MLFCS

Solving system of linear equations using Gaussian Elimination

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Lecture attendance code:

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Today's plan

- Systems of linear Equations
- Solving system of linear equations using Gaussian elimination
- Compact way of doing Gaussian elimination
- Some extensions

Let us start with a simple example

rational number

Consider the equation $ax = b$ where x is a variable and a, b are rational numbers. How many solutions does this equation have?

- Exactly one solution → $a = 5$ $b = 10$ $x = 2$
- No solutions → $a = 0$ $b = 10$
- Infinitely many solutions → $a = 0$ & $b = 0$
- Don't know
- Question is not well-defined $a = 2$ $b = 3$

2 variables and 2 linear equations

How many solutions does the following system of linear equations have over the set of rational numbers?

$$3(x+2y) = 3(4) \longrightarrow \begin{cases} x + 2y = 4 \\ 3x + 6y = 12 \end{cases} \qquad \left(x, \frac{4-x}{2}\right) \text{ is infinite set of solutions}$$

How about this next system?

$$\begin{cases} x + y = 6 \\ 0x + 0y = 102 \end{cases}$$

2 variables and 2 linear equations

How many solutions does the following system of linear equations have over the set of rational numbers?

$$\begin{cases} 2x - y = 0 \\ x + y = 6 \end{cases}$$

$x = 2$; $y = 4$

Take a minute and solve this individually!

If you are done, verify your solution is correct by plugging it back into both equations.

Next slide: slightly harder example

How many solutions does the following system of linear equations have over the set of rational numbers?

$$2x - y = 0 \quad \text{—— } 1$$
$$x + 3y = 6 \quad \text{—— } 2$$

Take a minute and solve this individually!

If you are done, verify your solution is correct by plugging it back into both equations.

Substitution: From 1, you get $y = 2x$

Put this in 2 to get $6 = x + 3y$

$= x + 6x$

$\Rightarrow 6 = 7x$

Avoid

How did you solve this?

Elimination: Multiply 1 by 3 $\rightarrow$ $6x - 3y = 0$ —— 3

Add 2 & 3 to get $7x = 6$

Use

3 variables: how many equations should we have?

What if we have only 2 equations?

$$\begin{cases} 3x + 2y + z = 6 \\ x + 3y + 2z = 4 \end{cases}$$

Show that this has infinite number of solutions

What if we have 4 equations?

$$\begin{cases} 3x + 2y + z = 6 \\ x + 3y + 2z = 4 \\ 2x + 3y + 3z = 9 \\ x + y + z = 1 \end{cases}$$

$x = -6$
$y = -4$
$z = 11$

Each equation imposes a constraint
Each variable gives a "degree of freedom"

Interesting case is when number of variables is equal to number of equations

3 variables and 3 linear equations

$$x_1 + 5x_2 - 2x_3 = -11 \quad \text{—— } ①$$

$$3x_1 - 2x_2 + 7x_3 = 5 \quad \text{—— } ②$$

$$-2x_1 - x_2 - x_3 = 0 \quad \text{—— } ③$$

Solve this individually!

If you are done, verify your solution is correct by plugging it back into all three equations.

Want to eliminate x_2.

Multiply ③ by 2 to get $-4x_1 - 2x_2 - 2x_3 = 0$ —— ④

Subtract ④ from ② to get $\boxed{7x_1 + 9x_3 = 5}$ —— ⑤

Multiply ③ by 5 to get $-10x_1 - 5x_2 - 5x_3 = 0$ —— ⑥

Add ⑥ & ① to get $\boxed{-9x_1 - 7x_3 = -11}$

Solve the two equations in red boxes (like last slide)

n variables and n linear equations

The Gaussian elimination method

- Base case is one variable and one equation, i.e., $ax = b$
- Eliminate any variable to get $(n-1)$ equations in $(n-1)$ variables
  - We can choose which one to eliminate!
- Perform this recursively till you reach the base case.
  - Somewhere in the middle you might reach a special case
  - Special cases: No solution, infinitely many solutions, ...
  - Otherwise you will find the unique solution!

Think about how we did the above steps for example from previous slide:

$$\begin{cases} x_1 + 5x_2 - 2x_3 = -11 \\ 3x_1 - 2x_2 + 7x_3 = 5 \\ -2x_1 - x_2 - x_3 = 0 \end{cases}$$

Example with 4 variables & 4 equations

Solve this system of linear equations:

$$x_1 + 5x_2 - 2x_3 + 3x_4 = -11$$

$$3x_1 - 2x_2 + 7x_3 + x_4 = 5$$

$$-2x_1 - x_2 - x_3 - 2x_4 = 0$$

$$5x_1 + 3x_2 + 4x_3 - x_4 = 13$$

Exercise Sheet

Today's plan

- Systems of linear Equations
- Solving system of linear equations using Gaussian elimination
- Compact way of doing Gaussian elimination
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Compact way of doing Gaussian Elimination

We will use a matrix for the computation

- Matrix is a two-dimensional array

Consider the system of equations:

$$2x_1 - x_2 = 0$$

$$x_1 + x_2 = 6$$

We write this in compact form as :

$$\begin{array}{ccc} x_1 & x_2 & \\ \end{array}$$

$$\left(\begin{array}{cc|c} 2 & -1 & 0 \\ 1 & 1 & 6 \end{array}\right)$$

We now allow to perform some operations on this matrix to do Gaussian elimination:

- Rearrange rows → just reading equation in a different order
- Multiply a row by any rational number → multiplying an equation by a rational number
- Add/subtract any row from another

Why are these operations OK?

Compact way of doing Gaussian Elimination

Consider the system of equations:

$$2x_1 - x_2 = 0$$

$$x_1 + x_2 = 6$$

$$\begin{array}{c|cc|c} & x_1 & x_2 & \\ \hline R_1 & 2 & -1 & 0 \\ R_2 & 1 & 1 & 6 \end{array}$$

$$\begin{array}{c|cc|c} & x_1 & x_2 & \\ \hline R_1+R_2 & 3 & 0 & 6 \\ R_2 & 1 & 1 & 6 \end{array}$$

Operations allowed are:

- Rearrange rows
- Multiply a row by any rational number
- Add/subtract any row from another

Row R_1 gives $3x_1 = 6 \Rightarrow \boxed{x_1 = 2}$

Row R_2 gives $x_1 + x_2 = 6$ $\boxed{x_2 = 4}$

Compact way of doing Gaussian Elimination

Consider the system of equations:

$$3x_1 + x_2 + 5x_3 = +3$$

$$-3x_1 + x_2 - 2x_3 = -5$$

$$3x_1 - x_2 + 7x_3 = 10$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 3 & 1 & 5 & 3 \\ -3 & 1 & -2 & -5 \\ 3 & -1 & 7 & 10 \end{array}\right)$$

$$\downarrow$$

$$\begin{array}{c} R_1 \\ R_2+R_3 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 3 & 1 & 5 & 3 \\ 0 & 0 & 5 & 5 \\ 3 & -1 & 7 & 10 \end{array}\right)$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3-R_1 \end{array} \left(\begin{array}{ccc|c} 3 & 1 & 5 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & -2 & 2 & 7 \end{array}\right)$$

Operations allowed are:

- Rearrange rows
- Multiply a row by any rational number
- Add/subtract any row from another

Exercise: get this into row echelon form

Compact way of doing Gaussian Elimination

Aim is to get the matrix into row echelon form:

$$\left(\begin{array}{ccc|c} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & \bullet & * \end{array}\right)$$

$$\left(\begin{array}{ccc|c} \bullet & * & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right)$$

Compact way of doing Gaussian Elimination

Aim is to get the matrix into row echelon form:

$$\left(\begin{array}{cccc|c} \bullet & * & * & * & * \\ 0 & \bullet & * & * & * \\ 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & \bullet & * \end{array}\right)$$

$$\left(\begin{array}{cccc|c} \bullet & * & * & 0 & * \\ 0 & \bullet & * & 0 & * \\ 0 & 0 & \bullet & 0 & * \\ 0 & 0 & 0 & 1 & * \end{array}\right)$$

$$\left(\begin{array}{cccc|c} \bullet & * & 0 & 0 & * \\ 0 & \bullet & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \end{array}\right)$$

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Not very useful in practice!

A basic operation is addition, subtraction, multiplication or division of two rational numbers

Given n variables and n linear equations, how many "basic" operations do we need to implement Gaussian elimination?

$O(n^3)$

Iterative methods often used instead!

- Start with a solution satisfying one/some of the given equations.
- Keep adjusting it till you reach close to the correct solution

What about non-linear equations?

$$10x^2 + y = 13$$

$$11\ 5x + y^{1.3} = 7$$

Summary of the lecture

- Gaussian elimination method to solve system of linear equations
  - Be careful of special cases (no solutions, infinite solutions, etc.)
  - Eliminate one variable to recursively reach the base case of one variable
  - Plug in your solution back into each of the given equations to verify!
- Compact way to do Gaussian elimination
  - Express in matrix form
  - Use three operations to convert matrix into row echelon form
  - This is the same as the earlier method of eliminating!
- Cannot do something similar to Gaussian elimination for solving non-linear equations