Shapley-Scarf Markets with Objective Indifferences

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Abstract

In many object allocation problems, some of the objects are indistinguishable from each other. For example, in a college dormitory, rooms in the same building with the same floor plan are effectively identical. In such cases, it is reasonable to assume that agents are indifferent between identical objects, and matching mechanisms in these settings should account for the agents' indifferences. Top trading cycles with fixed tie-breaking (TTC) has been suggested to deal with indifferences in object allocation problems. Unfortunately, under general indifferences, TTC is neither Pareto efficient nor group strategy-proof. Furthermore, it may not select an allocation in the core of the market, even when the core is non-empty. However, when indifferences are agreed upon by all agents (which we call "objective indifferences"), TTC maintains Pareto efficiency, group strategy-proofness, and core selection. Further, we characterize objective indifferences as the most general setting where TTC maintains maintains these important properties.

1 Introduction

Important markets including living donor organ transplants, public housing assignments, and school choice can be modeled as Shapley-Scarf markets: each agent is endowed with an indivisible object and has preferences over the set of objects. Monetary transfers are not allowed, and participants have property rights to their own endowments. The goal is to re-allocate these objects among the agents to achieve efficiency and stability. The usual stability notion is the core: an allocation is in the core if no subset of agents would prefer to trade their endowments among themselves. In the original setting of Shapley and Scarf (1974), agents

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have strict preferences over the houses, and Gale's *top trading cycles* (TTC) algorithm finds an allocation in the core. Roth and Postlewaite (1977) further show that the core is non-empty, unique, and Pareto efficient. Roth (1982) shows that TTC is strategy-proof; Moulin (1995), Bird (1984), Sandholtz and Tai (2024), and Pápai (2000) show that it is group strategy-proof. These properties make TTC an attractive algorithm for practical applications.

However, the assumption that preferences are strict is strong. In particular, if any objects are essentially identical, agents should naturally be indifferent between them. Consider the problem of assigning students to college dormitories. It seems reasonable to assume that two units with the same floor plan in the same building are equivalent in the eyes of a student. For example, the undergraduate dorm application process at UC Berkeley applies this same logic. On housing applications, students rank their five most preferred *housing complex* $\times$ *floor plan* pairs. It is implicitly assumed that students have strict preferences over *housing complex* $\times$ *floor plan* pairs, but are indifferent between dorm units of the same type. Beyond college dormitory assignment problems, there are a host of real-world object assignment problems (military occupational specialty assignment, school choice, class assignment, etc.) that can be modeled with a similar structure.

Motivated by these examples, we study a model of Shapley-Scarf markets where there may be indistinguishable copies (which we will call "houses") of the objects (which we will call "types" or "house types"). Our model restricts agents to be indifferent between houses of the same type, but never indifferent between houses of different types. We call these preferences "objective indifferences." Objective indifferences is a minimal model of indifferences, capturing the most basic and plausible form of indifferences.

In the fully general setting where agents' preferences may contain indifferences, *TTC with fixed tie-breaking* has been proposed as an intuitive extension. Ties in the agents' preference rankings are broken by some external rule, then TTC is run on the resulting strict preference rankings. Ehlers (2014) is the first to formalize this procedure for Shapley-Scarf markets, though similar ideas date to earlier work. For example, Abdulkadoroglu and Sönmez (2003) discuss how to break ties between students in the school choice setting. However, TTC with fixed tie-breaking has several drawbacks. First, TTC with fixed tie-breaking is not Pareto efficient nor group strategy-proof. In fact, there is an inherent tension between these two properties: Ehlers (2002) shows that under general indifferences, there does not exist a Pareto efficient and group strategy-proof mechanism for Shapley-Scarf markets. Second, even when the core of a market is non-empty, TTC with fixed tie-breaking may not select a core allocation.

Objective indifferences adds structure to the case of general indifferences by constraining any indifferences to be universal among agents. We show that objective indifferences characterizes maximal domains on which TTC with fixed tie-breaking is Pareto efficient, group strategy-proof, and core selecting. This statement contains three distinct results. First, TTC with fixed tie-breaking recovers these three properties on the restriction of objective indifferences. Second, any more general setting causes TTC to lose Pareto efficiency and core selection. Third, any setting in which TTC maintains either of these properties is a subset of an objective indifferences domain. The analogous results hold for group strategy-proofness among preference domains that include both directions of any strict ranking. The last two results are remarkable – objective indifferences and its subsets are all of the settings on which TTC with fixed tie-breaking is Pareto efficient, core selecting, and group strategy-proof.

While our findings are not necessarily prescriptive nor proscriptive, they may help policymakers decide whether TTC is a good solution for their setting. Consider school choice in San Francisco, which uses a lottery system to assign school seats at most public schools. While current details are not readily available, Abdulkadiroglu, Featherstone, Niederle, Pathak, and Roth designed a system using TTC.[1] At some schools, there are seats dedicated to language immersion programs and other seats that are intended for general education. For instance, at West Portal Elementary School, there are roughly 120 seats, approximately 25% of which are for Cantonese immersion.[2] Families separately rank seats by $school \times language$. If all families' preferences can be described by objective indifferences, then TTC maintains its most important properties.

However, the real situation may be more complicated. Some families may be indifferent between bilingual and regular seats, while other families have strict preferences. For example, a family whose children are already bilingual in Cantonese may be indifferent between the two types of seats at West Portal, while another family may have a strict preference for cultural community through the Cantonese bilingual program.[3] If so, then TTC with fixed tie breaking is no longer Pareto efficient, core-selecting, nor group strategy-proof. Indeed, it appears San Francisco is ready to change its school choice mechanism away from TTC.[4] While there are certainly a variety of reasons San Francisco finds TTC unsatisfactory,[5] our results can partially rationalize events.

To our knowledge, the first use of the term "objective indifferences" was due to Bogomolnaia and Moulin (1999), who (almost tangentially) note that their probabilistic serial mechanism for random object assignment can accommodate this case. Fekete, Skutella, and Woeginger (2003) and Cechlárová and Schlotter (2010) deal with competitive equilibrium in Shapley-Scarf markets with objective indifferences. Objective indifferences is similar to models where objects have capacities, as is sometimes applied in school choice; for example, see Morrill (2015) or Abdulkadoroglu and Sönmez (2003). However, there are important differences which we illustrate in Appendix B. Object capacities give rise to a priority structure rather than an endowment structure, which changes the way TTC operates. More importantly, our emphasis here is on the objective indifferences setting as a domain of preferences, as will be clear in the results.

Others have have studied Shapley-Scarf markets with indifferences. Ehlers (2014) gives a characterization of TTC with fixed tie-breaking under general indifferences. Quint and Wako (2004) are the first to provide an algorithm finding the strict core in the presence of indifferences. Alcalde-Unzu and Molis (2011) and Jaramillo and Manjunath (2012) provide strategy-proof, Pareto efficient, and core selecting families of trading cycle mechanisms. Aziz and de Keijzer (2012) and Plaxton (2013) propose further generalizations. Fundamentally,

[1] See the blog post by Al Roth: https://marketdesigner.blogspot.com/2010/09/san-francisco-school-choice-goes-in.html. As he notes, the team were not privy to the implementation or resulting data.

[2] https://web.archive.org/web/20250422170224/https://www.sfchronicle.com/bayarea/article/sfusd-competitive-public-schools-20252957.php

[3] The relative non-competitiveness of language immersion seats in the SFUSD appears to be a source of anxiety for parents. Seats are further divided into already-bilingual and not already bilingual. While West Portal receives 7.7 requests per open seat, its Cantonese immersion program for already bilingual students receives 7.2 requests per open seat, and 13.3 requests per open seat for students not already bilingual. That is, parents who simply want a seat at West Portal may apply for and receive already-bilingual seats, preventing parents with true desires for these seats from receiving them. Perhaps noticing this, parents have been agitating for expansion of language immersion programs, particularly for Mandarin, although we can only speculate on private motives.

[4] https://web.archive.org/web/20250524170000/https://www.sfusd.edu/schools/enroll/student-assignment-policy/student-assignment-changes

[5] Pathak (2017) writes "The difficulty in explaining TTC compared to DA is also a reason that Recovery School District in New Orleans switched mechanisms after one year with DA."

the challenge for any mechanism in a Shapley-Scarf market with indifferences is determining which trading cycles to execute from among the many potential trading cycles that indifferences may induce. Using a fixed tie-breaking rule is intuitive and easy to implement, but as Ehlers (2014) shows, it comes at the expense of certain desirable properties, including Pareto efficiency and core selection. Though we too study Shapley-Scarf markets with indifferences, we place additional structure on the agents' indifferences and demonstrate how this resolves many of the challenges that indifferences pose.

Our paper makes important contributions to the research in Shapley-Scarf markets along two lines. First, we contribute to the understanding of TTC. While there already exist TTC-like mechanisms that deal with indifferences, TTC is more intuitive and more widely known. Our paper captures exactly when its most important properties are maintained – under objective indifferences. We thus rationalize its use in many settings where the Shapley-Scarf model is actually applicable, such as in housing or school assignment. Second, we contribute to the understanding and awareness of the objective indifferences setting. While many have implicitly incorporated this model (e.g., objects with capacities), few have explicitly considered objective indifferences as a domain of preferences. Indeed, literature in mechanism design often considers only strict preferences or the full domain of indifferences. We show that objective indifferences adds an interesting and important intermediate case. It realistically models many object allocation problems and has the potential to preserve important properties relative to general indifferences.

Section 2 presents the formal notation. Section 3 explains TTC with fixed tie-breaking. Section 4 provides the main results. Section 5 concludes. Proofs are in Appendix A.

2 Model

In this section, we present the model primitives. First we recount the classical Shapley and Scarf (1974) setting. Afterwards we introduce the "objective indifferences" domain.

We first present the classical Shapley-Scarf model. Let $N = \{1, \dots, n\}$ be a finite set of agents, with generic member i. Let $H = \{h_1, \dots, h_n\}$ be a set of houses, with generic member h. Every agent is endowed with one object, given by a bijection $w : N \to H$. An allocation is an assignment of an object to each agent, also a bijection $x : N \to H$. The set of all allocations given (N, H) is $X(N, H)$, or simply X. For any $i \in N$, we use w_i as shorthand for $w(i)$ and x_i for $x(i)$. Similarly, for any $Q \subseteq N$ we use w_Q for $w(Q) = \{w_i : i \in Q\}$ and x_Q for $x(Q) = \{x_i : i \in Q\}$.

Each agent i has some preference ordering over H, denoted R_i. That is, R_i is a transitive, complete, reflexive binary relation. As usual, we let P_i denote the strict relation and I_i denote the indifference relation associated with R_i. A set of allowable preference orderings in a problem is denoted $\mathcal{R}$, which call a **domain**. A preference profile is $R = (R_1, ..., R_n) \in \mathcal{R}^n$. For any $Q \subseteq N$ and preference profile R, we denote $R_Q = (R_q)_{q \in Q}$. In this paper we restrict attention to settings where allowable preference profiles are drawn from some $\mathcal{R}^n$. That is, every agent has the same set of possible preference relations.

If $\mathcal{R}$ is the set of strict preference relations over H, then this is the **strict preferences domain**. To demonstrate the notation, the strict preference domain can be denoted

$$\mathcal{R}_{\text{strict}} = \{R_i : hI_ih' \iff h = h'\}$$

If $\mathcal{R}$ is the set of weak preference relations over H, it is the **general indifferences domain**.

Our main domain is **objective indifferences**. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_K\}$ be a partition of H. An element H_k of a partition is a **block**. Given H and $\mathcal{H}$, let $\eta : H \to \mathcal{H}$ be the mapping from a house to the partition element containing it; that is, $\eta(h) = H_k$ if $h \in H_k$. Given $\mathcal{H}$, we denote the objective indifferences domain as

$$\mathcal{R}(\mathcal{H}) := \{R_i : hI_ih' \iff \eta(h) = \eta(h')\}$$

We sometimes suppress $(\mathcal{H})$ from the notation when context makes it clear. Note that an agent is indifferent between houses in the same block of $\mathcal{H}$ and has strict preferences between houses in different blocks. Since all $i \in N$ draw from the same set of allowable preferences, these indifferences are "objective"; all agents must be indifferent between houses in the same block. Thus, we sometimes refer to the "indifference classes" of the domain with the understanding that every agent shares the same indifference classes. The next example illustrates.

Example 1. Let $H = \{h_1, h_2, h_3\}$ and $\mathcal{H} = \{\{h_1, h_2\}, \{h_3\}\}$. Then the two possible preference orderings in $\mathcal{R}(\mathcal{H})$ are given by

R_α	R_β
h_1, h_2	h_3
h_3	h_1, h_2

That is, $h_1I_\alpha h_2P_\alpha h_3$ and $h_3P_\beta h_1I_\beta h_2$. Given a set of agents $N = \{1, 2, 3\}$, objective indifferences requires $R_i \in \{R_\alpha, R_\beta\}$ for each $i \in N$.

2.1 Mechanisms

This subsection recounts formalities on mechanisms and top trading cycles. Familiar readers may safely skip this subsection.

A **market** is a tuple (N, H, w, R). A **mechanism** is a function $f : \mathcal{R}^n \to X$; given a preference profile, it produces an allocation. When it is unimportant or clear from context, we suppress inputs from the notation. For any $i \in N$, let $f_i(R)$ denote i's allocated house under $f(R)$. Similarly, for any $Q \subseteq N$, let $f_Q(R) = \{f_i(R) : i \in Q\}$.

Fix a mechanism f, a tuple (N, H, w), and a preference domain $\mathcal{R}$. We work with the following axioms.

A mechanism is Pareto efficient if it always selects Pareto efficient allocations.

Pareto efficiency (PE). For all $R \in \mathcal{R}^n$, there is no other allocation $x \in X$ such that $x_iR_if_i(R)$ for all $i \in N$ and $x_iP_if_i(R)$ for at least one $i \in N$.

Group strategy-proofness requires that no coalition of agents can collectively improve their outcomes by submitting false preferences. Note that in the following definition, we require both the true preferences and misreported preferences to come from the preference domain $\mathcal{R}$.[6]

Group strategy-proofness (GSP). For all $R \in \mathcal{R}^n$, there do not exist $Q \subseteq N$ and $R' = (R'_Q, R_{-Q}) \in \mathcal{R}^n$ such that $f_i(R') R_i f_i(R)$ for all $i \in Q$ with $f_i(R') P_i f_i(R)$ for at least one $i \in Q$.

Individual rationality models the constraint of voluntary participation. It requires that agents receive a house they weakly prefer to their endowment.

Individual rationality (IR). For all $R \in \mathcal{R}^n$, $f_i(R) R_i w_i$ for all $i \in N$.

We also define the core of a market: an allocation is in the core if there is no subset of agents who could benefit from trading their endowments among themselves.

Definition 1. An allocation x is **blocked** if there exists a coalition $Q \subseteq N$ and allocation y such that $y_Q = w_Q$ and $y_i R_i x_i$ for all $i \in Q$, with $y_i P_i x_i$ for at least one $i \in Q$. An allocation x is in the **core** of the market if it is not blocked.

The weak core requires that all members of a potential coalition are strictly better off.

Definition 2. An allocation x is **weakly blocked** if there exists a coalition $Q \subseteq N$ and allocation y such that $y_Q = w_Q$ and $y_i P_i x_i$ for all $i \in Q$. An allocation x is in the **weak core** if it is not weakly blocked.

Of course, the weak core also contains the core. The (weak) core property models the restriction imposed by property rights. For example, in housing allocation, existing tenants might have the right to stay in their status quo allocations. Notice that individual rationality excludes blocking coalitions of size 1. The last axiom is core-selecting.

Core-selecting (CS). For all $R \in \mathcal{R}$, if the core of the market is non-empty then $f(R)$ is in the core.

2.2 Maximality

In Section 4, we characterize maximal domains on which TTC with fixed tie-breaking satisfies the axioms. By a "maximal" domain, we mean the following.

Definition 3. A domain $\mathcal{R}$ is **maximal** for Pareto efficiency and a class of mechanisms F if

1. every $f \in F$ is Pareto efficient on $\mathcal{R}$, and
2. for any $\tilde{\mathcal{R}} \supset \mathcal{R}$, there is some $f \in F$ that is *not* Pareto efficient on $\tilde{\mathcal{R}}$.

The definition for any axiom besides Pareto efficiency is analogous. Notice that this definition of maximality depends on both the axiom and the class of mechanisms, which differs from elsewhere in the literature.[7] We define maximality for a class of mechanisms, since our claims deal with TTC for *all* tie-breaking rules. Again note that we only consider preference profiles drawn from $\mathcal{R}^n$, which is common.

[6]This differs from other studies, e.g., Ehlers (2002).

[7]Commonly in the literature, a domain is maximal for Pareto efficiency if there exists *some* mechanism which is Pareto efficient on it, and none on any larger domain.

3 Top trading cycles with fixed tie-breaking

In this paper, we analyze top trading cycles (TTC) with fixed tie-breaking on the domains defined in the previous section. For an extensive history of TTC, we refer the reader to Morrill and Roth (2024). We briefly define TTC and TTC with fixed tie-breaking.

Algorithm 1. ***Top Trading Cycles***. *Consider a market (N, H, w, R) under strict preferences. Draw a graph with N as nodes.*

1. *Draw an arrow from each agent i to the owner (endowee) of his favorite remaining object.*
2. *There must exist at least one cycle; select one of them. For each agent in this cycle, give him the object owned by the agent he is pointing at. Remove these agents from the graph.*
3. *If there are remaining agents, repeat from step 1.*

We denote the resulting allocation as $\mathrm{TTC}(R)$.

TTC is well-defined only with strict preferences, as Step 1 requires a unique favorite object. In practice, a **fixed tie-breaking profile** $\succ$ is often proposed to resolve indifferences. Given N, let $\succ = (\succ_1, \ldots, \succ_n)$, where each $\succ_i$ is a strict linear order over N. This linear order will be used to break indifferences between objects (based on their owners). For any preference relation R_i and tie-breaking rule $\succ_i$, let $R_{i,\succ}$ be given by the following. For any $j \neq j'$, let $w_j R_{i,\succ} w_{j'}$ if

1. $w_j P_i w_{j'}$, or
2. $w_j I_i w_{j'}$ and $j \succ_i j'$

and $w_j R_{i,\succ} w_j$.

Then $R_{i,\succ}$ is a strict linear order over the individual houses. Note that $R_{i,\succ}$ is actually $R_{i,\succ_i}$ as it depends only on i's tie-break rule, but we suppress this at minimal risk of confusion. Example 2 illustrates how we combine an agent's preferences and the tie-breaking rule to construct tie-broken preferences.

Example 2. Let $N = \{1, 2, 3, 4\}$. Agent 1's preferences R_1 and tie-breaking rule $\succ_1$ are shown below. In our lists of preference relations, each line represents an indifference class, and the houses on any line are strictly preferred to houses on lines below them. For example, the representation of R_1 below indicates $w_3 I_1 w_4 P_1 w_1 I_1 w_2$.

R_1		$\succ_1$		$R_{1,\succ_1}$
w_3, w_4		1		w_3
w_1, w_2	+	2	$\rightarrow$	w_4
		3		w_1
		4		w_2

Since $w_3 I_1 w_4$ and $3 \succ_1 4$, we have $w_3 P_{1,\succ} w_4$. Likewise, since $w_1 I_1 w_2$ and $1 \succ_1 2$, we have $w_1 P_{1,\succ} w_2$. Therefore agent 1's complete tie-broken preferences $R_{1,\succ}$ are given by $w_3\ P_{1,\succ}\ w_4\ P_{1,\succ}\ w_1\ P_{1,\succ}\ w_2$.

Given a preference profile $R \in \mathcal{R}^n$ and a tie-breaking profile $\succ$, let $R_{\succ} = (R_{1,\succ}, \ldots, R_{n,\succ})$. **TTC with fixed tie-breaking ($\mathbf{TTC}_{\succ}$)** is $\text{TTC}_{\succ}(R) \equiv \text{TTC}(R_{\succ})$. That is, the tie-breaking profile is used to generate strict preferences, and TTC is applied to the resulting strict preference profile. Formally, each tie-breaking profile $\succ$ generates a different $\text{TTC}_{\succ}$ mechanism, so TTC with fixed tie-breaking is a class of mechanisms, $\{\text{TTC}_{\succ}\}_{\succ}$. For a given R and $\succ$, we use $TTC_{\succ}(R)$ to refer both to the step-by-step procedure of $\text{TTC}_{\succ}$ and to the final allocation it generates. The following example illustrates how $\text{TTC}_{\succ}$ works in the objective indifferences domain.

Example 3. Let $N = \{1, 2, 3, 4\}$. The preference profile R and tie-breaking profile $\succ = (\succ_1, \succ_2, \succ_3, \succ_4)$ are shown below. R and $\succ$ are combined as shown in Example 2 to construct the tie-broken preference profile $R_{\succ}$. Recall that $\text{TTC}_{\succ}(R)$ is equivalent to $\text{TTC}(R_{\succ})$.

R_1	R_2	R_3	R_4
w_2, w_3	w_1	w_1	w_2, w_3
w_1	w_2, w_3	w_4	w_4
w_4	w_4	w_2, w_3	w_1

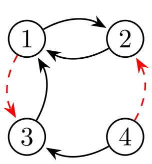
Preference profile R

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$
2	1	3	3
1	2	2	1
3	3	1	2
4	4	4	3

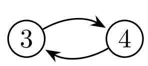
Tie-breaking profile $\succ$

$R_{1,\succ_1}$	$R_{2,\succ_2}$	$R_{3,\succ_3}$	$R_{4,\succ_4}$
w_2	w_1	w_1	w_3
w_3	w_2	w_4	w_2
w_1	w_3	w_3	w_4
w_4	w_4	w_2	w_1

Tie-broken preference profile $R_{\succ}$



Step 1 of $\text{TTC}_{\succ}(R)$:
Each agent points to the owner of their favorite house according to their *tie-broken* preferences, represented by black arrows. Red dashed arrows represent indifferences between w_2 and w_3. Agents 1 and 2 form a cycle and therefore swap houses.



Step 2 of $\text{TTC}_{\succ}(R)$:
After removing the agents assigned in Step 1, the remaining agents (3 and 4) point to the owner of their favorite remaining house. They form a cycle, and therefore swap houses. Since every agent has been assigned to a house, the $\text{TTC}_{\succ}$ procedure ends. The resulting allocation is $x = (w_2, w_1, w_4, w_3)$.

4 Results

With general indifferences, $\text{TTC}_{\succ}$ mechanisms are not Pareto efficient, core-selecting, nor group strategy-proof. We provide simple examples below to illustrate these failures. However, we show that in the objective indifferences domain, $\text{TTC}_{\succ}$ mechanisms satisfy all three properties. We also show that expanding beyond this domain results in the loss of each property. Strikingly, any domain satisfying any of these properties must be a subset of objective indifferences. That is, objective indifferences characterizes the set of maximal domains on which $\text{TTC}_\succ$ mechanisms are PE and CS and characterizes the set of "symmetric-maximal" (defined below) domains on which $\text{TTC}_\succ$ mechanisms are GSP.

4.1 Pareto efficiency and core-selecting

When we relax the assumption of strict preferences and allow for general indifferences, $\text{TTC}_\succ$ loses two of its most appealing properties: Pareto efficiency and core-selecting. However, in the intermediate case of objective indifferences, $\text{TTC}_\succ$ retains these two properties. Moreover, on *any* larger domain, $\text{TTC}_\succ$ loses both Pareto efficiency and core-selecting. Thus, we show that it is not indifferences per se, but rather *subjective* evaluations of indifferences which cause $\text{TTC}_\succ$ to lose these properties.

We first demonstrate that $\text{TTC}_\succ$ mechanisms are not Pareto efficient under general indifferences. Example 4 gives the simplest case.

Example 4. Let $N = \{1, 2\}$. The preference profile $R = (R_1, R_2)$, tie-breaking profile $\succ = (\succ_1, \succ_2)$, and tie-broken preference profile $R_\succ = (R_{1,\succ}, R_{2,\succ})$ are shown below.

R_1	R_2
w_1, w_2	w_1
	w_2

Preference profile R

$\succ_1 = \succ_2$
1
2

Tie-breaking profile $\succ$

$R_{1,\succ_1}$	$R_{2,\succ_2}$
w_1	w_1
w_2	w_2

Tie-broken preference profile $R_\succ$

The $\text{TTC}_\succ$ allocation is $x = (w_1, w_2)$, which is Pareto dominated by $y = (w_2, w_1)$.

This example demonstrates the underlying reason that $\text{TTC}_\succ$ fails PE under general indifferences: tie-breaking rules may not take advantage of Pareto gains made possible by the agents' indifferences. However, under objective indifferences, if any agent is indifferent between two houses, then all agents are indifferent between those two houses. Consequently, objective indifferences rules out situations like the one shown in Example 4.

Under general indifferences, the set of core allocations may not be a singleton; there may be no core allocations or there may be multiple. As Example 4 demonstrates, even when the core of the market is non-empty, $\text{TTC}_\succ$ may still fail to select a core allocation.[8] However, under objective indifferences, if the core of a market is non-empty, then $\text{TTC}_\succ$ always selects a core allocation.

In fact, the objective indifferences setting characterizes the entire set of maximal domains on which $\text{TTC}_\succ$ mechanisms are Pareto efficient or core-selecting. That is, if all $\text{TTC}_\succ$ mechanisms are PE or CS on a domain $\mathcal{R}$, then it must be a subset of some objective indifferences domain. Conversely, for any superset of an objective indifferences domain, there is some $\text{TTC}_\succ$ mechanism (that is, some $\succ$) that is not PE or CS.

Theorem 1. *The following are equivalent:*

1. *$\mathcal{R}$ is an objective indifferences domain.*

[8] It is straightforward to see that $y = (w_2, w_1)$ is in the core of the market and $x = (w_1, w_2)$ is not.

2. *$\mathcal{R}$ is a maximal domain on which $TTC_\succ$ mechanisms are Pareto efficient.*

3. *$\mathcal{R}$ is a maximal domain on which $TTC_\succ$ mechanisms are core-selecting.*

Proof. Appendix A.1. ☐

The full proof is in the appendix, but the intuition is simple. The objective indifferences domain precludes possibilities such as Example 4, and any larger domain inevitably introduces the possibility of such a pair.

It follows from Sönmez (1999) that under objective indifferences, the core of a market is *essentially single-valued* when it exists. That is, for any two allocations x and y in the core of a market, we have $x_i I_i y_i$ for all agents i. In our proof of Theorem 1, we also prove this claim directly. Since the core is essentially single-valued, under objective indifferences the core can be interpreted as a unique mapping from agents to house types. In other words, the core allocations are permutations of one another, where in each core allocation an agent always receives a house from the same indifference class.

Corollary 1. *If x and y are in the core of an objective indifferences market, then $x_i I_i y_i$ for all $i \in N$.*

Proof. Appendix A.1. ☐

Though all TTC$_\succ$ mechanisms are core-selecting under objective indifferences, the core of the market may still be empty, as the following simple example shows.

Example 5. Let $N = \{1, 2, 3\}$. It is easy to verify that for the preference profile $R = (R_1, R_2, R_3)$ shown below, there are no core allocations.

$$\underbrace{\begin{array}{cc} R_1 & R_2 \\ \hline w_1, w_2 & w_1 \\ & w_2 \end{array}}_{\text{Preference profile } R} \qquad \underbrace{\begin{array}{c} \succ_1 = \succ_2 \\ \hline 1 \\ 2 \end{array}}_{\text{Tie-breaking profile } \succ} \qquad \underbrace{\begin{array}{cc} R_{1,\succ_1} & R_{2,\succ_2} \\ \hline w_1 & w_1 \\ w_2 & w_2 \end{array}}_{\text{Tie-broken preference profile } R_\succ}$$

Any allocation x such that $x_1 \in \{w_1, w_2\}$ is blocked by $Q = \{1, 3\}$ and $y_Q = (w_3, w_1)$. Similarly, any allocation x such that $x_1 = w_3$ is blocked by $Q = \{1, 2\}$ and $y_Q = (w_2, w_1)$.

Even when the core of an objective indifferences market is empty, all TTC$_\succ$ mechanisms select an allocation in the weak core of the market. This is a known fact even under general indifferences; we reproduce it here and provide a proof for convenience.

Proposition 1. *For any market, the weak core is non-empty and $TTC_\succ$ mechanisms select an allocation in the weak core.*

Proof. Appendix A.1. ☐

4.2 Group strategy-proofness

$\text{TTC}_\succ$ also loses group strategy-proofness once we move from strict preferences to weak preferences. However, in the intermediate case of objective indifferences, $\text{TTC}_\succ$ recovers group strategy-proofness. Further, $\text{TTC}_\succ$ mechanisms are not GSP in any larger "symmetric" domain. We say that a domain $\mathcal{R}$ is symmetric if, when $h_1 P_i h_2$ is possible, then so is $h_2 P_i h_1$. We will informally argue that this is not an onerous modeling restriction.

First we present a simple example demonstrating that under general indifferences, $\text{TTC}_\succ$ mechanisms are not group strategy-proof. Notice that in Example 4, agent 1 can misreport $w_2 R_1' w_1$ to benefit agent 2 without harming himself. As a less trivial example, we also provide Example 6.

Example 6. Let $N = \{1, 2, 3\}$ and let $Q = \{1, 3\}$. For the preference profile $R = (R_1, R_2, R_3)$ and tie-breaking profile $\succ = (\succ_1, \succ_2, \succ_3)$ shown below, the $\text{TTC}_\succ$ allocation is $x = (w_2, w_1, w_3)$. However, if agent 1 were to report R_1', then for $R' = (R_1', R_2, R_3)$ the $\text{TTC}_\succ$ allocation is $y = (w_3, w_2, w_1)$. Note that $y_3 P_3 x_3$ and $y_1 I_1 x_1$, so $\text{TTC}_\succ$ is not GSP.

R_1	R_2	R_3
w_2, w_3	w_1	w_1
w_1	w_2	w_2
	w_3	w_3

Preference profile R

R_1'	R_2	R_3
w_3	w_1	w_1
w_2	w_2	w_2
w_1	w_3	w_3

Preference profile R'

$\succ_1 = \succ_2 = \succ_3$
1
2
3

Tie-breaking profile $\succ$

Objective indifferences excludes situations like Example 6 in two ways. First, it eliminates the possibility that one agent is indifferent between two houses while another agent has a strict preference. Second, it constrains the possible set of misreports available to a manipulating coalition, since agents can *only* report indifference among all houses of the same type. Our next result characterizes the set of symmetric-maximal domains on which $\text{TTC}_\succ$ mechanisms are GSP.

Before presenting our result, we define "symmetric" and "symmetric-maximal" domains.

Definition 4. A domain $\mathcal{R}$ is **symmetric** if for any $h_1, h_2 \in H$, if there exists $R_i \in \mathcal{R}$ such that $h_1 P_i h_2$, then there also exists $R_i' \in \mathcal{R}$ such that $h_2 P_i' h_1$.

Definition 5. A domain $\mathcal{R}$ is **symmetric-maximal** for Pareto efficiency and a class of mechanisms F if

1. $\mathcal{R}$ is symmetric, and
2. for any symmetric $\tilde{\mathcal{R}} \supset \mathcal{R}$, there is some $f \in F$ that is *not* Pareto efficient on $\tilde{\mathcal{R}}$.

The definition for any axiom besides Pareto efficiency is again analogous.

In most practical applications, symmetry is a natural restriction to place on the domain. If it is possible that agents might report strictly preferring some house h to another house h', we should not preclude the possibility they strictly prefer h' to h. Indeed, a central principle of mechanism design is that preferences are unknown and must be elicited. It is easy to see that objective indifferences domains are symmetric. Compared to maximality, symmetric-maximality restricts the possible expansions of objective indifferences domains that we must consider. The symmetry restriction is relatively weak; it does not enforce strict preferences nor general indifferences, as the next example illustrates.

Example 7. Let $H = \{h_1, h_2, h_3, h_4\}$ and $\mathcal{H} = \{\{h_1\}, \{h_2, h_3\}\}$. Consider the objective indifferences domain $\mathcal{R}(\mathcal{H}) = \{h_1Ph_2Ih_3, h_2Ih_3Ph_1\}$. Let $R_\alpha = h_1Ph_2Ph_3$ and $R_\beta = h_1Ph_3Ph_2$. Then $\mathcal{R}(\mathcal{H}) \cup \{R_\alpha\}$ is not a symmetric domain, since it allows h_2Ph_3 but not h_3Ph_2. However, $\mathcal{R}(\mathcal{H}) \cup \{R_\alpha, R_\beta\}$ *is* a symmetric domain. Further, notice that $\mathcal{R}(\mathcal{H}) \cup \{R_\alpha, R_\beta\}$ is not general indifferences nor strict preferences. Nor does it belong to some other objective indifferences domain, as it allows agents to report both h_2Ih_3 and h_2Ph_3.

Our second main result is that objective indifferences domains are the only symmetric-maximal domains on which every TTC$_\succ$ is group strategy-proof.

Theorem 2. *$\mathcal{R}$ is a symmetric-maximal domain on which TTC$_\succ$ mechanisms are group strategy-proof if and only if $\mathcal{R}$ is an objective indifferences domain.*

Proof. Appendix A.2. □

Our proof uses similar reasoning to the proof that TTC is group strategy-proof under strict preferences contained in Sandholtz and Tai (2024). Any coalition requires a "first mover" to misreport, but this agent must receive an inferior house to the one he originally received. Under objective indifferences, this "inferior" house must actually be inferior according to his true preferences. Under more general indifferences, the situation in Example 4 again inevitably arises.

In the following example, we note that objective indifferences domains are *not* maximal domains on which TTC$_\succ$ mechanisms are GSP.

Example 8. Let $N = \{1, 2\}$, $H = \{h_1, h_2\}$, and $\mathcal{H} = \{\{h_1, h_2\}\}$. Suppose $\mathcal{R}' = \mathcal{R}(\mathcal{H}) \cup \{h_1Ph_2\} = \{h_1Ih_2, h_1Ph_2\}$. That is, expand the objective indifferences domain induced by $\mathcal{H}$ to include the ordering h_1Ph_2. Note that this expanded domain is not symmetric, since $\mathcal{R}'$ does not also contain the preference ordering h_2Ph_1.

Let $\succ = ((1 \succ_1 2), (1 \succ_2 2))$. We will show that for any market (N, H, w, R), TTC$_\succ$ is group strategy-proof. It is straightforward to show that the same is true for the remaining 3 possible tie-breaking profiles.

Without loss of generality, assume $w_i = h_i$. If both agents have the same preferences, then there is clearly no profitable group manipulation. Consider the following two possible preference profiles:

R_1	R_2
h_1	h_1, h_2
h_2	

or

R_1	R_2
h_1, h_2	h_1
	h_2

In the first case, the TTC$_\succ$ allocation is $x = (h_1, h_2)$, so both agents receive one of their most preferred houses. Therefore, it is not possible for either agent to strictly improve. In the second case, the TTC$_\succ$ allocation is $x = (h_1, h_2)$. It would benefit agent 2 for agent 1 to rank h_2 above h_1, since agent 1 is indifferent between h_1 and h_2. However, this is not possible since $(h_2Ph_1) \notin \mathcal{R}'$.

Even though objective indifferences domains are not maximal for GSP, we have already argued that the symmetric restriction is not onerous. In most mechanism design settings, it is strange to allow participants to strictly prefer one object to another, but not vice versa.[9]

5 Conclusion

Our main set of results show that objective indifferences domains are the maximal domains on which $\text{TTC}_{\succ}$ mechanisms are Pareto efficient, core-selecting, and group strategy-proof. While it may not be surprising that objective indifferences preserves these properties, it is quite remarkable that the maximal domains on which $\text{TTC}_{\succ}$ satisfies these three distinct properties essentially coincide.

In markets where one could reasonably assume that any indifferences are shared among all agents, $\text{TTC}_{\succ}$ is a sensible choice of mechanism. Regardless of the tie-breaking rule, $\text{TTC}_{\succ}$ will be Pareto efficient, core-selecting, and group strategy-proof. While a number of other mechanisms generalize TTC and retain various properties in general indifferences, $\text{TTC}_{\succ}$ remains computationally efficient and easier to explain. However, in any more general domain of preferences, TTC inevitably loses some of its appeal. That is not to say that TTC with tie-breaking should be avoided in more general settings. We simply characterize exactly when TTC retains its most desirable properties.

Our paper opens several new lines of inquiry. First, we believe that studying matching markets with constrained indifferences is an exciting avenue for future research. In many real-world matching markets, agents have indifferences, but often with a structure imposed by the particular market. Adding structure to general indifferences is not only theoretically interesting, but could improve policy choices. For instance, there may be tradeoffs in the selection of the partition $\mathcal{H}$ given the set of objects H. In some cases, there may be some ambiguity: are two dorms with the same floor plan but on different floors of the same building equivalent? Inappropriately combining indifference classes might lead to efficiency losses in the spirit of Example 4. On the other hand, splitting indifference classes might allow group manipulations like in Example 6. We leave formal results as future work. We also leave an axiomatic characterization of $\text{TTC}_{\succ}$ on objective indifferences domains as future work.

[9]One reasonable exception is contracts where the objects may include payments; e.g., a seat at a school with or without a scholarship.

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Appendix A Proofs

We provide proofs for the results in the main text. Note that individual rationality (IR) of $\text{TTC}_{\succ}$ follows immediately from IR of TTC and the fact that $TTC_{\succ}(R) \equiv TTC(R_{\succ})$. That is, TTC is IR according to $R_{\succ}$, which implies $\text{TTC}_{\succ}$ is IR according to R.

Given a market (N, H, w, R) and mechanism $\text{TTC}_{\succ}$, let $S_k(R)$ be the agents in the kth cycle executed in the process of $\text{TTC}_{\succ}(R)$. Denote $\bar{S}_{k-1}(R) = \cup_{\ell=1}^{k-1} S_\ell(R)$. Where the risk of confusion is minimal (generally when not dealing with strategy-proofness), we suppress the dependence on R and simply refer to S_k and $\bar{S}_{k-1}$. While $\text{TTC}_{\succ}(R)$ may execute cycles in different orders, the cycles are always the same for a given market, so the order will be unimportant.

We will appeal to the following fact.

Fact 1. *Fix a market (N, H, w, R) and a tie-breaking profile $\succ$. Let $x = TTC_{\succ}(R)$. If $i \in S_k$ and $x_j P_i x_i$, then $j \in \bar{S}_{k-1}$.*

In words, if i strictly prefers some object to x_i, that object must have been assigned in an earlier cycle. Fact 1 follows from the definitions: $x_j P_i x_i$ implies $x_j P_{i,\succ} x_i$, and $\text{TTC}_{\succ}(R) \equiv \text{TTC}(R_{\succ})$. If a better object than x_i is available, i will point at it; thus i cannot be assigned to x_i until the better object is gone.

We also make use of the following technical lemma. We do not think it is of interest on its own, but it will be useful in the proceeding proofs.

Lemma 1. *Fix (N, H, w) and a domain $\mathcal{R}$. For any (not necessarily distinct) preference relations $R_\alpha, R_\beta, R_\gamma \in \mathcal{R}$ and (not necessarily distinct) houses $h_1, h_2 \in H$, suppose A and B satisfy*

$$\{i \in N : w_i P_\alpha h_1\} \subseteq A \subseteq \{i \in N : w_i R_\alpha h_1\}$$

and

$$\{i \in A^c : w_i P_\beta h_2\} \subseteq B \subseteq \{i \in A^c : w_i R_\beta h_2\}.$$

That is, A includes all agents whose endowments are strictly preferred to h_1 according to R_α, and perhaps some whose endowments are indifferent. Among the remainder, B includes all those whose endowments are strictly preferred to h_2 according to R_β, and perhaps some whose endowments are indifferent.

Let $R_i = R_\alpha$ for all $i \in A$ and $R_i = R_\beta$ for all $i \in B$. Let $\succ$ be any tie-breaking profile such that for all $i \in N$, $i \succ_i j$ for all $j \neq i$. That is, every agent's tie-breaking rule prioritizes himself first. Denote $x := TTC_{\succ}(R)$. Then $x_i = w_i$ for all $i \in A \cup B$. If in addition, $R_i = R_\gamma$ for all $i \in N \backslash \{A \cup B\}$, then $x = w$.

Proof of Lemma 1. We first note that since $i \succ_i j$ for all $i \neq j$, an immediate consequence of $\text{TTC}_{\succ}$ is that if $x_i I_i w_i$, then $x_i = w_i$. To see this, observe that if w_i is available, i will never point to any other $h I_i w_i$.

First we show that $x_i = w_i$ for all $i \in A$. Toward a contradiction, suppose that $U = \{i \in A : x_i \neq w_i\}$ (for "upper") is non-empty. IR and the above fact imply $U \equiv \{i \in A : x_i P_i w_i\}$. Fix some $i \in U$ such that $w_i R_\alpha w_{i'}$ for all $i' \in U$; informally, i's endowment is one of U's favorites among their endowments. Since $i \in A$, by construction $R_i = R_\alpha$; therefore $x_i P_i w_i$ implies $x_i P_\alpha w_i$. Consider $j \in N$ endowed with x_i; i.e.,

$x_i = w_j$. We have $(x_i =) w_j P_\alpha w_i$ and $w_i P_\alpha h_1$ by construction, so $j \in A$. We also have $x_j \neq w_j$, so $j \in U$. But since $w_j P_\alpha w_i$, $j \in U$ contradicts the assumption that $w_i R_\alpha w_{i'}$ for all $i' \in U$.

Next we show that $x_i = w_i$ for all $i \in B$. This follows nearly the same logic as above. Suppose that $U = \{i \in B : x_i \neq w_i\} \equiv \{i \in B : x_i P_i w_i\}$ is non-empty. Fix some $i \in U$ such that $w_i R_\beta w_{i'}$ for all $i' \in U$. Since $i \in B$, by construction $R_i = R_\beta$; therefore, $x_i P_i w_i$ implies $x_i P_\beta w_i$. Consider $j \in N$ endowed with x_i; i.e., $x_i = w_j$. It must be that $j \in A^c$, because we showed that $x_i = w_i$ for all $i \in A$ and $x_j \neq w_j$. We have $w_j P_\beta w_i$ and $w_i P_\beta h_2$ by construction, so $j \in B$. We also have $x_j \neq w_j$, so $j \in U$. But since $w_j P_\beta w_i$, $j \in U$ contradicts the assumption that $w_i R_\beta w_{i'}$ for all $i' \in U$.

Then we are left with agents $i \in N \backslash \{A \cup B\}$, who must re-allocate their endowments among themselves. If they all have the same preferences, IR requires $x_i I_i w_i$ for each $i \in N \setminus (A \cup B)$. By construction of $\succ$, we have $x_i = w_i$ as desired. □

Appendix A.1 Pareto efficiency and core-selecting

Theorem 1. *The following are equivalent:*

1. $\mathcal{R}$ is an objective indifferences domain.

2. $\mathcal{R}$ is a maximal domain on which $\text{TTC}_\succ$ mechanisms are Pareto efficient.

3. $\mathcal{R}$ is a maximal domain on which $\text{TTC}_\succ$ mechanisms are core-selecting.

Fix (N, H, w). The result is trivial for $|N| = 1$, so assume $|N| \geq 2$.

Proof of 1 $\Longleftrightarrow$ 2. First we show that for any objective indifferences domain, all $\text{TTC}_\succ$ mechanisms are PE. Fix some tie-breaking rule $\succ$. Let $\mathcal{H}$ be any partition of H and let $R \in \mathcal{R}(\mathcal{H})^n$. If $\mathcal{H} = \{H\}$, the result is trivial, so suppose the partition has at least two blocks.

Let $x = \text{TTC}_\succ(R)$, and suppose that some feasible allocation y Pareto dominates x. Let $U = \{i \in N : y_i P_i x_i\}$ be the set of agents who are strictly better off under y, which must be non-empty. Let k be the first step in the process of $\text{TTC}_\succ(R)$ that an agent in U is assigned. Formally, $\bar{S}_{k-1} \cap U = \emptyset$ and $S_k \cap U \neq \emptyset$. Consider some $j \in S_k \cap U$. We have $y_j P_j x_j$, so Fact 1 implies that $\{i : x_i \in \eta(y_j)\} \subseteq \bar{S}_{k-1}$. That is, anyone assigned to a house in $\eta(y_j)$ was assigned before j. Because $\eta(y_j)$ is finite and $\eta(y_j) \neq \eta(x_j)$, there must be an agent $\ell \in \bar{S}_{k-1}$ for whom $x_\ell \in \eta(y_j)$ but $y_\ell \notin \eta(y_j)$. In words, in order to assign j to y_j, someone who originally received an object in $\eta(y_j)$ under x must receive something else under y. Therefore $\neg(y_\ell I_\ell x_\ell)$. Since y Pareto dominates x, it must be that $y_\ell P_\ell x_\ell$. But then $\ell \in U$, and ℓ was assigned before cycle k, a contradiction.

We now turn to the maximality claim. We show that for any domain $\tilde{\mathcal{R}}$ where $\tilde{\mathcal{R}} \nsubseteq \mathcal{R}(\mathcal{H})$ for any partition $\mathcal{H}$ of H, there is some $\text{TTC}_\succ$ mechanism which is not PE on $\tilde{\mathcal{R}}$. If $\tilde{\mathcal{R}} \nsubseteq \mathcal{R}(\mathcal{H})$ for any $\mathcal{H}$, then $\tilde{\mathcal{R}}$ must contain two preference orderings, R_α and R_β, such that for some $h_1, h_2 \in H$ we have $h_1 I_\alpha h_2$ but $h_1 P_\beta h_2$. The proof seeks to isolate a pair like in Example 4. Taking only the existence of $R_\alpha, R_\beta \in \tilde{\mathcal{R}}$ for granted, we find a preference profile $R \in \tilde{\mathcal{R}}^n$ and tie-breaking profile $\succ$ such that $\text{TTC}_\succ(R)$ is not PE. Without loss of generality, let $w_i = h_i$ for all $i \in N$. Define $A = \{i \in N : w_i R_\alpha w_1\} \setminus \{2\}$. Let the preference profile R be given by

$$R_i = \begin{cases} R_\alpha & \text{(having } h_1 I_\alpha h_2) & \text{if } i \in A := \{i \in N : w_i R_\alpha w_1\} \setminus \{2\} \\ R_\beta & \text{(having } h_1 P_\beta h_2) & \text{if } i \in A^c. \end{cases}$$

Note that $1 \in A$ and $2 \in A^c$, so $R_1 = R_\alpha$ and $R_2 = R_\beta$. Except for agent 2, if an agent's endowment is preferred to w_1 according to R_α, he is grouped into A and has preference ranking R_α. The rest have preference ranking R_β.

Take any tie-breaking profile $\succ$ such that for all $i \in N$, $i \succ_i j$ for all $j \neq i$. Observe that the preference profile R, tie-breaking rule $\succ$, and sets A, $B := \{i \in A^c : w_i R_\beta h_2\}$, and $C := N \backslash \{A \cup B\}$ satisfy the conditions of Lemma 1 (where $R_\beta = R_\gamma$). Denote $x := \text{TTC}_\succ(R)$. Then by Lemma 1, $x = w$. However, note that $w_2 I_1 w_1$ and $w_1 P_2 w_2$, so x is Pareto dominated by $y = (w_2, w_1, w_3, ..., w_n)$. $\square$

Proof of (1) $\iff$ (3). First we show that for any objective indifferences domain, $\text{TTC}_\succ$ mechanisms are CS. Fix some tie-breaking rule $\succ$. Let $\mathcal{H}$ be any partition of H and let $R \in \mathcal{R}(\mathcal{H})^n$. If $\mathcal{H} = \{H\}$, the result is trivial, so suppose the partition has at least two blocks. Suppose that the core of (N, H, w, R) is non-empty and contains some allocation y. Let $x = \text{TTC}_\succ(R)$. It suffices to show that $x_i I_i y_i$ for all $i \in N$. We proceed by induction on the steps of $\text{TTC}_\succ(R)$.

Step 1. By definition of $\text{TTC}_\succ$, for all $i \in S_1$ we have $x_i R_i h$ for all $h \in H$; that is, agents in the first cycle receive (one of) their favorite houses. Therefore $x_i R_i y_i$ for all $i \in S_1$. Suppose there is some $j \in S_1$ such that $x_j P_j y_j$. Then S_1 and x block y, contradicting the assumption that y is in the core. Thus, $x_i I_i y_i$ for all $i \in S_1$.

Step k. Assume that $x_i I_i y_i$ for all $i \in \bar{S}_{k-1}$. Suppose that $y_j P_j x_j$ for some $j \in S_k$. By construction of objective indifferences, $\eta(y_j) \neq \eta(x_j)$. By Fact 1, $\{i \in N : x_i \in \eta(y_j)\} \subseteq \bar{S}_{k-1}$. Because $\eta(y_j)$ is finite, if $\eta(y_j) \neq \eta(x_j)$, there must be an agent ℓ in $\bar{S}_{k-1}$ for whom $x_\ell \in \eta(y_j)$ but $y_\ell \notin \eta(y_j)$. Therefore $\neg(y_\ell I_\ell x_\ell)$, contradicting the assumption that $x_i I_i y_i$ for all $i \in \bar{S}_{k-1}$. Thus $x_i R_i y_i$ for all $i \in S_k$. Now suppose there is some $j \in S_k$ such that $x_j P_j y_j$. Then S_k and x block y, contradicting that y is in the core.

Thus $x_i I_i y_i$ for all $i \in N$, so x must also be in the core as desired. (Since y was an arbitrary allocation in the core, this also proves Corollary 1.)

We now turn to the maximality claim. We show that for any domain $\tilde{\mathcal{R}}$ where $\tilde{\mathcal{R}} \nsubseteq \mathcal{R}(\mathcal{H})$ for any partition $\mathcal{H}$, there is some $\text{TTC}_\succ$ which is not CS on $\tilde{\mathcal{R}}$. As previously, if $\tilde{\mathcal{R}} \nsubseteq \mathcal{R}(\mathcal{H})$ for any $\mathcal{H}$, then $\tilde{\mathcal{R}}$ must contain two orderings, R_α and R_β, such that for some $h_1, h_2 \in H$ we have $h_1 I_\alpha h_2$ but $h_1 P_\beta h_2$. Without loss of generality, let h_1 be R_β's highest ranked house such that $h_1 I_\alpha h_2$ and $h_1 P_\beta h_2$ are true. Formally,

$$h_1 R_\beta h \text{ for all } h \in H \text{ such that } h I_\alpha h_2. \qquad (*)$$

Also without loss of generality, let $w_i = h_i$ for all $i \in N$.

The following construction is the same as in the previous part. Define $A = \{i \in N : w_i R_\alpha w_1\} \setminus \{2\}$ and consider the preference profile where $R_i = R_\alpha$ for all $i \in A$ and $R_i = R_\beta$ for all $i \in A^c$. Let $x = \text{TTC}_\succ(R)$. It follows from Lemma 1 that $x = w$. As before, x is Pareto dominated by $y = (w_2, w_1, w_3, ..., w_n)$, so x is not in the core (x is blocked by the grand coalition N and y).

It remains to show that y is in the core. Toward a contradiction, suppose there is a coalition Q and allocation z that blocks y. Let $U = \{i \in Q : z_i P_i y_i\}$ be the strict improvers under z, which must be non-empty. We proceed in steps denoted by the following claims.

Claim 1. $U \cap A = \emptyset$.

Intuitively, if any agent in A strictly prefers a house to his assignment, this house is also in w_A, preventing mutual improvements within A. Toward a contradiction, suppose $U_A := U \cap A$ is non-empty; note that all agents in U_A have the preference ranking R_α. Take $i \in U_A$ such that $z_i R_\alpha z_{i'}$ for all $i' \in U_A$. That is, i is the best off improver in U_A (or one of the best off improvers). Observe that $z_i P_\alpha w_i$ since $i \in U$ and $w_i R_\alpha w_1$ since $i \in A$. Thus by construction of R_α and A, we have $J := \{j \in Q : w_j I_\alpha z_i\} \subseteq A$. That is, objects equivalent to z_i are also owned by members of A. Also, J is nonempty since $z_i \in w_J$. Further, note $1 \notin J$ since $z_i P_\alpha w_1$. Agent 1 is the only agent in A who does not receive his endowment, so $y_j = w_j$ for all $j \in J$.

Since w_J is finite and $i \notin J$, in order for i to receive $z_i \in w_J$, some $j \in J$ must have $z_j \notin w_J$. But then since $j \in Q \cap A$, it must be that $z_j P_\alpha w_j (= y_j)$. Then $j \in U_A$, contradicting the assumption that $z_i R_\alpha z_{i'}$ for all $i' \in U_A$.

Claim 2. $Q \cap A \neq \emptyset$.

The proof shows that the coalition Q cannot generate improvements with w_Q if all its members have the same preference R_β. Toward a contradiction, suppose $Q \subseteq A^c$; then $R_i = R_\beta$ for all $i \in Q$. Let $i \in U$ be such that $z_i R_\beta z_{i'}$ for all $i' \in U$; that is, i is the best off improver.

Now consider $J := \{j \in Q : w_j I_\beta z_i\} \subseteq A^c$. Since w_J is finite and $i \notin J$, in order for i to receive $z_i \in w_J$, some $j \in J$ must have $z_j \notin w_J$. Then since $j \in Q$, it must be $z_j P_\beta w_j$. If $j \neq 2$, then $y_j = w_j$ so $z_j P_\beta y_j$. But then $j \in U$, contradicting $z_i R_\beta z_{i'}$ for all $i' \in U$.

Now suppose $j = 2$. Since $w_1 \in w_A$, we have $z_2 \neq w_1 (= y_2)$. Then $z_2 R_\beta w_1$ and $z_2 \neq w_1$. That is, agent 2 must receive a house besides w_1 endowed to $Q \subseteq A^c$. Consider $L := \{\ell \in Q : w_\ell R_\beta w_1\} \subseteq A^c$, and note that $2 \notin L$ since $w_1 P_\beta w_2$. Since w_L is finite, in order for $z_2 \in w_L$, some $\ell \in L$ must receive $z_\ell \notin w_L$. Then it must be that $w_\ell P_\beta z_\ell$. Since $\ell \in A^c$ and $\ell \neq 2$, we have $y_\ell = w_\ell$. This gives $y_\ell P_\beta z_\ell$, contradicting that every agent in Q weakly improves under z.

Claim 3. Claims 1 and 2 cannot both be true.

Without loss of generality, assume Q forms a single trading cycle under z.[10] Since Q contains agents in both A and A^c in the same cycle, there exists some agent $\bar{b} \in A^c$ such that $z_{\bar{b}} \in w_A$ and some agent $\bar{a} \in A$ such that $z_{\bar{a}} \in w_{A^c}$. It must be that $z_{\bar{a}} = w_2$, since by construction $w_{\bar{a}} P_\alpha w_i$ for any $i \in A^c \setminus \{2\}$. For the same reason, $\bar{a}$ must be the only agent in $A \cap Q$ who receives a house from w_{A^c}.

[10] If the agents in Q formed two or more trading cycles, then some cycle S must contain an agent i such that $z_i P_i y_i$, and this cycle S suffices to form a blocking coalition.

Since only one agent in $A \cap Q$ receives a house from w_{A^c}, only one agent from $A^c \cap Q$ receives a house from w_A. We can represent the trading cycle Q as

$$a_1 \to a_2 \to ... \to a_m \to \bar{a} \to 2 \to b_1 \to ... \to \bar{b} \to a_1$$

where $a_1, ..., a_m, \bar{a} \in A$ and $2, b_1, ..., \bar{b} \in A^c$, and $z_{a_1} = w_{a_2}$, etc.

Since $U \cap A = \emptyset$, no $a \in \{a_1, ..., a_m, \bar{a}\}$ strictly improves under z; that is, $z_a I_\alpha y_a$ for each a. Recall that $y_i I_\alpha w_i$ for all $i \in A$, since either 1) $y_i = w_i$ or 2) $i = 1$, $y_1 = w_2$ and $w_2 I_1 w_1$. Thus $w_{a_1} I_\alpha w_{a_2} I_\alpha \cdots I_\alpha w_2$.

So $z_b R_\beta y_b$ for all $b \in \{2, b_1, ..., \bar{b}\}$, with a strict relation for at least one b. Since $y_2 = w_1$, we have $w_{b_1} R_\beta w_1$. For all other b, we have $y_b = w_b$, giving us $w_{b_2} R_\beta w_{b_1}$, ..., $w_{a_1} R_\beta w_{\bar{b}}$. At least one of these relations is strict; combining these gives $w_{a_1} P_\beta w_1$.

Then $w_{a_1} I_\alpha w_2$ and $w_{a_1} P_\beta w_1$. But this contradicts the assumption (*).

Thus we have that y is in the core, as desired. □

TTC$_\succ$ always selects an allocation in the weak core. This is a known property, but we give a short proof for convenience.

Proposition 1. *For any market, the weak core is non-empty and TTC$_\succ$ mechanisms select an allocation in the weak core.*

Proof. Fix a market (N, H, w, R) and a tie-breaking profile $\succ$. Denote $x := \text{TTC}_\succ(R) \equiv \text{TTC}(R_\succ)$. Toward a contradiction, suppose there exists a coalition $Q \subseteq N$ and allocation y that weakly blocks x. That is, $y_Q = w_Q$ and $y_i P_i x_i$ for all $i \in Q$. Then we also have $y_i P_{i,\succ} x_i$ for all $i \in Q$, since the tie-breaking procedure preserves strict comparisons. Thus y and Q form a weakly blocking coalition against x *according to the strict preferences* $R_\succ$. This violates the core property of TTC with strict preferences; thus TTC$_\succ$ always selects a mechanism in the weak core.

Since TTC$_\succ$ always produces an allocation, this also proves the existence of a weak core allocation. □

Appendix A.2 Group strategy-proofness

Theorem 2. *$\mathcal{R}$ is a symmetric-maximal domain on which TTC$_\succ$ mechanisms are group strategy-proof if and only if it is an objective indifferences domain.*

Before proving Theorem 2, we review an important property of TTC$_\succ$ and state a useful lemma. Let $L(h, R_i) = \{h' \in H : h R_i h'\}$ be the lower contour set of a preference ranking R_i at house h.

Monotonicity (MON). A mechanism f is **monotone** if $f(R) = f(R')$ for any preference profiles R and R' such that $L(f_i(R), R_i) \subseteq L(f_i(R), R'_i)$ for all $i \in N$.

That is, a mechanism f is monotone if, whenever any set of agents move their allocations upwards in their preference rankings, the allocation remains the same. It is straightforward to show that TTC is monotone for strict preferences (e.g., Takamiya (2001)). Since $\text{TTC}_\succ(R) \equiv \text{TTC}(R_\succ)$ for any R and $\succ$, it follows directly that TTC$_\succ$ mechanisms are monotone.

The following result is an immediate consequence of Lemma 1 from Sandholtz and Tai (2024) and the fact that $\text{TTC}_{\succ}(R) \equiv \text{TTC}(R_{\succ})$.

Lemma 2 (Sandholtz and Tai (2024))**.** *For any R, R', let $x = TTC_{\succ}(R)$ and $y = TTC_{\succ}(R')$. Suppose there is some i such that $y_i P_{i,\succ} x_i$. Then there exists some agent j and house h such that $h P'_{j,\succ} x_j$ and $x_j P_{j,\succ} h$.*

In other words, if i is to receive a better house, some other agent j must misreport his preferences and rank a worse house above his original allocation.

We now provide our proof of Theorem 2.

Proof of Theorem 2. Fix (N, H, w). The result is trivial for $|N| = 1$, so assume $|N| \geq 2$. First we show that for any objective indifferences domain, $\text{TTC}_{\succ}$ mechanisms are GSP. Fix some tie-breaking rule $\succ$. Let $\mathcal{H}$ be any partition of H and let $R \in \mathcal{R}(\mathcal{H})^n$. If $\mathcal{H} = \{H\}$, the result is trivial, so suppose assume that $\mathcal{H}$ has at least two blocks. Suppose $Q \subseteq N$ reports R'_Q where $R' = (R'_Q, R_{-Q}) \in \mathcal{R}(\mathcal{H})^n$. Let $y = \text{TTC}_{\succ}(R')$. We will show that if $y_i P_i x_i$ for some $i \in Q$, then $x_j P_j y_j$ for some $j \in Q$.

Let $R'' = (R''_Q, R_{-Q})$ be the preference profile in $\mathcal{R}(\mathcal{H})^n$ such that for each $i \in Q$, R''_i top-ranks the houses in $\eta(y_i)$ and otherwise preserves the ordering of R_i. That is, for any $h \in \eta(y_i)$ and $h' \notin \eta(y_i)$, $h P''_i h'$; otherwise, $h R''_i h'$ if and only if $h R_i h'$. Let $z = TTC_{\succ}(R'')$. By monotonicity of $\text{TTC}_{\succ}$, $z = y$. Fix $i \in Q$ such that $y_i P_i x_i$. Since $z = y$, we have $z_i P_i x_i$ and consequently $z_i P_{i,\succ} x_i$. Applying Lemma 2, there must be some $j \in Q$ and $h \in H$ such that $x_j P_{j,\succ} h$ but $h P''_{j,\succ} x_j$. Note that $h \notin \eta(x_j)$; if it were, then for any $R, R'' \in \mathcal{R}(\mathcal{H})^n$, $x_j P_{j,\succ} h$ if and only if $x_j P''_{j,\succ} h$. Therefore, $x_j P_j h$ and $h P''_j x_j$.[11] The only change from R_j to R''_j is to top-rank the houses in $\eta(y_j)$, so it must be that $h \in \eta(y_j)$. But then $x_j P_j y_j$, as desired.

Next we show that for any symmetric domain $\tilde{\mathcal{R}}$ where $\tilde{\mathcal{R}} \not\subseteq \mathcal{R}(\mathcal{H})$ for any partition $\mathcal{H}$ of H, $\text{TTC}_{\succ}$ mechanisms are not GSP on $\tilde{\mathcal{R}}$. As before, if $\tilde{\mathcal{R}} \not\subseteq \mathcal{R}(\mathcal{H})$ for any $\mathcal{H}$, then $\tilde{\mathcal{R}}$ must contain two orderings, R_α and R_β, such that for some $h_1, h_2 \in H$ we have $h_1 I_\alpha h_2$ but $h_1 P_\beta h_2$. The symmetric requirement also necessitates that $\tilde{\mathcal{R}}$ contains some R_γ such that $h_2 P_\gamma h_1$. Taking only the existence of $R_\alpha, R_\beta, R_\gamma \in \tilde{\mathcal{R}}$ for granted, we find a preference profile $R \in \tilde{\mathcal{R}}^n$ and tie-breaking profile $\succ$ such that $\text{TTC}_{\succ}(R)$ is not GSP.

Without loss of generality, assume $w_i = h_i$ for all $i \in N$. Define $A = \{i \in N : w_i R_\alpha w_1\} \setminus \{2\}$, $B = \{i \in A^c : w_i R_\beta w_1\} \cup \{2\}$, and $C = N \setminus (A \cup B)$. Consider the preference profile $R \in \tilde{\mathcal{R}}^n$ where

$$R_i = \begin{cases} R_\alpha & (\text{having } w_1 I_\alpha w_2) \quad \text{if } i \in A := \{i \in N : w_i R_\alpha w_1\} \setminus \{2\} \\ R_\beta & (\text{having } w_1 P_\beta w_2) \quad \text{if } i \in B := \{i \in A^c : w_i R_\beta w_1\} \cup \{2\} \\ R_\gamma & (\text{having } w_2 P_\gamma w_1) \quad \text{if } i \in C := (A \cup B)^c \end{cases}$$

Note that $1 \in A$ and $2 \in B$, so $R_1 = R_\alpha$ and $R_2 = R_\beta$. Let $\succ$ be any tie-breaking profile such that for all $i \in N$, $i \succ_i j$ for all $j \neq i$. Additionally, let agent 2's tie-breaking order be $2 \succ_2 1 \succ_2 \cdots$. Let $x = \text{TTC}_{\succ}(R)$.

It follows directly from Lemma 1 that $x = w$. Also observe that $2 \in B$ so $w_1 P_2 w_2$ and therefore $w_1 P_2 x_2$. Now suppose that agent 1 misreports $R'_1 = R_\gamma$. Let $R' = (R'_1, R_{-1})$ and let $y = TTC_{\succ}(R')$.

Claim 4. $y_1 = w_2$ and $y_2 = w_1$.

[11] This is where the restriction to objective indifferences is used. Under general indifferences, this is not necessarily true.

We can apply Lemma 1 to get $y_i = w_i$ for all $i \in (A \cup B) \setminus \{1, 2\}$. We will further show that $y_i = w_i$ for all $i \in C$ such that $w_i P_\gamma w_2$. The logic is similar to Claim 1. Toward a contradiction, suppose $U = \{i \in C : w_i P_\gamma w_2, y_i \neq w_i\} = \{i \in C : w_i P_\gamma w_2, y_i P_\gamma w_i\}$ is non-empty. Note that $1 \notin U$ since $w_2 P_\gamma w_1$. Also note that $2 \notin U$ since $2 \notin C$. Fix $i \in U$ such that $y_i R_\gamma y_{i'}$ for all $i' \in U$; that is, i is the best off member of U. Consider j endowed with y_i; that is, $w_j = y_i$. Since $y_j \neq w_j$, it must be $j \notin (A \cup B) \setminus \{1, 2\}$, since $y_i = w_i$ for all $i \in (A \cup B) \setminus \{1, 2\}$. Then, since $y_j \neq w_j$ and $w_j P_\gamma y_i P_\gamma w_2$, it must be that $j \in U$. This contradicts the assumption that $y_i R_\gamma y_{i'}$ for all $i' \in U$.

We have shown that $y_i = w_i$ for all $i \in (A \cup B) \setminus \{1, 2\}$ and for all $\{i \in C : w_i P_\gamma w_2\}$. Consider the set of remaining agents, $N' = \{1, 2\} \cup \{i \in C : w_2 R_\gamma w_i\}$. Note that since $R_2 = R_\beta$, we have $w_1 P_{2,\succ} w_i$ for all $i \in N'$, $i \neq 1$. Also, by our choice of $\succ_1$, since $R_1 = R_\gamma$ we have $w_2 P_{1,\succ} w_i$ for all $i \in N'$, $i \neq 2$. Therefore, $y_1 = w_2$ and $y_2 = w_1$. Suppose not. Since $\text{TTC}(R_\succ) \equiv \text{TTC}_\succ(R)$, we violate the core property of TTC with respect to the strict preferences $R_\succ$.

Observe that $w_2 I_1 w_1$ and $w_1 P_2 w_2$. Thus, via the misrepresentation R', we have $y_1 I_1 x_i$ but $y_2 P_2 x_2$, showing $\text{TTC}_\succ(R)$ is not GSP. $\square$

Appendix B Matching with capacities and priorities

We briefly note that TTC in the objective indifferences setting is not identical to TTC where the objects have capacities and priorities. Intuitively, in Shapley-Scarf markets with objective indifferences, the fixed tie-breaking rule determines which object an agent points at. Conversely, a priority ranking determines which agent an object points at. Consider the following example.

Example 9. Let the set of schools be $H = \{A, B, C\}$, with C having 2 seats. Let the students be $N = \{a, b, c_1, c_2\}$, where a is "endowed" with A, and so on. Below are a possible priority ranking for the school choice setting and a possible tie-breaking profile for the Shapley-Scarf setting.

A	B	C
a	b	c_1
b	a	c_2
c_1	c_2	a
c_1	c_1	b

School priority for school choice setting

vs

$\succ_a$	$\succ_b$	$\succ_{c_1}$	$\succ_{c_{c_2}}$
c_1	c_2	c_1	c_1
c_2	c_1	c_2	c_2
a	a	a	a
b	b	b	b

Tie-breaking profile for Shapley-Scarf setting

Consider the preference profiles $R = (R_a, R_b, R_{c_1}, R_{c_2})$ and $R' = (R'_a, R'_b, R'_{c_1}, R'_{c_2})$, shown below.

R_a	R_b	R_{c_1}	R_{c_2}
C	C	A	A
A	A	B	B
B	B	C	C

Preference profile R'

R'_a	R'_b	R'_{c_1}	R'_{c_2}
C	C	B	B
A	A	A	A
B	B	C	C

Preference profile R'

Note that under R and R', both c_1 and c_2 have the same preferences. TTC with school priorities results in $(A : c_1, B : c_2, C : ab)$ and $(A : c_2, B : c_1, C : ab)$ under R and R' respectively. Note that c_1 gets his preferred school in either case, since his priority at school C is higher than c_2's priority at school C. In fact, in the school choice setting, since c_1 has higher priority at C than c_2 has, whenever c_1 and c_2 have the same preferences c_1 will weakly prefer his assignment to c_2's assignment. By contrast, in the Shapley-Scarf setting, TTC$_{\succ}$ results in $(A : c_1, B : c_2, C : ab)$ and $(A : c_1, B : c_2, C : ab)$ under R and R' respectively. Now, c_1 does not necessarily always get his top choice when c_1 and c_2 have the same preferences. Under R', c_2 gets his top choice at the expense of c_1, since $c_2 \succ_b c_1$.