Optimizing Profits for Airlines via Strategic Overbooking

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Math 4043 Applied Mathematics

December 2023

Non-refundable reservations allow companies to maximize profits by overselling. For example, event and concert promoters can sell tickets for seats that are not available. They anticipate that some ticket holders will not show up, and then the company can rearrange seats needed. The same can be done with any non-refundable and reserved service or product. This includes car rentals, tours and excursions, hotel reservations, and the example we will explore mathematically, airline seats.

Two hundred seats on a plane shouldn't stop airlines from selling 205 tickets. Airlines are aware that for any given flight, passengers might change plans at the last minute, arrive too late to board their flight, or fail to make it through security. The airline can oversell tickets and rearrange people as necessary before the plane takes off. This approach, of course, runs the risk that more passengers will arrive at the gate intending to board than there are seats on the plane. In this case, the solution is simple: reimburse any extra passengers and provide them with a generous reimbursement to quell any dissatisfaction.

Define the following:

Variables:

- N Seats available
- T Ticket price per passengers
- S Number of tickets sold
- X Passengers arriving at the gate ready to board
- p Probability a passenger makes it to the gate and is ready to board
- R Reimbursement given to passengers who give up their seat

Probability Functions:

- $P_X(x)$: Probability of x passengers arriving at the gate ready to board
- $P_R(r)$: Probability of a passenger accepting reimbursement r

Work with the assumption that the airline has sold tickets the fair market value and that it is approximately 20 minutes until boarding occurs. All passengers who will board the flight have made it to the gate and are ready. If more passengers than seats are available on the flight, then provide these extra passengers with a reimbursement of constant value of R. This allows us to develop a function to determine the flight's profit from ticket sales.

Profit: Overbooked Passengers Receive Equal Reimbursements R

$$\text{Profit} = \begin{cases} S \cdot T - \overbrace{(X - N)}^{\text{Extra Passengers}} \cdot R & \text{if } X > N, \quad \textit{Overbooked passengers} \\ S \cdot T & \text{if } X \leq N, \quad \textit{Enough seats} \end{cases} \tag{1}$$

While it is clear that profits can be maximized by increasing S (tickets sold) and decreasing R (reimbursement given to the passengers), this is not realistic. The airline must provide passengers whose tickets were voided with a reimbursement of at least the amount of the original ticket price T. Establish a policy that the minimum reimbursement, $\min\{R\}$, is the ticket price T, plus some additional amount.

$$\min\{R\} = T + \text{Additional Amount} \tag{2}$$

Note that this reimbursement may include an alternative ticket, pure cash, and a free hotel stay. Whatever the airline decides, the statistician's only concern is the reimbursement's monetary value R.

Assume that the distribution of X passengers arriving at the airport ready to board follows the normal distribution. Each passenger equates to a binary event of whether or not they made to the airport ready. For simplicity, we assume that each of these binary events occur independently.

Expected Profit: Overbooked Passengers Receive Equal Reimbursements R

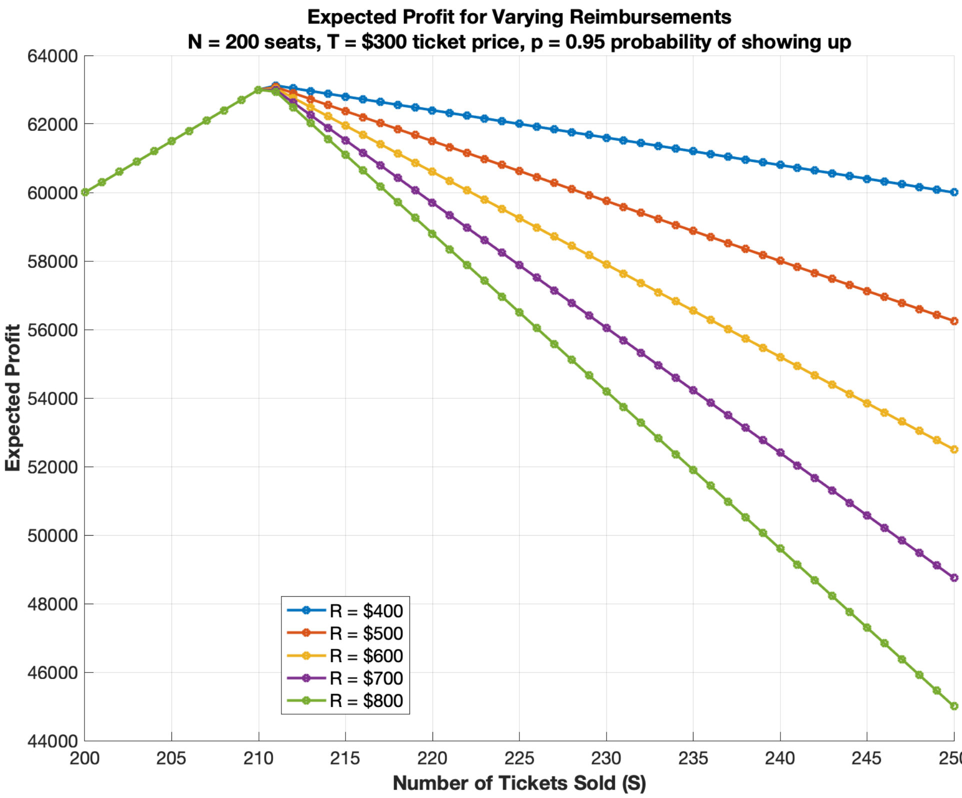
Then, the probability a passenger makes it to the gate ready to board is:

$$P_X(X) = \binom{S}{x} p^x (1-p)^{S-x} \tag{3}$$

Now, the expected profit for the flight is:

$$\begin{aligned} E[\text{Profit}] &= \sum_{x=0}^{S} \overbrace{\text{Profit}}^{\text{From } x \text{ persons}} \cdot P_X(x) \\ &= \sum_{x=0}^{N} S \cdot T \cdot P_X(x) + \sum_{x=N+1}^{S} (S \cdot T - R \cdot (X - N)) \cdot P_X(x) \\ &= \underbrace{\sum_{x=0}^{N} S \cdot T \cdot \binom{S}{x} p^x (1-p)^{S-x}}_{\text{Expected profit from selling } N \text{ tickets for } N \text{ seats}} + \underbrace{\sum_{x=N+1}^{S} (S \cdot T - R \cdot ((X - N))) \cdot \binom{S}{x} p^x (1-p)^{S-x}}_{\text{Expected profit from oversold seats}} \end{aligned} \tag{4}$$

To maximize profits, find S such that $\frac{d}{dS} E[\text{Profit}] = 0$.



N Seats Available $= 200$
T Ticket Price $= \$300$
p P(Arriving) $= 0.95$

Figure 1: S Tickets Sold versus Expected Profit

What about the Airline's Reputation?

Choosing a reimbursement of R that negates the reputational cost is not feasible. Instead, the reputational cost can be diminished to zero by redesigning how the reimbursements work. Instead of forcing passengers to take a reimbursement, create a bidding system. Airline staff will announce a reimbursement package of value R, and passengers may willingly accept it. If not enough passengers accept, the reimbursement's value is raised. This process continues until $X - N$ passengers accept a reimbursement.

A caveat is that when passengers accept a reimbursement only to see another passenger accept a higher reimbursement, they will feel cheated, thus making this face-saving exercise of giving large reimbursements pointless. So, we fix this by providing all overbooked passengers who accepted a reimbursement, the highest reimbursement offered for this flight.
For example, imagine the airline is running the reimbursement bidding system, and a passenger accepts a $500 reimbursement, but the airline needs another passenger to accept it. They raise the value of the reimbursement, and an additional passenger takes it. Now, every passenger will receive this highest reimbursement.

Let $P_R(r)$ be the probability of the reimbursement r being accepted by any passenger. One decent generalization of P_R is of the form $\tanh(R)$ as follows:

$$P_R(R = x) = \frac{1 + \tanh(\lambda(x - \text{shift} \cdot T))}{2} \tag{5}$$

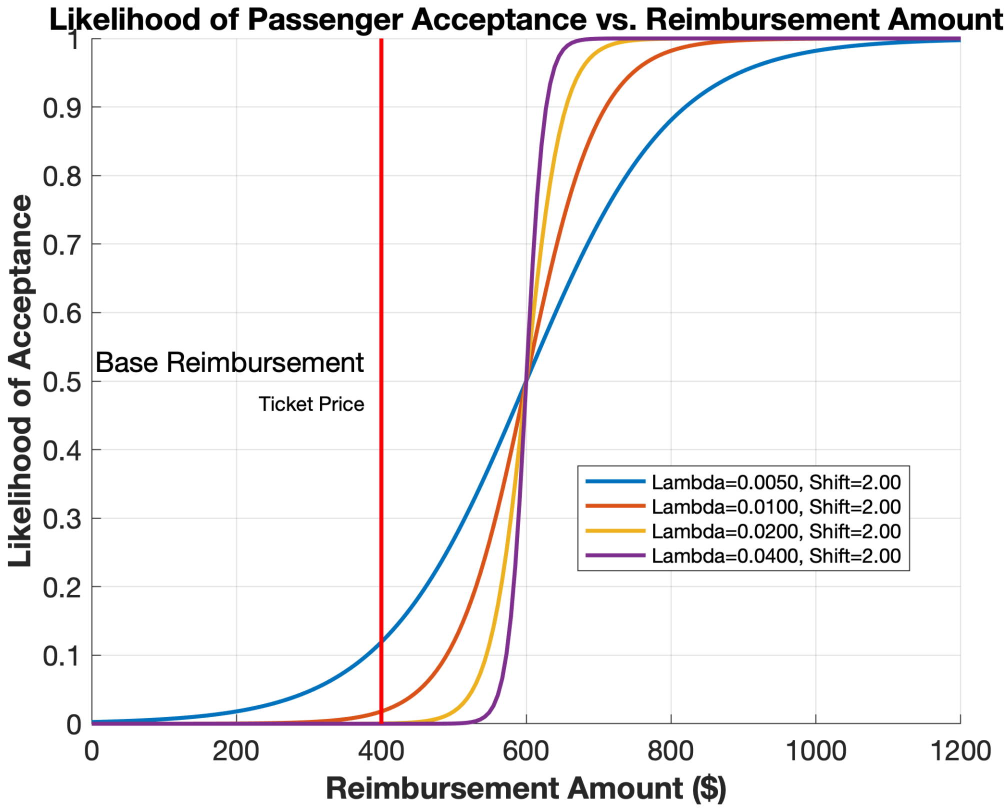


Figure 2: Parameters are varied to reflect how likely passengers are to voluntarily accept reimbursements.

Airlines determine the function that best describes how likely their passengers are to accept a reimbursement.

We will allow passengers to voluntarily accept the reimbursements in the form of a bidding process. Start the reimbursement bidding at R_0 and increase it by ΔR until $X - N$ passengers accept the reimbursement. Recursively define R_k, the k^{th} reimbursement amount, as follows:

$$R_k = R_0 + k \cdot \Delta R \tag{6}$$

Expected Profit: Overbooked Passengers Voluntarily Accept Reimbursements $\boldsymbol{R}$

Now we can define the expected profit:

$$
\begin{aligned}
E[\text{Profit}] &= S \cdot T - E[(X-N)\cdot R] \\
&= S \cdot T - \sum_{x=N+1}^{S} P_X(x) \cdot \boxed{EL(x)}
\end{aligned}
$$

Expected Loss $\boxed{EL}$

- P_i is the probability of a passenger accepting the reimbursement R_i, $P_R(R_i)$
- R_i is the amount of reimbursement offered in the i-th iteration
- X is the number of passengers who show up with the intention to board.

We need to recursively define expected loss. Our expected loss will be the product of: the probability P_0 of accepting R_0, the number of people who we need to accept R_0 (which is $X-N$ passengers), and the actual reparation R_0. But, we need to consider all scenarios where a reparation was not accepted. For the next step, multiply the probability of the first event not happening $(1-P_0)$ by $P_1 \cdot (X-N) \cdot R_1$ and move the recursive logic forward.

$$
\boxed{EL(X\text{passengers})} = \overbrace{\overbrace{P_R(R_0)(X-N)R_0}^{\text{Event 1}} + \overbrace{(1-P_R(R_0))}^{\overline{P(\text{Event 1})}}\,\overbrace{P_R(R_1)(X-N)R_1}^{\text{Event2}} + \overbrace{(1-P_R(R_0))}^{\overline{\text{Event 1}}}\,\overbrace{(1-P_R(R_1)}^{\overline{\text{Event 2}}}\,\overbrace{P_R(R_2)(X-N)R_2}^{\text{Event 3}}}^{\sum \text{ Expected Loss from each Event}} +
$$

$$
(1-P_R(R_0)(1-P_R(R_1))(1-P_R(R_2))P_R(R_3)(X-N)R_3 + \ldots
$$

$$
= (X-N)\left(P_R(R_0)R_0 + (1-P_R(R_0))\left(P_R(R_1)R_1 + (1-P_R(R_1))\left(P_R(R_2)R_2 + (1-P_R(R_2))\left(P_R(R_3)R_3 + \ldots\right)\right)\right)\right)
$$

$$
= \boxed{\sum_{i=0}^{\infty}\left(\prod_{k=0}^{i-1}(1-P_R(R_k))\right)P_R(R_i)(X-N)R_i}
$$

Then

$$
= S \cdot T - \sum_{x=N+1}^{S} P_X(x) \cdot \boxed{\left(\sum_{i=0}^{\infty}\left(\prod_{k=0}^{i-1}(1-P_R(R_k))\right)P_R(R_i)(x-N)R_i\right)} \tag{7}
$$

This formula can be implemented into code using an expected profit function and an additional recursively defined expected loss function.

This recursive expected loss is encased in a summation indexed by i. While the theoretical result is a summation $i \to \infty$, these values i represent the bidding rounds. It is unlikely an airline would want to announce reimbursements more than a few times, and for any $P_R(r)$, the rounds past $i=3$ lead to an indistinguishable change in expected loss anyway. Thus, $i=4$ is a reasonable summation limit.
To understand this recursion, start at the outermost level $i=4$, representing the 4^{th} bidding processes that could be needed to decrease the number of passengers boarding the flight. You must determine the expected loss at this round, 4, and then add it to the expected loss found at round 3, all the way to round, or recursive level, zero.

Level 4 Loss: Level 3 Loss + $\prod_{k=0}^{3}(1-P_R(R_k)) \cdot P_R(R_4) \cdot (x-N) \cdot R_4$

Level 3 Loss: Level 2 Loss + $\prod_{k=0}^{2}(1-P_R(R_k)) \cdot P_R(R_3) \cdot (x-N) \cdot R_3$

Level 2 Loss: Level 1 Loss + $\prod_{k=0}^{1}(1-P_R(R_k)) \cdot P_R(R_2) \cdot (x-N) \cdot R_2$

Level 1 Loss: Level 0 Loss + $(1-P_R(R_0)) \cdot P_R(R_1) \cdot (x-N) \cdot R_1$

Level 0 Loss: $P_R(R_0) \cdot (x-N) \cdot R_0$

Base Case: Loss $= 0$

Figure 3: Recursion from $i=4$

This is the actual code for the recursive expected loss function:

Parameter description for the expectedLoss function

```
j:      Number of overbooked passengers
N:      Number of available seats
P_R:    Probability function for the likelihood of any passenger accepting reimbursement R
lambda: Parameter for the P_R function
shift:  Horizontal shift parameter for the P_R function
R_0:    Initial reimbursement amount
deltaR: Increment in reimbursement amount each round
T:      Ticket price
level:  Recursive level for the loss calculation
```

Function to calculate expected loss due to overbooking.

```
function EL = expectedLoss(j, N, P_R, lambda, shift, R_0, deltaR, T, level)
    % Calculate the expected loss using the recursive loss function
    EL = (j - N) * recursiveLoss(P_R, lambda, shift, R_0, deltaR, T, level);
end
```

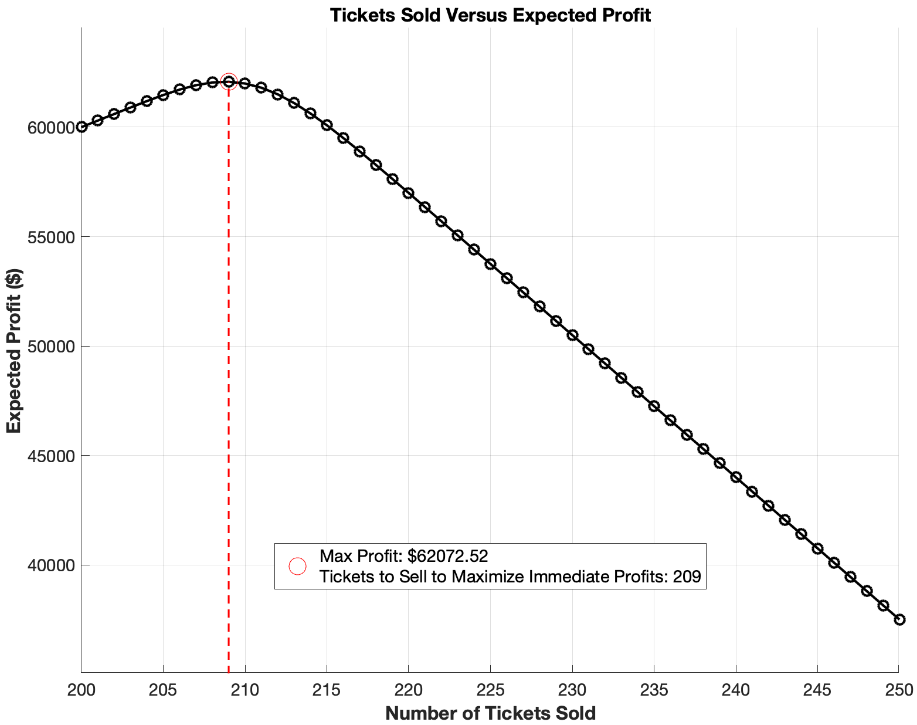
Recursive function to assist in calculating loss due to overbooking.

```
function loss = recursiveLoss(P_R, lambda, shift, R_0, deltaR, T, level)
    if level < 0 % Note: We start the recursion at level zero.
        loss = 0; % Base case
    else
        % Calculate the probability of a passenger accepting the current reimbursement offer
        P_R_current = P_R(R_0 + level * deltaR, lambda, shift, T);

        % Calculate the cumulative loss up to the previous level
        loss_previous = recursiveLoss(P_R, lambda, shift, R_0, deltaR, T, level - 1);

        % Combine the loss at the current level with the loss from the previous level
        loss = (1 - P_R_current) * loss_previous + P_R_current * (R_0 + level * deltaR);
    end
end
```

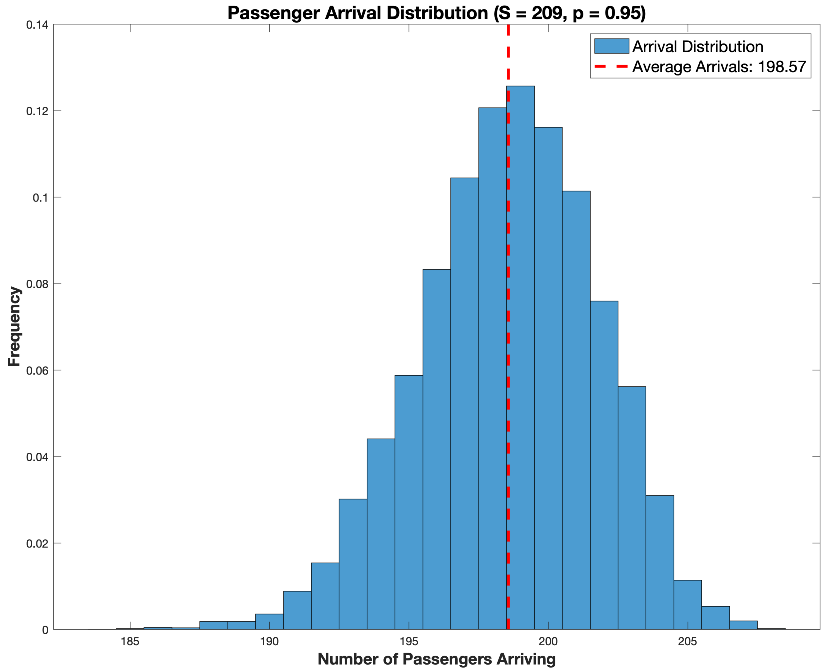
The following represents expected profit for varying numbers of tickets sold S.



Seats :	$N = 200$	Reparation Increment :	$\Delta R = \$100$
Ticket Price :	$T = \$300$	Acceptance Parameter :	$\lambda = 0.01$
Show-up Probability :	$p = 0.95$	Shift Parameter :	$\text{shift} = 2$
Base Reparation :	$R_0 = T + 100 = 400$		

Figure 4: S Tickets Sold versus Expected Profit With Bidding System

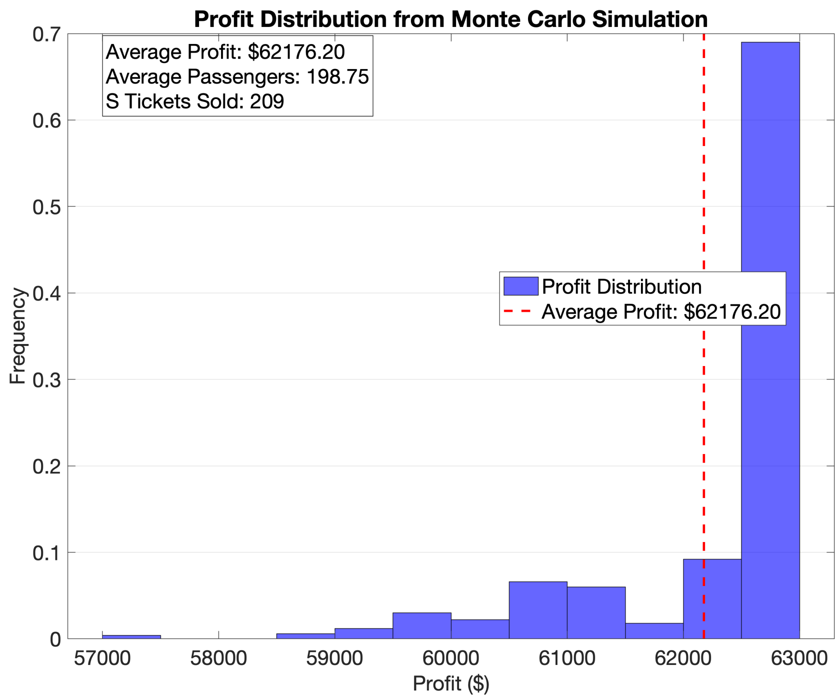
Using this system, we are able to identify the optimal number of S tickets to sell. In this case, we found it is most optimal to sell 209 tickets instead of 200. To understand this result, run a simulation using the likelihood of passengers arriving at the airport. It is clear the total number of passengers arriving for each flight is normally distributed using a Monte Carlo simulation.



N Seats Available $= 200$
T Ticket Price $= 300$
p Likelihood of arriving $= 0.95$
$\boldsymbol{S}$ **Tickets Sold** $\mathbf{= 209}$

Figure 5: Number of passengers arriving when $S = 209$ tickets are sold

Now, plot a Monte Carlo simulation for the expected profit. For each trial, sell the optimal number of tickets $S_{optimal} = 209$ and see the expected yield over $1,000$ simulations with randomness described by P_X ad P_R.



N Seats Available $= 200$
T Ticket Price $= 300$
p Likelihood of arriving $= 0.95$
$\boldsymbol{S}$ **Tickets Sold** $\mathbf{= 209}$

Figure 6: Expected profit when $S = 209$ tickets are sold

Change the number of tickets S sold to be higher and see how the expected profits decrease, since more passengers will have to be paid a reimbursement.

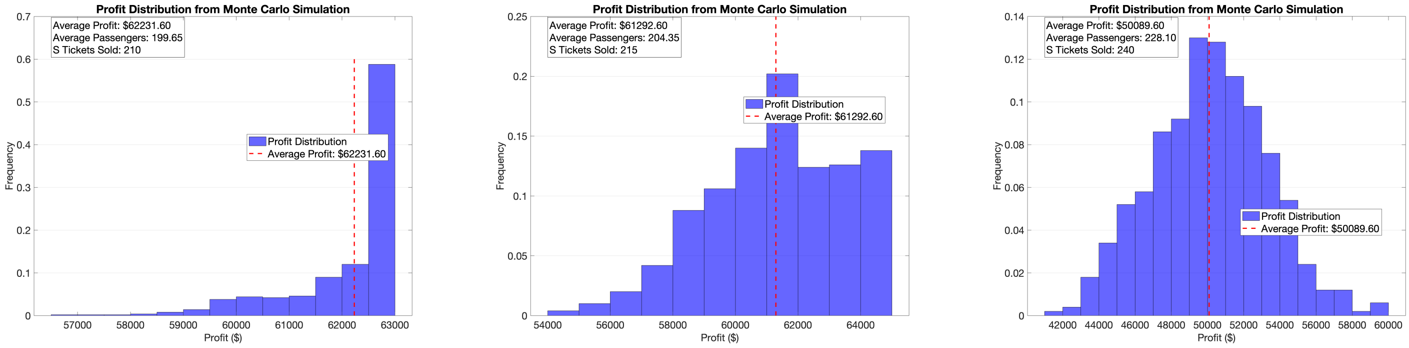


Figure 7: Expected profit for varying numbers of sold tickets

The distribution of expected profits skews right (decrease in expectation) as the number of tickets sold exceeds $S_{optimal}$.

Airlines maximize profits by relying on chance and customer greed. Reputation can be maintained while simultaneously overbooking flights. This paper explored how expected profits could be mathematically derived when this bidding system for reimbursements is implemented. These results can be expanded to and utilized in many industries where products are explicitly non-refundable and reserved.