#### **MLFCS**

Solving system of linear equations using Gaussian Elimination

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# Lecture attendance code:

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#### Today's plan

- Systems of linear Equations
- Solving system of linear equations using Gaussian elimination
- Compact way of doing Gaussian elimination
- Some extensions



#### Let us start with a simple example

#### rational number

Consider the equation ax = b where x is a variable and a, b are rational numbers. How many solutions does this equation have?

- Exactly one solution  $\rightarrow \alpha = 5$  b = 16 d = 2
- No solutions q = 0 b = 0
- ▶ Don't know
- ▶ Question is not well-defined  $\alpha = 2$  b = 3

#### 2 variables and 2 linear equations

How many solutions does the following system of linear equations have over the set of rational numbers?

$$3(n+2y)=3(4) \longrightarrow \begin{cases} x+2y=4 \\ 3x+6y=12 \end{cases}$$
 is infinite set of solutions

How about this next system?

$$\begin{cases}
x + y = 6 \\
0x + 0y = 102
\end{cases}$$

#### 2 variables and 2 linear equations

How many solutions does the following system of linear equations have over the set of rational numbers?

$$\begin{cases} 2x - y = 0 \\ x + y = 6 \end{cases}$$
  $\chi = 2$   $\chi = 4$ 

Take a minute and solve this individually!

If you are done, verify your solution is correct by plugging it back into both equations.

Next slide: slightly harder example

#### 2 variables and 2 linear equations

How many solutions does the following system of linear equations have over the set of rational numbers?

$$2x - y = 0$$

$$x + 3y = 6$$

Take a minute and solve this individually!

If you are done, verify your solution is correct by plugging it back

into both equations.

Substintion: From (1), you get 
$$y = 2\pi$$

Put this in (2) to get  $6 = \pi + 3y$ 
 $= \pi + 6\pi$ 

How did you solve this?

> Elimination: Maltiply (1) by 3 -> 67-3y=0-(3)

Add (2) & 5 get 7n=6

#### 3 variables: how many equations should we have?

What if we have only 2 equations?

$$3x + 2y + z = 6$$
$$x + 3y + 2z = 4$$

What if we have 4 equations?

$$3x + 2y + z = 6$$

$$x + 3y + 2z = 4$$

$$2x + 3y + 3z = 9$$

$$x + y + z = 1$$

$$x + y + z = 1$$

$$x + y + z = 1$$

Each equation imposes a constraint

Each variable gives a "degree of freedom"

Therefy Case is when number of variables is equal to number of equations

#### 3 variables and 3 linear equations

$$x_1 + 5x_2 - 2x_3 = -11$$

$$3x_1 - 2x_2 + 7x_3 = 5$$

$$-2x_1 - x_2 - x_3 = 0$$

Solve this individually!

If you are done, verify your solution is correct by plugging it back into all three equations.

Mant to eliminate 
$$n_2$$
.

Multiply 3 by 2 to get  $-4n_1-2n_2-2n_3=0$  — 6

Subtract 6 firm © to get  $7n_1+9n_3=5$  — 5

Multiply 3 by 5 to get  $-10n_1-5n_2-5n_3=0$  — 6

Add 6 & 0 to get  $-9n_1-7n_3=-11$ 

Solve the two equations in red boxes (like (ref slide)

#### n variables and n linear equations

#### The Gaussian elimination method

- **Base case** is one variable and one equation, i.e., ax = b
- Eliminate any variable to get (n-1) equations in (n-1)variables
  - We can choose which one to eliminate!
- Perform this recursively till you reach the base case.
  - Somewhere in the middle you might reach a special case
  - Special cases: No solution, infinitely many solutions, ...
  - Otherwise you will find the unique solution!

Think about how we did the above steps for example from previous slide:

$$x_1 + 5x_2 - 2x_3 = -11$$

$$3x_1 - 2x_2 + 7x_3 = 5$$

$$-2x_1 - x_2 - x_3 = 0$$





# Example with 4 variables & 4 equations

Solve this system of linear equations:

$$x_1 + 5x_2 - 2x_3 + 3x_4 = -11$$
$$3x_1 - 2x_2 + 7x_3 + x_4 = 5$$
$$-2x_1 - x_2 - x_3 - 2x_4 = 0$$
$$5x_1 + 3x_2 + 4x_3 - x_4 = 13$$

Exercise Sheet



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We will use a matrix for the computation

Matrix is a two-dimensional array

Consider the system of equations:

$$2x_1 - x_2 = 0$$

$$x_1 + x_2 = 6$$
We write this in compact form as:
$$\begin{pmatrix} x_1 - x_2 & 0 \\ x_2 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_2 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_2 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_2 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_2 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_2 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_2 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_2 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_2 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_2 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_1 - x_2 & 0 \\ x_2 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_1 - x_2 & 0 \\ x_2 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_2 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_1 - x_2 & 0 \\ x_2 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_1 - x_2 & 0 \\ x_2 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_2 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_2 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_2 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_2 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_1 - x_2 & 0 \\ x_2 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_1 - x_2 & 0 \\ x_1 - x_2 & 0 \\ x_2 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_1 - x_2 & 0 \\ x_2 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_1 - x_1 & 0 \\ x_1 - x_2 & 0 \\ x_1 - x_1 & 0 \\ x_1 - x$$

We now allow to perform some operations on this matrix to do Gaussian elimination:

- ▶ Rearrange rows → just reality quation in a different orda
   ▶ Multiply a row by any rational number → Multiplying an equation
   ▶ Add/subtract any row from another

Why are these operations OK?

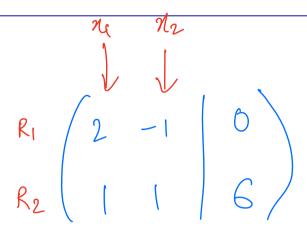




#### Consider the system of equations:

$$2x_1-x_2=0$$

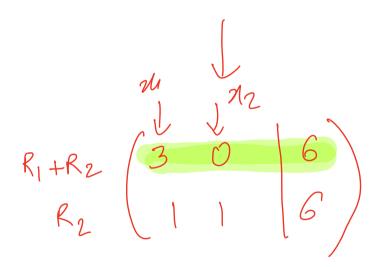
$$x_1 + x_2 = 6$$



#### Operations allowed are:

- ► Rearrange rows
- Multiply a row by any rational number
- Add/subtract any row from another

Row R, gives 
$$324 = 5 = 544 = 2$$
  
Row Ry gives  $242 = 6$   $12 = 4$ 





Consider the system of equations:

f equations: 
$$3x_1 + x_2 + 5x_3 = +3$$

$$-3x_1 + x_2 - 2x_3 = -5$$

$$3x_1 - x_2 + 7x_3 = 10$$

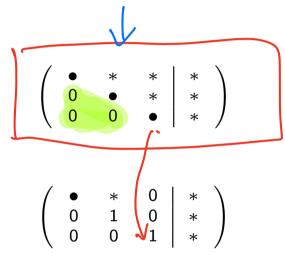
Operations allowed are:

- ▶ Rearrange rows
- Multiply a row by any rational number
- Add/subtract any row from another

$$\begin{array}{c} R_{1} \\ R_{2} \\ R_{3} - R_{1} \end{array} \begin{pmatrix} 3 & 1 & 5 \\ 0 & 0 & 5 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}$$



Aim is to get the matrix into row echelon form:



$$\left( egin{array}{ccc|ccc} 1 & 0 & 0 & * \ 0 & 1 & 0 & * \ 0 & 0 & 1 & * \end{array} 
ight)$$

Aim is to get the matrix into row echelon form:

$$\left(\begin{array}{cccc|cccc}
\bullet & * & * & 0 & * \\
0 & \bullet & * & 0 & * \\
0 & 0 & \bullet & 0 & * \\
0 & 0 & 0 & 1 & * \\
\end{array}\right)$$

$$\left(\begin{array}{cccc|cccc}
\bullet & * & 0 & 0 & * \\
0 & \bullet & 0 & 0 & * \\
0 & 0 & 1 & 0 & * \\
0 & 0 & 0 & 1 & *
\end{array}\right)$$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & | & * \\
0 & 1 & 0 & 0 & | & * \\
0 & 0 & 1 & 0 & | & * \\
0 & 0 & 0 & 1 & | & *
\end{pmatrix}$$

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#### Not very useful in practice!

A basic operation is addition, subtraction, multiplication or division of two rational numbers

Given n variables and n linear equations, how many "basic" operations do we need to implement Gaussian elimination?

Iterative methods often used instead!

- Start with a solution satisfying one/some of the given equations.
- ► Keep adjusting it till you reach close to the correct solution

#### What about non-linear equations?

$$10 \text{ n}^2 \text{ ty} = 13$$
 $11 \text{ 5n} + \text{y}^{1.3} = 7$ 

#### Summary of the lecture

- Gaussian elimination method to solve system of linear equations
  - Be careful of special cases (no solutions, infinite solutions, etc.)
  - ▶ Eliminate one variable to recursively reach the base case of one variable
  - Plug in your solution back into each of the given equations to verify!
- Compact way to do Gaussian elimination
  - Express in matrix form
  - Use three operations to convert matrix into row echelon form
  - ▶ This is the same as the earlier method of eliminating!
- Cannot do something similar to Gaussian elimination for solving non-linear equations

