Fields and real numbers

1 Coprime numbers

Two positive natural numbers are *coprime* when their highest common factor is 1. For example 8 and 31 are coprime, but 8 and 30 are not. Coprimeness is significant whenever we have cycles. Example:

- In Coprimeland, weeks are 8 days long and months are 31 days long. Over a cycle of 8×31 days, each day-date combination happens exactly once.
- In Factorland, weeks are 8 days long and months are 30 days long. Some day-date combinations never happen.

2 Rational numbers

We've seen that $\mathbb{Z}$ with addition and multiplication forms a commutative ring. A *rational number* is one that can be expressed as $\frac{m}{n}$, for integers m and n, with $n \neq 0$. For example $\frac{-37}{5}$. For a positive rational number, we can make this representation unique by requiring m and n to be positive and coprime.

The set of rational numbers is written **Q**. Any sum, difference or product of rational numbers is rational. In fact, $\mathbb{Q}$ forms a *field*, i.e., not only do all the commutative ring laws hold, but every non-zero number has a multiplicative inverse:

$$a \times a^{-1} \quad = \quad 1 \quad (a \neq 0)$$

From the field laws, we can derive the *multiplicative cancellation* law: if $a \neq 0$ and $a \times x = a \times y$, then $x = y$. (This also holds in $\mathbb{Z}$, which isn't a field.) We define division via $a \div b \stackrel{\text{def}}{=} a \times b^{-1}$.

3 Real numbers

The set of real numbers is written $\mathbb{R}$. Like $\mathbb{Q}$, it is a field. Examples of real numbers are $-\sqrt{17}$ and π.

4 Real intervals

We write real intervals using a comma. For example:

- $[-3, 5)$ is the set of real numbers n such that $-3 \leqslant n < 5$.
- $[-3, 5]$ is the set of real numbers n such that $-3 \leqslant n \leqslant 5$.
- $[23, 23)$ is the set of real numbers n such that $23 \leqslant n < 23$. This is the empty set.
- $[23, \infty)$ is the set of real numbers n such that $23 \leqslant n < \infty$.
- $\mathbb{R}$ is $(-\infty, \infty)$.

5 Exponentiation and logarithm

I'm assuming you've seen these before, but revision is useful. Fill in the following chart:

$$\begin{aligned}
10^3 &= ? \\
10^{-3} &= ? \\
10^0 &= ? \\
25^{\frac{1}{2}} &= ? \\
25^{\frac{3}{2}} &= ? \\
25^{-\frac{3}{2}} &= ? \\
\log_{10} 100000 &= ? \\
\log_{10} 0.0001 &= ? \\
\log_{10} 1 &= ?
\end{aligned}$$

Warning: a^n is only defined when $a > 0$ or $n \in \mathbb{N}$. Likewise, $\log_a b$ is only defined when $a > 0$. Here are some properties of exponentiation:

$$\begin{aligned}
a^0 &= 1 \\
a^{m+n} &= a^m \times a^n \\
a^{m-n} &= a^m \div a^n \\
a^{mn} &= (a^m)^n
\end{aligned}$$

And some properties of logarithm. (Thinking $a = 10$ may be helpful.)

$$\begin{aligned}
\log_a 1 &= 0 \\
\log_a(c \times d) &= \log_a c + \log_a d \\
\log_a(c \div d) &= \log_a c - \log_a d \\
\log_b a &= 1 \div \log_a b \\
\log_a c &= \log_a b \times \log_b c
\end{aligned}$$

To illustrate the last one, $\log_{10} c = 2 \times \log_{100} c$. Putting $c = 1,000,000$ gives $6 = 2 \times 3$.

6 Floor and ceiling

The floor and ceiling operations are often used. The floor of a, written $\lfloor a \rfloor$, is a rounded down to an integer. The ceiling of a, written $\lceil a \rceil$, is a rounded up to an integer. For example:

$$\begin{aligned}
\lfloor 7.3 \rfloor &= ? \\
\lceil 7.3 \rceil &= ? \\
\lfloor -7.3 \rfloor &= ? \\
\lceil -7.3 \rceil &= ?
\end{aligned}$$

In fact, $\lfloor a \rfloor$ is just the same as $a \bmod 1$.

7 The radix point

In base ten, the notation 235.76 represents the number

$$2 \times 10^2 + 3 \times 10^1 + 5 \times 10^0 + 7 \times 10^{-1} + 6 \times 10^{-2}$$

The same idea works in other bases. For example, in binary, the notation 1011.00101 represents the number

$$2^3 + 2^1 + 2^0 + 2^{-3} + 2^{-5}$$

which is $11\frac{5}{32}$. Typically a real number has an expansion that continues forever. For example:

$$\pi = 3.14159\ldots$$

A representation of a rational number ends in a recurring sequence of digits, whereas a representation of an irrational number does not. Beware that representations ending in $\dot{0}$ or $\dot{9}$ are not unique, e.g. $7.32\dot{9} = 7.33$.

8 Scientific notation

Let's recall base ten scientific notation. Any positive number can be uniquely expressed as $m \times 10^n$, where m is in the range $[1, 10)$ and n is an integer. This may be written as $m\,\mathrm{E}\,n$, and we call m the *mantissa* and n the *exponent*. For example Avogadro's constant is $6.022\,\mathrm{E}\,23$. Note that the mantissa has just one digit before the point, which is either 1,2,3,4,5,6,7,8 or 9.

This can be adapted to other bases (greater than one), such as base two. Any positive number can be expressed as $m \times 2^n$, where m is in the range $[1, 2)$ and n is an integer. The digit before the point is necessarily 1, so it doesn't need to be stored.

9 Floating-point

As we have seen scientific notation is a precise way to represent every positive real number. But it has two practical limitations:

- The mantissa has infinitely many digits after the point.
- The exponent can be a very large or very small integer.

Accordingly, to represent positive real numbers in memory, we make two compromises:

- We round the mantissa to a fixed number of digits.
- We restrict the exponent to a fixed range of integers.

For example, we might allocate 16 bits, with the first 8 bits representing the mantissa rounded to 8 places after the point, and second 8 bits representing the exponent n in the range $[-2^7 \mathinner{..} 2^7)$. For the latter, we do not use complement notation, but use a *bias* of 2^7. That means we represent n by $n + 2^7$, which is in the range $[0 \mathinner{..} 2^8)$. This representation is called *8+8-bit floating-point*.

In this case, what does the bit-pattern

1011 0010 1101 0110

represent? The mantissa is 1.10110010_2, which is

$$2^0 + 2^{-1} + 2^{-3} + 2^{-4} + 2^{-7} = 1\frac{89}{128}$$

The exponent is

$$\begin{aligned} 11010110_2 - 128 &= 64 + 16 + 4 + 2 \\ &= 86 \end{aligned}$$

So the number is $1\frac{89}{128} \times 2^{86}$.

Although the limited exponent range might be an issue, most of the problems arise from the rounding of the mantissa. A case in point is when we add a big number a to a small number b, resulting in a. To illustrate, in base ten with the mantissa rounded to 3 decimal places, we have

$$3.404 \times 10^8 + 7.191 \times 10^2 = 3.404007191 \times 10^8$$

which rounds to 3.404×10^8.

Formats used in practice (such as IEEE-754 format used for the `float` type in Java) allocate a *sign bit* so that they can represent both positive and negative real numbers. They also have special bit-patterns, to represent 0 and various overflow situations known as NaN (not a number).