# Mathematical and Logical Foundations of Computer Science

## Lecture 6 - Propositional Logic (Classical Reasoning)

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(some slides were adapted from Rajesh Chitnis' slides)

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# Where are we?

- Symbolic logic
- **Propositional logic**
- Predicate logic

# Today

- Classical Reasoning
- Constructive vs. Classical Natural Deduction

### Further reading

- Chapter 5 of
  http://leanprover.github.io/logic_and_proof/

# Recap: Propositional logic syntax

**Syntax:**

$$P ::= a \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \neg P$$

**Two special atoms:**

- $\top$ which stands for True
- $\bot$ which stands for False

**We also introduced four connectives:**

- $P \wedge Q$: we have a proof of both $P$ and $Q$
- $P \vee Q$: we have a proof of at least one of $P$ and $Q$
- $P \rightarrow Q$: if we have a proof of $P$ then we have a proof of $Q$
- $\neg P$: stands for $P \rightarrow \bot$

# Recap: Proofs

**Natural Deduction**

introduction/elimination rules

natural proofs

$$\dfrac{\dfrac{}{A}\,1 \\ \vdots \\ B}{A \to B}\ 1\ [\to I]$$

# Classical Reasoning

The proof systems we have seen so far are sometimes called **constructive** or **intuitionistic**, i.e., **proofs** can be viewed as **programs**:

- A proof of $A \wedge B$ can be viewed as a **pair** of a proof of $A$ and a proof of $B$
- A proof of $A \rightarrow B$ can be viewed as a **procedure** which transforms evidence for $A$ into evidence for $B$
- A proof of $A \vee B$ is either a proof of $A$ or a proof of $B$, which indicates which one it is

There are other proof systems, called **classical**, which

- rely on Boolean truth values
- introduce additional reasoning principles

# Classical Reasoning: Proof by Contradiction

A typical classical reasoning principal is the **“proof by contradiction”** proof technique

**Example: Euclid’s proof of infinitude of primes**

- Assume the negation: Suppose there are only finitely many primes, say $p_1, p_2, \ldots, p_r$
- Consider the number $n = (p_1 \times p_2 \times \ldots \times p_r) + 1$
- Then $n$ cannot be a prime (by assumption)
- But none of the primes $p_1, p_2, \ldots, p_r$ can divide $n$
- Contradiction

**Proof by Contradiction:**

- If $\neg A \rightarrow \perp$ then $A$
- That is, $\neg\neg A \vdash A$

# Negation of a negation is?

Can we deduce $A$ and $\neg\neg A$ from each other? That is, are they equivalent?

One direction is easy: $A \vdash \neg\neg A$

Here is the proof:

$$\dfrac{\dfrac{A \qquad \dfrac{}{\neg A}\,1}{\bot}\,[\neg E]}{\neg\neg A}\,1\ [\neg I]$$

Can we show the other direction, i.e., $\neg\neg A \vdash A$?

Not using the current set of inference rules we have!

# Classical vs. Intutionistic Reasoning in Natural Deduction

### Two more (equivalent) assumptions/rules

### Law of Excluded Middle (LEM)

- For each $A$, we can always prove one of $A$ or $\neg A$
- i.e., $\vdash A \vee \neg A$
- E.g., we can assume every even natural number $> 2$ is the sum of two primes, or not, without knowing which one is true

### Double Negation Elimination (DNE)

- $\neg\neg A \vdash A$
- Equivalently, $(\neg A) \to \bot \vdash A$
- "proof by contradiction"

# Classical vs. Intutionistic Reasoning in Natural Deduction

### Two more (equivalent) assumptions/rules

As rules:

$$\frac{}{A \vee \neg A}\ [LEM] \qquad \frac{\neg\neg A}{A}\ [DNE]$$

Classical reasoning allows using these two rules

We so far have not used them, and were therefore using what is called **constructive** or **intuitionistic** logic

# LEM implies DNE

Assuming $A \vee \neg A$, infer $\neg\neg A \vdash A$

Here is a proof:

$$\dfrac{A \vee \neg A \qquad \dfrac{\dfrac{}{A}\;1}{A \to A}\;1\ [\to I] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\neg A}\;2 \qquad \neg\neg A}{\bot}\;[\neg E]}{A}\;[\bot\ E]}{\neg A \to A}\;2\ [\to I]}{A}\;[\vee E]$$

# DNE implies LEM

Assuming $\neg\neg A \vdash A$, infer $\vdash A \vee \neg A$

Here is a proof:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\neg(A \vee \neg A)$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\neg(A \vee \neg A)$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$A$}
\RightLabel{$[\vee I_L]$}
\UnaryInfC{$A \vee \neg A$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{2 $[\neg I]$}
\UnaryInfC{$\neg A$}
\RightLabel{$[\vee I_R]$}
\UnaryInfC{$A \vee \neg A$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{1 $[\neg I]$}
\UnaryInfC{$\neg\neg(A \vee \neg A)$}
\RightLabel{$[DNE]$}
\UnaryInfC{$A \vee \neg A$}
\end{prooftree}
$$

# Contrapositive

Given an implication $A \to B$, the formula $\neg B \to \neg A$ is called the **"contrapositive"**

Can we prove that an implication implies its contrapositive?

$$A \to B \vdash \neg B \to \neg A$$

Here is a proof (intuitionistic):

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to B \quad \dfrac{}{A}\ 2}{B}\ [\to E] \qquad \dfrac{}{\neg B}\ 1}{\bot}\ [\neg E]}{\neg A}\ 2\ [\neg I]}{\neg B \to \neg A}\ 1\ [\to I]$$

The other direction holds in classical logic (next slide)

# Contrapositive

Given an implication $A \to B$, the formula $\neg B \to \neg A$ is called the **"contrapositive"**

Can we prove that an implication follows from its contrapositive?
$\neg B \to \neg A \vdash A \to B$

Here is a proof (classical):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A}\,1 \qquad \dfrac{\neg B \to \neg A \qquad \dfrac{}{\neg B}\,2}{\neg A}\,[\to E]}{\bot}\,[\neg E]}{\neg\neg B}\,2\ [\neg I]}{B}\,[DNE]}{A \to B}\,1\ [\to I]$$

We used DNE, and hence this proof uses classical reasoning!

# Classical Reasoning Through Examples

We will present classical proofs of:

- $(A \to B) \vee (B \to A)$
- $(\neg B \to \neg A) \to (A \to B)$

We saw a classical proof of $(\neg B \to \neg A) \to (A \to B)$ before using DNE – we will present an alternative proof that uses LEM instead

Which we will prove in classical Natural Deduction.

# Example 1

Provide a classical Natural Deduction proof of $(A \to B) \lor (B \to A)$

$$
\dfrac{
\dfrac{}{A \lor \neg A}\,[LEM]
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{A}\,1}{B \to A}\,2\;[\to I]}{(A \to B) \lor (B \to A)}\,[\lor I_R]}{A \to (A \to B) \lor (B \to A)}\,1\;[\to I]
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A}\,3 \quad \dfrac{}{A}\,4}{\bot}\,[\neg E]}{B}\,[\bot\, E]}{A \to B}\,4\;[\to I]}{(A \to B) \lor (B \to A)}\,[\lor I_L]}{\neg A \to (A \to B) \lor (B \to A)}\,3\;[\to I]
}{(A \to B) \lor (B \to A)}\,[\lor E]
$$

Hypotheses:

- hyp. $1$: $A$
- hyp. $2$: $B$
- hyp. $3$: $\neg A$
- hyp. $4$: $A$

# Example 2

Provide a classical Natural Deduction proof of
$(\neg B \to \neg A) \to (A \to B)$
Here is a proof:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$[LEM]$}
\UnaryInfC{$B \vee \neg B$}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$B$}
\RightLabel{3 $[\to I]$}
\UnaryInfC{$B \to B$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\neg B \to \neg A$}
\AxiomC{}
\RightLabel{4}
\UnaryInfC{$\neg B$}
\RightLabel{$[\to E]$}
\BinaryInfC{$\neg A$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$A$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{$[\bot E]$}
\UnaryInfC{$B$}
\RightLabel{4 $[\to I]$}
\UnaryInfC{$\neg B \to B$}
\RightLabel{$[\vee E]$}
\TrinaryInfC{$B$}
\RightLabel{2 $[\to I]$}
\UnaryInfC{$A \to B$}
\RightLabel{1 $[\to I]$}
\UnaryInfC{$(\neg B \to \neg A) \to (A \to B)$}
\end{prooftree}
$$

Hypotheses:

- hyp. $1$: $\neg B \to \neg A$
- hyp. $2$: $A$
- hyp. $3$: $B$
- hyp. $4$: $\neg B$

# Conclusion

### What did we cover today?

- Classical Reasoning
- Constructive vs. Classical Natural Deduction

### Further reading

- Chapter 5 of `http://leanprover.github.io/logic_and_proof/`
- "Proofs and Types", Girard, Taylor, and Lafont, Chapter 5

### Next time

- Propositional logic's (classical) semantics