Shapley-Scarf Markets with Objective Indifferences

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Abstract

In many object allocation problems, some of the objects may be indistinguishable from each other. For example, in a college dormitory, rooms in the same building with the same floor plan are effectively identical. In such cases, it is reasonable to assume that agents are indifferent between identical objects, and matching mechanisms in these settings should account for the agents' indifferences. Top trading cycles (TTC) with fixed tie-breaking has been suggested and used in practice to deal with indifferences in object allocation problems. Under general indifferences, TTC with fixed tie-breaking is neither Pareto efficient nor group strategy-proof. Furthermore, it may not select an allocation in the core of the market, even when the core is non-empty. We introduce a new setting, objective indifferences, in which any indifferences are shared by all agents. In this setting, which includes strict preferences as a special case, TTC with fixed tie-breaking maintains Pareto efficiency, group strategy-proofness, and core selection. Further, we characterize objective indifferences as the most general setting where TTC with fixed tie-breaking maintains these important properties.

1 Introduction

Important markets including living donor organ transplants, public housing assignments, and school choice can be modeled as Shapley-Scarf markets: each agent is endowed with an indivisible object and has preferences over the set of objects. Monetary transfers are disallowed, and participants have property rights to their own endowments. The goal is to re-allocate these objects among the agents to achieve efficiency and

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stability. The usual stability notion is the core: an allocation is in the core if no subset of agents would prefer to trade their endowments among themselves. In the original setting of Shapley and Scarf (1974), agents have strict preferences over the houses, and Gale's *top trading cycles* (TTC) algorithm finds an allocation in the core. Roth and Postlewaite (1977) further show that the core is non-empty, unique, and Pareto efficient. Roth (1982) shows that TTC is strategy-proof; Bird (1984), Moulin (1995), Pápai (2000), and Sandholtz and Tai (2024) show that it is group strategy-proof. These properties make TTC an attractive algorithm for practical applications.

However, the assumption that preferences are strict is quite strong. In particular, if any objects are essentially identical, agents should naturally be indifferent between them. For example, consider the problem of assigning students to college dormitory units. It seems reasonable to assume that two units with the same floor plan in the same building are basically equivalent from a student's perspective. In fact, the undergraduate on-campus housing application process at UC Berkeley applies this same logic. For first-year undergraduates, there are seven possible housing complexes (Unit 1, Unit 2, Unit 3, Stern, Foothill, Clark Kerr, and Blackwell), each with a variety of possible room configurations (double-occupancy room, triple-occupancy room, double-occupancy room in a 4-person suite, etc.). On housing applications, incoming first-year students rank their five most preferred *housing complex* $\times$ *floor plan* pairs. Partly, this is due to the practical challenges associated with collecting and aggregating students' preferences over the thousands of available dormitory units. More importantly, it demonstrates how it is implicitly assumed that students have strict preferences over *housing complex* $\times$ *floor plan* pairs, but are indifferent between dormitory units of the same type. Beyond college dormitory assignment problems, there are a host of real-world object assignment problems (military occupational specialty (MOS) assignment, school choice, etc.) that can be modeled with a similar structure.

We therefore present a model of Shapley-Scarf markets where there may be indistinguishable copies (which we will call "houses") of the objects (which we will call "types" or "house types"). Our model restricts agents to be indifferent between houses of the same type, but never indifferent between houses of different types. We call these preferences "objective indifferences." We see objective indifferences as a minimal model of indifferences, capturing the most basic and plausible form of indifferences.

In the fully general setting where agents' preferences may contain indifferences, *TTC with fixed tie-breaking* is often used in practice; ties in preference rankings are broken by some external rule. For example, Abdulkadiroglu and Sönmez (2003) propose something similar in the setting of school choice with priorities. However, TTC with fixed tie-breaking is not Pareto efficient nor group strategy-proof. In fact, there is an inherent tension between these two properties: Ehlers (2002) shows that when agents have weak preferences, there does not exist a Pareto efficient and group strategy-proof mechanism in Shapley-Scarf markets. With weak preferences, the core of the market may be empty or non-unique. But even when the core of a market is non-empty, TTC with fixed tie-breaking may not select a core allocation.

Objective indifferences adds structure to the case of general indifferences by constraining any indifferences to be universal among agents. While the core may still be empty, it is essentially single-valued when it is non-empty. That is, for any allocations in the core, all agents are indifferent between their allocated objects. In our setting, this means that every core allocation assigns an agent to the same house type, though not necessarily to the exact same house. Therefore, under objective indifferences, the core can be thought of as a unique mapping from agents to house types. Moreover, we show that in Shapley-Scarf markets with objective indifferences, TTC with fixed tie-breaking recovers Pareto efficiency and group strategy-proofness. It also selects a core allocation when the core is non-empty, and selects an allocation in the weak core otherwise. In fact, the objective indifferences setting is the most general setting such that TTC with fixed tie-breaking maintains any of these properties.

Others have have studied Shapley-Scarf markets with indifferences. Ehlers (2014) shows that in the general indifferences setting, a mechanism is individually rational, strategy-proof, weakly efficient, non-bossy, and consistent mechanism if and only if it is TTC for some fixed tie-breaking rule. Alcalde-Unzu and Molis (2011) and Jaramillo and Manjunath (2012) propose new families of trading cycle mechanisms, Top Trading Absorbing Sets (TTAS) and Top Cycle Rules (TCR), respectively, for the general indifferences setting. Both TTAS and TCR mechanisms are strategy-proof, Pareto efficient, and core selecting. Aziz and de Keijzer (2012) develop an even more general family of mechanisms, Generalized Absorbing Top Trading Cycles (GATTC), containing both TTAS and TCR as subclasses. GATTC mechanisms are Pareto efficient and core selecting, but are not generally strategy-proof. Plaxton (2013) defines a new subclass of GATTC mechanisms which are strategy-proof, Pareto efficient, core selecting, and run in $O(n^3)$-time, a substantial improvement over TCR mechansims, which run in $O(n^6)$-time, and TTAS mechanisms, which do not run in polynomial time. Fundamentally, the challenge for any mechanism in a Shapley-Scarf market with indifferences is determining which trading cycles to execute from among the many potential trading cycles that indifferences may induce. Using a fixed tie-breaking rule is intuitive and easy to implement, but as Ehlers (2014) shows, it comes at the expense of certain desirable properties, including Pareto efficiency and core selection. Though we too study Shapley-Scarf markets with indifferences, we place additional structure on the agents' indifferences and demonstrate how this resolves many of the challenges that indifferences pose.

Our paper makes several important new contributions to the literature on Shapley-Scarf markets. First, it defines and explores a new domain of preferences that accurately captures many real-world scenarios where this model is applied. Second, it characterizes the most general setting where TTC has no obvious drawbacks, in the sense that it retains all of the properties that make it so appealing under strict preferences. Third, it illustrates the underlying reason why weak preferences cause TTC to lose these properties: it is not indifferences per se, but *subjective* indifferences that may differ across agents.

Section 2 presents the formal notation. Section 3 explains TTC with fixed tie-breaking. Section 4 provides the main results. Section 5 concludes. Proofs of our results can be found in Appendix A.

2 Model

We present the model primitives. First we recount the classical Shapley and Scarf (1974) domain. Afterwards we introduce our "objective indifferences" domain.

We now present the general model of a Shapley-Scarf market. Let $N = \{1, \ldots, n\}$ be a finite set of agents, with generic member i. Let $H = \{h_1, \ldots, h_n\}$ be a set of houses, with generic member h. Every agent is endowed with one object, given by a bijection $w : N \to H$. The set of all endowments is $W(N, H)$, or W for short. An allocation is an assignment of an object to each agent, given by a bijection $x : N \to H$. The set of all allocations is $X(N, H)$, or simply X. For any $i \in N$, we use w_i and x_i as shorthand notation for $w(i)$ and $x(i)$ respectively. Similarly, for any $Q \subseteq N$ we use w_Q and x_Q as shorthand notation for $w(Q) = \{w(i) : i \in Q\}$ and $x(Q) = \{x(i) : i \in Q\}$ respectively.

A set $\mathcal{R}^n$ of possible preference profiles is a **domain**. Note that we restrict attention in this paper to domains that can be expressed as $\mathcal{R}^n$ for some set of preference relations $\mathcal{R}$ over H. That is, every agent has the same set of possible preference relations. If $\mathcal{R}$ is the set of strict preference relations over H, then $\mathcal{R}^n$ is the classical **strict preferences domain**. If $\mathcal{R}$ is the set of weak preference relations over H, then $\mathcal{R}^n$ is the classical **general indifferences domain**.

Our main domain is objective indifferences. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_K\}$ be a partition of H. An element H_k of a partition is a **block**. Given H and $\mathcal{H}$, let $\eta : H \to \mathcal{H}$ be the mapping from a house to the partition element containing it; that is, $\eta(h) = H_k$ if $h \in H_k$. From any strict linear order $>$ over $\mathcal{H}$, we derive a preference relation $R_>$ over H: for $h, h' \in H$, we say that $hR_>h'$ if $\eta(h) > \eta(h')$ or $\eta(h) = \eta(h')$.

Let $\mathcal{R}(\mathcal{H}) := \{R_>\}_>$ be the set of all $R_>$ given $\mathcal{H}$. Given $\mathcal{H}$, $\mathcal{R}(\mathcal{H})^n$ is an **objective indifferences domain**. We sometimes suppress $(\mathcal{H})$ from the notation when context makes it clear. Note that all agents are indifferent between houses in the same block of $\mathcal{H}$ and have strict preferences between houses in different blocks. Thus, we may refer to the "indifference classes" for the domain with the understanding that everyone shares the same indifference classes. As usual, we let P_i and I_i denote the strict relation and indifference relation associated with a preference relation $R_i \in \mathcal{R}(\mathcal{H})$. For any $Q \subseteq N$ and preference profile R, we use R_Q to denote $\{R_i : i \in Q\}$.

2.1 Mechanisms

This subsection recounts formalities on mechanisms and top trading cycles. Familiar readers may safely skip this subsection.

A **market** is a tuple (N, H, w, R). A **mechanism** is a function $f : \mathcal{R}^n \to X$; given a preference profile, it produces an allocation. When it is unimportant or clear from context, we suppress inputs from the notation. For any $i \in N$, let $f_i(R)$ denote i's allocated house under $f(R)$. Similarly, for any $Q \subseteq N$, let $f_Q(R) = \{f_i(R) : i \in Q\}$. Fix a mechanism f, a market (N, H, w), and a preference domain $\mathcal{R}^n$. We work with the following axioms.

A mechanism is Pareto efficient if it always produces Pareto efficient allocations.

Pareto efficiency (PE). For all $R \in \mathcal{R}^n$, there is no other allocation $x \in X$ such that $x_i R_i f_i(R)$ for all $i \in N$ and $x_i P_i f_i(R)$ for at least one $i \in N$.

Group strategy-proofness requires that no coalition of agents can collectively improve their outcomes by submitting false preferences. Note that in the following definition, we require both the true preferences and misreported preferences to come from the preference domain $\mathcal{R}^n$.

Group strategy-proofness (GSP). For all $R \in \mathcal{R}^n$, there do not exist $Q \subseteq N$ and $R' = (R'_Q, R_{-Q}) \in \mathcal{R}^n$ such that $f_i(R') R_i f_i(R)$ for all $i \in Q$ with $f_i(R') P_i f_i(R)$ for at least one $i \in Q$.

Individual rationality models the constraint of voluntary participation. It requires that agents receive a house they weakly prefer to their endowment.

Individual rationality (IR). For all $R \in \mathcal{R}^n$, $f_i(R) R_i w_i$ for all $i \in N$.

We also define the core of a market: an allocation is in the core if there is no subset of agents who could benefit from trading their endowments among themselves.

Definition 1. An allocation x is **blocked** if there exists a coalition $Q \subseteq N$ and allocation x' such that $x'_Q = w_Q$ and $x'_i R_i x_i$ for all $i \in Q$, with $x'_i P_i x_i$ for at least one $i \in Q$.

Definition 2. An allocation x is in the **core** of the market if it is not blocked.

The core property models the restriction imposed by property rights. Notice that individual rationality excludes blocking coalitions of size 1. The last axiom is core-selecting.

Core-selecting (CS). For all $R \in \mathcal{R}$, if the core of the market is non-empty then $f(R)$ is in the core.

In Section 4, we characterize maximal domains on which TTC with fixed tie-breaking satisfies the axioms. By a "maximal" domain, we mean the following.

Definition 3. A domain $\mathcal{R}^n$ is **maximal** for an axiom A and a class of rules F if

1. each $f \in F$ is A on $\mathcal{R}^n$, and
2. for any $\tilde{\mathcal{R}}^n \supset \mathcal{R}^n$, there is some $f \in F$ that is *not* A on $\tilde{\mathcal{R}}^n$.

Note that this definition of maximality depends on both the axiom and the class of rules, which differs from elsewhere in the literature. Typically, a maximal domain for some property is the largest possible domain on which *some* rule exists which satisfies the desired property. We focus on a specific class of rules: top trading cycles with fixed tie-breaking. Again note that we only consider domains that can be written as $\mathcal{R}^n$, which is common.

3 Top trading cycles with fixed tie-breaking

In this paper, we analyze top trading cycles (TTC) with fixed tie-breaking on the domains defined in the previous section. For an extensive history of TTC, we refer the reader to Morill and Roth (2024). We briefly define TTC and TTC with fixed tie-breaking.

Algorithm 1. ***Top Trading Cycles****. Consider a market (N, H, w, R) under strict preferences. Draw a graph with N as nodes.*

1. *Draw an arrow from each agent i to the owner (endowee) of his favorite remaining object.*
2. *There must exist at least one cycle; select one of them. For each agent in this cycle, give him the object owned by the agent he is pointing at. Remove these agents from the graph.*
3. *If there are remaining agents, repeat from step 1.*

We denote the resulting allocation as $\mathrm{TTC}(R)$.

TTC is well-defined only with strict preferences, as Step 1 requires a unique favorite object. In practice, a **fixed tie-breaking profile** $\succ$ is often used to resolve indifferences. Given N, let $\succ = (\succ_1, \ldots, \succ_n)$, where each $\succ_i$ is a strict linear order over N. This linear order will be used to break indifferences between objects (based on their owners). For any preference relation R_i and tie-breaking rule $\succ_i$, let $R_{i,\succ_i}$ be given by the following. For any $j \neq j'$, let $w_j R_{i,\succ_i} w_{j'}$ if

1. $w_j P_i w_{j'}$, or
2. $w_j I_i w_{j'}$ and $j \succ_i j'$

Then $R_{i,\succ_i}$ is a strict linear order over the individual houses. Example 1 illustrates how we combine an agent's preferences and the tie-breaking rule to construct tie-broken preferences.

Example 1. Let $N = \{1, 2, 3, 4\}$. Agent 1's preferences R_1 and tie-breaking rule $\succ_1$ are shown below. In our visual representations of preference relations, each line represents an indifference class, and the houses on any line are strictly preferred to houses on lines below them. For example, the representation of R_1 below indicates that $w_3 \ I_1 \ w_4 \ P_1 \ w_1 \ I_1 \ w_2$.

$$\begin{array}{c} R_1 \\ \hline w_3, w_4 \\ w_1, w_2 \\ \\ \\ \end{array} \quad + \quad \begin{array}{c} \succ_1 \\ \hline 1 \\ 2 \\ 3 \\ 4 \end{array} \quad \rightarrow \quad \begin{array}{c} R_{1,\succ_1} \\ \hline w_3 \\ w_4 \\ w_1 \\ w_2 \end{array}$$

Since $w_3 I_1 w_4$ and $3 \succ_1 4$, we have $w_3 P_{1,\succ_1} w_4$. Likewise, since $w_1 I_1 w_2$ and $1 \succ_1 2$, we have $w_1 P_{1,\succ_1} w_2$. Therefore, agent 1's complete tie-broken preferences $R_{1,\succ_1}$ are given by $w_3 \ P_{1,\succ_1} \ w_4 \ P_{1,\succ_1} \ w_1 \ P_{1,\succ_1} \ w_2$.

Given a preference profile $R \in \mathcal{R}^n$ and a tie-breaking profile $\succ$, let $R_\succ = (R_{1,\succ_1}, \ldots, R_{n,\succ_n})$. **TTC with fixed tie-breaking (TTC$_\succ$)** is $\text{TTC}_\succ(R) \equiv \text{TTC}(R_\succ)$. That is, the tie-breaking profile is used to generate strict preferences, and TTC is applied to the resulting strict preference profile. Formally, each tie-breaking profile $\succ$ generates a different TTC$_\succ$ rule. The following example illustrates how TTC$_\succ$ works in the objective indifferences domain.

Example 2. Let $N = \{1,2,3,4\}$. The preference profile $R = (R_1, R_2, R_3, R_4)$ and tie-breaking profile $\succ = (\succ_1, \succ_2, \succ_3, \succ_4)$ are shown below. R and $\succ$ are combined as shown in Example 1 to construct the tie-broken preference profile $R_\succ = (R_{1,\succ_1}, R_{2,\succ_2}, R_{3,\succ_3}, R_{4,\succ_4})$. Recall that $\text{TTC}_\succ(R)$ is equivalent to $\text{TTC}(R_\succ)$.

R_1	R_2	R_3	R_4
w_2, w_3	w_1	w_1	w_2, w_3
w_1	w_2, w_3	w_4	w_4
w_4	w_4	w_2, w_3	w_1

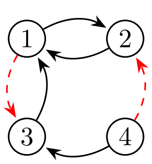
Preference profile R

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$
2	1	3	3
1	2	2	1
3	3	1	2
4	4	4	3

Tie-breaking profile $\succ$

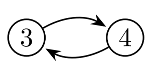
$R_{1,\succ_1}$	$R_{2,\succ_2}$	$R_{3,\succ_3}$	$R_{4,\succ_4}$
w_2	w_1	w_1	w_3
w_3	w_2	w_4	w_2
w_1	w_3	w_3	w_4
w_4	w_4	w_2	w_1

Tie-broken preference profile $R_\succ$



Step 1 of $\text{TTC}_\succ(R)$:

Each agent points to the owner of their favorite house according to their *tie-broken* preferences, represented by the black arrows. The red dashed arrows are only shown to emphasize that agents 1 and 4 are indifferent between their top choices, w_2 and w_3. Agents 1 and 2 form a cycle, and therefore swap houses.



Step 2 of $\text{TTC}_\succ(R)$:

After removing the agents assigned in Step 1, the remaining agents (3 and 4) point to the owner of their favorite remaining house. They form a cycle, and therefore swap houses. Since every agent has been assigned to a house, the TTC$_\succ$ procedure ends. The resulting allocation is $x = (w_2, w_1, w_4, w_3)$.

4 Results

In the general indifferences domain, TTC$_\succ$ mechanisms are not Pareto efficient, core-selecting, nor group strategy-proof. We give some simple examples below to illustrate these failures. However, we show that in the objective indifferences domain, TTC$_\succ$ mechanisms satisfy all three properties. Furthermore, we show that objective indifferences characterizes the set of maximal domains on which TTC$_\succ$ mechanisms are PE and CS, and characterizes the set of "symmetric-maximal" domains on which TTC$_\succ$ mechanisms are GSP.

4.1 Pareto efficiency and core-selecting

When we relax the assumption of strict preferences and allow for general indifferences, $\text{TTC}_\succ$ loses two of its most appealing properties: Pareto efficiency and core-selecting. However, in the intermediate case of objective indifferences, $\text{TTC}_\succ$ retains these two properties. Moreover, on *any* larger domain, $\text{TTC}_\succ$ loses both Pareto efficiency and core-selecting. Thus, we show that it is not indifferences per se, but rather *subjective* evaluations of indifferences which cause $\text{TTC}_\succ$ to lose these properties.

We first demonstrate that $\text{TTC}_\succ$ mechanisms are not Pareto efficient under general indifferences. Example 3 gives the simplest case.

Example 3. Let $N = \{1, 2\}$. The preference profile $R = (R_1, R_2)$, tie-breaking profile $\succ = (\succ_1, \succ_2)$, and tie-broken preference profile $R_\succ = (R_{1,\succ_1}, R_{2,\succ_2})$ are shown below.

$$\underbrace{\begin{array}{cc} R_1 & R_2 \\ \hline w_1, w_2 & w_1 \\ & w_2 \end{array}}_{\text{Preference profile } R} \qquad \underbrace{\begin{array}{c} \succ_1 = \succ_2 \\ \hline 1 \\ 2 \end{array}}_{\text{Tie-breaking profile } \succ} \qquad \underbrace{\begin{array}{cc} R_{1,\succ_1} & R_{2,\succ_2} \\ \hline w_1 & w_1 \\ w_2 & w_2 \end{array}}_{\text{Tie-broken preference profile } R_\succ}$$

The $\text{TTC}_\succ$ allocation is $x = (w_1, w_2)$, which is Pareto dominated by $x' = (w_2, w_1)$ since

$$(x'_1 =)\, w_2 I_1 w_1 \,(= x_1) \quad \text{and} \quad (x'_2 =)\, w_1 P_2 w_2 \,(= x_2).$$

This example demonstrates the underlying reason that $\text{TTC}_\succ$ fails PE under general indifferences: tie-breaking rules may not take advantage of Pareto gains made possible by the agents' indifferences. However, under objective indifferences, if any agent is indifferent between two houses, then all agents are indifferent between those two houses. Consequently, objective indifferences rules out situations like the one shown in Example 3.

Under general indifferences, the set of core allocations may not be a singleton; there may be no core allocations or there may be multiple. As Example 3 demonstrates, even when the core of the market is non-empty, $\text{TTC}_\succ$ may still fail to select a core allocation.[1] However, under objective indifferences, if the core of a market is non-empty then $\text{TTC}_\succ$ mechanisms always select a core allocation.

In fact, the objective indifferences setting characterizes the entire set of maximal domains on which $\text{TTC}_\succ$ mechanisms are Pareto efficient or core-selecting. That is, if all $\text{TTC}_\succ$ mechanisms are PE or CS on a domain $\mathcal{R}^n$, then it must be a weak subset of some objective indifferences domain. Conversely, for any superset of an objective indifferences domain, there is some $\text{TTC}_\succ$ mechanism that is not PE or CS.

Theorem 1. *The following are equivalent:*

[1] It is straightforward to see that $x' = (w_2, w_1)$ is in the core of the market and $x = (w_1, w_2)$ is not.

1. $\mathcal{R}^n$ is an objective indifferences domain.

2. $\mathcal{R}^n$ is a maximal domain on which $TTC_{\succ}$ mechanisms are Pareto efficient.

3. $\mathcal{R}^n$ is a maximal domain on which $TTC_{\succ}$ mechanisms are core-selecting.

Proof. Appendix A.1. □

The full proof is in the appendix, but the intuition is simple. The objective indifferences domain precludes possibilities such as Example 3, and any larger domain inevitably introduces the possibility of such a pair.

It follows from Sönmez (1999) that under objective indifferences, the core of a market is *essentially single-valued* when it exists. That is, for any two allocations x and y in the core of a market, we have $x_i I_i y_i$ for all agents i. In our proof of Theorem 1, we also prove this claim directly. Since the core is essentially single-valued, under objective indifferences the core can be thought of as a unique mapping from agents to house types. In other words, the core allocations are permutations of one another where agents may be assigned to different houses, but always receive houses of the same type.

Corollary 1. *For any two allocations $x \neq y$ in the core of an objective indifferences market, $x_i I_i y_i$ for all $i \in N$.*

Proof. Appendix A.1. □

Though all TTC$_{\succ}$ mechanisms are core-selecting under objective indifferences, the core of the market may be empty, as the following simple example shows.

Example 4. Let $N = \{1, 2, 3\}$. It is easy to verify that for the preference profile $R = (R_1, R_2, R_3)$ shown below, there are no core allocations.

R_1	R_2	R_3
w_2, w_3	w_1	w_1
w_1	w_2, w_3	w_2, w_3

Any allocation x such that $x_1 \in \{w_1, w_2\}$ is blocked by $Q = \{1, 3\}$ and $x' = (w_3, w_1)$. Similarly, any allocation x such that $x_1 = w_3$ is blocked by $Q = \{1, 2\}$ and $x' = (w_2, w_1)$.

When the core of an objective indifferences market is empty, all TTC$_{\succ}$ mechanisms select an allocation in the **weak core** of the market. In fact, even under general indifferences, the weak core is non-empty and TTC$_{\succ}$ mechanisms select an allocation in the weak core.

Definition 4. An allocation x is **weakly blocked** if there exists a coalition $Q \subseteq N$ and allocation x' such that $x'_Q = w_Q$ and $x'_i P_i x_i$ for all $i \in Q$.

Definition 5. An allocation x is in the **weak core** of the market if it is not weakly blocked.

Proposition 1. *For any market, the weak core is non-empty and* $TTC_\succ$ *mechanisms select an allocation in the weak core.*

Proof. Appendix A.1. ∎

4.2 Group strategy-proofness

$TTC_\succ$ also loses group strategy-proofness once we move from strict preferences to weak preferences. However, in the intermediate case of objective indifferences, $TTC_\succ$ recovers group strategy-proofness. Further, $TTC_\succ$ mechanisms are not GSP in any larger "symmetric" domain. We say that a domain is symmetric if, when $h_1 P_i h_2$ is possible, then so is $h_2 P_i h_1$. We will informally argue that this is not an onerous modeling restriction.

First we present a simple example demonstrating that under general indifferences, $TTC_\succ$ mechanisms are not group strategy-proof. Example 5 shows how an agent can break his own indifference to benefit a coalition member without harming himself.

Example 5. Let $N = \{1, 2, 3\}$ and let $Q = \{1, 3\}$. For the preference profile $R = (R_1, R_2, R_3)$ and tie-breaking profile $\succ = (\succ_1, \succ_2, \succ_3)$ shown below, the $TTC_\succ$ allocation is $x = (w_2, w_1, w_3)$. However, if agent 1 were to report R'_1, then for $R' = (R'_1, R_2, R_3)$ the $TTC_\succ$ allocation is $x' = (w_3, w_2, w_1)$. Note that $x'_3 P_3 x_3$ and $x'_1 I_1 x_1$, so $TTC_\succ$ is not GSP.

R_1	R_2	R_3
w_2, w_3	w_1	w_1
w_1	w_2	w_2
	w_3	w_3

Preference profile R

R'_1	R_2	R_3
w_3	w_1	w_1
w_2	w_2	w_2
w_1	w_3	w_3

Preference profile R'

$\succ_1 = \succ_2 = \succ_3$
1
2
3

Tie-breaking profile $\succ$

Objective indifferences excludes situations like Example 5 in two ways. First, it eliminates the possibility that one agent is indifferent between two houses while another agent has a strict preference. Second, it constrains the possible set of misreports available to a manipulating coalition, since agents can *only* report indifference among all houses of the same type.[2] Our next result characterizes the set of symmetric-maximal domains on which $TTC_\succ$ mechanisms are GSP.

Before presenting our result, we must define "symmetric" and "symmetric-maximal" domains.

Definition 6. A domain $\mathcal{R}^n$ is **symmetric** if for any $h_1, h_2 \in H$, if there exists $R_i \in \mathcal{R}$ such that $h_1 P_i h_2$, then there also exists $R'_i \in \mathcal{R}$ such that $h_2 P'_i h_1$.

Definition 7. A domain $\mathcal{R}^n$ is **symmetric-maximal** for an axiom A and a class of rules F if

1. $\mathcal{R}^n$ is symmetric,

[2] The constraint on agents' reports is an important difference from Ehlers (2002).

2. each $f \in F$ is A on $\mathcal{R}^n$, and

3. for any symmetric $\tilde{\mathcal{R}}^n \supset \mathcal{R}^n$, there is some $f \in F$ that is *not* A on $\tilde{\mathcal{R}}^n$.

In practical applications, symmetry is a natural restriction to place on the domain; if it is possible that agents might report strictly preferring some house h to another house h', we should not preclude the possibility they strictly prefer h' to h. Indeed, the point of mechanism design is that preferences are unknown and must be solicited. It is easy to see that objective indifferences domains are symmetric. Relative to maximality, symmetric-maximality restricts the possible expansions of objective indifferences domains that we must consider.

Theorem 2. *$\mathcal{R}^n$ is a symmetric-maximal domain on which $TTC_{\succ}$ mechanisms are group strategy-proof if and only if it is an objective indifferences domain.*

Proof. Appendix A.2. □

Our proof uses similar reasoning to the proof that TTC is group strategy-proof under strict preferences contained in Sandholtz and Tai (2024). Any coalition requires a "first mover" to misreport, but this agent must receive an inferior house to the one he originally received. In the following example, we note that objective indifferences domains are *not* maximal domains on which $\text{TTC}_{\succ}$ mechanisms are GSP.

Example 6. Let $N = \{1, 2\}$, $H = \{h_1, h_2\}$, and $\mathcal{H} = \{\{h_1, h_2\}\}$. Suppose $\mathcal{R}' = \mathcal{R}(\mathcal{H}) \cup (h_1 P h_2) = \{(h_1 I h_2), (h_1 P h_2)\}$. That is, expand the objective indifferences domain induced by $\mathcal{H}$ to include the ordering $(h_1 P h_2)$. Note that this expanded domain is not symmetric, since $\mathcal{R}'$ does not also contain the preference ordering $(h_2 P h_1)$.

Let $\succ = ((1 \succ_1 2), (1 \succ_2 2))$. We will show that for any market (N, H, w, R), $\text{TTC}_{\succ}$ is group strategy-proof. It is straightforward to show that the same is true for the remaining 3 possible tie-breaking profiles.

Without loss of generality, assume $w_i = h_i$. If both agents have the same preferences, then there is clearly no profitable group manipulation. Consider the following two possible preference profiles:

R_1	R_2
h_1	h_1, h_2
h_2	

or

R_1	R_2
h_1, h_2	h_1
	h_2

In the first case, the $\text{TTC}_{\succ}$ allocation is $x = (h_1, h_2)$, so both agents receive one of their most preferred houses. Therefore, it is not possible for either agent to strictly improve. In the second case, the $\text{TTC}_{\succ}$ allocation is $x = (h_1, h_2)$. It would benefit agent 2 for agent 1 to rank h_2 above h_1, since agent 1 is indifferent between h_1 and h_2. However, this is not possible since $(h_2 P h_1) \notin \mathcal{R}'$.

5 Conclusion

Our main set of results show that objective indifferences domains are maximal domains on which $\text{TTC}_{\succ}$ mechanisms are Pareto efficient, core selecting, and group strategy-proof. It is remarkable that the maximal domains on which $\text{TTC}_{\succ}$ satisfies these three distinct properties (essentially) coincide. We therefore view objective indifferences domains as the most general possible setting where $\text{TTC}_{\succ}$ can be applied without any tradeoffs. Moreover, we interpret our results as showing that it *subjective* indifferences, not indifferences themselves, which cause issues for TTC when we relax the assumption of strict preferences.

Therefore, in markets where one could reasonably assume that any indifferences are shared among all agents, $\text{TTC}_{\succ}$ is a sensible choice of mechanism. Even when the market imposes constraints on the possible tie-breaking rules, it is guaranteed that $\text{TTC}_{\succ}$ will be PE, CS, and GSP regardless of which tie-breaking rule is chosen. Moreover, $\text{TTC}_{\succ}$ is computationally efficient, as well as easy to explain and implement.

We do not believe our results imply that $\text{TTC}_{\succ}$ should be avoided in settings beyond objective indifferences, or that market designers should only allow objective indifference preference reports. Consider school choice in San Francisco, which uses a lottery system to assign school seats at most public schools. While current details are not readily available, Abdulkadiroglu, Featherstone, Niederle, Pathak, and Roth designed a system using TTC.[3] At some schools, there are seats dedicated to language immersion programs and other seats that are intended for general education. For instance, at West Portal Elementary School, there are roughly 120 seats, approximately 25% of which are for Cantonese immersion.[4] The Cantonese immersion seats at West Portal are further divided into seats reserved for children who are already bilingual and students who do not yet speak Cantonese. Suppose families' preferences over schools can be described by objective indifferences. That is, suppose families care only about which school they attend, and are indifferent about what kind of seat they receive. Our results suggest that $\text{TTC}_{\succ}$ is an excellent candidate mechanism for this setting.

However, the real situation may be more complicated. Perhaps some families are indifferent between bilingual and regular seats, while other families have strict preferences for one type of seat or the other. For example, a family whose children are already bilingual in Cantonese may be indifferent between the two types of seats at West Portal, while another family may have a strict preference for cultural community through the Cantonese bilingual program. In this case, our results show that $\text{TTC}_{\succ}$ mechanisms are no longer PE, CS, nor GSP. However, the policy implications are not clear. While there are mechanisms for the general indifferences case, these mechanisms may have other tradeoffs, such as increased computational or cognitive complexity. Our results lay out exactly the situations where $\text{TTC}_{\succ}$ is Pareto efficient, core-selecting, and group strategy-proof. However, the results do not necessarily proscribe its use outside of these settings. Rather, one could view this set of results as rationalizing the use of $\text{TTC}_{\succ}$ in many settings where $\text{TTC}_{\succ}$

[3] See the blog post by Al Roth: https://marketdesigner.blogspot.com/2010/09/san-francisco-school-choice-goes-in.html. As he notes, the team were not privy to the implementation or resulting data.

[4] https://web.archive.org/web/20250422170224/https://www.sfchronicle.com/bayarea/article/sfusd-competitive-public-schools-20252957.php

has actually been suggested and applied.

Our paper opens interesting new lines of inquiry. First, we believe that studying matching markets with constrained indifferences is an exciting avenue for future research. In many real-world matching markets, agents have indifferences, but often with a certain structure imposed by the specific market. Understanding how adding structure to the case of general indifferences may affect matching problems is not only theoretically interesting, but could improve policy choices. For instance, there may be tradeoffs in the selection of the partition $\mathcal{H}$ given the set of objects H. In some cases, there may be some ambiguity: are two dorms with the same floor plan but on different floors of the same building equivalent? Inappropriately combining indifference classes might lead to efficiency losses in the spirit of Example 3. On the other hand, splitting indifference classes might allow group manipulations like in Example 5. We leave formal results as future work. We also leave an axiomatic characterization of $\text{TTC}_{\succ}$ on objective indifferences domains as future work.

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Appendix A Proofs

We provide proofs for the results in the main text. Given a market and $\text{TTC}_{\succ}(R)$, denote $S_k(R)$ as the kth cycle executed in $\text{TTC}_{\succ}(R)$.[5] Note that individual rationality (IR) of $\text{TTC}_{\succ}$ follows immediately from IR of TTC and the fact that $TTC_{\succ}(R) \equiv TTC(R_{\succ})$.

We will appeal to the following fact: let $x = \text{TTC}_{\succ}(R)$; if $i \in S_{\ell}(R)$ and $hP_i x_i$, then h must have been assigned at some step before step ℓ. This follows from the definitions – $hP_i x_i$ implies $hP_{i,\succ} x_i$, and $\text{TTC}_{\succ}(R) \equiv \text{TTC}(R_{\succ})$. Under $\text{TTC}(R_{\succ})$, an object h such that $hP_{i,\succ} x_i$ must have been assigned prior to step ℓ, otherwise i would have pointed to h's owner instead of at x_i's owner.

We also make use of the following lemma.

Lemma 1. *Fix a market (N, H, w) and a domain $\mathcal{R}^n$. For any two preference relations $R_*, R_{**} \in \mathcal{R}$ and houses $h_1, h_2 \in H$, let*

$$\{i \in N : w_i P_* h_1\} \subseteq A \subseteq \{i \in N : w_i R_* h_1\}$$

and let

$$B = \{i \in A^c : w_i R_{**} h_2\}.$$

Let $\succ$ be any tie-breaking profile such that $i \succ_i j$ for all $i \neq j$. Fix a preference profile $R \in \mathcal{R}^n$ and let $x = TTC_{\succ}(R)$. If $R_i = R_*$ for all $i \in A$ and $R_i = R_{**}$ for all $i \in B$, then $x_i = w_i$ for all $i \in A \cup B$.

Proof of Lemma 1. First we show that $x_i = w_i$ for all $i \in A$. Toward a contradiction, suppose that $W = \{i \in A : x_i \neq w_i\}$ is non-empty. Take any $i \in W$ such that $w_i R_* w_j$ for all $j \in W$. By individual rationality of x, $x_i R_i w_i$. Also, since $i \succ_i j$ for all $j \neq i$, we know that if $x_i I_i w_i$ then $x_i = w_i$. Therefore, it must be that $x_i P_i w_i$. Since $R_i = R_*$, $x_i P_* w_i$. Suppose $x_i = w_j$. Note that $j \in A$, since $i \in A$ and $w_j P_* w_i$. Also, $x_j \neq w_j$, so $j \in W$. But this contradicts our assumption that $w_i R_* w_j$ for all $j \in W$.

Next we show that $x_i = w_i$ for all $i \in B$. Toward a contradiction, suppose that $W = \{i \in B : x_i \neq w_i\}$ is non-empty. Take any $i \in W$ such that $w_i R_{**} w_j$ for all $j \in W$. By individual rationality of x, $x_i R_i w_i$. Also, since $i \succ_i j$ for all $j \neq i$, we know that if $x_i I_i w_i$ then $x_i = w_i$. Therefore, it must be that $x_i P_i w_i$. Since $R_i = R_{**}$, $x_i P_{**} w_i$. Suppose $x_i = w_j$. We know that $j \notin A$, because $x_j \neq w_j$ and we showed that $x_i = w_i$ for all $i \in A$. Therefore, $j \in B$, since $i \in B$ and $w_j P_{**} w_i$. Also, $j \in W$. But this contradicts our assumption that $w_i R_{**} w_j$ for all $j \in W$. $\square$

Appendix A.1 Pareto efficiency and core-selecting

Theorem 1. *The following are equivalent:*

1. *$\mathcal{R}^n$ is an objective indifferences domain.*

[5] Note that S_k may not be unique, since multiple cycles may appear in step 2 of Algorithm 1.

2. $\mathcal{R}^n$ *is a maximal domain on which* $TTC_\succ$ *mechanisms are Pareto efficient.*

3. $\mathcal{R}^n$ *is a maximal domain on which* $TTC_\succ$ *mechanisms are core-selecting.*

The result is trivial for $|N| = 1$, so assume $|N| \geq 2$. First we show that statements (1) and (2) are equivalent.

Proof of (1) $\iff$ (2). First we show that for any objective indifferences domain, $\text{TTC}_\succ$ mechanisms are PE. Consider any (N, H, w) and fix some tie-breaking rule $\succ$. Let $\mathcal{H}$ be any partition of H and let $R \in \mathcal{R}(\mathcal{H})^n$. If $\mathcal{H} = \{H\}$, the result is trivial, so suppose the partition has at least two blocks. Let $x = TTC_\succ(R)$, and suppose that some feasible allocation y Pareto dominates x. Let $W = \{i : y_i P_i x_i\}$ be the set of agents who strictly improve under y, which must be non-empty. Let $i \in W$ be the first agent in W assigned during the process of $TTC_\succ(R)$. If $i \in S_k(R)$ and $y_i P_i x_i$, then i) $\eta(y_i) \neq \eta(x_i)$, and ii) y_i was assigned prior to step k. Therefore, there must be an agent j in $\cup_{\ell=1}^{k-1} S_\ell(R)$ for whom $x_j \in \eta(y_i)$ but $y_j \notin \eta(y_i)$. Since y Pareto dominates x, this implies $y_j P_j x_j$. But then $j \in W$, a contradiction.

Next we show that for any domain $\tilde{\mathcal{R}}^n$ where $\tilde{\mathcal{R}}^n \nsubseteq \mathcal{R}(\mathcal{H})^n$ for any $\mathcal{H}$, $\text{TTC}_\succ$ mechanisms are not PE on $\tilde{\mathcal{R}}^n$. Fix (N, H). Without loss of generality, assume $w_i = h_i$. If $\tilde{\mathcal{R}}^n \nsubseteq \mathcal{R}(\mathcal{H})^n$ for any $\mathcal{H}$, then $\tilde{\mathcal{R}}$ must contain two orderings R_*, R_{**}, such that for some $h_1, h_2 \in H$, we have $h_1 I_* h_2$ but $h_1 P_{**} h_2$.

Taking only the existence of $R_*, R_{**} \in \tilde{\mathcal{R}}$ for granted, we find a preference profile $R \in \tilde{\mathcal{R}}^n$ and tie-breaking profile $\succ$ such that $TTC_\succ(R)$ is not PE. Define $A = \{i \in N : w_i R_* w_1\} \setminus \{2\}$; note that $1 \in A$ and $2 \in A^c$. Define the preference profile R such that

$$R_i = \begin{cases} R_* & \text{if } i \in A \\ R_{**} & \text{if } i \in A^c. \end{cases}$$

Take any tie-breaking profile $\succ$ such that $i \succ_i j$ for all $j \neq i$. Let $x = \text{TTC}_\succ(R)$. It follows directly from Lemma 1 that $x = w$. However, note that $w_2 I_1 w_1 (= x_1)$ and $w_1 P_2 w_2 (= x_2)$, so x is Pareto dominated by $y = (w_2, w_1, w_3, ..., w_n)$. □

We now turn to the second part of the proof.

Proof of (1) $\iff$ (3). First we show that for any objective indifferences domain, $\text{TTC}_\succ$ mechanisms are CS. Consider any (N, H, w) and fix some tie-breaking rule $\succ$. Let $\mathcal{H}$ be any partition of H and let $R \in \mathcal{R}(\mathcal{H})$. If $\mathcal{H} = \{H\}$, the result is trivial, so suppose the partition has at least two blocks.

Suppose that the core of (N, H, w, R) is non-empty and contains some allocation y. Denote $x = TTC_\succ(R)$. We will show that $x_i I_i y_i$ $(\star)$ for all i by induction on the steps of $TTC_\succ(R)$.

$\boxed{\text{Step 1}}$ All $i \in S_1(R)$ received one of their top-ranked objects, so $x_i R_i y_i$. Suppose $(\star)$ is not true for $S_1(R)$. Then there is some $i \in S_1(R)$ such that $x_i P_i y_i$. But then $S_1(R)$ and x block y, a contradiction.

$\boxed{\text{Step k}}$ Suppose that ($\star$) is true for all steps before k. Suppose for some $i \in S_k(R)$ we have $y_i P_i x_i$. Then y_i was assigned before step k. Further, $\eta(y_i) \neq \eta(x_i)$. (Otherwise, it could not be that $y_i P_i x_i$.) So if y_i is assigned to agent i under y, there must be an agent j in $\cup_{\ell=1}^{k-1} S_\ell(R)$ for whom $x_j \in \eta(y_i)$ but $y_j \notin \eta(y_i)$. But then it cannot be that $y_j I_j x_j$, a contradiction.[6] Thus we have that $x_i R_i y_i$ for all $i \in S_k(R)$. Suppose ($\star$) is not true for $S_k(R)$. Then there is some $i \in S_k(R)$ such that $x_i P_i y_i$. But then $S_k(R)$ and x block y, a contradiction.

Thus $x_i I_i y_i$ for all i, so x must also be in the core. (Since y was an arbitrary allocation in the core, this also proves Corollary 1.)

Next we show that for any domain $\tilde{\mathcal{R}}^n$ where $\tilde{\mathcal{R}}^n \nsubseteq \mathcal{R}(\mathcal{H})^n$ for any $\mathcal{H}$, $\text{TTC}_\succ$ mechanisms are not CS on $\tilde{\mathcal{R}}^n$. Fix (N, H). If $\tilde{\mathcal{R}}^n \nsubseteq \mathcal{R}(\mathcal{H})^n$ for any $\mathcal{H}$, then $\tilde{\mathcal{R}}$ must contain two orderings R_*, R_{**}, such that for some $h_1, h_2 \in H$, we have $h_1 I_* h_2$ but $h_1 P_{**} h_2$. Without loss of generality, assume that $h_1 R_{**} h$ for all h such that $h I_* h_2$. Also, without loss of generality, assume $w_i = h_i$ for all $i \in N$.

Once again, let $A = \{i \in N : w_i R_* w_1\} \setminus \{2\}$ and consider the preference profile $R \in \tilde{\mathcal{R}}^n$ where $R_i = R_*$ for all $i \in A$ and $R_i = R_{**}$ for all $i \in A^c$. Let $x = TTC_\succ(R)$. It follows directly from Lemma 1 that $x = w$. However, as we noted earlier, x is Pareto dominated by $y = (w_2, w_1, w_3, ..., w_n)$, and is therefore blocked by N and y. It remains to show that y is in the core of the market.

Toward a contradiction, suppose there is a coalition Q and allocation z that blocks y. Let $W = \{i \in Q : z_i P_i y_i\}$, which must be non-empty. Note that since x is individually rational and y Pareto dominates x, for each $i \in W$ we have $z_i P_i w_i$. More specifically, for any $i \in W \cap A$ we have $z_i P_* w_i$ and for any $i \in W \cap A^c$ we have $z_i P_{**} w_i$.

First we show that $W_A := W \cap A = \emptyset$. Toward a contradiction, suppose W_A is non-empty and take any $i \in W_A$ such that $w_i R_* w_j$ for all $j \in W_A$. Since $z_i P_* w_i$ and $z_Q = w_Q$, there must exist some agent $j \in Q$ for whom $w_j I_* z_i$ but $\neg(w_j I_* z_j)$. Note that $j \in A \setminus \{1\}$, since $w_j P_* w_i$ and $i \in A$. Thus, $y_j = w_j$. Then $\neg(w_j I_* z_j)$ and $j \in Q$ imply that $z_j P_* y_j$; that is, $j \in W_A$. But since $w_j P_* w_i$, this contradicts our assumption that $w_i R_* w_j$ for all $j \in W_A$. So $W_A = \emptyset$.

Next we show that $Q \cap A \neq \emptyset$. Toward a contradiction, suppose that $Q \subseteq A^c$. Take any $i \in W$ such that $w_i R_{**} w_j$ for all $j \in W$. Since $z_i P_{**} w_i$ and $z_Q = w_Q$, there must exist some agent $j \in Q$ such that $w_j I_{**} z_i$ but $\neg(w_j I_{**} z_j)$. Since $R_j = R_{**}$ and $z_j R_{**} w_j$, this means that $z_j P_{**} w_j$. But then $j \in W$, contradicting $w_i R_{**} w_j$ for all $j \in W$.

Without loss of generality, assume that the agents in Q form a single trading cycle under z. Since Q forms a single trading cycle and contains agents in A and A^c, there must exist an agent $\bar{b} \in A^c$ such that $z_{\bar{b}} \in w(A)$ and an agent $\bar{a} \in A$ such that $z_{\bar{a}} \in w(A^c)$. In fact, $\bar{a}$ is the only agent in A who receives a house from an agent in A^c. Recall that for every $i \in A^c \setminus \{2\}$, $w_1 P_* w_i$. Therefore, if there exists some agent $a \neq \bar{a}$ in A such that $z_a \in w(A^c)$, then either $w_1 P_* z_{\bar{a}}$ or $w_1 P_* z_a$. But since $a, \bar{a} \in A$, $w_a R_* w_1$ and $w_{\bar{a}} R_* w_1$, contradicting individual rationality of z. It follows that $z_{\bar{a}} = w_2$. Moreover, $\bar{b}$ must be the only agent in B

[6] This is where objective indifferences is used – this claim fails in general indifferences.

to receive a house from an agent in A. Therefore, we can write Q as

$$a_1 \to ... \to \bar{a} \to 2 \to b_1 \to ... \to \bar{b} \to a_1$$

where all $a_1, ..., \bar{a} \in A$ and $2, b_1, ..., \bar{b} \in A^c$.

Since $W \subseteq A^c$ and by individual rationality of z, we know that $z_a I_* y_a I_* w_a$ for $a = a_1, ..., \bar{a}$. In particular, since $z_{\bar{a}} = w_2$, we know that $w_{\bar{a}} I_* w_2$. Applying the same reasoning for all other agents in $Q \cap A$, we have that $w_{a_1} I_* w_2$. Also, we know that $z_b R_{**} y_b$ for $b = 2, b_1, ..., \bar{b}$, with strict preference for at least one b since $W \neq \emptyset$. Recall that $y_2 = w_1$ and $y_b = w_b$ for $b = b_1, ..., \bar{b}$. Therefore, since each $b = 2, b_1, ..., \bar{b}$ receives the house of the agent they are pointing at, we have that $w_{a_1} P_{**} w_1$. However, this contradicts our assumption that $1 R_{**} h$ for all $h I_* w_2$. □

We show a proof that $\text{TTC}_{\succ}$ selects the weak core.

Proposition 1. *For any market (N, H, w, R), the weak core is non-empty and $TTC_{\succ}$ selects an allocation in the weak core.*

Proof. Let $x = \text{TTC}_{\succ}(R)$. Toward a contradiction, suppose there exists a coalition $Q \subseteq N$ and allocation y that strongly blocks x. That is, $y_Q = w_Q$ and $y_i P_i x_i$ for all $i \in Q$. Among the agents in Q, let i be the agent assigned at the earliest step of $TTC_{\succ}(R)$. Say that $i \in S_k(R)$. Since $y_Q = w_Q$, we know that $y_i = w_j$ for some $j \in Q$. But $y_i P_i x_i$ implies $j \in S_\ell(R)$ for some $\ell < k$, contradicting that i was the first agent in Q to be assigned during $TTC_{\succ}(R)$. □

Appendix A.2 Group strategy-proofness

We first review an important property of $\text{TTC}_{\succ}$ and state a useful lemma. Let $L(h, R_i) = \{h' \in H : h R_i h'\}$ be the lower contour set of a preference ranking R_i at house h.

Monotonicity (MON). A rule f is **monotone** if, for any R and R' such that $L(f_i(R), R_i) \subseteq L(f_i(R), R'_i)$ for all i, then $f(R) = f(R')$.

That is, a rule f is monotone if, whenever any set of agents move up their allocations in their rankings, the allocation remains the same. It is straightforward to show that TTC is monotone for strict preferences; e.g. Takamiya (2001). Then, since $TTC_{\succ}(R) \equiv TTC(R_{\succ})$ for any R and $\succ$, it follows directly that $\text{TTC}_{\succ}$ is monotone.

The following result is adapted from Sandholtz and Tai (2024), who show it for TTC with strict preferences.

Lemma 2 (Sandholtz and Tai, 2024)**.** *For any R, R', let $x = TTC_{\succ}(R)$ and $y = TTC_{\succ}(R')$. Suppose there is some i such that $y_i P_{i,\succ} x_i$. Then there exists some agent j and house h such that $h P'_{j,\succ} x_j$ and $x_j P_{j,\succ} h$.*

Theorem 2. *$\mathcal{R}^n$ is a symmetric-maximal domain on which $TTC_\succ$ mechanisms are group strategy-proof if and only if it is an objective indifferences domain.*

Proof. First we show that for any objective indifferences domain, $\mathrm{TTC}_\succ$ mechanisms are GSP. Consider any (N, H, w) and fix some tie-breaking rule $\succ$. Let $\mathcal{H}$ be any partition of H and let $R \in \mathcal{R}(\mathcal{H})$. If $|N| = 1$ or $\mathcal{H} = \{H\}$, the result is trivial, so suppose that $|N| \geq 2$ and that the partition has at least two blocks. Without loss of generality, assume $w_i = h_i$ for all $i \in N$.

Suppose $Q \subseteq N$ reports R'_Q where $R' = (R'_Q, R_{-Q}) \in \mathcal{R}(\mathcal{H})^n$. Let $y = \mathrm{TTC}_\succ(R')$. We will show that if $y_i P_i x_i$ for some $i \in Q$, then $x_j P_j y_j$ for some $j \in Q$.

Let R'' be the preference profile in $\mathcal{R}(\mathcal{H})^n$ such that each R''_i top-ranks $\eta(y_i)$ and otherwise preserves the ordering of R_i. Let $z = TTC_\succ(R'')$. By monotonicity of $\mathrm{TTC}_\succ$, $z = y$. Therefore $z_i P_i x_i$, and consequently, $z_i P_{i,\succ_i} x_i$. Applying Lemma 2, there must be some $j \in Q$ and $h \in H$ such that $x_j P_{j,\succ_j} h$ but $h P''_{j,\succ_j} x_j$. Note that $h \notin \eta(x_j)$; if it were, then for any $R, R'' \in \mathcal{R}(\mathcal{H})^n$, $x_j P_{j,\succ_j} h$ if and only if $x_j P''_{j,\succ_j} h$. Therefore, $x_j P_j h$ and $h P''_j x_j$.[7] The only change from R_j to R''_j is to top-rank $\eta(y_j)$, so it must be that $h \in \eta(y_j)$. But then $x_j P_j y_j$, as desired.

Next we show that for any symmetric domain $\tilde{\mathcal{R}}^n$ where $\tilde{\mathcal{R}}^n \not\subseteq \mathcal{R}(\mathcal{H})^n$ for any $\mathcal{H}$, $\mathrm{TTC}_\succ$ is not GSP on $\tilde{\mathcal{R}}^n$. Fix (N, H). Without loss of generality, let $w_i = h_i$ for all $i \in N$. If $\tilde{\mathcal{R}}^n \not\subseteq \mathcal{R}(\mathcal{H})^n$, then $\tilde{\mathcal{R}}$ must contain two orderings R_*, R_{**} such that for some $h_1, h_2 \in H$ we have $h_1 I_* h_2$ but $h_1 P_{**} h_2$. The symmetric requirement also necessitates that $\tilde{\mathcal{R}}$ contains some R_{***} such that $h_2 P_{***} h_1$. Taking only the existence of $R_*, R_{**}, R_{***} \in \tilde{\mathcal{R}}$ for granted, we find a preference profile $R \in \tilde{\mathcal{R}}^n$ and tie-breaking profile $\succ$ such that $TTC_\succ(R)$ is not GSP.

Let $A = \{i \in N : w_i R_* w_1\} \setminus \{2\}$, let $B = \{i \in A^c : w_i R_{**} w_1\} \cup \{2\}$, and let $C = N \setminus (A \cup B)$. Note that $1 \in A$ and $2 \in B$. Consider the preference profile $R \in \tilde{\mathcal{R}}^n$ where

$$R_i = \begin{cases} R_* & \text{if } i \in A \\ R_{**} & \text{if } i \in B \\ R_{***} & \text{if } i \in C. \end{cases}$$

Let $\succ$ be any tie-breaking profile such that $i \succ_i j$ for all $j \neq i$. Also, let $2 \succ_1 j$ for all $j \neq 1, 2$. Let $x = TTC_\succ(R)$.

Claim 1. *$x_1 = w_1$ and $w_1 P_2 x_2$.*

Proof of Claim 1. It follows directly from Lemma 1 that $x_i = w_i$ for all $i \in (A \cup B) \setminus \{2\}$. Specifically, $x_1 = w_1$. Let $x_2 = w_j$. Note that $j \in C \cup \{2\}$, so $w_1 P_{**} w_j$. Thus, $w_1 P_2 x_2$. □

Now suppose that agent 1 misreports $R'_1 = R_{***}$. Let $R' = (R'_1, R_{-1})$ and let $y = TTC_\succ(R')$.

[7] This is where the restriction to objective indifferences is used. Under general indifferences, this is not necessarily true.

Claim 2. $y_1 = w_2$ *and* $y_2 = w_1$.

Proof of Claim 2. It follows directly from Lemma 1 that $y_i = w_i$ for all $i \in (A \cup B) \setminus \{1, 2\}$. Moreover, $y_i = w_i$ for all $i \in C$ such that $w_i P_{***} w_2$. To see this, suppose $W = \{i \in C : w_i P_{***} w_2, y_i \neq w_i\}$ is non-empty. Take any $i \in W$ such that $w_i R_{***} w_j$ for all $j \in W$. By individual rationality of y, $y_i R_i w_i$. Also, since $i \succ_i j$ for all $j \neq i$, we know that if $y_i I_i w_i$ then $y_i = w_i$. Therefore, it must be that $y_i P_i w_i$. Since $R_i = R_{***}$, $y_i P_{***} w_i$. Suppose $y_i = w_j$. We know that $j \notin (A \cup B) \setminus \{1, 2\}$, because $y_j \neq w_j$ and we showed that $y_i = w_i$ for all $i \in (A \cup B) \setminus \{1, 2\}$. Moreover, $j \notin \{1, 2\}$ because $w_j P_{***} w_i P_{***} w_2 P_{***} w_1$. Therefore, $j \in C$. Also, $j \in W$. But this contradicts our assumption that $w_i R_{***} w_j$ for all $j \in W$.

Toward a contradiction, suppose that $y_1 \neq w_2$. It must be that $w_2 R_{***} y_1$, since we have shown that $y_i = w_i$ for all i such that $w_i P_{***} w_2$. Recall that $2 \succ_1 j$ for all $j \neq 1, 2$. Therefore, if $w_2 R_{***} y_1$, then agent 2 must have been assigned at an earlier step of $TTC_{\succ}(R')$ than agent 1 was; otherwise, agent 1 would have pointed at agent 2. Since w_1 still available when agent 2 was assigned, $y_2 R_{**} w_1$. But then $y_2 = w_j$ for some $j \in (A \cup B) \setminus \{2\}$, contradicting $y_i = w_i$ for all i in $(A \cup B) \setminus \{1, 2\}$. □

But then $(w_2 =) y_1 I_1 x_1 (= w_1)$ and $(w_1 =) y_2 P_2 x_2$ for $Q = \{1, 2\}$, so $TTC_{\succ}(R)$ is not GSP. □

Appendix B Relation to school choice with priorities

We briefly note that TTC in the objective indifferences setting is not identical to TTC in the school choice with priorities setting. Intuitively, in objective indifferences the fixed tie-breaking rule determines for i whom to point at; conversely, a school priority determines who points at i. Consider an example with 3 schools and 4 students.

Example 7. Let the set of schools (objects) be $H = \{A, B, C\}$, with C having two slots. Let the students be $N = \{a, b, c_1, c_2\}$, where a is "endowed" with A, and so on.

Let the school priorities be given by

A	B	C
a	b	c_1
b	a	c_2
c_1	c_2	a
c_1	c_1	b

Alternatively, let a fixed tie-breaking rule $\succ$ be given by

$\succ_a$	$\succ_b$	$\succ_{c_1}$	$\succ_{c_2}$
c_1	c_2	c_1	c_1
c_2	c_1	c_2	c_2

Finally, compare two alternatives for student preferences

R_a	R_b	R_{c_1}	R_{c_2}		R'_a	R'_b	R'_{c_1}	R'_{c_2}
C	C	A	A	and	C	C	B	B
A	A	B	B		A	A	A	A
B	B	C	C		B	B	C	C

TTC with school priorities results in Ac_1, Bc_2, Cab and Ac_2, Bc_1, Cab under R and R' respectively. Crucially, c_1 gets the preferred school in either case, since it depends on school C's priority. TTC$_\succ$ results in Ac_1, Bc_2, Cab and Ac_1, Bc_2, Cab under R and R' respectively. Either c_1 or c_2 will get the more preferred school, since it depends on a's or b's tie-breaking rule.