Lyapunov Analysis of the Natural System

$$V(x) := x^T P x \quad \text{where } P \text{ solves} \quad J^T P + P J = -I$$

$$\dot{V}(x) = -x^T (1 - 2P(A \circ X_r))x$$

$$\|x\|_2 < r := \frac{1}{2\sqrt{n}\,\|P\|_{i,2}\,\|A\|_F} \implies -\dot{V}(x) \text{ LPDF}$$

$$x \in \Omega : \{\bar{x} \mid V(\bar{x}) < \lambda_{min}(P)r^2\}$$

Translation to Equilibria of Interest

$$z = x - f \implies \dot{z} = (z + f) \circ (r + A(z + f))$$

$$\dot{z} = (f \circ A)z + (z \circ A)z = J'z + g(z)$$

$$\boxed{z \in \Omega_z : \{\bar{z} \mid V(\bar{z}) < \lambda_{min}(P_z)r_z^2\}}$$