MLFCS

Vector Spaces: Span, Linear (in)dependence & Basis

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Lecture attendance code:

Code: TO BE ADDED

Today's plan

- Recall: Span of a set of vectors
- Recall: Linear (in)dependence of a set of vectors

- Definition: Basis of a vector space
- Our go-to examples of two-tuples or three-tuples of rationals
  - (standard) basis for these vector spaces
  - size of basis for these vector spaces
  - inner product for these vector spaces
  - orthogonal basis for these vector spaces

Vector space V over a field F

Let V be a set of vectors and F be any field. Then V is said to be a vector space over the field F if the following conditions hold:

For any vectors $\vec{u}, \vec{v}, \vec{w} \in V$ and any scalars $r, s \in F$

(1) Commutativity of vector addition: $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$
(2) Associativity of vector addition: $(\vec{u} \oplus \vec{v}) \oplus \vec{w} = \vec{u} \oplus (\vec{v} \oplus \vec{w})$
(3) Existence of Additive identity: $\vec{0} \oplus \vec{v} = \vec{v}$
(4) Existence of additive inverse: for each $\vec{x}$, there exists $\overrightarrow{-x}$ such that $\vec{x} \oplus \overrightarrow{-x} = \vec{0}$
(5) Associativity of multiplication of scalar & vector: $r(s\vec{v}) = (rs)\vec{v}$
(6) Distributivity of scalar sums: $(r + s)\vec{v} = r\vec{v} \oplus s\vec{v}$
(7) Distributivity of vector sums: $r(\vec{u} \oplus \vec{v}) = r\vec{u} \oplus r\vec{v}$
(8) Existence of identity of multiplication of scalar & vector: $1\vec{v} = \vec{v}$

First, we need to define two operations for above 8 conditions to make sense:

- <u>Vector addition</u>: for each $\vec{u}, \vec{v}$ a vector from V is assigned to $\vec{u} \oplus \vec{v}$
- <u>Multiplication of a scalar by a vector</u>: for each $s \in F$ and $\vec{v} \in V$, a vector from V is assigned to $s\vec{v}$

Note that multiplication of two vectors is not defined here!

Go-to examples of a vector space

The set of 2-tuples of rational numbers is a vector space over the rational numbers:

- The set of 2-tuples of rational numbers is defined as $\mathbb{Q}^2 := \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : a, b \in \mathbb{Q} \right\}$

The set of 3-tuples of rational numbers is a vector space over the rational numbers:

- The set of 3-tuples of rational numbers is defined as $\mathbb{Q}^2 := \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} : a, b, c \in \mathbb{Q} \right\}$

- To verify each of the 8 conditions for the above two examples to be vector spaces, you will just need to use the fact that $\mathbb{Q}$ is a field.
  - Details for first example given in solution of Q4 of Week 8 exercise sheet

Span of a set of vectors

- Let V be a vector space over a field F.
- Let $\vec{v_1}, \vec{v_2}$ be two vectors in V
- We define $\text{Span}(\vec{v_1}, \vec{v_2}) := \{r_1\vec{v_1} \oplus r_2\vec{v_2} \mid r_1, r_2 \in F\}$
  - All possible linear combinations of $\vec{v_1}$ and $\vec{v_2}$
  - Span of $\vec{v_1}$ and $\vec{v_2}$
- Exercise: $\text{Span}(\vec{v_1}, \vec{v_2})$ is a vector space over the field F.
  - Simple verification of 8 conditions & using the fact that V is a vector space over F
  - Details presented in solution of Q3 in Week 9 exercise sheet

In general, given any set of vectors from V we can define its span.

How to check if a given vector belongs to a span?

Consider the vector space $\mathbb{Q}^3$ over $\mathbb{Q}$ of 3-tuples of rational numbers.

- Consider the vectors $\vec{v_1} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} 5 \\ 6 \\ 10 \end{pmatrix}, \vec{v_3} = \begin{pmatrix} 11 \\ 300 \\ 14 \end{pmatrix}$ from $\mathbb{Q}^3$
- I want to check if the vector $\vec{w} = \begin{pmatrix} 41 \\ 12 \\ 110 \end{pmatrix}$ belongs to $\mathrm{Span}(\vec{v_1}, \vec{v_2}, \vec{v_3})$
- How can we do that?
- Recall that $\mathrm{Span}(\vec{v_1}, \vec{v_2}, \vec{v_3}) := \left\{ r_1\vec{v_1} \oplus r_2\vec{v_2} \oplus r_3\vec{v_3} \mid r_1, r_2, r_3 \in F \right\}$

See solutions of Q1 and Q2 of Week 9 exercise sheet for more examples about spans

Linear (in)dependence of a set of vectors

- Let V be a vector space over a field F.
- A set of vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n} \in V$ is linearly independent if $r_1\vec{v_1} \oplus r_2\vec{v_2} \oplus \ldots \oplus r_n\vec{v_n} = \vec{0}$ implies $r_1 = r_2 = \ldots = r_n = 0$
  - Otherwise, the set of vectors is said to be linearly dependent
- Does $\vec{0}$ always belongs to the span of any set of vectors?
  -
- Therefore, a set of vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n} \in V$ is linearly independent if the only way to obtain $\vec{0}$ in its span is by taking all the scalars to be 0
- Question: Can $\vec{0}$ belong to any set of linearly independent vectors?
  -
- Next slide: How can we check if a given set of vectors is linearly independent or not?

Checking linear independence of a set of vectors

Consider the vector space $\mathbb{Q}^3$ over $\mathbb{Q}$ of 3-tuples of rational numbers.

- Consider the vectors $\vec{v_1} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} 5 \\ 6 \\ 10 \end{pmatrix}, \vec{v_3} = \begin{pmatrix} 11 \\ 300 \\ 14 \end{pmatrix}$ from $\mathbb{Q}^3$
- Want to check if these three vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are linearly independent
- How can we do that?
-

See solution of Q4 of Week 9 exercise sheet for how to show two given vectors in the vector space of 2-tuples of rational numbers are not linearly independent

Basis of a vector space

Let V be a vector space over a field F.

Definition: A set of vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n} \in V$ forms a basis if **both** the following two conditions are satisfied:

- $\text{Span}(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}) = V$
  - That is, every vector in V can be represented as a linear combination of the given vectors
  - If $\text{Span}(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}) = V$ then the set of vectors $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ is called as a spanning set
- The set of vectors $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ is linearly independent
  - That is, $r_1\vec{v_1} \oplus r_2\vec{v_2} \oplus \ldots \oplus r_n\vec{v_n} = \vec{0}$ implies $r_1 = r_2 = \ldots = r_n = 0$

Consider our go-to examples of the vector spaces:

- $\mathbb{Q}^2$ over $\mathbb{Q}$, i.e., 2-tuples of rational numbers over the field of rational numbers
- $\mathbb{Q}^3$ over $\mathbb{Q}$, i.e., 3-tuples of rational numbers over the field of rational numbers

Two questions about these (go-to) examples of vector spaces:

- Do these vector spaces have a basis?
- If yes, can you find one?

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  - That is, $r_1\vec{v_1} \oplus r_2\vec{v_2} \oplus \ldots \oplus r_n\vec{v_n} = \vec{0}$ implies $r_1 = r_2 = \ldots = r_n = 0$

Two theorems about basis of vector spaces

- Theorem 1: Every vector space has a basis
- Theorem 2: Every basis of a vector space has the same number of vectors
  - The number of vectors in a basis of a vector space is called as its dimension

Proofs of Theorem 1 and Theorem 2 are beyond the scope of this module!

Questions:

- What is the dimension of the vector space $\mathbb{Q}^2$ of two-tuples of rational numbers over the field of rational numbers?
  - Answer:
- What is the dimension of the vector space $\mathbb{Q}^3$ of three-tuples of rational numbers over the field of rational numbers?
  - Answer:

Definition: Inner product of two vectors

Recall (slide 4 of this lecture) that we did not define multiplication of two vectors for (general) vector spaces!

However, we can define (inner) product of two vectors for our two go-to examples of vector spaces:

- Vector space $\mathbb{Q}^2$ of two-tuples of rational numbers over the field of rational numbers $\mathbb{Q}$
  - If $\vec{u} = \binom{u_1}{u_2}$ and $\vec{v} = \binom{v_1}{v_2}$, then the inner product of $\vec{u}$ and $\vec{v}$ is defined as $\vec{u} \cdot \vec{v} = (u_1 \times v_1) + (u_2 \times v_2)$
  - <u>Example</u>:
- Vector space $\mathbb{Q}^3$ of two-tuples of rational numbers over the field of rational numbers $\mathbb{Q}$
  - If $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ u_2 \\ u_3 \end{pmatrix}$, then the inner product of $\vec{u}$ and $\vec{v}$ is defined as $\vec{u} \cdot \vec{v} = (u_1 \times v_1) + (u_2 \times v_2) + (u_3 \times v_3)$
  - <u>Example</u>:

Definition: Orthogonal basis

We can define what it means for a basis to be orthogonal if there is some notion of inner product between vectors

Hence, we can define orthogonal basis for our two go-to examples of vector spaces:

- Vector space $\mathbb{Q}^2$ of two-tuples of rational numbers over the field of rational numbers $\mathbb{Q}$
  - A basis $\{\vec{u}, \vec{v}\}$ for $\mathbb{Q}^2$ over $\mathbb{Q}$ is orthogonal if $\vec{u} \cdot \vec{v} = 0$
  - Question: Does this vector space have an orthogonal basis?
- Vector space $\mathbb{Q}^3$ of two-tuples of rational numbers over the field of rational numbers $\mathbb{Q}$
  - A basis $\{\vec{u}, \vec{v}, \vec{w}\}$ for $\mathbb{Q}^3$ over $\mathbb{Q}$ is orthogonal if $\vec{u} \cdot \vec{v} = 0$, $\vec{u} \cdot \vec{w} = 0$ and $\vec{v} \cdot \vec{w} = 0$
  - Question: Does this vector space have an orthogonal basis?

What is the advantage of an orthogonal basis?

Consider the vector space $\mathbb{Q}^2$ of two-tuples of rational numbers over the field of rational numbers $\mathbb{Q}$

- Consider the orthogonal basis given by $\vec{u} = \binom{1}{1}$ and $\vec{v} = \binom{1}{-1}$ is orthogonal
- I want to find how $\vec{w} = \binom{11}{-9}$ can be represented as linear combination of
    - Since $\{\vec{u}, \vec{v}\}$ form a basis we know $\vec{w} \in \mathrm{Span}(\vec{u}, \vec{v})$
- Suppose $\vec{w} = r\vec{u} \oplus s\vec{v}$. How can we find r and s easily?

Summary of the lecture

- Recall: Span of a set of vectors
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