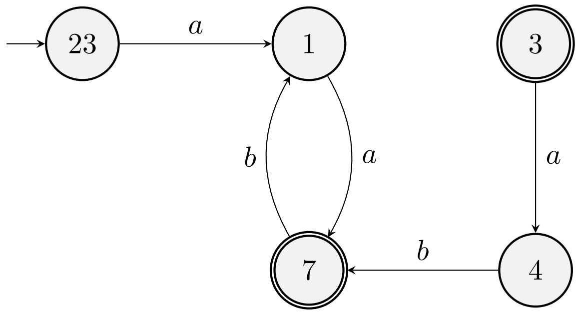
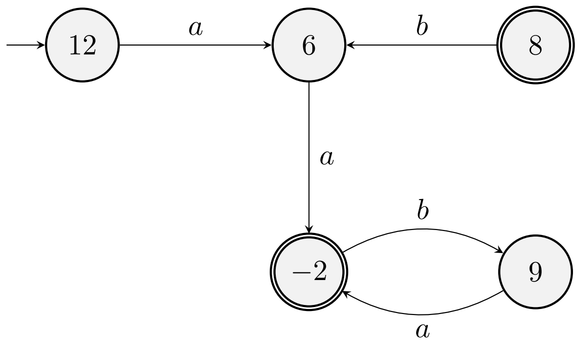
# Exercise sheet for Week 3

**Exercise 1.** *Here are three automata over the alphabet $\Sigma = \{a, b\}$ are equivalent. Test algorithmically the equivalence of (1)–(2), and the equivalence of (1)–(3). If you obtain a negative answer, you should give a word that's accepted by one automaton but not the other.*

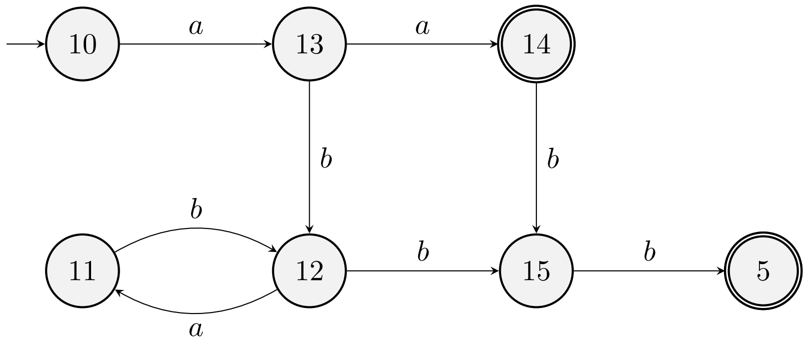
*1.*




*2.*

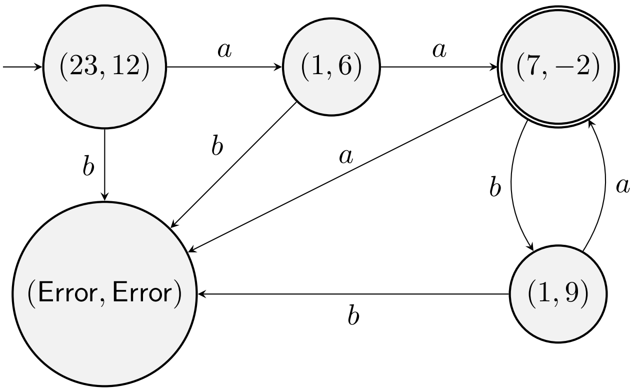



*3.*

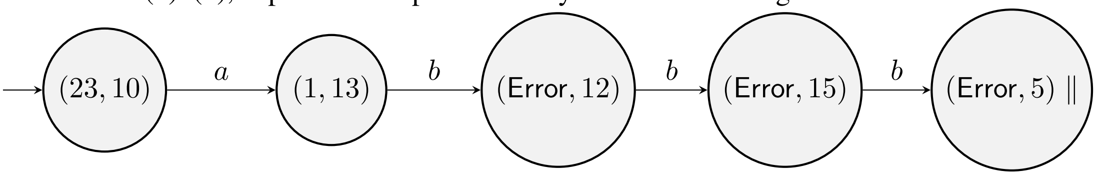



**Solution**

For automata (1)–(2), exploration of all possibilities yields the following:



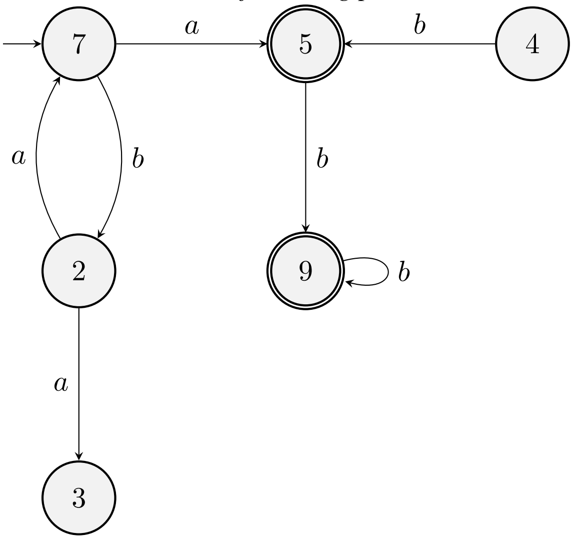

so the automata are equivalent.
For automata (1)–(3), exploration of possibilities yields the following:




When we reach $(\mathsf{Error}, 5)$ via the word $abbb$, the left component is rejecting and the right component is accepting. So this word is rejected by the left automaton and accepted by the right automaton. So the automata are not equivalent.

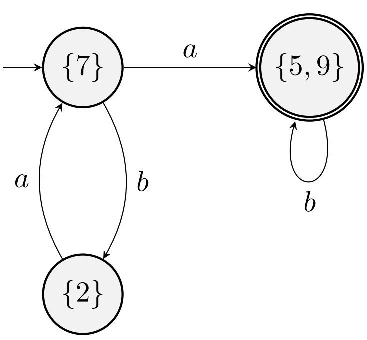
(Other diagrams are possible depending on your exploration order.)

**Exercise 2.** *Minimize the following partial DFA, and then prove that the partial DFA you have obtained is minimal.*




**Solution**

- State $4$ is unreachable, and hence can be removed.
- State $3$ is hopeless, and hence can be removed.
- States $5$ and $9$ are equivalent: both are accepting (accept $\varepsilon$), and accept words $b^n$ and reject any word containing an $a$. So they are unified.

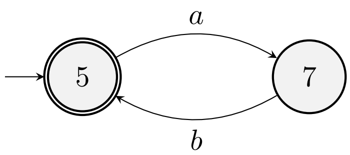



Proof of minimality:

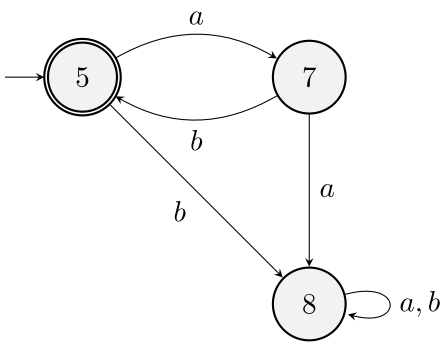
- The state $\{7\}$ is reachable via $\varepsilon$ and hopeful via $a$.
- The state $\{2\}$ is reachable via $b$ and hopeful via $aa$.
- The state $\{5, 9\}$ is reachable via $a$ and hopeful via $\varepsilon$.
- The states $\{7\}$ and $\{2\}$ are inequivalent, as $a$ is accepted by $\{7\}$ but not by $\{2\}$.
- The states $\{7\}$ and $\{5, 9\}$ are inequivalent, as $\varepsilon$ is accepted by $\{5, 9\}$ but not by $\{7\}$.
- The states $\{2\}$ and $\{5, 9\}$ are inequivalent, as $\varepsilon$ is accepted by $\{5, 9\}$ but not by $\{2\}$.

**Exercise 3.** *The alphabet is $\{a, b\}$. Give a two-state partial DFA for the regex $(ab)^*$. Convert it into a DFA, then obtain a DFA for the set of all words that are* **not** *matched by $(ab)^*$.*

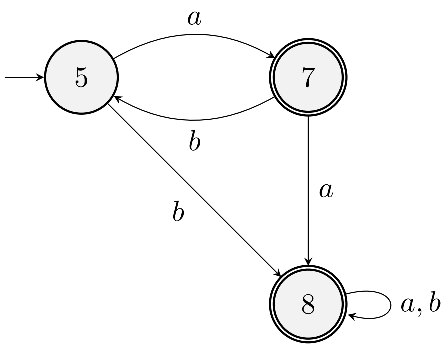
**Solution**
Here is a two-state partial DFA for the regex $(ab)^*$.




Adding an error state 8 gives the following:




For the complement of its language, we use the following:




**Exercise 4.** *Show that the set $\mathbb{N} + \mathbb{N}$ is countably infinite.*

**Solution**
The following is a bijection between $\mathbb{N}$ and $\mathbb{N} + \mathbb{N}$.

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ (0,0) & (1,0) & (0,1) & (1,1) & (0,2) & (1,2) & \cdots \end{array}$$

**Exercise 5.** *Consider the following language over the alphabet $\Sigma = \{a, b\}$:*

$$L = \{w | w \text{ contains the same number of } a\text{'s and } b\text{'s}\}$$

*Show that $L$ is non-regular.*

**Solution**

- Suppose that we are given a DFA $D$ that recognizes $L$.
- Consider $x_n$ the state of $D$ reached after reading $a^n$. State $x_n$ accepts the word $b^n$, but not the word $b^m$ for $m < n$.
- Hence all $x_n$ are inequivalent to $x_m$ for $m < n$.
- Hence the DFA $D$ has infinitely many different states, a contradiction to its assumed finiteness.

**Exercise 6.** *Are the following languages over $\Sigma = \{a, b\}$ regular ? Why (not)?*

1. $L = \{a^m b^n | m > n\}$
2. $L = \{a^m b^n | m < n\}$
3. $L = \{w | \textit{length}(w) \textit{ is a square number}\}$

**Solution**

1. $L$ is non-regular. Proof:
   - Suppose that we are given a DFA $D$ that recognizes $L$.
   - Consider the state $x_n$ of $D$ reached after reading $a^{n+1}$. State $x_n$ accepts the word $b^n$, but not the word $b^m$ for $m > n$.
   - Hence all $x_n$ are inequivalent to $x_m$ for $m > n$.
   - Hence the DFA $D$ has infinitely many different states, a contradiction to its assumed finiteness.
2. $L$ is non-regular. Proof:
   - Suppose that we are given a DFA $D$ that recognizes $L$.
   - For $n > 1$, consider the state $x_n$ of $D$ reached after reading $a^{n-1}$. State $x_n$ accepts the word $b^n$, but not the word $b^m$ for $m < n$.
   - Hence all $x_n$ are inequivalent to $x_m$ for $1 < m < n$.
   - Hence the DFA $D$ has infinitely many different states, a contradiction to its assumed finiteness.
3. $L$ is non-regular. Proof:
   - Suppose that we are given a DFA $D$ that recognizes $L$.
   - Consider the state $x_n$ of $D$ reached after reading $a^{(n^2)}$. State $x_n$ accepts the word $a^{(2n+1)}$, but not the word $a^{(2m+1)}$ for $m < n$. This is because the next square number after $n^2$ is $(n+1)^2 = n^2 + 2n + 1$.
   - Hence all $x_n$ are inequivalent to $x_m$ for $m < n$.
   - Hence the DFA $D$ has infinitely many different states, a contradiction to its assumed finiteness.

**Exercise 7.** *For any string $w = w_1 w_2 \ldots w_n$, the* ***reverse of*** *$w$, written $w^R$, is the string $w$ in reverse order, $w_n \ldots w_2 w_1$. For any language $L$, let $L^R = \{w^R | w \in L\}$. Show that if $L$ is regular, so is $L^R$.*

**Solution**
Suppose $L$ is recognized by a regex $E$. We construct a new regex $E^R$ that recognizes $L^R$, by induction:

1. If $E$ is a character $x$ of the alphabet, we set $E^R$ to be $x$.
2. If $E$ is $E_0 | E_1$, we set $E^R$ to be $E_0^R | E_1^R$.
3. If $E$ is $E_0 E_1$, we set $E^R$ to be $E_1^R E_0^R$. (This is the only thing that changes between $E$ and $E^R$.)
4. If $E$ is $(E_0)^*$, we set $E^R$ to be $(E_0^R)^*$.

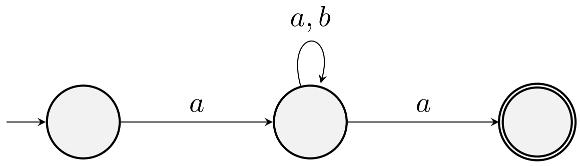
**Exercise 8.** *Let $\Sigma = \{a, b\}$.*

1. *Let $L_1 = \{a^k u a^k | k \geq 1$ and $u \in \Sigma^*\}$. Show that $L_1$ is regular.*

2. *Let* $L_2 = \{a^k bua^k | k \geq 1$ *and* $u \in \Sigma^*\}$*. Show that* $L_2$ *is not regular.*

**Solution**

1. Note that $L_1$ is equivalently written as $L_1 = \{ava | v \in \Sigma^*\}$. (We need at least one $a$ at the beginning and one at the end, the others are absorbed into $v$.) An automaton for this is easy to build:




2. Suppose that we have a DFA accepting this language. For any $n \in \mathbb{N}$, let $x_n$ be the state reached from the initial state after reading $a^n b$. For $m < n$, if we start at $x_m$ and read in $a^m$ we reach an accepting state, but if we start at $x_n$ and read in $a^m$ we reach a rejecting state, so $x_m$ is not equivalent to $x_n$. Hence there are infinitely many states, contradicting the assumed finiteness of the DFA.