Mathematical and Logical Foundations of Computer Science

Lecture 10 - Propositional Logic (Wrap-up)

Vincent Rahli

(some slides were adapted from Rajesh Chitnis' slides)

University of Birmingham

Where are we?

- Symbolic logic
- **Propositional logic**
- Predicate logic

Today

- Syntax of propositional logic
- Natural Deduction
- Classical reasoning
- Semantics
- Equivalences
- Provability/Validity

Syntax & Informal Semantics

Syntax:

$$P ::= a \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \neg P$$

Lower-case letters are atoms: p, q, r, etc.

Upper-case letters are (meta-)variables: P, Q, R, etc.

Two special atoms:

- $\top$ which stands for True
- $\bot$ which stands for False

We also introduced four connectives:

- $P \wedge Q$: we have a proof of both P and Q
- $P \vee Q$: we have a proof of at least one of P and Q
- $P \rightarrow Q$: if we have a proof of P then we have a proof of Q
- $\neg P$: stands for $P \rightarrow \bot$

Syntax

Example of propositions:

- "if x is a number then it is even or odd"
  - atom p: "x is a number"
  - atom q: "x is even"
  - atom r: "x is odd"
  - $p \to q \vee r$
- "if x is even then it is not odd"
  - atom p: "x is even"
  - atom q: "x is odd"
  - $p \to \neg q$
- "if $a = b$ and $b = c$ then $a = c$"
  - atom p: "$a = b$"
  - atom q: "$b = c$"
  - atom r: "$a = c$"
  - $(p \wedge q) \to r$
  - or equivalently: $p \to q \to r$

Precedence & Associativity

Precedence: in decreasing order of precedence $\neg, \wedge, \vee, \rightarrow$.

For example:

- $\neg P \vee Q$ means $(\neg P) \vee Q$
- $P \wedge Q \vee R$ means $(P \wedge Q) \vee R$
- $P \wedge Q \rightarrow Q \wedge P$ means $(P \wedge Q) \rightarrow (Q \wedge P)$

Associativity: all operators are right associative

For example:

- $P \vee Q \vee R$ means $P \vee (Q \vee R)$.
- $P \wedge Q \wedge R$ means $P \wedge (Q \wedge R)$.
- $P \rightarrow Q \rightarrow R$ means $P \rightarrow (Q \rightarrow R)$.

However use parentheses around compound formulas for clarity.

Constructive Natural Deduction

Constructive Natural Deduction rules:

$$\frac{\bot}{A}\ [\bot E] \qquad \frac{}{\top}\ [\top I] \qquad \frac{\begin{array}{c}\overline{A}^{\,1}\\ \vdots \\ B\end{array}}{A \to B}\ 1\ [\to I] \qquad \frac{A \to B \quad A}{B}\ [\to E]$$

$$\frac{\begin{array}{c}\overline{A}^{\,1}\\ \vdots \\ \bot\end{array}}{\neg A}\ 1\ [\neg I] \qquad \frac{\neg A \quad A}{\bot}\ [\neg E]$$

$$\frac{A}{A \vee B}\ [\vee I_L] \qquad \frac{A}{B \vee A}\ [\vee I_R] \qquad \frac{A \vee B \quad A \to C \quad B \to C}{C}\ [\vee E]$$

$$\frac{A \quad B}{A \wedge B}\ [\wedge I] \qquad \frac{A \wedge B}{B}\ [\wedge E_R] \qquad \frac{A \wedge B}{A}\ [\wedge E_L]$$

Classical Reasoning

Classical Natural Deduction includes all the Constructive Natural Deduction rules, plus:

$$\frac{}{A \vee \neg A}\ [LEM] \qquad \frac{\neg\neg A}{A}\ [DNE]$$

Semantics

A **valuation** ϕ assigns **T** or **F** with each atom

A valuation is **extended** to all formulas as follows:

- $\phi(\top) = \mathbf{T}$
- $\phi(\bot) = \mathbf{F}$
- $\phi(A \vee B) = \mathbf{T}$ iff either $\phi(A) = \mathbf{T}$ or $\phi(B) = \mathbf{T}$
- $\phi(A \wedge B) = \mathbf{T}$ iff both $\phi(A) = \mathbf{T}$ and $\phi(B) = \mathbf{T}$
- $\phi(A \rightarrow B) = \mathbf{T}$ iff $\phi(B) = \mathbf{T}$ whenever $\phi(A) = \mathbf{T}$
- $\phi(\neg A) = \mathbf{T}$ iff $\phi(A) = \mathbf{F}$

Satisfaction & validity:

- Given a valuation ϕ, we say that ϕ **satisfies** A if $\phi(A) = \mathbf{T}$
- A is **satisfiable** if there exists a valuation ϕ on atomic propositions such that $\phi(A) = \mathbf{T}$
- A is **valid** if $\phi(A) = \mathbf{T}$ for all possible valuations ϕ

Truth Tables

We can use **truth tables** to check whether propositions are valid:

A	B	$A \vee B$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

A	B	$A \wedge B$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

P	Q	$P \rightarrow Q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

A	$\neg A$
T	**F**
F	**T**

A proposition is (semantically) valid if the last column in its truth table only contains **T**

Validity

These techniques can be used to prove the validity of propositions:

- a **Natural Deduction** proof (syntactic validity)
- a **truth table** with only **T** in the last column (semantical validity)

We saw that:

- a formula A is provable in **Natural Deduction**
- **iff** A is **semantically valid**

This is true about the classical versions of these deduction systems

Logical equivalences

Let $A \leftrightarrow B$ be defined as $(A \to B) \land (B \to A)$

- it means that A and B are logically equivalent
- this is called a "bi-implication"
- read as "A if and only if B"

We will now prove:

- Distributivity of $\land$ over $\lor$:
  $(A \land (B \lor C)) \leftrightarrow ((A \land B) \lor (A \land C))$
- Double negation elimination as an equivalence: $\neg\neg A \leftrightarrow A$

You can also try proving the distributivity of $\lor$ over $\land$:
$(A \lor (B \land C)) \leftrightarrow ((A \lor B) \land (A \lor C))$

Provability/Validity

Provide a constructive Natural Deduction proof of the following equivalence: $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$

Left-to-right implication:

$$
\dfrac{
\dfrac{\dfrac{}{A \wedge (B \vee C)}\ 1}{B \vee C}\ [\wedge E_R]
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge (B \vee C)}\ 1}{A}\ [\wedge E_L] \quad \dfrac{}{B}\ 2}{A \wedge B}\ [\wedge I]}{(A \wedge B) \vee (A \wedge C)}\ [\vee I_L]}{B \to (A \wedge B) \vee (A \wedge C)}\ 2\ [\to I]
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge (B \vee C)}\ 1}{A}\ [\wedge E_L] \quad \dfrac{}{C}\ 3}{A \wedge C}\ [\wedge I]}{(A \wedge B) \vee (A \wedge C)}\ [\vee I_R]}{C \to (A \wedge B) \vee (A \wedge C)}\ 3\ [\to I]
}{
\dfrac{(A \wedge B) \vee (A \wedge C)}{(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))}\ 1\ [\to I]
}\ [\vee E]
$$

Provability/Validity

Right-to-left implication:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{(A \wedge B) \vee (A \wedge C)}\,1 \quad \Pi_1 \quad \Pi_2}{A}\,[\vee E] \qquad \dfrac{\dfrac{}{(A \wedge B) \vee (A \wedge C)}\,1 \quad \Pi_3 \quad \Pi_4}{B \vee C}\,[\vee E]}{A \wedge (B \vee C)}\,[\wedge I]}{((A \wedge B) \vee (A \wedge C)) \to (A \wedge (B \vee C))}\,1\ [\to I]
$$

where Π_1 is:

$$
\dfrac{\dfrac{\dfrac{}{A \wedge B}\,2}{A}\,[\wedge E_L]}{(A \wedge B) \to A}\,2\ [\to I]
$$

where Π_2 is:

$$
\dfrac{\dfrac{\dfrac{}{A \wedge C}\,3}{A}\,[\wedge E_L]}{(A \wedge C) \to A}\,3\ [\to I]
$$

where Π_3 is:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge B}\,4}{B}\,[\wedge E_R]}{B \vee C}\,[\vee I_L]}{(A \wedge B) \to (B \vee C)}\,4\ [\to I]
$$

where Π_4 is:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge C}\,5}{C}\,[\wedge E_R]}{B \vee C}\,[\vee I_R]}{(A \wedge C) \to (B \vee C)}\,5\ [\to I]
$$

Provability/Validity

Prove that $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$ is valid using a truth table

A	B	C	$B \vee C$	$A \wedge (B \vee C)$	$A \wedge B$	$A \wedge C$	$(A \wedge B) \vee (A \wedge C)$
T	**T**	**T**	**T**	**T**	**T**	**T**	**T**
T	**T**	**F**	**T**	**T**	**T**	**F**	**T**
T	**F**	**T**	**T**	**T**	**F**	**T**	**T**
T	**F**	**F**	**F**	**F**	**F**	**F**	**F**
F	**T**	**T**	**T**	**F**	**F**	**F**	**F**
F	**T**	**F**	**T**	**F**	**F**	**F**	**F**
F	**F**	**T**	**T**	**F**	**F**	**F**	**F**
F	**F**	**F**	**F**	**F**	**F**	**F**	**F**

The 5th and last columns are identical, so the two formulas are equivalent

Provability/Validity

Provide a classical Natural Deduction proof of the following equivalence: $\neg\neg A \leftrightarrow A$

$$
\dfrac{
\dfrac{\dfrac{\dfrac{}{\neg\neg A}\;1}{A}\;[DNE]}{\neg\neg A \to A}\;1\;[\to I]
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A}\;3 \quad \dfrac{}{A}\;2}{\bot}\;[\neg E]}{\neg\neg A}\;3\;[\neg I]}{A \to \neg\neg A}\;2\;[\to I]
}{\neg\neg A \leftrightarrow A}\;[\wedge I]
$$

Provability/Validity

Prove that $\neg\neg A \leftrightarrow A$ is valid using a truth table

A	$\neg A$	$\neg\neg A$
T	**F**	**T**
F	**T**	**F**

The 1st and last columns are identical, so the two formulas are equivalent

Conclusion

What did we cover today?

- Syntax of propositional logic
- Natural Deduction
- Classical reasoning
- Semantics
- Equivalences
- Provability/Validity

Next time?

- Predicate logic (syntax)