1./

$$\beta_G^3 = \big((X_{0,0} \wedge X_{1,1}) \vee (X_{0,0} \wedge X_{1,2}) \vee (X_{0,1} \wedge X_{1,2}) \vee (X_{0,1} \wedge X_{1,3})\big) \wedge$$
$$\big((X_{1,0} \wedge X_{2,1}) \vee (X_{1,0} \wedge X_{2,2}) \vee (X_{1,1} \wedge X_{2,2}) \vee (X_{1,1} \wedge X_{2,3})\big) \wedge$$
$$\big((X_{0,0} \wedge X_{2,1}) \vee (X_{0,0} \wedge X_{2,2}) \vee (X_{0,1} \wedge X_{2,2}) \vee (X_{0,1} \wedge X_{2,3})\big)$$

0^{th}	1^{st}	1^{st}	2^{nd}	0^{th}	2^{nd}
0	1	0	1	0	1
0	2	1	2	0	2
1	2	1	3	1	2
1	3	0	2	1	3

I have highlighted values which will never be chosen, regardless of other nodes, and have skeptically included them, even though they will never evaluate true in the full β. The reason these nodes will never evaluate true is that for $X_{0,1} \wedge X_{2,2}$ and $X_{0,0} \wedge X_{2,1}$ we require a number between 1 and 2, and 0 and 1 respectively. For $X_{1,0} \wedge X_{2,1}$ and $X_{1,0} \wedge X_{2,2}$ we require a number smaller than 0 for the 0^{th} position. For Q2 I have constructed generalised constraints to deal with this, but not included for answer for β_G as I think it's outside of whats asked.

2/

$$\beta_G^k = \bigwedge_{i<j<k} \left(\bigvee_{[(p,q)\in E \mid p<q]} (X_{i,p} \wedge X_{j,q}) \right)$$

For $n=4$ and $k=3$:

$p-(n-k) \leq i \leftarrow$ Bans

$q-p \geq j-i \leftarrow$ Bans

$p \geq i,\ q \geq j \leftarrow$ Bans

These 3 constraints ban edges that don't make sense, but even with only the constraint $p<q$, only the assignments that return a clique will return β as true, but the constraints mean that validating an answer is more efficient.

3/

For β_G^k, $\bigwedge_{i<j<k} \left(\bigvee_{[(p,q)\in E \mid p<q]} (X_{i,p} \wedge X_{j,q}) \right)$

This loops through all edges in E, hence $|E|$.

This loops through j from $0-(k-1)$ times, so k and through i 0, then 1, then 2, up to $(k-1)$ times. So $k(k-1)$ at a maximum.

And as each pair has 2 atoms, we include a constant of 2, giving us: $|\beta_G^k| = 2|E|(k^2-k)$

For $\mathbb{C}_G^k$, $\bigwedge_{i<k} \left(\bigvee_{p<n} X_{i,p} \right) \wedge \bigwedge_{i<k} \neg \left(\bigvee_{p<q<n} (X_{i,p} \wedge X_{i,q}) \right)$

This loops p up to n times and i up to k times, giving us nk.

This loops through q n times and p $(n-1)$ times, with 2 atoms each loop, giving us $2n(n-1)$

This loops through i k times, giving us k

In total we have, $|\mathbb{C}_G^k| = nk + 2kn(n-1)$

So $|\varphi_G^k| = nk + 2kn(n-1) + 2|E|(k^2-k)$.

We know that $k \leq n$, and it can be shown that $|E|$ is at most $\frac{n^2-n}{2}$,

giving us $|\varphi_G^k| = n^2 + 2n^3 - n^2 + 2\frac{n^2-n}{2}(n^2-n)$

$= 2n^3 + n^4 - 2n^3 + n^2$

$= n^4 + n^2$

So, $P(n) = n^4 + n^2$ is an upper bound for the number of atoms in φ_G^k

So there exists a polynomial such that $P(n) \geq |\varphi_G^k|$

4/ If it were the case that $P=NP$, then the would be a way to not only validate whether a solution represents a clique in polytime, but also generate a valid solution within polynomial time. This would mean that given a graph G and clique size k, we could ascertain whether the graph contains a clique of size k in polynomial time, along with all other SAT problems.