stokes:

$$\frac{\partial \overline{\boldsymbol{u}}}{\partial t} + \frac{\partial \boldsymbol{u}'}{\partial t} + \boldsymbol{\nabla}\cdot((\overline{\boldsymbol{u}}+\boldsymbol{u}')(\overline{\boldsymbol{u}}+\boldsymbol{u}')) = -\frac{1}{\rho}\boldsymbol{\nabla}\overline{p} - \frac{1}{\rho}\boldsymbol{\nabla}p' + \nu\nabla^2\overline{\boldsymbol{u}} + \nu\nabla^2\boldsymbol{u}'$$

$$\boldsymbol{\nabla}\cdot\overline{\boldsymbol{u}} + \boldsymbol{\nabla}\cdot\boldsymbol{u}' = 0$$

Further Reynolds average these to get
the Reynolds-averaged Navier–Stokes (RANS) equations:

$$\boldsymbol{\nabla}\cdot(\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}) = -\frac{1}{\rho}\boldsymbol{\nabla}\overline{p} + \nu\nabla^2\overline{\boldsymbol{u}} + \boxed{\boldsymbol{\nabla}\cdot(-\overline{\boldsymbol{u}'\boldsymbol{u}'})}$$

$$\boldsymbol{\nabla}\cdot\overline{\boldsymbol{u}} = 0$$

unknown (unclosed)
Reynolds stresses

cf. steady (laminar)
Navier–Stokes:

$$\boldsymbol{\nabla}\cdot(\boldsymbol{u}\,\boldsymbol{u}) = -\frac{1}{\rho}\boldsymbol{\nabla}p + \nu\nabla^2\boldsymbol{u}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$$

Reynolds-averaged equation for 2D flow in the mean

RANS:

$$\boldsymbol{\nabla}\cdot(\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}) = -\frac{1}{\rho}\boldsymbol{\nabla}\overline{p} + \nu\nabla^2\overline{\boldsymbol{u}} + \boldsymbol{\nabla}\cdot(-\overline{\boldsymbol{u}'\boldsymbol{u}'})$$

$$\boldsymbol{\nabla}\cdot\overline{\boldsymbol{u}} = 0$$

2D (Reynolds-averaged) mean flow:

$$\overline{w} = 0, \quad \frac{\partial}{\partial z} = 0$$

Exact (but unclosed) equations

$$\overline{u}\frac{\partial\overline{u}}{\partial x} + \overline{v}\frac{\partial\overline{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial\overline{p}}{\partial x} + \nu\frac{\partial^2\overline{u}}{\partial x^2} + \nu\frac{\partial^2\overline{u}}{\partial y^2} \boxed{+ \frac{\partial(-\overline{u'u'})}{\partial x} + \frac{\partial(-\overline{v'u'})}{\partial y}}$$

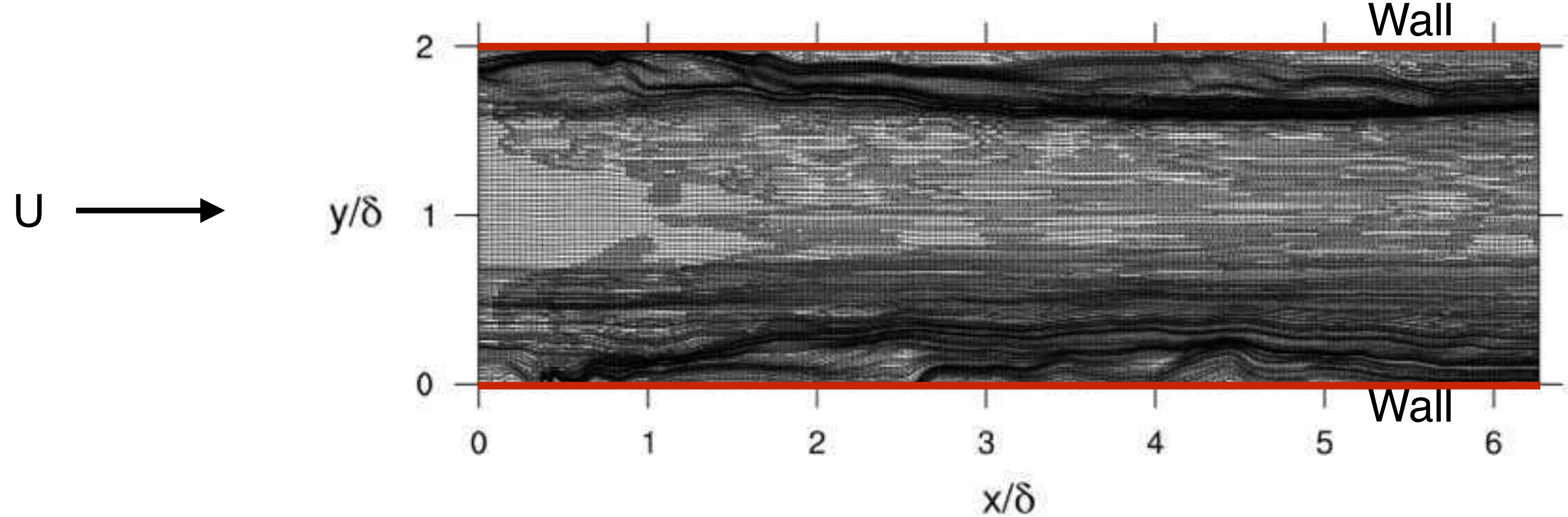
$$\overline{u}\frac{\partial\overline{v}}{\partial x} + \overline{v}\frac{\partial\overline{v}}{\partial y} = -\frac{1}{\rho}\frac{\partial\overline{p}}{\partial y} + \nu\frac{\partial^2\overline{v}}{\partial x^2} + \nu\frac{\partial^2\overline{v}}{\partial y^2} \boxed{+ \frac{\partial(-\overline{u'v'})}{\partial x} + \frac{\partial(-\overline{v'v'})}{\partial y}}$$

$$\frac{\partial\overline{u}}{\partial x} + \frac{\partial\overline{v}}{\partial y} = 0$$

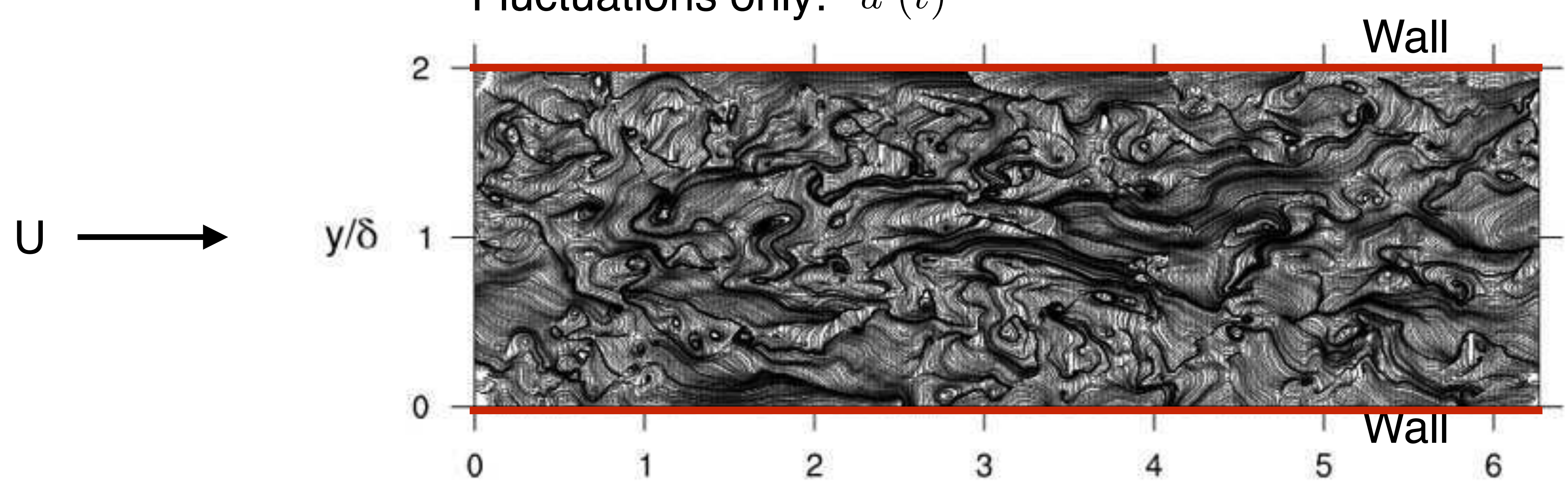
unknown (unclosed)
Reynolds stresses

Turbulent channel flow streamlines

Full flow: $u(t) = \overline{u} + u'(t)$



Fluctuations only: $u'(t)$



Interpretation of Reynolds stresses using control volume

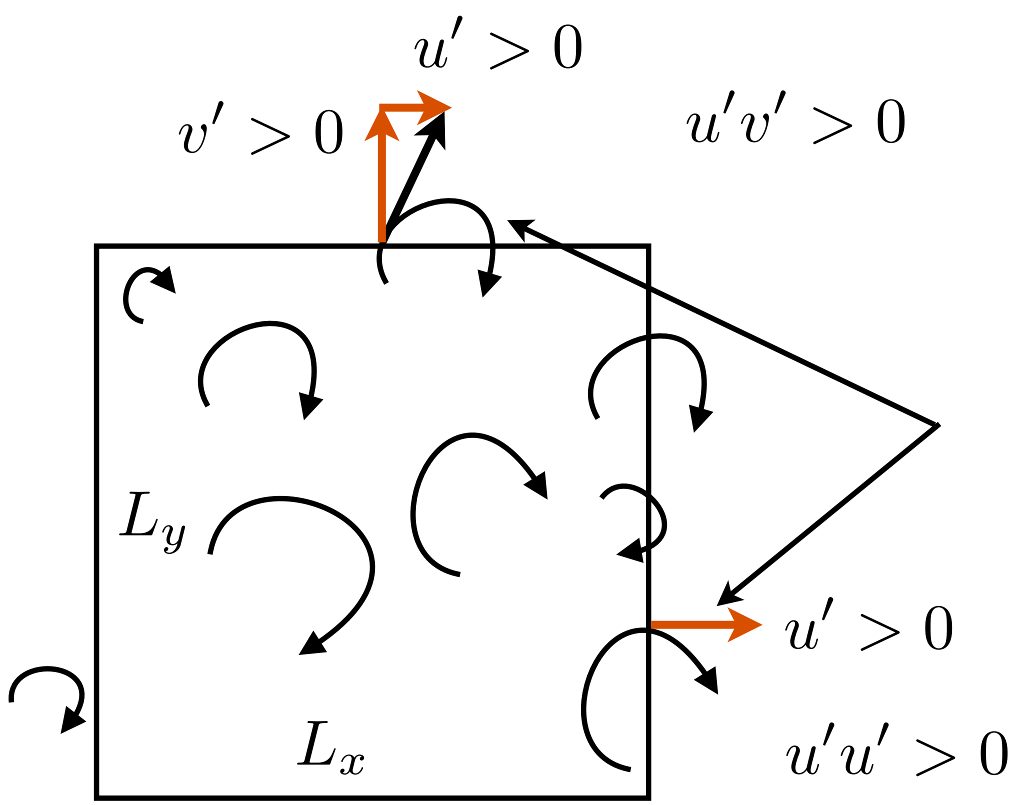
The mean flow perceives correlations in fluctuations due to turbulent eddies as force (integrand is force per unit area = stress).

$$\frac{\partial}{\partial t}\int_0^{L_x}\int_0^{L_y}\int_0^{L_z}\rho u\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = 0 = \cdots + \left[\int_0^{L_y}\int_0^{L_z} -\rho\overline{u'u'}\,\mathrm{d}z\,\mathrm{d}y\right]_{x=0}^{x=L_x} + \left[\int_0^{L_x}\int_0^{L_z} -\rho\overline{v'u'}\,\mathrm{d}z\,\mathrm{d}x\right]_{y=0}^{y=L_y}$$

Rate of change of x-momentum in control volume

Force. SI units: kg/m³ m²/s² m² = N

Force. SI units: kg/m³ m²/s² m² = N



Eddies that carry positive u fluctuation out of control volume tends to reduce x-momentum (drag).
They carry a negative sign on the right-hand side of the momentum equation.

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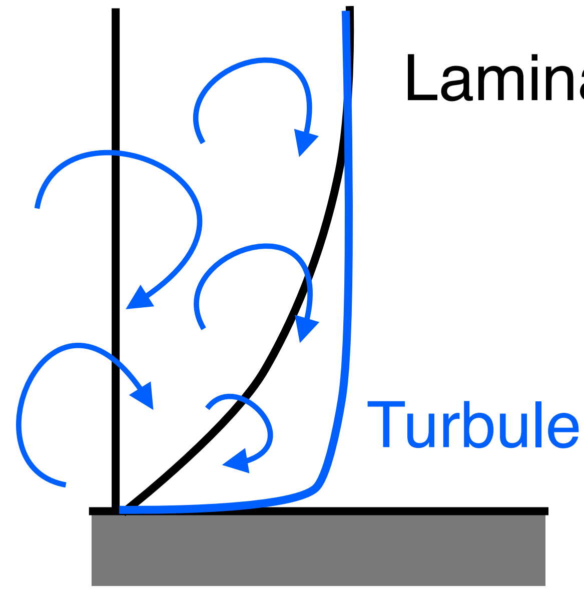
MCEN90018 Advanced Fluid Dynamics

Lecture BL13: Millikan overlap log law
21 April 2016

Effect of turbulence on mean flow

Mean profiles:



Turbulent eddies make the mean profile flatter because they are more effective than viscosity in bringing faster fluid down and slower fluid up: $-\overline{v'u'} > 0$

Efficient mixing:

That is, the Reynolds shear stress is higher than the viscous shear stress: $-\overline{v'u'} > \nu \dfrac{\partial \overline{u}}{\partial y}$

Reynolds-averaged boundary-layer approximation

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \nu\frac{\partial^2 \overline{u}}{\partial x^2} + \nu\frac{\partial^2 \overline{u}}{\partial y^2} + \frac{\partial(-\overline{u'u'})}{\partial x} + \frac{\partial(-\overline{v'u'})}{\partial y}$$

$$\overline{u}\frac{\partial \overline{v}}{\partial x} + \overline{v}\frac{\partial \overline{v}}{\partial y} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial y} + \nu\frac{\partial^2 \overline{v}}{\partial x^2} + \nu\frac{\partial^2 \overline{v}}{\partial y^2} + \frac{\partial(-\overline{u'v'})}{\partial x} + \frac{\partial(-\overline{v'v'})}{\partial y}$$

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0$$

Thin layer: $X \gg Y$

Scale for turbulent fluctuations: $u' = O(U_t), \quad v' = O(U_t), \quad w' = O(U_t)$

$$O(\frac{U^2}{X}) + O(\frac{VU}{Y}) = O(\frac{U^2}{X}) + O(\frac{\nu U}{X^2}) + O(\frac{\nu U}{Y^2}) + O(\frac{U_t^2}{X}) + O(\frac{U_t^2}{Y})$$

$$O(\frac{UV}{X}) + O(\frac{V^2}{Y}) = O(\frac{U^2}{Y}) + O(\frac{\nu V}{X^2}) + O(\frac{\nu V}{Y^2}) + O(\frac{U_t^2}{X}) + O(\frac{U_t^2}{Y})$$

$$O(\frac{U}{X}) + O(\frac{V}{Y}) = 0$$

(continuity same as for laminar)

Reynolds-averaged boundary-layer approximation

Use $V = U\dfrac{Y}{X}$ and multiply by $\dfrac{X}{U^2}$ and $\dfrac{Y}{U^2}$ respectively:

$$O(1) + O(1) = O(1) + O(\frac{\nu}{UX}) + O(\frac{X^2}{Y^2}\frac{\nu}{UX}) + O(\frac{U_t^2}{U^2}) + O(\frac{X}{Y}\frac{U_t^2}{U^2})$$

$$O(\frac{Y^2}{X^2}) + O(\frac{Y^2}{X^2}) = O(1) + O(\frac{Y^2}{X^2}\frac{\nu}{UX}) + O(\frac{\nu}{UX}) + O(\frac{Y}{X}\frac{U_t^2}{U^2}) + O(\frac{U_t^2}{U^2})$$

Want to keep last term (turbulent mixing of x-momentum) in x-momentum: $\dfrac{X}{Y} = \dfrac{U^2}{U_t^2}$

$$O(1) + O(1) = O(1) + O(\frac{\nu}{UX}) + O(\frac{X^2}{Y^2}\frac{\nu}{UX}) + O(\frac{Y}{X}) + O(1)$$

$$O(\frac{Y^2}{X^2}) + O(\frac{Y^2}{X^2}) = O(1) + O(\frac{Y^2}{X^2}\frac{\nu}{UX}) + O(\frac{\nu}{UX}) + O(\frac{Y^2}{X^2}) + O(\frac{Y}{X})$$

Want to keep effect of viscosity very near the wall (where even turbulence can't act): $\dfrac{X^2}{Y^2} = \dfrac{UX}{\nu}$

$$O(1) + O(1) = O(1) + O(\frac{Y^2}{X^2}) + O(1) + O(\frac{Y}{X}) + O(1)$$

$$O(\frac{Y^2}{X^2}) + O(\frac{Y^2}{X^2}) = O(1) + O(\frac{Y^4}{X^4}) + O(\frac{Y^2}{X^2}) + O(\frac{Y^2}{X^2}) + O(\frac{Y}{X})$$

Reynolds-averaged boundary-layer approximation

Send $X \gg Y$:

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \nu\frac{\partial^2 \overline{u}}{\partial y^2} + \boxed{\frac{\partial(-\overline{v'u'})}{\partial y}} + \frac{\partial(-\overline{u'u'})}{\partial x}$$

$$0 = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial y} + \frac{\partial(-\overline{v'v'})}{\partial y}$$

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0$$

“2nd-order”

Exactly the same as laminar flow except for (unclosed) Reynolds shear stress gradient.

Check: does not change von Karman momentum integral equation.

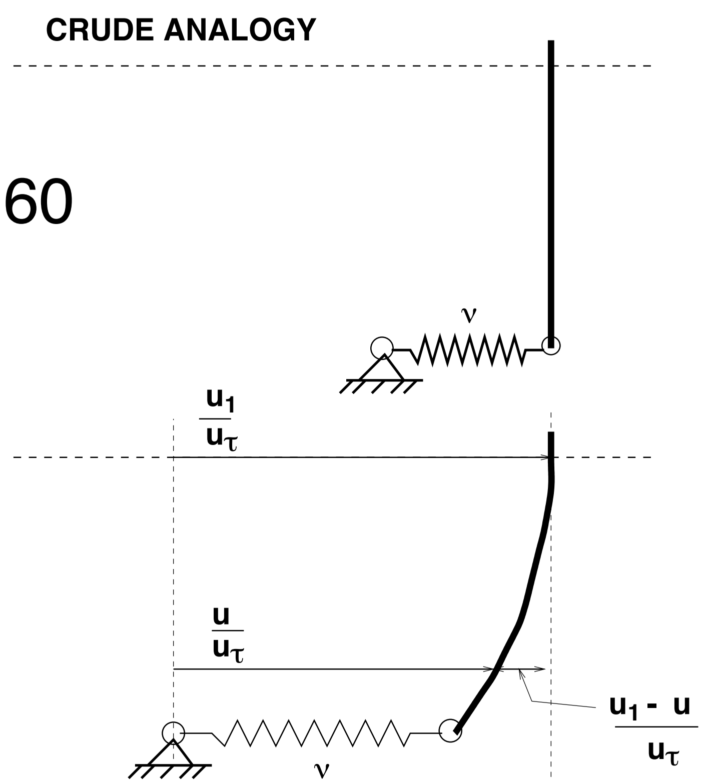
Dimensional analysis to obtain turbulent mean velocity profile (law of the wall)

Parameters: $\dfrac{\mathrm{d}\overline{u}}{\mathrm{d}y}$ $\quad y \quad$ $u_\tau \equiv \sqrt{\nu\left(\dfrac{\mathrm{d}\overline{u}}{\mathrm{d}y}\right)_{y=0}}$ $\quad \nu \quad$ δ

Same as skin friction (wall shear stress)

Same as freestream velocity (in fact, the boundary-layer thickness is defined by the freestream velocity).

Chong, p. 60



Net deflection is function of both spring and beam, but shape of beam is function of beam only.

Townsend: Viscosity sets the skin friction, but once the skin friction is set, the outer flow only sees the skin friction (not viscosity).

Dimensional analysis to obtain turbulent mean velocity profile (law of the wall)

Parameters:	$\dfrac{\mathrm{d}\overline{u}}{\mathrm{d}y}$	y	$u_\tau \equiv \sqrt{\nu\left(\dfrac{\mathrm{d}\overline{u}}{\mathrm{d}y}\right)_{y=0}}$	ν	δ
Units:	$\dfrac{1}{T}$	L	$\dfrac{L}{T}$	$\dfrac{L^2}{T}$	L

Number of Pi products = number of parameters – number of units

3 = 5 – 2

Pi products: $\dfrac{y}{u_\tau}\dfrac{\mathrm{d}\overline{u}}{\mathrm{d}y}$ $\quad \dfrac{u_\tau y}{\nu}$ $\quad \dfrac{y}{\delta}$

In general:

$$\frac{y}{u_\tau}\frac{\mathrm{d}\overline{u}}{\mathrm{d}y} = \Phi\left(\frac{u_\tau y}{\delta},\ \frac{y}{\delta}\right)$$

Melbourne School of Engineering
MCEN90018 Advanced Fluid Dynamics

Lecture BL14: Resistance laws
22 April 2016

Dimensional analysis to obtain turbulent mean velocity profile (law of the wall)

Pi products: $\dfrac{y}{u_\tau}\dfrac{\mathrm{d}\overline{u}}{\mathrm{d}y}$ $\qquad \dfrac{u_\tau y}{\nu}$ $\qquad \dfrac{y}{\delta}$

In a layer where viscosity does not matter: $\nu/u_\tau \ll y$ then $\dfrac{y}{u_\tau}\dfrac{\mathrm{d}\overline{u}}{\mathrm{d}y} = f\left(\dfrac{y}{\delta}\right)$

(inner layer)

$$\Leftrightarrow \quad \frac{\overline{u}(\delta) - \overline{u}(y)}{u_\tau} = F(y/\delta) = \int_{(y/\delta)}^{1} \frac{f(y'/\delta)}{(y'/\delta)}\,\mathrm{d}(y'/\delta)$$

Dimensional analysis to obtain turbulent mean velocity profile (law of the wall)

Pi products: $\dfrac{y}{u_\tau}\dfrac{\mathrm{d}\overline{u}}{\mathrm{d}y}$ $\dfrac{u_\tau y}{\nu}$ $\dfrac{y}{\delta}$

In a layer where viscosity does not matter: $\nu/u_\tau \ll y$ then $\dfrac{y}{u_\tau}\dfrac{\mathrm{d}\overline{u}}{\mathrm{d}y} = f\left(\dfrac{y}{\delta}\right)$

(inner layer)

$$\Leftrightarrow \quad \frac{\overline{u}(\delta)-\overline{u}(y)}{u_\tau} = F(y/\delta) = \int_{(y/\delta)}^{1} \frac{f(y'/\delta)}{(y'/\delta)}\,\mathrm{d}(y'/\delta)$$

In a layer where boundary-layer thickness does not matter:

(outer layer) $y \ll \delta$ then $\dfrac{y}{u_\tau}\dfrac{\mathrm{d}\overline{u}}{\mathrm{d}y} = g\left(\dfrac{yu_\tau}{\nu}\right)$

$$\Leftrightarrow \quad \frac{\overline{u}(y)-\overline{u}(0)}{u_\tau} = G(yu_\tau/\nu) = \int_{0}^{yu_\tau/\nu} \frac{g(y'u_\tau/\nu)}{(y'u_\tau/\nu)}\,\mathrm{d}(y'u_\tau/\nu)$$

Dimensional analysis to obtain turbulent mean velocity profile (law of the wall)

Pi products: $\dfrac{y}{u_\tau}\dfrac{\mathrm{d}\overline{u}}{\mathrm{d}y}$ $\qquad \dfrac{u_\tau y}{\nu}$ $\qquad \dfrac{y}{\delta}$

In a layer where viscosity does not matter: $\nu/u_\tau \ll y$ then $\dfrac{y}{u_\tau}\dfrac{\mathrm{d}\overline{u}}{\mathrm{d}y} = f\left(\dfrac{y}{\delta}\right)$

(inner layer)

$$\Leftrightarrow \quad \frac{\overline{u}(\delta)-\overline{u}(y)}{u_\tau} = F(y/\delta) = \int_{(y/\delta)}^{1} \frac{f(y'/\delta)}{(y'/\delta)}\,\mathrm{d}(y'/\delta)$$

In a layer where boundary-layer thickness does not matter:

(outer layer) $y \ll \delta$ then $\dfrac{y}{u_\tau}\dfrac{\mathrm{d}\overline{u}}{\mathrm{d}y} = g\left(\dfrac{yu_\tau}{\nu}\right)$

$$\Leftrightarrow \quad \frac{\overline{u}(y)-\overline{u}(0)}{u_\tau} = G(yu_\tau/\nu) = \int_{0}^{yu_\tau/\nu} \frac{g(y'u_\tau/\nu)}{(y'u_\tau/\nu)}\,\mathrm{d}(y'u_\tau/\nu)$$

In a layer where both viscosity and boundary-layer thickness do not matter:

(overlap layer) $\nu/u_\tau \ll y \ll \delta$ then $\dfrac{y}{u_\tau}\dfrac{\mathrm{d}\overline{u}}{\mathrm{d}y} = \dfrac{1}{\kappa} \quad \Leftrightarrow \quad \dfrac{\overline{u}(y)}{u_\tau} = \dfrac{1}{\kappa}\log y + \mathrm{const.}$

Inner and outer coordinates

Define inner and outer coordinates: $y^+ \equiv \frac{y u_\tau}{\nu}$ $\eta \equiv \frac{y}{\delta}$ $\overline{u}^+ \equiv \frac{\overline{u}}{u_\tau}$

In a layer where viscosity does not matter: $1 \ll y^+$ then $\eta \frac{\mathrm{d}\overline{u}^+}{\mathrm{d}\eta} = f(\eta)$

(also called the defect law) $\Leftrightarrow \quad \overline{u}^+(\delta) - \overline{u}^+(y) = F(\eta) = \int_\eta^1 \frac{f(\eta_1)}{\eta_1}\,\mathrm{d}\eta_1$

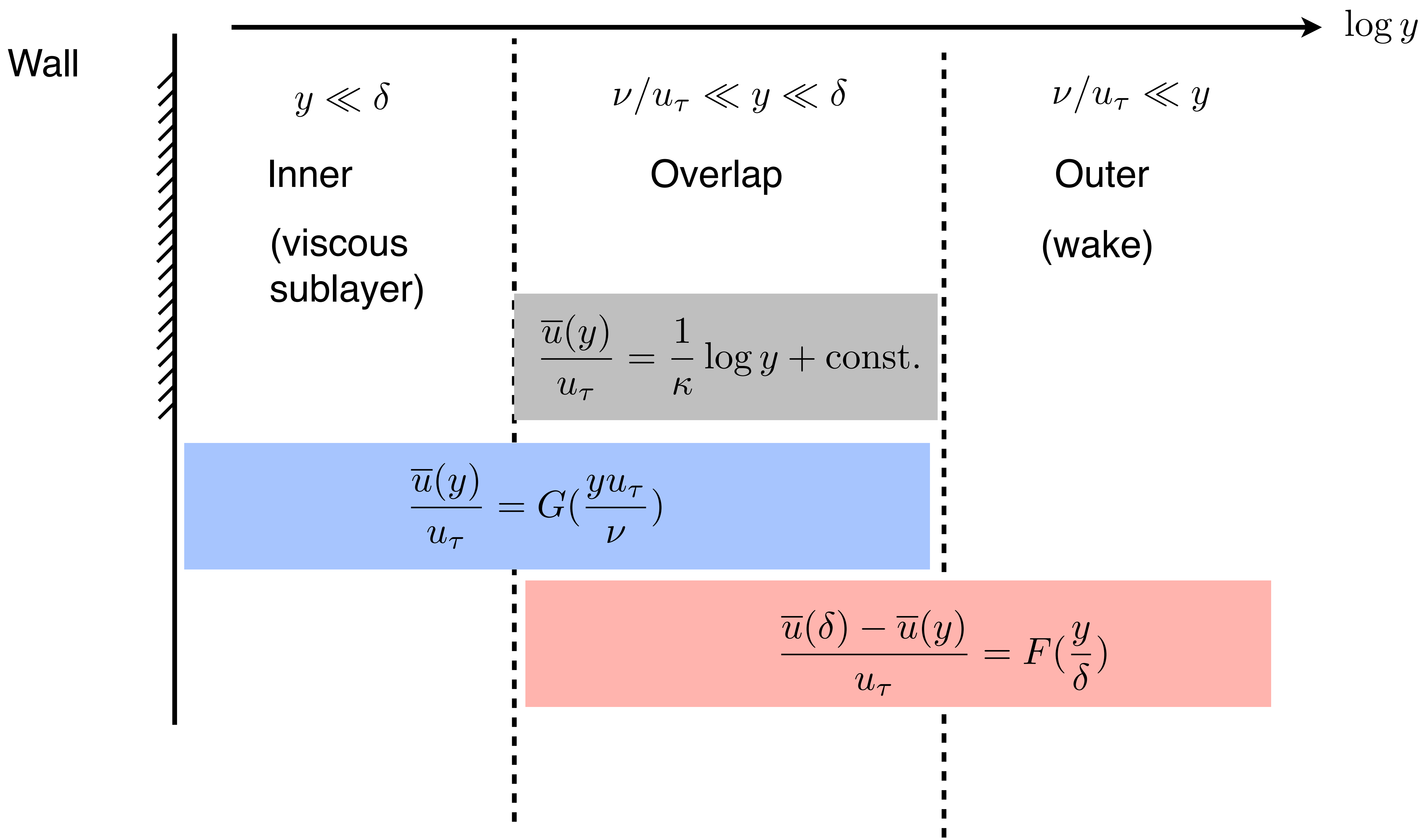
In a layer where boundary-layer thickness does not matter:

$\eta \ll 1$ then $y^+ \frac{\mathrm{d}\overline{u}^+}{\mathrm{d}y^+} = g(y^+)$

$$\Leftrightarrow \quad \overline{u}^+(y) = G(y^+) = \int_0^{y+} \frac{g(y_1^+)}{y_1^+}\,\mathrm{d}y_1^+$$

Anatomy of a turbulent mean velocity profile

In general: $\dfrac{y}{u_\tau}\dfrac{\mathrm{d}\overline{u}}{\mathrm{d}y} = \Phi\left(\dfrac{y u_\tau}{\nu}, \dfrac{y}{\delta}\right)$



Anatomy of a turbulent mean velocity profile

In general: $$\frac{y}{u_\tau}\frac{\mathrm{d}\overline{u}}{\mathrm{d}y} = \Phi\left(\frac{y u_\tau}{\nu}, \frac{y}{\delta}\right)$$

$\log y$

Wall

$y \ll \delta$	$\nu/u_\tau \ll y \ll \delta$	$\nu/u_\tau \ll y$
Inner	Overlap	Outer
(viscous sublayer)		(wake)

$$\frac{\overline{u}(y)}{u_\tau} = \frac{1}{\kappa}\log y + \text{const.}$$

$$\frac{\overline{u}(y)}{u_\tau} = \frac{1}{\kappa}\log\left(\frac{y u_\tau}{\nu}\right) + A$$

$$\frac{\overline{u}(\delta) - \overline{u}(y)}{u_\tau} = -\frac{1}{\kappa}\log\left(\frac{y}{\delta}\right) + B$$

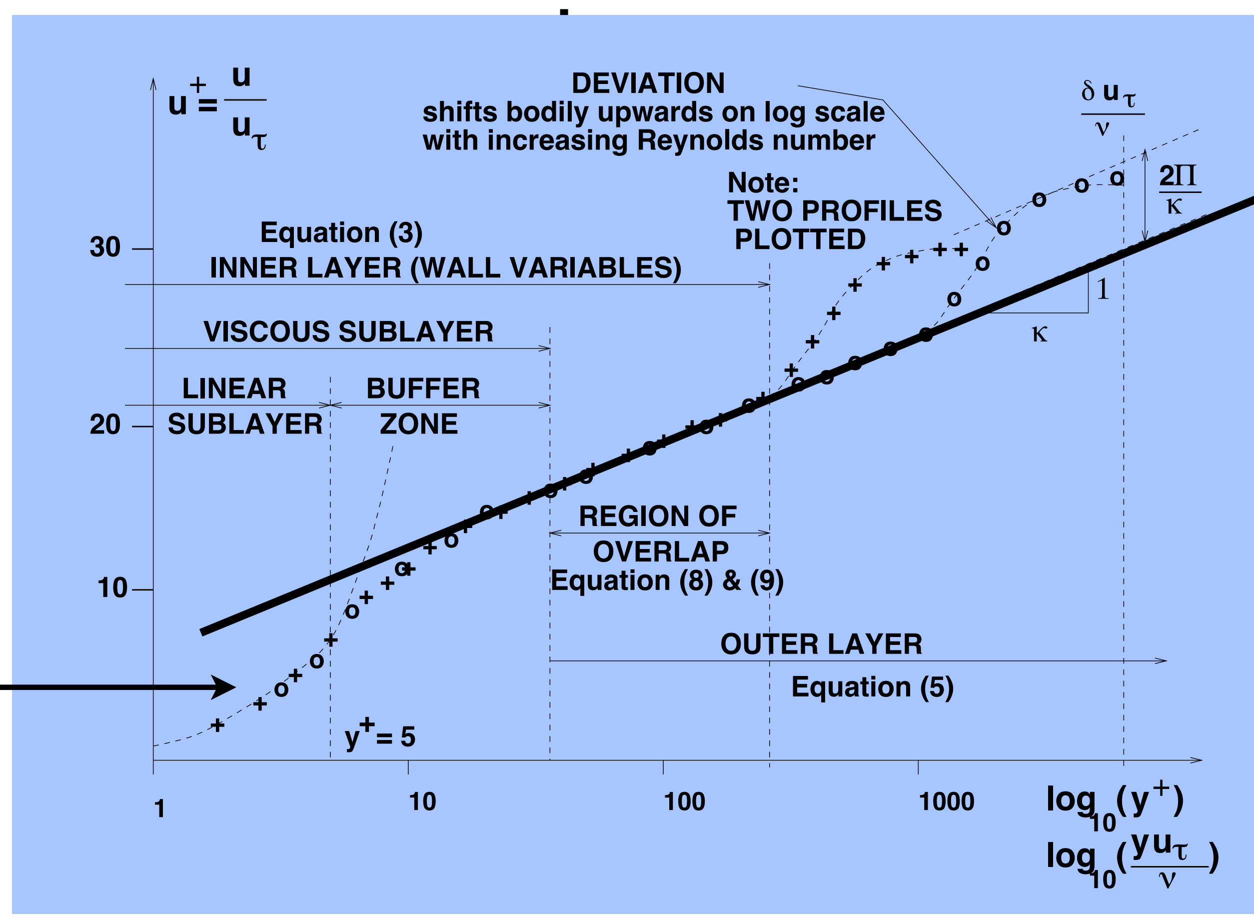
Anatomy of a turbulent mean velocity profile

In general: $\dfrac{y}{u_\tau}\dfrac{\mathrm{d}\overline{u}}{\mathrm{d}y} = \Phi(\dfrac{yu_\tau}{\nu}, \dfrac{y}{\delta})$

$\log y$

Wall

$y \ll \delta$	$\nu/u_\tau \ll y \ll \delta$	$\nu/u_\tau \ll y$
Inner	Overlap	Outer
(viscous sublayer)		(wake)



Linear viscous sublayer from expanding Taylor series

$$\frac{\overline{u}(y)}{u_\tau} = \frac{1}{\kappa}\log(\frac{yu_\tau}{\nu}) + A$$

A is universal for all wall flows

Anatomy of a turbulent mean velocity profile

In general: $$\frac{y}{u_\tau}\frac{\mathrm{d}\overline{u}}{\mathrm{d}y} = \Phi(\frac{yu_\tau}{\nu}, \frac{y}{\delta})$$

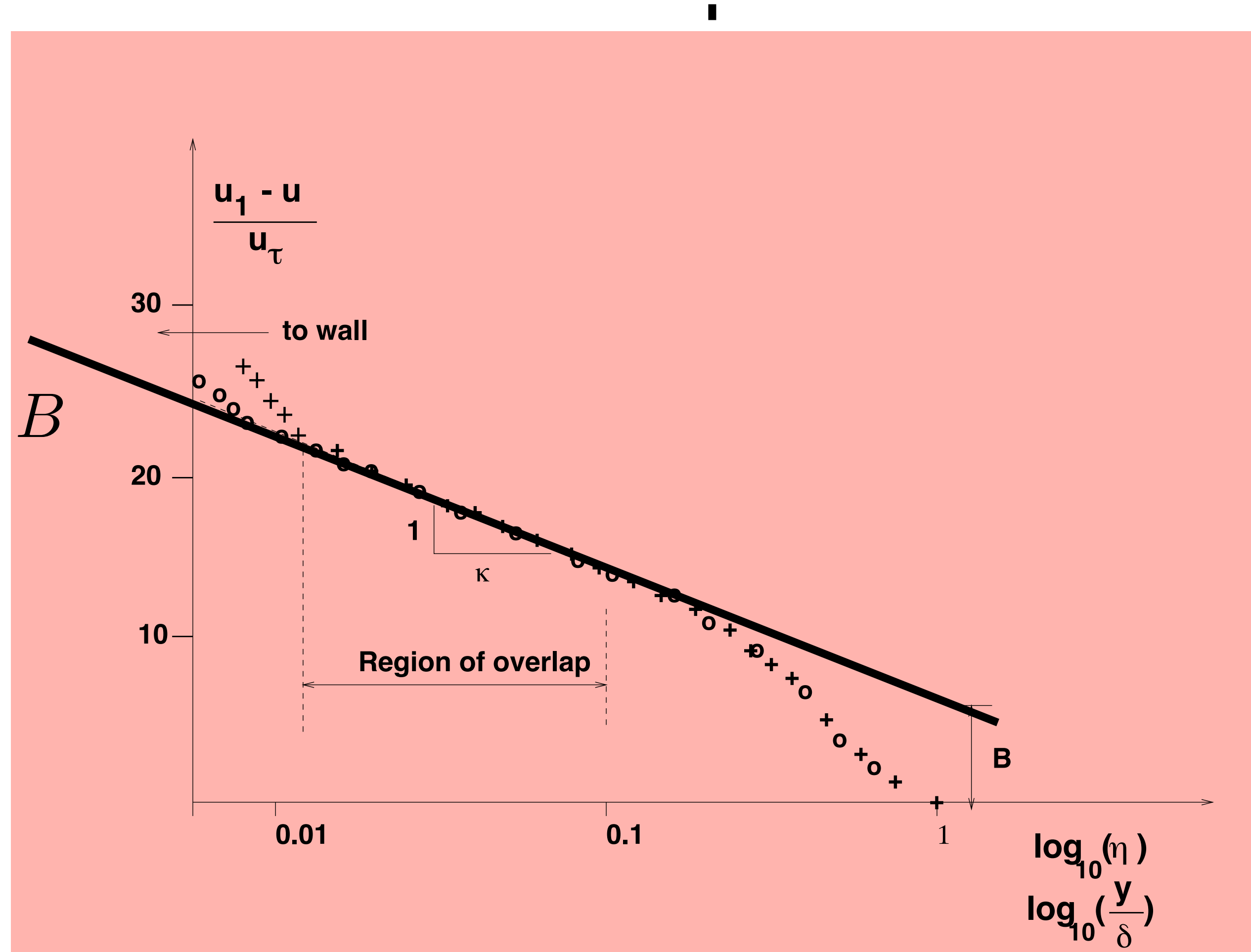
$\log y$

Wall

$y \ll \delta$	$\nu/u_\tau \ll y \ll \delta$	$\nu/u_\tau \ll y$
Inner	Overlap	Outer
(viscous sublayer)		(wake)

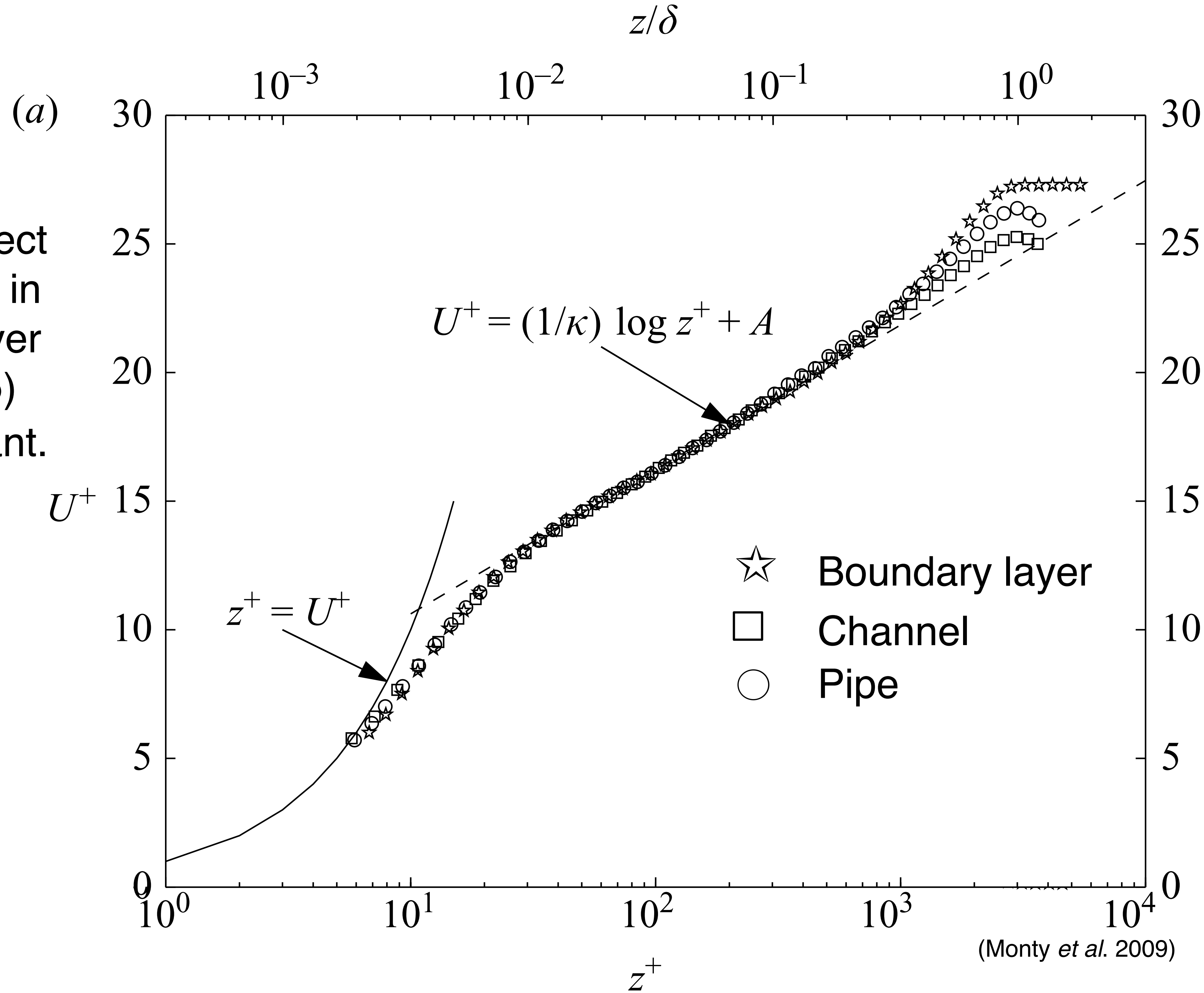
$$\frac{\overline{u}(\delta) - \overline{u}(y)}{u_\tau} = -\frac{1}{\kappa}\log(\frac{y}{\delta}) + B$$

B, unlike A, depends on type of flow (boundary layers, pipes, channels)



Log law - inner scaling

Collapse because effect of geometry in the inner layer (when y ≪ δ) is insignificant.



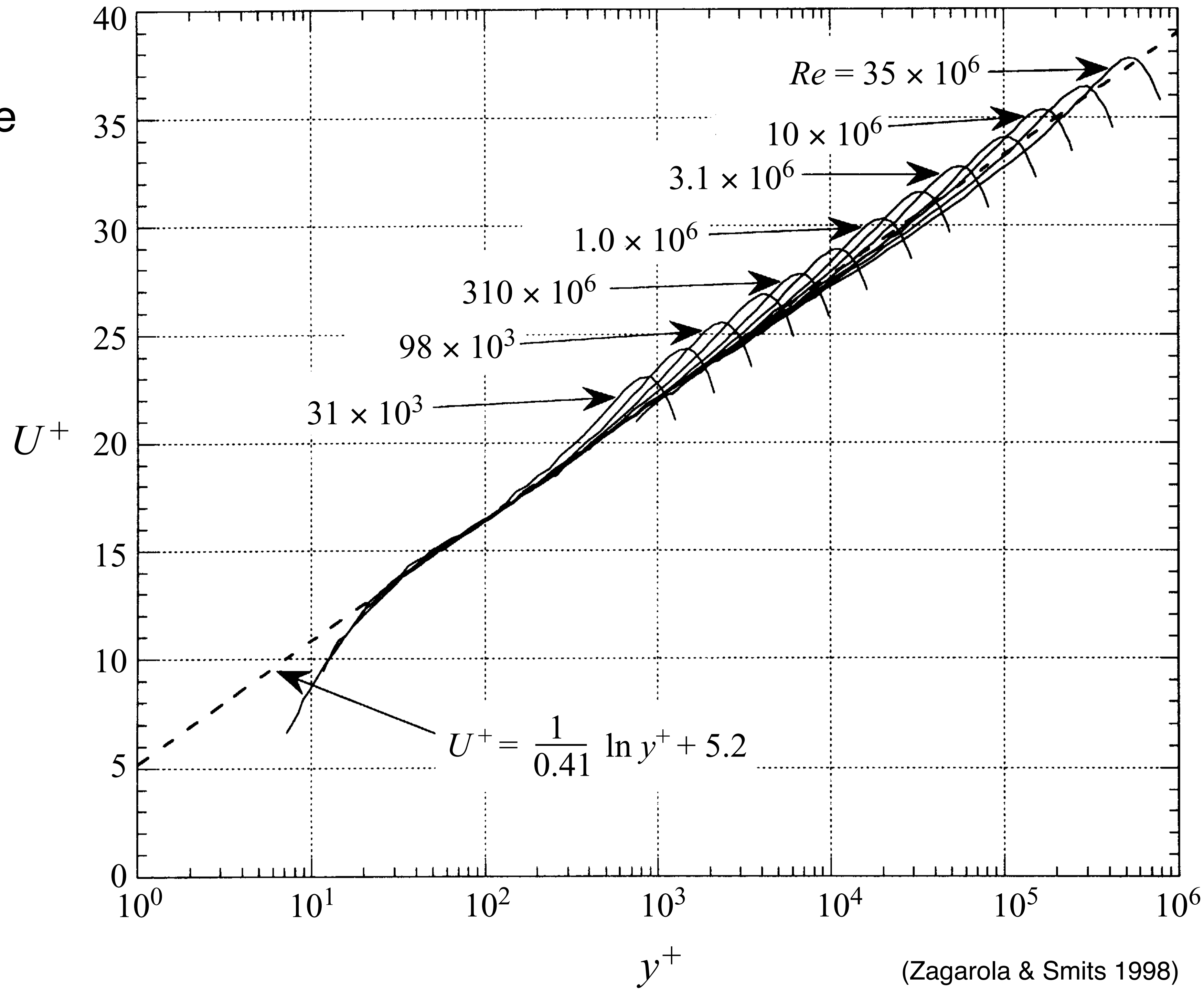
Experimental facilities

Princeton
Superpipe

http://www.princeton.edu/mae/people/faculty/smits/homepage/superpipe.jpg

Log law - inner scaling

Princeton Superpipe



(Zagarola & Smits 1998)

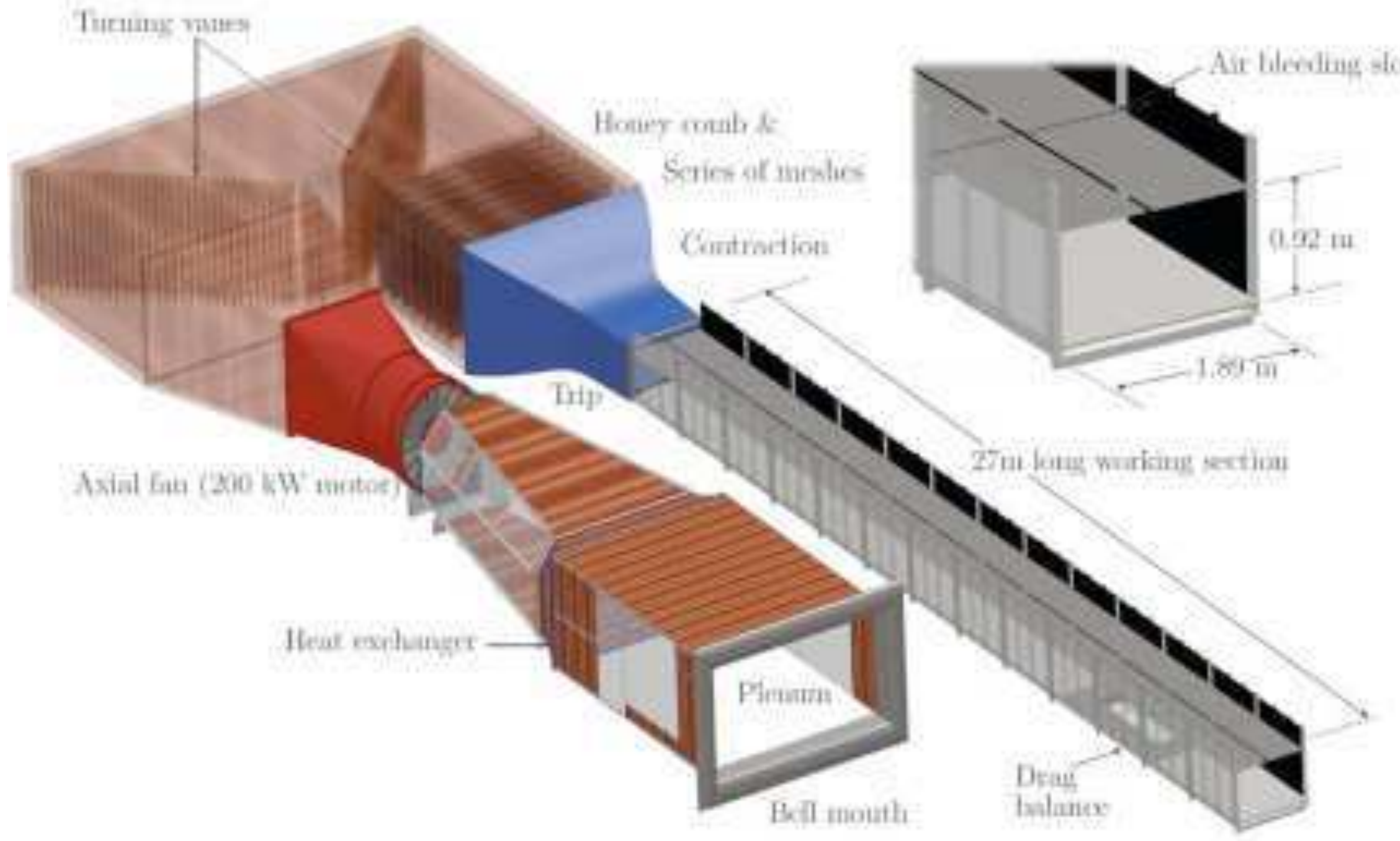
Experimental facilities

Large
Cavitation
Channel (LCC)
Memphis

http://www.memphis.edu/herffhighlights/fall11/images/LCC_INT.jpg

Experimental facilities

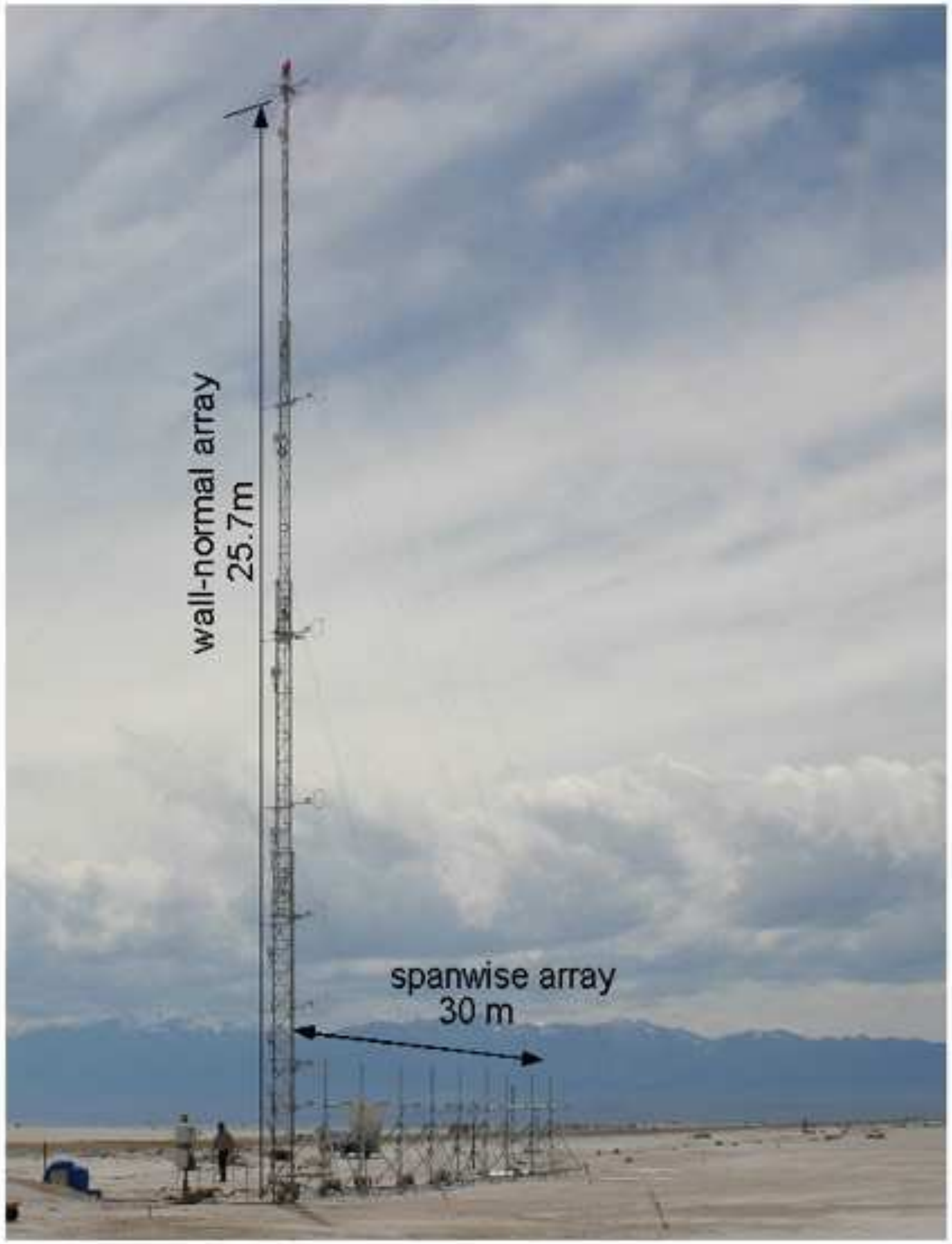
Melbourne
University
High
Reynolds
Number
Wind
Tunnel




SLTEST
Utah
desert

Experimental facilities

SLTEST
Utah
desert



Experimental facilities

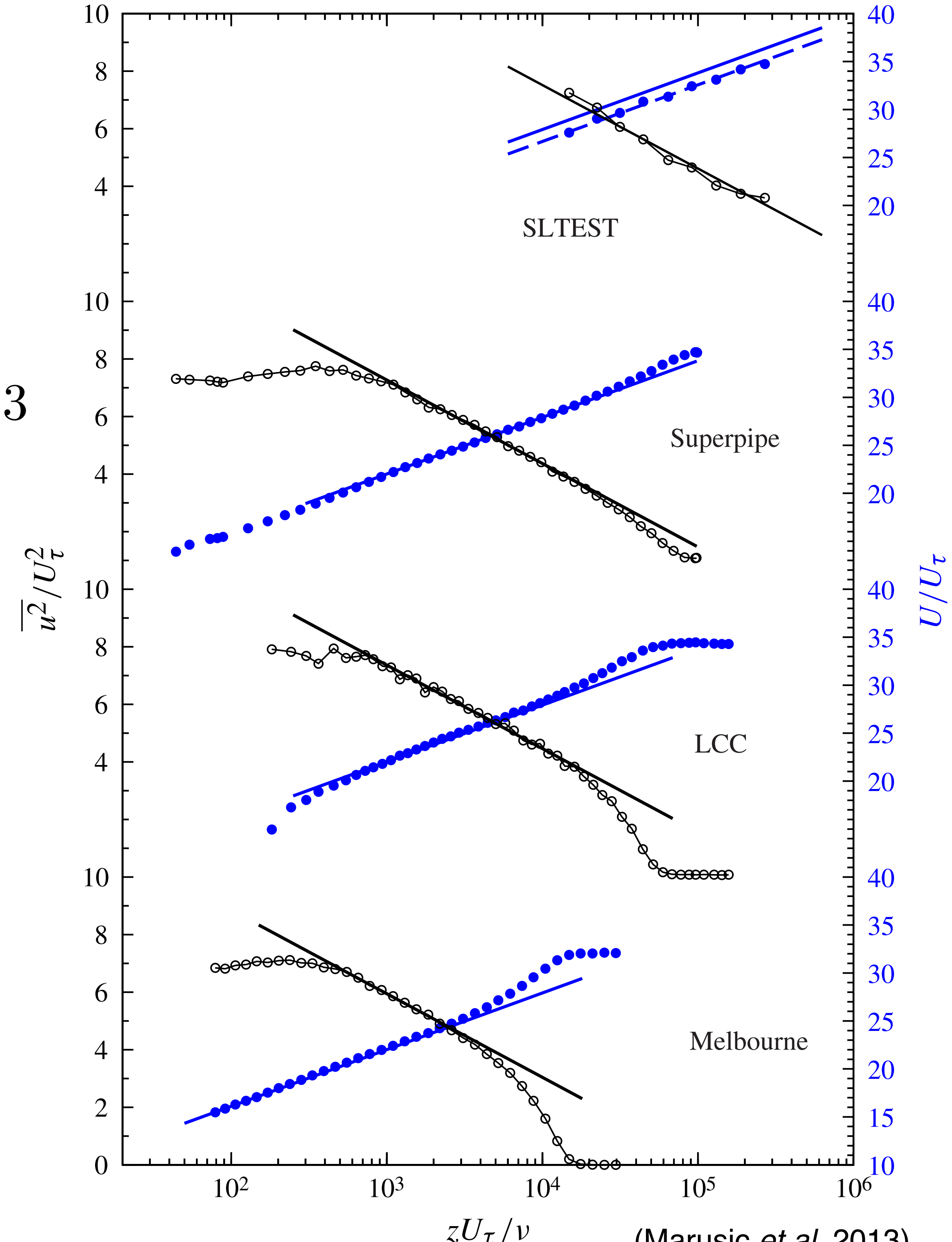
SLTEST
Utah

Log law - inner scaling

Boundary layer (Mean velocity profile given by blue curves)

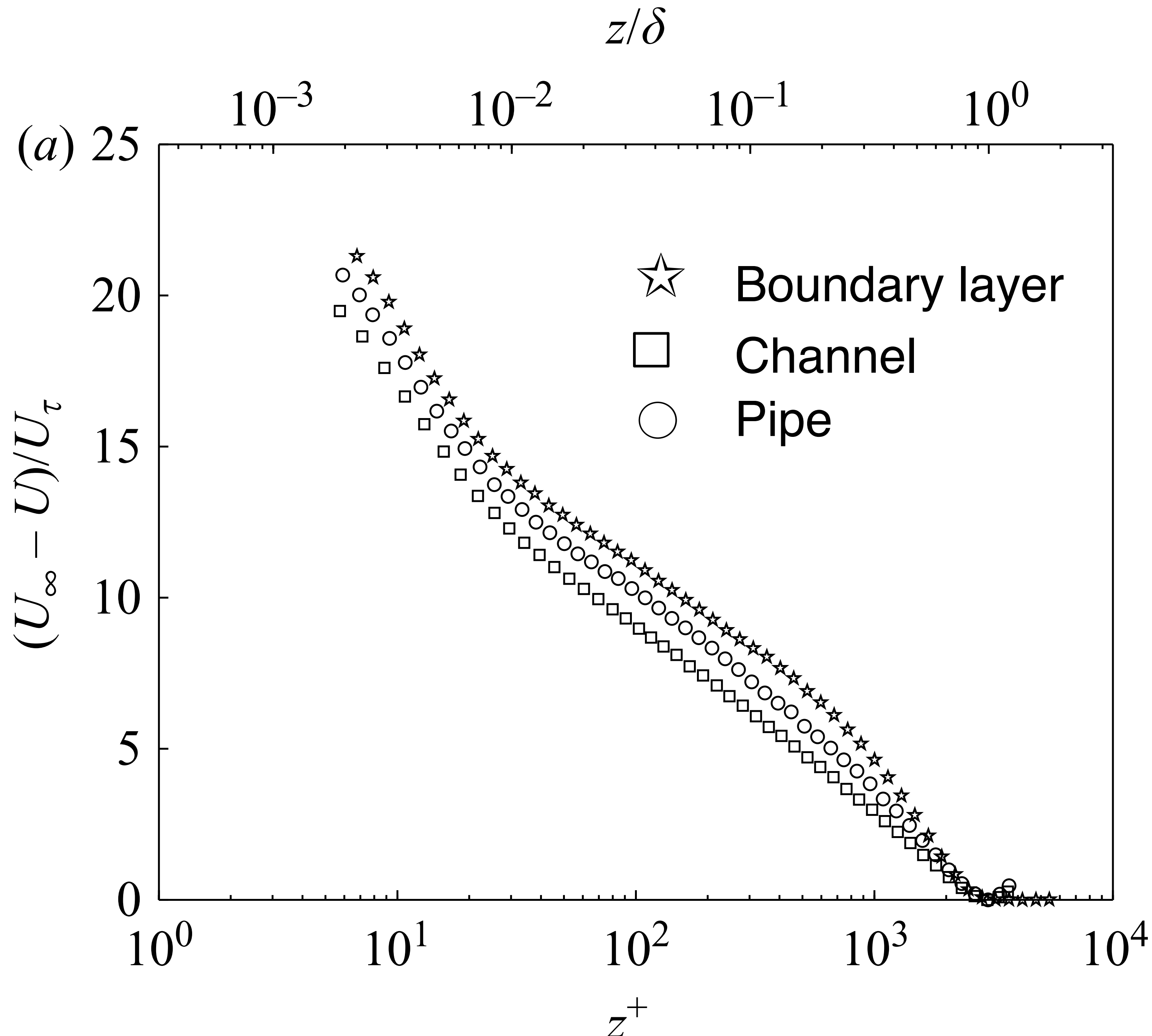
$$\kappa = 0.39, \quad A = 4.3$$

$$\overline{u}^{+} = \frac{1}{\kappa}\log(z^{+}) + A$$



(Marusic *et al.* 2013)

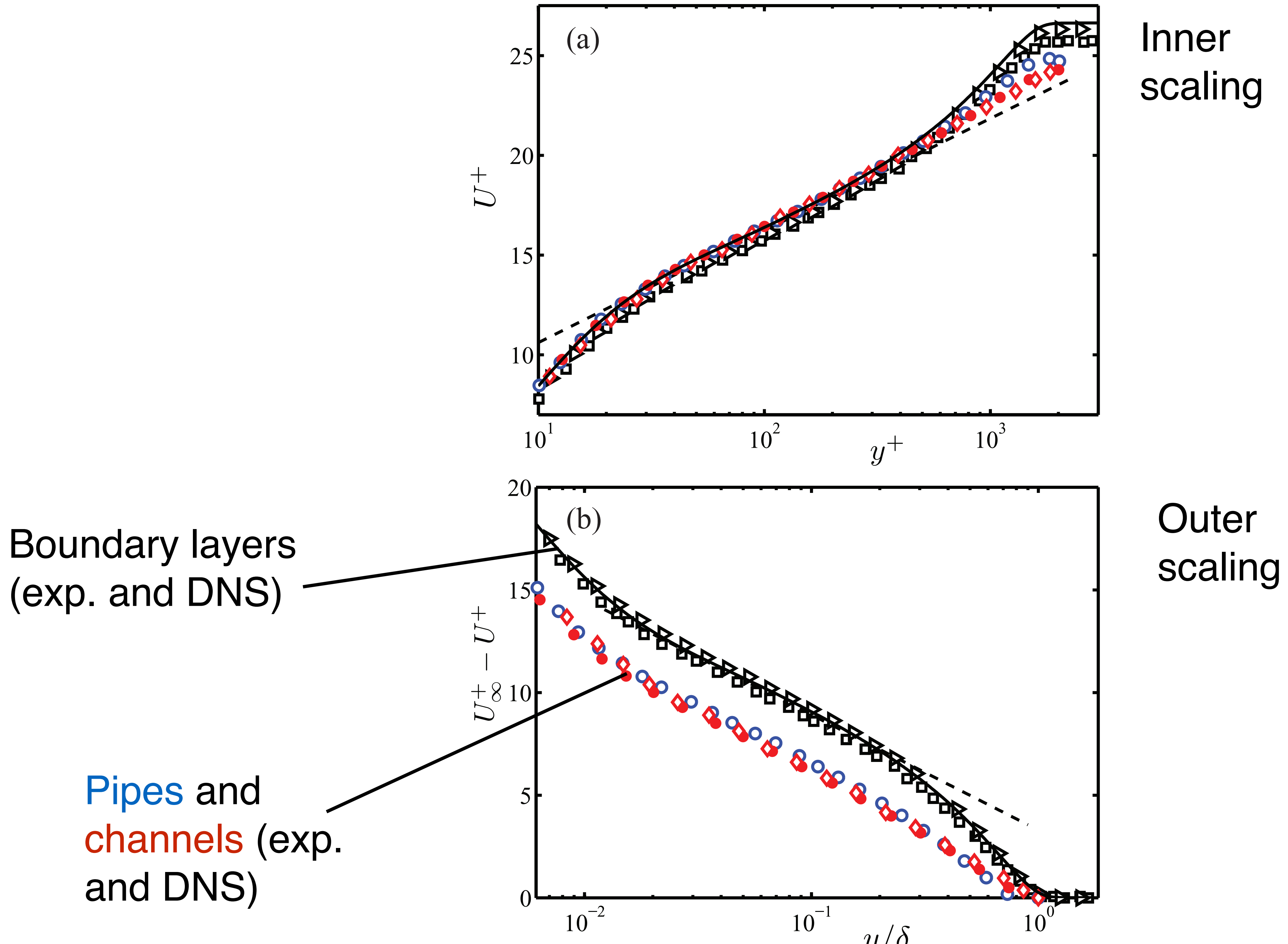
Log law - outer scaling



(Monty *et al.* 2009)

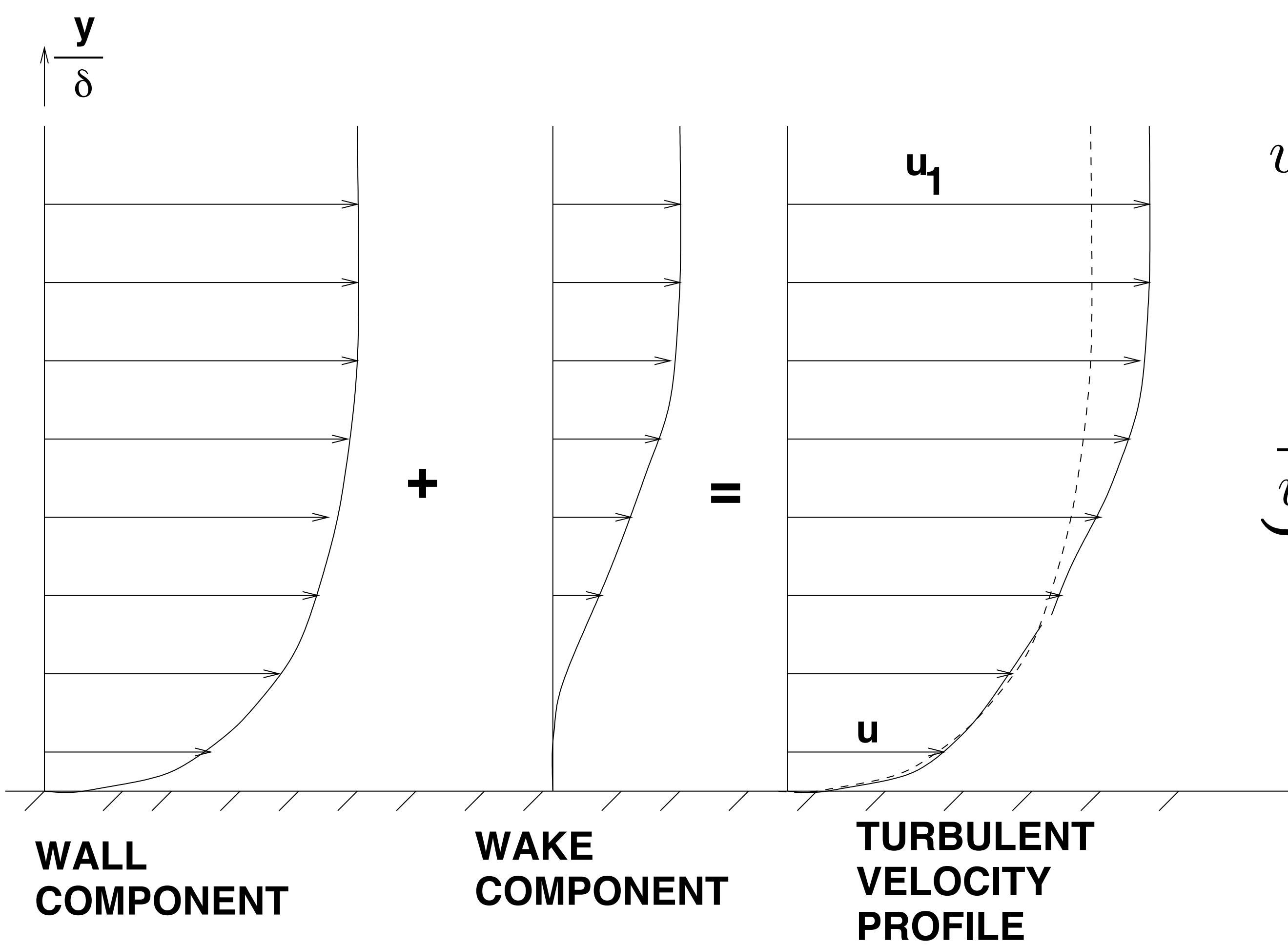
No collapse because effect of geometry in the outer layer (when y = O(δ)) is significant. But log layers are apparent in all.

Log law - inner and outer scaling. Experiments and direct numerical simulation (DNS)



(Sillero *et al.* 2014)

Composite profile



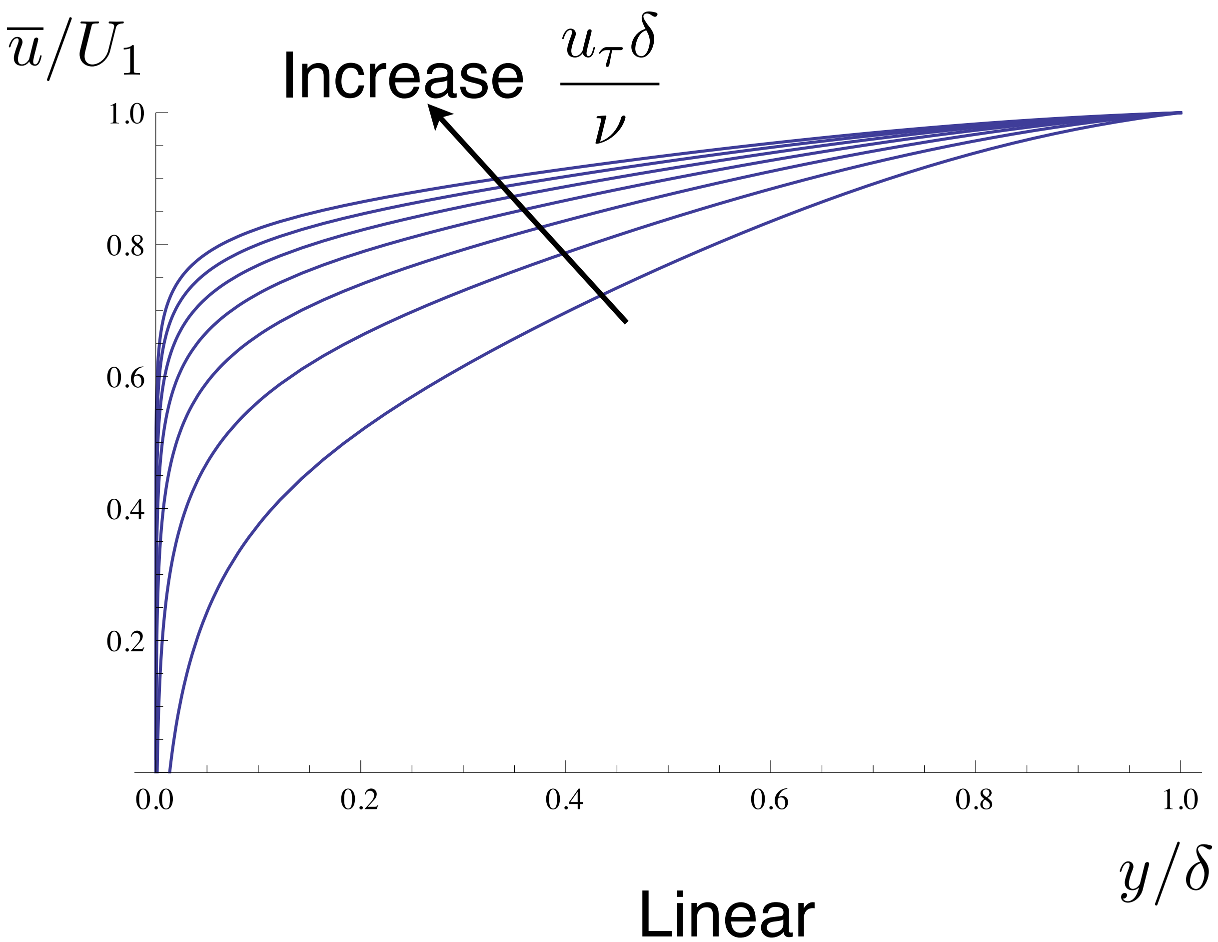
Functional form is from curve fit

$$w\left(\frac{y}{\delta}\right) = 1 - \cos\left(\pi\frac{y}{\delta}\right)$$

$$\underbrace{\frac{u}{u_\tau}}_{} = \underbrace{\frac{1}{\kappa} ln\left[\left(\frac{y u_\tau}{\nu}\right)\right] + A}_{\text{Law of the wall}} + \underbrace{\frac{\Pi}{\kappa} w\left(\frac{y}{\delta}\right)}_{\text{Law of the wake}}$$

For zero pressure gradient turbulent boundary layers: $\Pi \approx 0.55$

Composite profile



Functional form is from curve fit

$$w\left(\frac{y}{\delta}\right) = 1 - \cos\left(\pi\frac{y}{\delta}\right)$$

$$\underbrace{\frac{u}{u_\tau}} = \underbrace{\frac{1}{\kappa} ln\left[\left(\frac{yu_\tau}{\nu}\right)\right] + A}_{\text{Law of the wall}} + \underbrace{\frac{\Pi}{\kappa} w\left(\frac{y}{\delta}\right)}_{\text{Law of the wake}}$$

For zero pressure gradient turbulent boundary layers: $\Pi \approx 0.55$

Resistance/Friction law

Relates Reynolds number and skin-friction coefficient:

Inner

$$\frac{\overline{u}(y)}{u_\tau} = \frac{1}{\kappa}\log(\frac{y u_\tau}{\nu}) + A$$

Outer

$$\frac{\overline{u}(\delta) - \overline{u}(y)}{u_\tau} = -\frac{1}{\kappa}\log(\frac{y}{\delta}) + B$$

Eliminate: $\overline{u}(y)$ $\longrightarrow$ $\dfrac{\overline{u}(\delta)}{u_\tau} = \dfrac{1}{\kappa}\log(\dfrac{u_\tau \delta}{\nu}) + A + B$

Use:

$$U_1 = \overline{u}(\delta),$$

$$Re = \frac{U_1 \delta}{\nu},$$

$$\frac{c_f'}{2} = \frac{\tau_0}{\rho U_1^2} = \frac{u_\tau^2}{U_1^2}$$

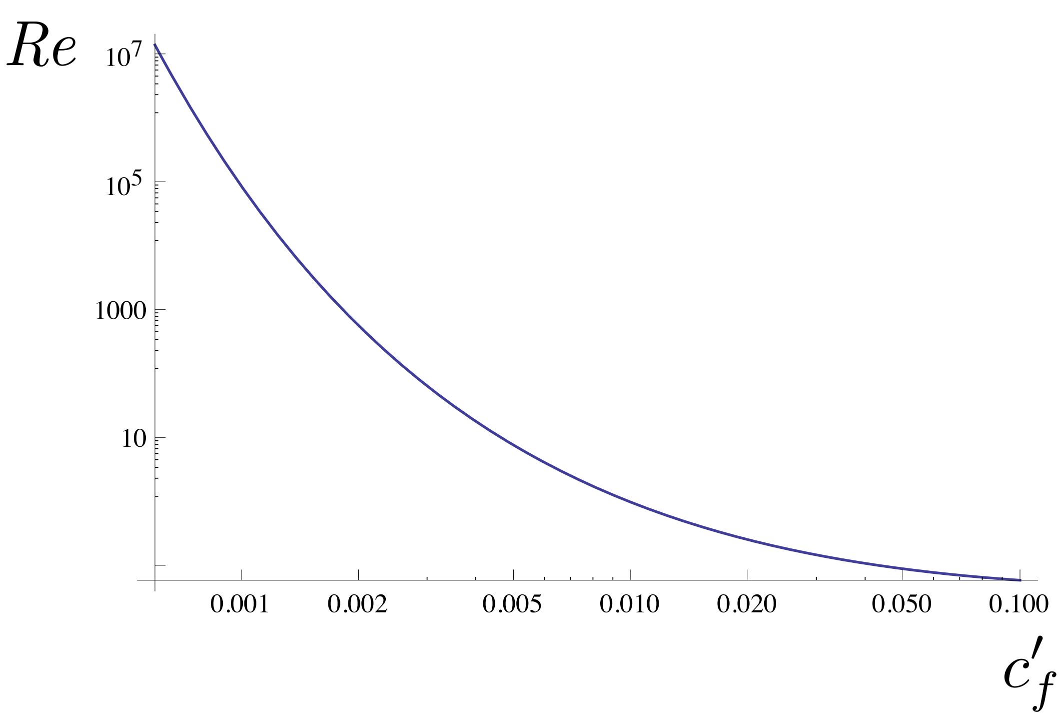
$\longrightarrow$

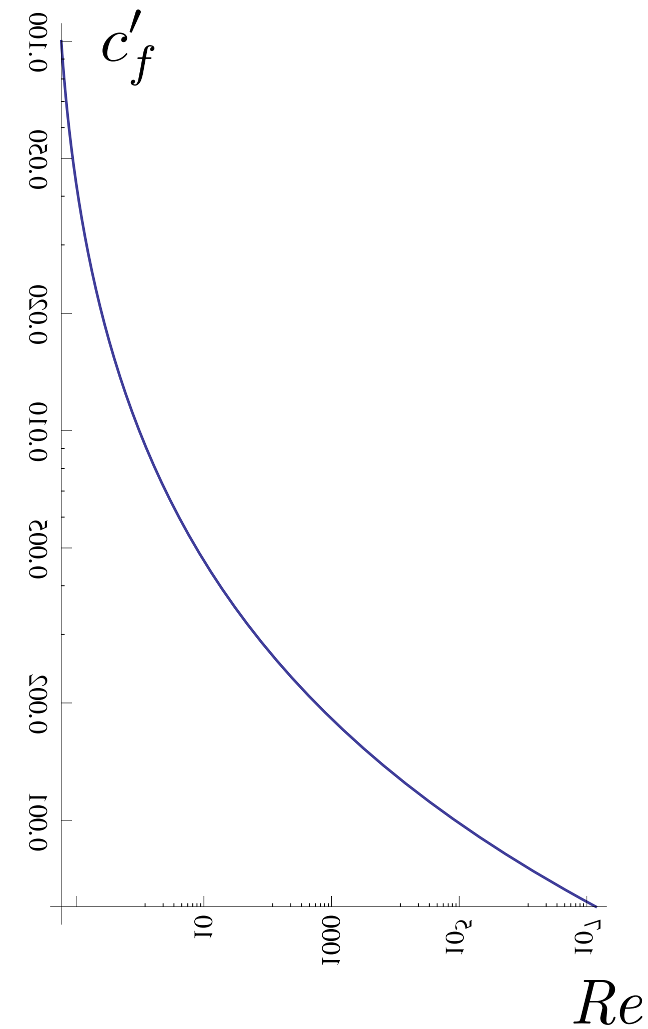
$$\sqrt{2/c_f'} = \frac{1}{\kappa}\log(\sqrt{c_f'/2}\, Re) + A + B$$

Implicit equation for skin-friction coefficient given Reynolds number.

Resistance/Friction law

$$\sqrt{2/c'_f} = \frac{1}{\kappa}\log(\sqrt{c'_f/2}\,Re) + A + B$$





c_f' decreases with increasing Re.
Compare with c_p.

Resistance/Friction law

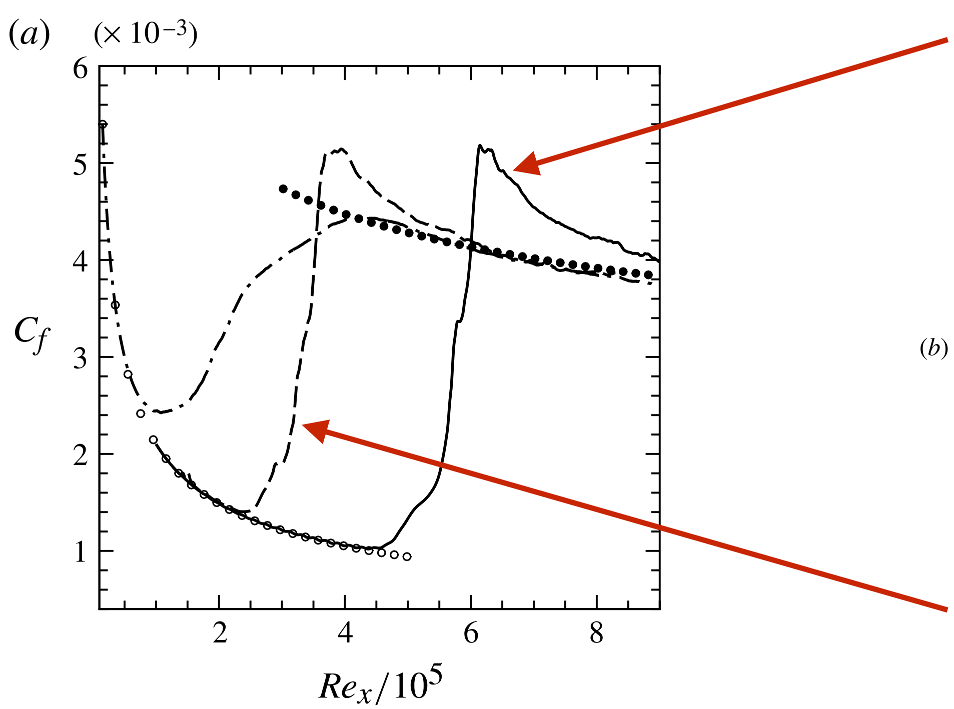
Many versions:

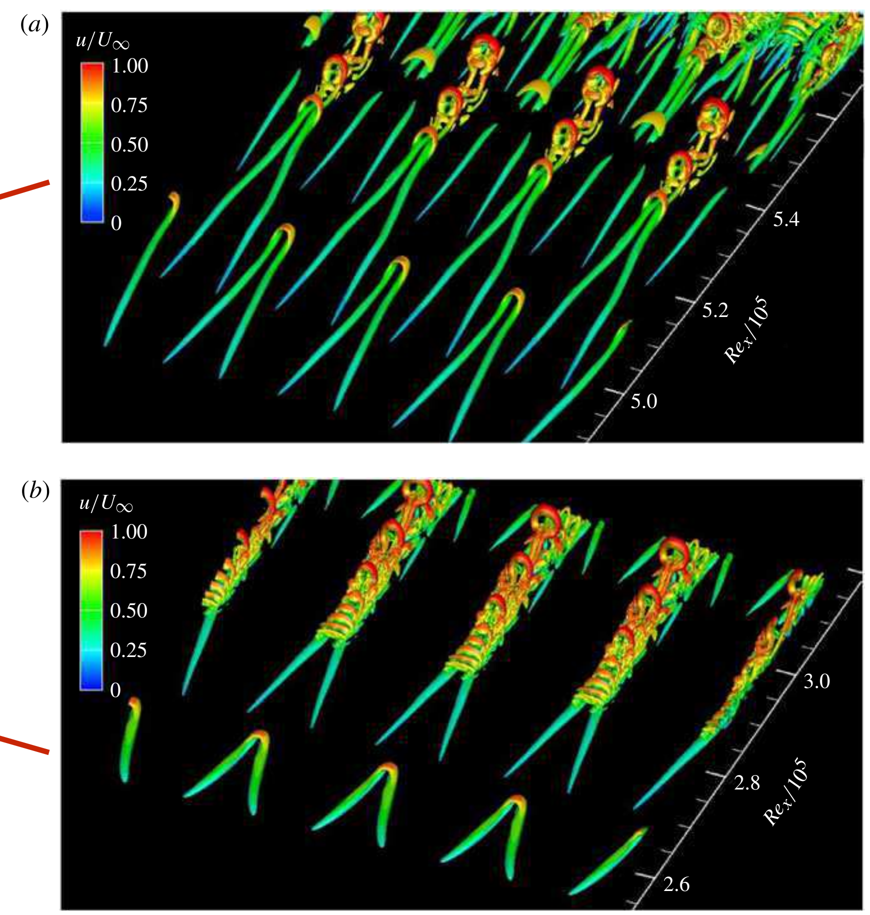
$$c_f'(U_\infty \delta/\nu)$$
$$c_f'(U_{\mathrm{bulk}} \delta/\nu)$$
$$c_f'(u_\tau \delta/\nu)$$
$$c_f'(U_\infty x/\nu)$$
$$c_f'(U_\infty \delta^*/\nu)$$
$$c_f'(U_\infty \theta/\nu)$$
$$\vdots$$

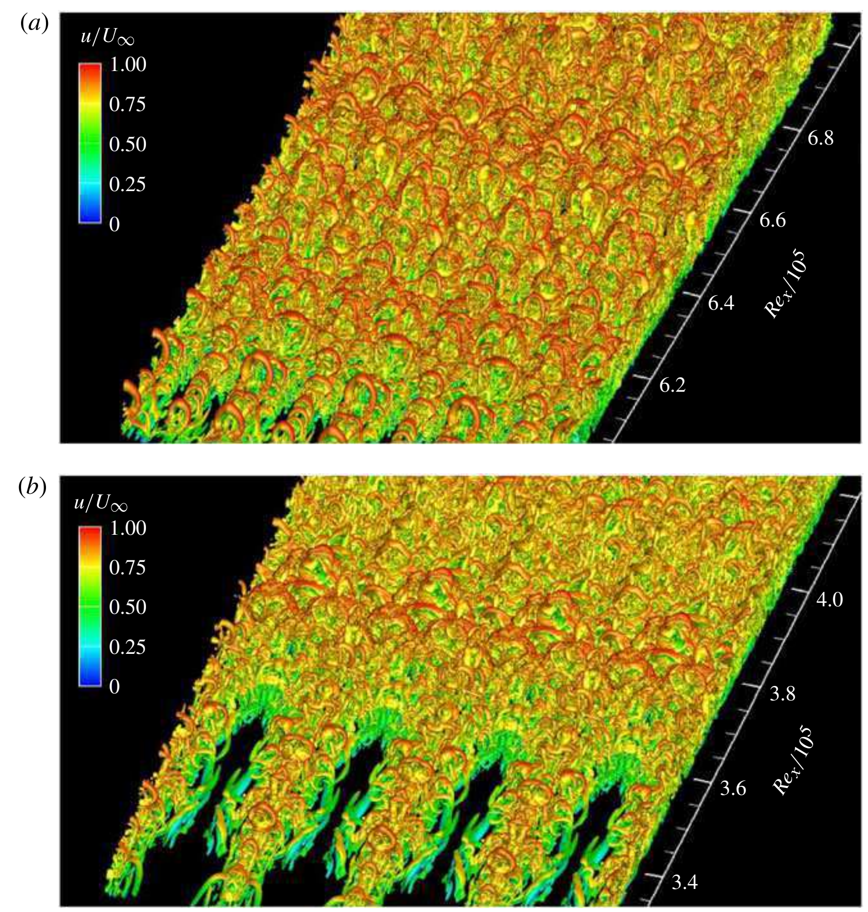
δ could be boundary-layer thickness, half-channel height or pipe radius.

U_∞ could be free-stream or centreline velocity.

Skin friction on zero-pressure-gradient boundary layer through transition







Sayadi, Hamman & Moin (2013)

Future of aircraft aerodynamics

IATA
Technology Roadmap
4th Edition
June 2013

Group	Concept	Technology	Applicability to aircraft program	Fuel Reduction Benefits	Current development status (TRL #)	Availability of technology (calculated)
Aerodynamics	Advanced Wingtip Devices	Wingtip Fence	retrofit	1 to 3%	9	2012
		Blended Winglet / Sharklets	retrofit	3 to 6%	9	2012
		Raked Wingtip	retrofit	3 to 6%	9	2012
		Split Winglets with scimitar tips	retrofit	2 to 6%	7	2022
		Spiroid Wingtip	after 2020	2 to 6%	7	2022
	High Lift Devices	High-Lift / Low-Noise Devices	after 2020	1 to 3%	4	2026
		Variable Camber Trailing Edge	before 2020	1 to 2%	9	2012
		Dropped Spoiler	before 2020	1 to 2%	9	2012
		Hinge-less Flap	after 2030	1 to 2%	3	2027
	Drag Reduction Coatings	Drag reduction coatings	retrofit	< 1%	9	2012
		Turbulent Flow Drag Coatings (Riblets)	retrofit	1%	8	2015
		Aircraft Graphic Films	retrofit	1%	9	2012
	Natural Laminar Flow		after 2020	5 to 10%	7	2022
	Hybrid Laminar Flow		after 2020	10 to 15%	7	2022
	Variable Camber with existing control surfaces		before 2020	1 to 3%	8	2015
	Variable Camber with new control surfaces		after 2020	1 to 5%	5	2024

Future of aircraft aerodynamics

(Bechert & Bartenwerfer 1989)

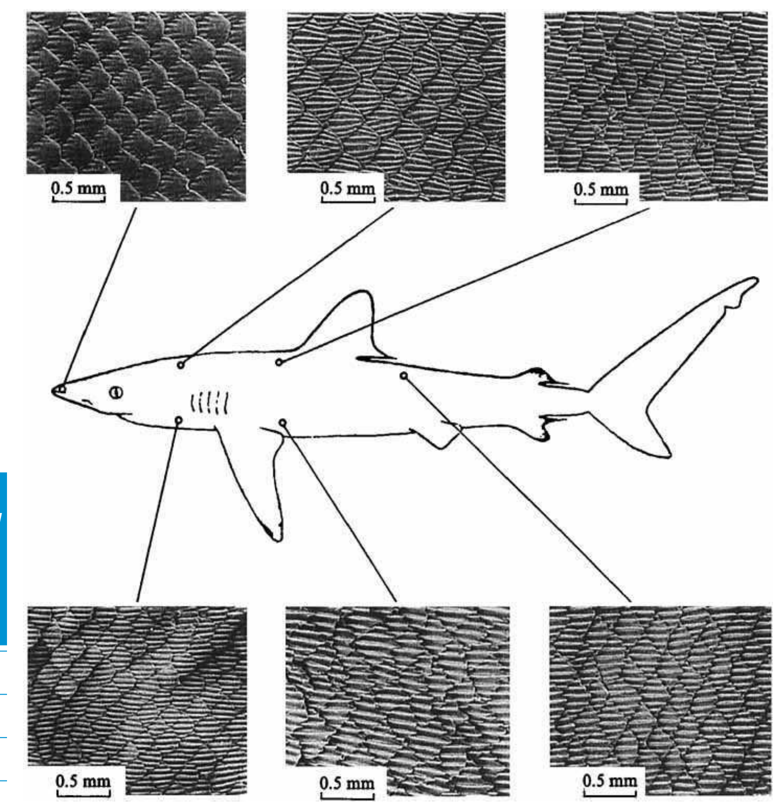


FIGURE 1. Silky shark, *Carcharhinus falciformis*, 2.27 m length.

Group	Concept	Technology	Applicability to aircraft program			
Aerodynamics	Advanced Wingtip Devices	Wingtip Fence	retrofit			
		Blended Winglet / Sharklets	retrofit			
		Raked Wingtip	retrofit			
		Split Winglets with scimitar tips	retrofit			
		Spiroid Wingtip	after 2020	2 to 6%	7	2022
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	Variable Camber with new control surfaces		after 2020	1 to 5%	5	2024

Melbourne School of Engineering
MCEN90018 Advanced Fluid Dynamics

Lectures RG01 and RG02: Roughness 1 and 2
10 and 12 May 2016

Examples of rough turbulent boundary layers

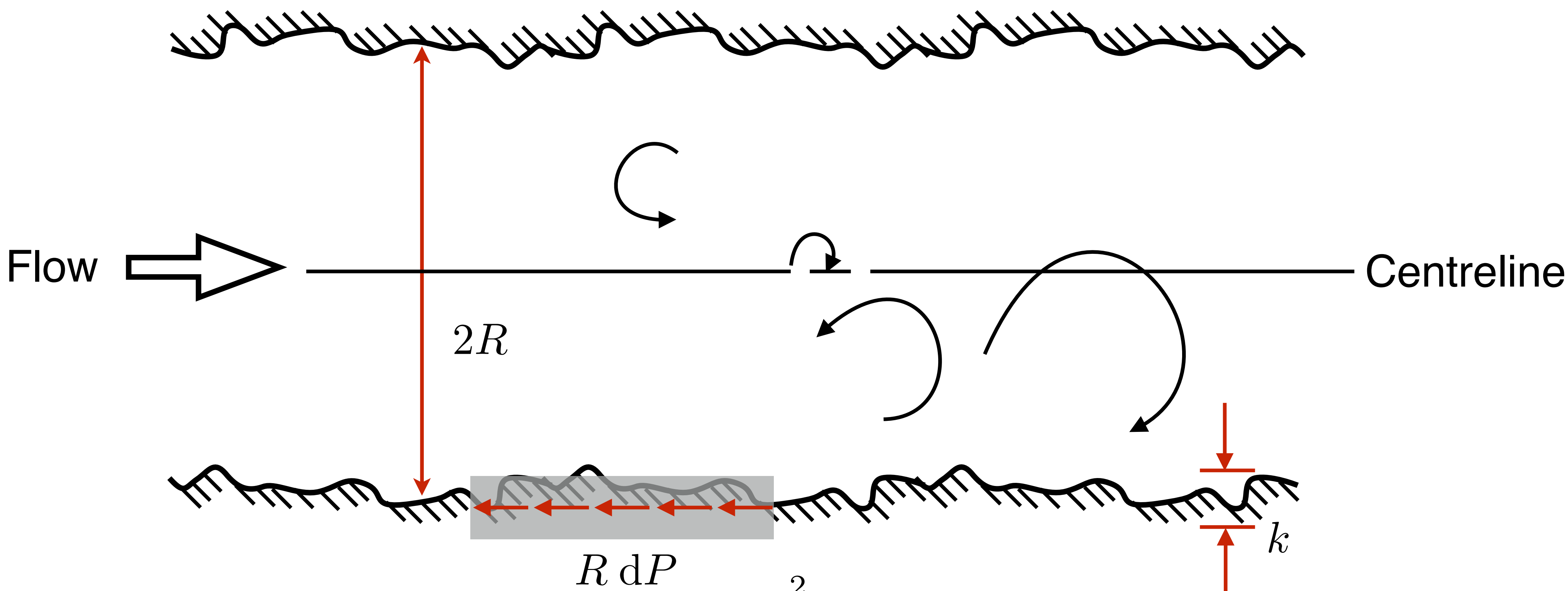
http://en.wikipedia.org/wiki/File:Typhoon3.jpg

http://www.sciencemediacentre.co.nz/2012/01/30/underwater-sound-increases-mussel-biofouling/

http://commons.wikimedia.org/wiki/File:Ocean_waves.jpg

http://en.wikipedia.org/wiki/List_of_tallest_buildings_in_Melbourne

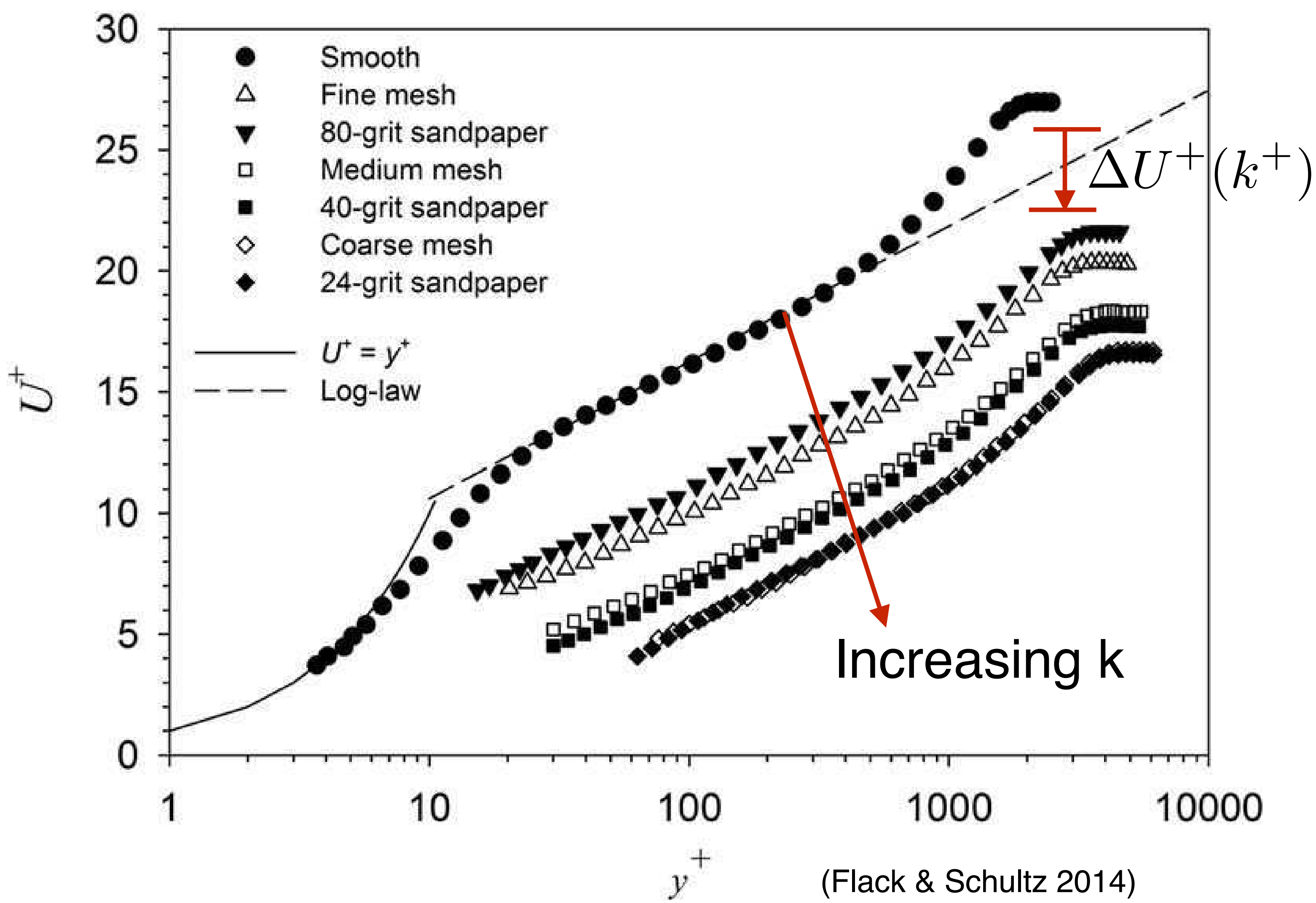
Rough pipe flow



$$\tau_0 = -\frac{R}{2}\frac{\mathrm{d}P}{\mathrm{d}x} = \rho u_\tau^2$$

Shear stress = (Viscous + pressure drag force)/pipe area.

Roughness retards the flow



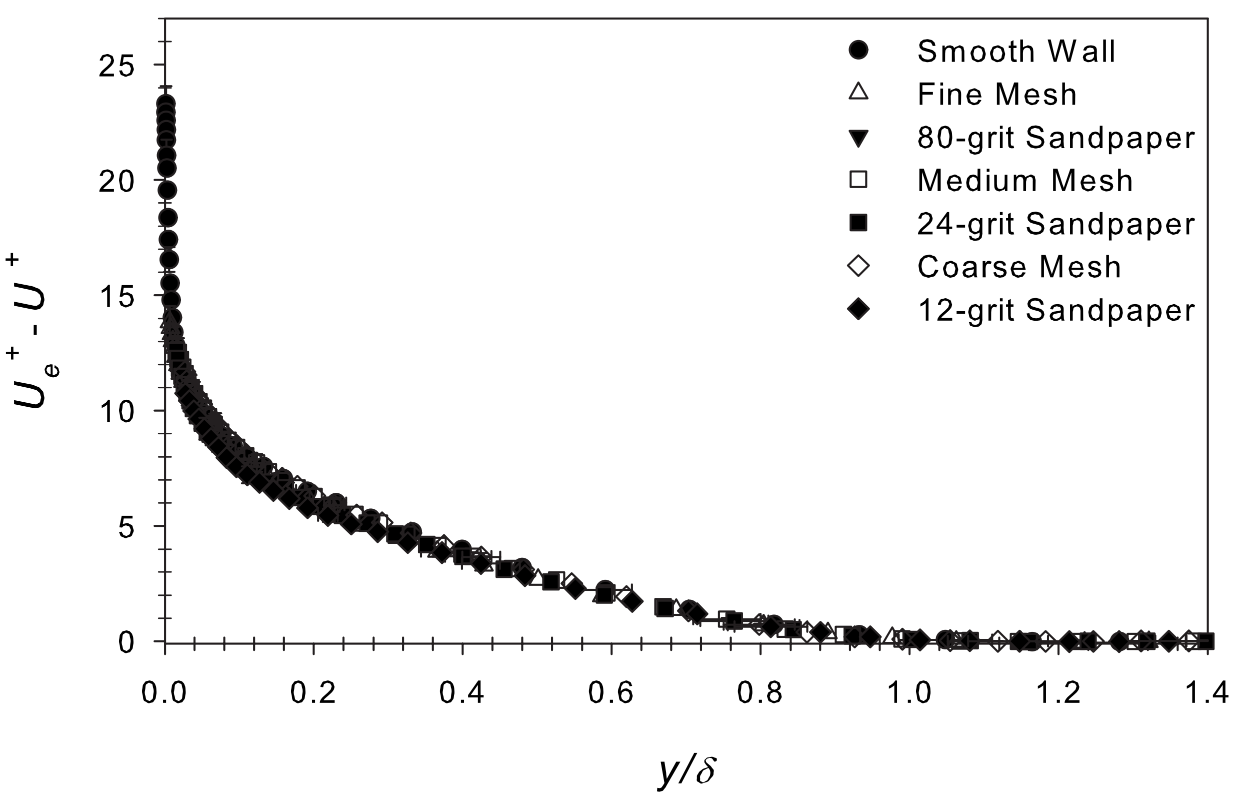
(Flack & Schultz 2014)

(Roughness function)

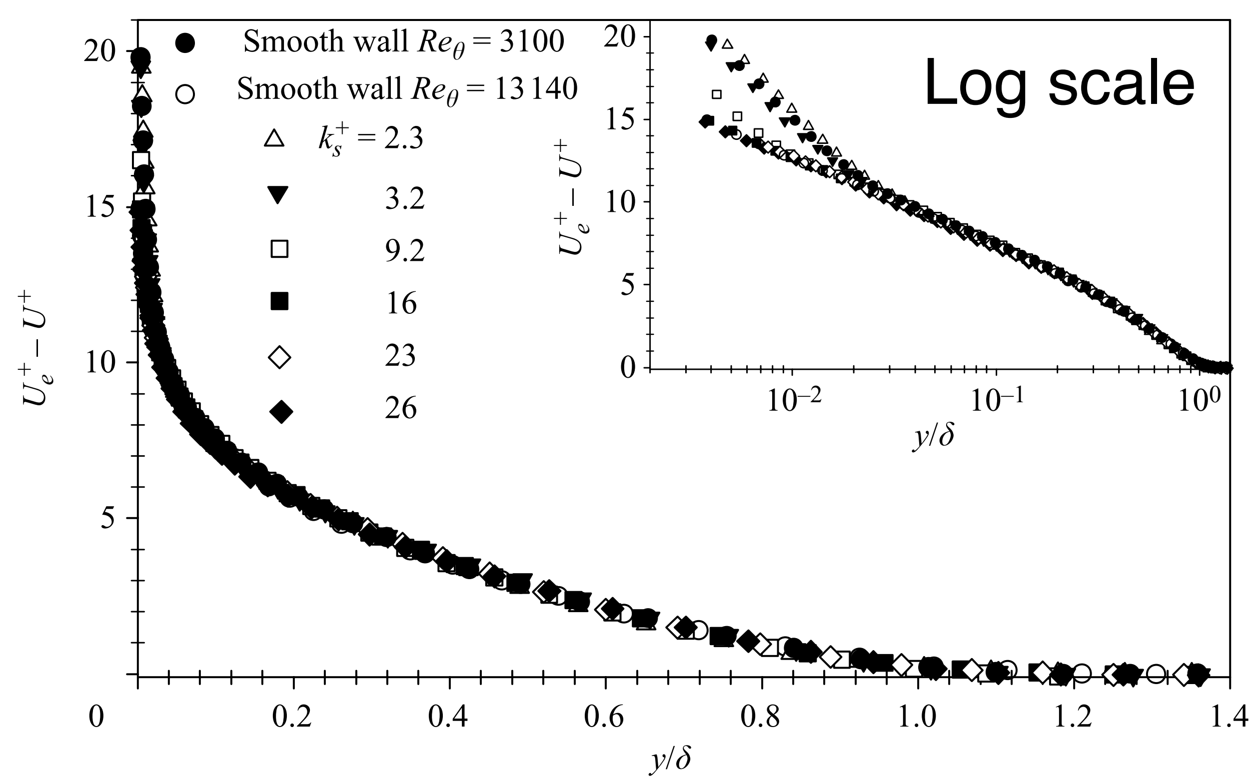
Parallel shift down to log region.

Actually, these are boundary-layer data, but theory is the same for pipe.

Roughness does not affect outer region



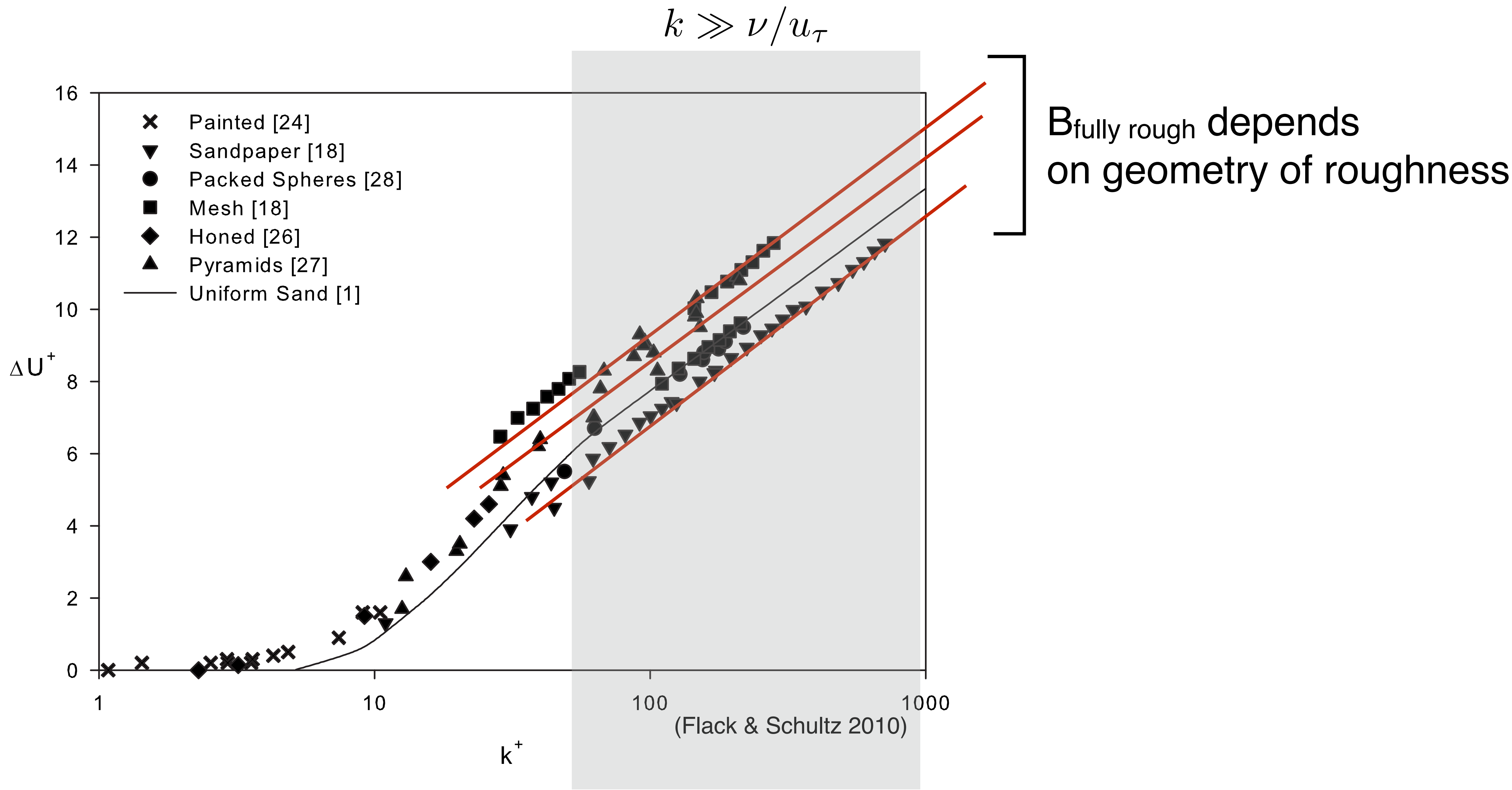
(Flack & Schultz 2010)



(Flack, Schultz and Connelly 2007)

$$\Rightarrow \frac{U_{CL} - \overline{u}}{u_\tau} = F(y/R)$$

Roughness function versus k⁺

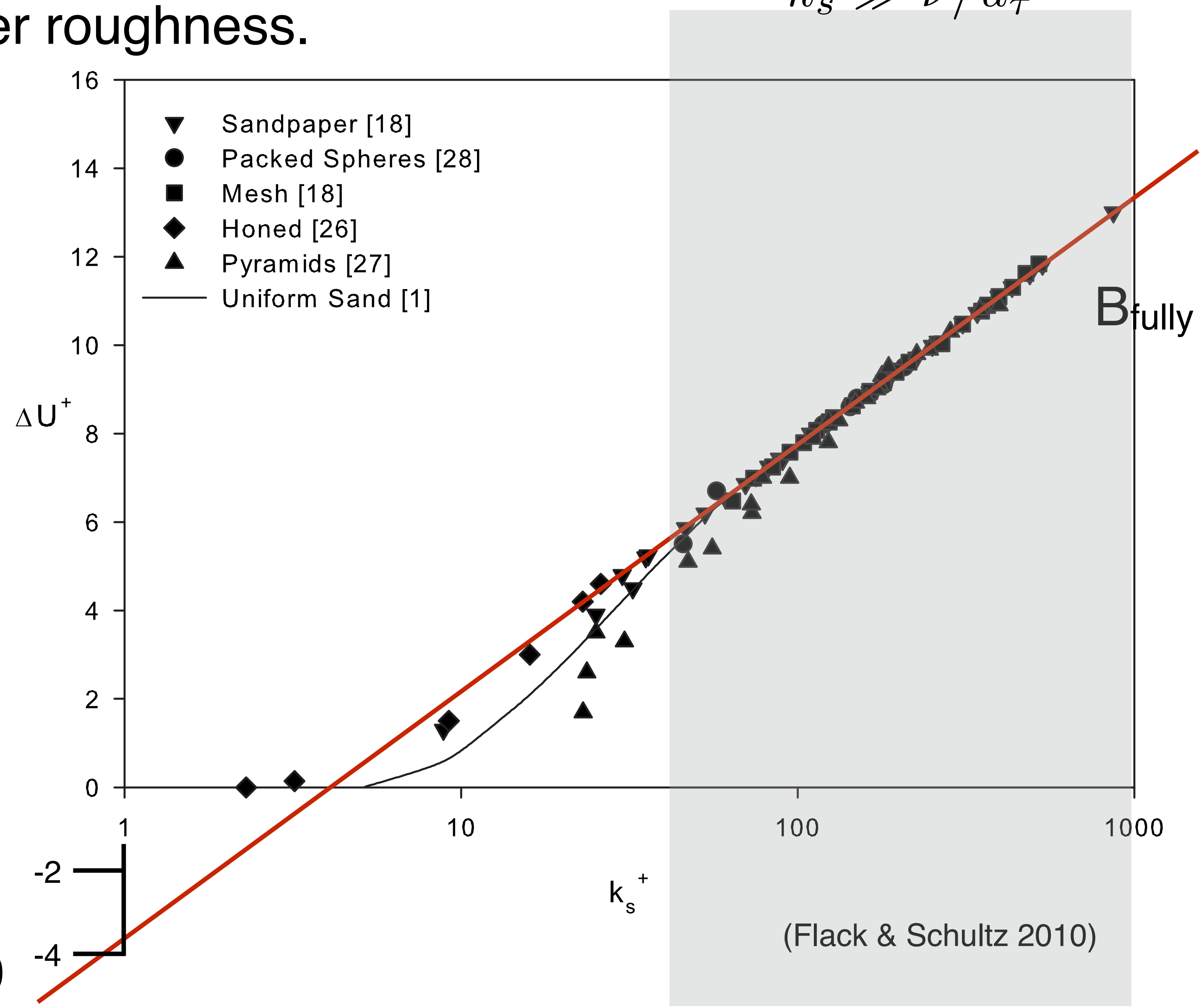


$$\Delta U^+ = \frac{1}{\kappa}\log k^+ - B_{\text{fully rough}} + A$$

Roughness function versus k_s^+

Set k_s = const. × k to get equivalent sandpaper roughness.

$k_s \gg \nu/u_\tau$



$B_{\text{fully rough}}$ (sandpaper) = 8.5

Intercept
≈ −8.5+5.0
= −3.5

$$\Delta U^+ = \frac{1}{\kappa} \log k_s^+ - 8.5 + A$$

Common k_s

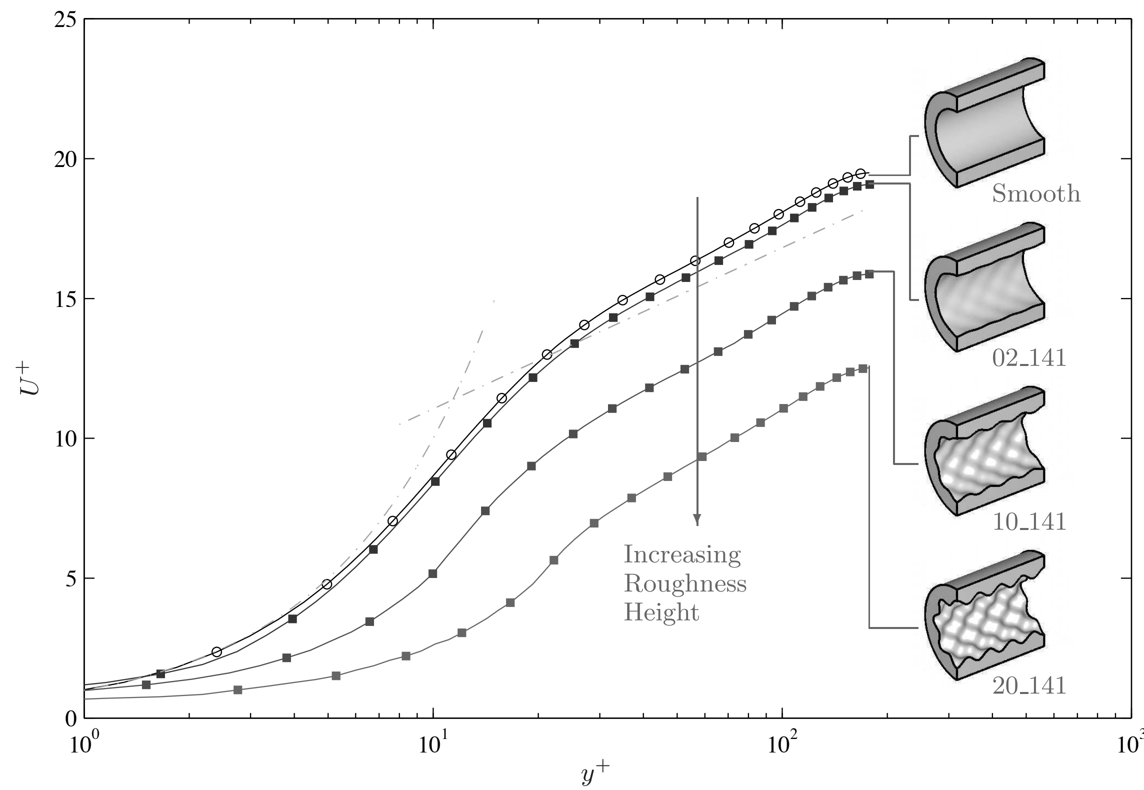
TABLE 6-1
Roughness of commercial pipes

Type of pipe	Equivalent roughness k_s, ft
Glass	0.000001
Drawn tubing	0.000005
Steel wrought iron	0.00015
Asphalted cast iron	0.0004
Galvanized iron, new	0.0005
3 years old	0.0009
Cast iron	0.00085
Wood stave	0.0006–0.003
Concrete	0.001–0.01
Riveted steel	0.003–0.03

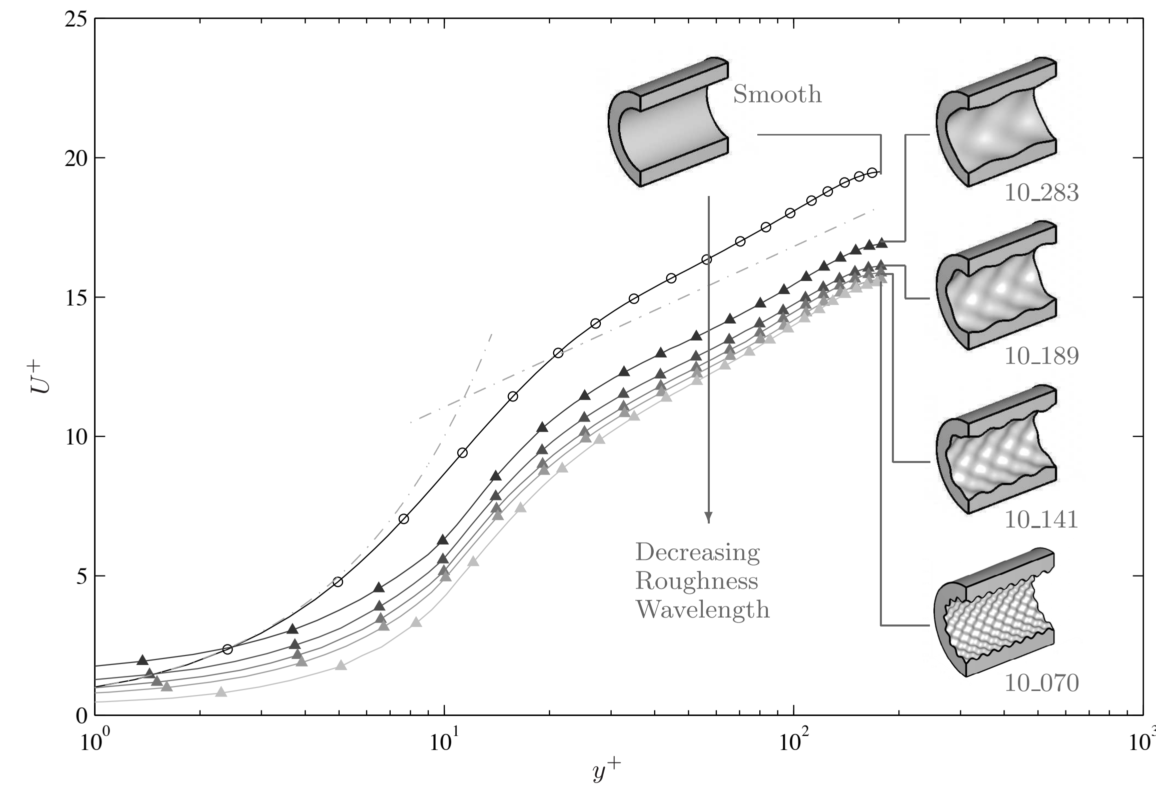
(White)

Finding k_s is not trivial

Varying height

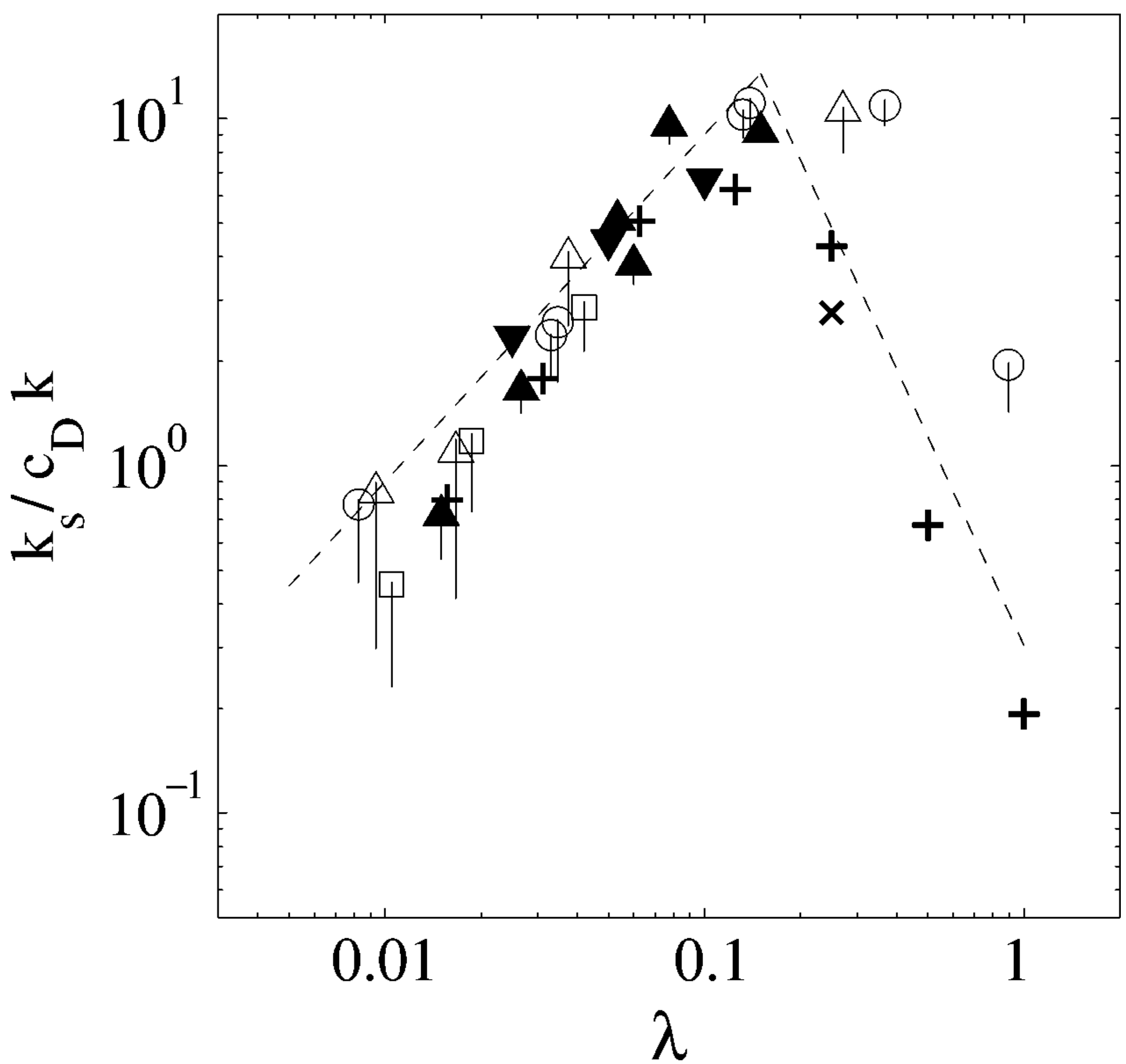


Varying wavelength



(Chan *et al.* 2015)

Finding k_s is not trivial

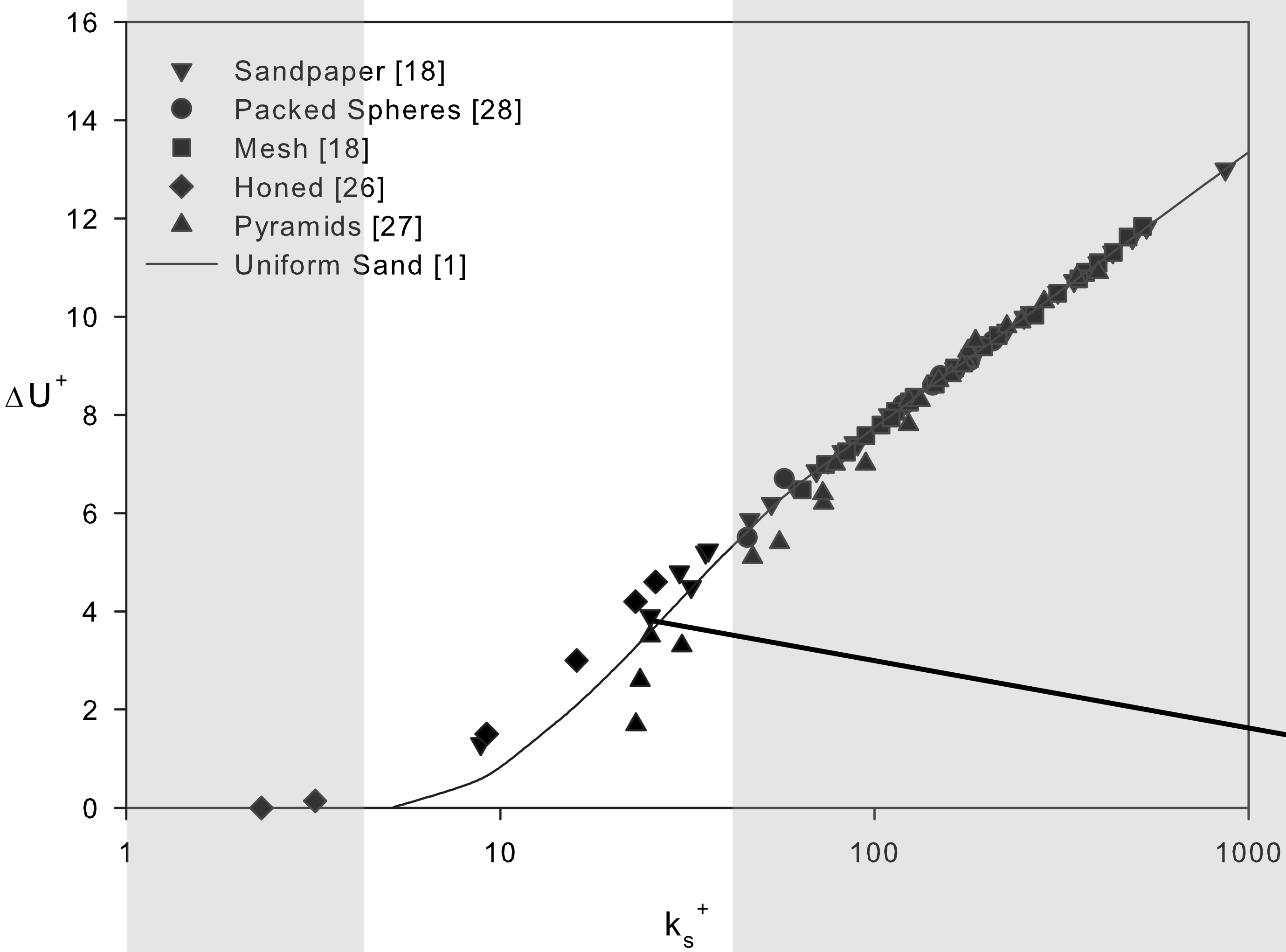


~ height/wavelength

Sandpaper roughness regimes

$k_s < 5\nu/u_\tau$

$70\nu/u_\tau < k_s$



(Flack & Schultz 2010)

Smooth

Transitionally rough

Fully rough

Scatter shows roughness transition depends on roughness geometry

Composite profiles

$$\overline{u}^+ = \frac{1}{\kappa}\log y^+ + A - \Delta U^+ + \frac{\Pi}{\kappa}w(y/\delta)$$

$$\overline{u}^+ = \frac{1}{\kappa}\log(y/k_s) + 8.5 + \frac{\Pi}{\kappa}w(y/\delta)$$

Engineering

Fully rough

$$\overline{u}^+ = \frac{1}{\kappa}\log(y/y_0) + \frac{\Pi}{\kappa}w(y/\delta)$$

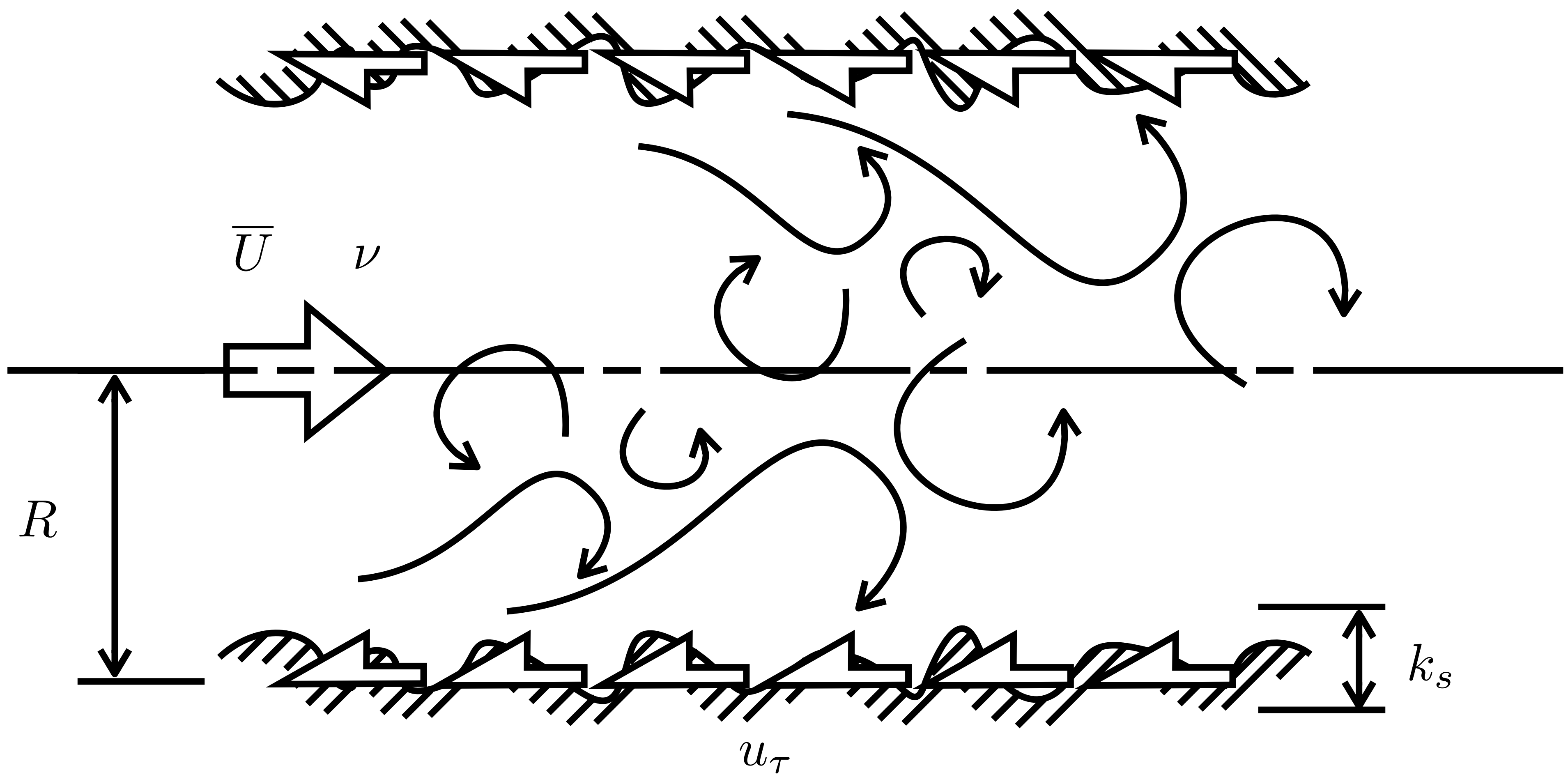
$$\frac{y_0}{k_s} = \mathrm{e}^{-\kappa 8.5}$$

$$\approx 0.033$$

Meteorology

$$w(1) = 2$$

Moody chart



Moody chart

$\overline{U}$ ν

R

k_s

u_τ

5 quantities – 2 units = 3 dimensionless numbers

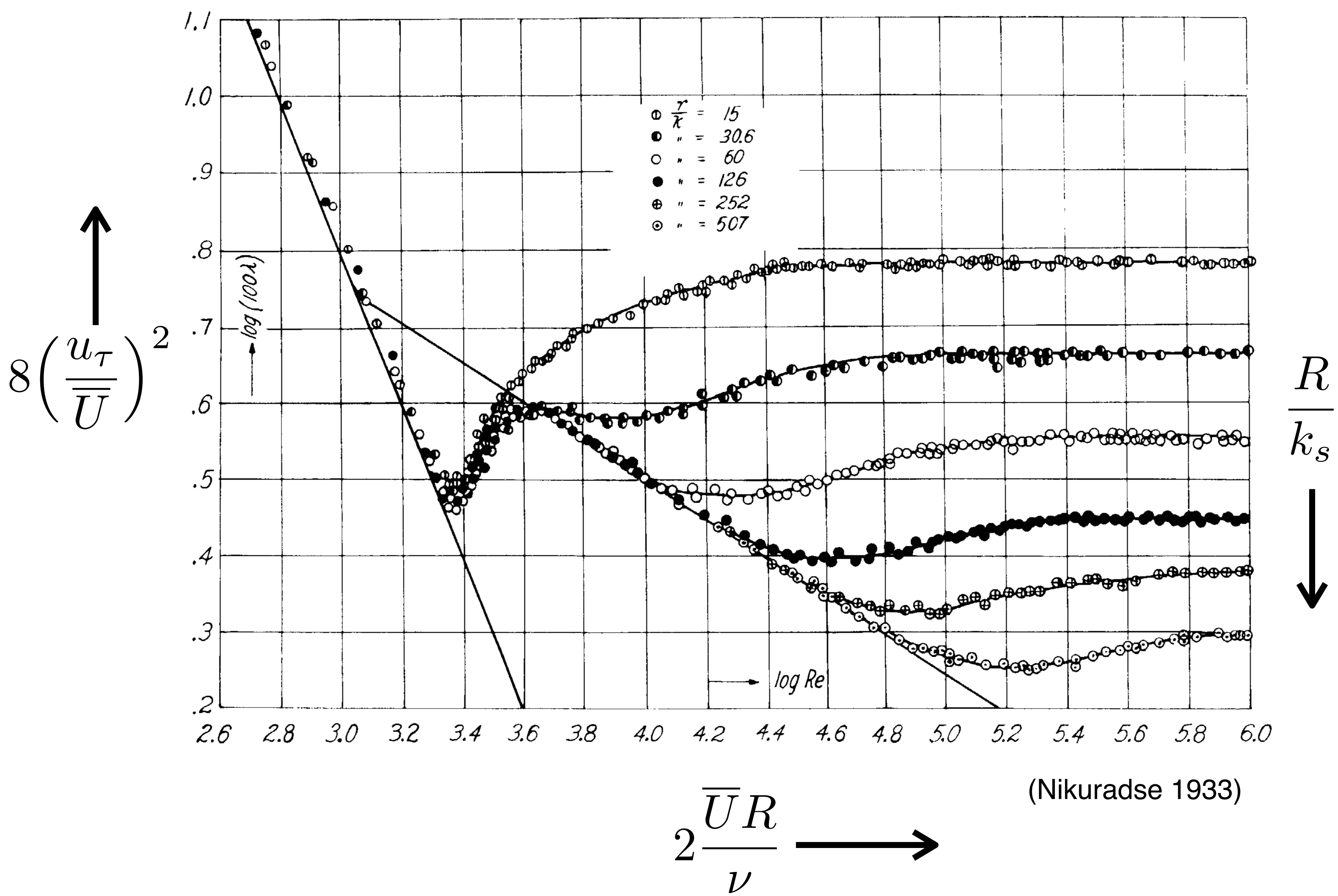
Moody chart

$$8\left(\frac{u_\tau}{\overline{U}}\right)^2 = F\left(2\frac{\overline{U}R}{\nu}, \frac{R}{k_s}\right)$$

5 quantities – 2 units = 3 dimensionless numbers

Moody chart

$$8\left(\frac{u_\tau}{\overline{U}}\right)^2 = F\left(2\frac{\overline{U}R}{\nu}, \frac{R}{k_s}\right)$$



(Nikuradse 1933)

Moody chart

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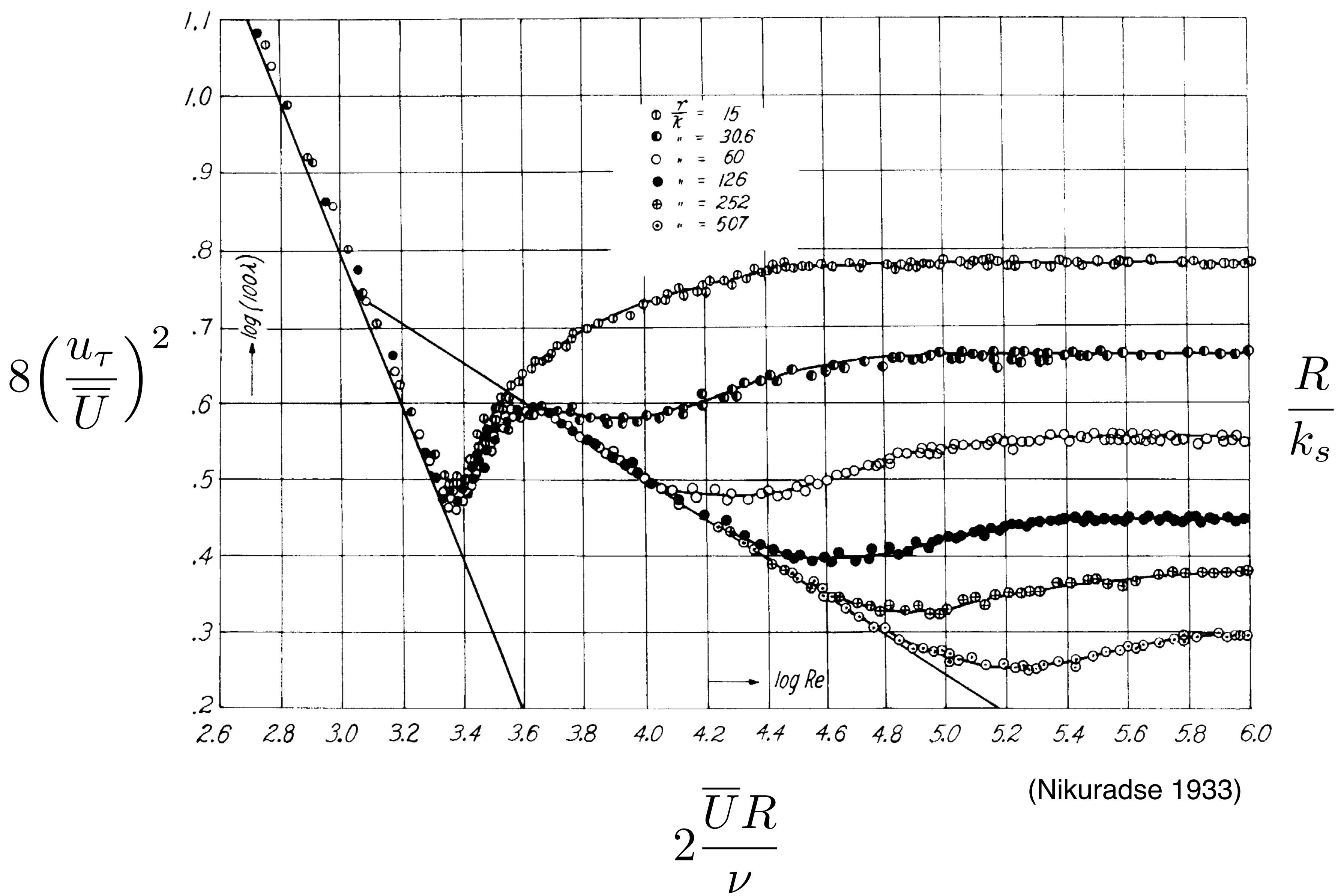
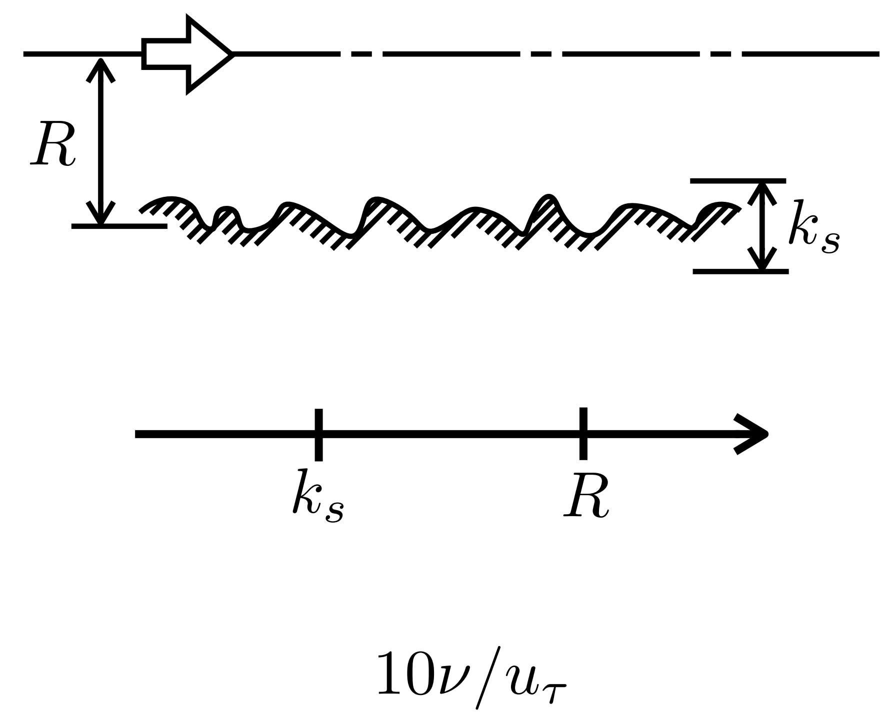


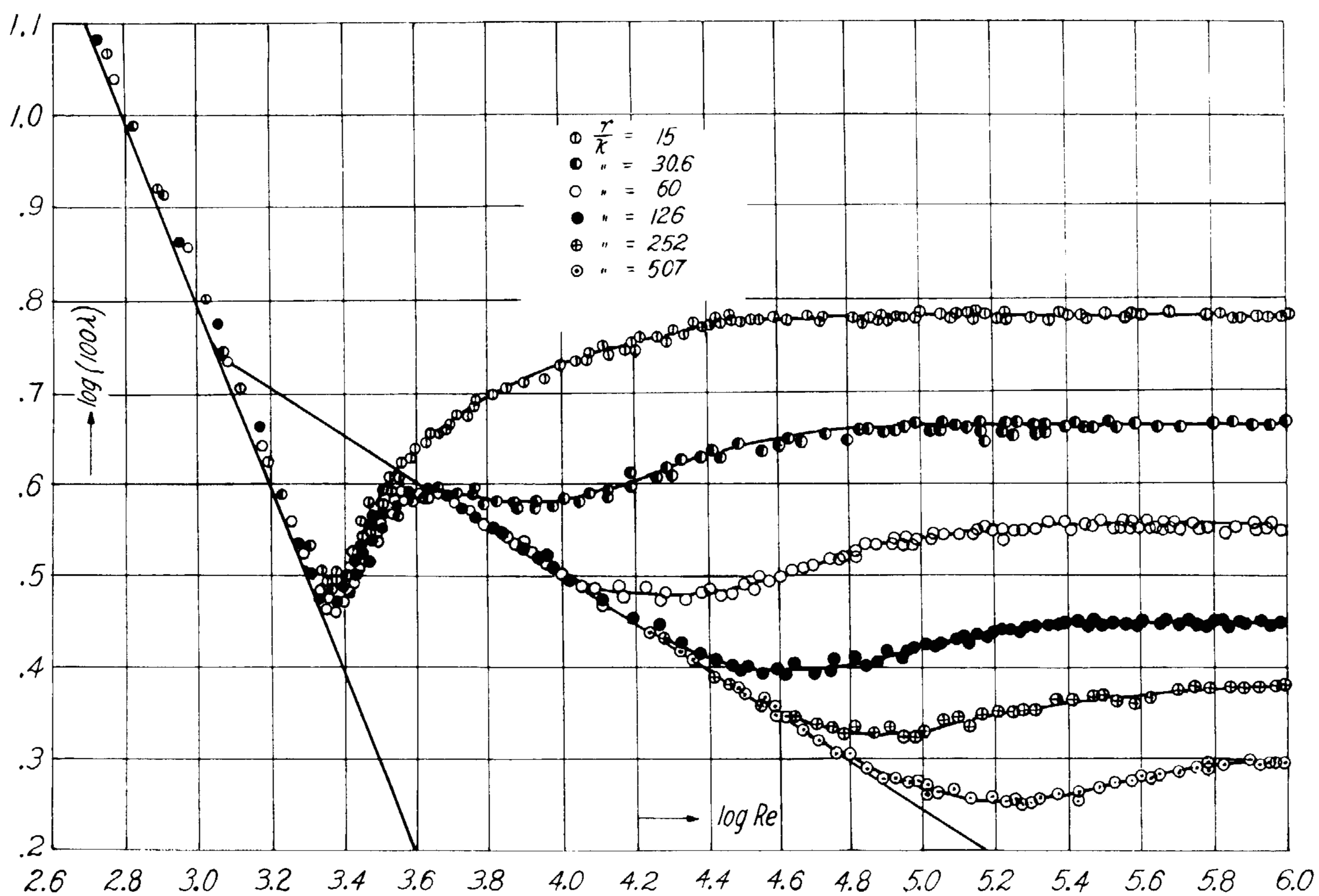
(Nikuradse 1933)



Moody chart

$$8\left(\frac{u_\tau}{\overline{U}}\right)^2 = F\left(2\frac{\overline{U}R}{\nu}, \frac{R}{k_s}\right)$$

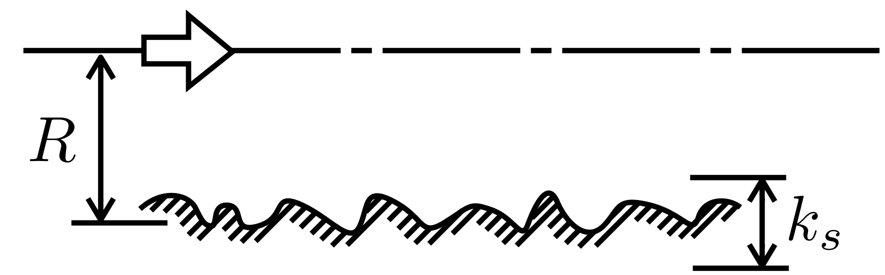
$$8\left(\frac{u_\tau}{\overline{U}}\right)^2$$

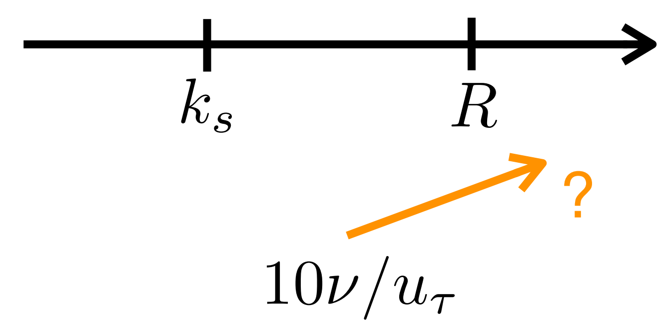


$$\frac{R}{k_s}$$

$$2\frac{\overline{U}R}{\nu}$$

(Nikuradse 1933)

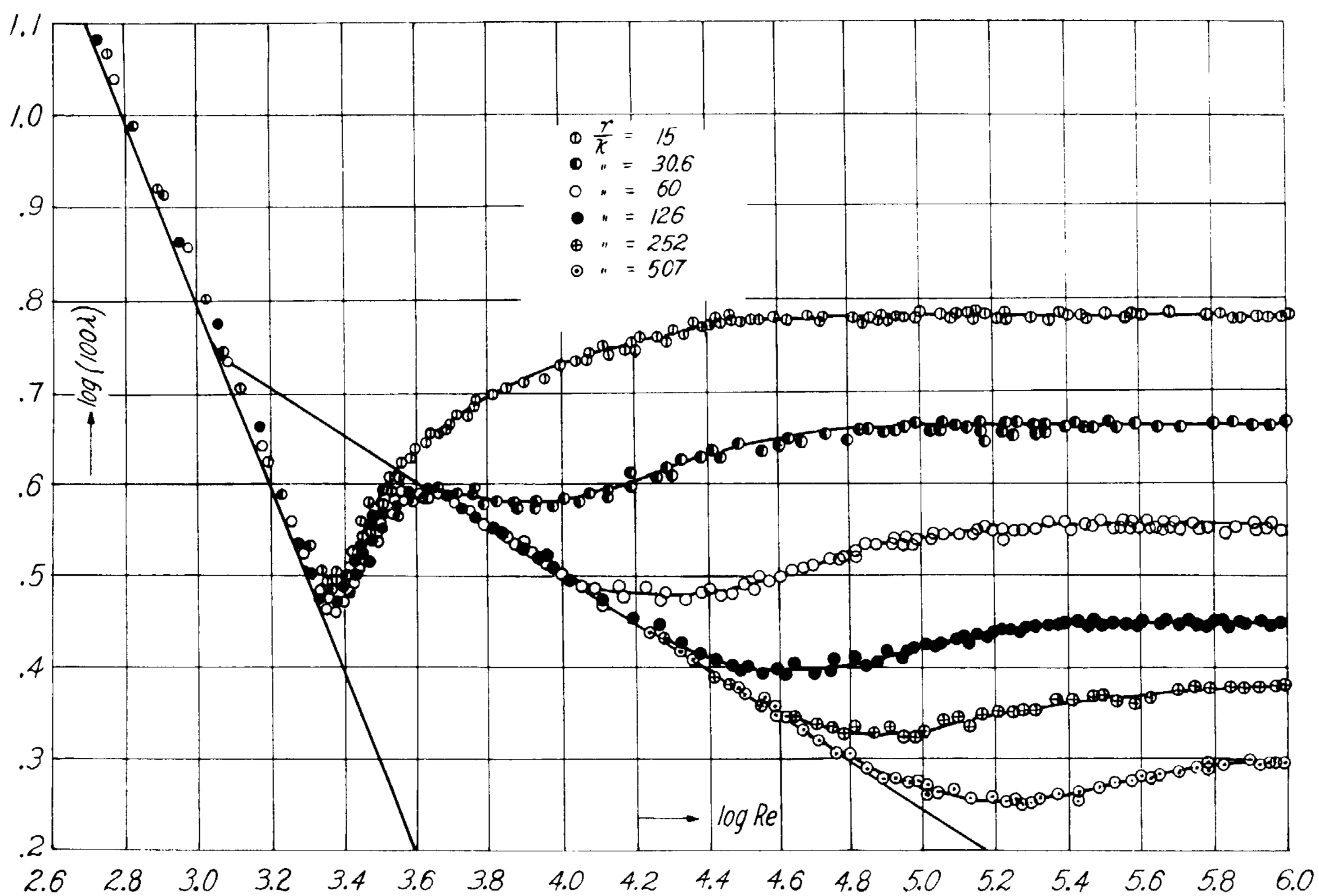




Moody chart

$$8\left(\frac{u_\tau}{\overline{U}}\right)^2 = F\left(2\frac{\overline{U}R}{\nu}, \frac{R}{k_s}\right)$$

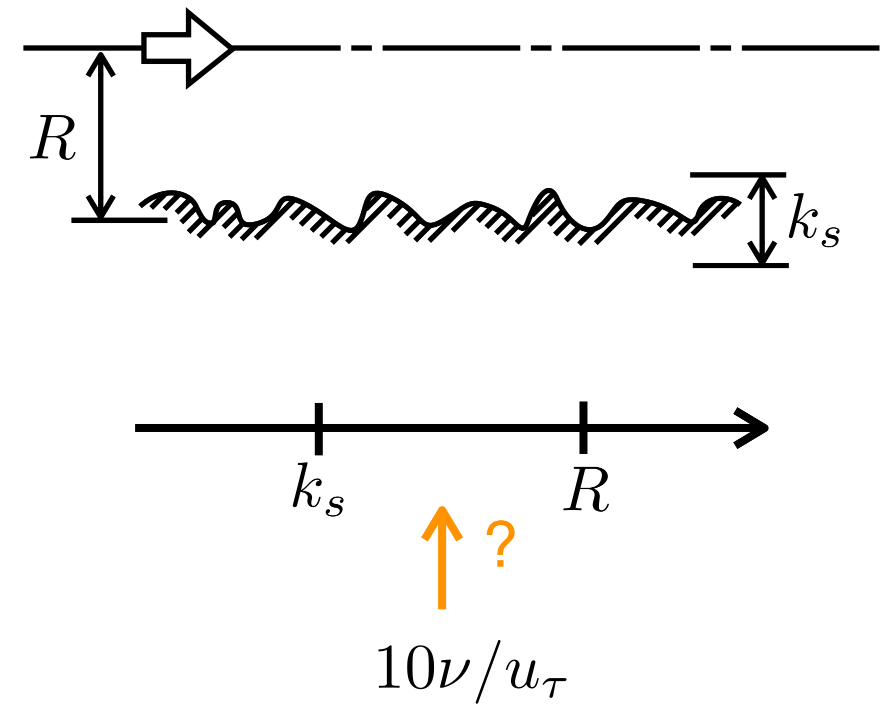
$8\left(\frac{u_\tau}{\overline{U}}\right)^2$



$\frac{R}{k_s}$

$2\frac{\overline{U}R}{\nu}$

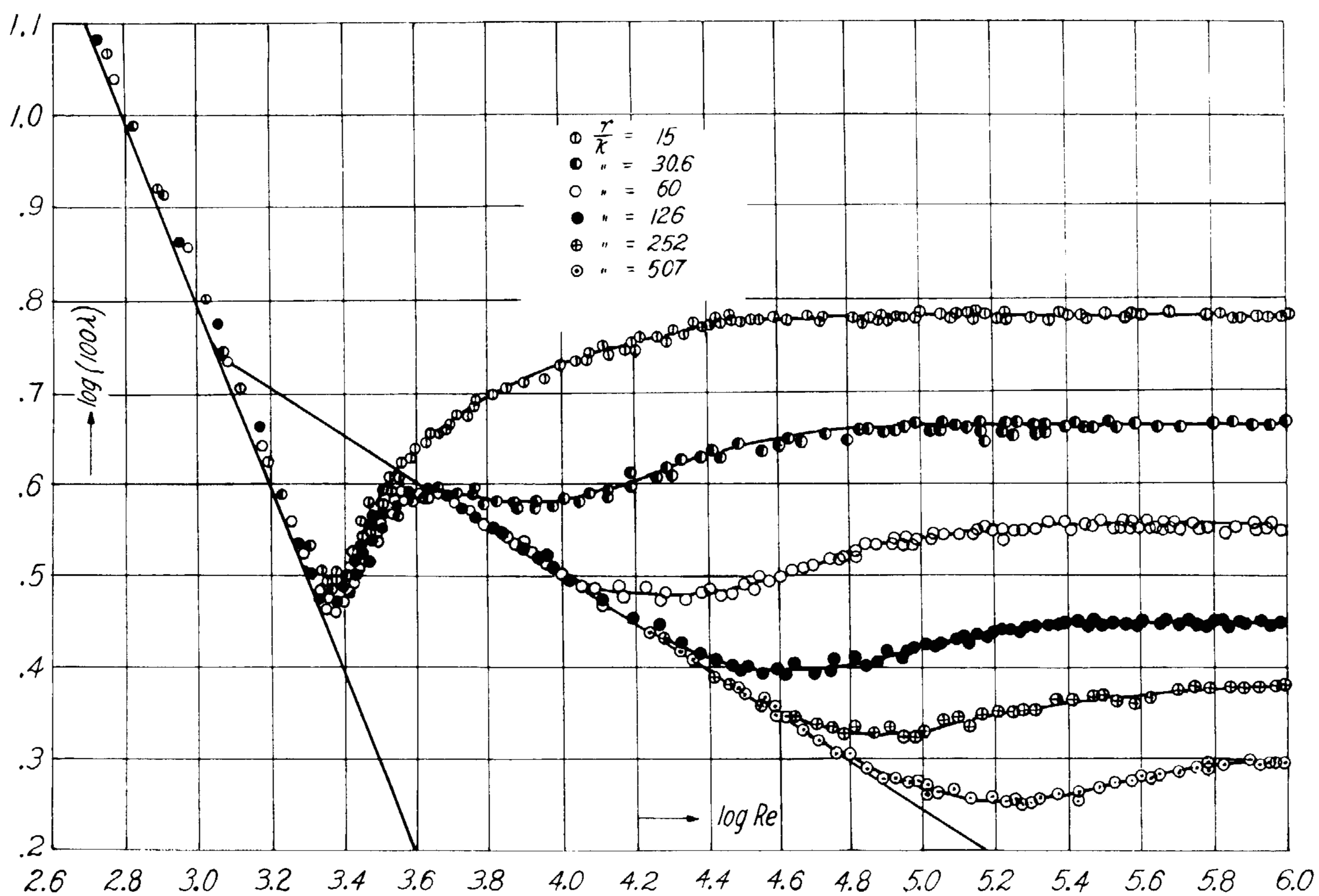
(Nikuradse 1933)



Moody chart

$$8\left(\frac{u_\tau}{\overline{U}}\right)^2 = F\left(2\frac{\overline{U}R}{\nu}, \frac{R}{k_s}\right)$$

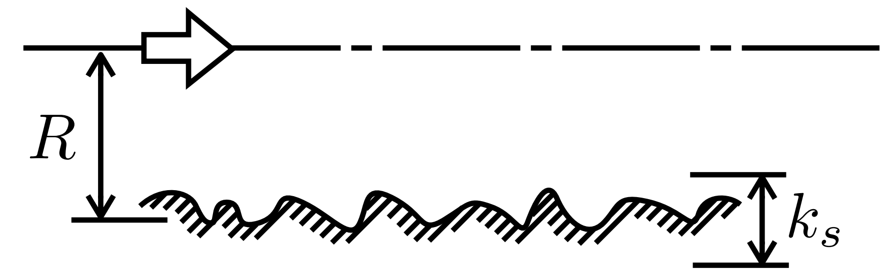
$8\left(\frac{u_\tau}{\overline{U}}\right)^2$

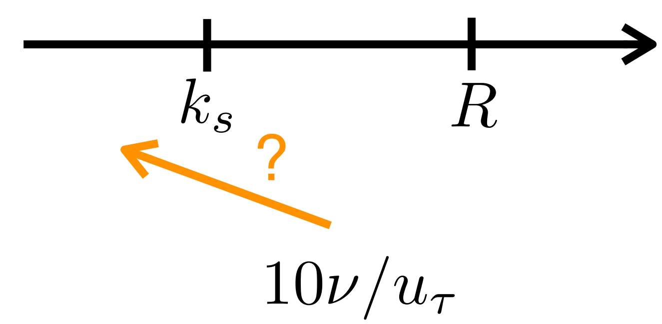


$\frac{R}{k_s}$

(Nikuradse 1933)

$2\frac{\overline{U}R}{\nu}$

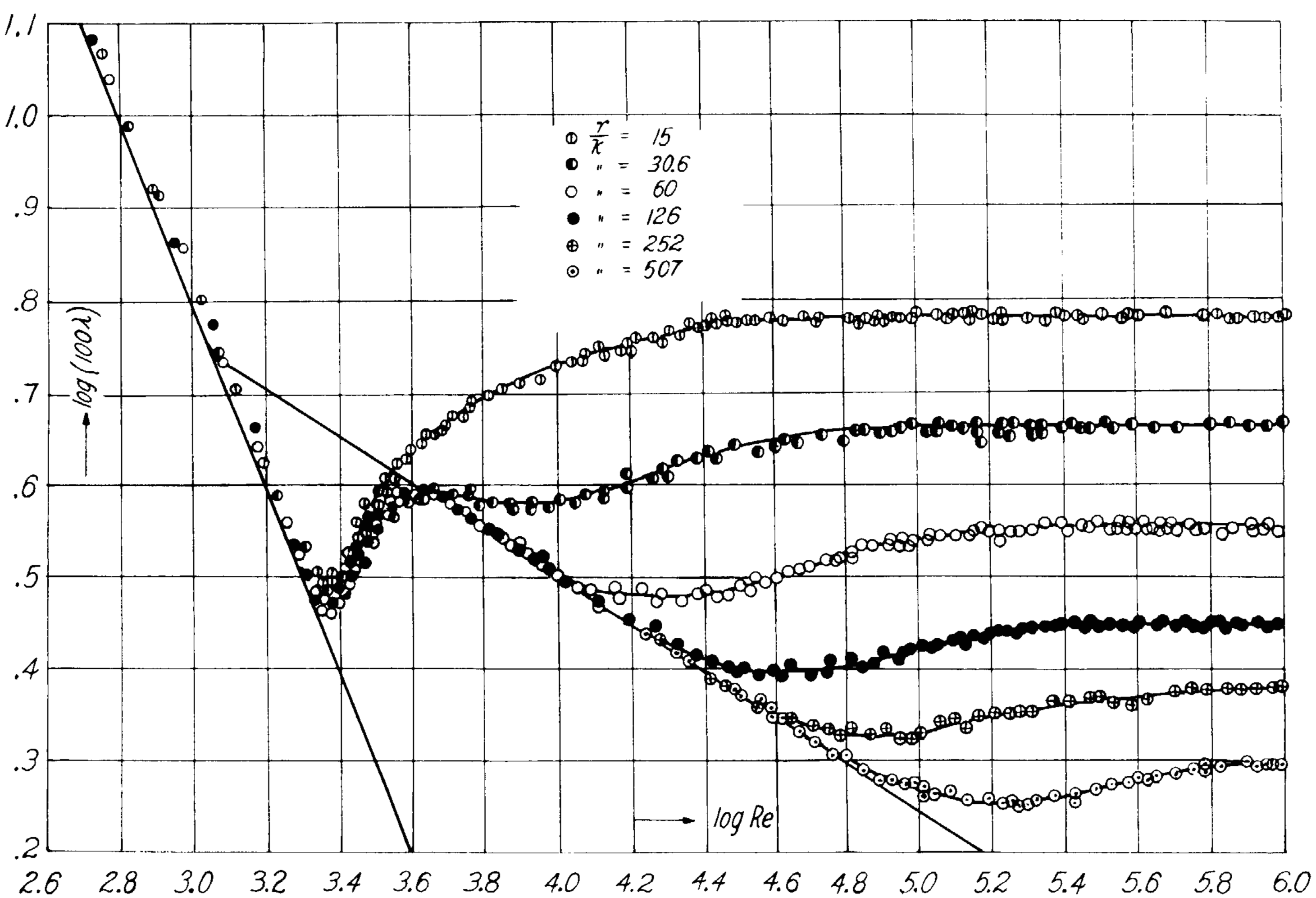




Moody chart

$$8\left(\frac{u_\tau}{\overline{U}}\right)^2 = F\left(2\frac{\overline{U}R}{\nu}, \frac{R}{k_s}\right)$$

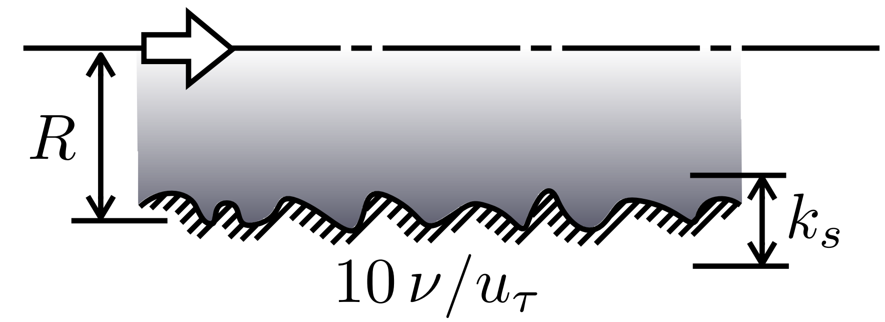
$$8\left(\frac{u_\tau}{\overline{U}}\right)^2$$

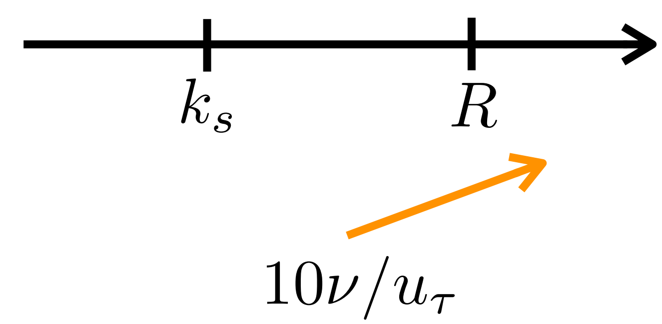


$$\frac{R}{k_s}$$

$$2\frac{\overline{U}R}{\nu}$$

(Nikuradse 1933)





Moody chart

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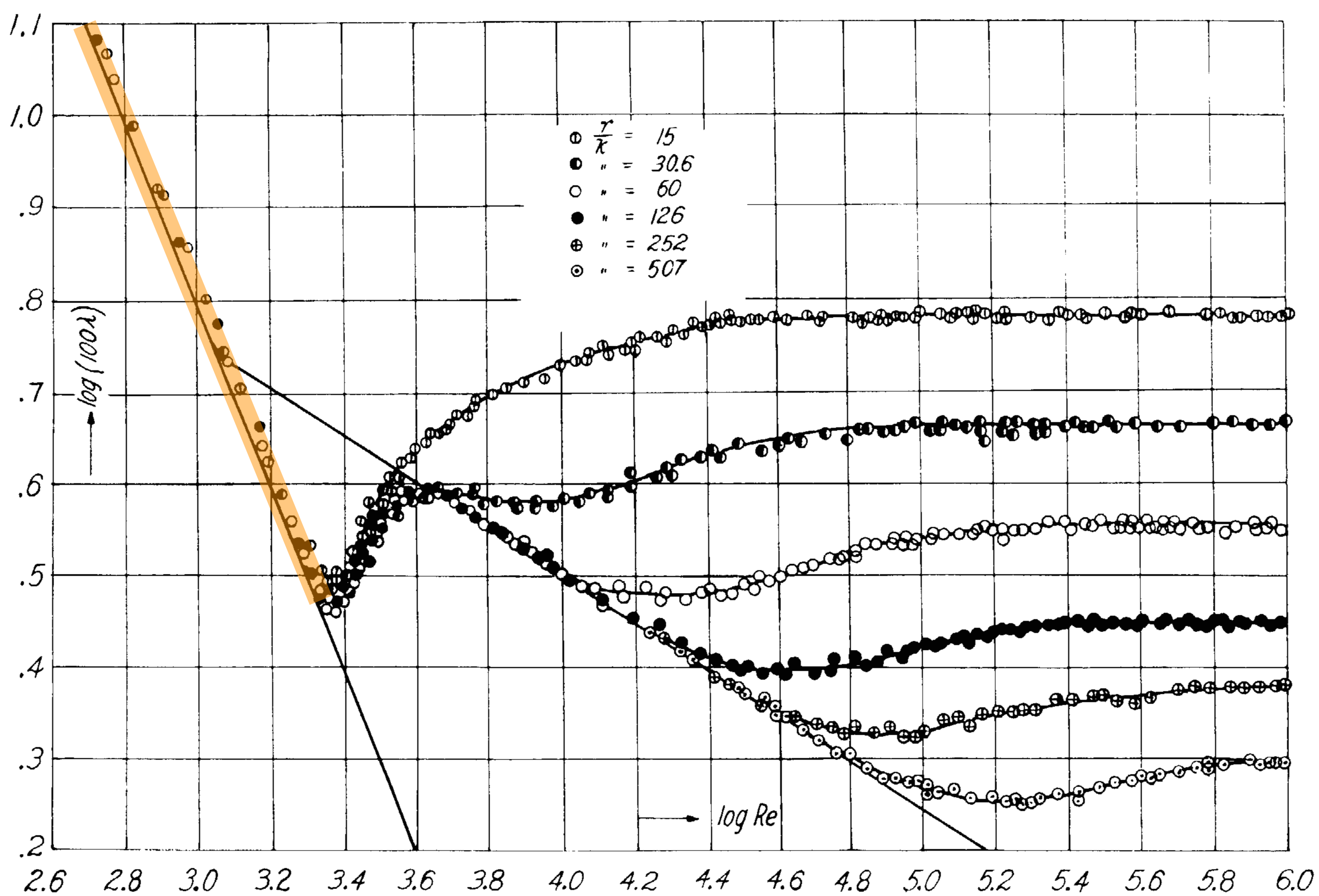

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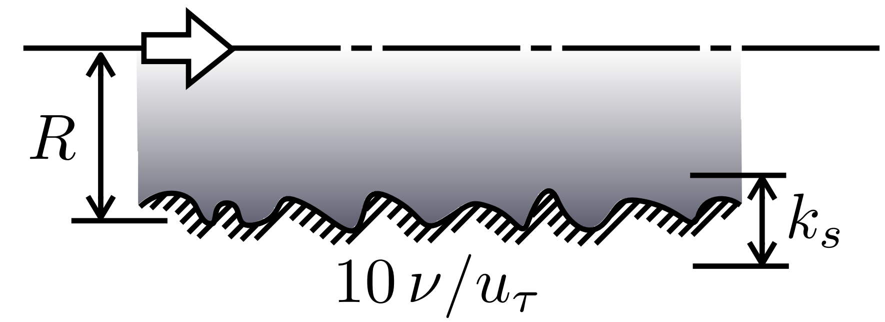
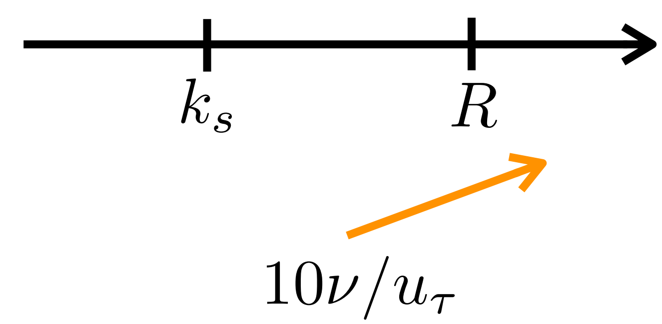
(Nikuradse 1933)





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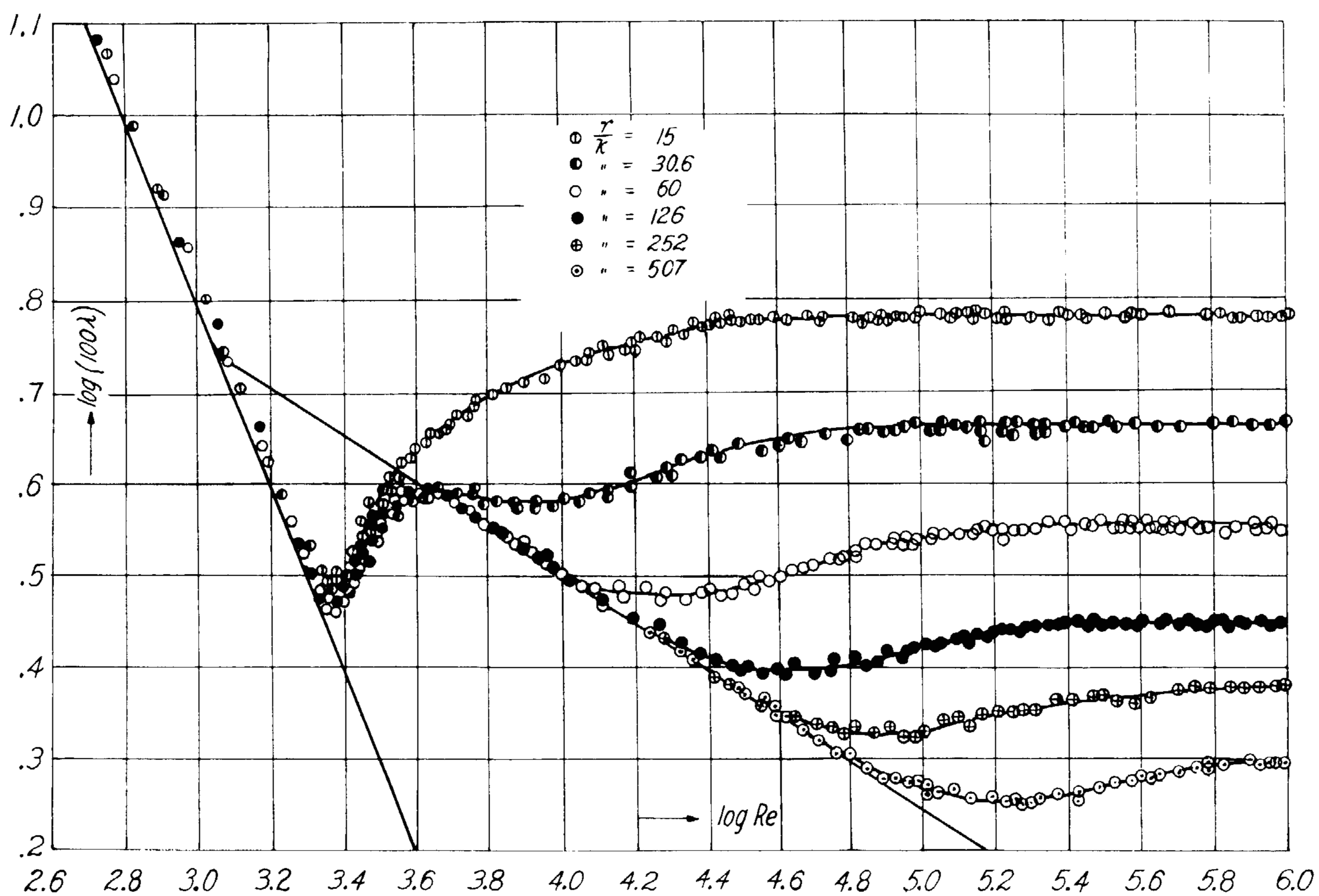

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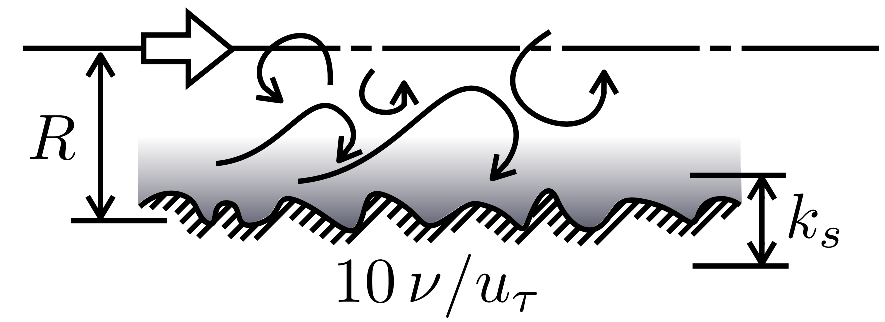

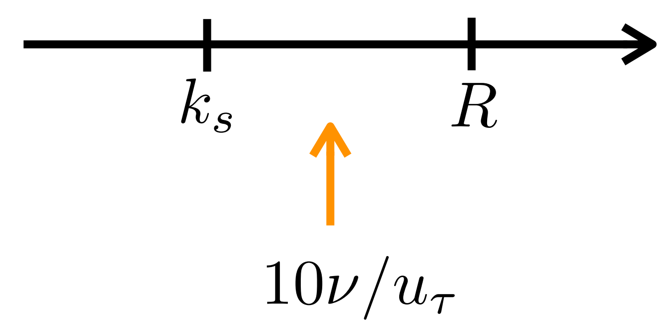

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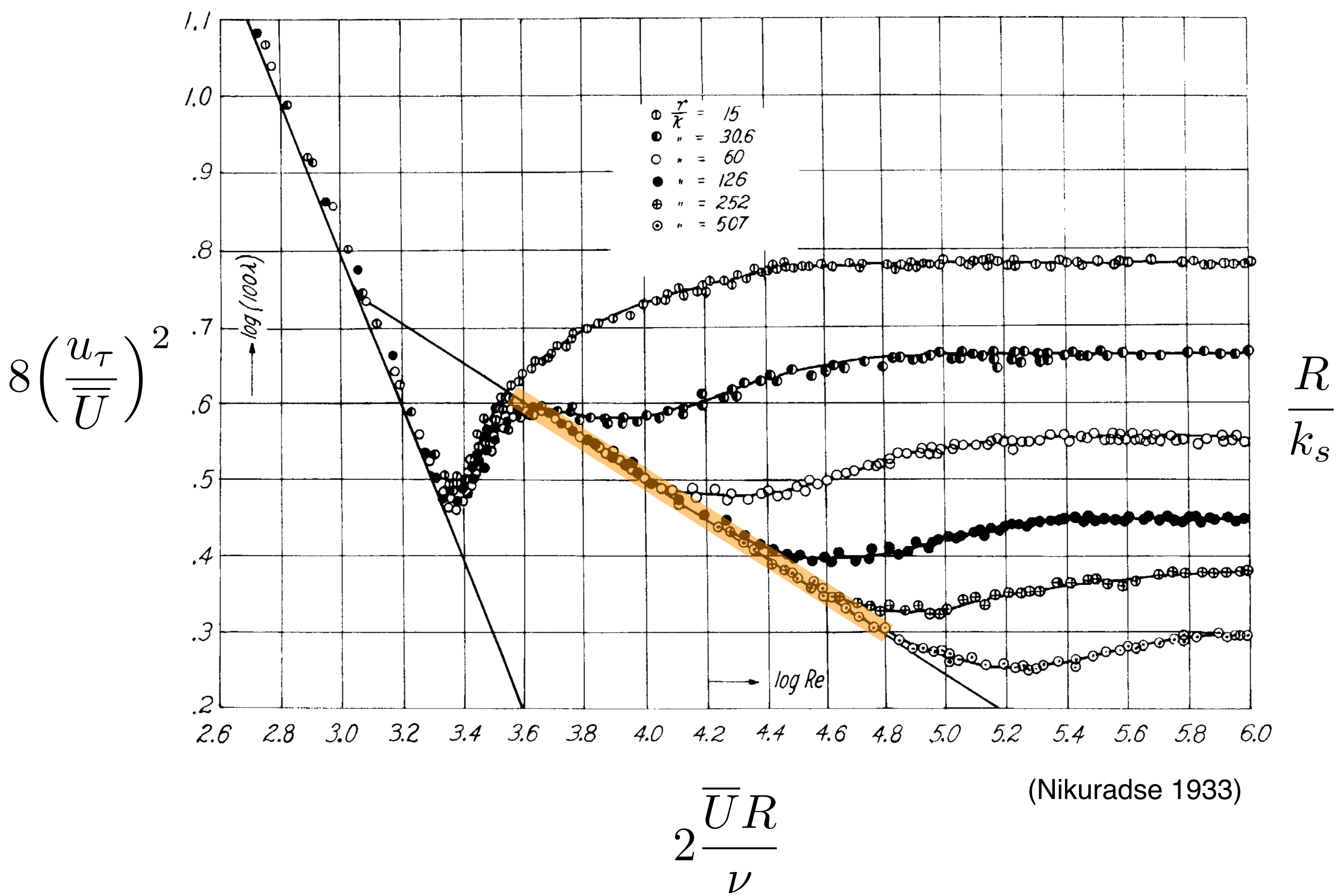
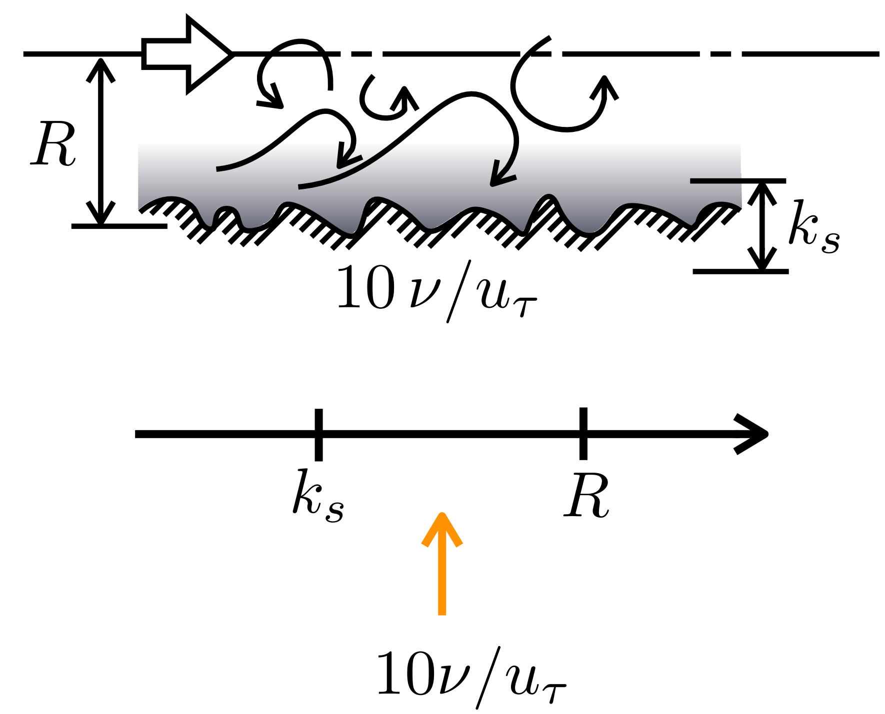


(Nikuradse 1933)



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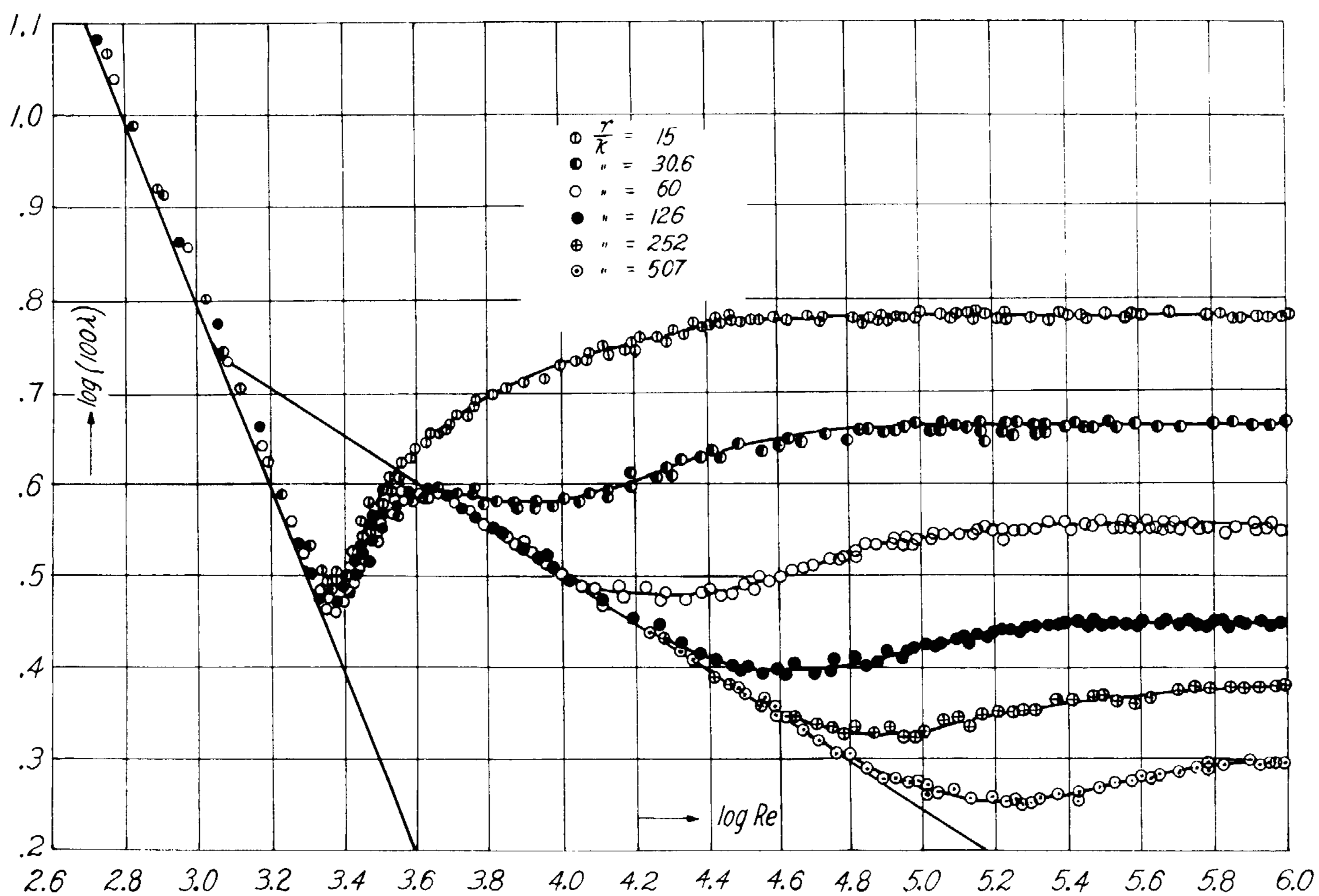

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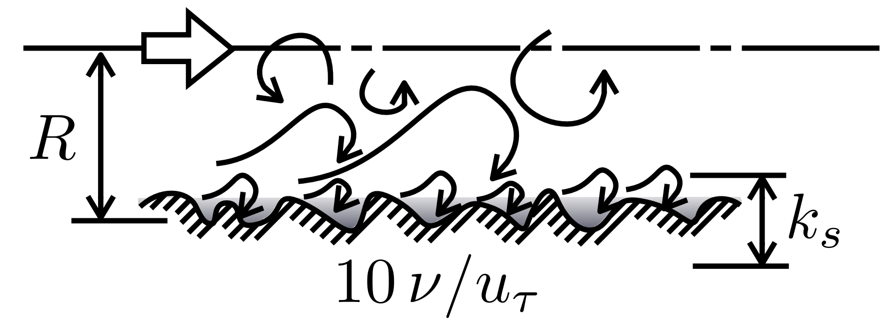
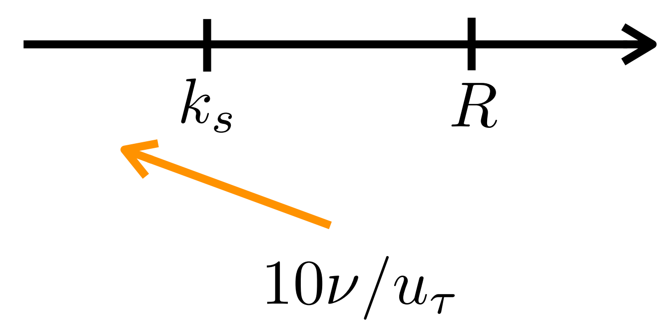
(Nikuradse 1933)





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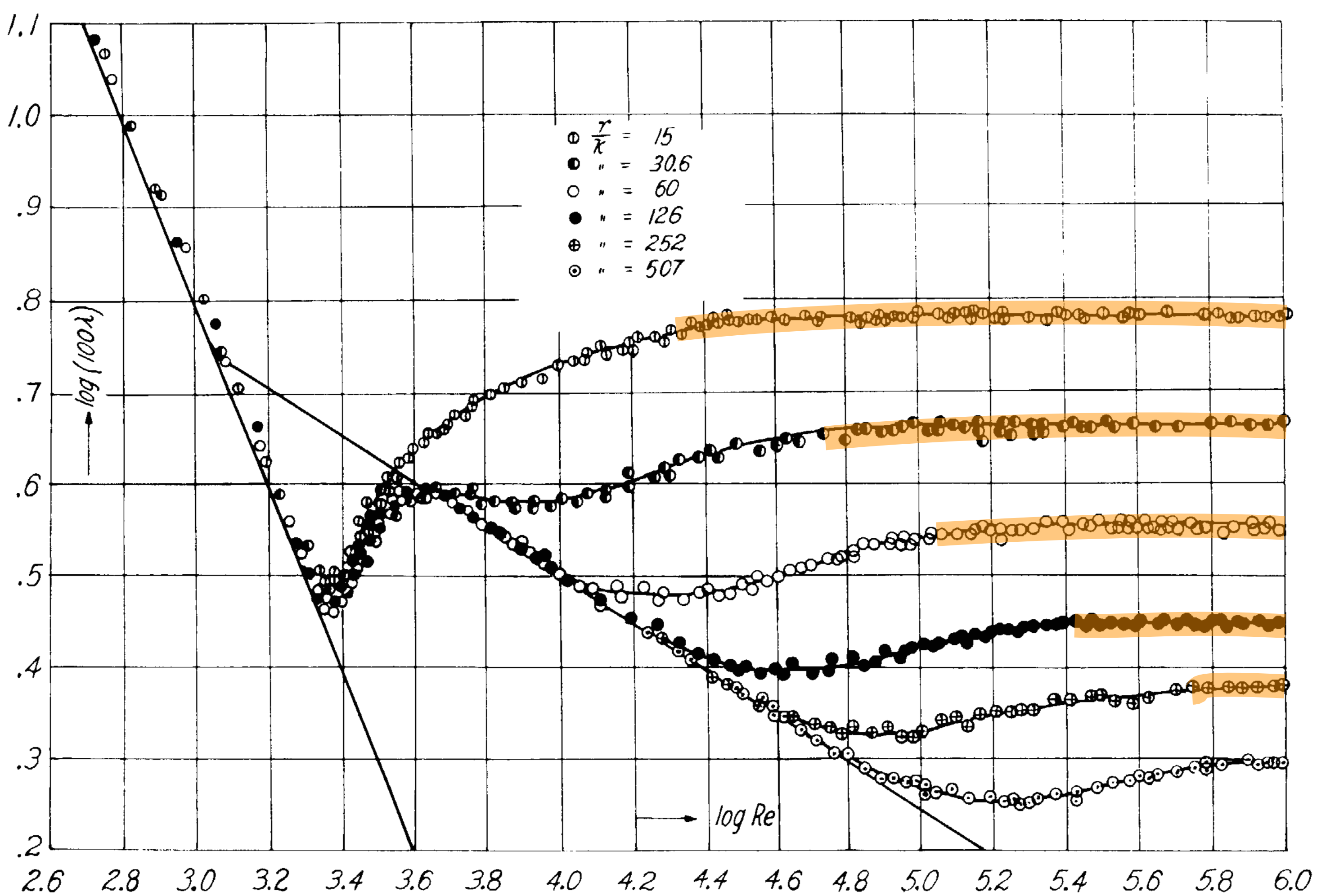

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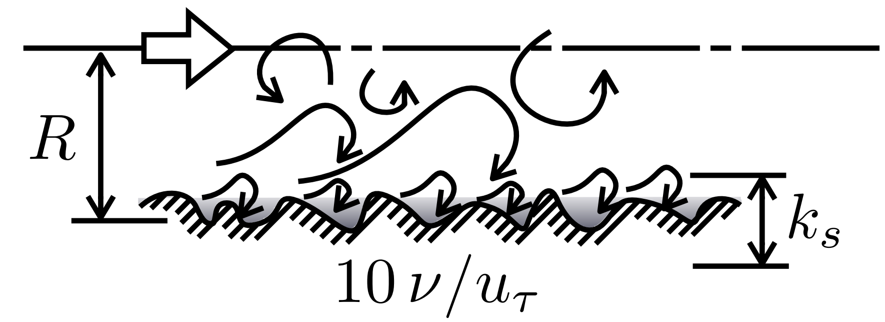
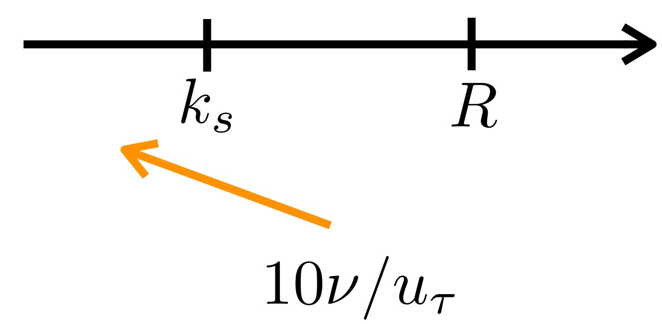
(Nikuradse 1933)

$$2\frac{\overline{U}R}{\nu}$$





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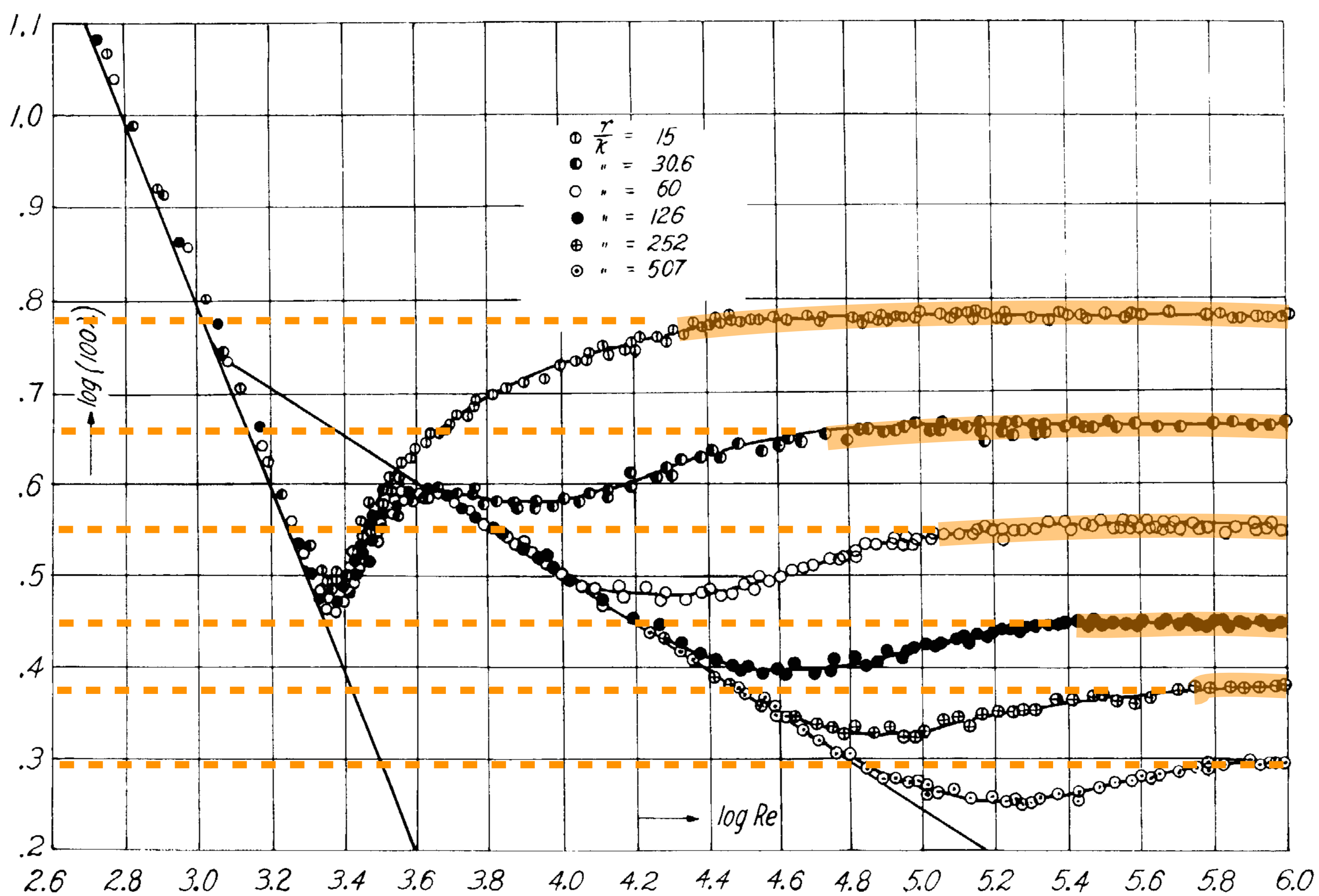

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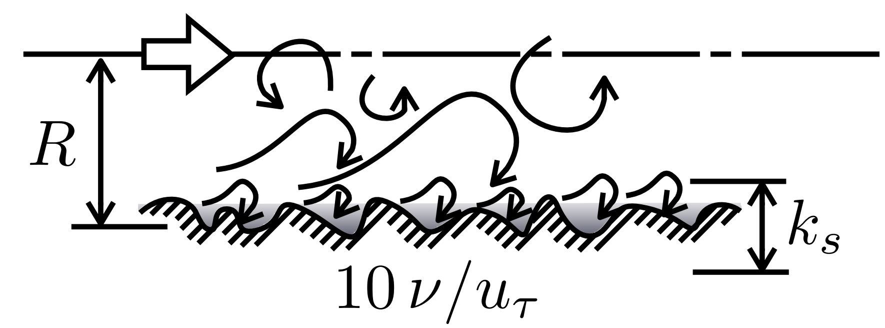

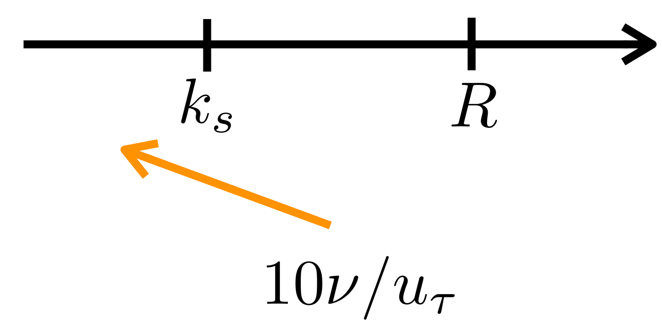

(Nikuradse 1933)

$$2\frac{\overline{U}R}{\nu}$$

Moody chart

$$8\left(\frac{u_\tau}{\overline{U}}\right)^2 = F\left(2\frac{\overline{U}R}{\nu}, \frac{R}{k_s}\right)$$

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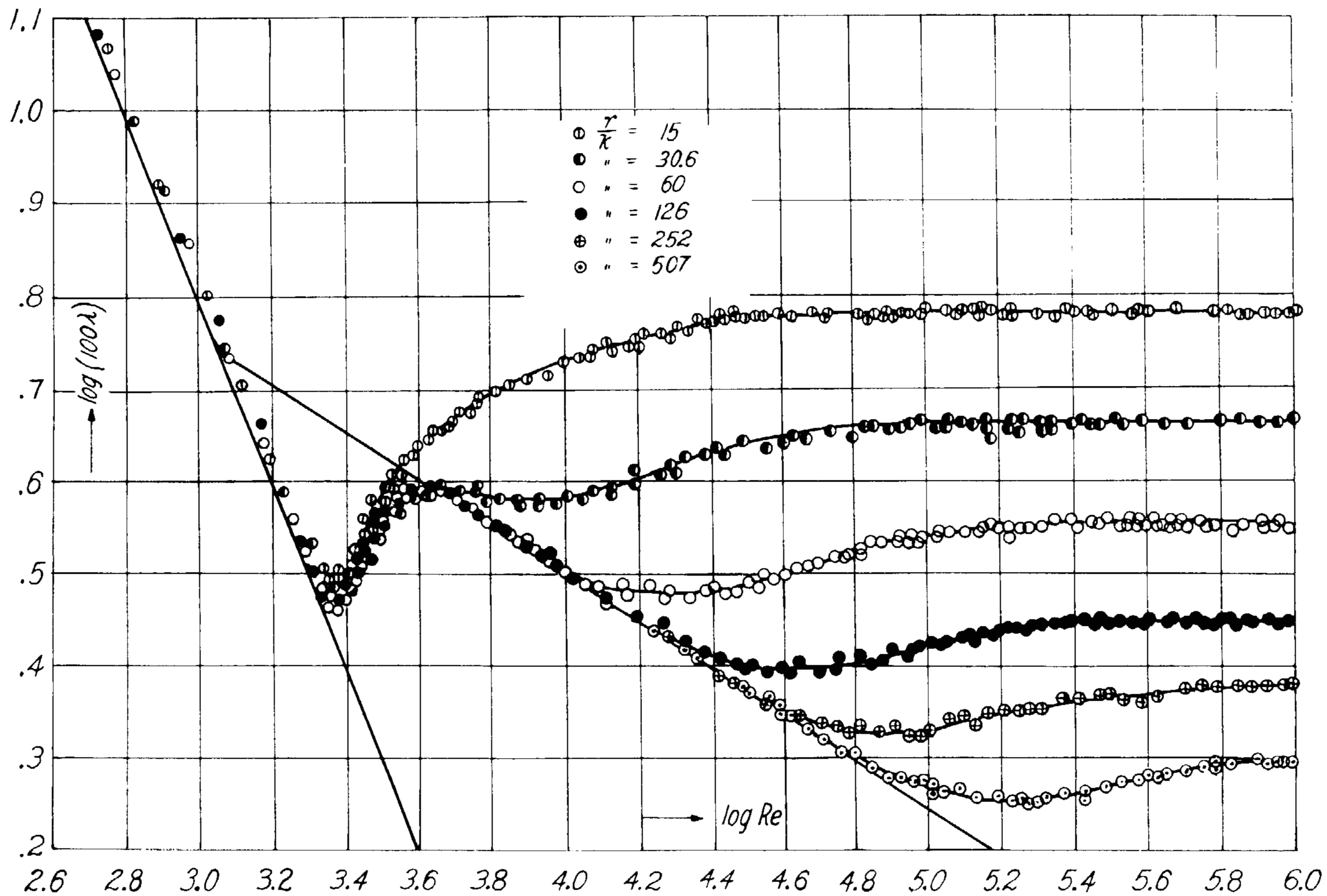

$$\frac{R}{k_s}$$

$$2\frac{\overline{U}R}{\nu}$$

(Nikuradse 1933)

Moody chart

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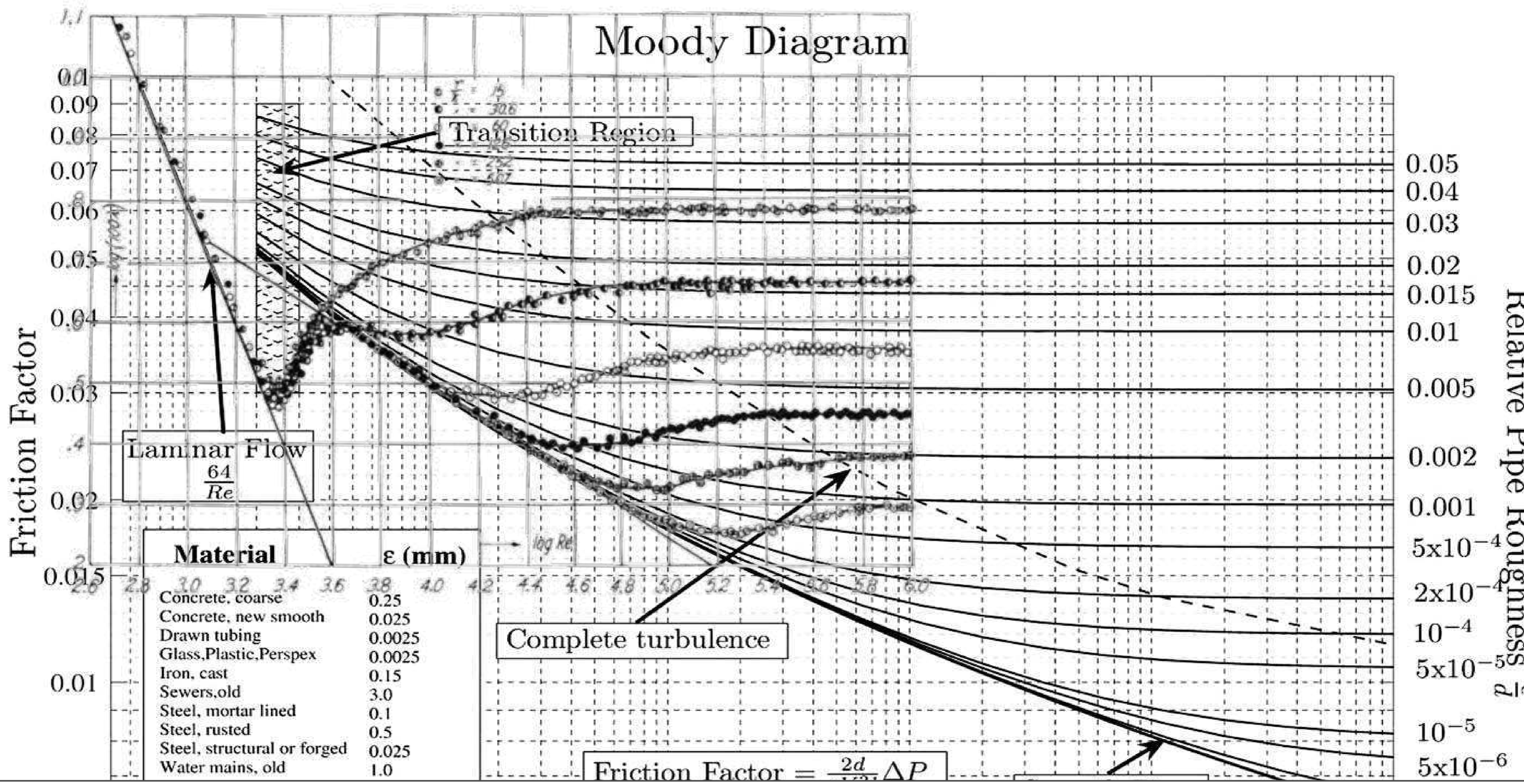
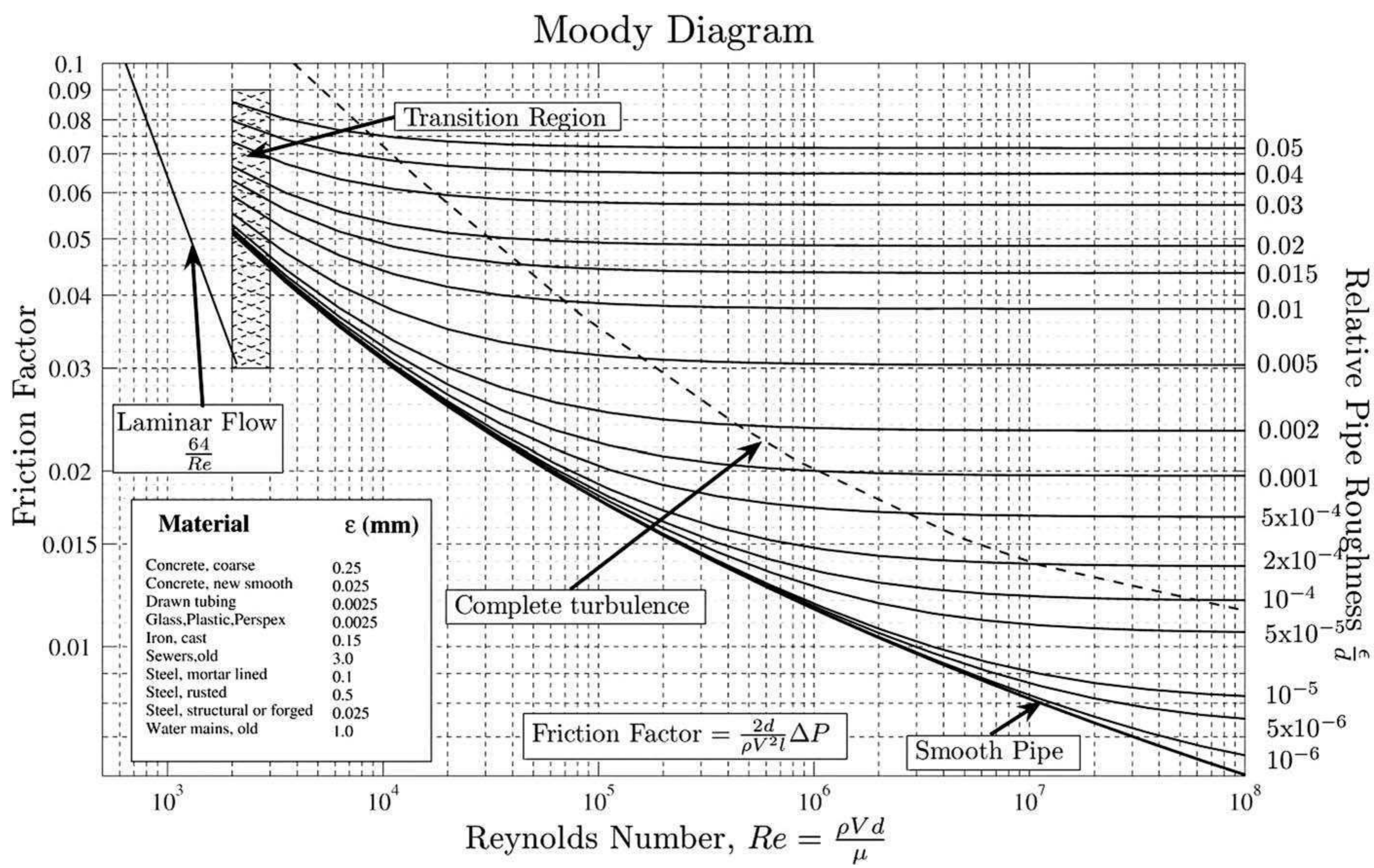


Material	ε (mm)
Concrete, coarse	0.25
Concrete, new smooth	0.025
Drawn tubing	0.0025
Glass,Plastic,Perspex	0.0025
Iron, cast	0.15
Sewers,old	3.0
Steel, mortar lined	0.1
Steel, rusted	0.5
Steel, structural or forged	0.025
Water mains, old	1.0

Moody chart

$$8\left(\frac{u_\tau}{\overline{U}}\right)^2 = F\left(2\frac{\overline{U}R}{\nu}, \frac{R}{k_s}\right)$$



(Flack & Schultz 2014)

Fully rough friction law

$$\frac{\overline{u}}{u_\tau} = \frac{1}{\kappa}\log\frac{y}{k_s} + 8.5$$

$$\frac{U_{CL} - \overline{u}}{u_\tau} = -\frac{1}{\kappa}\log\frac{y}{R} + B_{\text{pipe}}$$

$$\frac{U_{CL}}{u_\tau} = \sqrt{\frac{2}{c_f}} = \frac{1}{\kappa}\log\frac{R}{k_s} + B_{\text{pipe}} + 8.5$$

Independent on Re!

Compare with smooth turbulent friction law:

$$\sqrt{\frac{2}{c_f}} = \frac{1}{\kappa}\log\big(\sqrt{\frac{c_f}{2}}Re\big) + \text{const.}$$

Homework: better to express these in terms of U_{bulk}.

Moody diagram

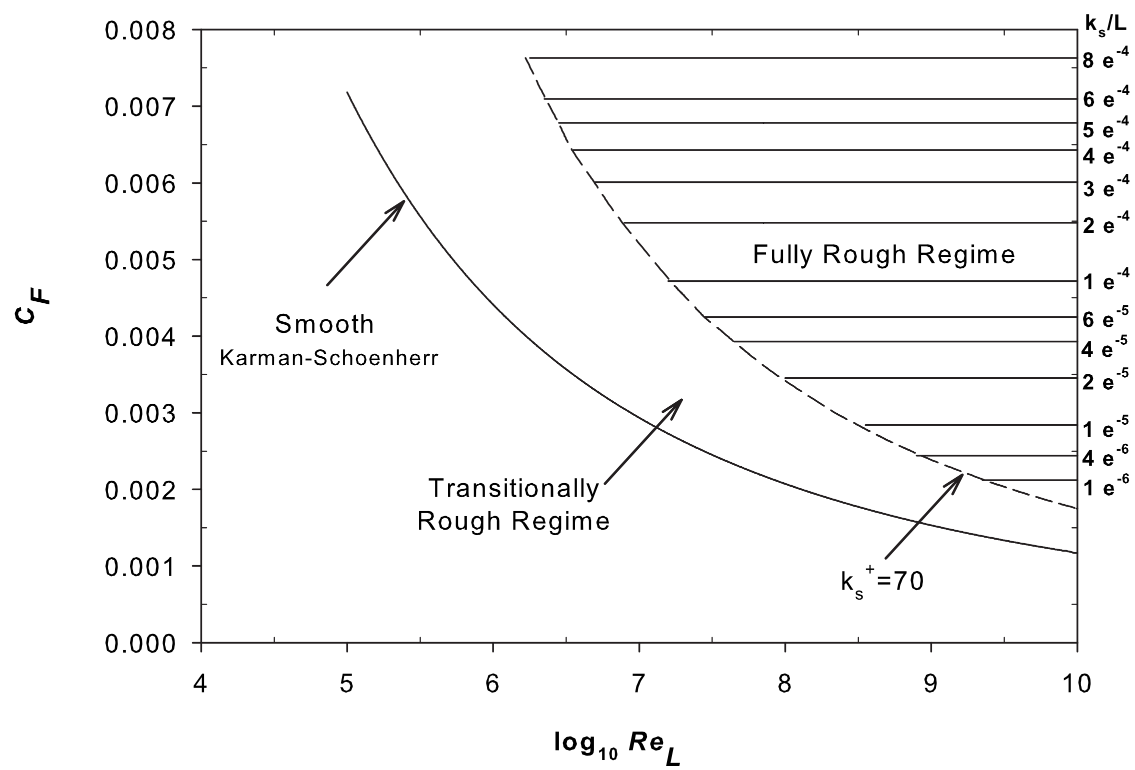
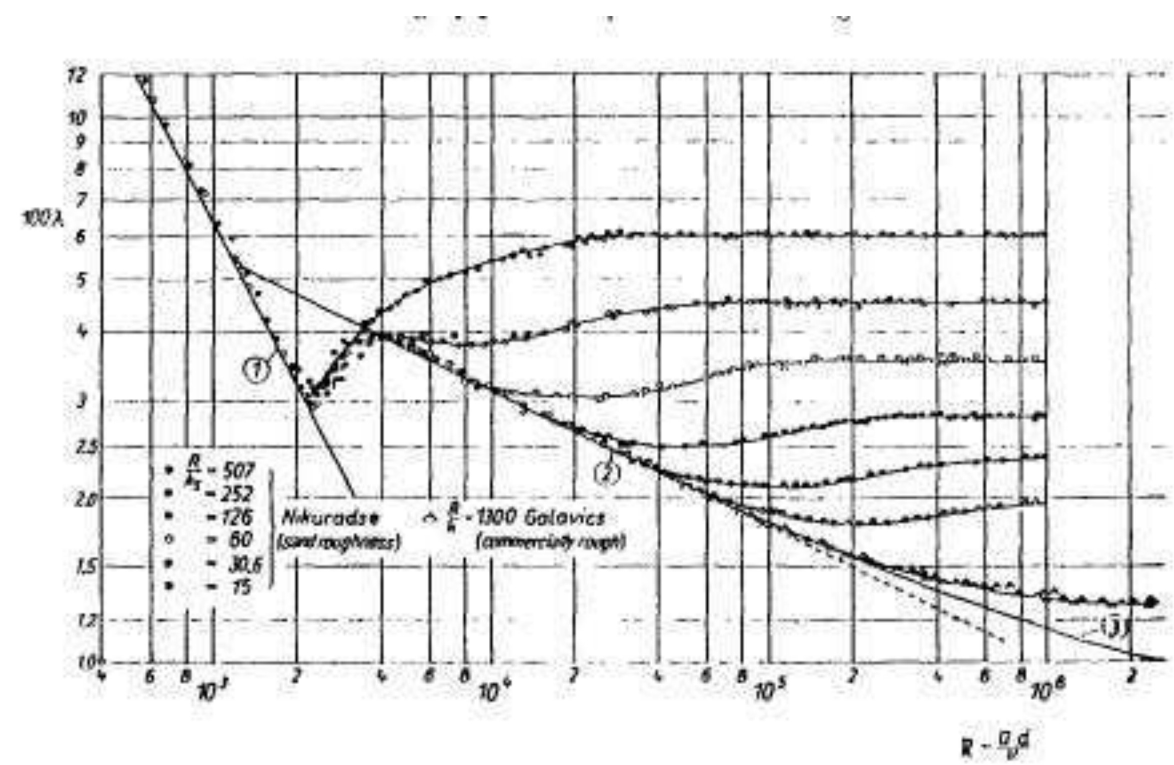


Fig. 6 Overall frictional drag coefficient in the fully rough regime

(Flack & Schultz 2010)



(Schlichting)

"Smoothness" depends on Re!