

Mathematical and Logical Foundations of Computer Science

Predicate Logic (Equivalences)

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(some slides were adapted from Rajesh Chitnis' slides)

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Where are we?

- ▶ Symbolic logic
- ▶ Propositional logic
- ▶ **Predicate logic**

Today

Equivalences:

- ▶ in Natural Deduction
- ▶ using semantics

Further reading:

- ▶ Chapter 8 of
http://leanprover.github.io/logic_and_proof/

Recap: Syntax

The syntax of predicate logic is defined by the following grammar:

$$t ::= x \mid f(t, \dots, t)$$

$$P ::= p(t, \dots, t) \mid \neg P \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \forall x.P \mid \exists x.P$$

where:

- ▶ x ranges over variables
- ▶ f ranges over function symbols
- ▶ $f(t_1, \dots, t_n)$ is a well-formed term only if f has arity n
- ▶ p ranges over predicate symbols
- ▶ $p(t_1, \dots, t_n)$ is a well-formed formula only if p has arity n

The pair of a collection of function symbols, and a collection of predicate symbols, along with their arities, is called a **signature**.

The scope of a quantifier extends as far right as possible. E.g., $P \wedge \forall x.p(x) \vee q(x)$ is read as $P \wedge \forall x.(p(x) \vee q(x))$

Recap: Substitution

Substitution is defined recursively on terms and formulas:
 $P[x \backslash t]$ substitute all the free occurrences of x in P with t .

$$\begin{array}{ll} x[x \backslash t] & = t \\ x[y \backslash t] & = x \\ (f(t_1, \dots, t_n))[x \backslash t] & = f(t_1[x \backslash t], \dots, t_n[x \backslash t]) \\ (p(t_1, \dots, t_n))[x \backslash t] & = p(t_1[x \backslash t], \dots, t_n[x \backslash t]) \\ \hline (\neg P)[x \backslash t] & = \neg P[x \backslash t] \\ (P_1 \wedge P_2)[x \backslash t] & = P_1[x \backslash t] \wedge P_2[x \backslash t] \\ (P_1 \vee P_2)[x \backslash t] & = P_1[x \backslash t] \vee P_2[x \backslash t] \\ (P_1 \rightarrow P_2)[x \backslash t] & = P_1[x \backslash t] \rightarrow P_2[x \backslash t] \\ \hline (\forall x. P)[x \backslash t] & = \forall x. P \\ (\exists x. P)[x \backslash t] & = \exists x. P \\ (\forall y. P)[x \backslash t] & = \forall y. P[x \backslash t], \text{ if } y \notin \text{fv}(t) \\ (\exists y. P)[x \backslash t] & = \exists y. P[x \backslash t], \text{ if } y \notin \text{fv}(t) \end{array}$$

The additional **conditions** ensure that **free variables do not get captured**.

These conditions can always be met by silently renaming bound variables before substituting.

Recap: \forall & \exists elimination and introduction rules

Natural Deduction rules for quantifiers:

$$\begin{array}{c}
 \frac{P[x \backslash y]}{\forall x.P} \quad [\forall I] \qquad \frac{\forall x.P}{P[x \backslash t]} \quad [\forall E] \qquad \frac{P[x \backslash t]}{\exists x.P} \quad [\exists I] \qquad \frac{\exists x.P \quad \begin{array}{c} \overline{P[x \backslash y]}^1 \\ \vdots \\ Q \end{array}}{Q}^1 \quad [\exists E]
 \end{array}$$

Condition:

- ▶ for $[\forall I]$: y must not be free in any not-yet-discharged hypothesis or in $\forall x.P$
- ▶ for $[\forall E]$: $\mathbf{fv}(t)$ must not clash with $\mathbf{bv}(P)$
- ▶ for $[\exists I]$: $\mathbf{fv}(t)$ must not clash with $\mathbf{bv}(P)$
- ▶ for $[\exists E]$: y must not be free in Q or in not-yet-discharged hypotheses or in $\exists x.P$

Recap: Example of a proof

here is a proof of $(\forall z.p(z)) \rightarrow \forall x.p(x) \vee q(x)$.

$$\frac{\frac{\frac{\overline{\quad} 1}{\forall z.p(z)} [\forall E]}{p(y)} [\vee I_L]}{\frac{p(y) \vee q(y)}{\forall x.p(x) \vee q(x)} [\forall I]} 1 [\rightarrow I]$$

Conditions:

- ▶ y does not occur free in not-yet-discharged hypotheses or in $\forall x.p(x) \vee q(x)$
- ▶ y does not clash with bound variables in $p(z)$

Recap: Models

Models: a model provides the interpretation of all symbols

Given a **signature** $\langle\langle f_1^{k_1}, \dots, f_n^{k_n} \rangle, \langle p_1^{j_1}, \dots, p_m^{j_m} \rangle\rangle$

- ▶ of function symbols f_i of arity k_i , for $1 \leq i \leq n$
- ▶ of predicate symbols p_i of arity j_i , for $1 \leq i \leq m$

a **model** is a structure $\langle D, \langle \mathcal{F}_{f_1}, \dots, \mathcal{F}_{f_n} \rangle, \langle \mathcal{R}_{p_1}, \dots, \mathcal{R}_{p_m} \rangle \rangle$

- ▶ of a non-empty domain D
- ▶ interpretations \mathcal{F}_{f_i} for function symbols f_i
- ▶ interpretations \mathcal{R}_{p_i} for predicate symbols p_i

Models of predicate logic replace **truth assignments** for propositional logic

Variable valuations:

- ▶ a partial function v
- ▶ that map variables to D
- ▶ i.e., a mapping of the form $x_1 \mapsto d_1, \dots, x_n \mapsto d_n$

Recap: Semantics of Predicate Logic

Given a **model** M with domain D and a **variable valuation** v :

- ▶ $\llbracket t \rrbracket_v^M$ gives meaning to the term t w.r.t. M and v
- ▶ $\models_{M,v} P$ gives meaning to the formula P w.r.t. M and v

Meaning of terms:

- ▶ $\llbracket x \rrbracket_v^M = v(x)$
- ▶ $\llbracket f(t_1, \dots, t_n) \rrbracket_v^M = \mathcal{F}_f(\langle \llbracket t_1 \rrbracket_v^M, \dots, \llbracket t_n \rrbracket_v^M \rangle)$

Meaning of formulas:

- ▶ $\models_{M,v} p(t_1, \dots, t_n)$ iff $\langle \llbracket t_1 \rrbracket_v^M, \dots, \llbracket t_n \rrbracket_v^M \rangle \in \mathcal{R}_p$
- ▶ $\models_{M,v} \neg P$ iff $\not\models_{M,v} P$
- ▶ $\models_{M,v} P \wedge Q$ iff $\models_{M,v} P$ and $\models_{M,v} Q$
- ▶ $\models_{M,v} P \vee Q$ iff $\models_{M,v} P$ or $\models_{M,v} Q$
- ▶ $\models_{M,v} P \rightarrow Q$ iff $\models_{M,v} Q$ whenever $\models_{M,v} P$
- ▶ $\models_{M,v} \forall x. P$ iff for every $d \in D$ we have $\models_{M,(v,x \mapsto d)} P$
- ▶ $\models_{M,v} \exists x. P$ iff there exists a $d \in D$ such that $\models_{M,(v,x \mapsto d)} P$

Recap: Logical equivalences for Propositional Logic

The same equivalences hold as in Propositional Logic:

- ▶ De Morgan's law (I): $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$
- ▶ De Morgan's law (II): $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$
- ▶ Implication elimination: $(A \rightarrow B) \leftrightarrow (\neg A \vee B)$
- ▶ Commutativity of \wedge : $(A \wedge B) \leftrightarrow (B \wedge A)$
- ▶ Commutativity of \vee : $(A \vee B) \leftrightarrow (B \vee A)$
- ▶ Associativity of \wedge : $((A \wedge B) \wedge C) \leftrightarrow (A \wedge (B \wedge C))$
- ▶ Associativity of \vee : $((A \vee B) \vee C) \leftrightarrow (A \vee (B \vee C))$
- ▶ Distributivity of \wedge over \vee : $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$
- ▶ Distributivity of \vee over \wedge : $(A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C))$
- ▶ Double negation elimination: $(\neg\neg A) \leftrightarrow A$
- ▶ Idempotence: $(A \wedge A) \leftrightarrow A$ and $(A \vee A) \leftrightarrow A$

Logical Equivalences

In addition, the following hold (some hold only classically):

- ▶ $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$
- ▶ $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$
- ▶ $(\neg \forall x.A) \leftrightarrow (\exists x. \neg A)$
- ▶ $(\neg \exists x.A) \leftrightarrow (\forall x. \neg A)$
- ▶ $(\forall x.A) \leftrightarrow A$ if $x \notin \text{fv}(A)$
- ▶ $(\exists x.A) \leftrightarrow A$ if $x \notin \text{fv}(A)$
- ▶ $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \text{fv}(B)$
- ▶ $(\exists x.A \wedge B) \leftrightarrow ((\exists x.A) \wedge B)$ if $x \notin \text{fv}(B)$
- ▶ $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \text{fv}(B)$
- ▶ $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \text{fv}(B)$
- ▶ $(\forall x.A \rightarrow B) \leftrightarrow (A \rightarrow \forall x.B)$ if $x \notin \text{fv}(A)$
- ▶ $(\exists x.A \rightarrow B) \leftrightarrow (A \rightarrow \exists x.B)$ if $x \notin \text{fv}(A)$

Logical Equivalences

As before to prove a logical equivalence $A \leftrightarrow B$, we will prove:

- ▶ that we can derive B from A
- ▶ that we can derive A from B

We will prove:

- ▶ $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$
- ▶ $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$
- ▶ $(\neg \forall x.A) \leftrightarrow (\exists x. \neg A)$
- ▶ $(\neg \exists x.A) \leftrightarrow (\forall x. \neg A)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\begin{array}{c}
 \frac{\forall x.A \wedge B}{A[x \backslash y] \wedge B[x \backslash y]} \quad [\forall E] \quad \frac{\forall x.A \wedge B}{A[x \backslash y] \wedge B[x \backslash y]} \quad [\forall E] \\
 \frac{A[x \backslash y]}{\forall x.A} \quad [\wedge E_L] \quad \frac{B[x \backslash y]}{\forall x.B} \quad [\wedge E_R] \\
 \frac{\forall x.A \quad \forall x.B}{(\forall x.A) \wedge (\forall x.B)} \quad [\wedge I]
 \end{array}$$

- ▶ pick y such that it does not occur in A or B
- ▶ y must not be free in $\forall x.A \wedge B$ or in $\forall x.A$
- ▶ y must not clash with $\text{bv}(A \wedge B)$
- ▶ y must not be free in $\forall x.A \wedge B$ or in $\forall x.B$
- ▶ y must not clash with $\text{bv}(A \wedge B)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\frac{\frac{\frac{(\forall x.A) \wedge (\forall x.B)}{\forall x.A} [\wedge E_L] \quad \frac{\frac{\forall x.A}{A[x \setminus y]} [\forall E]}{A[x \setminus y] \wedge B[x \setminus y]} [\wedge I] \quad \frac{\frac{(\forall x.A) \wedge (\forall x.B)}{\forall x.B} [\wedge E_R] \quad \frac{\frac{\forall x.B}{B[x \setminus y]} [\forall E]}{B[x \setminus y]} [\wedge I]}{\forall x.A \wedge B} [\forall I]$$

- ▶ pick y such that it does not occur in A or B
- ▶ y must not be free in $(\forall x.A) \wedge (\forall x.B)$ or in $\forall x.A \wedge B$
- ▶ y must not clash with $\text{bv}(A)$
- ▶ y must not clash with $\text{bv}(B)$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\begin{array}{c}
 \frac{\frac{\frac{}{A[x \backslash y]} \quad 2}{\exists x.A} [\exists I]}{(\exists x.A) \vee (\exists x.B)} [\vee I_L] \quad \frac{\frac{\frac{}{B[x \backslash y]} \quad 3}{\exists x.B} [\exists I]}{(\exists x.A) \vee (\exists x.B)} [\vee I_R] \\
 \frac{A[x \backslash y] \vee B[x \backslash y] \quad 1 \quad A[x \backslash y] \rightarrow (\exists x.A) \vee (\exists x.B) \quad 2 \quad [\rightarrow I] \quad B[x \backslash y] \rightarrow (\exists x.A) \vee (\exists x.B) \quad 3 \quad [\rightarrow I]}{(\exists x.A) \vee (\exists x.B)} [\vee E] \\
 \frac{\exists x.A \vee B \quad (\exists x.A) \vee (\exists x.B)}{(\exists x.A) \vee (\exists x.B)} 1 \quad [\exists E]
 \end{array}$$

- ▶ pick y such that it does not occur in A or B
- ▶ 1: $A[x \backslash y] \vee B[x \backslash y]$
- ▶ 2: $A[x \backslash y]$
- ▶ 3: $B[x \backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

[illegible]

- ▶ 1: $\exists x.A$
- ▶ pick y such that it does not occur in A or B
- ▶ 2: $A[x \backslash y]$
- ▶ 3: $\exists x.B$
- ▶ 4: $B[x \backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\frac{\frac{\frac{\overline{A[x\backslash y]}^1}{\exists x.A} [\exists I]}{\neg\exists x.A \quad \exists x.A} [\neg E]}{\perp} \quad \frac{\perp}{\neg A[x\backslash y]}^1 [\neg I] \quad \frac{\neg A[x\backslash y]}{\forall x.\neg A} [\forall I]$$

- ▶ pick y such that it does not occur in A
- ▶ 1: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\begin{array}{c}
 \frac{\frac{\frac{\forall x.\neg A}{\neg A[x\backslash y]} \quad [\forall E] \quad \frac{}{A[x\backslash y]} \quad 2}{\perp} \quad [\neg E]}{\frac{\frac{\exists x.A \quad 1}{\perp} \quad 2 \quad [\exists E]}{\neg\exists x.A} \quad 1 \quad [\neg I]}
 \end{array}$$

- ▶ 1: $\exists x.A$
- ▶ pick y such that it does not occur in A
- ▶ 2: $A[x\backslash y]$

Logical Equivalences

As before: if $(P \leftrightarrow Q \text{ or } Q \leftrightarrow P)$ and P occurs in A , then replacing P by Q in A leads to a formula B , such that $A \leftrightarrow B$

Also,

Semantical equivalence: two formulas P and Q are equivalent if for all models M and valuations v , $\models_{M,v} P$ iff $\models_{M,v} Q$

Logical Equivalences

Example: prove $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$

- ▶ if $\models_{M,v} \neg\exists x.A$ then $\models_{M,v} \forall x.\neg A$
 - ▶ to prove: $\models_{M,v} \forall x.\neg A$, i.e., for every $d \in D$ it is not the case that $\models_{M,v,x \mapsto d} A$
 - ▶ assume $d \in D$ and $\models_{M,v,x \mapsto d} A$, and prove a contradiction
 - ▶ assumption: $\models_{M,v} \neg\exists x.A$, i.e., it is not the case that there exists a $e \in D$ such that $\models_{M,v,x \mapsto e} A$
 - ▶ contradiction! there is one: take $e = d$
- ▶ if $\models_{M,v} \forall x.\neg A$ then $\models_{M,v} \neg\exists x.A$
 - ▶ to prove: $\models_{M,v} \neg\exists x.A$, i.e., it is not the case that there exists a $e \in D$ such that $\models_{M,v,x \mapsto e} A$
 - ▶ assume that there exists a $e \in D$ such that $\models_{M,v,x \mapsto e} A$, and prove a contradiction
 - ▶ assumption: $\models_{M,v} \forall x.\neg A$, i.e., for every $d \in D$ it is not the case that $\models_{M,v,x \mapsto d} A$
 - ▶ therefore, instantiating this assumption with e : it is not the case that $\models_{M,v,x \mapsto e} A$
 - ▶ contradiction!

Conclusion

What did we cover today?

- ▶ Equivalence using Natural Deduction
- ▶ Equivalences using semantics

Further reading:

- ▶ Chapter 8 of
http://leanprover.github.io/logic_and_proof/

Next time?

- ▶ Predicate Logic – Equivalences