Sets and boolean algebras

1 Lists, streams and subsets

Given a set A, we can obtain some more sets.

- A^* is the set of all lists of elements of A. For example, $\mathbb{N}^*$ is the set of all lists of natural numbers, such as $[6, 2, 6, 7]$ or $]$. For example, if A is the set of characters, then A^* is the set of all strings.
- A^ω is the set of all *streams* (infinite sequences) of elements of A. For example, $\mathbb{N}^\omega$ is the set of all streams of natural numbers, such as

$$1, 4, 9, 16, \ldots$$
$$3, 3, 2, 4, 4, 3, 5, 5, 4, \ldots$$

  For example, the decimal representation of π after the point is the stream $1, 4, 1, 5, 9, \ldots$. This belong to the set $[0 \mathinner{..} 10)^\omega$. Another example: my program runs forever and keeps printing messages, so its output is a stream of strings, belonging to the set String^ω.

- $\mathcal{P}A$ is the set of all subsets of A. This is called the *powerset* of A. For example, $\mathcal{P}\mathbb{N}$ is the set of all sets of natural numbers, such as $\{13, 15, 29\}$ and $\{4, 6, 8, 10, \ldots\}$.

2 The algebra of subsets

Let X be a set, e.g., the set of all people in the world. Then we have several operations on $\mathcal{P}X$. Firstly, for any subsets A and B, we have subsets $A \cup B$ and $A \cap B$. Secondly, for each subset A, we define its *complement* (relative to X) as follows:

$$\overline{X} \stackrel{\text{def}}{=} X \setminus A$$

Now let's look at the laws that these operations satisfy.

$$\begin{array}{rclrcll}
A \cup \emptyset & = & A & A \cap X & = & A & \text{(neutral elements)} \\
A \cup X & = & X & A \cap \emptyset & = & \emptyset & \text{(annihilators)} \\
A \cup B & = & B \cup A & A \cap B & = & B \cap A & \text{(commutativity)} \\
(A \cup B) \cup C & = & A \cup (B \cup C) & (A \cap B) \cap C & = & A \cap (B \cap C) & \text{(associativity)} \\
A \cup A & = & A & A \cap A & = & A & \text{(idempotence)} \\
(A \cup B) \cap B & = & B & (A \cap B) \cup B & = & B & \text{(absorption)} \\
\multicolumn{6}{c}{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)} & \text{(distributivity of } \cap \text{ over } \cup) \\
\multicolumn{6}{c}{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)} & \text{(distributivity of } \cup \text{ over } \cap) \\
A \cup \overline{A} & = & X & A \cap \overline{A} & = & \emptyset & \text{(complements)} \\
\overline{A \cup B} & = & \overline{A} \cap \overline{B} & \overline{A \cap B} & = & \overline{A} \cup \overline{B} & \text{(de Morgan laws)} \\
\multicolumn{6}{c}{\overline{\overline{A}} = A} & \text{(double complement)}
\end{array}$$

Too many to remember! but each of them quite easy to prove. Let's do this for the de Morgan law $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

$$\begin{array}{rcl}
x \in \overline{A \cup B} & \Longrightarrow & x \notin A \cup B \\
& \Longrightarrow & x \notin A \text{ and } x \notin B \\
& \Longrightarrow & x \in \overline{A} \text{ and } x \in \overline{B} \\
& \Longrightarrow & x \in \overline{A} \cap \overline{B}
\end{array}$$

This shows $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$. For the other inclusion, we can read the transformations above from bottom to top, and the proof is complete.

The subset operations introduced in this chapter are closely connected to logic. If we have subsets A and Beach defined by a condition, as in

$$A = \{x \in X \mid x \text{ satisfies condition } c\} \quad \text{and} \quad B = \{x \in X \mid x \text{ satisfies condition } d\}$$

Then $A \cup B$ is defined by

$$A \cup B = \{x \in X \mid x \text{ satisfies condition } c \textbf{ or } d\}$$

Likewise, $A \cap B$ is defined by

$$A \cap B = \{x \in X \mid x \text{ satisfies condition } c \textbf{ and } d\}$$

Complement, finally, is given by negation:

$$\overline{A} = \{x \in X \mid x \text{ does } \textbf{not} \text{ satisfy condition } c\}$$

The logical operators are written with symbols very similar to those for the corresponding subset operations: We write $\vee$ for "or", $\wedge$ for "and", and $\neg$ for "not". The laws of set operations are all valid for the logical connectives as well, if we also remember that the role of the empty set $\emptyset$ is filled by the logical constant false, while the ambient set X corresponds to true. Let's write them out (again):

$$\begin{array}{rclrcll}
A \vee \mathsf{false} & = & A & A \wedge \mathsf{true} & = & A & \text{(neutral elements)} \\
A \vee \mathsf{true} & = & \mathsf{true} & A \wedge \mathsf{false} & = & \mathsf{false} & \text{(annihilators)} \\
A \vee B & = & B \vee A & A \wedge B & = & B \wedge A & \text{(commutativity)} \\
(A \vee B) \vee C & = & A \vee (B \vee C) & (A \wedge B) \wedge C & = & A \wedge (B \wedge C) & \text{(associativity)} \\
A \vee A & = & A & A \wedge A & = & A & \text{(idempotence)} \\
(A \vee B) \wedge B & = & B & (A \wedge B) \vee B & = & B & \text{(absorption)} \\
\multicolumn{6}{c}{A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)} & \text{(distributivity of } \wedge \text{ over } \vee) \\
\multicolumn{6}{c}{A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)} & \text{(distributivity of } \vee \text{ over } \wedge) \\
A \vee \neg A & = & \mathsf{true} & A \wedge \neg A & = & \mathsf{false} & \text{(complements)} \\
\neg(A \vee B) & = & \neg A \wedge \neg B & \neg(A \wedge B) & = & \neg A \vee \neg B & \text{(de Morgan laws)} \\
\multicolumn{6}{c}{\neg\neg A = A} & \text{(double negation)}
\end{array}$$

Boolean algebras. Generally, any set that's equipped with operations $\vee$, $\wedge$, and $\neg$, and constants false and true, and satisfies these laws, is called a **boolean algebra**. Thus, every power set can be viewed as a boolean algebra.

A very simple example is the boolean algebra of *truth values*, which is just $\{0, 1\}$. After we set false to be 0 and true to be 1, the boolean algebra laws tell us exactly how the operations work. As an example:

$$\begin{array}{rcll}
0 \vee 1 & = & \mathsf{false} \vee \mathsf{true} & \text{(definition)} \\
 & = & \mathsf{true} & \text{(annihilation)} \\
 & & & \\
\neg 0 & = & \neg\mathsf{false} & \text{(definition)} \\
 & = & \neg\mathsf{false} \vee \mathsf{false} & \text{(neutral element)} \\
 & = & \mathsf{true} & \text{(complements)}
\end{array}$$

If we do this for all possible combinations we get the well-known truth tables:

$$\begin{array}{c|cc} \vee & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array} \qquad \begin{array}{c|cc} \wedge & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \qquad \begin{array}{c|cc} \neg & 0 & 1 \\ \hline & 1 & 0 \end{array}$$