MLFCS

Vector Spaces: Definition, Examples & Span

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Lecture attendance code:

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Today's plan

- Definition of a vector space (revisited)
- Some examples of vector spaces
- Detour: why consider vectors?
- Span of a set of vectors
- Linear (in)dependence of a set of vectors
- [redacted]

Recall definition of a field F (Week 2)

F is a set with $+$ and $\times$

$+$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\times$

$a + 0 = a$	$a \times 1 = a$	(neutral elements)
$a + b = b + a$	$a \times b = b \times a$	(commutativity)
$(a + b) + c = a + (b + c)$	$(a \times b) \times c = a \times (b \times c)$	(associativity)
$a + (-a) = 0$	$a \times a^{-1} = 1 \ (a \neq 0)$	(inverses)

$a \times (b + c) = a \times b + a \times c$ (distributivity of $\times$ over $+$)

Examples of fields:

- $\{0, 1\}$
- Set of rational numbers $\mathbb{Q}$
- Set of real numbers $\mathbb{R}$

Vector space V over a field F

F has two operations $+$ and $\times$

Let V be a set of vectors and F be any field. Then V is said to be a vector space over the field F if the following conditions hold:

For any vectors $\vec{u}, \vec{v}, \vec{w} \in V$ and any scalars $r, s \in F$

(1) Commutativity of vector addition: $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$
(2) Associativity of vector addition: $(\vec{u} \oplus \vec{v}) \oplus \vec{w} = \vec{u} \oplus (\vec{v} \oplus \vec{w})$
(3) Existence of Additive identity: $\vec{0} \oplus \vec{v} = \vec{v}$
(4) Existence of additive inverse: for each $\vec{x}$, there exists $\overrightarrow{-x}$ such that $\vec{x} \oplus \overrightarrow{-x} = \vec{0}$
(5) Associativity of multiplication of scalar & vector: $r(s\vec{v}) = (rs)\vec{v}$
(6) Distributivity of scalar sums: $(r + s)\vec{v} = r\vec{v} \oplus s\vec{v}$
(7) Distributivity of vector sums: $r(\vec{u} \oplus \vec{v}) = r\vec{u} \oplus r\vec{v}$
(8) Existence of identity of multiplication of scalar & vector: $1\vec{v} = \vec{v}$

First, we need to define two operations for above 8 conditions to make sense:

- Vector addition: for each $\vec{u}, \vec{v}$ a vector from V is assigned to $\vec{u} \oplus \vec{v}$
- Multiplication of a scalar by a vector: for each $s \in F$ and $\vec{v} \in V$, a vector from V is assigned to $s\vec{v}$

Multiplication of two scalars is given by $\times$ from F

Note that multiplication of two vectors is not defined here!

Some consequences of the vector space conditions

Let V be a vector space over a field F. Then for every $s \in F$ and $\vec{v} \in V$, we have

- Additive identity $\vec{0}$ is unique
- $0\vec{v} = \vec{0}$
- $s\vec{0} = \vec{0}$
- $(-1)\vec{v} = \overrightarrow{-v}$
- $s\vec{v} = \vec{0}$ implies $s = 0$ or $\vec{v} = \vec{0}$

Harder but not very creative!

Prove each of the above five using the 8 conditions given to us:

(1) Commutativity of vector addition: $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$

(2) Associativity of vector addition: $(\vec{u} \oplus \vec{v}) \oplus \vec{w} = \vec{u} \oplus (\vec{v} \oplus \vec{w})$

(3) Existence of Additive identity: $\vec{0} \oplus \vec{v} = \vec{v}$

(4) Existence of additive inverse: for each $\vec{x}$, there exists $\overrightarrow{-x}$ such that $\vec{x} \oplus \overrightarrow{-x} = \vec{0}$

(5) Associativity of multiplication of scalar & vector: $r(s\vec{v}) = (rs)\vec{v}$

(6) Distributivity of scalar sums: $(r + s)\vec{v} = r\vec{v} \oplus s\vec{v}$

(7) Distributivity of vector sums: $r(\vec{u} \oplus \vec{v}) = r\vec{u} \oplus r\vec{v}$

(8) Existence of identity of multiplication of scalar & vector: $1\vec{v} = \vec{v}$

Example 1 of a vector space

$V = F$

Every field F is a vector space over itself!

F comes with $+$ and $\times$

$\oplus$ is same as $+$

Take $F = \mathbb{Q}$ and verify each of the 8 conditions:

(1) Commutativity of vector addition: $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$

(2) Associativity of vector addition: $(\vec{u} \oplus \vec{v}) \oplus \vec{w} = \vec{u} \oplus (\vec{v} \oplus \vec{w})$

(3) Existence of Additive identity: $\vec{0} \oplus \vec{v} = \vec{v}$

(4) Existence of additive inverse: for each $\vec{x}$, there exists $\overrightarrow{-x}$ such that $\vec{x} \oplus \overrightarrow{-x} = \vec{0}$

(5) Associativity of multiplication of scalar & vector: $r(s\vec{v}) = (rs)\vec{v}$

(6) Distributivity of scalar sums: $(r + s)\vec{v} = r\vec{v} \oplus s\vec{v}$

(7) Distributivity of vector sums: $r(\vec{u} \oplus \vec{v}) = r\vec{u} \oplus r\vec{v}$

(8) Existence of identity of multiplication of scalar & vector: $1\vec{v} = \vec{v}$

You will need to use the fact that $\mathbb{Q}$ is a field.

Example 2 of a vector space

$\begin{pmatrix} a \\ b \end{pmatrix} \oplus \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$

$r \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} r \times a \\ r \times b \end{pmatrix}$

+ and × come from $\mathbb{Q}$

The set of 2-tuples of rational numbers is a vector space over the rational numbers:

- The set of 2-tuples of rational numbers is defined as $\mathbb{Q}^2 := \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : \ a, b \in \mathbb{Q} \right\}$

Verify each of the 8 conditions for $\mathbb{Q}^2$ to be a vector space over $\mathbb{Q}$:

Week 8 Ex Sheet

(1) Commutativity of vector addition: $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$

(2) Associativity of vector addition: $(\vec{u} \oplus \vec{v}) \oplus \vec{w} = \vec{u} \oplus (\vec{v} \oplus \vec{w})$

(3) Existence of Additive identity: $\vec{0} \oplus \vec{v} = \vec{v}$

(4) Existence of additive inverse: for each $\vec{x}$, there exists $\overrightarrow{-x}$ such that $\vec{x} \oplus \overrightarrow{-x} = \vec{0}$

(5) Associativity of multiplication of scalar & vector: $r(s\vec{v}) = (rs)\vec{v}$

(6) Distributivity of scalar sums: $(r+s)\vec{v} = r\vec{v} \oplus s\vec{v}$

(7) Distributivity of vector sums: $r(\vec{u} \oplus \vec{v}) = r\vec{u} \oplus r\vec{v}$

(8) Existence of identity of multiplication of scalar & vector: $1\vec{v} = \vec{v}$

You will just need to use the fact that $\mathbb{Q}$ is a field.

Example n of a vector space $(n \geq 1)$

The set of n-tuples of rational numbers is a vector space over the rational numbers:

$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

- The set of n-tuples of rational numbers is defined as
$$\mathbb{Q}^n := \left\{ (a_1, a_2, \ldots, a_n) \; : \; a_1, a_2, \ldots, a_n \in \mathbb{Q} \right\}$$

Verify each of the 8 conditions for $\mathbb{Q}^n$ to be a vector space over $\mathbb{Q}$:

(1) Commutativity of vector addition: $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$

(2) Associativity of vector addition: $(\vec{u} \oplus \vec{v}) \oplus \vec{w} = \vec{u} \oplus (\vec{v} \oplus \vec{w})$

(3) Existence of Additive identity: $\vec{0} \oplus \vec{v} = \vec{v}$

(4) Existence of additive inverse: for each $\vec{x}$, there exists $\overrightarrow{-x}$ such that $\vec{x} \oplus \overrightarrow{-x} = \vec{0}$

(5) Associativity of multiplication of scalar & vector: $r(s\vec{v}) = (rs)\vec{v}$

(6) Distributivity of scalar sums: $(r + s)\vec{v} = r\vec{v} \oplus s\vec{v}$

(7) Distributivity of vector sums: $r(\vec{u} \oplus \vec{v}) = r\vec{u} \oplus r\vec{v}$

(8) Existence of identity of multiplication of scalar & vector: $1\vec{v} = \vec{v}$

(✱) Go-to example is ($\mathbb{Q}^2$ over $\mathbb{Q}$) or ($\mathbb{Q}^3$ over $\mathbb{Q}$)

You will just need to use the fact that $\mathbb{Q}$ is a field.

Detour: Why consider tuples of rational numbers?

why consider (a,b,c) instead of just say $a \in \mathbb{Q}$

If you have not seen tuples of rational numbers such as $\mathbb{Q}^2$ or $\mathbb{Q}^3$, then you might be wondering why would one want to consider this

- Why not just stick with the set $\mathbb{Q}$?

Some potential applications/advantages of using tuples:

- Netflix

  AMAZON

  Co-ordinate system ($\mathbb{R}^2$ or $\mathbb{R}^3$)

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Span of a set of vectors

- Let V be a vector space over a field F. $\rightarrow$ $F = \mathbb{Q}$
- Let $\vec{v_1}, \vec{v_2}$ be two vectors in V
- We define $\text{Span}(\vec{v_1}, \vec{v_2}) := \{r_1\vec{v_1} \oplus r_2\vec{v_2} \mid r_1, r_2 \in F\}$
  - All possible linear combinations of $\vec{v_1}$ and $\vec{v_2}$
  - Span of $\vec{v_1}$ and $\vec{v_2}$

Does $10\vec{v_1}$ belong to $\text{Span}(\vec{v_1}, \vec{v_2})$?

Yes, pick $r_1 = 10$ & $r_2 = 0$

$r_1\vec{v_1} \oplus r_2\vec{v_2} = 10\vec{v_1} + 0\vec{v_2} = 10\vec{v_1} + \vec{0}$

$= 10\vec{v_1}$

- Exercise: $\text{Span}(\vec{v_1}, \vec{v_2})$ is a vector space over the field F.
  - Simple verification of 8 conditions & using the fact that V is a vector space over F

week 9 ex sheet

$(r_1\vec{v_1} \oplus r_2\vec{v_2}) \oplus (s_1\vec{v_1} \oplus s_2\vec{v_2}) = (r_1+s_1)\vec{v_1} + (r_2+s_2)\vec{v_2}$

$s(r_1\vec{v_1} \oplus r_2\vec{v_2}) = sr_1\vec{v_1} + sr_2\vec{v_2}$

In general, given any set of vectors from V we can define its span.

How to check if a given vector belongs to a span?

Consider the vector space $\mathbb{Q}^3$ over $\mathbb{Q}$.

- Consider the vectors $\vec{v_1} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$, $\vec{v_2} = \begin{pmatrix} 5 \\ 6 \\ 10 \end{pmatrix}$, $\vec{v_3} = \begin{pmatrix} 11 \\ 300 \\ 14 \end{pmatrix}$ from $\mathbb{Q}^3$
- I want to check if the vector $\vec{w} = \begin{pmatrix} 41 \\ 12 \\ 110 \end{pmatrix}$ belongs to $\mathrm{Span}(\vec{v_1}, \vec{v_2}, \vec{v_3})$
- How can we do that?
- Recall that $\mathrm{Span}(\vec{v_1}, \vec{v_2}, \vec{v_3}) := \left\{ r_1 \vec{v_1} \oplus r_2 \vec{v_2} \oplus r_3 \vec{v_3} \mid r_1, r_2, r_3 \in F \right\}$ $F = \mathbb{Q}$

Is $\vec{w}$ in $\mathrm{Span}(\vec{v_1}, \vec{v_2}, \vec{v_3})$?

Do there exist $r_1, r_2, r_3 \in \mathbb{Q}$ s.t $\vec{w} = r_1 \vec{v_1} \oplus r_2 \vec{v_2} \oplus r_3 \vec{v_3}$?

Do there exist $r_1, r_2, r_3 \in \mathbb{Q}$ s.t $\begin{pmatrix} 41 \\ 12 \\ 110 \end{pmatrix} = r_1 \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} + r_2 \begin{pmatrix} 5 \\ 6 \\ 10 \end{pmatrix} + r_3 \begin{pmatrix} 11 \\ 300 \\ 14 \end{pmatrix}$

s.t
$$\begin{aligned} 41 &= r_1 + 5r_2 + 11r_3 \\ 12 &= 3r_1 + 6r_2 + 300r_3 \\ 110 &= 4r_1 + 10r_2 + 14r_3 \end{aligned}$$

Gaussian Elimination!!

Linear (in)dependence of a set of vectors

- Let V be a vector space over a field F.
- A set of vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n} \in V$ is linearly independent if $r_1\vec{v_1} \oplus r_2\vec{v_2} \oplus \ldots \oplus r_n\vec{v_n} = \vec{0}$ implies $r_1 = r_2 = \ldots = r_n = 0$
  - Otherwise, the set of vectors is said to be linearly dependent
- Does $\vec{0}$ always belongs to the span of any set of vectors?
  - Yes! Take $r_1 = r_2 = r_3 = \ldots = r_n = 0$. Then $r_1\vec{v_1} \oplus r_2\vec{v_2} \oplus \ldots \oplus r_n\vec{v_n}$ $= 0\vec{v_1} + 0\vec{v_2} + \ldots + 0\vec{v_n}$ $= \vec{0} + \vec{0} + \ldots + \vec{0}$ $= \vec{0}$
- Therefore, a set of vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n} \in V$ is linearly independent if the only way to obtain $\vec{0}$ in its span is by taking all the scalars to be 0
- Question: Can $\vec{0}$ belong to any set of linearly independent vectors? Say $\vec{v_1} = \vec{0}$
  - No! Take $r_1 = 10$ and $r_2 = r_3 = \ldots = r_n = 0$
- Next slide: How can we check if a given set of vectors is linearly independent or not?

Checking linear independence of a set of vectors

Consider the vector space $\mathbb{Q}^3$ over Q.

- Consider the vectors $\vec{v_1} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} 5 \\ 6 \\ 10 \end{pmatrix}, \vec{v_3} = \begin{pmatrix} 11 \\ 300 \\ 14 \end{pmatrix}$ from $\mathbb{Q}^3$
- Want to check if these three vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are linearly independent

  Are there $r_1, r_2, r_3 \in \mathbb{Q}$ s.t. $r_1\vec{v_1} + r_2\vec{v_2} + r_3\vec{v_3} = \vec{0}$ and not all of r_1, r_2, r_3 are 0?
- How can we do that?
-

$$r_1 \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} + r_2 \begin{pmatrix} 5 \\ 6 \\ 10 \end{pmatrix} + r_3 \begin{pmatrix} 11 \\ 300 \\ 14 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If $\exists$ a solution with not all three of r_1, r_2, r_3 being 0
Then linearly dependent
otherwise linearly independent.

Summary of the lecture

- Definition of a vector space (revisited)
- Some examples of vector spaces
  - Every field over itself
  - $\mathbb{Q}^2$ over $\mathbb{Q}$
  - $\mathbb{Q}^3$ over $\mathbb{Q}$
- Detour: why consider vectors?
- Span of a set of vectors
  - How to check if a given vector belongs to span of a set of vectors?
- Linear (in)dependence of a set of vectors
- [redacted]