MLFCS

Introduction to Vector Spaces

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Lecture attendance code:

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Today's plan

- (recalling) definition of a field
- Definition of a vector space
- Some examples of vector spaces
- Detour: why consider vectors?
- Span of a set of vectors

Recall definition of a field F (Week 2)

- Set F is closed under $+$ & $\times$

$$
\begin{array}{rclrcll}
a+0 & = & a & a\times 1 & = & a & \text{(neutral elements)} \\
a+b & = & b+a & a\times b & = & b\times a & \text{(commutativity)} \\
(a+b)+c & = & a+(b+c) & (a\times b)\times c & = & a\times(b\times c) & \text{(associativity)} \\
a+(-a) & = & 0 & a\times a^{-1} & = & 1\ \ (a\neq 0) & \text{(inverses)}
\end{array}
$$

$$a\times(b+c) = a\times b + a\times c \qquad \text{(distributivity of } \times \text{ over } +\text{)}$$

Examples of fields:

- $\{0,1\}$

  $0+1=1$ $\quad 0\times 1=0$

  $1+1=0$ $\quad 1\times 1=1$

  $0+0=0$
- Set of rational numbers $\mathbb{Q}$
- Set of real numbers $\mathbb{R}$

Vector space V over a field F

elements of V are called vectors
elements of F are called scalars

$(+, \times) \longrightarrow +_F$ & $\times_F$

Let V be a set of vectors and F be any field. Then V is said to be a vector space over the field F if the following conditions hold:

For any vectors $\vec{u}, \vec{v}, \vec{w} \in V$ and any scalars $r, s \in F$

(1) Commutativity of vector addition: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(2) Associativity of vector addition: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

(3) Existence of Additive identity: $\vec{0} + \vec{v} = \vec{v}$

(4) Existence of additive inverse: for each $\vec{x}$, there exists $\overrightarrow{-x}$ such that $\vec{x} + \overrightarrow{-x} = \vec{0}$

(5) Associativity of scalar multiplication: $r(s\vec{v}) = (rs)\vec{v}$

(6) Distributivity of scalar sums: $(r + s)\vec{v} = r\vec{v} + s\vec{v}$

(7) Distributivity of vector sums: $r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$

(8) Existence of Scalar multiplication identity: $1\vec{v} = \vec{v}$

First, we need to define two operations for above 8 conditions to make sense:

- Vector addition: for each $\vec{u}, \vec{v}$ a vector from V is assigned to $\vec{u} + \vec{v}$
- Scalar multiplication: for each $s \in F$ and $\vec{v} \in V$, a vector from V is assigned to $s\vec{v}$

Some consequences of the vector space conditions

Let V be a vector space over a field F. Then for every $s \in F$ and $\vec{v} \in V$, we have

- Additive identity $\vec{0}$ is unique
- $0\vec{v} = \vec{0}$
- $s\vec{0} = \vec{0}$
- $(-1)\vec{v} = \overrightarrow{-v}$
- $s\vec{v} = \vec{0}$ implies $s = 0$ or $\vec{v} = \vec{0}$

Exercise Sheet

Suppose $\exists$ another additive identity say $\vec{a}$

Then, $\forall \vec{v} \in V$ we have $\vec{a} + \vec{v} = \vec{v}$ — (9)

Put $\vec{v} = \vec{0}$ in (9) to get $\vec{a} + \vec{0} = \vec{0}$ — (10)

$\vec{a} + \vec{0} = \vec{a}$ by (3)

Prove each of the above five using the 8 conditions given to us:

(1) Commutativity of vector addition: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(2) Associativity of vector addition: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

(3) Existence of Additive identity: $\vec{0} + \vec{v} = \vec{v}$

(4) Existence of additive inverse: for each $\vec{x}$, there exists $\overrightarrow{-x}$ such that $\vec{x} + \overrightarrow{-x} = \vec{0}$

(5) Associativity of scalar multiplication: $r(s\vec{v}) = (rs)\vec{v}$

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Example 1 of a vector space

vector addition is just + in F
scalar multiplication is just × in F

Every field F is a vector space over itself!

$V = F$: $\vec{u} + \vec{v}$

$= u \underset{F}{+} v$

Take $F = \mathbb{Q}$ and verify each of the 8 conditions:

(1) Commutativity of vector addition: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(2) Associativity of vector addition: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

(3) Existence of Additive identity: $\vec{0} + \vec{v} = \vec{v}$

(4) Existence of additive inverse: for each $\vec{x}$, there exists $\overrightarrow{-x}$ such that $\vec{x} + \overrightarrow{-x} = \vec{0}$

(5) Associativity of scalar multiplication: $r(s\vec{v}) = (rs)\vec{v}$

(6) Distributivity of scalar sums: $(r + s)\vec{v} = r\vec{v} + s\vec{v}$

(7) Distributivity of vector sums: $r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$

(8) Existence of Scalar multiplication identity: $1\vec{v} = \vec{v}$

You will need to use the fact that $\mathbb{Q}$ is a field.

Example 2 of a vector space

$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$

vector addition

The set of 2-tuples of rational numbers is a vector space over the rational numbers:

- The set of 2-tuples of rational numbers is defined as $\mathbb{Q}^2 := \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : a, b \in \mathbb{Q} \right\}$

V

scalar multiplication $r\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ra \\ rb \end{pmatrix}$

Verify each of the 8 conditions for $\mathbb{Q}^2$ to be a vector space over $\mathbb{Q}$:

(1) Commutativity of vector addition: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(2) Associativity of vector addition: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

(3) Existence of Additive identity: $\vec{0} + \vec{v} = \vec{v}$

(4) Existence of additive inverse: for each $\vec{x}$, there exists $\overrightarrow{-x}$ such that $\vec{x} + \overrightarrow{-x} = \vec{0}$

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(8) Existence of Scalar multiplication identity: $1\vec{v} = \vec{v}$

You will just need to use the fact that $\mathbb{Q}$ is a field.