Hierarchical Dirichlet Process for Switching Linear Dynamical Systems

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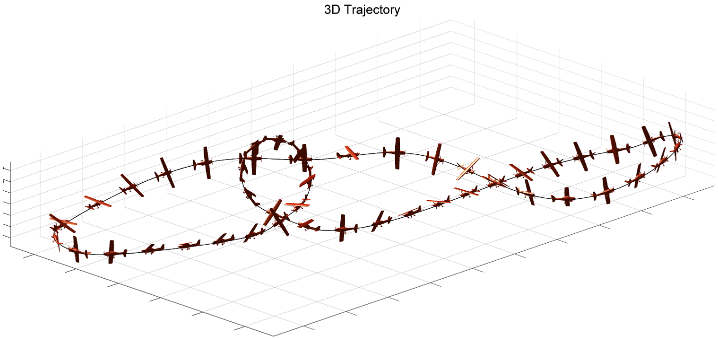
Overview

How to Estimate?

Fighter pilot's

- Maneuvering style
- Maneuvering strategy

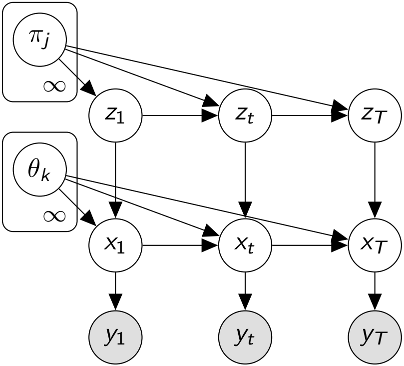


Switching Linear Dynamical System

$$\{\mathbf{A}, \mathbf{B}\} \in \mathcal{A} \times \mathcal{B} = \mathbf{\Theta}$$

$$z_t \sim Markov(\mathbf{\Pi})$$
$$x_t | z_t = \mathbf{A}_{z_t} x_{t-1} + \mathbf{B}_{z_t}$$
$$y_t | x_t \sim F(x_t)$$

Dimension?



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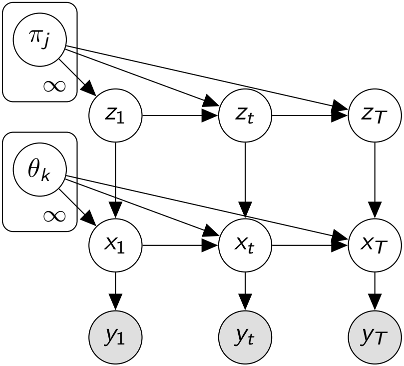
Dimension?

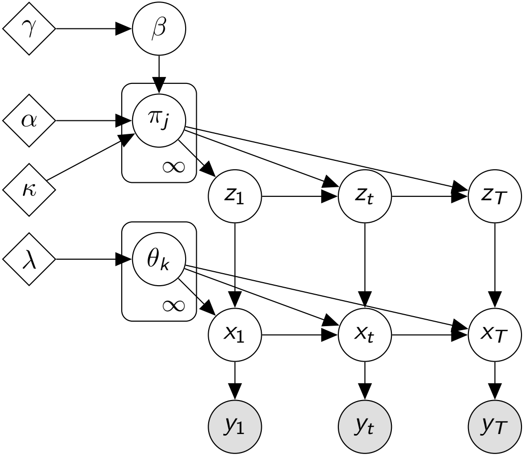
1. $dim(\mathcal{Z}) = |\mathcal{Z}|$
2. $dim(\mathbf{\Pi}) = O(|\mathcal{Z}|^2)$
3. $dim(\mathbf{\Theta}) = O(d_x^2 * |\mathcal{Z}|)$



Keep Calm...

and put a prior...

- $\pi_j \sim DP(\alpha, \beta)$
- But....

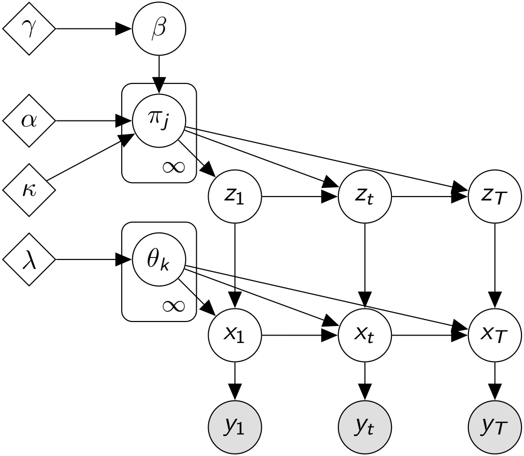


What does HDP mean in HMM?

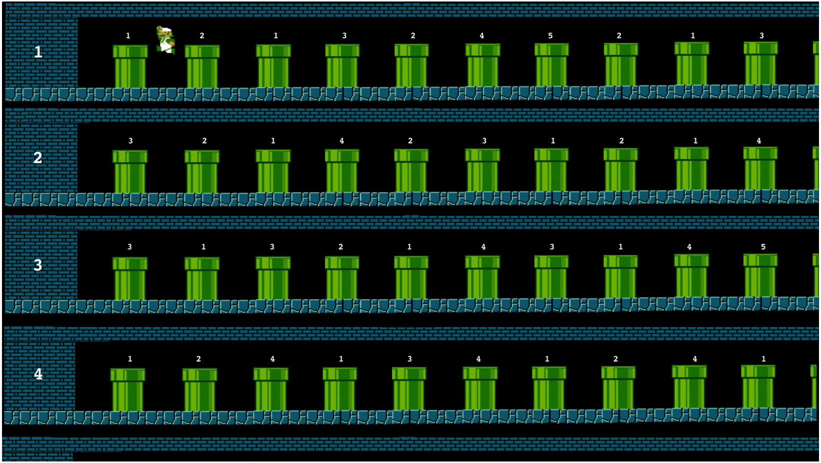
Restaurant Analogy?

- Global Dish Menu?
- Local Dish Menu?
- Restaurant?
- Dish?
- Customer?

Better Analogy?



HDP in HMM: Mario's Warp Pipe Process



HDP in SLDS: Inference Overview

Need to sample $(\mathbf{z}, \beta, \mathbf{\Pi})$ and $(\mathbf{x}, \mathbf{\Theta})$ jointly:

- Sample $\mathbf{z}, \beta, \mathbf{\Pi} | \mathbf{\Theta}, \mathbf{x}$
  1. Sample $\mathbf{z} \sim p(\mathbf{z} | \mathbf{\Pi}, \mathbf{\Theta}, \mathbf{x})$
     - In contrast to conditional sampler in original HDP $p(z_t = k | \mathbf{z}_{(-t)}, \beta) = p(z_t = k | \mathbf{y}) f(y_t | \mathbf{y}_{-(t)})$
     - efficiently algorithm available for Markov Model (Forward-backward Message Passing)
  2. Sample β, π_k through standard CRF:
     - $\beta | \mathbf{z} \sim Dir(\mathbf{m}, \gamma)$
     - $\pi_k | \beta, \mathbf{z} \sim Dir(\alpha \beta_k + n_k)$
- Sample $\mathbf{x}, \mathbf{\Theta} | \mathbf{z}, \beta, \mathbf{\Pi}$

Linear Dynamical Systems

For times $t = 1 \dots T$, we are given data $y_t \in \mathbb{R}^n$. We assume y_t is a noise observation of a hidden, continuous state $x_t \in \mathbb{R}^d$, with $d \geq n$, and that x_t is a markov chain. The probability model is:

$$x_t | x_{t-1} \sim \mathcal{N}(Ax_{t-1} + B, \Sigma)$$
$$y_t | x_t \sim \mathcal{N}(Cx_t, R)$$

We assume C is known.

Switching Linear Dynamical Systems

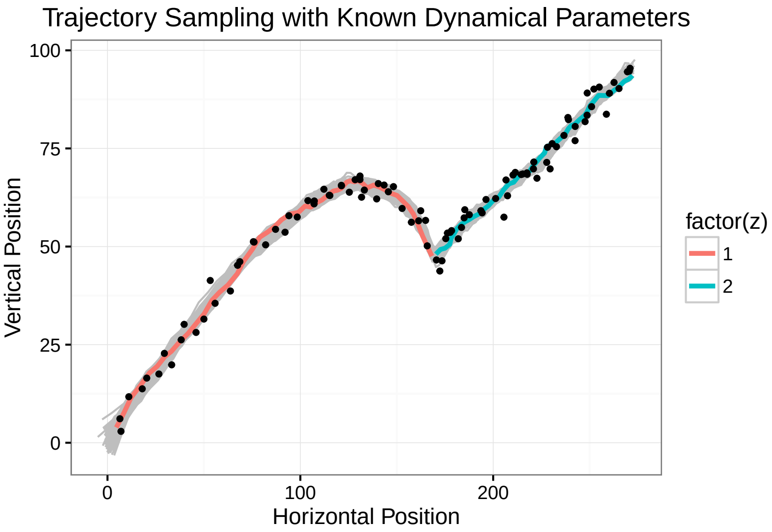
We now assume at each time t there is a hidden mode indexed by $z_t \in \{1, \ldots, K\}$ that determines the dynamical regime. The probability model is:

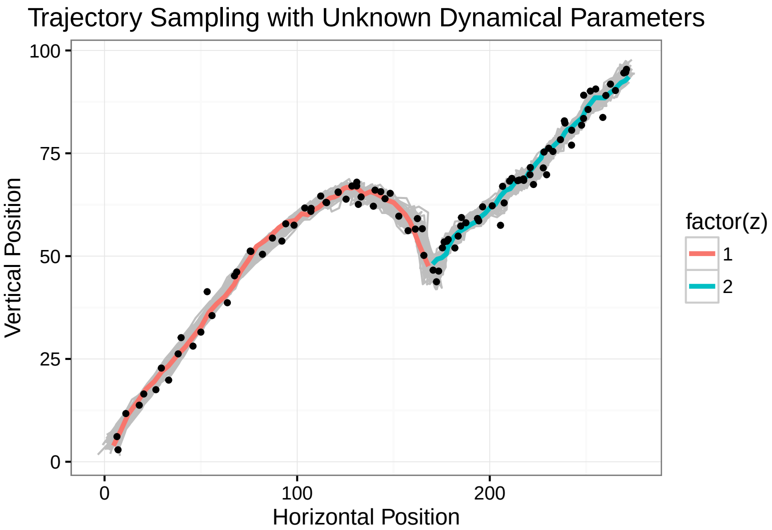
$$
\begin{aligned}
x_t | x_{t-1}, z_t &\sim \mathcal{N}(A^{(z_t)} x_{t-1} + B^{(z_t)}, \Sigma^{(z_t)}) \\
y_t | x_t &\sim \mathcal{N}(C x_t, R)
\end{aligned}
$$

In the standard SLDS, $K < \infty$. In the HDP-HMM-SLDS, $K = \infty$.

Algorithms for Inference

- Conditional on known dynamical parameters A, B, Σ and $z_{1:T}$, the hidden states $x_{1:T}$ are obtained via a **Kalman sampler**.
- Conditional on known hidden states $x_{1:T}$, the hidden modes $z_{1:T}$ are obtained via the **Forward-Backward** (sampling) algorithm.
- Conditional on known $x_{1:T}, z_{1:T}$, the dynamical parameters are obtained via **multivariate linear regressions** for each of the modes.


Trajectory Sampling with Known Dynamical Parameters
Vertical Position
100
75
50
25
0
0
100
200
Horizontal Position
factor(z)
1
2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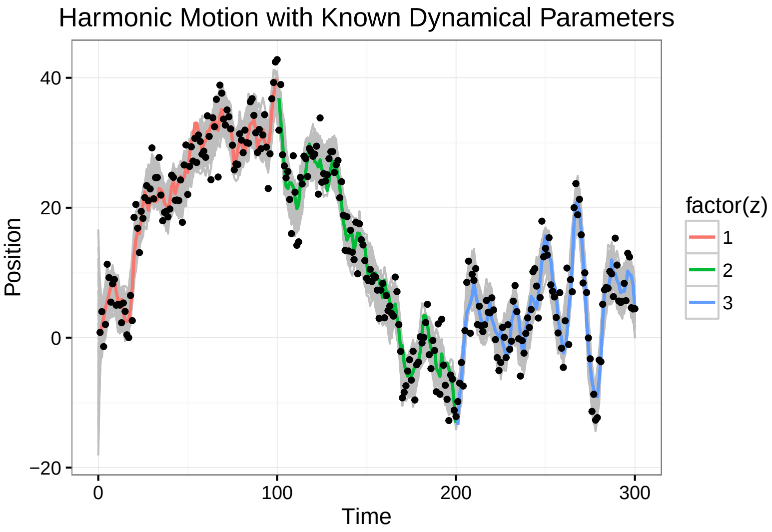
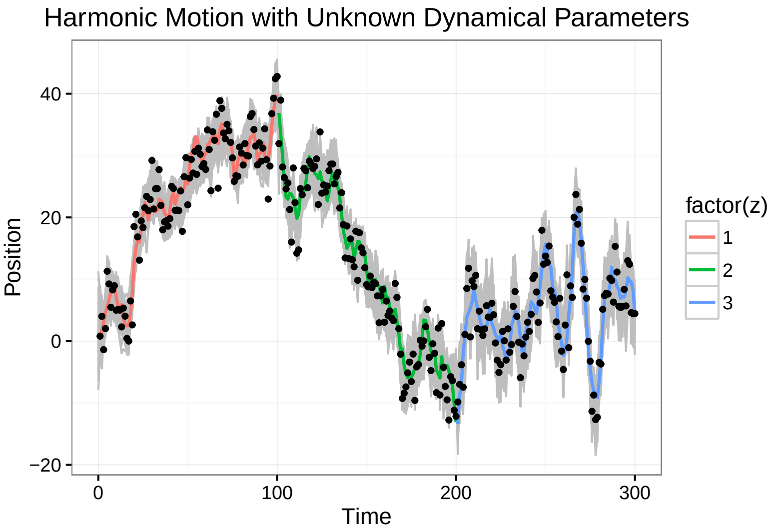

Harmonic Motion with Known Dynamical Parameters
Position
Time
40
20
0
−20
0
100
200
300
factor(z)
1
2
3


Harmonic Motion with Unknown Dynamical Parameters
Position
Time
factor(z)
1
2
3
40
20
0
−20
0
100
200
300

Lessons Learned

1. Kalman sampler great when dynamical parameters known.
2. When dynamical parameters unknown, choice of prior is crucial.
3. "Uninformative" hyperparameters didn't work. Need regularization.
4. Non-stationary time series present problems for empirical bayes.
5. Higher dimensional state space is more flexible but may overfit.