The Knight's Tour

ECEN 2703 Final Project Report

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Introduction

For the final project, we chose to use python to create a program which simulates a "Knight's Tour". What is a Knight's Tour? This is a set of moves a chess knight can take on a chess board to cover every single square exactly once and only once, using only the valid "two in one direction, one in another" move which a knight uses in chess. Not only that, in certain situations, it is possible for the knight to complete its tour and end at a square from where it can jump back to the start square, a closed tour, or have a closed tour in which one corner square is omitted, a corner closed tour, an unrestricted tour is called "open".

The scope of our project was to find a tour for any sized chess board in which one exists in linear time, with the user choosing both dimensions of the board. Also, the user can choose "closed", "open", or "corner" to create a path which follows the rules of said tour. We wanted our program to create a path for the knight which was stored inside a list, where each item in the list is the square which the knight will move to. In order to actually show what the path looks like, we made a print function which shows the board and which move number corresponded to each square, and also an animation that tracks the knight as it travels.

The reason this topic interested us is because we both play chess, and chess has been very popular among people our age, partially due to the Netflix series "The Queen's Gambit", a show centered around a chess prodigy. The knight's tour was originally researched by Euler, on a standard 8x8 chessboard and other small boards in the 1700's. As time has passed, this problem has interested many mathematicians, and as technology grew, there were new ways to create and solve the knight's tour using programming, AI, and algorithms. The knight's tour can be reduced to finding a hamiltonian path or cycle, an NP complete problem. Chess boards will at a minimum contain dozens of squares and hundreds of edges when constructing the graph, this makes solving the problem particularly difficult with brute force methodology such as breadth/depth first searches.

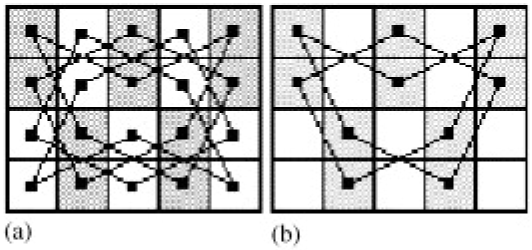
The Algorithm

Initially a heuristic approach was taken, in which a standard recursive depth first search of the graph is done with the choice of which move to take being determined by a heuristic. For knights tours, the Warnsdorff's heuristic is commonly used, this orders the moves by which one has the least following moves, second order tie breaking was implemented in which ties were decided by the sum of each moves available at the next layer down, this often allows boards to be solved in linear time. This approach worked reasonably well, allowing boards up to a couple thousand squares to be solved reasonably quickly. This heuristic doesn't specifically look for closed tours and thus some modifications had to be made to find them, specifically forcing the last square to have the start square as one of its edges, to further improve success, the starting square is chosen from the center as the heuristic searches “outward”, this increased the odds of ending near the start. This was a good start allowing us to solve closed boards up to 30x30 reasonably fast and open tours up to 50x50 until recursion depth issues were reached. To solve the recursion issue, implementing the recursion manually through a stack allowed open tours up to 160x160 to be solved, this was later reverted to recursive as we didn’t need large boards for the final algorithm and the code looked much nicer on the eyes. This approach can be found in HeuristicApproach.py.

As a side note the reason for this approach breaking down on large boards is actually quite interesting. On larger boards, the tie breaking will, more often than not, still result in ties and so “wrong” moves will be made early on. Further levels of tie breaking can be used, but there also exist perfectly symmetrical positions in which no amount of tie breaking will help and at a certain point the amount of levels looked into will become extremely inefficient and will practically be running a breadth first search every move. Multiple proposals to improve the heuristic exist, yet none work for every board, and none allow for closed boards in any other way than being an improved DFS.

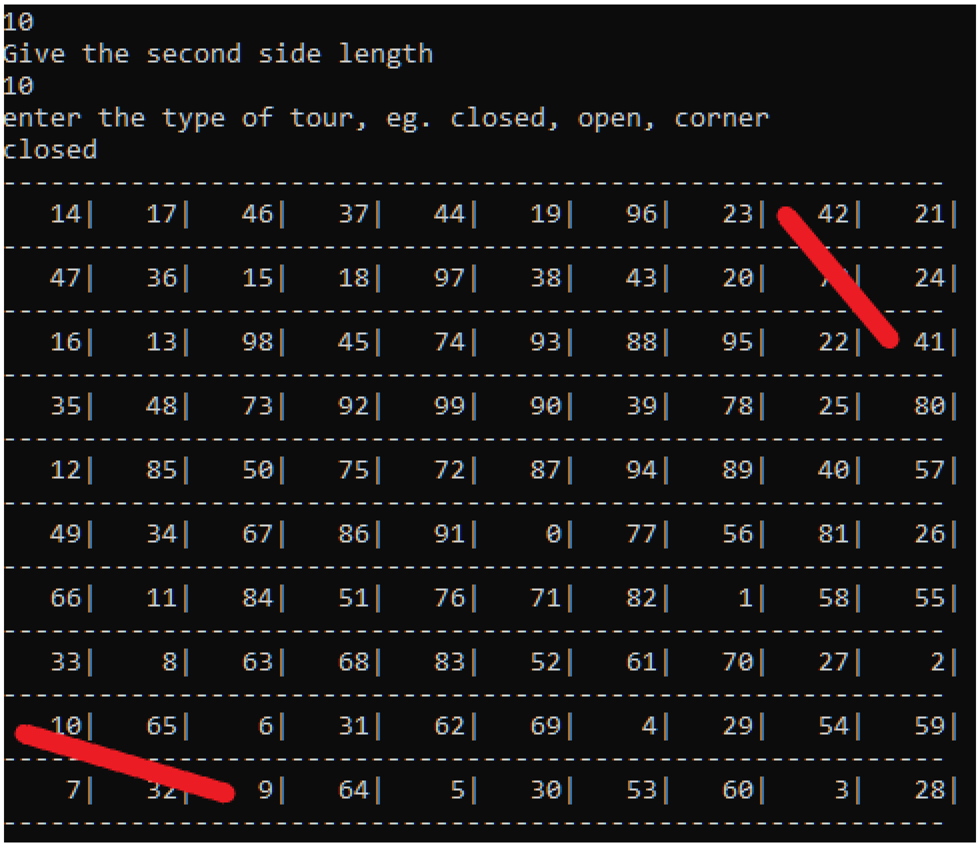
From here we wanted to implement traversal of a cubic chessboard in which the surfaces are the boards. For a multitude of reasons, our current heuristic approach wasn’t viable for this. We wanted to produce much larger boards and being able to stitch together all the faces, required the ability to generate much larger closed tours and other specific kinds of tours that will be explained later. This new algorithm ended up taking much longer than expected and thus we weren’t able to implement the cubic boards; however, as far as we can tell, this is the only runnable program that can compute open and closed tours of arbritarilly sized square and nonsquare boards in linear time that we could find. The algorithm itself was developed by Shun-Shii Lin and Chung-Liang Wei in 2005 [1] as a further improvement on an algorithm created by Ian Parberry which was able to find tours on square boards. In the paper, closed and open tours are separated, however they can be combined with only one extra case and so that is how ours is implemented in code.

As a basis let’s discuss certain unique kinds of tours that will be important, a double loop knights tour is a tour consisting of two closed tours that do not overlap. The only board where this will be important is 4x5, said board is shown below.



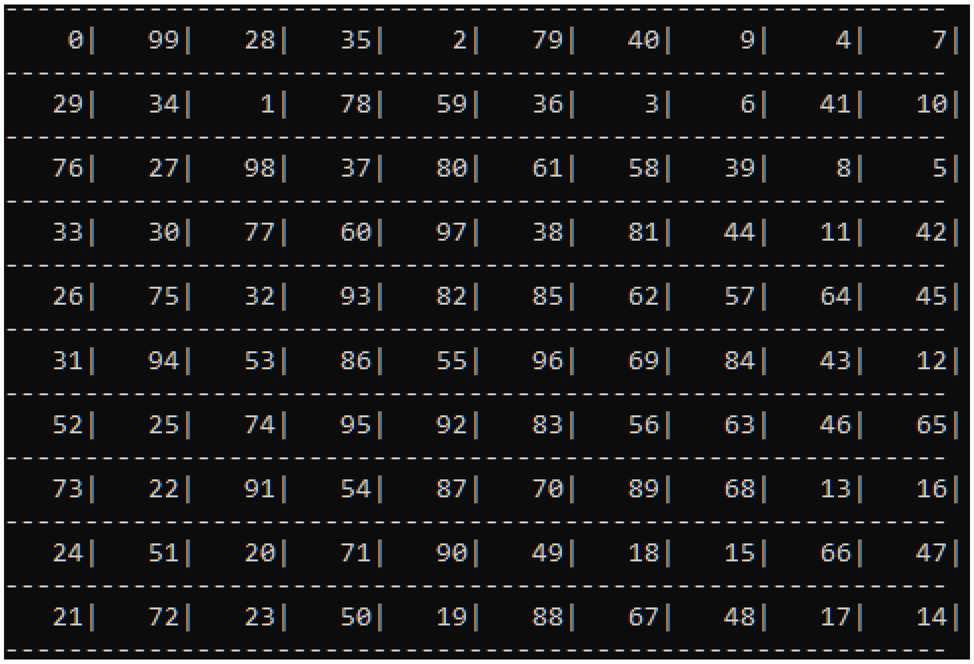
[1]

The next unique tour is a structured tour, a structured tour is a tour in which each corner has a unique pattern, for the algorithm, a fully structured tour is never needed, only partially structured tours, when a structured tour is mentioned from here on, it will refer to a partially structured tour, an example of which is shown below, specifically those two corner moves will have to be present in the path for it to be considered partially structured.



```
10
Give the second side length
10
enter the type of tour, eg. closed, open, corner
closed
-------------------------------------------------------------
   14|   17|   46|   37|   44|   19|   96|   23|   42|   21|
-------------------------------------------------------------
   47|   36|   15|   18|   97|   38|   43|   20|[illegible]|   24|
-------------------------------------------------------------
   16|   13|   98|   45|   74|   93|   88|   95|   22|   41|
-------------------------------------------------------------
   35|   48|   73|   92|   99|   90|   39|   78|   25|   80|
-------------------------------------------------------------
   12|   85|   50|   75|   72|   87|   94|   89|   40|   57|
-------------------------------------------------------------
   49|   34|   67|   86|   91|    0|   77|   56|   81|   26|
-------------------------------------------------------------
   66|   11|   84|   51|   76|   71|   82|    1|   58|   55|
-------------------------------------------------------------
   33|    8|   63|   68|   83|   52|   61|   70|   27|    2|
-------------------------------------------------------------
   10|   65|    6|   31|   62|   69|    4|   29|   54|   59|
-------------------------------------------------------------
    7|   32|    9|   64|    5|   30|   53|   60|    3|   28|
-------------------------------------------------------------
```

Lastly a stretched tour is a structured open tour in which the start square is the top left corner and the end square is on either side of the start. An example is shown below. These only exist when n*m is even.



```
-------------------------------------------------------------
    0|   99|   28|   35|    2|   79|   40|    9|    4|    7|
-------------------------------------------------------------
   29|   34|    1|   78|   59|   36|    3|    6|   41|   10|
-------------------------------------------------------------
   76|   27|   98|   37|   80|   61|   58|   39|    8|    5|
-------------------------------------------------------------
   33|   30|   77|   60|   97|   38|   81|   44|   11|   42|
-------------------------------------------------------------
   26|   75|   32|   93|   82|   85|   62|   57|   64|   45|
-------------------------------------------------------------
   31|   94|   53|   86|   55|   96|   69|   84|   43|   12|
-------------------------------------------------------------
   52|   25|   74|   95|   92|   83|   56|   63|   46|   65|
-------------------------------------------------------------
   73|   22|   91|   54|   87|   70|   89|   68|   13|   16|
-------------------------------------------------------------
   24|   51|   20|   71|   90|   49|   18|   15|   66|   47|
-------------------------------------------------------------
   21|   72|   23|   50|   19|   88|   67|   48|   17|   14|
-------------------------------------------------------------
```

For reasons that will be discussed later, stretched boards will be required a lot, and often with the same sizes. This allowed for some optimization, using our heuristic approach, which by nature of the heuristic will always produce structured corners where possible, we are able to generate a catalog of n*m stretched boards for 4<= n <=10 and 4<= m <= 10. Certain boards, specifically some where n or m are 4, don't exist and so aren't an issue when not generated. A select few others required some manipulation of other tours to generate, such as rotations and mirroring. To generate stretched tours, a few modifications to the original approach were made, as getting a specific end square, especially one so close to the start, which is in a corner, is even more challenging than a normal closed tour. Further pruning of the move tree was required, first removing the end square from all possible moves except the second to last, and second removing all the squares that can jump to the last square when less than half the board has been traversed. This gave us all the required boards.

Constructing Tours

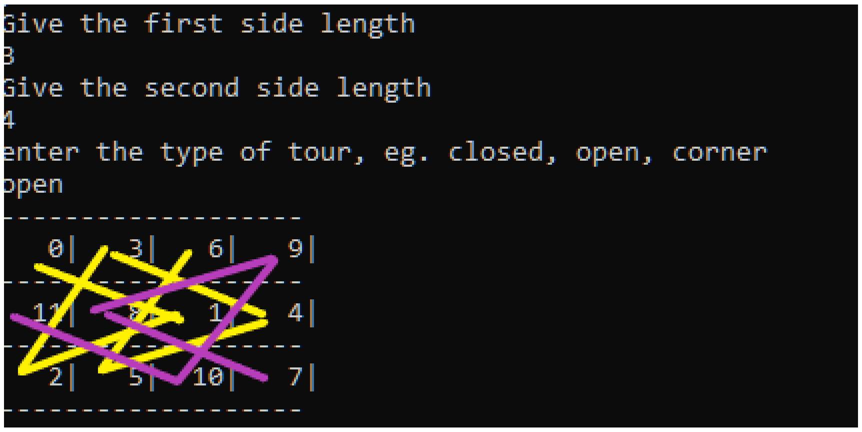
From here on n will always be less than m, the function will always be called when the dimensions are sorted so that this is the case, recovering the opposite dimensions simply requires a rotation plus a mirror depending on what the required outcome is. There are five cases to be considered when constructing knights tours.

Case One: n<=10 and m<=10

For this case, if a stretched tour is requested, the catalog will be referenced, all other tours will be found by calling the heuristic approach, cataloging all other tours is inefficient as, at most, only one will be required regardless of board size.

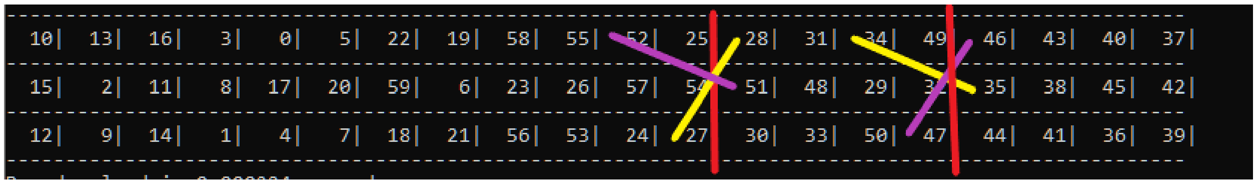
Case Two: n == 3 and (m>=10 for closed, or (m==4 or m>=7 for open))

A 3*k board with the desired type will be constructed where $k = (m - 7)\ mod\ 4 + 7$ for open or $k = (m - 9)\ mod\ 4 + 9$ for any other type. The remaining squares will be evenly separated into 3*4 stretched boards. The 3*4 stretched boards can be separated into the two paths shown below.



```
Give the first side length
3
Give the second side length
4
enter the type of tour, eg. closed, open, corner
open
------------------
   0|   3|   6|   9|
------------------
  11|   8|   1|   4|
------------------
   2|   5|  10|   7|
------------------
```

From here the 3*k board can be stitched together with the rest along the structured corners as shown below. Yellow represents the path going forward and purple represents going back, as can be seen it alternates between both of the above paths going forwards and the reverse of the paths going backwards.

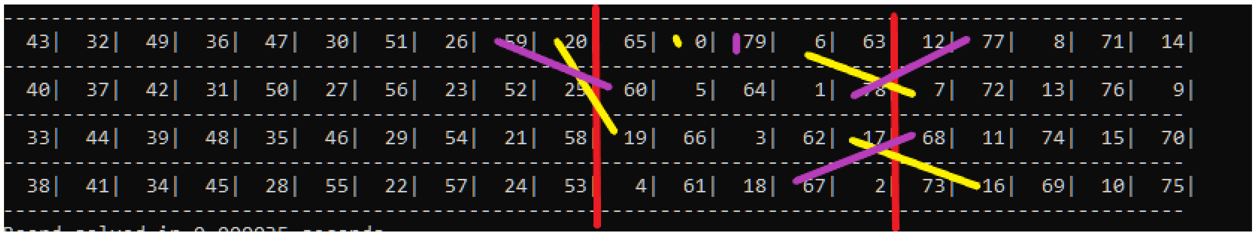


```
-----------------------------------------------------------------------------------------------------
  10|  13|  16|   3|   0|   5|  22|  19|  58|  55|  52|  25|  28|  31|  34|  49|  46|  43|  40|  37|
-----------------------------------------------------------------------------------------------------
  15|   2|  11|   8|  17|  20|  59|   6|  23|  26|  57|  54|  51|  48|  29|  32|  35|  38|  45|  42|
-----------------------------------------------------------------------------------------------------
  12|   9|  14|   1|   4|   7|  18|  21|  56|  53|  24|  27|  30|  33|  50|  47|  44|  41|  36|  39|
-----------------------------------------------------------------------------------------------------
```

Doing this works for arbitrarily long boards, finding the 3*k path can essentially be considered constant time and the rest of the construction consists of doing two list operations per 12 squares.

Case Three: n == 4 and m>10

This special case only pertains to open boards as closed and corner closed do not exist on 4*m boards. Using the double loop board discussed above, we can construct boards in a similar fashion to 3*m boards, this time a 4*k stretched board will be requested and mirrored across the n axis so as to have the start and end at the top right corner, where $k = (m - 6) \bmod 5 + 6$. From here we will find both the forward and backward path at the same time by iterating through the double loop boards and constructing the paths as shown below.

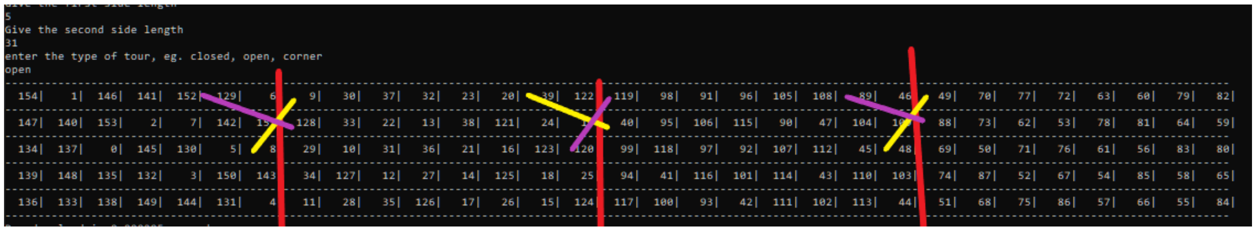


```
-----------------------------------------------------------------------------------------------------
  43|  32|  49|  36|  47|  30|  51|  26|  59|  20|  65|   0|  79|   6|  63|  12|  77|   8|  71|  14|
-----------------------------------------------------------------------------------------------------
  40|  37|  42|  31|  50|  27|  56|  23|  52|  25|  60|   5|  64|   1|  78|   7|  72|  13|  76|   9|
-----------------------------------------------------------------------------------------------------
  33|  44|  39|  48|  35|  46|  29|  54|  21|  58|  19|  66|   3|  62|  17|  68|  11|  74|  15|  70|
-----------------------------------------------------------------------------------------------------
  38|  41|  34|  45|  28|  55|  22|  57|  24|  53|   4|  61|  18|  67|   2|  73|  16|  69|  10|  75|
-----------------------------------------------------------------------------------------------------
```

The path also follows a similar alternating structure as the previous case, to better understand how this case works, it's much easier to see if you run the visualiser on a 4*30 board. This case is slightly less efficient than the previous, requiring 4 list operations per 20 squares.

Case Four: n<=10 and m>10

This case pertains to all boards, here the board will be split into two, n*m1 and n*m2, where $m1 = (m//4) * 2 + m \bmod 2$and $m2 = m - m1$. This partitioning guarantees that only m1 could possibly be odd and will only be so if m is also odd. From here we can recursively call this function to find a stretched n*m2 board and whatever is requested for n*m1. The partitioning guarantees that at most one board could be odd and the rest even, guaranteeing that we will be able to construct stretched tours from the rest. Below shows how the tours are stitched together.



```
Give the first side length
5
Give the second side length
31
enter the type of tour, eg. closed, open, corner
open
------------------------------------------------------------------------------------------------------------------------------------------------------------------
  154|    1|  146|  141|  152|  129|    6|    9|   30|   37|   32|   23|   20|   39|  122|  119|   98|   91|   96|  105|  108|   89|   46|   49|   70|   77|   72|   63|   60|   79|   82|
------------------------------------------------------------------------------------------------------------------------------------------------------------------
  147|  140|  153|    2|    7|  142|  151|  128|   33|   22|   13|   38|  121|   24|   19|   40|   95|  106|  115|   90|   47|  104|  109|   88|   73|   62|   53|   78|   81|   64|   59|
------------------------------------------------------------------------------------------------------------------------------------------------------------------
  134|  137|    0|  145|  130|    5|    8|   29|   10|   31|   36|   21|   16|  123|  120|   99|  118|   97|   92|  107|  112|   45|   48|   69|   50|   71|   76|   61|   56|   83|   80|
------------------------------------------------------------------------------------------------------------------------------------------------------------------
  139|  148|  135|  132|    3|  150|  143|   34|  127|   12|   27|   14|  125|   18|   25|   94|   41|  116|  101|  114|   43|  110|  103|   74|   87|   52|   67|   54|   85|   58|   65|
------------------------------------------------------------------------------------------------------------------------------------------------------------------
  136|  133|  138|  149|  144|  131|    4|   11|   28|   35|  126|   17|   26|   15|  124|  117|  100|   93|   42|  111|  102|  113|   44|   51|   68|   75|   86|   57|   66|   55|   84|
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```
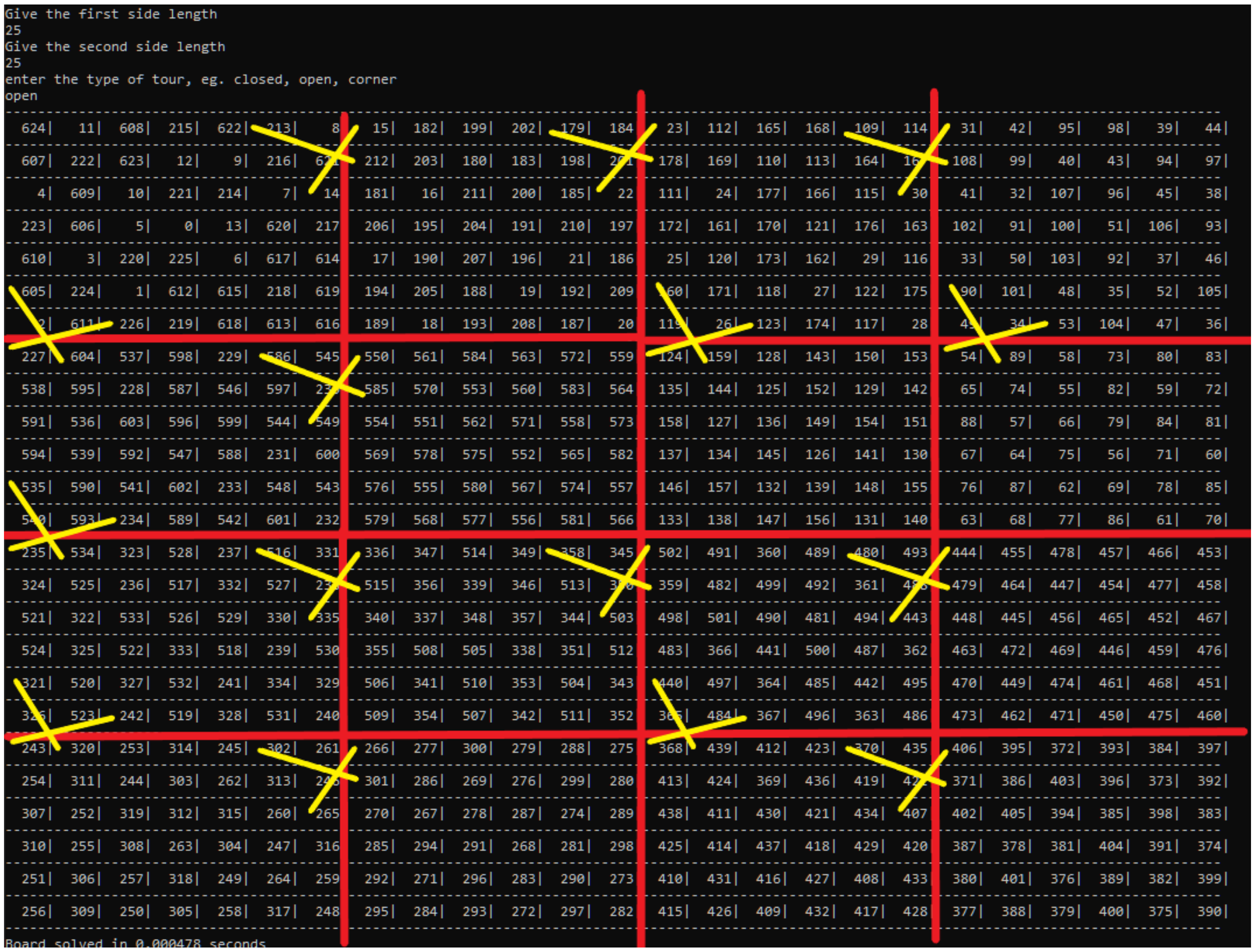

Case 5: n>10 and m>10

Here we can separate the board much the same way as the previous case but this time into four different boards that are generated recursively. Following an identical partitioning rule, only the top left board will be unable to be stretched, this allows us to construct any tour we want in a similar way to the previous case. The stitching together of boards is shown below.



```
Give the first side length
25
Give the second side length
25
enter the type of tour, eg. closed, open, corner
open
------------------------------------------------------------------------------------------------------------------------------
  624|   11|  608|  215|  622|  213|    8|   15|  182|  199|  202|  179|  184|   23|  112|  165|  168|  109|  114|   31|   42|   95|   98|   39|   44|
------------------------------------------------------------------------------------------------------------------------------
  607|  222|  623|   12|    9|  216|  621|  212|  203|  180|  183|  198|  201|  178|  169|  110|  113|  164|  167|  108|   99|   40|   43|   94|   97|
------------------------------------------------------------------------------------------------------------------------------
    4|  609|   10|  221|  214|    7|   14|  181|   16|  211|  200|  185|   22|  111|   24|  177|  166|  115|   30|   41|   32|  107|   96|   45|   38|
------------------------------------------------------------------------------------------------------------------------------
  223|  606|    5|    0|   13|  620|  217|  206|  195|  204|  191|  210|  197|  172|  161|  170|  121|  176|  163|  102|   91|  100|   51|  106|   93|
------------------------------------------------------------------------------------------------------------------------------
  610|    3|  220|  225|    6|  617|  614|   17|  190|  207|  196|   21|  186|   25|  120|  173|  162|   29|  116|   33|   50|  103|   92|   37|   46|
------------------------------------------------------------------------------------------------------------------------------
  605|  224|    1|  612|  615|  218|  619|  194|  205|  188|   19|  192|  209|  160|  171|  118|   27|  122|  175|   90|  101|   48|   35|   52|  105|
------------------------------------------------------------------------------------------------------------------------------
    2|  611|  226|  219|  618|  613|  616|  189|   18|  193|  208|  187|   20|  119|   26|  123|  174|  117|   28|   49|   34|   53|  104|   47|   36|
------------------------------------------------------------------------------------------------------------------------------
  227|  604|  537|  598|  229|  586|  545|  550|  561|  584|  563|  572|  559|  124|  159|  128|  143|  150|  153|   54|   89|   58|   73|   80|   83|
------------------------------------------------------------------------------------------------------------------------------
  538|  595|  228|  587|  546|  597|  230|  585|  570|  553|  560|  583|  564|  135|  144|  125|  152|  129|  142|   65|   74|   55|   82|   59|   72|
------------------------------------------------------------------------------------------------------------------------------
  591|  536|  603|  596|  599|  544|  549|  554|  551|  562|  571|  558|  573|  158|  127|  136|  149|  154|  151|   88|   57|   66|   79|   84|   81|
------------------------------------------------------------------------------------------------------------------------------
  594|  539|  592|  547|  588|  231|  600|  569|  578|  575|  552|  565|  582|  137|  134|  145|  126|  141|  130|   67|   64|   75|   56|   71|   60|
------------------------------------------------------------------------------------------------------------------------------
  535|  590|  541|  602|  233|  548|  543|  576|  555|  580|  567|  574|  557|  146|  157|  132|  139|  148|  155|   76|   87|   62|   69|   78|   85|
------------------------------------------------------------------------------------------------------------------------------
  540|  593|  234|  589|  542|  601|  232|  579|  568|  577|  556|  581|  566|  133|  138|  147|  156|  131|  140|   63|   68|   77|   86|   61|   70|
------------------------------------------------------------------------------------------------------------------------------
  235|  534|  323|  528|  237|  516|  331|  336|  347|  514|  349|  358|  345|  502|  491|  360|  489|  480|  493|  444|  455|  478|  457|  466|  453|
------------------------------------------------------------------------------------------------------------------------------
  324|  525|  236|  517|  332|  527|  238|  515|  356|  339|  346|  513|  350|  359|  482|  499|  492|  361|  488|  479|  464|  447|  454|  477|  458|
------------------------------------------------------------------------------------------------------------------------------
  521|  322|  533|  526|  529|  330|  335|  340|  337|  348|  357|  344|  503|  498|  501|  490|  481|  494|  443|  448|  445|  456|  465|  452|  467|
------------------------------------------------------------------------------------------------------------------------------
  524|  325|  522|  333|  518|  239|  530|  355|  508|  505|  338|  351|  512|  483|  366|  441|  500|  487|  362|  463|  472|  469|  446|  459|  476|
------------------------------------------------------------------------------------------------------------------------------
  321|  520|  327|  532|  241|  334|  329|  506|  341|  510|  353|  504|  343|  440|  497|  364|  485|  442|  495|  470|  449|  474|  461|  468|  451|
------------------------------------------------------------------------------------------------------------------------------
  326|  523|  242|  519|  328|  531|  240|  509|  354|  507|  342|  511|  352|  365|  484|  367|  496|  363|  486|  473|  462|  471|  450|  475|  460|
------------------------------------------------------------------------------------------------------------------------------
  243|  320|  253|  314|  245|  302|  261|  266|  277|  300|  279|  288|  275|  368|  439|  412|  423|  370|  435|  406|  395|  372|  393|  384|  397|
------------------------------------------------------------------------------------------------------------------------------
  254|  311|  244|  303|  262|  313|  246|  301|  286|  269|  276|  299|  280|  413|  424|  369|  436|  419|  422|  371|  386|  403|  396|  373|  392|
------------------------------------------------------------------------------------------------------------------------------
  307|  252|  319|  312|  315|  260|  265|  270|  267|  278|  287|  274|  289|  438|  411|  430|  421|  434|  407|  402|  405|  394|  385|  398|  383|
------------------------------------------------------------------------------------------------------------------------------
  310|  255|  308|  263|  304|  247|  316|  285|  294|  291|  268|  281|  298|  425|  414|  437|  418|  429|  420|  387|  378|  381|  404|  391|  374|
------------------------------------------------------------------------------------------------------------------------------
  251|  306|  257|  318|  249|  264|  259|  292|  271|  296|  283|  290|  273|  410|  431|  416|  427|  408|  433|  380|  401|  376|  389|  382|  399|
------------------------------------------------------------------------------------------------------------------------------
  256|  309|  250|  305|  258|  317|  248|  295|  284|  293|  272|  297|  282|  415|  426|  409|  432|  417|  428|  377|  388|  379|  400|  375|  390|
------------------------------------------------------------------------------------------------------------------------------
Board solved in 0.000478 seconds
```

For these last two cases, you can think of the board as being a smaller board where each square is one of the smaller boards except with each requiring 2-4 operations when they are visited rather than just one, as we methodically jump from board to board the required time to construct the tour is linear.

What's Next?

From the boards we can generate, we would be able to construct surface area tours fairly easily for cubes that are bigger than roughly 3*3*4 by using the same concepts from the above algorithm. The cube would be separated into l*(2m+2n), n*m, n*m boards where l<=n<=m. This guarantees that the middle partition is even and thus can be closed, if n*m is even, we can create two stretched tours and combine at the corners much like above. If n*m is odd we can do some clever manipulation around the corners of the boards to generate tours with the n*m boards being corner closed. Smaller cubic boards would either have to be cataloged or generated in some other way. Improving the current program could be done through multithreading in the last two cases, however in terms of speed, the print function seems to be the largest bottleneck on memory and time.

Conclusion

As far as accomplishing our goals of finding knight's tours and displaying that, we certainly accomplished them. The tours can be found extremely quickly, even with very large boards.. They can be displayed in the python terminal using ASCII characters for any size, and can be simulated as long as neither side is more than 30 long, due to the large window size required, and the time it takes to simulate. We achieved everything we wanted with this project, strengthened Discrete Math knowledge, Python expertise, and even some graphic processing.

Running the Program

Make sure to have pygame installed, the program will try to do so through pip if you don't, however it's likely best that you install it yourself. To run our project, inside the submitted folder, run driver.py and follow the instructions in the terminal.

References

[1] Shun-Shii Lin, Chung-Liang Wei
Optimal algorithms for constructing knight's tours on arbitrary n×m chessboards,
Discrete Applied Mathematics, Volume 146, Issue 3, 2005, Pages 219-232,
(https://www.sciencedirect.com/science/article/pii/S0166218X04003488)