RELATIONS AND FUNCTIONS

MATHEMATICAL AND LOGICAL FOUNDATIONS OF COMPUTER SCIENCE

Muhammad Usman Ilyas, Ph.D.
Computer Science

UNIVERSITY OF BIRMINGHAM DUBAI

RELATIONS

- Definition:

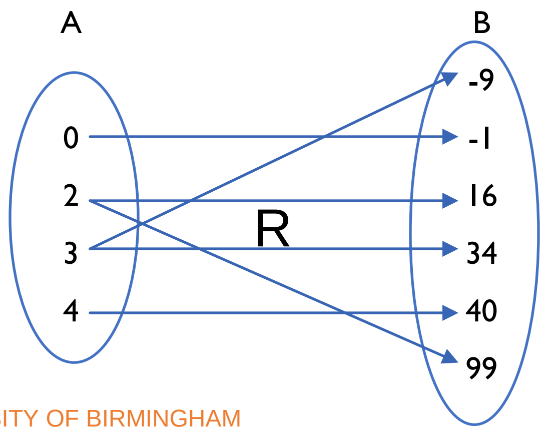
  A relation R is a mapping from elements of one set (the *domain*) to elements of another (the *codomain*).

- A relation can be represented in a number of ways.

RELATIONS

- A (binary) relation R from a set A to a set B can be represented using…

  A) Arrow representation:



B) A ordered pairs that are a subset of $A \times B$:

$\{(3,-9),(0,-1),(2,16),(3,34),(4,40),(2,99)\}$

C) Tabular form:

A	B
3	-9
0	-1
2	16
3	34
4	40
2	99

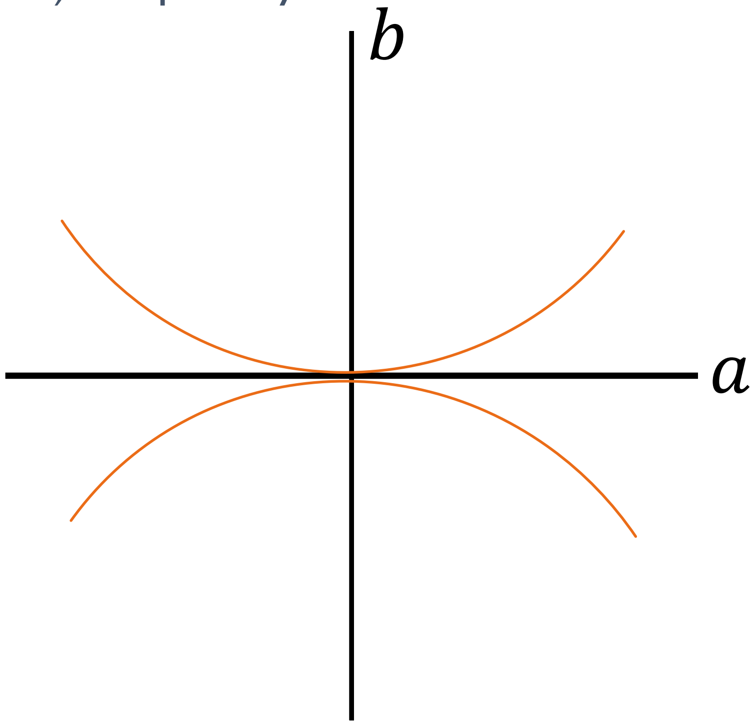
RELATIONS

D) Set builder notation:

$$\{(a, b) | b = \pm a^2, a \in \mathbb{R}\}$$

<u>Question:</u> Is this not a "function"?

E) Graphically



N-ARY RELATIONS

- Just like tuples can be n-ary (i.e., may have more than two components) relations may also be n-ary.

- Example:

A 3-Ary relation,

$$\{(3,-9,3),(0,-1,4),(2,16,11),(3,34,10),(4,40,-3),(2,16,1)\}$$

And represented in tabular form:

A	B	C
3	-9	3
0	-1	4
2	16	11
3	34	10
4	40	-3
2	16	1

COMPOSITION OF RELATIONS

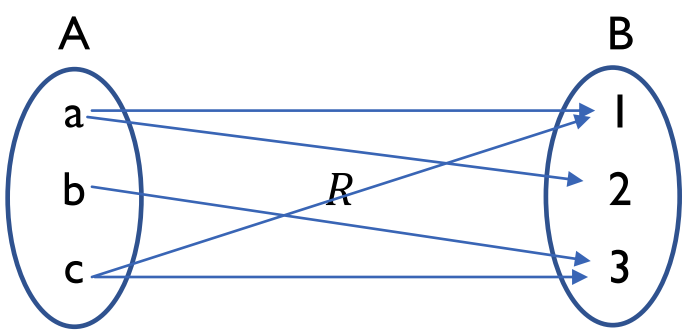
- The composition of relations $R \subseteq A \times B$ and relation $S \subseteq B \times C$ is denoted $S \circ R$ and is defined as,

$$S \circ R = \left\{\{x, z\} | \exists y \in B: (x, y) \in R \wedge (y, z) \in S\right\}.$$

Example:

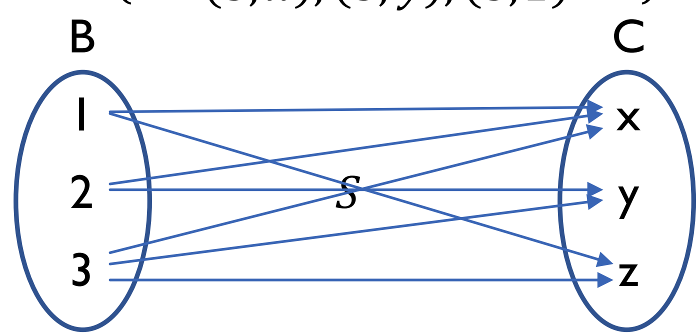
1. Relation R is a subset of $A \times B$.

$R = \{(a, 1), (a, 2), (b, 3), (c, 1), (c, 3)\}$



2. Relation S is a subset of $B \times C$.

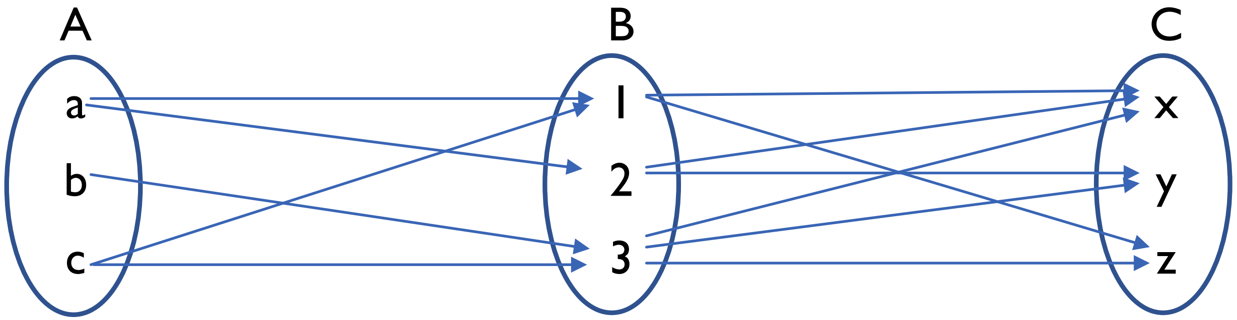
$$S = \begin{Bmatrix} (1, x), (1, z), (2, x), (2, y), \\ (3, x), (3, y), (3, z) \end{Bmatrix}$$



COMPOSITION OF RELATIONS

Then the composition of relations R and S is denoted by $S \circ R$ and can be represented in set and arrow notation as,

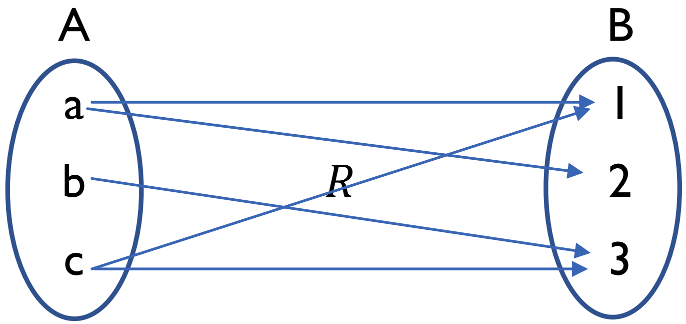
$$S \circ R = \left\{ \begin{array}{c} (a,x),(a,z),(a,y), \\ (b,x),(b,y),(b,z), \\ (c,x),(c,y),(c,z) \end{array} \right\} \subseteq A \times C$$



INVERSE RELATIONS

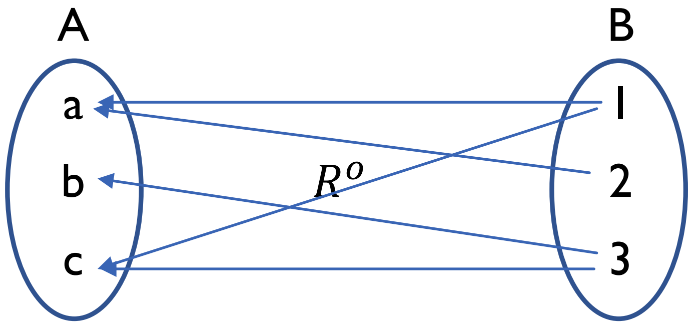
Consider the relation R is a subset of $A \times B$ from before,…

$$R = \begin{Bmatrix} (a,1),(a,2),(b,3), \\ (c,1),(c,3) \end{Bmatrix}$$



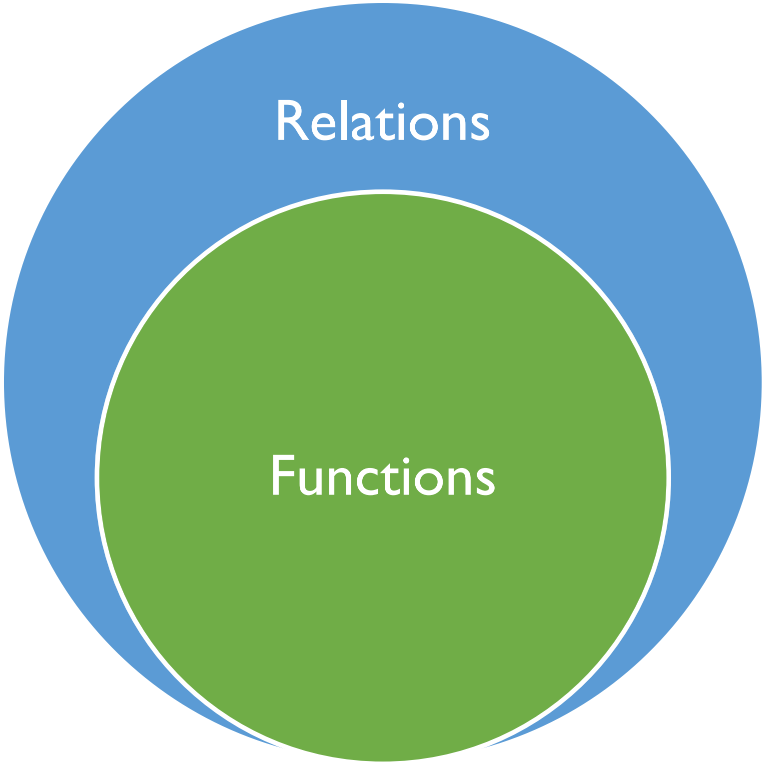
then the inverse relation R^o is a subset of $B \times A$.

$$R^o = \begin{Bmatrix} (1,a),(2,a),(3,b), \\ (1,c),(3,c) \end{Bmatrix}$$



RELATIONS & FUNCTIONS

- A function is special kind of relation.
- Every function is a relation, but every relation does not qualify as a function.



FUNCTIONS

- A function f from a set A to a set B is an assignment of exactly one element of b ∈ B to each element of a ∈ A. We write,

  $$f(a) = b \quad \text{or} \quad f\colon a \mapsto b$$

  if b is the unique element of B assigned by the function f to the element a of A.

- If f is a function from A to B, we write

  $$f\colon A \to B$$

Note: Here, "→" has nothing to do with implication (*if … then*) from the section on predicate logic

CIRCLING BACK …

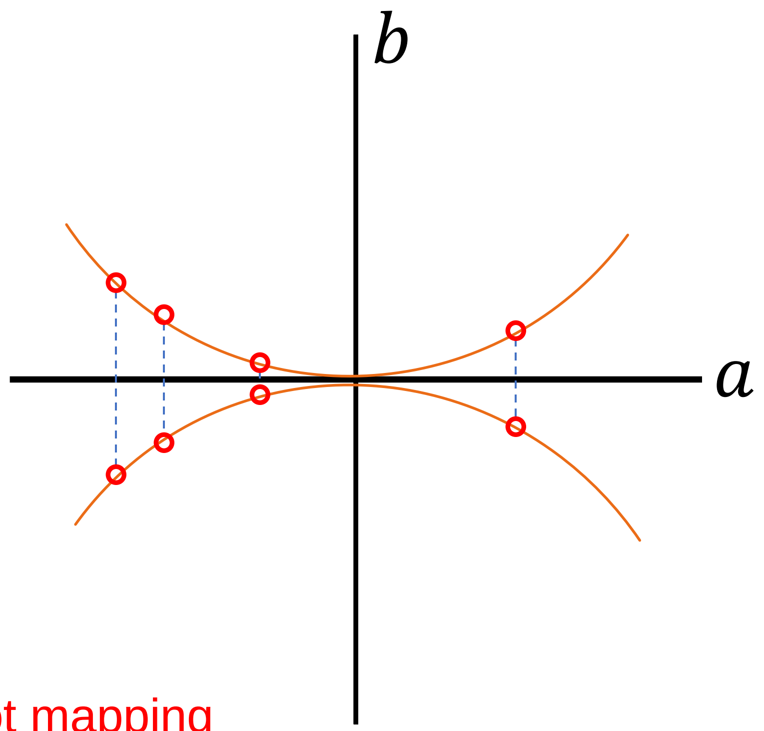
- So, then, is this a function?

No!

- A function f from a set A to a set B is an assignment of exactly one element of b $\in B$ to each element of a $\in A$. We write,

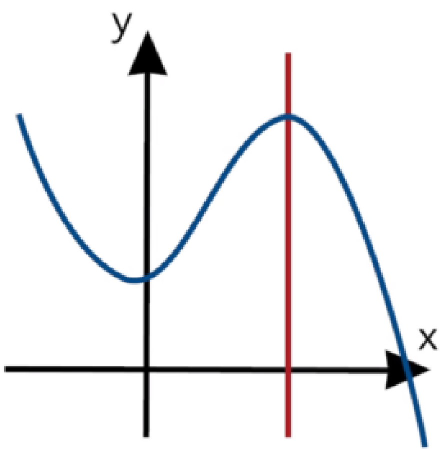
$$f(a) = b \qquad \text{or} \qquad f\colon a \mapsto b$$

… if b is the unique element of B assigned by the function f to the element a of A.

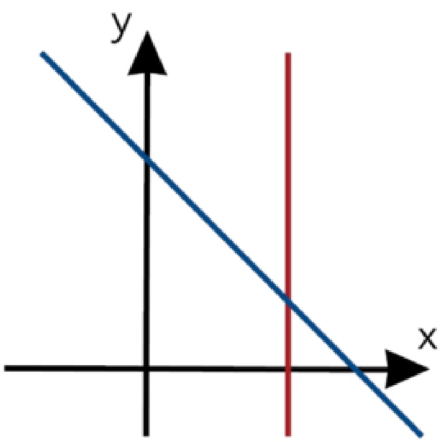


$f(a)$ is not mapping to a unique value of b

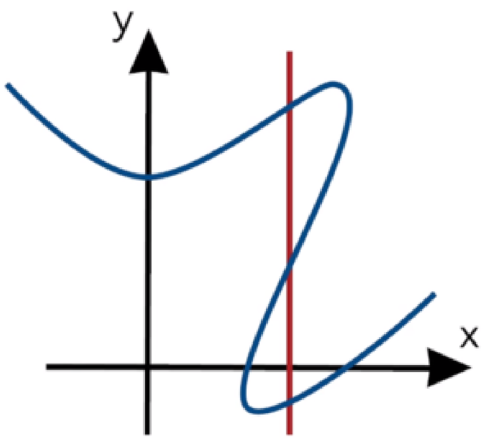
FUNCTION OR RELATION?



Function
$(f: x \mapsto y)$



Function
$(f: x \mapsto y)$



Not a Function / Relation

Ref: https://www.mometrix.com/academy/definition-of-a-function/

FUNCTIONS

- A relation must fulfill two conditions to qualify as a function $f: A \to B$.

1. It must be ***single-valued***, i.e., it must map each input to at most one image $\mathrm{b} \in B$.
2. It must be ***total***, i.e., it must map each possible input to at least one image $\mathrm{b} \in B$.

Taken together, 1 and 2 mean each value $a \in A$ must be mapped to exactly one value $b \in B$.

- Example:

For $f, g: \mathbb{R} \to \mathbb{R}$ …

- $f: x \mapsto x^2$ is a total function
- $g: x \mapsto \frac{1}{x}$ is not a total function (because it is not defined for $x = 0$)

Note: $g(x)$ is also called a partial function

TERMINOLOGY

- If $f: A \to B$, we say that A is the domain of f and B is the codomain of f.
- If $f(a) = b$, we say that the element b is the image of element a, and a is a pre-image of b.
- The range of $f: A \to B$ is the set of all images of elements of A.
- We say that $f: a \to b$ maps a $\in A$ to b $\in B$.
- For a subset of the codomain $Y \subset B$, the subset $f^{-1}(Y) \subset A$ is called the pre-image or inverse image.

FUNCTIONS

- Let us take a look at the function $f : P \rightarrow C$ with
- $P = \{Linda, Max, Kathy, Peter\}$
- $C = \{Boston, New\ York, Hong\ Kong, Moscow\}$

- $f(Linda) = Moscow$
- $f(Max) = Boston$
- $f(Kathy) = Hong\ Kong$
- $f(Peter) = New\ York$

- Here, the range of f is C.

FUNCTIONS

- Let us re-specify f as follows:

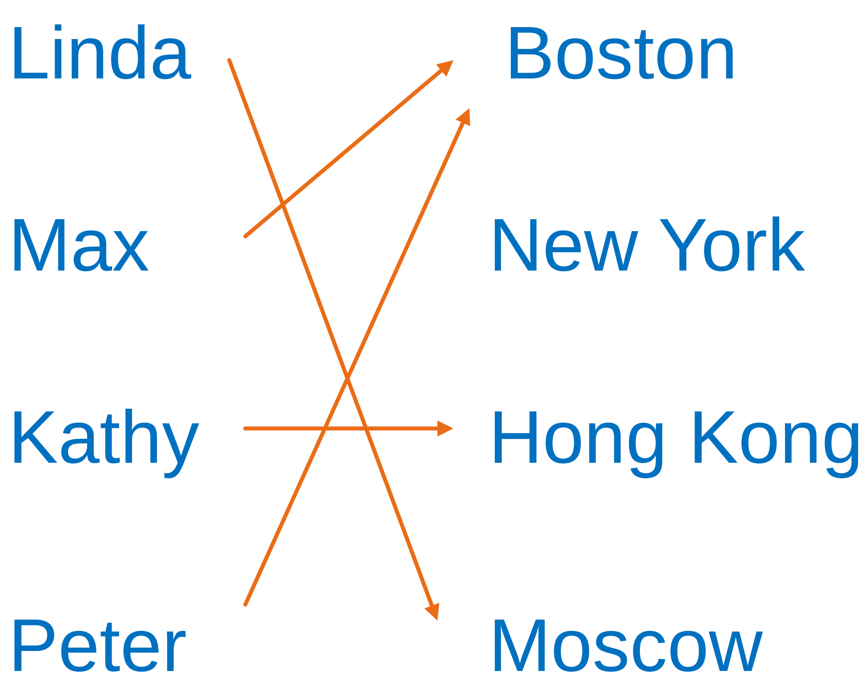
- $f(Linda) = Moscow$
- $f(Max) = Boston$
- $f(Kathy) = Hong\ Kong$
- $f(Peter) = Boston$

- Is f still a function? yes
- What is its range? $\{Moscow, Boston, Hong\ Kong\}$

FUNCTIONS

- Other ways to represent f :

x	$f(x)$
Linda	Moscow
Max	Boston
Kathy	Hong Kong
Peter	Boston



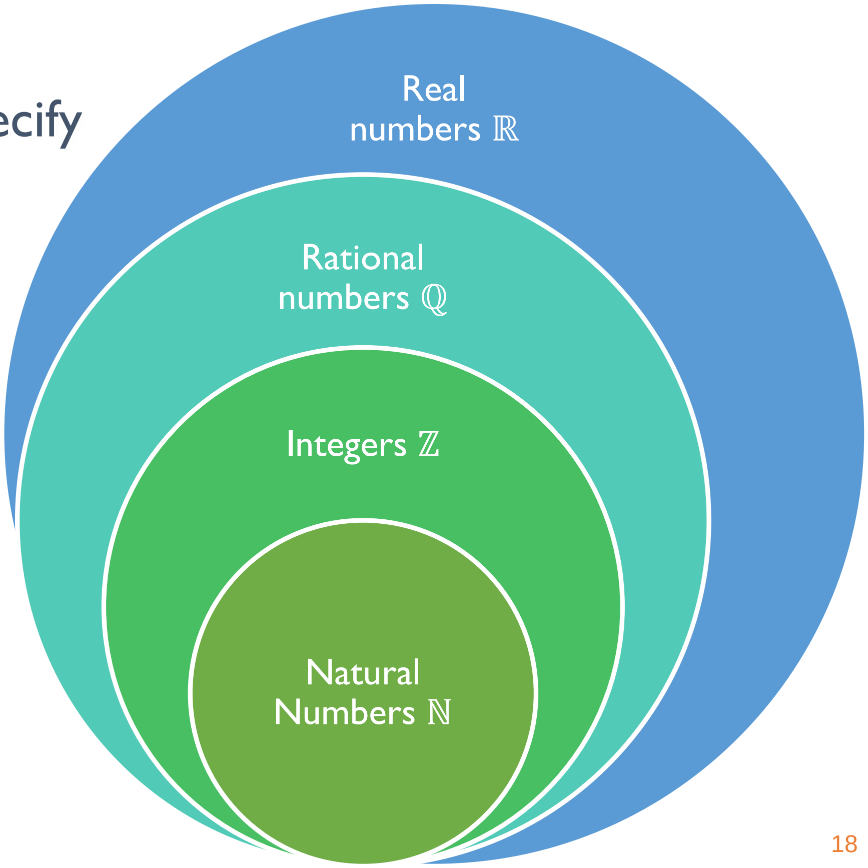
FUNCTIONS

- If the domain of our function f is large, it is often convenient to specify f with a formula, e.g.:

$$f: \mathbb{R} \to \mathbb{R}$$

$$f(x) = 2x$$

- This leads to:
- $f(1) = 2$
- $f(3) = 6$
- $f(-3) = -6$
- …



FUNCTIONS

- Let us look at the following well-known function:
- $f(Linda) = Moscow$
- $f(Max) = Boston$
- $f(Kathy) = Hong\ Kong$
- $f(Peter) = Boston$

- What is the image of $S = \{Linda, Max\}$?

$$f[S] = \{Moscow, Boston\}$$

- What is the image of $S = \{Max, Peter\}$?

$$f[S] = \{Boston\}$$

PROPERTIES OF FUNCTIONS

- A function $f : A \to B$ is said to be one-to-one (or *injective*), if and only if

$$\forall x, y \in A\ (f(x) = f(y) \to x = y)$$

- In other words: f is one-to-one if and only if it does not map two distinct elements of A onto the same element of B.

PROPERTIES OF FUNCTIONS

And again…

- $f(Linda) = Moscow$
- $f(Max) = Boston$
- $f(Kathy) = Hong\ Kong$
- $f(Peter) = Boston$

- Is f one-to-one?

- No, Max and Peter are mapped onto the same element of the image.

- $g(Linda) = Moscow$
- $g(Max) = Boston$
- $g(Kathy) = Hong\ Kong$
- $g(Peter) = New\ York$

- Is g one-to-one?

- Yes, each element is assigned a unique element of the image.

PROPERTIES OF FUNCTIONS

- How can we prove that a function f is one-to-one?
- Whenever you want to prove something, first take a look at the relevant definition(s):

$$\forall x, y \in A (f(x) = f(y) \rightarrow x = y)$$

- **Example:**

$$f: \mathbb{R} \rightarrow \mathbb{R}$$
$$f(x) = x^2$$

- Disproof by counterexample:

$f(3) = f(-3)$, but $3 \neq -3$, so f is not one-to-one.

PROPERTIES OF FUNCTIONS

- … and yet another example:

$$f: \mathbb{R} \to \mathbb{R}$$
$$f(x) = 3x$$

- One-to-one: $\forall x, y \in A (f(x) = f(y) \to x = y)$
- To show: $f(x) \neq f(y)$ whenever $x \neq y$

$\quad x \neq y$

$\Leftrightarrow 3x \neq 3y$

$\Leftrightarrow f(x) \neq f(y)$,

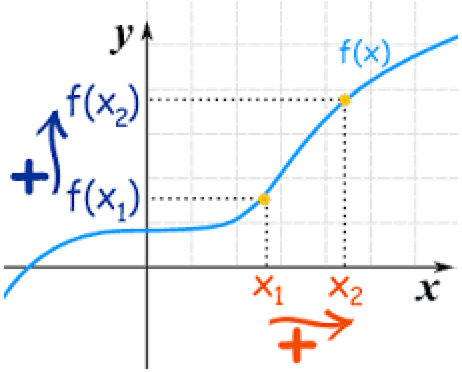
So, if $x \neq y$, then $f(x) \neq f(y)$, that is, f is one-to-one.

PROPERTIES OF FUNCTIONS

- A function $f: A \to B$ with $A, B \subseteq \mathbb{R}$ is called strictly increasing, if

  $$\forall x, y \in A\ (x < y \to f(x) < f(y)),$$

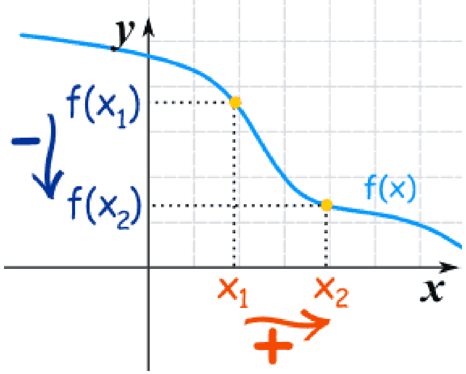
- Example: $\forall\ x_1, x_2\ (x_1 < x_2 \to f(x_1) < f(x_2))$



- A function $f: A \to B$ with $A, B \subseteq \mathbb{R}$ is called strictly decreasing, if

  $$\forall x, y \in A\ (x < y \to f(x) > f(y)).$$

- Example: $\forall\ x_1, x_2\ (x_1 < x_2 \to f(x_1) > f(x_2))$



Obviously, a function that is either strictly increasing or strictly decreasing is one-to-one.

PROPERTIES OF FUNCTIONS

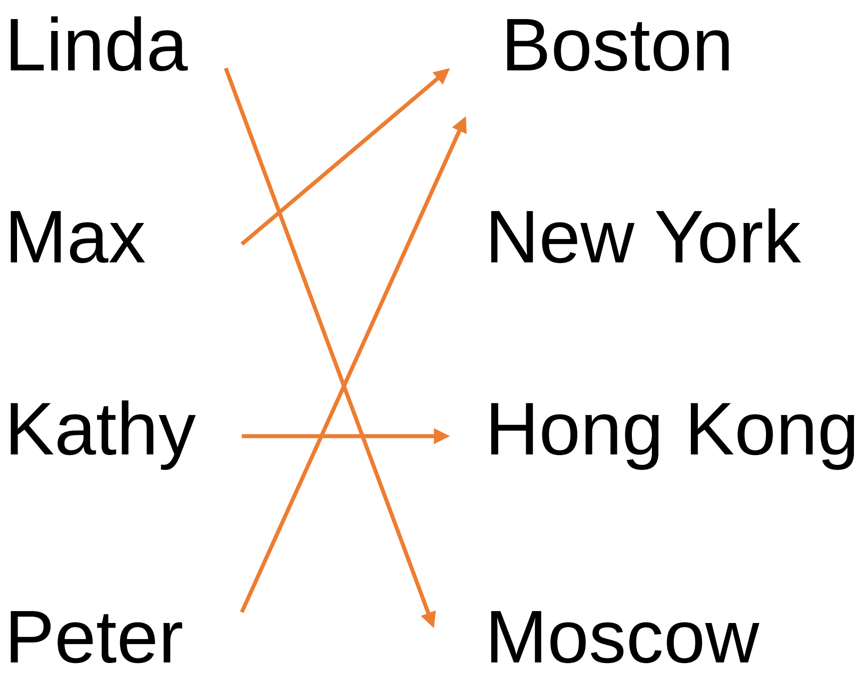
- A function $f : A \rightarrow B$ is called onto, or *surjective*, if and only if $\forall b \in B$ there is an element $a \in A$ with $f(a) = b$.
- In other words, f is onto iff its range is its entire codomain.
- A function f: $A \rightarrow B$ is a one-to-one correspondence, or a *bijection*, if and only if it is both one-to-one and onto.
- Obviously, if $f : A \rightarrow B$ is a bijection then
  - Either they are both finite with equal cardinality $|A| = |B|$
  - Or they are both infinite.

PROPERTIES OF FUNCTIONS

Examples:

- In the following examples, we use the arrow representation to illustrate functions $f: A \rightarrow B$.
- In each example, the complete sets A and B are shown.

PROPERTIES OF FUNCTIONS



Is f injective?

No.

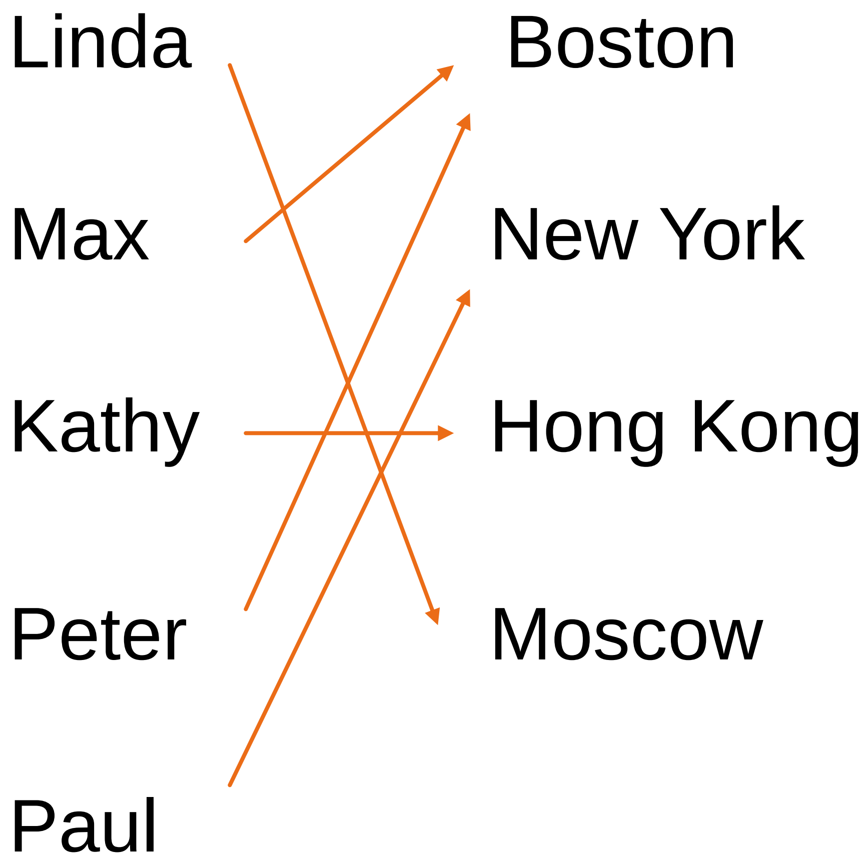
Is f surjective?

No.

Is f bijective?

No.

PROPERTIES OF FUNCTIONS



Is f injective?

No.

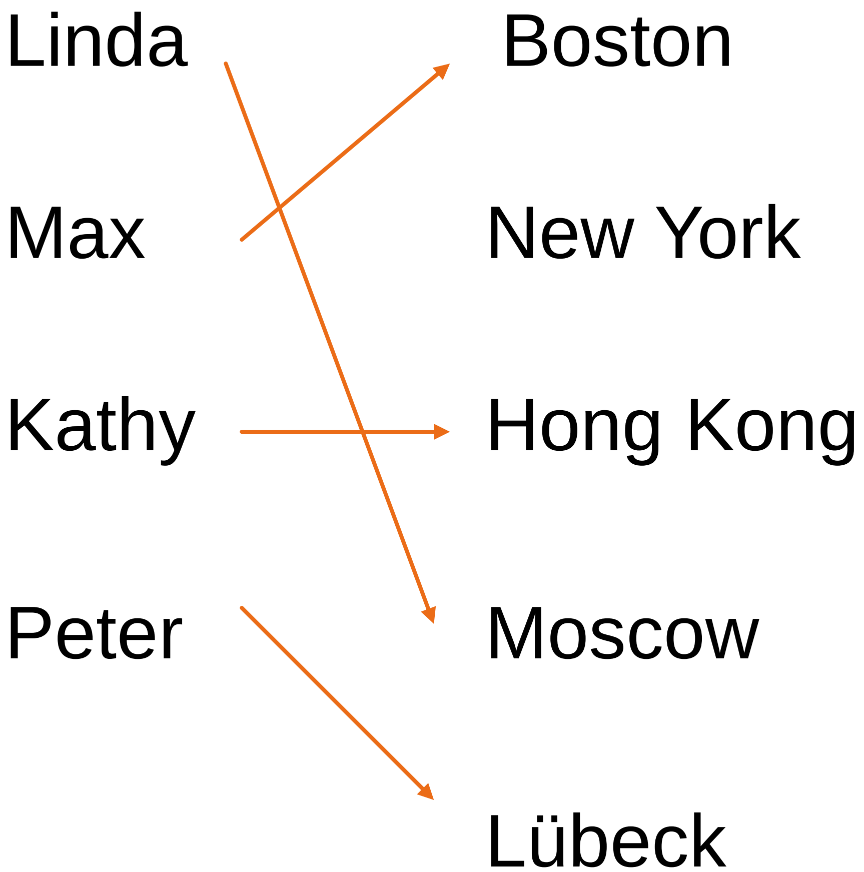
Is f surjective?

Yes.

Is f bijective?

No.

PROPERTIES OF FUNCTIONS



Is f injective?

Yes.

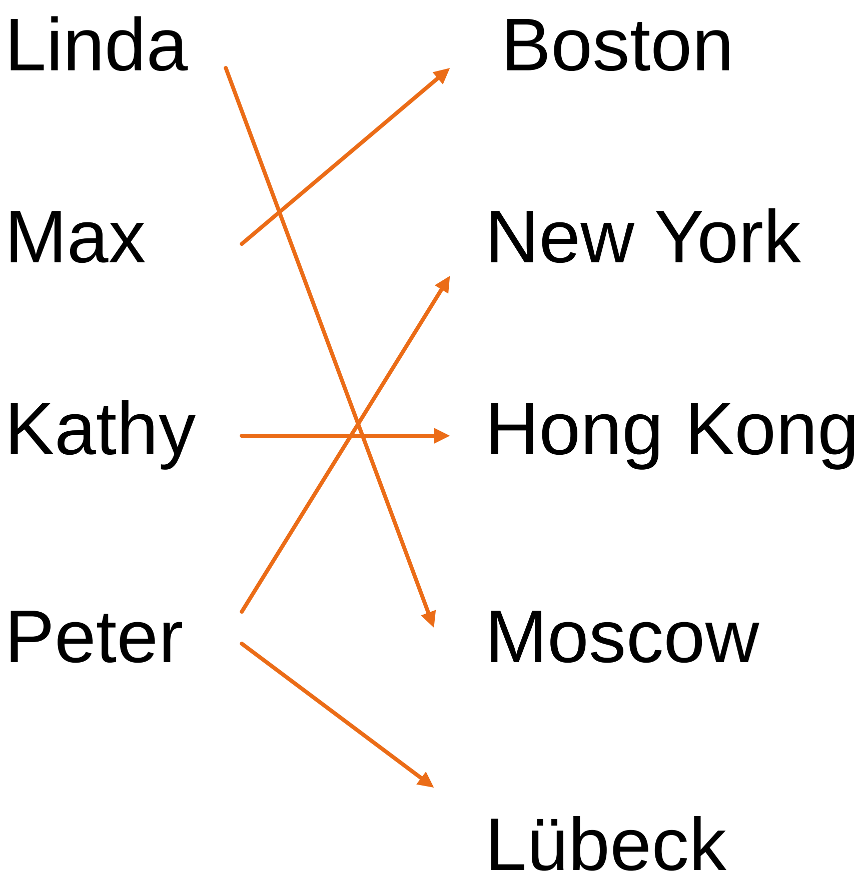
Is f surjective?

No.

Is f bijective?

No.

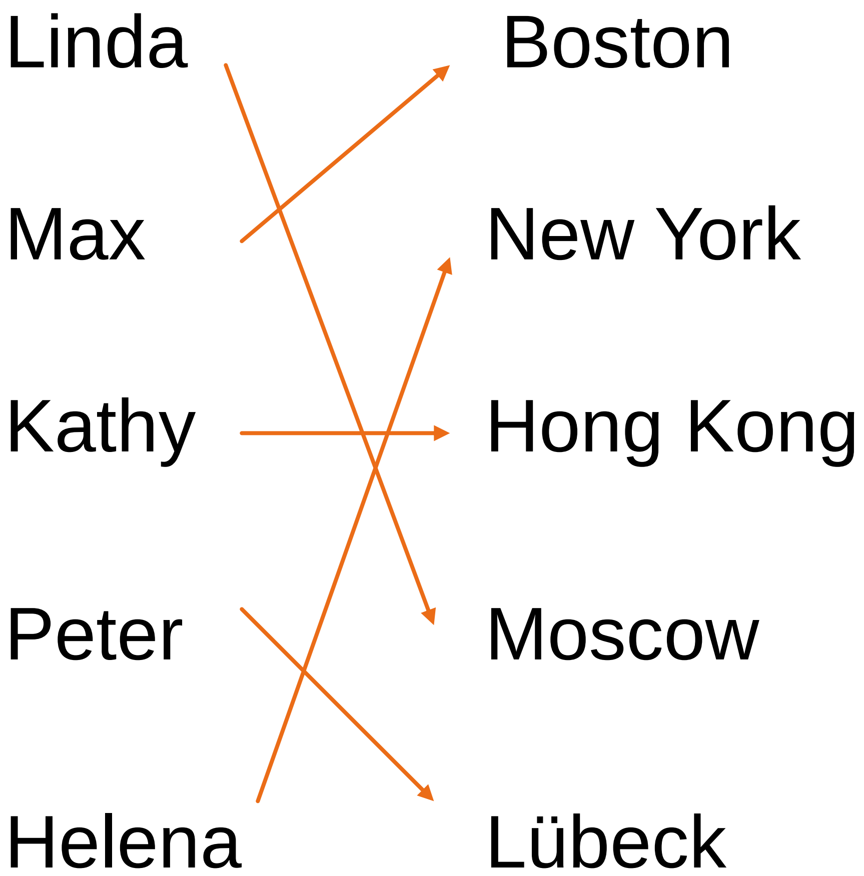
PROPERTIES OF FUNCTIONS



Is f injective?

No! f is not even a function! (just a relation)

PROPERTIES OF FUNCTIONS



Is f injective?

Yes.

Is f surjective?

Yes.

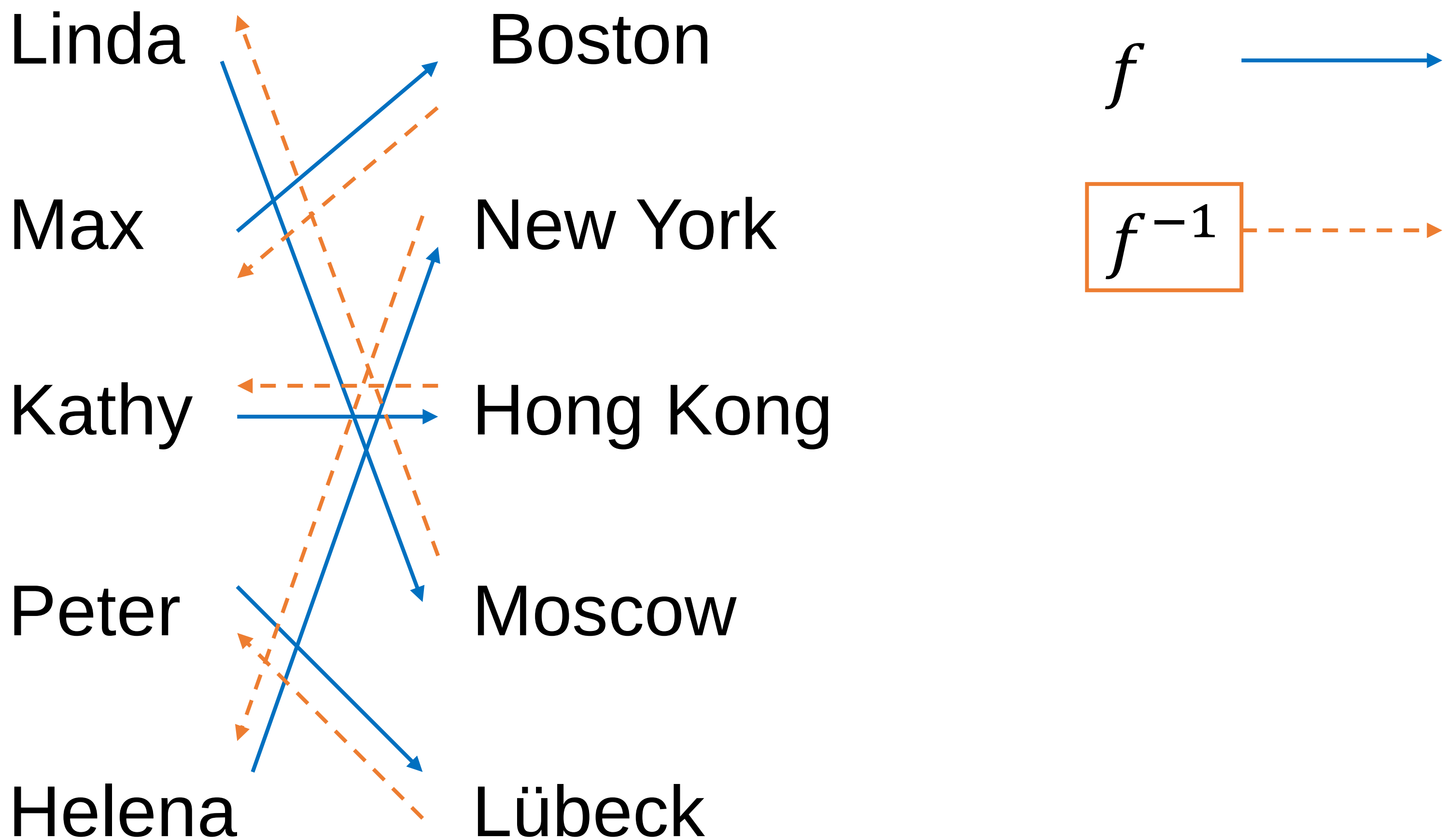
Is f bijective?

Yes.

INVERSION

- An interesting property of bijections is that they have an inverse function.
- The inverse function of the bijection $f\colon A \rightarrow B$ is the function $f^{-1}\colon B \rightarrow A$ with
- $f^{-1}(b) = a$ whenever $f(a) = b$.

INVERSION


Linda
Max
Kathy
Peter
Helena
Boston
New York
Hong Kong
Moscow
Lübeck
f
$\boxed{f^{-1}}$

INVERSION

Example:

- $f(Linda) = Moscow$
- $f(Max) = Boston$
- $f(Kathy) = Hong\ Kong$
- $f(Peter) = Lübeck$
- $f(Helena) = New\ York$

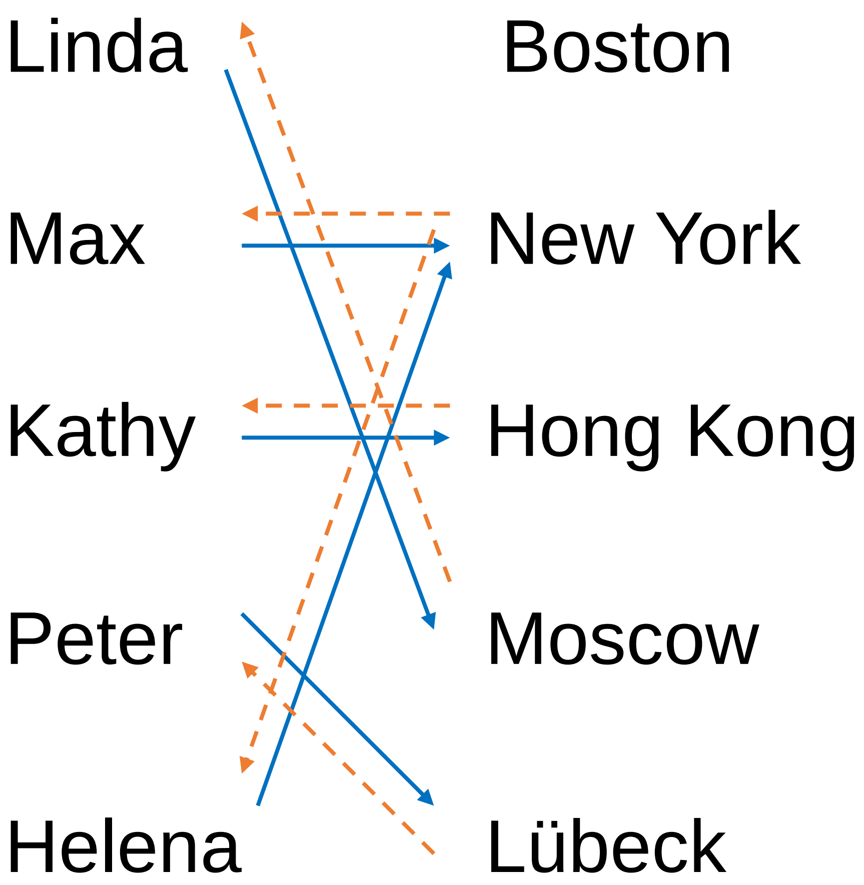
- Clearly, f is bijective.

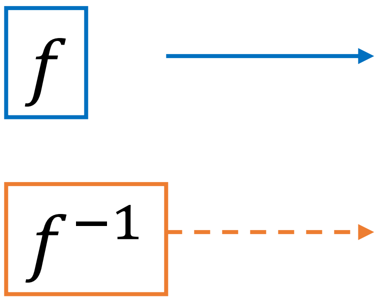
- The inverse function f^{-1} is given by:

- $f^{-1}(Moscow) = Linda$
- $f^{-1}(Boston) = Max$
- $f^{-1}(Hong\ Kong) = Kathy$
- $f^{-1}(Lübeck) = Peter$
- $f^{-1}(New\ York) = Helena$

- Inversion is only possible for bijections
(= *invertible* functions)

INVERSION





- $f^{-1}: C \rightarrow P$ is <u>not a function</u>, because …

…it is not defined for all elements of C and assigns two images to the pre-image New York.

COMPOSITION

- The composition of two functions $f: A \to B$ and $g: B \to C$, denoted by $g \circ f$, is defined by

$$(g \circ f)(a) = g(f(a))$$

- This means that
  - **first**, function f is applied to element $a \in A$,
    mapping it to an element of $\mathrm{b} \in B$,
  - **then**, function g is applied to this element $b \in B$,
    mapping it to an element $\mathrm{c} \in C$.
  - **Therefore**, the composite function maps
    from A to C.

COMPOSITION

- **Example:**

$$f(x) = 7x - 4, g(x) = 3x,$$
$$f: \mathbb{R} \to \mathbb{R}, g: \mathbb{R} \to \mathbb{R}$$

$$(f \circ g)(5) = f(g(5)) = f(15) = 105 - 4 = 101$$
$$(f \circ g)(x) = f(g(x)) = f(3x) = 21x - 4$$

COMPOSITION

- Composition of a bijection and its inverse:

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$$

The composition of a function and its inverse is the identity function $i(x) = x$.