Safe Envelopes for High-Dimension Reachability by Linear Hopf Solutions with Antagonistic Error

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November 6, 2024

Contents

Preliminaries - Hamilton-Jacobi Reachability

System

Consider a system,

$$\dot{x} = f_x(x, \tau) + h_1(x, \tau)u + h_2(x, \tau)d \triangleq f(x, u, d, \tau), \tag{1}$$

where $\tau \in [t, T]$, u and d are drawn from convex sets $\mathcal{U} \subset \mathbb{R}^{n_u}$, $\mathcal{D} \subset \mathbb{R}^{n_d}$, and (23) is Lipschitz continuous in (x, u, d) and continuous in t.

Differential Game Problem Statement

Suppose a game is defined by,

$$P(x, u(\cdot), d(\cdot), t) = J(x(T)) + \int_t^T L(u(\tau), d(\tau)) d\tau. \tag{2}$$

where Player I (actions u) and Player II (actions d) compete by optimizing (2), in a zero-sum fashion.

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where Player I (actions u) and Player II (actions d) compete by optimizing (2), in a zero-sum fashion.

Reach game $\triangleq$ (Player I, Player II) have objectives $(\min_u P,\ \max_d P)$
Avoid game $\triangleq$ (Player I, Player II) have objectives $(\max_u P,\ \min_d P)$

(Note, we use $\cdot, \cdot^-$ to define analogous *Reach* & *Avoid* game objects resp.)

Differential Game Problem Statement

Given a target set $\mathcal{T} \in \mathbb{KR}^{n_x}$ to reach or avoid, let the terminal cost $J : \mathbb{R}^{n_x} \mapsto \mathbb{R}$ be defined by

$$\begin{cases} J(x) < 0 & \text{for} \quad x \in \mathcal{T} \setminus \partial\mathcal{T} \\ J(x) = 0 & \text{for} \quad x \in \partial\mathcal{T} \\ J(x) > 0 & \text{for} \quad x \notin \mathcal{T} \end{cases}$$

such that is is convex and lower semicontinuous.

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such that is is convex and lower semicontinuous.

Running costs $L, L^- : \mathbb{R}^{n_u} \times \mathbb{R}^{n_d} \mapsto \mathbb{R}$ solely constrain the inputs,

$$\begin{aligned} L(u,d) &= \mathcal{I}_{\mathcal{U}}(u) - \mathcal{I}_{\mathcal{D}}(d), \\ L^-(u,d) &= \mathcal{I}_{\mathcal{D}}(d) - \mathcal{I}_{\mathcal{U}}(u), \end{aligned} \tag{3}$$

where $\mathcal{I}_{\mathcal{C}}(c) = \{0 \text{ if } c \in \mathcal{C}, +\infty \text{ else}\}$.

Game Cost and Value Properties

For trajectory $x(\cdot)$ satisfying the dynamics, if $u(\cdot) \in \mathbb{U} : [t, T] \mapsto \mathcal{U}$, $d(\cdot) \in \mathbb{D} : [t, T] \mapsto \mathcal{D}$,

$$P(x, u(\cdot), d(\cdot), t) \leq 0 \iff x(T) \in \mathcal{T}. \tag{4}$$

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Consider the optimal values of the games $V, V^- : \mathbb{R}^{n_x} \times (-\infty, T] \mapsto \mathbb{R}$ defined

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These analogously satisfy,

$$\begin{aligned} V(x,t) \leq 0 &\iff x \in \mathcal{R}(\mathcal{T}, t), \\ V^{-}(x,t) \leq 0 &\iff x \in \mathcal{R}^{-}(\mathcal{T}, t). \end{aligned} \tag{6}$$

Reach and Avoid Sets

where $\mathcal{R}(\mathcal{T}, t), \mathcal{R}^{-}(\mathcal{T}, t) \in \mathbb{KR}^{n_x}$ are the *Reach* and *Avoid* sets resp., formally given by

$$\begin{aligned}\mathcal{R}(\mathcal{T}, t) \triangleq \{x \mid\ & \exists u(\cdot) \in \mathbb{U}\ \forall d(\cdot) \in \mathbb{D} \text{ s.t.} \\ & x(\cdot) \text{ satisfies } (23) \wedge x(T) \in \mathcal{T}\}, \\ \mathcal{R}^{-}(\mathcal{T}, t) \triangleq \{x \mid\ & \exists d(\cdot) \in \mathbb{D}\ \forall u(\cdot) \in \mathbb{U} \text{ s.t.} \\ & x(\cdot) \text{ satisfies } (23) \wedge x(T) \in \mathcal{T}\}.\end{aligned} \tag{7}$$

Reachability versus Feasibility

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In contrast, the feasible set $\mathcal{S} \in \mathbb{KR}^{n_x}$ is given by,

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and captures all trajectories which might arrive at the target, irrespective of player objectives. This set is useful in this work because $\mathcal{S} \supset \mathcal{R}, \mathcal{R}^-$.

Hamilton-Jacobi-Bellman PDE

Theorem (Evans 84, (1))

The value function V defined in (5) is the viscosity solution to,

$$\begin{aligned} \frac{\partial V}{\partial t} + H(x, \nabla_x V, t) = 0 \qquad & \text{on } \mathbb{R}^{n_x} \times [t, T], \\ V(x, T) = J(x(T)) \qquad & \text{on } \mathbb{R}^{n_x} \end{aligned} \tag{10}$$

where the Hamiltonian $H : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times [t, T] \mapsto \mathbb{R}$ is

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Additionally, the optimal control is given by,

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This identically applies to V^- with $H^-(x, p, t) = \max_u \min_d p \cdot f(x, u, d, t)$.

Preliminaries - Hopf Formula

Forwards to Backwards change

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$$\begin{aligned} -\frac{\partial \phi}{\partial t} + H(x, \nabla_x \phi, t) = 0 \qquad & \text{on } \mathbb{R}^{n_x} \times [0,t], \\ \phi(x,0) = J(x) \qquad & \text{on } \mathbb{R}^{n_x} \end{aligned} \tag{13}$$

with Hamiltonian

$$H(x,p,t) = \max_{u \in \mathcal{U}} \min_{d \in \mathcal{D}} -p \cdot f(x,u,d,T-t). \tag{14}$$

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For systems lacking state dependence $f(x,u,d,t) \equiv f(u,d,t)$, H also lacks state dependence $H(x,p,t) \equiv H(p,t)$. In this setting, there is a solution to the PDE ...

The Hopf Formula

Theorem (Rublev 00, (2))

Assume that $J(x)$ is convex and that $H(t,p)$ is pseudoconvex in p and satisfies (B.i-B.iii) in [(2)], then the minimax-viscosity solution of (13) is given by

$$\phi(x,t) = -\min_{p \in \mathbb{R}^{n_x}} \left\{ J^\star(p) - x \cdot p + \int_0^t H(p,\tau)d\tau \right\} \tag{15}$$

where $J^\star(p) : \mathbb{R}^{n_x} \to \mathbb{R} \cup \{+\infty\}$ is the Fenchel-Legendre transform (i.e. convex-conjugate) of a convex, proper, lower semicontinuous function $J : \mathbb{R}^{n_x} \to \mathbb{R}$ defined by

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Remark. The minimax-viscosity and viscosity solutions coincide when $H(p,\tau)$ is convex in p for $\tau \in [0,t]$.

In an *Avoid* Game, minimax viscosity is a safe estimate

Remark. The minimax-viscosity solution given by the Hopf formula is always less than or equal to the viscosity solution [(3)].

Hence, in an avoid problem, ϕ^- (the Avoid Hopf value) guarantees a conservative lower bound of V^- for any H^- satisfying Thm. 2. [1]

[1] This yields a potentially non-convex objective, thus one should employ ADMM, known to solve similar non-convex problems [(4)].

Hopf with Linear Systems

Requiring "state-independence" is restrictive. It happens that, however, any linear system,

$$\dot{x} = A(\tau)x + B_1(\tau)u + B_2(\tau)d \tag{17}$$

may be transformed with the mapping $z(\tau) \triangleq \Phi(\tau)x(\tau)$ where $\Phi(\tau)$ defined by the *fundamental matrix* $\dot{\Phi} = -\Phi(\tau)A(\tau), \Phi(0) = I$. If $A(\tau) \equiv A$ then $\Phi(\tau) = \exp(-\tau A)$.

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In the $\mathcal{Z}$-space,

$$\begin{aligned} z(\tau) = \Phi(\tau)x(\tau) \implies \dot{z} &= \dot{\Phi}(\tau)x + \Phi(\tau)\dot{x} \\ &= \Phi(\tau)(-A(\tau)x) + \Phi(\tau)\dot{x} \\ &= \Phi(\tau)(B_1(\tau)u + B_2(\tau)d) \end{aligned} \tag{18}$$

and thus may be solved by the Hopf formula.

Hopf, Linear Systems, $\mathcal{Z}$-space

After the change(s) of variables, the *Reach* $\mathcal{Z}$-Hamiltonian takes the form,

$$\begin{aligned} H_{\mathcal{Z}}(p,t) &= \max_{u\in\mathcal{U}} \min_{d\in\mathcal{D}} -p\cdot\Phi(T-t)\big(B_1(T-t)u + B_2(T-t)d\big), \\ &= \max_{u\in\mathcal{U}}\{p\cdot R_1(\tau)u\} - \max_{d\in\mathcal{D}}\{p\cdot R_2(\tau)d\} \\ &= \mathcal{I}^{\star}_{\mathcal{U}}(R_1^{\top}(\tau)p) - \mathcal{I}^{\star}_{\mathcal{D}}(R_2^{\top}(\tau)p) \end{aligned} \tag{19}$$

where $R_i(\tau) \triangleq -\Phi(T-t)B_i(T-t)$.

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where $R_i(\tau) \triangleq -\Phi(T-t)B_i(T-t)$.

If $\mathcal{U}$ and $\mathcal{D}$ are defined by norms, then $\mathcal{I}^{\star}_{\mathcal{U}}$ and $\mathcal{I}^{\star}_{\mathcal{D}}$ are the corresponding dual norms. This is useful for fast computation and proving the convexity of H (*but may be ignored, if confusing*).

Safe Envelopes by Linear Hopf Solutions

Running Example (Van der Pol)

To illustrate theoretical results, we consider the system,

$$\dot{x} = \begin{bmatrix} x_1 \\ \mu(1 - x_1^2)x_2 - x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u + d) \tag{20}$$

with $\mu = 1$, and inputs

$$u \in \mathcal{U} \triangleq \{|u| \leq 1\}, \quad d \in \mathcal{D} \triangleq \{|d| \leq 1/2\}, \tag{21}$$

and a target set defined by,

$$\mathcal{T} \triangleq \{||(x - c_{\mathcal{T}})||_{Q_{\mathcal{T}}} \leq 1\}, \tag{22}$$

for parameters $c_{\mathcal{T}} \in \mathbb{R}^{n_x}, Q_{\mathcal{T}} \in \mathbb{S}^{n_x}_{++}$. Note,
$||(x - c_{\mathcal{T}})||_{Q_{\mathcal{T}}} = ||(x - c_{\mathcal{T}})^\top Q_{\mathcal{T}}^{-1}(x - c_{\mathcal{T}})||_2$.

Comparing Nonlinear and Linear Systems

Recall, in general, the system takes the form,

$$\dot{x} = f_x(x,\tau) + h_1(x,\tau)u + h_2(x,\tau)d \triangleq f(x,u,d,\tau). \tag{23}$$

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Consider a continuous, linear function

$$l(x,u,d,\tau) \triangleq A(\tau)x + B_1(\tau)u + B_2(\tau)d, \tag{24}$$

which will serve as an approximate model of (23) which we may solved with Hopf.

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which will serve as an approximate model of (23) which we may solved with Hopf.

Across $\mathbb{R}^{n_x}$, their difference may be unbounded. But, in the feasible set $\mathcal{S}$, since $(\mathcal{T} \times \mathcal{U} \times \mathcal{D} \times [t,T])$ is compact, by Filippov's Lemma (5),

$$\delta^* \triangleq \max_{x\in\mathcal{S}} ||(f-l)(x,u,d,\tau)|| \quad (u,d) \in \mathcal{U} \times \mathcal{D} \tag{25}$$

is finite.

Comparing Nonlinear and Linear Systems

Hence, for $x \in \mathcal{S} \subset \mathbb{R}^{n_x}$,

$$\dot{x} = f(x,u,d) = l(x,u,d) + \varepsilon \quad ||\varepsilon|| \leq \delta^*, \tag{26}$$

and

$$H(x,p,t) = H_l(x,p,t) + p \cdot \varepsilon, \tag{27}$$

where $H_l(x,p,t) \triangleq \min_{u\in\mathcal{U}} \max_{d\in\mathcal{D}}\{p \cdot l(x,u,d)\}$.

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Hence, for $x \in \mathcal{S} \subset \mathbb{R}^{n_x}$,

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Although ε is nonlinear and state-dependent, we upper-bound this relationship to use of the Hopf formula, assuming the worst for safety.

Game with Antagonistic Error

Consider another antagonistic player with actions

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The corresponding Hamiltonians for the *Reach* and *Avoid* games resp. are,

$$H_{\delta^*}(x,p,t) \triangleq H_l(x,p,t) + \max_{\varepsilon \in \mathcal{E}} p \cdot \varepsilon, \tag{29}$$

$$H^-_{\delta^*}(x,p,t) \triangleq H_l(x,p,t) + \min_{\varepsilon \in \mathcal{E}} p \cdot \varepsilon. \tag{30}$$

Remark. $H_{\delta^*} = \max_{\varepsilon \in \mathcal{E}} H$ and hence H_{δ^*} is the *envelope* of the Hamiltonian H in the parlance of optimization and PDEs (6).

Safe Upper & Lower Values for *Reach* & *Avoid* games

Lemma

For $V(x,t)$ defined in (5), arising from target $\mathcal{T}$ and dynamics f with players confined to $\mathcal{U}, \mathcal{D}$, then

$$V(x,\tau) \leq V_{\delta^*}(x,\tau) \quad x \in \mathcal{S}, \tau \in [t,T] \tag{31}$$

where $V_{\delta^*}(x,t)$ is defined as the value of the Reach game with dynamics l and an additional antagonistic player, confined to $\mathcal{E}$, s.t. $V_{\delta^*}(x,t)$ is the solution to the PDE in (10) with Hamiltonian H_{δ^*}.

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In the Avoid game, analogously

$$V(x,\tau) \geq V^-_{\delta^*}(x,\tau) \quad x \in \mathcal{S}, \tau \in [t,T] \tag{32}$$

s.t. $V^-_{\delta^*}(x,t)$ is the solution to the PDE in (10) with Hamiltonian $H^-_{\delta^*}$.

Safe Upper & Lower Values for *Reach* & *Avoid* games

Proof.

From [(1)], the viscosity solution corresponds to

$$V(x,T) = \inf_x \left\{ J(x(T)) + \int_t^T \mathcal{L}(x(\tau), f(x(\tau))d\tau \right\}$$

$$V_{\delta^*}(x,T) = \inf_x \left\{ J(x(T)) + \int_t^T \mathcal{L}_{\delta^*}(x(\tau), f(x(\tau))d\tau \right\}$$

where $\mathcal{L}$, $\mathcal{L}_{\delta^*}$ are the Lagrangians of the system,

$$\mathcal{L} = \inf_x \{x \cdot p + H(x,p)\}, \; \mathcal{L}_{\delta^*} = \inf_x \{x \cdot p + H_{\delta^*}(x,p)\}.$$

By definition $H \leq H_{\delta^*}$, so $\forall p$, $\mathcal{L} \leq x \cdot p + H(x,p) \leq x \cdot p + H_{\delta^*}(x,p)$, and thus $\mathcal{L} \leq \mathcal{L}_{\delta^*}$.

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$$V_{\delta^*}(x,T) = \inf_x \left\{ J(x(T)) + \int_t^T \mathcal{L}_{\delta^*}(x(\tau), f(x(\tau))d\tau \right\}$$

and thus $\mathcal{L} \leq \mathcal{L}_{\delta^*}$. Then,

$$\begin{aligned} \forall x,\ V(x,t) &\leq J(x(T)) + \int_t^T \mathcal{L}(x(\tau), f(x(\tau)) \\ &\leq J(x(T)) + \int_t^T \mathcal{L}_{\delta^*}(x(\tau), f(x(\tau)). \end{aligned}$$

Hence, $V(x,t) \leq V_\delta^*(x,t)$. □

Subsets & Supersets for *Reach* & *Avoid* games

Corollary

In the Reach game, at any time $\tau \in [t, T]$,

$$\mathcal{R}_{\delta^*}(\mathcal{T}, \tau) \subset \mathcal{R}(\mathcal{T}, \tau) \tag{33}$$

and for any $x \in \mathcal{R}_{\delta^*}(\mathcal{T}, \tau)$, $u^*_{\delta^*}(\cdot)$ *derived from (12) with* V_{δ^*} *will yield* $x(T) \in \mathcal{T}$ *under the true, nonlinear dynamics (23) despite any disturbances* $d(\cdot) \in \mathbb{D}$.

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Analogously, for the Avoid game,

$$\mathcal{R}^-_{\delta^*}(\mathcal{T}, \tau) \supset \mathcal{R}^-(\mathcal{T}, \tau) \tag{34}$$

and $\forall x \notin \mathcal{R}^-_{\delta^*}(\mathcal{T}, \tau)$, *the trajectory* $x(\cdot)$ *arising from* $u^*_{\delta^*}(\cdot)$ *solved from (12) with* $V^-_{\delta^*}$ *will yield* $x(T) \notin \mathcal{T}$ *under the true, nonlinear dynamics despite any* $d(\cdot) \in \mathbb{D}$.

Subsets & Supersets for *Reach* & *Avoid* games

Proof.

The crucial property of the game value (6) guarantees,

$$x \in \mathcal{R}_{\delta*}(\mathcal{T}, \tau) \iff V_{\delta*}(x, \tau) \leq 0 \implies V(x, \tau) \leq 0 \iff x \in \mathcal{R}(\mathcal{T}, \tau).$$

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By definition $x \in \mathcal{R}_{\delta *}(\mathcal{T}, \tau)$ implies

$$\begin{aligned} \exists u^*_{\delta *}(\cdot) \in \mathbb{U}, \forall \varepsilon(\cdot) \in \mathcal{E}, \forall d(\cdot) \in \mathbb{D}, \text{ s.t.} \\ x(T) \in \mathcal{T} \text{ and } \dot{x} = l(x, u^*_{\delta *}, d) + \varepsilon. \end{aligned}$$

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Assuming that $\mathcal{S}$ is sufficiently large, for the trajectory $x(\cdot)$ arising from $u^*_{\delta*}(\cdot)$,

$$\forall \tau \in [t, T], x(\tau) \in \mathcal{R}_{\delta*}(\mathcal{T}, \tau) \subset \mathcal{R}(\mathcal{T}, \tau) \subset \mathcal{S},$$

hence, $\exists \varepsilon(\cdot) \subset \mathcal{E}$ s.t.

$$\dot{x} = l(x, u^*_{\delta*}, d) + \varepsilon = f(x, u^*_{\delta*}, d).$$

□

Safe Envelopes in the VanderPol System

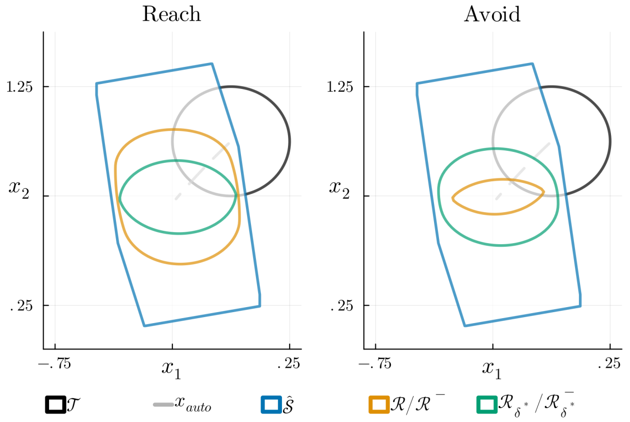

Figure: For a target $\mathcal{T}$ (black), true reachable sets $\mathcal{R}/\mathcal{R}^-$ (gold) are solved for a *Reach* and *Avoid* game resp. at $t = [0.26]$. The corresponding reachable sets of the safe envelope $\mathcal{R}_{\delta*}/\mathcal{R}^-_{\delta*}$ (green) based on the constant Taylor-series linearization error are also solved. The feasible sets $\mathcal{S}$ (blue) are over-approximated via DI methods (7) (8).

Limitations

Convexity of the δ^*-Hamiltonian

To ensure the δ^*-minimax solution is not below the true value in a *Reach* game, convexity of the δ^*-Hamiltonian is required.

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Assume the linear Hamiltonians H_l, H_l^-, are convex. These are transformed by the addition of error s.t.,

$$\begin{aligned} H_{l,\mathcal{Z}} &= \mathcal{I}^\star_{\mathcal{U}}(R_1^\top(t)p) - \mathcal{I}^\star_{\mathcal{D}}(R_2^\top(t)p) \Rightarrow H_{\delta^*,\mathcal{Z}} = H_{l,\mathcal{Z}} - \mathcal{I}^\star_{\mathcal{E}}(-\Phi(T-t)^{-\top}p) \\ H^-_{l,\mathcal{Z}} &= \mathcal{I}^\star_{\mathcal{D}}(R_2^\top(t)p) - \mathcal{I}^\star_{\mathcal{U}}(R_2^\top(t)p) \Rightarrow H^-_{\delta^*,\mathcal{Z}} = H^-_{l,\mathcal{Z}} + \mathcal{I}^\star_{\mathcal{E}}(-\Phi(T-t)^{-\top}p). \end{aligned} \tag{35}$$

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1. *Reach* game, H_{δ^*} will not necessarily be convex and must be checked for a specific definition of $\mathcal{U}, \mathcal{D}$ and observed error $\mathcal{E}$.

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Nonlinear $\dot{x}_i$ must have control in 2P *Reach*

Corollary

Any state x_i*, with evolution* $\dot{x}_i = f_i(x, u, d)$ *s.t.* $\delta_i^* = |f - l_i| > 0$ *and* $\forall u \in \mathcal{U}, f_i(x, u, d) = f_i(x, d)$*, will have a non-convex hamiltonian.*

Thus, in a two-player Reach game, any state with nonlinear f_i *and thus* $\delta_i^* > 0$ *must have at least some control, or the guarantees of Lemma 3 will not hold.*

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Note, the dubin's car is thus immediately problematic. On the other hand, integrator chains are fine.

Poor Linear Models are Valueless

Regardless of convexity, a large maximum error will be too conservative for meaningful results (empty *Reach* set, infinite *Avoid* set).

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Remark. Lem. 3 does not guarantee that $\mathcal{R}_{\delta^*}(\mathcal{T}, t)$ is non-empty. In fact, mapping (26) to the form (19) yields a simple heuristic.

E.g. if $\mathcal{U}$ is defined by a ball with $Q \triangleq r_{\mathcal{U}} I$ and $\exists \tau$ for $t < \tau$,

$$\delta^* > r_{\mathcal{U}} ||B_1^\top \Phi(t)^{-\top}|| \implies \lim_{t \to -\infty} \mathcal{R}_{\delta^*}(\mathcal{T}, t) = \emptyset, \tag{36}$$

despite that $\mathcal{R}(\mathcal{T}, t) \neq \emptyset$ in general. The condition ensures a globally-positive Hamiltonian $H_{\delta^*}(x, p, t) \geq 0$.

Methods for Tighter Envelopes

Time-varying Feasibility and Error

To use the state-independent Hopf formula, the error δ^* cannot be a function of x, however, we may define a time-varying error bound $\delta^*(\tau) : [t, T] \mapsto \mathbb{R}$ that implicitly captures a spatial relationship for a tighter bound on the error.

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Corollary

(Time-varying Error) Let the maximum antagonistic error be defined by,

$$\delta^*(\tau) \triangleq \max_{x \in \mathcal{S}(\tau)} ||(f - l)(x, u, d)||, \tag{37}$$

where $\mathcal{S}(\tau) : [t, T] \mapsto \mathbb{KR}^{n_x}$ *maps a time to the compact set containing all trajectories for time interval* $[\tau, T]$*.*

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where $\mathcal{S}(\tau) : [t, T] \mapsto \mathbb{KR}^{n_x}$ maps a time to the compact set containing all trajectories for time interval $[\tau, T]$. Then for reach and avoid games resp.,

$$\begin{aligned} \mathcal{R}_{\delta^*}(\mathcal{T}, t) \subset \mathcal{R}_{\delta^*(\tau)}(\mathcal{T}, t) \subset \mathcal{R}(\mathcal{T}, t), \\ \mathcal{R}^-_{\delta^*}(\mathcal{T}, t) \supset \mathcal{R}^-_{\delta^*(\tau)}(\mathcal{T}, t) \supset \mathcal{R}(\mathcal{T}, t). \end{aligned} \tag{38}$$

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If $\mathcal{S}(\tau)$ is evaluated sparsely at $\{s_i\}_{s_1=t}^{s_n=T}$ then the function $\hat{\delta}^*(\tau) \triangleq \{\delta^*(s_i^*) \mid s_i^* \triangleq \max_{s_i \leq \tau} s_i\}$ will yield the same intermediate result.

Time-Varying Error Envelopes in Van der Pol

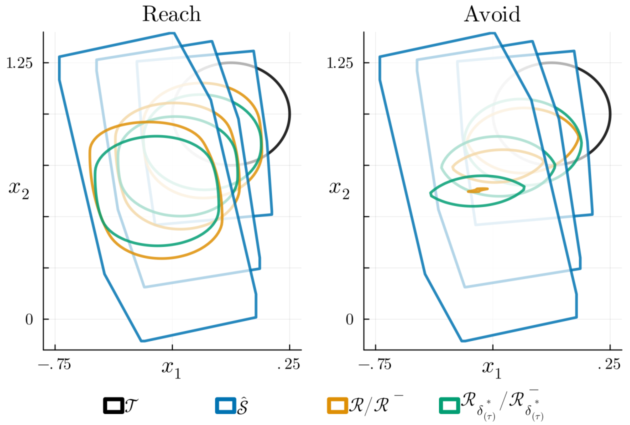

Figure: For a target $\mathcal{T}$ (black), true reachable sets $\mathcal{R}/\mathcal{R}^-$ (gold) are solved for a *Reach* and *Avoid* game resp. at $t = [0.13, 0.26, 0.39]$. The corresponding reachable sets of the safe envelope $\mathcal{R}_{\delta^*(\tau)}/\mathcal{R}^-_{\delta^*(\tau)}$ (green) based on the TV Taylor-series linearization error are also solved.

Single-Player Error Sets

To assume the set $\mathcal{S}$ contains all trajectories is sufficient but superfluous. Ultimately, the error need only be bounded on the trajectories evolving from the optimal strategy of Player I for any input of Player II (*in the BRT*).

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Corollary

Suppose $\mathcal{S}_{\mathcal{U}}, \mathcal{S}_{\mathcal{D}} \subset \mathbb{KR}^{n_x}$ *are defined as in (9) for control-only and disturbance-only settings resp., and let the maximum errors on these sets be defined as* $\delta^*_{\mathcal{U}}$ *and* $\delta^*_{\mathcal{D}}$*. Then for Reach and Avoid games resp.,*

$$\begin{aligned} \mathcal{R}_{\delta^*}(\mathcal{T}, t) \subset \mathcal{R}_{\delta^*_{\mathcal{U}}}(\mathcal{T}, t) \subset \mathcal{R}(\mathcal{T}, t), \\ \mathcal{R}^-_{\delta^*}(\mathcal{T}, t) \supset \mathcal{R}^-_{\delta^*_{\mathcal{D}}}(\mathcal{T}, t) \supset \mathcal{R}(\mathcal{T}, t). \end{aligned} \tag{39}$$

Single-Player Error Envelopes in Van der Pol

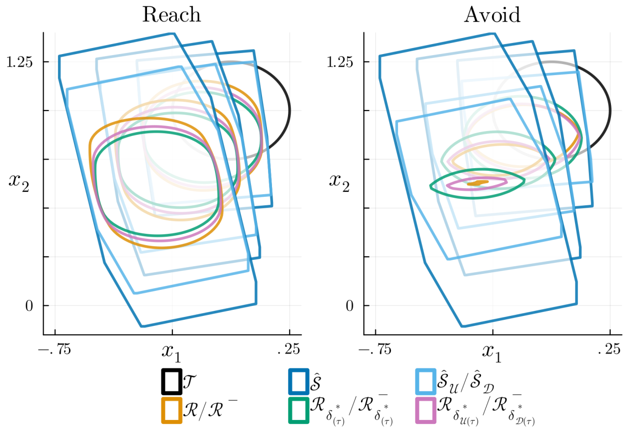

Figure: The reachable sets of the safe envelope $\mathcal{R}_{\delta^*_{\mathcal{U}}(\tau)}/\mathcal{R}^-_{\delta^*_{\mathcal{D}}(\tau)}$ (pink) based on the time-varying Taylor-series linearization error for single-player feasible sets $\mathcal{S}_{\mathcal{U}}, \mathcal{S}_{\mathcal{D}}$ (light blue) are also solved.

Error for Ensemble of Linear Models

Variations of the linear problem may offer variations in the safe envelope, yielding more information.

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Lemma

Let $\{l_i(x,u,d) = A_i x + B_{1,i} u + B_{2,i} d\}_i$ *be a set of linear models and* $\delta_i \triangleq \max_{x \in \mathcal{S}} ||(f - l_i)|s|$ *be their respective errors. Then for reach and avoid games,*

$$\bigcup_i \mathcal{R}_{\delta_i^*}(\mathcal{T}, t) \subset \mathcal{R}(\mathcal{T}, t) \subset \bigcap_i \mathcal{R}^-_{\delta_i^*}(\mathcal{T}, t). \tag{40}$$

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Note, the sub-problems corresponding to each linear model may be solved independently, and hence the load of the total ensemble solution scales in a parallelizeable, linear fashion with respect to number of models.

Ensemble Error Envelopes in Van der Pol

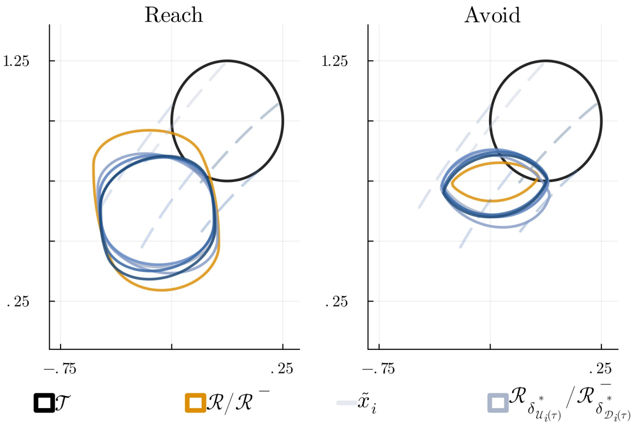

Figure: The linear ensemble envelopes $\mathcal{R}_{\delta^*_{\mathcal{U}_i}(\tau)}/\mathcal{R}^-_{\mathcal{D}_i}\delta^*(\tau)$ (blues) based on the ensemble trajectories (blue dashed), with time-varying, single-player Taylor-series linearization error are solved. The union and intersection of ensemble envelopes in *Reach* and *Avoid* games resp. are also safe envelopes

Target Partitioning Envelopes

One may recall the proposed work hinges on the transformation of the space-dependent error to a *uniformly* applied quantity.

Hence, splitting the target frees some divisions from the maximum error in another.[2]

[2] This procedure could be repeated *ad infinitum*, reducing $\varepsilon \to 0$ for $\mathcal{T}_i = B(x, \varepsilon)$ to converge to a space-optimal formulation of [(9)].

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Suppose $\{\mathcal{T}_i\}_i$ is a partition of $\mathcal{T}$ such that $\cup_i \mathcal{T}_i \subseteq \mathcal{T}$. Let $\mathcal{S}_i$ be defined as in (9) with $x(T) \in \mathcal{T}_i$, and let δ_i^* be the corresponding error for l. For the Reach game,

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For $x \in \partial\mathcal{T}_i$, *if* $(\min_{y\in\mathcal{T}} ||x - y|| \geq L(t)) \vee (\exists j, x \in \mathcal{T}_j, \min_{y\in\partial\mathcal{T}_j} ||x - y|| \geq L(t))$, *we say the partition "*$L(t)$*-overlaps" and the following property holds,*

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Target Partitioning Envelopes in Van der Pol

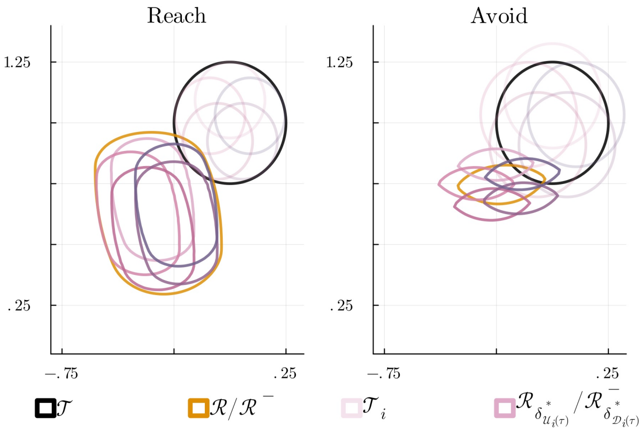

Figure: The partition envelopes $\mathcal{R}_{\delta^*_{\mathcal{U}_i}(\tau)}/\mathcal{R}^-_{\mathcal{D}_i}\delta^*(\tau)$ (pinks) based on the partitioned target (faint pinks), with time-varying, single-player Taylor-series linearization error are solved. The union of partition envelopes in *Reach* and *Avoid* games is also a safe envelope.

Forward-Backward Feasibility and Error

Finally, there is a substantial improvement in the special case when considering the reachability or avoidability of a local region $\mathcal{X}_g \subset \mathcal{X} \subset \mathbb{R}^{n_x}$, e.g. a single state.

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Lemma

Let $\mathcal{S}_g \triangleq \mathcal{S}(\mathcal{X}_g, -t)$ be the feasible set of $\mathcal{X}_g$ at $|t|$ time in the future and δ_g^* be the error. Furthermore, let $\bar{\mathcal{T}} \triangleq \mathcal{T} \cap \mathcal{S}_g$ be its intersection with the target and $\bar{\mathcal{S}} \triangleq \mathcal{S}(\bar{\mathcal{T}}, t)$ be the backwards feasible set of the intersection. Suppose $\bar{\mathcal{S}}_g \triangleq \bar{\mathcal{S}} \cap \mathcal{S}_g$ and $\bar{\delta}_g^*$ to be the error.

Then for Reach and Avoid games resp.,

$$\begin{aligned} V_{\delta^*}(x,t) \geq V_{\delta_g^*}(x,t) \geq V_{\bar{\delta}_g^*}(x,t) \geq V(x,t), \\ V^-_{\delta^*}(x,t) \leq V^-_{\delta_g^*}(x,t) \leq V^-_{\bar{\delta}_g^*}(x,t) \leq V(x,t). \end{aligned} \quad x \in \mathcal{X}_g \tag{43}$$

Forward-Backward Feasibility Illustrative Example

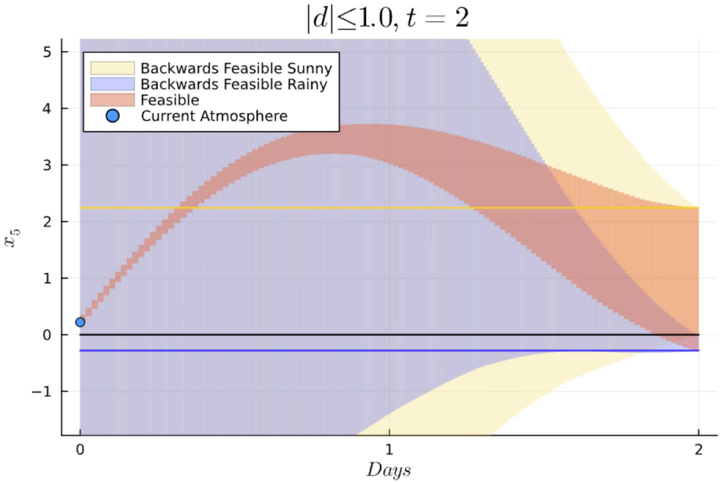

Figure: Forward-Backward feasibility sets in the Lorenz 96 Model.

Computation of $\mathcal{S}$ and δ^* *a priori*

There are several developed DI approaches to the problem of safely approximating $\hat{\mathcal{S}} \supset \mathcal{S}$ (10; 11; 7; 12). At simplest, one may consider the "Lipschitz tube" that bounds the flow (12), however, this is unlikely to yield a tight superset.

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This may be non-convex. However, if l is the Taylor series linearization, then we may upper-bound with convex objective.

Multi-Agent Dubin's Pursuit-Evasion

Consider N Dubin's cars in the relative frame of an additional car [(13; 14)]. Each agent's relative state, $x_i = [x_{\Delta_i}, y_{\Delta_i}, \theta_{\Delta_i}]$, evolves,

$$\dot{x}_i = \begin{bmatrix} -v_a + v_b \cos(\theta_{\Delta_i}) + a y_{\Delta_i} \\ v_b \sin(\theta_{\Delta_i}) - a x_{\Delta_i} \\ b_i - a \end{bmatrix} \tag{45}$$

where $a (\in [-1, 1])$ and b_i are the controls of the central agent and the $i \in [1, N]$ agent resp., and v_a and $v_b (= 3)$ are their velocities resp.

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where $a(\in [-1, 1])$ and b_i are the controls of the central agent and the $i \in [1, N]$ agent resp., and v_a and $v_b(= 3)$ are their velocities resp.

In the game, the central agent seeks to avoid all N other agents which simultaneously aim to reach it. Note, the combined system has $x \in \mathbb{R}^{3N}$.

Multi-Agent Dubin's Pursuit-Evasion

Note, the linearization of this system takes the form (14),

$$\begin{aligned}\dot{x} = &\begin{bmatrix} A_1 & & \cdots & 0 \\ & A_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & A_N \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} + \begin{bmatrix} B_{a_1} \\ B_{a_2} \\ \vdots \\ B_{a_N} \end{bmatrix} a \\ &+ \begin{bmatrix} B_{b_1} & & \cdots & 0 \\ & B_{b_2} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & B_{b_N} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_N \end{bmatrix},\end{aligned} \tag{46}$$

where $(A, B_a, B_b) \triangleq (D_x f, D_a f, D_b f)|_{\tilde{x},0,\tilde{b}}$ and ε is the residual error.

The error is derived by linearizing around the iLQG trajectory $\tilde{x}(\tau)$ without pursuer action, and solving the feasible pursuer set for the initial condition given by $x_i(0) = [10, 0, \pm\pi]$.

Multi-Agent Dubin's Pursuit-Evasion

Capture may occur iff the central agent is close ($r = 1.5$) to one pursuer and near the entire team ($r_f = 5$).

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Hence, the target for each pair (central agent, team member) takes the form of an ellipsoid, given by

$$\Omega_i \triangleq \{x \mid x^\top W_i^{-1} x \leq 1\} \tag{47}$$

where $W_i \in \mathbb{R}^{3N\times 3N}$ is a block-diagonal matrix of diagonal matrices with values of r^2 in the i-th block's first and second diagonals and θ_{max}^2 in the third, and r_f^2 in the j-th block's first and second diagonals θ_{max}^2 in the third for all $j \neq i$.

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The joint system has a target then defined by $\Omega = \cup \Omega_i$, which max-plus algebras certify may be solved by the Hopf formula independently (15).

Multi-Agent Dubin's Pursuit-Evasion

Note, this game may be non-convex, however, in the *Avoid* context a conservative lower-value $V_{\delta*}$ may be solved despite this.

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We use the Hopf formula with the linearized system and antagonistic error to solve when the central agent may evade capture, given by,

$$\min_{t\in[0,1]} V(x_0, t) = V^*(x_0) > 0. \tag{48}$$

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We will solve the surface of $V^*(x_0(\theta_a); v_a, \max|b|) = 0$ i.e. w.r.t. parameters $v_a, \max|b|$ and θ_a.

Multi-Agent Dubin's Pursuit-Evasion

Let x_0 be defined s.t. the agents form a regular N-polygon around the evader, oriented relative to the evader angle θ_a.

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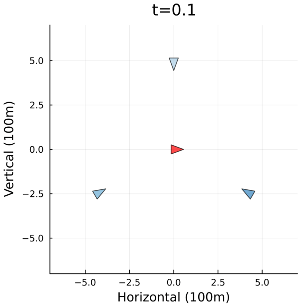
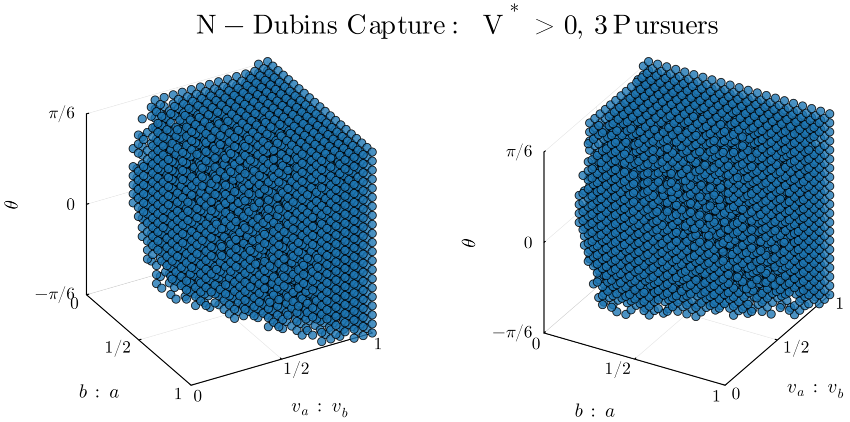
Let x_0 be defined s.t. the agents form a regular N-polygon around the evader, oriented relative to the evader angle θ_a. Hence,

$$\begin{bmatrix} x_{\Delta_i,0} \\ y_{\Delta_i,0} \end{bmatrix} = R_{\text{rot}}\left((i-1)\frac{2\pi}{N} + \theta_a\right)\begin{bmatrix} 0 \\ r_p \end{bmatrix}$$
$$\theta_{\Delta_i,0} = (i-1)\frac{2\pi}{N} + \theta_a - \pi/2 \tag{49}$$

where $r_p = v_b$ is the initial distance between each agent and the central agent.

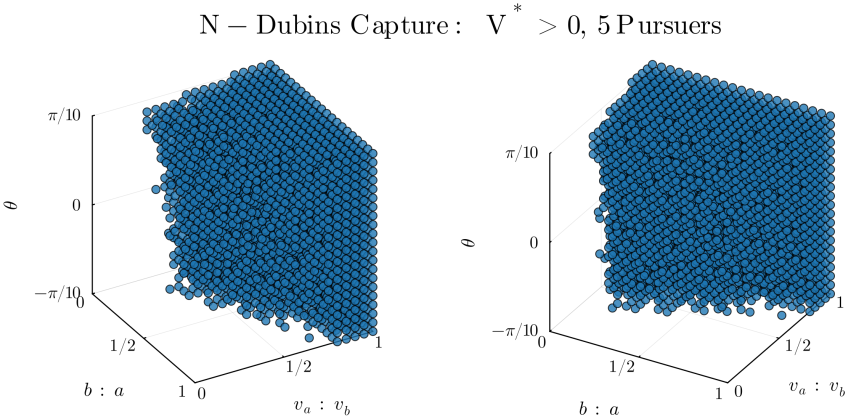


Multi-Agent Dubin's Pursuit-Evasion



Mean time to compute $V^*(x_0; a, b, v_a, v_b)$ for $N = 3$ (9 dimensions)is 5.9 s/agent.

Multi-Agent Dubin's Pursuit-Evasion



Mean time to compute $V^*(x_0; a, b, v_a, v_b)$ for $N = 5$ (15 dimensions) is 8.0 s/agent.

Thank you!

This research group - people, undergrad and advisor - is truly awesome.

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