MLFCS

Matrices: Definition, Examples & Matrix Multiplication

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Lecture attendance code:

Code: TO BE ADDED

Today's plan

- Definition of a matrix
- Vector space of all $(m \times n)$ matrices with entries from $\mathbb{Q}$
  - Need to define addition of two matrices
  - Need to define multiplication of a matrix by a scalar
- Definition of matrix multiplication
- Identity for square matrices
- Computing inverse of a matrix

What is a matrix?

For us, matrix is a two-dimensional array whose entries come from a field (say $\mathbb{Q}$ or $\mathbb{R}$)

- We can read it row-wise
- or column-wise

Size of a matrix

- Number of rows
- Number of columns

How to denote entries of a matrix?

Addition of two matrices over the field $\mathbb{Q}$

We can add two matrices if they have the same size

- Number of rows must be same
- Number of columns must also be the same

This is done by adding the corresponding entries from each matrix!

Example:

Multiplying a matrix by a scalar over the field $\mathbb{Q}$

Let A be any $(m \times n)$ matrix with entries from $\mathbb{Q}$

For any $r \in \mathbb{Q}$, we define the product of the scalar $r \in Q$ and the matrix A as the $(m \times n)$ matrix $B := rA$ obtained as follows:

- Each entry of B is obtained by multiplying the corresponding entry of A by r
- That is, for each $1 \leq i \leq m$; $1 \leq j \leq n$ we define $b_{i,j} = r \times a_{i,j}$

Example:

Vector space of matrices over the field $\mathbb{Q}$

Another example of a vector space:

- Fix any $m, n \geq 1$
- Then the set of all $(m \times n)$ matrices whose entries are from $\mathbb{Q}$
  - Each vector in this vector space is a $(m \times n)$ matrix
  - Each scalar in this vector space is a rational number

To show this is a vector space, we need to verify the following 8 conditions

For any vectors $\vec{u}, \vec{v}, \vec{w} \in V$ and any scalars $r, s \in F$

(1) Commutativity of vector addition: $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$
(2) Associativity of vector addition: $(\vec{u} \oplus \vec{v}) \oplus \vec{w} = \vec{u} \oplus (\vec{v} \oplus \vec{w})$
(3) Existence of Additive identity: $\vec{0} \oplus \vec{v} = \vec{v}$
(4) Existence of additive inverse: for each $\vec{x}$, there exists $\overrightarrow{-x}$ such that $\vec{x} \oplus \overrightarrow{-x} = \vec{0}$
(5) Associativity of multiplication of scalar & vector: $r(s\vec{v}) = (rs)\vec{v}$
(6) Distributivity of scalar sums: $(r + s)\vec{v} = r\vec{v} \oplus s\vec{v}$
(7) Distributivity of vector sums: $r(\vec{u} \oplus \vec{v}) = r\vec{u} \oplus r\vec{v}$
(8) Existence of identity of multiplication of scalar & vector: $1\vec{v} = \vec{v}$

First, we need to define two operations for above 8 conditions to make sense:

- <u>Vector addition</u>: for each $\vec{u}, \vec{v}$ a vector from V is assigned to $\vec{u} \oplus \vec{v}$
- <u>Multiplication of a scalar by a vector</u>: for each $s \in F$ and $\vec{v} \in V$, a vector from V is assigned to $s\vec{v}$

Definition of matrix multiplication

We now define matrix multiplication:

- Let A be an $m \times n$ matrix whose entry in row i and column j is given by a_{ij}
- Let B be an $n \times p$ matrix whose entry in row j and column k is given by b_{jk}
- Then the result of multiplying A and B is a $(n \times p)$ matrix C whose entry c_{ik} in row i and column k is given by

$$c_{ik} \quad = \quad \sum_{j=1}^{n} a_{ij} \times b_{jk} \quad = \quad (a_{i1} \times b_{1k}) + (a_{i2} \times b_{2k}) + (a_{i3} \times b_{3k}) + \ldots + (a_{in} \times b_{nk})$$

- That is, the entry in row i and column k of the matrix AB is defined to be the inner product of row i of A with column k of B

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Matrix multiplication is not commutative!

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  - For two matrices A and B, it is possible to have $AB \neq BA$

But matrix multiplication is associative!

- Let A be a $(m \times n)$ matrix
- Let B be a $(n \times p)$ matrix
- Let C be a $(p \times s)$ matrix
- Then $A(BC) = (AB)C$

Sanity checks:

- BC is well-defined and a $(n \times s)$ matrix
- So $A(BC)$ is well-defined and a $(m \times s)$ matrix
- AB is well-defined and a $(m \times p)$ matrix
- So $(AB)C$ is well-defined and a $(m \times s)$ matrix

- Proof is not very hard, but we will not cover it in this module!

Identity matrix for (2×2) matrices

Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Let A be any (2×2) matrix over the field of rational numbers

- Show that $AI = A = IA$

Identity matrix for (3×3) matrices

Let $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Let A be any (3×3) matrix over the field of rational numbers

- Show that $AI = A = IA$

Inverse of a (2×2) matrix

Let A be a (2×2) matrix.

Then a matrix B is said to be inverse of A if $BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

How can we find inverse of a specific (2×2) matrix, say $A = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$?

Inverse of a matrix need not always exist!

Show that the (2×2) matrix $A = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$ does not have an inverse.

Need to show that there is no (2×2) matrix B with entries which are rational numbers such that $BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Inverse of a (3×3) matrix

Consider the system of equations:

$$2x_1 - x_2 = 0$$

$$x_1 + x_2 = 6$$

Operations allowed were:

- Rearrange rows
- Multiply a row by any rational number
- Add/subtract any row from another

Consider the system of equations:

$$3x_1 + x_2 + 5x_3 = -3$$

$$-3x_1 + x_2 - 2x_3 = -5$$

$$3x_1 - x_2 + 7x_3 = 10$$

Operations allowed are:

- Rearrange rows
- Multiply a row by any rational number
- Add/subtract any row from another

Summary of the lecture

- Definition of a matrix
- Vector space of all $(m \times n)$ matrices with entries from $\mathbb{Q}$
  - Need to define addition of two matrices
  - Need to define multiplication of a matrix by a scalar
- Definition of matrix multiplication
  - Not always well-defined!
  - Is not commutative in general
  - Is always associative
- Identity for square matrices
- Computing inverse of a matrix
  - Can be computed using Gaussian elimination