Mathematical and Logical Foundations of Computer Science

Lecture 8 - Propositional Logic (Equivalences & Normal Forms)

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(some slides were adapted from Rajesh Chitnis' slides)

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Where are we?

- Symbolic logic
- **Propositional logic**
- Predicate logic

Today

- Logical Equivalences
- Proving logical Equivalences in Natural Deduction
- Proving logical Equivalences using truth tables
- Normal forms

Further reading:

- Chapter 3 of
  http://leanprover.github.io/logic_and_proof/

Recap: Propositional logic syntax

Syntax:

$$P ::= a \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \neg P$$

Lower-case letters are atoms: p, q, r, etc.

Upper-case letters stand for any proposition: P, Q, R, etc.

Two special atoms:

- $\top$ which stands for True
- $\perp$ which stands for False

We also introduced four connectives:

- $P \wedge Q$: we have a proof of both P and Q
- $P \vee Q$: we have a proof of at least one of P and Q
- $P \rightarrow Q$: if we have a proof of P then we have a proof of Q
- $\neg P$: stands for $P \rightarrow \perp$

Recap: Proofs

Natural Deduction

introduction/elimination rules

natural proofs

$$\dfrac{\begin{array}{c}\overline{A}^{\,1} \\ \vdots \\ B\end{array}}{A \to B}\ 1\ [\to I]$$

Recap: Classical Reasoning

Two (equivalent) classical rules

Law of Excluded Middle (LEM)

- $\vdash A \vee \neg A$
- We will write LEM for $A \vee \neg A$

Double Negation Elimination (DNE)

- "proof by contradiction"
- $\neg\neg A \vdash A$
- Equivalently, $(\neg A) \rightarrow \bot \vdash A$
- Equivalently, $\vdash (\neg\neg A) \rightarrow A$
- We will write DNE for $(\neg\neg A) \rightarrow A$

Classical system:

- Classical Natural Deduction with LEM and DNE rules

Recap: Semantics

Semantics for “implies”

$$\phi(A \to B) = \textbf{T} \text{ iff } \phi(B) = \textbf{T} \text{ whenever } \phi(A) = \textbf{T}$$

Truth table for “implies”

P	Q	$P \to Q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

Logical equivalences

Let $A \leftrightarrow B$ be defined as $(A \to B) \wedge (B \to A)$

- it means that A and B are logically equivalent
- A and B have the same semantics
- $\phi(A) = \mathbf{T}$ if and only if $\phi(B) = \mathbf{T}$
- A is provable if and only if B is provable
- this is called a "bi-implication"
- read as "A if and only if B"

Example: we showed that DNE and LEM are equivalent

We have already proved:

- DNE $\to$ LEM
- LEM $\to$ DNE

It is then straightforward to derive a proof of DNE $\leftrightarrow$ LEM

Logical equivalences

Another example: we showed that implications are classically equivalent to their contrapositives (in classical logic)

We have proved:

- $(A \to B) \to (\neg B \to \neg A)$ in intuitionistic logic
- $(\neg B \to \neg A) \to (A \to B)$ in classical logic

It is then straightforward to derive a proof of $(A \to B) \leftrightarrow (\neg B \to \neg A)$ in classical Natural Deduction

Equivalences are for example useful in proofs to “replace” a formula by another equivalent formula

We will now present some standard ones

Logical equivalences

We are going to prove:

- De Morgan's law (I): $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$
- De Morgan's law (II): $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$
- implication elimination: $(A \to B) \leftrightarrow (\neg A \vee B)$

Some of these proofs are **intuitionistic**, while some are **classical**

In addition you can try to prove:

- Commutativity of $\wedge$: $(A \wedge B) \leftrightarrow (B \wedge A)$
- Commutativity of $\vee$: $(A \vee B) \leftrightarrow (B \vee A)$
- Associativity of $\wedge$: $((A \wedge B) \wedge C) \leftrightarrow (A \wedge (B \wedge C))$
- Associativity of $\vee$: $((A \vee B) \vee C) \leftrightarrow (A \vee (B \vee C))$
- Distributivity of $\wedge$ over $\vee$: $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$
- Distributivity of $\vee$ over $\wedge$: $(A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C))$
- Double negation elimination: $(\neg\neg A) \leftrightarrow A$
- Idempotence: $(A \wedge A) \leftrightarrow A$ and $(A \vee A) \leftrightarrow A$

Logical equivalences

As our Natural Deduction equivalence proofs will all be as follows:

$$\dfrac{\dfrac{\dfrac{\overline{A}^{\,1}}{\vdots} \quad B}{A \to B}\ 1\ [\to I] \qquad \dfrac{\dfrac{\overline{B}^{\,2}}{\vdots} \quad A}{B \to A}\ 2\ [\to I]}{A \leftrightarrow B}\ [\wedge I]$$

then, we will focus on proving

- $A \vdash B$ (left-to-right implication)
- $B \vdash A$ (right-to-left implication)

De Morgan's Laws (I): Negation of OR

Show the logical equivalence $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$ in Natural Deduction

We first prove the left-to-right implication:
$\neg(A \vee B) \vdash (\neg A \wedge \neg B)$

Here is a proof:

$$\dfrac{\dfrac{\dfrac{\neg(A \vee B) \quad \dfrac{\overline{A}^{\,1}}{A \vee B}\,[\vee I_L]}{\bot}\,[\neg E]}{\neg A}\,1\,[\neg I] \qquad \dfrac{\dfrac{\neg(A \vee B) \quad \dfrac{\overline{B}^{\,2}}{A \vee B}\,[\vee I_R]}{\bot}\,[\neg E]}{\neg B}\,2\,[\neg I]}{\neg A \wedge \neg B}\,[\wedge I]$$

Proof only uses intuitionistic rules!

Other direction on the next slide

De Morgan's Laws (I): Negation of OR

Show the logical equivalence $\neg(A \lor B) \leftrightarrow (\neg A \land \neg B)$ in Natural Deduction

We now prove the right-to-left implication:
$(\neg A \land \neg B) \vdash \neg(A \lor B)$

Here is a proof:

$$\dfrac{\dfrac{\dfrac{}{A \lor B}\,1 \qquad \dfrac{\dfrac{\dfrac{}{A}\,2 \quad \dfrac{\neg A \land \neg B}{\neg A}\,[\land E]}{\bot}\,[\neg E]}{A \to \bot}\,2\,[\to I] \qquad \dfrac{\dfrac{\dfrac{}{B}\,3 \quad \dfrac{\neg A \land \neg B}{\neg B}\,[\land E]}{\bot}\,[\neg E]}{B \to \bot}\,3\,[\to I]}{\bot}\,[\lor E]}{\neg(A \lor B)}\,1\,[\neg I]$$

Again, we only used intuitionistic rules!

De Morgan's Laws (II): Negation of AND

Show the logical equivalence $\neg(A \wedge B) \leftrightarrow \neg A \vee \neg B$ in Natural Deduction

We first prove the right-to-left implication: $\neg A \vee \neg B \vdash \neg(A \wedge B)$

Here is a proof:

$$
\dfrac{
\dfrac{
\neg A \vee \neg B
\qquad
\dfrac{
\dfrac{
\dfrac{}{\neg A}\,{}^{2}
\qquad
\dfrac{\dfrac{}{A \wedge B}\,{}^{1}}{A}\,[\wedge E_L]
}{\bot}\,[\neg E]
}{\neg A \to \bot}\,2\ [\to I]
\qquad
\dfrac{
\dfrac{
\dfrac{}{\neg B}\,{}^{3}
\qquad
\dfrac{\dfrac{}{A \wedge B}\,{}^{1}}{B}\,[\wedge E_R]
}{\bot}\,[\neg E]
}{\neg B \to \bot}\,3\ [\to I]
}{\bot}\,[\vee E]
}{\neg(A \wedge B)}\,1\ [\neg I]
$$

Proof uses intuitionistic rules!

De Morgan's Laws (II): Negation of AND

Show the logical equivalence $\neg(A \wedge B) \leftrightarrow \neg A \vee \neg B$ in Natural Deduction

We now prove the left-to-right implication: $\neg(A \wedge B) \vdash \neg A \vee \neg B$

Here is a proof (classical—we use DNE thrice):

$$
\begin{prooftree}
\AxiomC{$\neg(A \wedge B)$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$\neg A$}
\RightLabel{$[\vee I_L]$}
\UnaryInfC{$\neg A \vee \neg B$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\neg(\neg A \vee \neg B)$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{2 $[\neg I]$}
\UnaryInfC{$\neg\neg A$}
\RightLabel{$[DNE]$}
\UnaryInfC{A}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$\neg B$}
\RightLabel{$[\vee I_R]$}
\UnaryInfC{$\neg A \vee \neg B$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\neg(\neg A \vee \neg B)$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{3 $[\neg I]$}
\UnaryInfC{$\neg\neg B$}
\RightLabel{$[DNE]$}
\UnaryInfC{B}
\RightLabel{$[\wedge I]$}
\BinaryInfC{$A \wedge B$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{1 $[\neg I]$}
\UnaryInfC{$\neg\neg(\neg A \vee \neg B)$}
\RightLabel{$[DNE]$}
\UnaryInfC{$\neg A \vee \neg B$}
\end{prooftree}
$$

Expressing $\to$ using $\neg$ and $\vee$

Show the logical equivalence: $A \to B \leftrightarrow \neg A \vee B$

We first prove the left-to-right implication $A \to B \vdash \neg A \vee B$

Here is a proof (classical—it uses LEM):

$$
\dfrac{
\dfrac{}{A \vee \neg A}\,[LEM]
\quad
\dfrac{\dfrac{\dfrac{\dfrac{}{A}^{1} \quad A \to B}{B}\,[\to E]}{\neg A \vee B}\,[\vee I_R]}{A \to (\neg A \vee B)}\,1\,[\to I]
\quad
\dfrac{\dfrac{\dfrac{}{\neg A}^{2}}{\neg A \vee B}\,[\vee I_L]}{\neg A \to (\neg A \vee B)}\,2\,[\to I]
}{\neg A \vee B}\,[\vee E]
$$

The other direction holds intuitionistically (next slide)

Expressing $\rightarrow$ using $\neg$ and $\vee$

Show the logical equivalence: $A \rightarrow B \leftrightarrow \neg A \vee B$

We now prove the right-to-left implication $\neg A \vee B \vdash A \rightarrow B$

Here is a proof (intuitionistic):

$$
\dfrac{\dfrac{\neg A \vee B \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\neg A}\,2 \quad \dfrac{}{A}\,1}{\bot}\,[\neg E]}{B}\,[\bot\, E]}{\neg A \rightarrow B}\,2\ [\rightarrow I] \qquad \dfrac{\dfrac{}{B}\,3}{B \rightarrow B}\,3\ [\rightarrow I]}{B}\,[\vee E]}{A \rightarrow B}\,1\ [\rightarrow I]
$$

Logical equivalences using truth tables

Classically, two formulas are logically equivalent if they have the same semantics.

I.e., they have the same truth values for all valuations.

E.g., an implication and its contrapositive are logically equivalent:

Show that $(A \to B) \leftrightarrow (\neg B \to \neg A)$ using a truth table

A	B	$A \to B$	$\neg B$	$\neg A$	$\neg B \to \neg A$
T	**T**	**T**	**F**	**F**	**T**
T	**F**	**F**	**T**	**F**	**F**
F	**T**	**T**	**F**	**T**	**T**
F	**F**	**T**	**T**	**T**	**T**

The two formulas are equivalent because the two columns for $A \to B$ and $\neg B \to \neg A$ are identical

Normal forms

Among the formulas equivalent to a given formula, some are of particular interest:

- **Conjunctive Normal forms** (CNF)
  - $(A \vee B \vee C) \wedge (D \vee X) \wedge (\neg A)$
  - ANDs of ORs of literals (atoms or negations of atoms)
  - A **clause** in this context is a disjunction of literals
- **Disjunctive Normal Form** (DNF)
  - $(P \wedge Q \wedge A) \vee (R \wedge \neg Q) \vee (\neg A)$
  - ORs of ANDs of literals
  - A **clause** in this context is a conjunction of literals

All the variables above and the ones used in the rest of this lecture stand for atomic propositions

Every formula can be expressed in DNF

Every proposition is equivalent to a formula in DNF (OR of ANDs)!

Can you find propositions in DNF that are logically equivalent to:

- $(A \land \neg B \land \neg C) \lor X$

  Already in DNF

- Z

  Already in DNF

- $A \to B$

  Logically equivalent to $\neg A \lor B$

- $\neg(A \land B)$

  Logically equivalent (by De Morgan's law) to $\neg A \lor \neg B$

Every formula can be expressed in CNF

Every proposition is equivalent to a formula in CNF (AND of ORs)!

Can you find propositions in CNF that are logically equivalent to:

- $(A \vee \neg B \vee \neg C) \wedge X$
  Already in CNF
- Z
  Already in CNF
- $A \rightarrow B$
  Logically equivalent to $\neg A \vee B$
- $\neg(A \vee B)$
  Logically equivalent (by De Morgan's law) to $\neg A \wedge \neg B$

Every proposition can be expressed in DNF

Every proposition can be expressed in DNF (ORs of ANDs)!

Express $(P \to Q) \wedge Q$ in DNF

We do it using a truth table

P	Q	$(P \to Q)$	$(P \to Q) \wedge Q$
T	**T**	**T**	**T**
T	**F**	**F**	**F**
F	**T**	**T**	**T**
F	**F**	**T**	**F**

- Enumerate all the **T** rows from the conclusion column
  - Row 1 gives $P \wedge Q$
  - Row 3 gives $\neg P \wedge Q$
- Take **OR** of these formulas
- **Final answer** is $(P \wedge Q) \vee (\neg P \wedge Q)$

Every formula can be expressed in CNF

Every proposition can be expressed in CNF (ANDs of ORs)!

Express $(P \to Q) \wedge Q$ in CNF

We do it by using a truth table

P	Q	$(P \to Q)$	$(P \to Q) \wedge Q$
T	**T**	**T**	**T**
T	**F**	**F**	**F**
F	**T**	**T**	**T**
F	**F**	**T**	**F**

- Enumerate all the **F** rows from the conclusion column
  - Row 2 gives $P \wedge \neg Q$
  - Row 4 gives $\neg P \wedge \neg Q$
- Do **AND** of negations of each of these formulas
- We obtain $\neg(P \wedge \neg Q) \wedge \neg(\neg P \wedge \neg Q)$
- **Finally**: equivalent to $(\neg P \vee Q) \wedge (P \vee Q)$ by De Morgan

Making use of equivalences to convert to CNF/DNF

If $P \leftrightarrow Q$ and P occurs in A, then replacing P by Q in A leads to a proposition B, such that $A \leftrightarrow B$

Example:

- consider the formula $P \to Q \to (P \wedge Q)$
- we know that classically $(Q \to (P \wedge Q)) \leftrightarrow (\neg Q \vee (P \wedge Q))$
- this is an instance of $(A \to B) \leftrightarrow (\neg A \vee B)$
- when replacing $Q \to (P \wedge Q)$ by $\neg Q \vee (P \wedge Q)$ in $P \to Q \to (P \wedge Q)$, we obtain $P \to (\neg Q \vee (P \wedge Q))$
- $P \to Q \to (P \wedge Q)$ and $P \to (\neg Q \vee (P \wedge Q))$ are equivalent

Making use of equivalences to convert to CNF/DNF

We can convert a formula to an equivalent formula in CNF or DNF using the equivalences presented above (slide 10)

Example: express $(P \to Q) \wedge Q$ in CNF using known equivalences

- $(P \to Q) \wedge Q$ $(\boxed{P \to Q}) \wedge Q$
- $\leftrightarrow (\neg P \vee Q) \wedge Q$ – using $(A \to B) \leftrightarrow (\neg A \vee B)$

Example: express $\neg(P \wedge \neg Q) \wedge \neg(\neg P \wedge \neg Q)$ in CNF using known equivalences

- $\neg(P \wedge \neg Q) \wedge \neg(\neg P \wedge \neg Q)$ $\boxed{\neg(P \wedge \neg Q)} \wedge \neg(\neg P \wedge \neg Q)$
- $\leftrightarrow (\neg P \vee \neg\neg Q) \wedge \neg(\neg P \wedge \neg Q)$ $(\neg P \vee \neg\neg Q) \wedge \boxed{\neg(\neg P \wedge \neg Q)}$ – using de Morgan
- $\leftrightarrow (\neg P \vee \neg\neg Q) \wedge (\neg\neg P \vee \neg\neg Q)$ $(\neg P \vee \boxed{\neg\neg Q}) \wedge (\neg\neg P \vee \neg\neg Q)$ – using de Morgan
- $\leftrightarrow (\neg P \vee Q) \wedge (\neg\neg P \vee \neg\neg Q)$ $(\neg P \vee Q) \wedge (\boxed{\neg\neg P} \vee \neg\neg Q)$ – using double negation elim.
- $\leftrightarrow (\neg P \vee Q) \wedge (P \vee \neg\neg Q)$ $(\neg P \vee Q) \wedge (P \vee \boxed{\neg\neg Q})$ – using

Making use of equivalences to convert to CNF/DNF

Example: express $(P \to Q) \wedge Q$ in DNF using known equivalences

- $(P \to Q) \wedge Q$
- $\leftrightarrow (\neg P \vee Q) \wedge Q$ – using $(A \to B) \leftrightarrow (\neg A \vee B)$
- $\leftrightarrow Q \wedge (\neg P \vee Q)$ – using commutativity of $\wedge$
- $\leftrightarrow (Q \wedge \neg P) \vee (Q \wedge Q)$ – using distributivity of $\wedge$ over $\vee$

Conclusion

What did we cover today?

- Logical Equivalences
- Proving logical Equivalences in Natural Deduction
- Proving logical Equivalences using truth tables
- Normal forms

Further reading:

- Chapter 3 of `http://leanprover.github.io/logic_and_proof/`

Next time

- SAT