Chapter 9
Matrices

9.1 Introduction

Matrices, like determinants, have their background in algebra and offer another way to represent and manipulate equations. Matrices can be added, subtracted and multiplied together, and even inverted, however, they must give the same result obtained through traditional algebraic techniques. A useful way to introduce the subject is via geometric transforms, which we examine first.

9.2 Geometric Transforms

Let $P(x, y)$ be a vertex on a 2D shape, then we can devise a *geometric transform* where $P(x, y)$ becomes $P'(x', y')$ on a second shape. For example, when the following transform is applied to every point on a shape, it is halved in size, relative to the origin:

$$\begin{aligned} x' &= 0.5x \\ y' &= 0.5y \end{aligned}$$

and this transform translates a shape horizontally by 4 units:

$$\begin{aligned} x' &= x + 4 \\ y' &= y. \end{aligned}$$

Figure 9.1 illustrates two successive transforms applied to the large green star centered at the origin. The first transform scales the star by a factor of 0.5 creating the smaller yellow star, which in turn is subjected to a horizontal translation of 4 units, creating the blue star.

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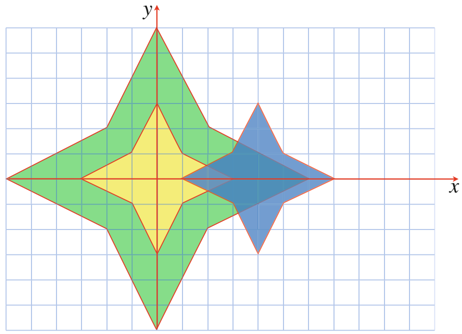

Fig. 9.1 A scale transform followed by a translate transform

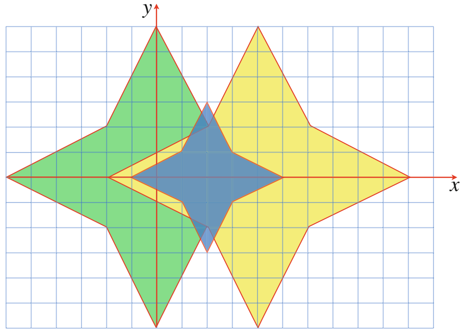

Fig. 9.2 A translate transform followed by a scale transform

Figure 9.2 starts with the same green star, but this time it is translated before being scaled. The final blue star ends up in a different position to the one shown in Fig. 9.1, which demonstrates the importance of transform order.

The algebra supporting the transforms in Fig. 9.1 comprises:

$$
\begin{aligned}
x' &= 0.5x \\
y' &= 0.5y \\
x'' &= x' + 4 \\
y'' &= y'
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
x'' &= 0.5x + 4 \\
y'' &= 0.5y
\end{aligned}
$$

whereas, the algebra supporting the transforms in Fig. 9.2 comprises:

$$\begin{aligned}
x' &= x + 4\\
y' &= y\\
x'' &= 0.5x'\\
y'' &= 0.5y'
\end{aligned}$$

which simplifies to

$$\begin{aligned}
x'' &= 0.5(x + 4)\\
y'' &= 0.5y
\end{aligned}$$

and reveals the difference between the two transform sequences.

9.3 Transforms and Matrices

Matrix notation was researched by the British mathematician Arthur Cayley around 1858. Cayley formalised matrix algebra, along with the American mathematicians Charles Peirce (1839–1914) and his father, Benjamin Peirce (1809–1880). Previously, Carl Gauss had shown that transforms were not commutative, i.e. $\mathbf{T}_1\mathbf{T}_2 \neq \mathbf{T}_2\mathbf{T}_1$, (where $\mathbf{T}_1$ and $\mathbf{T}_2$ are transforms) and matrix notation clarified such observations.

Consider the transform $\mathbf{T}_1$, where x and y are transformed into x' and y' respectively:

$$\mathbf{T}_1 = \begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \tag{9.1}$$

and a second transform $\mathbf{T}_2$, where x' and y' are transformed into x'' and y'' respectively:

$$\mathbf{T}_2 = \begin{cases} x'' = Ax' + By' \\ y'' = Cx' + Dy' \end{cases} . \tag{9.2}$$

Substituting (9.1) in (9.2) we get

$$\mathbf{T_3} = \begin{cases} x'' = A(ax + by) + B(cx + dy) \\ y'' = C(ax + by) + D(cx + dy) \end{cases}$$

which simplifies to

$$\mathbf{T_3} = \begin{cases} x'' = (Aa + Bc)x + (Ab + Bd)y \\ y'' = (Ca + Dc)x + (Cb + Dd)y \end{cases} . \tag{9.3}$$

Having derived the algebra for $\mathbf{T_3}$, let's examine matrix notation, where constants are separated from the variables. For example, the transform (9.4)

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \tag{9.4}$$

can be written in matrix form as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{9.5}$$

where (9.5) contains two different structures: two single-column matrices or column vectors

$$\begin{bmatrix} x' \\ y' \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \end{bmatrix},$$

and a 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Algebraically, (9.4) and (9.5) are identical, which dictates the way (9.5) is converted to (9.4). Therefore, using (9.5) we have x' followed by the "=" sign, and the sum of the products of the top row of constants a and b with the x and y in the last column vector:

$$x' = ax + by.$$

Next, we have y' followed by the "=" sign, and the sum of the products of the bottom row of constants c and d with the x and y in the last column vector:

$$y' = cx + dy.$$

As an example,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

is equivalent to

$$\begin{aligned} x' &= 3x + 4y \\ y' &= 5x + 6y. \end{aligned}$$

We can now write $\mathbf{T}_1$ and $\mathbf{T}_2$ using matrix notation:

$$\mathbf{T}_1 = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{9.6}$$

$$\mathbf{T}_2 = \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \tag{9.7}$$

and substituting (9.6) in (9.7) we have

$$\mathbf{T}_3 = \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{9.8}$$

But we have already computed $\mathbf{T_3}$ (9.3), which in matrix form is:

$$\mathbf{T_3} = \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} Aa + Bc & Ab + Bd \\ Ca + Dc & Cb + Dd \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{9.9}$$

which implies that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} Aa + Bc & Ab + Bd \\ Ca + Dc & Cb + Dd \end{bmatrix}$$

and demonstrates how matrices must be multiplied. Here are the rules for matrix multiplication:

$$\begin{bmatrix} A & B \\ \cdots & \cdots \end{bmatrix} \begin{bmatrix} a & \cdots \\ c & \cdots \end{bmatrix} = \begin{bmatrix} Aa + Bc & \cdots \\ \cdots & \cdots \end{bmatrix}.$$

1: The top left-hand corner element $Aa + Bc$ is the product of the top row of the first matrix by the left column of the second matrix.

$$\begin{bmatrix} A & B \\ \cdots & \cdots \end{bmatrix} \begin{bmatrix} \cdots & b \\ \cdots & d \end{bmatrix} = \begin{bmatrix} \cdots & Ab + Bd \\ \cdots & \cdots \end{bmatrix}.$$

2: The top right-hand element $Ab + Bd$ is the product of the top row of the first matrix by the right column of the second matrix.

$$\begin{bmatrix} \cdots & \cdots \\ C & D \end{bmatrix} \begin{bmatrix} a & \cdots \\ c & \cdots \end{bmatrix} = \begin{bmatrix} \cdots & \cdots \\ Ca + Dc & \cdots \end{bmatrix}.$$

3: The bottom left-hand element $Ca + Dc$ is the product of the bottom row of the first matrix by the left column of the second matrix.

$$\begin{bmatrix} \cdots & \cdots \\ C & D \end{bmatrix} \begin{bmatrix} \cdots & b \\ \cdots & d \end{bmatrix} = \begin{bmatrix} \cdots & \cdots \\ \cdots & Cb + Dd \end{bmatrix}.$$

4: The bottom right-hand element $Cb + Dd$ is the product of the bottom row of the first matrix by the right column of the second matrix.

Let's multiply the following matrices together:

$$\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} (2 \times 3 + 4 \times 7) & (2 \times 5 + 4 \times 9) \\ (6 \times 3 + 8 \times 7) & (6 \times 5 + 8 \times 9) \end{bmatrix} = \begin{bmatrix} 34 & 46 \\ 74 & 102 \end{bmatrix}.$$

9.4 Matrix Notation

Having examined the background to matrices, we can now formalise their notation.

A matrix is an array of numbers (real, imaginary, complex, etc.) organised in m rows and n columns, where each entry a_{ij} belongs to the ith row and jth column:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

It is also convenient to express the above definition as

$$\mathbf{A} = [a_{ij}]_{m\ n}.$$

9.4.1 Matrix Dimension or Order

The *dimension* or *order* of a matrix is the expression $m \times n$ where m is the number of rows, and n is the number of columns.

9.4.2 Square Matrix

A *square matrix* has the same number of rows as columns:

$$\mathbf{A} = [a_{ij}]_{n\ n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad \text{e.g.,} \quad \begin{bmatrix} 1 & -2 & 4 \\ 6 & 5 & 7 \\ 4 & 3 & 1 \end{bmatrix}.$$

9.4.3 Column Vector

A *column vector* is a matrix with a single column:

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \text{e.g.,} \quad \begin{bmatrix} 2 \\ 3 \\ 23 \end{bmatrix}.$$

9.4.4 Row Vector

A *row vector* is a matrix with a single row:

$$\begin{bmatrix} a_{11} \ a_{12} \cdots a_{1n} \end{bmatrix}, \quad \text{e.g.,} \quad \begin{bmatrix} 2 \ 3 \ 5 \end{bmatrix}.$$

9.4.5 Null Matrix

A *null matrix* has all its elements equal to zero:

$$\theta_n = [a_{ij}]_{n\,n} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \text{e.g.,} \quad \theta_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The null matrix behaves like zero when used with numbers, where we have, $0 + n = n + 0 = n$ and $0 \times n = n \times 0 = 0$, and similarly, $\theta + \mathbf{A} = \mathbf{A} + \theta = \mathbf{A}$ and $\theta\mathbf{A} = \mathbf{A}\theta = \theta$. For example,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

9.4.6 Unit Matrix

A *unit matrix* $\mathbf{I}_n$, is a square matrix with the elements on its diagonal a_{11} to a_{nn} equal to 1:

$$\mathbf{I}_n = [a_{ij}]_{n\,n} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \quad \text{e.g.,} \quad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The unit matrix behaves like the number 1 in a conventional product, where we have, $1 \times n = n \times 1 = n$, and similarly, $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$. For example,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

9.4.7 Trace

The *trace* of a square matrix is the sum of the elements on its diagonal a_{11} to a_{nn}:

$$\operatorname{Tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}.$$

For example, given

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad \text{then} \quad \operatorname{Tr}(\mathbf{A}) = 1 + 5 + 9 = 15.$$

The trace of a rotation matrix can be used to compute the angle of rotation. For example, the matrix to rotate a point about the origin is

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where

$$\operatorname{Tr}(\mathbf{A}) = 2\cos\theta$$

which means that

$$\theta = \arccos\left(\frac{\operatorname{Tr}(\mathbf{A})}{2}\right).$$

The three matrices for rotating points about the x-, y- and z-axis are respectively:

$$\mathbf{R}_{\alpha,x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\mathbf{R}_{\alpha,y} = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$\mathbf{R}_{\alpha,z} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and it is clear that

$$\mathrm{Tr}(\mathbf{R}_{\alpha,x}) = \mathrm{Tr}(\mathbf{R}_{\alpha,y}) = \mathrm{Tr}(\mathbf{R}_{\alpha,z}) = 1 + 2\cos\alpha$$

therefore,

$$\alpha = \arccos\left(\frac{\mathrm{Tr}(\mathbf{R}_{\alpha,x}) - 1}{2}\right).$$

9.4.8 Determinant of a Matrix

The *determinant* of a matrix is a scalar value computed from the elements of the matrix. The different methods for computing the determinant are described in Chap. 6. For example, using Sarrus's rule:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{then,} \quad \det\mathbf{A} = 45 + 84 + 96 - 105 - 48 - 72 = 0.$$

9.4.9 Transpose

The *transpose* of a matrix exchanges all row elements for column elements. The transposition is indicated by the letter 'T' outside the right-hand bracket.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

For example,

$$\begin{bmatrix} 1 & 2 & 4 \\ 6 & 5 & 7 \\ 4 & 3 & 1 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 3 \\ 4 & 7 & 1 \end{bmatrix},$$

and

$$\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix}.$$

To prove that $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$, we could develop a general proof using $n \times n$ matrices, but for simplicity, let's employ 3×3 matrices and assume the result generalises to higher dimensions. Given

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}, \quad \mathbf{B}^{\mathrm{T}} = \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{bmatrix}$$

then,

$$\mathbf{AB} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix}$$

$$(\mathbf{AB})^{\mathrm{T}} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} \\ a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \\ a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix}$$

and

$$\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} b_{11}a_{11} + b_{21}a_{12} + b_{31}a_{13} & b_{11}a_{21} + b_{21}a_{22} + b_{31}a_{23} & b_{11}a_{31} + b_{21}a_{32} + b_{31}a_{33} \\ b_{12}a_{11} + b_{22}a_{12} + b_{32}a_{13} & b_{12}a_{21} + b_{22}a_{22} + b_{32}a_{23} & b_{12}a_{31} + b_{22}a_{32} + b_{32}a_{33} \\ b_{13}a_{11} + b_{23}a_{12} + b_{33}a_{13} & b_{13}a_{21} + b_{23}a_{22} + b_{33}a_{23} & b_{13}a_{31} + b_{23}a_{32} + b_{33}a_{33} \end{bmatrix}$$

which confirms that $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$.

9.4.10 Symmetric Matrix

A *symmetric matrix* is a square matrix that equals its transpose: i.e., $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$. For example, $\mathbf{A}$ is a symmetric matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 3 \\ 4 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 3 \\ 4 & 3 & 6 \end{bmatrix}^{\mathrm{T}}.$$

In general, a square matrix $\mathbf{A} = \mathbf{S} + \mathbf{Q}$, where $\mathbf{S}$ is a symmetric matrix, and $\mathbf{Q}$ is an antisymmetric matrix. The symmetric matrix is computed as follows. Given a matrix $\mathbf{A}$ and its transpose $\mathbf{A}^{\mathrm{T}}$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix}$$

their sum is

$$\mathbf{A} + \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 2a_{11} & a_{12} + a_{21} & \dots & a_{1n} + a_{n1} \\ a_{12} + a_{21} & 2a_{22} & \dots & a_{2n} + a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} + a_{n1} & a_{2n} + a_{n2} & \dots & 2a_{nn} \end{bmatrix}.$$

By inspection, $\mathbf{A} + \mathbf{A}^{\mathrm{T}}$ is symmetric, and if we divide throughout by 2 we have

$$\mathbf{S} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^{\mathrm{T}}\right)$$

which is defined as the symmetric part of $\mathbf{A}$. For example, given

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

then

$$
\begin{aligned}
\mathbf{S} &= \frac{1}{2}\left(\mathbf{A}+\mathbf{A}^{\mathrm{T}}\right) \\
&= \begin{bmatrix} a_{11} & (a_{12}+a_{21})/2 & (a_{13}+a_{31})/2 \\ (a_{12}+a_{21})/2 & a_{22} & a_{23}+a_{32} \\ (a_{13}+a_{31})/2 & (a_{23}+a_{32})/2 & a_{33} \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & s_3/2 & s_2/2 \\ s_3/2 & a_{22} & s_1/2 \\ s_2/2 & s_1/2 & a_{33} \end{bmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
s_1 &= a_{23}+a_{32} \\
s_2 &= a_{13}+a_{31} \\
s_3 &= a_{12}+a_{21}.
\end{aligned}
$$

Using a real example:

$$
\mathbf{A} = \begin{bmatrix} 0 & 1 & 4 \\ 3 & 1 & 4 \\ 4 & 2 & 6 \end{bmatrix}, \quad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 0 & 3 & 4 \\ 1 & 1 & 2 \\ 4 & 4 & 6 \end{bmatrix}
$$

$$
\mathbf{S} = \begin{bmatrix} 0 & 2 & 4 \\ 2 & 1 & 3 \\ 4 & 3 & 6 \end{bmatrix}
$$

which equals its own transpose.

9.4.11 Antisymmetric Matrix

An *antisymmetric matrix* is a matrix whose transpose is its own negative:

$$
\mathbf{A}^{\mathrm{T}} = -\mathbf{A}
$$

and is also known as a *skew-symmetric matrix*.

As the elements of $\mathbf{A}$ and $\mathbf{A}^{\mathrm{T}}$ are related by

$$
a_{row,col} = -a_{col,row}.
$$

When $k = row = col$:

$$
a_{k,k} = -a_{k,k}
$$

which implies that the diagonal elements must be zero. For example, this is an antisymmetric matrix

$$\mathbf{A} = \begin{bmatrix} 0 & -2 & 4 \\ 2 & 0 & -3 \\ -4 & 3 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -2 & 4 \\ 2 & 0 & -3 \\ -4 & 3 & 0 \end{bmatrix}^{\mathrm{T}}.$$

The antisymmetric part is computed as follows. Given a matrix $\mathbf{A}$ and its transpose $\mathbf{A}^{\mathrm{T}}$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix}$$

their difference is

$$\mathbf{A} - \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 0 & a_{12} - a_{21} & \dots & a_{1n} - a_{n1} \\ -(a_{12} - a_{21}) & 0 & \dots & a_{2n} - a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -(a_{1n} - a_{n1}) & -(a_{2n} - a_{n2}) & \dots & 0 \end{bmatrix}.$$

It is clear that $\mathbf{A} - \mathbf{A}^{\mathrm{T}}$ is antisymmetric, and if we divide throughout by 2 we have

$$\mathbf{Q} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^{\mathrm{T}}\right).$$

For example:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} 0 & (a_{12} - a_{21})/2 & (a_{13} - a_{31})/2 \\ (a_{21} - a_{12})/2 & 0 & (a_{23} - a_{32})/2 \\ (a_{31} - a_{13})/2 & (a_{32} - a_{23})/2 & 0 \end{bmatrix}$$

and if we maintain some symmetry with the subscripts, we have

$$\begin{aligned} \mathbf{Q} &= \begin{bmatrix} 0 & (a_{12} - a_{21})/2 & -(a_{31} - a_{13})/2 \\ -(a_{12} - a_{21})/2 & 0 & (a_{23} - a_{32})/2 \\ (a_{31} - a_{13})/2 & -(a_{23} - a_{32})/2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & q_3/2 & -q_2/2 \\ -q_3/2 & 0 & q_1/2 \\ q_2/2 & -q_1/2 & 0 \end{bmatrix} \end{aligned}$$

where

$$
\begin{aligned}
q_1 &= a_{23} - a_{32} \\
q_2 &= a_{31} - a_{13} \\
q_3 &= a_{12} - a_{21}.
\end{aligned}
$$

Using a real example:

$$
\mathbf{A} = \begin{bmatrix} 0 & 1 & 4 \\ 3 & 1 & 4 \\ 4 & 2 & 6 \end{bmatrix}, \quad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 0 & 3 & 4 \\ 1 & 1 & 2 \\ 4 & 4 & 6 \end{bmatrix}
$$

$$
\mathbf{Q} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.
$$

Furthermore, we have already computed

$$
\mathbf{S} = \begin{bmatrix} 0 & 2 & 4 \\ 2 & 1 & 3 \\ 4 & 3 & 6 \end{bmatrix}
$$

and

$$
\mathbf{S} + \mathbf{Q} = \begin{bmatrix} 0 & 1 & 4 \\ 3 & 1 & 4 \\ 4 & 2 & 6 \end{bmatrix} = \mathbf{A}.
$$

9.5 Matrix Addition and Subtraction

As equations can be added and subtracted together, it follows that matrices can also be added and subtracted, as long as they have the same dimension. For example, given

$$
\mathbf{A} = \begin{bmatrix} 11 & 22 \\ 14 & -15 \\ 27 & 28 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ -4 & 5 \\ 1 & 8 \end{bmatrix}
$$

then

$$
\mathbf{A} + \mathbf{B} = \begin{bmatrix} 13 & 23 \\ 10 & -10 \\ 28 & 36 \end{bmatrix}, \quad \mathbf{A} - \mathbf{B} = \begin{bmatrix} 9 & 21 \\ 18 & -20 \\ 26 & 20 \end{bmatrix}.
$$

9.5.1 Scalar Multiplication

As equations can be scaled and factorised, it follows that matrixes can also be scaled and factorised.

$$\lambda\mathbf{A} = \lambda \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \dots & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \dots & \lambda a_{23} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \dots & \lambda a_{mn} \end{bmatrix}.$$

For example,

$$2\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}.$$

9.6 Matrix Products

We have already seen that matrices can be multiplied together employing rules that maintain the algebraic integrity of the equations they represent. And as matrices may be vectors, rectangular or square, we need to examine the products that are permitted. To keep the notation simple, the definitions and examples are restricted to a dimension of 3 or 3×3.

We begin with row and column vectors.

9.6.1 Row and Column Vectors

Given

$$\mathbf{A} = \begin{bmatrix} a & b & c \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

then

$$\mathbf{AB} = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = a\alpha + b\beta + c\gamma$$

which is a scalar and equivalent to the dot or scalar product of two vectors.

For example, given

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 10 \\ 30 \\ 20 \end{bmatrix}$$

then

$$\mathbf{AB} = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 10 \\ 30 \\ 20 \end{bmatrix} = 20 + 90 + 80 = 190.$$

Whereas,

$$\mathbf{BA} = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} = \begin{bmatrix} b_{11}a_{11} & b_{11}a_{12} & b_{11}a_{13} \\ b_{21}a_{11} & b_{21}a_{12} & b_{21}a_{13} \\ b_{31}a_{11} & b_{31}a_{12} & b_{31}a_{13} \end{bmatrix}.$$

For example,

$$\mathbf{BA} = \begin{bmatrix} 10 \\ 30 \\ 20 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 20 & 30 & 40 \\ 60 & 90 & 120 \\ 40 & 60 & 80 \end{bmatrix}.$$

The products **AA** and **BB** are not permitted.

9.6.2 Row Vector and a Matrix

Given

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{m1} & b_{m2} & b_{33} \end{bmatrix}$$

then

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{m1} & b_{m2} & b_{33} \end{bmatrix}$$
$$= \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}) & (a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}) & (a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33}) \end{bmatrix}.$$

The product **BA** is not permitted.

For example, given

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

then

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \\ &= \begin{bmatrix} (2+9+16) & (4+12+20) & (6+15+24) \end{bmatrix} \\ &= \begin{bmatrix} 27 & 36 & 45 \end{bmatrix}. \end{aligned}$$

9.6.3 Matrix and a Column Vector

Given

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}$$

then

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} \end{bmatrix}.$$

The product **BA** is not permitted.

For example, given

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}, \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

then

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2+6+12 \\ 6+12+20 \\ 8+15+24 \end{bmatrix} = \begin{bmatrix} 20 \\ 38 \\ 47 \end{bmatrix}.$$

9.6.4 Square Matrices

To clarify the products, lower-case Greek symbols are used with lower-case letters. Here are their names:

$$\begin{array}{lll} \alpha = \text{alpha}, & \beta = \text{beta}, & \gamma = \text{gamma}, \\ \lambda = \text{lambda}, & \mu = \text{mu}, & \nu = \text{nu}, \\ \rho = \text{rho}, & \sigma = \text{sigma}, & \tau = \text{tau}. \end{array}$$

Given

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ p & q & r \\ u & v & w \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} \alpha & \beta & \gamma \\ \lambda & \mu & \nu \\ \rho & \sigma & \tau \end{bmatrix}$$

then

$$\mathbf{AB} = \begin{bmatrix} a & b & c \\ p & q & r \\ u & v & w \end{bmatrix} \begin{bmatrix} \alpha & \beta & \gamma \\ \lambda & \mu & \nu \\ \rho & \sigma & \tau \end{bmatrix} = \begin{bmatrix} a\alpha + b\lambda + c\rho & a\beta + b\mu + c\sigma & a\gamma + b\nu + c\tau \\ p\alpha + q\lambda + r\rho & p\beta + q\mu + r\sigma & p\gamma + q\nu + r\tau \\ u\alpha + v\lambda + w\rho & u\beta + v\mu + w\sigma & u\gamma + v\nu + w\tau \end{bmatrix}$$

and

$$\mathbf{BA} = \begin{bmatrix} \alpha & \beta & \gamma \\ \lambda & \mu & \nu \\ \rho & \sigma & \tau \end{bmatrix} \begin{bmatrix} a & b & c \\ p & q & r \\ u & v & w \end{bmatrix} = \begin{bmatrix} \alpha a + \beta p + \gamma u & \alpha b + \beta q + \gamma v & \alpha c + \beta r + \gamma w \\ \lambda a + \mu p + \nu u & \lambda b + \mu q + \nu v & \lambda c + \mu r + \nu w \\ \rho a + \sigma p + \tau u & \rho b + \sigma q + \tau v & \rho c + \sigma r + \tau w \end{bmatrix}.$$

For example, given

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 8 \end{bmatrix}$$

then

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 28 & 34 & 40 \\ 52 & 64 & 76 \\ 76 & 92 & 112 \end{bmatrix}$$

and

$$\mathbf{BA} = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 31 & 40 & 49 \\ 49 & 64 & 89 \\ 67 & 88 & 109 \end{bmatrix}.$$

9.6.5 Rectangular Matrices

Given two rectangular matrices **A** and **B**, where **A** has a dimension $m \times n$, the product **AB** is permitted, if and only if, **B** has a dimension $n \times p$. The resulting matrix has a dimension $m \times p$. For example, given

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{bmatrix}$$

then

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{bmatrix} \\ &= \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{12} + a_{12}b_{22}) & (a_{11}b_{13} + a_{12}b_{23}) & (a_{11}b_{14} + a_{12}b_{24}) \\ (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{12} + a_{22}b_{22}) & (a_{21}b_{13} + a_{22}b_{23}) & (a_{21}b_{14} + a_{22}b_{24}) \\ (a_{31}b_{11} + a_{32}b_{21}) & (a_{31}b_{12} + a_{32}b_{22}) & (a_{31}b_{13} + a_{32}b_{23}) & (a_{31}b_{14} + a_{32}b_{24}) \end{bmatrix}. \end{aligned}$$

9.7 Inverse Matrix

A square matrix $\mathbf{A}_{nn}$ that is *invertible* satisfies the condition:

$$\mathbf{A}_{nn}\mathbf{A}_{nn}^{-1} = \mathbf{A}_{nn}^{-1}\mathbf{A}_{nn} = \mathbf{I}_n,$$

where $\mathbf{A}_{nn}^{-1}$ is unique, and is the *inverse matrix* of $\mathbf{A}_{nn}$. For example, given

$$\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}$$

then

$$\mathbf{A}^{-1} = \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix}$$

because

$$\mathbf{AA}^{-1} = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

A square matrix whose determinant is 0, cannot have an inverse, and is known as a *singular matrix*.

We now require a way to compute $\mathbf{A}^{-1}$, which is rather easy.
Consider two linear equations:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{9.10}$$

Let the inverse of

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

therefore,

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{9.11}$$

From (9.11) we have

$$ae + cf = 1 \tag{9.12}$$
$$be + df = 0 \tag{9.13}$$
$$ag + ch = 0 \tag{9.14}$$
$$bg + dh = 1. \tag{9.15}$$

Multiply (9.12) by d and (9.13) by c, and subtract:

$$\begin{aligned} ade + cdf &= d \\ bce + cdf &= 0 \\ ade - bce &= d \end{aligned}$$

therefore,

$$e = \frac{d}{ad - bc}.$$

Multiply (9.12) by b and (9.13) by a, and subtract:

$$\begin{aligned} abe + bcf &= b \\ abe + adf &= 0 \\ adf - bcf &= -b \end{aligned}$$

therefore,

$$f = \frac{-b}{ad - bc}.$$

Multiply (9.14) by d and (9.15) by c, and subtract:

$$\begin{aligned} adg + cdh &= 0 \\ bcg + cdh &= c \\ adg - bcg &= -c \end{aligned}$$

therefore,

$$g = \frac{-c}{ad - bc}.$$

Multiply (9.14) by b and (9.15) by a, and subtract:

$$\begin{aligned} abg + bch &= 0 \\ abg + adh &= a \\ adh - bch &= a \end{aligned}$$

therefore,

$$h = \frac{a}{ad - bc}.$$

We now have values for e, f, g and h, which are the elements of the inverse matrix. Consequently, given

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \mathbf{A}^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix},$$

then

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The inverse matrix permits us to solve a pair or linear equations as follows. Starting with

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

multiply both sides by the inverse matrix:

$$
\begin{aligned}
\mathbf{A}^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \mathbf{A}^{-1}\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} \\
\mathbf{A}^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \\
\begin{bmatrix} x \\ y \end{bmatrix} &= \mathbf{A}^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix} \\
\begin{bmatrix} x \\ y \end{bmatrix} &= \frac{1}{\det \mathbf{A}}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}.
\end{aligned}
$$

Although the elements of $\mathbf{A}^{-1}$ come from $\mathbf{A}$, the relationship is not obvious. However, if $\mathbf{A}$ is transposed, a pattern is revealed. Given

$$
\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then} \quad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}
$$

and placing $\mathbf{A}^{-1}$ alongside $\mathbf{A}^{\mathrm{T}}$, we have

$$
\mathbf{A}^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \quad \text{and} \quad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.
$$

The elements of $\mathbf{A}^{-1}$ share a common denominator ($\det \mathbf{A}$), which is placed outside the matrix, therefore, the matrix elements are taken from $\mathbf{A}^{\mathrm{T}}$ as follows. For any entry a_{ij} in $\mathbf{A}^{-1}$, mask out the ith row and jth column in $\mathbf{A}^{\mathrm{T}}$, and the remaining entry is copied to the ijth entry in $\mathbf{A}^{-1}$. In the case of e, it is d. For f, it is b, with a sign reversal. For g, it is c, with a sign reversal, and for h, it is a. The sign change is computed by the same formula used with determinants:

$$
(-1)^{i+j}.
$$

which generates this pattern:

$$
\begin{bmatrix} + & - \\ - & + \end{bmatrix}.
$$

You may be wondering what happens when a 3×3 matrix is inverted. Well, the same technique is used, but when the ith row and jth column in $\mathbf{A}^{\mathrm{T}}$ is masked out, it leaves behind a 2×2 determinant, whose value is copied to the ijth entry in $\mathbf{A}^{-1}$, with the appropriate sign change. We investigate this later on.

Let's illustrate this with an example. Given

$$
\begin{aligned}
42 &= 6x + 2y \\
28 &= 2x + 3y
\end{aligned}
$$

let

$$\mathbf{A} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

then $\det \mathbf{A} = 14$, therefore,

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \frac{1}{14} \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} 42 \\ 28 \end{bmatrix} \\ &= \frac{1}{14} \begin{bmatrix} 70 \\ 84 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ 6 \end{bmatrix}. \end{aligned}$$

which is the solution.

Now let's investigate how to invert a 3×3 matrix. Given three simultaneous equations in three unknowns:

$$\begin{aligned} x' &= ax + by + cz \\ y' &= dx + ey + fz \\ z' &= gx + hy + jz \end{aligned}$$

they can be written using matrices as follows:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Let

$$\mathbf{A}^{-1} = \begin{bmatrix} l & m & n \\ p & q & r \\ s & t & u \end{bmatrix}$$

therefore,

$$\begin{bmatrix} l & m & n \\ p & q & r \\ s & t & u \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{9.16}$$

From (9.16) we can write:

$$la + md + ng = 1 \tag{9.17}$$

$$lb + me + nh = 0 \tag{9.18}$$

$$lc + mf + nj = 0. \tag{9.19}$$

Multiply (9.17) by e and (9.18) by d, and subtract:

$$\begin{aligned} ael + dem + egn &= e \\ bdl + dem + dhn &= 0 \\ ael - bdl + egn - dhn &= e \\ l(ae - bd) + n(eg - dh) &= e. \end{aligned} \tag{9.20}$$

Multiply (9.18) by f and (9.19) by e, and subtract:

$$\begin{aligned} bfl + efm + fhn &= 0 \\ cel + efm + ejn &= 0 \\ bfl - cel + fhn - ejn &= 0 \\ l(bf - ce) + n(fh - ej) &= 0. \end{aligned} \tag{9.21}$$

Multiply (9.20) by $(fh - ej)$ and (9.21) by $(eg - dh)$, and subtract:

$$\begin{aligned} l(ae - bd)(fh - ej) + n(eg - dh)(fh - ej) &= e(fh - ej) \\ l(bf - ce)(eg - dh) + n(eg - dh)(fh - ej) &= 0 \\ l(ae - bd)(fh - ej) - l(bf - ce)(eg - dh) &= efh - e^2 j \\ l(aefh - ae^2 j - bdfh + bdej - befg + bdfh + ce^2 g - cdeh) &= efh - e^2 j \\ l(aefh - ae^2 j + bdej - befg + ce^2 g - cdeh) &= efh - e^2 j \\ l(afh + bdj + ceg - aej - cdh - bfg) &= fh - ej \\ l(aej + bfg + cdh - afh - bdj - ceg) &= ej - fh \end{aligned}$$

but $(aej + bfg + cdh - afh - bdj - ceg)$ is the Sarrus expansion for det **A**, therefore

$$l = \frac{ej - fh}{\det \mathbf{A}}.$$

An exhaustive algebraic analysis reveals:

$$\begin{aligned} l &= \frac{ej - fh}{\det \mathbf{A}}, & m &= -\frac{bj - ch}{\det \mathbf{A}}, & n &= \frac{bf - ce}{\det \mathbf{A}}, \\ p &= -\frac{dj - gf}{\det \mathbf{A}}, & q &= \frac{aj - gc}{\det \mathbf{A}}, & r &= -\frac{af - dc}{\det \mathbf{A}}, \\ s &= \frac{dh - ge}{\det \mathbf{A}}, & t &= -\frac{ah - gb}{\det \mathbf{A}}, & u &= \frac{ae - bd}{\det \mathbf{A}}, \end{aligned}$$

where

$$\mathbf{A}^{-1} = \begin{bmatrix} l & m & n \\ p & q & r \\ s & t & u \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}.$$

However, there does not appear to be an obvious way of deriving $\mathbf{A}^{-1}$ from $\mathbf{A}$. But, as we discovered with the 2×2 matrix, the transpose $\mathbf{A}^{\mathrm{T}}$ resolves the problem:

$$\mathbf{A}^{-1} = \begin{bmatrix} l & m & n \\ p & q & r \\ s & t & u \end{bmatrix}, \quad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & j \end{bmatrix}.$$

The elements for $\mathbf{A}^{-1}$ share a common denominator (det $\mathbf{A}$), which is placed outside the matrix, therefore, the matrix elements are taken from $\mathbf{A}^{\mathrm{T}}$ as follows. For any entry a_{ij} in $\mathbf{A}^{-1}$, mask out the ith row and jth column in $\mathbf{A}^{\mathrm{T}}$, and the remaining elements, in the form of a 2×2 determinant, is copied to the ijth entry in $\mathbf{A}^{-1}$. In the case of l, it is $(ej - hf)$. For m, it is $(bj - hc)$, with a sign reversal, and for n, it is $(bf - ec)$. The sign change is computed by the same formula used with determinants:

$$(-1)^{i+j},$$

which generates the pattern:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}.$$

With the above *aide-mémoire*, it is easy to write down the inverse matrix:

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} (ej - fh) & -(bj - ch) & (bf - ce) \\ -(dj - gf) & (aj - gc) & -(af - dc) \\ (dh - ge) & -(ah - gb) & (ae - bd) \end{bmatrix}.$$

This technique is known as the *Laplacian expansion* or the *cofactor expansion*, after Pierre-Simon Laplace. The matrix of minor determinants is called the *cofactor matrix* of $\mathbf{A}$, which permits the inverse matrix to be written as:

$$\mathbf{A}^{-1} = \frac{(\text{cofactor matrix of } \mathbf{A})^{\mathrm{T}}}{\det \mathbf{A}}.$$

Let's illustrate this solution with an example. Given

$$\begin{aligned} 18 &= 2x + 2y + 2z \\ 20 &= x + 2y + 3z \\ 7 &= y + z \end{aligned}$$

therefore,

$$\begin{bmatrix} 18 \\ 20 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

and

$$\begin{aligned} \det \mathbf{A} &= 4 + 2 - 2 - 6 = -2 \\ \mathbf{A}^{\mathrm{T}} &= \begin{bmatrix} 2 & 1 & 0 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix} \end{aligned}$$

therefore,

$$\mathbf{A}^{-1} = -\frac{1}{2} \begin{bmatrix} -1 & 0 & 2 \\ -1 & 2 & -4 \\ 1 & -2 & 2 \end{bmatrix}$$

and

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -1 & 0 & 2 \\ -1 & 2 & -4 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 18 \\ 20 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

which is the solution.

9.7.1 Inverting a Pair of Matrices

Having seen how to invert a single matrix, let's investigate how to invert of a pair of matrices.

Given two matrices $\mathbf{T}$ and $\mathbf{R}$, the product $\mathbf{TR}$ and its inverse $(\mathbf{TR})^{-1}$ must equal the identity matrix $\mathbf{I}$:

$$(\mathbf{TR})(\mathbf{TR})^{-1} = \mathbf{I}$$

and multiplying throughout by $\mathbf{T}^{-1}$ we have

$$\begin{aligned}\mathbf{T}^{-1}\mathbf{T}\mathbf{R}(\mathbf{T}\mathbf{R})^{-1} &= \mathbf{T}^{-1}\\ \mathbf{R}(\mathbf{T}\mathbf{R})^{-1} &= \mathbf{T}^{-1}.\end{aligned}$$

Multiplying throughout by $\mathbf{R}^{-1}$ we have

$$\begin{aligned}\mathbf{R}^{-1}\mathbf{R}(\mathbf{T}\mathbf{R})^{-1} &= \mathbf{R}^{-1}\mathbf{T}^{-1}\\ (\mathbf{T}\mathbf{R})^{-1} &= \mathbf{R}^{-1}\mathbf{T}^{-1}.\end{aligned}$$

Therefore, if $\mathbf{T}$ and $\mathbf{R}$ are invertible, then

$$(\mathbf{T}\mathbf{R})^{-1} = \mathbf{R}^{-1}\mathbf{T}^{-1}.$$

Generalising this result to a triple product such as **STR** we can reason that

$$(\mathbf{S}\mathbf{T}\mathbf{R})^{-1} = \mathbf{R}^{-1}\mathbf{T}^{-1}\mathbf{S}^{-1}.$$

9.8 Orthogonal Matrix

A matrix is *orthogonal* if its transpose is also its inverse, i.e., matrix **A** is orthogonal if

$$\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{-1}.$$

For example,

$$\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and

$$\mathbf{A}\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which implies that $\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{-1}$.

The following matrix is also orthogonal

$$\mathbf{A} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

because

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$$

and

$$\mathbf{A}\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Orthogonal matrices play an important role in rotations because they leave the origin fixed and preserve all angles and distances. Consequently, an object's geometric integrity is maintained after a rotation, which is why an orthogonal transform is known as a *rigid motion* transform.

9.9 Diagonal Matrix

A *diagonal matrix* is a square matrix whose elements are zero, apart from its diagonal:

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}.$$

The determinant of a diagonal matrix must be

$$\det \mathbf{A} = a_{11} \times a_{22} \times \cdots a_{nn}.$$

Here is a diagonal matrix with its determinant

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\det \mathbf{A} = 2 \times 3 \times 4 = 24.$$

The identity matrix **I** is a diagonal matrix with a determinant of 1.

9.10 Worked Examples

9.10.1 Matrix Inversion

Invert $\mathbf{A}$ and show that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_2$.

$$\mathbf{A} = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}.$$

Using

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

then $\det \mathbf{A} = 2$, and

$$\mathbf{A}^{-1} = \frac{1}{2} \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}.$$

Calculating $\mathbf{A}\mathbf{A}^{-1}$:

$$\mathbf{A}\mathbf{A}^{-1} = \frac{1}{2} \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

9.10.2 Identity Matrix

Invert $\mathbf{A}$ and show that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_3$.

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 1 \\ 5 & 6 & 7 \end{bmatrix}.$$

Using Sarrus's rule for $\det \mathbf{A}$:

$$\det \mathbf{A} = 28 + 15 + 24 - 40 - 12 - 21 = -6.$$

Therefore,

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 2 & 6 \\ 4 & 1 & 7 \end{bmatrix}$$

$$\mathbf{A}^{-1} = -\frac{1}{6}\begin{bmatrix} (14-6) & -(21-24) & (3-8) \\ -(7-5) & (14-20) & -(2-4) \\ (6-10) & -(12-15) & (4-3) \end{bmatrix}$$

$$= -\frac{1}{6}\begin{bmatrix} 8 & 3 & -5 \\ -2 & -6 & 2 \\ -4 & 3 & 1 \end{bmatrix}$$

and

$$\mathbf{A}\mathbf{A}^{-1} = -\frac{1}{6}\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 1 \\ 5 & 6 & 7 \end{bmatrix}\begin{bmatrix} 8 & 3 & -5 \\ -2 & -6 & 2 \\ -4 & 3 & 1 \end{bmatrix}$$

$$= -\frac{1}{6}\begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

9.10.3 Solving Two Equations Using Matrices

Solve the following equations using matrices.

$$20 = 2x + 3y$$
$$36 = 7x + 2y.$$

Let

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 7 & 2 \end{bmatrix}$$

therefore, det $\mathbf{A} = -17$, and

$$\mathbf{A}^{-1} = -\frac{1}{17}\begin{bmatrix} 2 & -3 \\ -7 & 2 \end{bmatrix}$$

therefore,

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{17}\begin{bmatrix} 2 & -3 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} 20 \\ 36 \end{bmatrix}$$

$$= -\frac{1}{17}\begin{bmatrix} 40 - 108 \\ -140 + 72 \end{bmatrix}$$

$$= -\frac{1}{17}\begin{bmatrix} -68 \\ -68 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

therefore, $x = y = 4$.

9.10.4 Solving Three Equations Using Matrices

Solve the following equations using matrices.

$$\begin{aligned} 10 &= 2x + y - z \\ 13 &= -x - y + z \\ 28 &= -x + 2y + z. \end{aligned}$$

Let

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix}.$$

Using Sarrus's rule for det $\mathbf{A}$:

$$\det \mathbf{A} = -2 - 1 + 2 + 1 - 4 + 1 = -3.$$

Therefore,

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 2 & -1 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & 1 \end{bmatrix}$$
$$\mathbf{A}^{-1} = -\frac{1}{3}\begin{bmatrix} (-1-2) & -(1+2) & (1-1) \\ -(-1+1) & (2-1) & -(2-1) \\ (-2-1) & -(4+1) & (-2+1) \end{bmatrix}$$
$$= -\frac{1}{3}\begin{bmatrix} -3 & -3 & 0 \\ 0 & 1 & -1 \\ -3 & -5 & -1 \end{bmatrix}$$

therefore,

$$\begin{aligned}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= -\frac{1}{3}\begin{bmatrix} -3 & -3 & 0 \\ 0 & 1 & -1 \\ -3 & -5 & -1 \end{bmatrix}\begin{bmatrix} 10 \\ 13 \\ 28 \end{bmatrix} \\
&= -\frac{1}{3}\begin{bmatrix} -30-39 \\ 13-28 \\ -30-65-28 \end{bmatrix} \\
&= -\frac{1}{3}\begin{bmatrix} -69 \\ -15 \\ -123 \end{bmatrix} \\
&= \begin{bmatrix} 23 \\ 5 \\ 41 \end{bmatrix}
\end{aligned}$$

therefore, $x = 23, \quad y = 5, \quad z = 41.$

9.10.5 Solving Two Complex Equations

Solve the following complex equations using matrices.

$$\begin{aligned}
7 + i8 &= 2x + y \\
-4 - i &= x - 2y.
\end{aligned}$$

Let

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$

therefore, $\det \mathbf{A} = -5$, and

$$\begin{aligned}
\mathbf{A}^{\mathrm{T}} &= \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \\
\mathbf{A}^{-1} &= -\frac{1}{5}\begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix}
\end{aligned}$$

therefore,

$$\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} &= -\frac{1}{5}\begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 7+i8 \\ -4-i \end{bmatrix} \\
&= -\frac{1}{5}\begin{bmatrix} -14 - i16 + 4 + i \\ -7 - i8 - 8 - i2 \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{5}\begin{bmatrix} -10 - i15 \\ -15 - i10 \end{bmatrix} \\
&= \begin{bmatrix} 2 + i3 \\ 3 + i2 \end{bmatrix}
\end{aligned}
$$

therefore, $x = 2 + i3, \quad y = 3 + i2.$

9.10.6 Solving Three Complex Equations

Solve the following complex equations using matrices.

$$
\begin{aligned}
0 &= x + y - z \\
3 + i3 &= 2x - y + z \\
-5 - i5 &= -x + y - 2z.
\end{aligned}
$$

Let

$$
\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & -2 \end{bmatrix}
$$

therefore, $\det \mathbf{A} = 2 - 1 - 2 + 1 - 1 + 4 = 3$, and

$$
\begin{aligned}
\mathbf{A}^{\mathrm{T}} &= \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -2 \end{bmatrix} \\
\mathbf{A}^{-1} &= \frac{1}{3}\begin{bmatrix} (2-1) & -(-2+1) & 0 \\ -(-4+1) & (-2-1) & -(1+2) \\ (2-1) & -(1+1) & (-1-2) \end{bmatrix}
\end{aligned}
$$

therefore,

$$
\begin{aligned}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \frac{1}{3}\begin{bmatrix} 1 & 1 & 0 \\ 3 & -3 & -3 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 3 + i3 \\ -5 - i5 \end{bmatrix} \\
&= \frac{1}{3}\begin{bmatrix} 3 + i3 \\ -9 - i9 + 15 + i15 \\ -6 - i6 + 15 + i15 \end{bmatrix} \\
&= \begin{bmatrix} 1 + i \\ 2 + i2 \\ 3 + i3 \end{bmatrix}
\end{aligned}
$$

therefore, $x = 1 + i, \quad y = 2 + i2, \quad z = 3 + i3.$

9.10.7 Solving Two Complex Equations

Solve the following complex equations using matrices.

$$\begin{aligned} 3 + i5 &= ix + 2y \\ 5 + i &= 3x - iy. \end{aligned}$$

Let

$$\mathbf{A} = \begin{bmatrix} i & 2 \\ 3 & -i \end{bmatrix}$$

therefore, set $\mathbf{A} = 1 - 6 = -5$, and

$$\begin{aligned} \mathbf{A}^{\mathrm{T}} &= \begin{bmatrix} i & 3 \\ 2 & -i \end{bmatrix} \\ \mathbf{A}^{-1} &= -\frac{1}{5}\begin{bmatrix} -i & -2 \\ -3 & i \end{bmatrix} \end{aligned}$$

therefore,

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= -\frac{1}{5}\begin{bmatrix} -i & -2 \\ -3 & i \end{bmatrix}\begin{bmatrix} 3 + i5 \\ 5 + i \end{bmatrix} \\ &= -\frac{1}{5}\begin{bmatrix} -i3 + 5 - 10 - i2 \\ -9 - i15 + i5 - 1 \end{bmatrix} \\ &= -\frac{1}{5}\begin{bmatrix} -5 - i5 \\ -10 - i10 \end{bmatrix} \\ &= \begin{bmatrix} 1 + i \\ 2 + i2 \end{bmatrix} \end{aligned}$$

therefore, $x = 1 + i, \quad y = 2 + i2.$

9.10.8 Solving Three Complex Equations

Solve the following complex equations using matrices.

$$\begin{aligned} 6 + i2 &= ix + 2y - iz \\ -2 + i6 &= 2x - iy + i2z \\ 2 + i10 &= i2x + iy + 2z. \end{aligned}$$

Let

$$\mathbf{A}=\begin{bmatrix} i & 2 & -i \\ 2 & -i & i2 \\ i2 & i & 2 \end{bmatrix}$$

therefore, $\det \mathbf{A} = 2-8+2+i2+i2-8=-12+i4$, and

$$\mathbf{A}^{\mathrm{T}}=\begin{bmatrix} i & 2 & i2 \\ 2 & -i & i \\ -i & i2 & 2 \end{bmatrix}$$

$$\mathbf{A}^{-1}=\frac{1}{-12+i4}\begin{bmatrix} (-i2+2) & -(4-1) & (i4+1) \\ -(4+4) & (i2-2) & -(-2+i2) \\ (i2-2) & -(-1-i4) & (1-4) \end{bmatrix}$$

$$=\frac{1}{-12+i4}\begin{bmatrix} 2-i2 & -3 & 1+i4 \\ -8 & -2+i2 & 2-i2 \\ -2+i2 & 1+i4 & -3 \end{bmatrix}$$

therefore,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\frac{1}{-12+i4}\begin{bmatrix} 2-i2 & -3 & 1+i4 \\ -8 & -2+i2 & 2-i2 \\ -2+i2 & 1+i4 & -3 \end{bmatrix}\begin{bmatrix} 6+i2 \\ -2+i6 \\ 2+i10 \end{bmatrix}$$

$$=\frac{1}{-12+i4}\begin{bmatrix} (2-i2)(6+i2)-3(-2+i6)+(1+i4)(2+i10) \\ -8(6+i2)+(-2+i2)(-2+i6)+(2-i2)(2+i10) \\ (-2+i2)(6+i2)+(1+i4)(-2+i6)-3(2+i10) \end{bmatrix}$$

$$=\frac{1}{-12+i4}\begin{bmatrix} 12+i4-i12+4+6-i18+2+i10+i8-40 \\ -48-i16+4-i12-i4-12+4+i20-i4+20 \\ -12-i4+i12-4-2+i6-i8-24-6-i30 \end{bmatrix}$$

$$=\frac{1}{-12+i4}\begin{bmatrix} -16-i8 \\ -32-i16 \\ -48-i24 \end{bmatrix}$$

multiply by the conjugate of $-12+i4$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\frac{-12-i4}{160}\begin{bmatrix} -16-i8 \\ -32-i16 \\ -48-i24 \end{bmatrix}$$

therefore,

$$x=\frac{1}{160}(-12-i4)(-16-i8)$$

$$
\begin{aligned}
&= \frac{1}{160}(192 + i64 + i96 - 32) \\
&= \frac{1}{160}(160 + i160) = 1 + i \\
y &= \frac{1}{160}(-12 - i4)(-32 - i16) \\
&= \frac{1}{160}(384 + i128 + i192 - 64) \\
&= \frac{1}{160}(320 + i320) = 2 + i2 \\
z &= \frac{1}{160}(-12 - i4)(-48 - i24) \\
&= \frac{1}{160}(576 + i192 + i288 - 96) \\
&= \frac{1}{160}(480 + i480) = 3 + i3
\end{aligned}
$$

therefore, $x = 1 + i, \quad y = 2 + i2, \quad z = 3 + i3.$