Bayesian Nonparametric Estimation of Switching Linear Dynamic System

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Contents

1 Problem Formulation

We consider the estimation of switching linear dynamics system (SLDS). SLDS is an state-space model in which at time t, an agent's observed state $y_t \in \mathbb{R}^{d_y}$ is an noisy and censored version of the underlying state $x_t \in \mathbb{R}^{d_x}$, whose movement is governed by an time-varying linear dynamic system. Namely:

$$x_t = A_t x_{t-1} + B_t$$
$$y_t = Cx_t + \epsilon_t$$

where $C_{d_y,d_x} = [\mathbf{I}_{d_y}\ \mathbf{0}_{d_x-d_y}]$ is a fixed "censoring matrix" that selects the first d_y elements of x_t, and $\epsilon_t \overset{iid}{\sim} N(0, \mathbf{R})$ is the noise of observation. Further, SLDS assumes the set of time-specific dynamics $\theta_t = \{A_t, B_t\}$ arise from a countable set $\mathbf{\Theta} = \mathcal{A} \times \mathcal{B}$ indexed by $\mathcal{Z}$, and define $z_t \in \mathcal{Z}$ the index of θ_t. Finally, SLDS assumes z_t follows an Markov process with transition matrix $\mathbf{\Pi}_{|\Theta|\times|\Theta|} = [\pi_1, \dots, \pi_z, \dots]^T$, such that:

$$z_t | z_{t-1} \sim \pi_{z_{t-1}}$$

Despite the Markovian assumption, SLDS is capable of modeling a diverse collection of phenomenon with complex temporal dependencies from maneuvering aircraft trajectory to financial time-series. For example, in order to use SLDS to analyze fighter pilot's combat style, we may denote $\mathbf{y}_t \in \mathbb{R}^3$ the observed position of the maneuvering fighter aircraft, which comes from a latent $\mathbf{x}_t \in \mathbb{R}^9$ comprised of position, speed and momentum in the 3D space. We can learn how pilot executes different maneuvers by estimating $\mathbf{\Theta}$ the countability finite set contain dynamics that describe the offensive maneuvers ("barrel roll attack", "lag roll", etc) and defensive maneuvers ("break", "last-ditch", etc). We can also learn the pilots' habit of "maneuvers combo's" by estimating $\mathbf{\Pi}_{|\Theta|\times|\Theta|}$ the transition matrix describing how pilots moves from one maneuver to the other.

However,

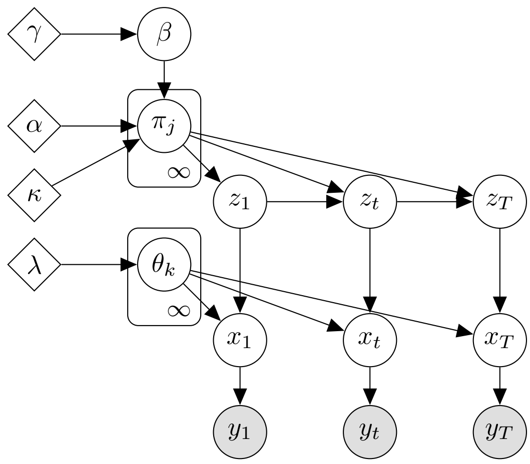


Figure 1: Graphical Model for Switching Linear Dynamics System

We also

2 Methods

2.1 HDP in Hidden Markov Model

3 Results

3.1 HDP-HMM

3.2 SLDS

3.3 HDP-SLDS

4 Conclusion and Future Direction

4.1 Sampling Hyperparameters

4.2 Automatic Relevance Determination

A Hierarchical Dirichlet Process

A.1 Model

Classic view:

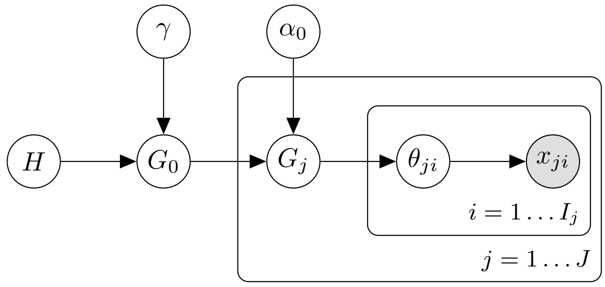
$$\begin{aligned} G_0|\gamma, H &\sim DP(\gamma, H) \\ G_j|\alpha_0, G_0 &\sim DP(\alpha_0, G_0) \\ \theta_{ji}|G_j &\sim G_j \\ x_{ji}|\theta_{ji} &\sim F(\theta_{ji}) \end{aligned}$$

where $P \sim DP(\alpha, G)$ adopts the stick breaking representation w.p. 1:

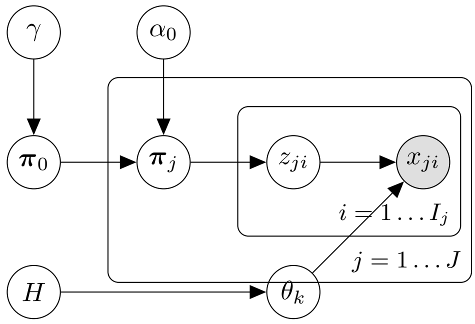
$$P = \sum_{k=1}^{\infty} \pi_k \delta_{\phi_k} \qquad \text{where:} \qquad \pi_k \sim GEM(\alpha), \quad \phi_k \sim G$$

Alternatively, one may describe the generative processes of π_k and θ_k separately as:

$$\begin{aligned} \boldsymbol{\pi}_0|\gamma &\sim GEM(\gamma) \qquad\qquad \theta_k|H \sim H \\ \boldsymbol{\pi}_j|\alpha_0, \boldsymbol{\pi}_0 &\sim DP(\alpha_0, \boldsymbol{\pi}_0) \\ \\ z_{ji}|\boldsymbol{\pi}_j &\sim \boldsymbol{\pi}_j \\ x_{ji}|z_{ji}, (\theta_k)_{k=1}^{\infty} &\sim F(\theta_{z_{ji}}) \end{aligned}$$



(a) Hierarchical Dirichlet Process



(b) Hierarchical Dirichlet Process

Figure 2: Hierarchical Dirichlet Process

A.2 Inference

Assuming conjugacy between H and F [1] and holding (γ, α_0) fixed,

we now describe a simplied Gibbs approach to sample parameters $(z_{ji}, m_{jk}, \boldsymbol{\pi}_0)$ from the Chinese Restaurant Franchise (see Appendix C) representation of the posterior, where the parameter z_{ji} are refered to respectively as customer-specific dish assignment, m_{jk} as dish-specific table count, and π_0 as global dish distribution. This particular method is referred to as "direct assignment" in Teh et al. [2006] since it circumvented the issue of bookkeeping for every t_{ij} (customer-specific table assignment) and k_{jt} (table-specific dish assignment) variables.

In each Gibbs iteration, denote $f_k^{-x_{ji}}(x_{ji}) = \frac{\int f(\mathbf{x}|\theta_k)h(\theta_k)d_{\theta_k}}{\int f(\mathbf{x}_{-(ji)}|\theta_k)h(\theta_k)d_{\theta_k}}$ the conditional distribution $x_{ji}|\mathbf{x}_{-(ji)}$ under $\theta = \theta_k$, and assume there are currently K dishes and T tables, we sample $(z_{ji}, m_{jk}, \boldsymbol{\pi}_0)$ iteratively as:

1. Sample $z_{ji} = k|\mathbf{z}_{-(ji)}, \mathbf{m}, \boldsymbol{\pi}_0$ from the distribution:

$$z_{ji} = k|\mathbf{z}_{-(ji)}, \mathbf{m}, \boldsymbol{\pi}_0 \propto \begin{cases} f_k^{-x_{ji}}(x_{ji}) * (n_{jk}^{-(ji)} + \alpha_0 \pi_{0,k}) & k \leq K \\ f_{K+1}^{-x_{ji}}(x_{ji}) * \alpha_0 \pi_{0,u} & k = K+1 \end{cases}$$

2. Sample $m_{jk} = m|\mathbf{z}, \mathbf{m}_{-(jk)}, \boldsymbol{\pi}_0$, by setting $m_{jk} = \sum_i I(t_{ji} = t_{new}|k_{jt_{new}} = k)$, we can sample t_{ji} from:

$$t_{ji} = t|k_{jt} = k, \mathbf{t}_{-(ji)}, \boldsymbol{\pi}_0 \propto \begin{cases} n_{jt}^{-(ji)} & t \leq T \\ \alpha_0 \pi_{0,k} & t = T+1 \end{cases}$$

and as in Fox [2009], sample $I(t_{ji} = t_{new}|k_{jt_{new}} = k)$ directly from:

$$Bern\Big(\frac{\alpha_0 \pi_{0,k}}{n_{jk} + \alpha_0 \pi_{0,k}}\Big)$$

3. Sample $\boldsymbol{\pi}_0$ from distribution:

$$\boldsymbol{\pi}_0 \sim Dir(m_1, \dots, m_K, \gamma)$$

[1] so we can integrate out the mixture component parameters

Algorithm 1 HDP, Gibbs Sampler through Direct Assignment

```
procedure hdp_gibbs_ds(K, y, (τ, μ, σ))
    α^0 = 0
    for p = 1 to MAX_ITER do
        α_0^p = (1 − μ/σ) α^{p−1} − 1/(σn) (K α^{p−1} − y)
        α^p = S_{τ/σ}(K, α_0^p)
    end for
    return f^{MAX_ITER} = (α^{MAX_ITER})^T k
end procedure
```

A.3 Application: Clustering Hierarchical Gaussian Data

Consider mixture of Gaussian data $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_K\}$ with $\mathbf{x}_k \overset{iid}{\sim} MVN(\boldsymbol{\theta}_{k,2\times 1}, \mathbf{I}_{2\times 2})$ with unknown mean $\boldsymbol{\theta}$. Assuming diffused Gaussian prior $\boldsymbol{\theta} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$, the form of likelihood F and base measure H are:

$$f(x_{ji}|\boldsymbol{\theta}_k) \propto exp(-\frac{1}{2\sigma^2}(x_{ji} - \boldsymbol{\theta}_k)^T(x_{ji} - \boldsymbol{\theta}_k))$$
$$h(\boldsymbol{\theta}_k) \propto exp(-\frac{1}{2\sigma_0^2}\boldsymbol{\theta}_k^T\boldsymbol{\theta}_k)$$

Then $f_k^{-x_{ji}}(x_{ji})$ should be:

$$f_k^{-x_{ji}}(x_{ji}) \sim N(\frac{n_k^{-(ji)}\sigma_0^2}{n_k^{-(ji)}\sigma_0^2 + \sigma^2}\bar{\mathbf{x}}_k^{-(ji)}, (1 + \frac{\sigma_0^2}{n_k^{-(ji)}\sigma_0^2 + \sigma^2})\mathbf{I})$$

B HDP for Hidden Markov Model

B.1 Hidden Markov Model

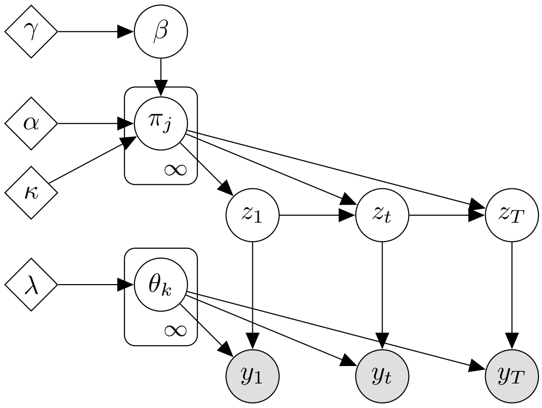


Figure 3: Hidden Markov Model

$$
\begin{aligned}
\beta|\gamma &\sim GEM(\gamma) \\
\pi_j|\beta,\alpha &\sim DP(\alpha,\beta) \\
\theta_k|H,\lambda &\sim H(\lambda) \\
\\
z_t|z_{t-1},\boldsymbol{\pi} &\sim \pi_{z_{t-1}} \\
y_t|z_t,\boldsymbol{\theta} &\sim F(\theta_{z_t})
\end{aligned}
$$

$$f_k(y_t) = p(y_t|\boldsymbol{\theta}_{z_t})p(z_t|z_{t-1})$$

B.2 Sticky HDP

Though flexible, the fact that HDP-HMM is deploying $\pi_k \sim DP(\alpha, \beta)$ leads to:

1. large posterior probability for unrealistically transition dynamics
2. once instantiated, the unrealistically transition dynamics will be reinforced by CRF

Sticky HDP address above issues by encouraging self-transition. More specifically, the base measure for π_k is augmented *a priori* from β to:

$$\pi_j \sim DP(\alpha+\kappa, \frac{\alpha\beta+\kappa\delta_j}{\alpha+\kappa})$$

B.3 Inference

Inference for HMM with Sticky HDP prior follows the sticky extension of CRF. For a observation y_t at time t, "restaurant" corresponds to the state z_t that y_t is at, and dishes at restaurant z_t indicates the potential states that y_{t+1} can transit to. To improve mixing rate of state sequence $\mathbf{z}$, we deploy the blocked sampler which uses a weak limit approximation of the infinite-dimension DP prior. More specifically, we assume there are L states, and β and π follows:

$$
\begin{aligned}
\beta|\gamma &\sim Dir(\frac{\gamma}{L},\dots,\frac{\gamma}{L}) \\
\pi_j|\alpha,\beta,\kappa &\sim Dir(\alpha\beta_1,\dots,\alpha\beta_j+\kappa,\dots,\alpha\beta_L)
\end{aligned}
$$

Define $\boldsymbol{\theta}_k$ as emission parameter for state k, we sample $(\mathbf{z},\mathbf{m},\boldsymbol{\pi}_0,\boldsymbol{\theta})$ as follows:

1. Sample z_t from the distribution:

$$z_t|\mathbf{z}_{-(ji)}, \mathbf{m}, \boldsymbol{\pi}_0, \boldsymbol{\theta} \sim f(z_t = k|\mathbf{y}, \mathbf{m}, \boldsymbol{\pi}_0, \boldsymbol{\theta})$$

   where $f(z_t = k|\mathbf{y})$ is calculated using the forward-backward message passing algorithm in B.3.1).
2. Sample m_{jk} through override correction:
   (a) Sample $m'_{jk} = \sum_i I(t_{ji} = t_{new}|k_{jt_{new}} = k)$, where:

$$I(t_{ji} = t_{new}|k_{jt_{new}} = k) \sim Bern\Big(\frac{\alpha\pi_{0,k} + \kappa\delta_j(k)}{n_{jk} + \alpha\pi_{0,k} + \kappa\delta_j(k)}\Big)$$

   (b) Sample override variable:

$$w_j \sim Binom\Big(m'_{jj}, \frac{\kappa}{\kappa + \alpha\boldsymbol{\pi}_{0,j}}\Big)$$

   (c) Finally calculate m_{jk} as:

$$m_{jk} = \begin{cases} m'_{ij} & j \neq k \\ m'_{jj} - w_j & j = k \end{cases}$$

3. Sample $\boldsymbol{\pi}_0$ from distribution:

$$\boldsymbol{\pi}_0 \sim Dir(\frac{\gamma}{L} + m_1, \ldots, \frac{\gamma}{L} + m_K)$$

4. Sample $\boldsymbol{\theta}$ from distribution:

$$\boldsymbol{\theta} \sim p(\boldsymbol{\theta}|\lambda, \mathbf{y})$$

B.3.1 Forward-backward Message Passing

The forward-backward algorithm provide an efficient method for computing node marginals $p(y_t)$. Define:

$$\begin{aligned} \text{Backward Message}: &\quad \beta_t(z_t) = p(\mathbf{y}_{T>t}|z_t) \\ \text{Forward Message}: &\quad \alpha_t(z_t) = p(\mathbf{y}_{T\leq t}, z_t) \\ \text{Joint Message}: &\quad \alpha_t(z_t)\beta_t(z_t) = p(\mathbf{y}, z_t) \end{aligned}$$

which can be alternatively defined using message m_{t_1,t_2}

$$\begin{aligned} \text{Backward Message}: &\quad \beta_t(z_t) = p(\mathbf{y}_{T>t}|z_t) = m_{t+1,t}(z_t) \\ \text{Forward Message}: &\quad \alpha_t(z_t) = p(y_t|z_t)p(\mathbf{y}_{T<t}, z_t) = p(y_t|z_t)m_{t-1,t}(z_t) \end{aligned}$$

.

These two types of messages can be computed β_t backward and α_t forward in time as:

$$\begin{aligned} \beta_{t-1} &= \sum_{z_t} p(y_t|z_t) \qquad p(z_t|z_{t-1})\beta_t(z_t) \qquad \text{with} \quad \beta_T(z_T) = 1 \\ \alpha_{t+1} &= \sum_{z_t} p(y_{t+1}|z_{t+1})p(z_{t+1}|z_t)\alpha_t(z_t) \qquad \text{with} \quad \alpha_1(z_1) = p(y_1, z_1) = p(y_1|z_1)\pi^0(z_1) \end{aligned}$$

Using the forward and backward messages, we can compute state assignment posterior as:

$$p(z_t|\mathbf{y}) = \frac{p(z_t, \mathbf{y})}{\sum_{z_t} p(z_t, \mathbf{y})} = \frac{\alpha_t(z_t)\beta_t(z_t) = p(\mathbf{y}, z_t)}{\sum_{z_t} \alpha_t(z_t)\beta_t(z_t) = p(\mathbf{y}, z_t)}$$

C Chinese Restaurant Franchise

A hierarchical analogy of Chinese Restaurant Process, the Chinese Restaurant Franchise offers a convenient scheme to sample from the posterior of cluster-specific θ's in HDP. This process draw below analogy:

- H as the dish distribution for all possible dishes in the world, with the types of possible dishes being $(\theta_k)_{k=1}^{\infty}$.
- $G_0 \sim DP(\gamma, H)$ as the dish distribution for the franchise
- $G_j \sim DP(\alpha_0, G_0)$ as the dish distribution for restaurant j in the franchise
- $\psi_{jt} \sim G_0$ as the dish served at table t in restaurant j.
  $k_{jt} \sim \pi_0$ as the index of dish choice for this table.
- $\theta_{ji} \sim G_j$ as the dish will be enjoyed by customer i in restaurant j.
  $t_{ji} \sim \pi_j$ as the index of table choice for this customer.

Integrating over G_j, the sampling scheme for subject-specific dish $\theta_{ji} \sim G_j$ is:

$$\theta_{ji}|\boldsymbol{\theta}_{j(-i)}, \alpha_0, G_0 \sim \sum_{k=1}^{K} \frac{n_{jt.}}{\alpha_0 + n_{j..}} \delta_{\psi_{jt}} + \frac{\gamma}{n_{j..} + \gamma} G_0$$

Integrating over G_0, the sampling scheme for table-specific dish $\psi_{jt} \sim G_0$ is:

$$\psi_{jk}|\Psi_{j(-k)}, \gamma, H \sim \sum_{k=1}^{K} \frac{m_{.k}}{\gamma + m_{..}} \delta_{\theta_k} + \frac{\gamma}{m_{..} + \gamma} H$$

D References

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Yee Whye Teh, Michael I. Jordan, Matthew J. Beal, and David M. Blei. Hierarchical Dirichlet Processes. *Journal of the American Statistical Association*, 101(476):1566–1581, 2006. ISSN 0162-1459. URL `http://www.jstor.org/stable/27639773`.

Emily Beth Fox. *Bayesian nonparametric learning of complex dynamical phenomena*. Thesis, Massachusetts Institute of Technology, 2009. URL `http://dspace.mit.edu/handle/1721.1/55111`.