Maths for Complexity Analysis

$$a^n = \underbrace{a \times a \times \ldots \times a}_{n \text{ times}}$$

$$a^m a^n = \underbrace{a \times \ldots \times a}_{m \text{ times}} \times \underbrace{a \times \ldots \times a}_{n \text{ times}} = a^{m+n}$$

$$(a^m)^n = \underbrace{a^m \times \cdots \times a^m}_{n \text{ times}} = a^{mn} = (a^n)^m$$

$$a^{(m^n)} = a^{\overbrace{m \times \cdots \times m}^{n \text{ times}}}$$

NOTE: $a^{(m^n)} \neq (a^m)^n$

$$a^0 = 1 \quad \text{because } a^0 a^1 = a^{0+1} = a^1 \Rightarrow a^0 = 1$$

$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad \text{because } \underbrace{a^{\frac{1}{n}} \times \cdots \times a^{\frac{1}{n}}}_{n \text{ times}} = a^{\frac{n}{n}} = a$$

$$a^{-n} = \frac{1}{a^n} \quad \text{because } a^{-n} a^n = a^0 = 1$$

Exponentials: Examples

$$
\begin{aligned}
2^3 &= 8 \\
10^2 &= 100 \\
10^1 &= 10 \\
10^0 &= 0^0 = 1 \\
9^{1/2} &= 3 \\
2^{-3} &= 1/8 \\
\sqrt{5} \times \sqrt{5} &= 5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 5^{\frac{1}{2}+\frac{1}{2}} = 5^1 = 5 \\
16^{3/2} &= (16^{1/2})^3 = 4^3 = 64
\end{aligned}
$$

Maths introduction: Logarithms

$\log_a b$ is the number you have to raise a to in order to get b:

$$\log_a b = c \text{ means that } a^c = b$$
$$\Rightarrow a^{\log_a b} = b$$

By applying $\log_a$ to both sides, and letting $c = \log_a b$, we get

$$\log_a \left(a^{\log_a b}\right) = \log_a b$$
$$\Rightarrow \log_a \left(a^c\right) = c \tag{1}$$

Thus $\log_a \bullet$ and $a^\bullet$ are inverses of each other and cancel.

$$\log_{10} 1000000 = 6$$
$$\log_{10} 0.0001 = -4$$
$$\log_2 32 = 5$$
$$\log_8 32 = \log_8(2^5) = \log_8\left((\sqrt[3]{8})^5\right) = 5/3$$

Maths introduction: Rules for Logarithms

$\log_a bc = \log_a b + \log_a c$:

$$\begin{aligned}
(bc) &= (b)(c) \\
\Rightarrow a^{\log_a bc} &= a^{\log_a b} a^{\log_a c} \\
&= a^{\log_a b + \log_a c} \\
\Rightarrow \log_a a^{\log_a bc} &= \log_a a^{\log_a b + \log_a c} \\
\Rightarrow \log_a bc &= \log_a b + \log_a c
\end{aligned}$$

$\log_a \frac{b}{c} = \log_a b - \log_a c$ (similarly)

$\log_a b^c = \log_a \underbrace{b \times \cdots \times b}_{c \text{ times}} = \underbrace{\log_a b + \cdots + \log_a b}_{c \text{ times}} = c \log_a b$

Maths introduction: Changing base of Logarithms

$\log_c x = (\log_c b)(\log_b x)$, First (impressive looking) proof:

$$\begin{aligned} x &= x \\ &= b^{\log_b x} \\ &= \left(c^{\log_c b}\right)^{\log_b x} \\ &= c^{(\log_c b)(\log_b x)} \\ \Rightarrow \log_c x &= \log_c c^{(\log_c b)(\log_b x)} \\ &= (\log_c b)(\log_b x) \end{aligned}$$

$\log_c x = (\log_c b)(\log_b x)$, Second, simpler proof:

$$\begin{aligned} x &= b^{\log_b x} \\ \Rightarrow log_c x &= log_c b^{\log_b x} \\ &= (\log_b x)(\log_c b) \end{aligned}$$

Maths introduction: More facts about Logarithms

$$\log_a a = 1$$

$$\log_a 1 = 0$$

$$\log_a x \text{ when } x \leq 0 \text{ is undefined}$$

$$\lim_{x \to 0^+} \log_a x = -\infty$$

$$\lim_{x \to +\infty} \log_a x = \infty$$