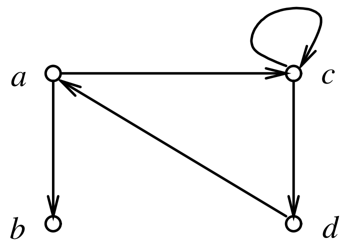
Relations on a set

1 Relations and directed graphs

Given a set A, we call a binary relation from A to itself an *endorelation* on A, and a function from A to itself an *endofunction* on A. For example, "sister" is an endorelation on the set of people, and "square" is an endofunction on the set of natural numbers. Often we drop the "endo" and just say 'relation on A" or "function on A".

A *directed graph* is a finite set A together with an endorelation R. They often appear in algorithmic problems and AI. The elements of A are called *vertices* or *nodes* and the ordered pairs $(x, y) \in R$ are called *edges*. Here isan example.



$$A = \{a,b,c,d\}$$

$$R = \{(a,c),(a,b),(c,d),(c,c),(d,a)\}$$

2 Equivalence relations

2.1 Definition

Consider the same-age relation on the set of all people. It is

- *reflexive*: every element x is related to itself.
- *symmetric*: if x is related to y, then y is related to x
- *transitive*: if x is related to y, and y to z, then x is related to z.

A relation with these three properties is called an *equivalence relation*. For example, the relation on $\mathbb{Z}$ given by congruence mod 23 is an equivalence relation. We often use symbols such as $\equiv$ or $\sim$ to indicate an equivalence relation.

2.2 Equivalence classes

When we have an equivalence relation $\equiv$ on a set A, each element x has an *equivalence class*—the set of all elements equivalent to it. For example, in the same-age of relation, my equivalence class is the set of all people with the same age as me. The equivalence class of x may be written $[x]_\equiv$. Thus we have

$$[x]_\equiv \quad \equiv \quad \{y \in A \mid x \equiv y\}$$

Note that we always have $x \in [x]_\equiv$. Also, if $x \equiv y$, then $[x]_\equiv = [y]_\equiv$. On the other hand, if $x \not\equiv y$, then $[x]_\equiv$ and $[y]_\equiv$ are disjoint.

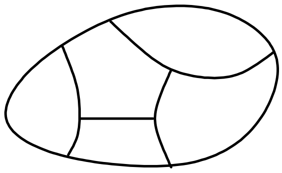
The set of all equivalence classes is written $A/\equiv$.

2.3 Partitions

A *partition* of a set A is a set $\mathcal{B}$ of subsets of A with the following properties:

- any two members of $\mathcal{B}$ are disjoint
- any element of A belongs to a member of $\mathcal{B}$.

Here's a picture:



Thus an equivalence relation $\equiv$ on A gives a partition $A/\equiv$ on A.

Conversely, any partition $\mathcal{B}$ on A gives an equivalence relation: two elements of A are related when they belong to the same member of $\mathcal{B}$.

This leads to a one-to-one correspondence between equivalence relations on A and partitions of A.

2.4 Kernel of a function

Whenever we have a function $f : A \to B$, its *kernel* is the equivalence relation on A, that relates two elements with the same image. For example, the kernel of the age function from the set of people to $\mathbb{N}$ is the same-age relation on the set of people. The kernel of f is written $\ker(f)$.

We then have a correspondence between $A/\mathsf{ker}(f)$ and the range of f. For example, the set of all people aged 19 corresponds to the number 19, because it's the set of all preimages of that number. There's no equivalence class corresponding to the number 5000, because that number isn't in the range of the age function, as it has no preimage.

3 Other kinds of relation

Again, let A be a set.

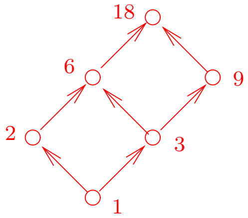
A relation on A is *irreflexive* when nothing is related to itself. For example, the relation $<$ on $\mathbb{N}$.

A relation on A is *antisymmetric* when for any x and y that are mutually related, we have $x = y$. For example, the relation $\leqslant$ on $\mathbb{N}$.

An *order* on A is a relation that's reflexive, transitive and antisymmetric. For example: for a set C, the subset relation on $\mathcal{P}C$ is an order.

Another example, for integers x and y, we say that x *divides* y when $y = nx$ for some integer x. The usual notation is $x|y$. This is a partial order on $\mathbb{N}$ but not on $\mathbb{Z}$.

We often use a *Hasse diagram* to depict an ordered set. For example, here is the set of positive factors of 18, ordered by the $|$ relation:



Given a relation R on A, we say that two elements x and y are said to be *comparable* when either $x = y$ or $(x, y) \in R$ or $(y, x) \in R$. A *linear order* is an order where any two elements are comparable. For example, the $\leqslant$ relation on $\mathbb{N}$. But the subset order on $\mathcal{P}\mathbb{N}$ is not linear because the subsets $\{2, 3, 4\}$ and $\{4, 5, 6, 7\}$ are incomprable.