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Transmission of Digital Signals

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3.1 Transmission of Digital Data

We now focus on the transmission of digital data over a baseband channel. Digital data have a broad spectrum with a significant low-frequency content. Baseband transmission of digital data requires the use of a low-pass channel with a bandwidth large enough to include the essential frequency content of the data stream. The channel is dispersive in that its frequency response deviates from that of an ideal low-pass filter. The result of data transmission over such a channel is that each received pulse is affected somewhat by adjacent pulses, thereby giving rise to intersymbol interference (ISI). ISI is a major source of bit errors in the reconstructed data stream at the receiver output. Another source of bit errors in a baseband data transmission system is the ubiquitous channel noise. Noise and ISI arise in the system simultaneously.

3.2 Detection of Binary Signals in Gaussian Noise

After the digital symbols are converted into electrical waveforms, they are transmitted through a channel. During a given signaling interval, T, a binary system transmits one of two waveforms, denoted $s_1(t)$ and $s_2(t)$. The signal, $r(t)$, received by the receiver is represented by these two equations:

$$r(t) = s_1(t) + n(t) \text{ for a binary 1} \quad 0 \leq t \leq T. \tag{3.1}$$

$$r(t) = s_2(t) + n(t) \text{ for a binary 0} \quad 0 \leq t \leq T. \tag{3.2}$$

In equations 3.1 and 3.2, $n(t)$ is a zero-mean additive Gaussian white noise, and T is the symbol duration.

We assume that the receiver has knowledge of the starting and ending times of each transmitted pulse; in other words, the receiver has prior knowledge of the pulse shape but not its polarity. Given the noisy signal, the receiver has to make a decision in each signaling interval as to whether the transmitted symbol is 1 or 0.

We refer to Figure 3.1 **step 1** involves reducing the received waveform to a single number $z(t = T)$. This operation can be performed by a linear filter followed by a sampler or optimally by a matched filter. The initial conditions of the filter are set to zero just before the arrival of each new symbol. At the end of a symbol duration T, the output of step 1 yields the sample $z(T)$:

$$z(T) = a_i(T) + n_0(T) \quad i = 1, 2, \tag{3.3}$$

where $a_i(T)$ is the signal component of $z(T)$ and where $n_0(T)$ is the noise component.

Since noise component $n_0(t)$ is a zero-mean Gaussian random variable, $z(T)$ is also a Gaussian random variable with a mean of either a_1 or a_2 depending on whether a binary 1 or 0 was sent. The probability density function (pdf) of the Gaussian random noise, n_0, can be expressed as:

$$p(n_0) = \frac{1}{\sigma_0\sqrt{2\pi}} e^{-1/2(n_0/\sigma_0)^2}, \tag{3.4}$$

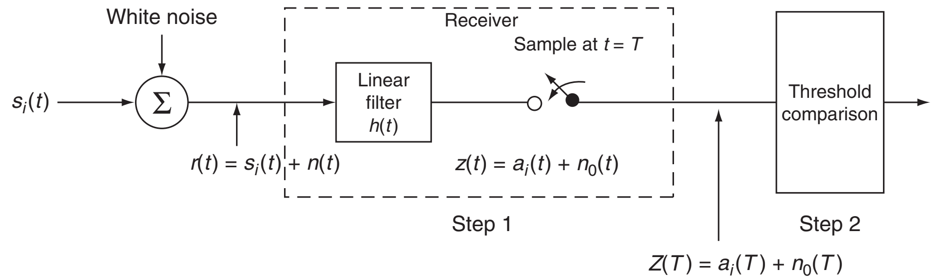

FIGURE 3.1 Detection of Binary Signals in Gaussian Noise

where, σ_0^2 is the noise variance.

The conditional pdfs, p(z s_1) and p(z s_2), can be given as:

$$p(z|s_1) = \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-a_1}{\sigma_0}\right)^2}. \quad (3.5)$$

$$p(z|s_2) = \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-a_2}{\sigma_0}\right)^2}. \quad (3.6)$$

These conditional pdfs are shown in Figure 3.2. The rightmost conditional pdf $p(z|s_1)$, shows the probability density of the detector output, $z(T)$, given that $s_1(t)$ was transmitted. Similarly, the leftmost conditional pdf $p(z|s_2)$, shows the probability density of the detector output $z(T)$, given $s_2(t)$ was transmitted.

Step 2 in signal detection process consists of comparing the $z(T)$ to a threshold level λ in block 2 of Figure 3.1 to estimate which signal, $s_1(t)$ or $s_2(t)$, has been transmitted. The filtering operation in block 1 does not depend on the decision criterion in block 2. Thus, the choice of how best to implement block 1 can be independent of the particular decision choice of the threshold setting, λ. After a received waveform, $r(t)$, is transformed to $z(T)$, the actual shape of the waveform is no longer important. The signal energy (not its shape) is the important parameter in the detection process. Thus, the detection analysis for **baseband signals** is the same as that for **bandpass signals**. The final step in block II is to make the detection.

A popular criterion for choosing the threshold level λ for the binary decision is based on minimizing the probability of error. It can be shown that if $p(s_1) = p(s_2)$, and if the likelihoods $p(z|s_i)$ $(i = 1, 2)$ are symmetrical, the optimum value of λ is given as:

$$\lambda_0 = \frac{a_1 + a_2}{2}, \quad (3.7)$$

where a_1 is the signal component of $z(T)$ when $s_1(t)$ is transmitted and where a_2 is the signal component of $z(T)$ when $s_2(t)$ is transmitted.

The threshold level, λ_0, represented by $(a_1 + a_2)/2$, is the optimum threshold to minimize the probability of making an incorrect decision for this important special case. The strategy is known as the **minimum error criterion**.

3.3 Error Probability

For the binary example in Figure 3.2, there are two ways in which errors can occur. An error, e, will occur when $s_1(t)$ is sent, and the channel noise results in the receiver output signal $z(T)$ being less than λ. The probability of such an occurrence is given as:

$$P(e|s_1) = \int_{-\infty}^{\lambda} p(z|s_1)dz. \quad (3.8)$$

Similarly, an error occurs when $s_2(t)$ is sent, and the channel noise results in the receiver output signal $z(T)$ being greater than λ. The probability of such an occurrence is given as:

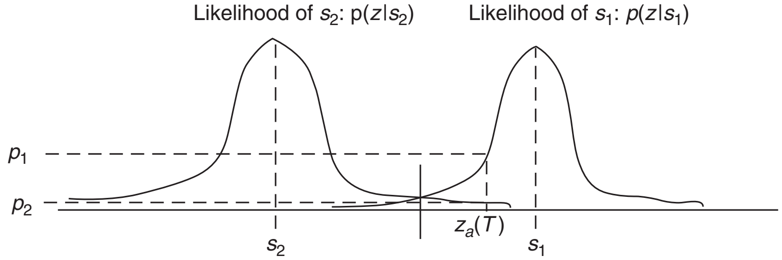

FIGURE 3.2 Conditional Probability Density Functions

$$P(e|s_2) = \int_{\lambda}^{\infty} p(z|s_2)dz. \tag{3.9}$$

Let α and $1 - \alpha$ denote the a prior probabilities of transmitting 0 and 1, respectively, and then the average probability of symbol error P_e in the receiver is given by:

$$P_e = \alpha P(e|s_1) + (1 - \alpha)P(e|s_2). \tag{3.10}$$

For the special case when 1 and 0 are equiprobable, we have $\alpha = 1/2$:

$$P_e = \frac{1}{2}[P(e|s_1) + P(e|s_2)]. \tag{3.11}$$

Because of the symmetry of pdf:

$$P_e = P(e|s_1) = P(e|s_2). \tag{3.12}$$

The average probability of symbol error with optimum λ_0:

$$P_e = \int_{(a_1+a_2)/2}^{\infty} \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-a_2}{\sigma_0}\right)^2} \tag{3.13}$$

Let:

$$u = (z - a_2)/\sigma_0,$$
$$\therefore \sigma_0 du = dz.$$
$$\therefore P_e = \int_{(a_1-a_2)/2\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du = Q\left(\frac{a_1 - a_2}{2\sigma_0}\right). \tag{3.14}$$

In equation 3.14 $(a_1 - a_2)$ is the difference of signal components at the filter output at time $t = T$, and the square of this difference signal is the instantaneous power of difference signal. The $Q(x)$ is the complementary error function, and it is defined as:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-u^2/2} du \approx \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}. \tag{3.15}$$

$$\left(\frac{S}{N}\right) = \frac{(a_1 - a_2)^2}{\sigma_0^2} = \frac{E_d}{\left(\frac{N_0}{2}\right)}. \tag{3.16}$$

The E_d is the energy of a difference signal at the filter input:

$$E_d = \int_0^T [s_1(t) - s_2(t)]^2 dt, \tag{3.17}$$

$$\therefore P_e = Q\left(\sqrt{\frac{E_d}{2N_0}}\right). \tag{3.18}$$

3.4 The Matched Filter

A **matched filter** is a linear filter designed to provide the maximum SNR at its output for a given transmitted symbol waveform. We refer to Figure 3.1 for the ratio of instantaneous signal power to average noise power at time $t = T$; out of the receiver block 1, the following results:

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2}. \tag{3.19}$$

We want to find the filter transfer function $H_0(f)$ to *maximize* equation 3.19. We express the signal, $a(t)$, at the filter output in terms of the filter transfer function, $H(f)$. The Fourier transform of the input signal will be:

$$a(t) = \int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi ft} dj, \tag{3.20}$$

where $S(f)$ is the Fourier transform of the input signal $s(t)$. With power spectral density of the input noise equal to $N_0/2$, we can express the output noise power as:

$$\sigma_0^2 = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df. \tag{3.21}$$

Using equations 3.20 and 3.21, we rewrite equation 3.19 as:

$$\left(\frac{S}{N}\right)_T = \frac{\left|\int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi fT} df\right|^2}{N_0/2 \int_{-\infty}^{\infty} |H(f)|^2 df}. \tag{3.22}$$

We must find that value of $H(f) = H_0(f)$ for which the maximum $(S/N)_T$ is achieved, by using **Schwarz's inequality**. One form of the inequality can be stated as:

$$\left|\int_{-\infty}^{\infty} f_1(x)f_2(x)dx\right|^2 \leq \int_{-\infty}^{\infty} |f_1(x)|^2 dx \int_{-\infty}^{\infty} |f_2(x)|^2 d.$$

The equality holds if $f_1(x) = k_2(x)^*$, where k is an arbitrary constant and * indicates complex conjugate. If we identify $H(f)$ with $f_1(x)$ and $S(f)e^{j2\pi ft}$ with $f_2(x)$, we can write:

$$\left|\int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi fT} df\right|^2 \leq \int_{-\infty}^{\infty} |H(f)|^2 df \int_{-\infty}^{\infty} |S(f)|^2 d(f). \tag{3.23}$$

Using equation 3.23 in equation 3.22, we get:

$$\left(\frac{S}{N}\right)_T \le \frac{2}{N_0}\int_{-\infty}^{\infty}|S(f)|^2 df. \quad (3.24)$$

$$\max\cdot\left(\frac{S}{N}\right)_T = \frac{2}{N_0}E. \quad (3.25)$$

The energy E of the input signal $s(t)$ is the following:

$$E = \int_{-\infty}^{\infty}|S(f)|^2 df \quad (3.26)$$

Thus, the maximum output $(S/N)_T$ depends on the input **signal energy** and power spectral density of the noise, *not* on the *particular shape* of the waveform that is used.

Therefore, the impulse response of a filter that produces the maximum output signal-to-noise ratio is the mirror image of the message signal $s(t)$, delayed by the symbol time duration, T.

3.5 Error Probability Performance of Binary Signaling

In the following sections, we cover the error probability performance of binary signaling.

3.5.1 Unipolar Signaling

$$s_1(t) = A \quad 0 \le t \le T \quad \text{for binary 1.}$$
$$s_2(t) = 0 \quad 0 \le t \le T \quad \text{for binary 0.}$$

$$E_d = \int_0^T A^2 dt = A^2 T.$$

$$\therefore P_e = Q\left(\sqrt{\frac{A^2T}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right),$$

where $E_b = A^2T/2$, the average energy per bit (see Figure 3.3).

3.5.2 Bipolar Signaling

$$s_1(t) = A \quad 0 \le t \le T \quad \text{for binary 1.}$$
$$s_2(t) = -A \quad 0 \le t \le T \quad \text{for binary 0.}$$

$$E_d = \int_0^T 4A^2 dt = 4A^2T.$$

$$\therefore P_e = Q\left(\sqrt{\frac{4A^2T}{2N_0}}\right) = Q\left(\sqrt{\frac{2A^2T}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right),$$

where the average energy per bit is $E_b = A^2T$ (see Figure 3.3).

3.6 Equalizer

In practical systems, the frequency response of the channel is not known with sufficient accuracy to allow for a receiver design that will compensate for ISI for all time. The filter for handling ISI at the receiver contains various parameters that are adjusted on the basis of measurements of the channel

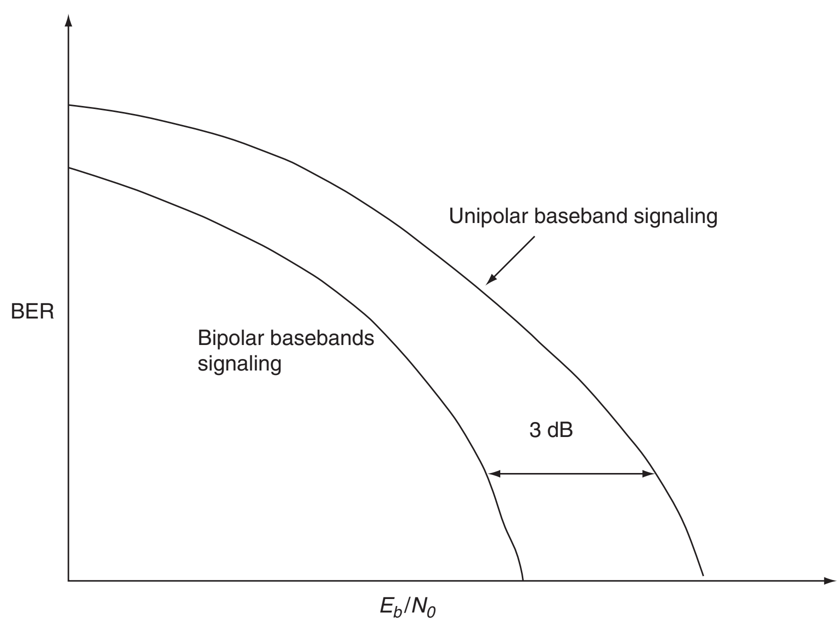

FIGURE 3.3 Bit Error Performance of Unipolar and Bipolar Signaling

characteristics. The process to correct channel-induced distortion is referred to as **equalization**. The adjustable filter is called an **equalizer**. Equalizers may be preset or adaptive. The parameters of a **preset equalizer** are adjusted by making measurements of channel impulse response and solving a set of equations for the parameters using these measurements. An **adaptive equalizer** is automatically adjusted by sending a known signal through the channel and allowing the equalizer to adjust its own parameters in response to this known signal.

A transversal filter—a delay line with T-second taps (where T is the symbol duration)—is a common choice for the equalizer. The outputs of the taps are amplified, summed, and fed to a decision device. The tap coefficients C_n are set to subtract the effects of interference from symbols that are adjacent in time to the desired symbol. The output samples y_k of the equalizer are written as:

$$y_k = \sum_{n=-N}^{N} C_n x_{k-n} \quad k = -2N, \ldots, 2N, \tag{3.27}$$

where:

$$\begin{pmatrix} y_{-2N} \\ \cdot \\ \cdot \\ y_0 \\ \cdot \\ \cdot \\ y_{2N} \end{pmatrix} = \{y_k\};\ C = \begin{pmatrix} C_{-N} \\ \cdot \\ C_0 \\ \cdot \\ C_N \end{pmatrix}, \text{ and}$$

$x = (2N+1) \times (2N+1)$ is the channel response matrix.

We can write the equation in matrix notation as:

$$y = Cx.$$

The impulse response is as follows:

$$h_E(t) = \sum_{n=-N}^{N} C_n \delta(t - nT). \tag{3.28}$$

The frequency response is as follows:

$$H_E(f) = \sum_{n=-N}^{N} C_n e^{-j2\pi nTf}. \tag{3.29}$$

Since there are only $2N+1$ unknown coefficients, it follows that only a finite number of interfering symbols can be nulled or forced to be zero:

$$y_k = 1 \text{ for } k = 0:$$
$$y_k = 0 \text{ for } k = \pm 1,\ \pm 2 \ldots \pm N.$$

The channel response matrix is given as:

$$[x] = \begin{bmatrix} x_{-N} & 0 & 0\cdot\cdot & 0 & 0 \\ x_{-N+1} & x_{-N} & 0\cdot\cdot & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ x_N & x_{N-1} & \cdots & x_{-N+1} & x_{-N} \\ 0 & 0 & 0\cdot\cdot & x_N & x_{N-1} \\ 0 & 0 & 0\cdot\cdot & 0 & x_N \end{bmatrix}. \tag{3.30}$$

The zero-forcing equations do not account for the effects of noise. In addition, the finite-length transversal filter equalizer can minimize worst-case ISI only if the peak distortion is less than 100% of the eye opening. Another type of equalizer is the minimum mean square error (MMSE) equalizer. In these equalizers, coefficients are selected to minimize the mean square error that consists of the sum of the square of all the ISI terms plus noise power at the equalizer output. The MMSE equalizer maximizes the signal-to-distortion ratio at its output within the constraints of equalizer length and delay.

Example 7

Consider a channel that uses a five-tap equalizer (see Figure 3.4) to correct ISI. The following measurements were made: $x(0) = 1.0$, $x(-1) = 0.2$, $x(-2) = 0.1$, $x(-3) = 0.05$, $x(-4) = -0.02$, $x(-5) = 0.01$, $x(1) = -0.1$, $x(2) = 0.1$, $x(3) = -0.05$, $x(4) = 0.02$, and $x(5) = 0.005$.

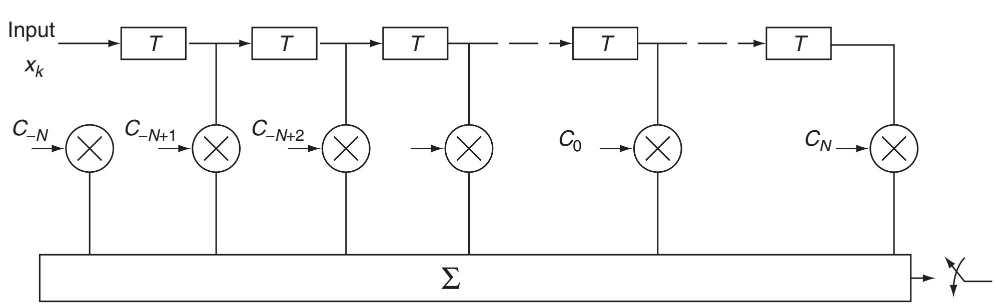

FIGURE 3.4 Tap Equalizer

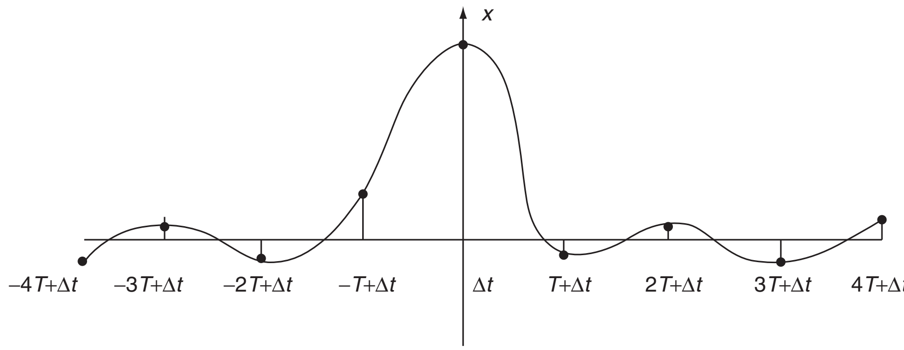

FIGURE 3.5

In addition, note the following results and equations:

$$\begin{bmatrix} x(0) & x(-1) & x(-2) & x(-3) & x(-4) \\ x(1) & x(0) & x(-1) & x(-2) & x(-3) \\ x(2) & x(1) & x(0) & x(-1) & x(-2) \\ x(3) & x(2) & x(1) & x(0) & x(-1) \\ x(4) & x(3) & x(2) & x(1) & x(0) \end{bmatrix} =$$

$$\begin{bmatrix} 1.0 & 0.2 & -0.1 & 0.05 & -0.02 \\ -0.1 & 1.0 & 0.2 & -0.1 & 0.05 \\ 0.1 & -0.1 & 1.0 & 0.2 & -0.1 \\ -0.05 & 0.1 & -0.1 & 1.0 & 0.2 \\ 0.02 & -0.05 & 0.1 & -0.01 & 1.0 \end{bmatrix} = [x].$$

$$[x]^{-1} = \begin{bmatrix} 0.966 & -0.170 & 0.117 & -0.083 & 0.056 \\ 0.118 & 0.945 & -0.158 & 0.112 & -0.083 \\ -0.091 & 0.133 & 0.937 & -0.158 & 0.117 \\ 0.028 & -0.095 & 0.133 & 0.945 & -0.170 \\ -0.002 & 0.028 & -0.091 & 0.118 & 0.966 \end{bmatrix}.$$

$$\{C\} = \begin{bmatrix} 0.117 \\ -0.158 \\ 0.945 \\ 0.133 \\ -0.091 \end{bmatrix} = \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \end{bmatrix}.$$

$$\begin{aligned} y_0 &= C_{-2} \cdot x_2 + C_{-1} \cdot x_1 + C_0 \cdot x_0 + C_1 \cdot x_{-1} + C_2 \cdot x_{-2} \\ &= 0.117 \times 0.1 + (-0.158) \times (-0.1) + 0.937 \times 1 \\ &\quad + 0.133 \times 0.2 - 0.091 \times -0.1 = 1.0. \end{aligned}$$