Bayesian Nonparametric Estimation of Switching Linear Dynamic System

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1 Problem Formulation

We consider the estimation of switching linear dynamics system (SLDS). SLDS is an state-space model in which at time t, an agent's observed state $y_t \in \mathbb{R}^{d_y}$ is an noisy and censored version of the underlying state $x_t \in \mathbb{R}^{d_x}$, whose movement is governed by an time-varying linear dynamic system. Namely:

$$x_t = A_t x_{t-1} + B_t$$
$$y_t = C x_t + \epsilon_t$$

where $C_{d_y,d_x} = [\mathbf{I}_{d_y}\ \mathbf{0}_{d_x - d_y}]$ is a fixed "censoring matrix" that selects the first d_y elements of x_t, and $\epsilon_t \overset{iid}{\sim} N(0, \mathbf{R})$ is the noise of observation. Further, SLDS assumes the set of time-specific dynamics $\theta_t = \{A_t, B_t\}$ arise from a countable set $\mathbf{\Theta} = \mathcal{A} \times \mathcal{B}$ indexed by $\mathcal{Z}$, and define $z_t \in \mathcal{Z}$ the index of θ_t. Finally, SLDS assumes z_t follows an Markov process with transition matrix $\mathbf{\Pi}_{|\Theta|\times|\Theta|} = [\pi_1, \ldots, \pi_z, \ldots]^T$, such that:

$$z_t | z_{t-1} \sim \pi_{z_{t-1}}$$

Despite the Markovian assumption, SLDS is capable of modeling a diverse collection of phenomenon with complex temporal dependencies from maneuvering aircraft trajectory to financial time-series. For example, in order to use SLDS to analyze fighter pilot's combat style, we may denote $\mathbf{y}_t \in \mathbb{R}^3$ the observed position of the maneuvering fighter aircraft, which comes from a latent $\mathbf{x}_t \in \mathbb{R}^9$ comprised of position, speed and momentum in the 3D space. We can learn how pilot executes different maneuvers by estimating $\mathbf{\Theta}$ the countability finite set contain dynamics that describe the offensive maneuvers ("barrel roll attack", "lag roll", etc) and defensive maneuvers ("break", "last-ditch", etc). We can also learn the pilots' habit of "maneuvers combo's" by estimating $\mathbf{\Pi}_{|\Theta|\times|\Theta|}$ the transition matrix describing how pilots moves from one maneuver to the other.

However, the flexible nature of SLDS also caused difficulty in estimation, in particular the dimension of transition matrix $\mathbf{\Pi}$ is $O(|\mathbf{\Theta}|^2)$ and can theoretically grow to infinity. To this regard, Bayesian nonparametric methods, in particular Hierarchical Dirichlet Process (HDP), achieves efficient inference and sparse solution to $\mathbf{\Pi}$ through global shrinkage on each state-specific transition distributions. In the rest of this report, we discuss how to properly adapt HDP into the estimation of $\mathbf{\Pi}$ matrix, and further how to integrate this method into the entire estimation process for the SLDS under Bayesian framework.

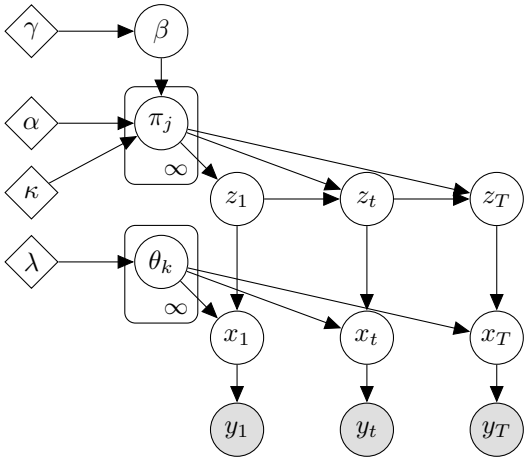

Figure 1: Graphical Model for Switching Linear Dynamics System

We also

2 Methods

2.1 HDP in Hidden Markov Model

3 Results

3.1 HDP-HMM

3.2 SLDS

3.3 HDP-SLDS

4 Conclusion and Future Direction

4.1 Sampling Hyperparameters

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A Hierarchical Dirichlet Process

A.1 Model

Classic view:

$$\begin{aligned} G_0|\gamma, H &\sim DP(\gamma, H) \\ G_j|\alpha_0, G_0 &\sim DP(\alpha_0, G_0) \\ \theta_{ji}|G_j &\sim G_j \\ x_{ji}|\theta_{ji} &\sim F(\theta_{ji}) \end{aligned}$$

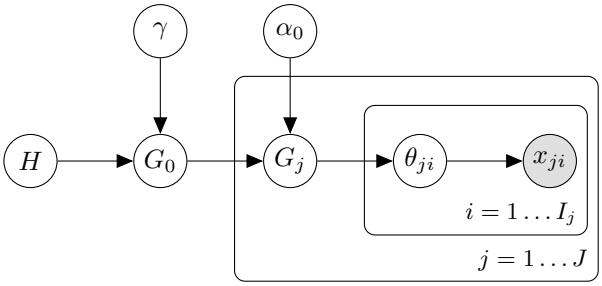
where $P \sim DP(\alpha, G)$ adopts the stick breaking representation w.p. 1:

$$P = \sum_{k=1}^{\infty} \pi_k \delta_{\phi_k} \qquad \text{where:} \qquad \pi_k \sim GEM(\alpha), \quad \phi_k \sim G$$

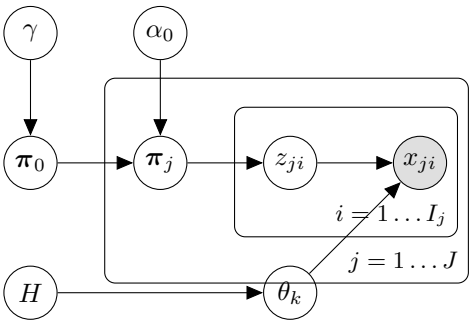
Alternatively, one may describe the generative processes of π_k and θ_k separately as:

$$\begin{aligned} \boldsymbol{\pi}_0|\gamma &\sim GEM(\gamma) & \theta_k|H \sim H \\ \boldsymbol{\pi}_j|\alpha_0, \boldsymbol{\pi}_0 &\sim DP(\alpha_0, \boldsymbol{\pi}_0) \end{aligned}$$

$$\begin{aligned} z_{ji}|\boldsymbol{\pi}_j &\sim \boldsymbol{\pi}_j \\ x_{ji}|z_{ji}, (\theta_k)_{k=1}^{\infty} &\sim F(\theta_{z_{ji}}) \end{aligned}$$



(a) Hierarchical Dirichlet Process



(b) Hierarchical Dirichlet Process

Figure 2: Hierarchical Dirichlet Process

A.2 Inference

Assuming conjugacy between H and F [1] and holding (γ, α_0) fixed,

we now describe a simplied Gibbs approach to sample parameters $(z_{ji}, m_{jk}, \boldsymbol{\pi}_0)$ from the Chinese Restaurant Franchise (see Appendix C) representation of the posterior, where the parameter z_{ji} are refered to respectively as customer-specific dish assignment, m_{jk} as dish-specific table count, and π_0 as global dish distribution. This particular method is referred to as "direct assignment" in Teh et al. [2006] since it circumvented the issue of bookkeeping for every t_{ij} (customer-specific table assignment) and k_{jt} (table-specific dish assignment) variables.

In each Gibbs iteration, denote $f_k^{-x_{ji}}(x_{ji}) = \frac{\int f(\mathbf{x}|\theta_k)h(\theta_k)d_{\theta_k}}{\int f(\mathbf{x}_{-(ji)}|\theta_k)h(\theta_k)d_{\theta_k}}$ the conditional distribution $x_{ji}|\mathbf{x}_{-(ji)}$ under $\theta = \theta_k$, and assume there are currently K dishes and T tables, we sample $(z_{ji}, m_{jk}, \boldsymbol{\pi}_0)$ iteratively as:

1. Sample $z_{ji} = k|\mathbf{z}_{-(ji)}, \mathbf{m}, \boldsymbol{\pi}_0$ from the distribution:

$$z_{ji} = k|\mathbf{z}_{-(ji)}, \mathbf{m}, \boldsymbol{\pi}_0 \propto \begin{cases} f_k^{-x_{ji}}(x_{ji}) * (n_{jk}^{-(ji)} + \alpha_0\pi_{0,k}) & k \leq K \\ f_{K+1}^{-x_{ji}}(x_{ji}) * \alpha_0\pi_{0,u} & k = K+1 \end{cases}$$

2. Sample $m_{jk} = m|\mathbf{z}, \mathbf{m}_{-(jk)}, \boldsymbol{\pi}_0$, by setting $m_{jk} = \sum_i I(t_{ji} = t_{new}|k_{jt_{new}} = k)$, we can sample t_{ji} from:

$$t_{ji} = t|k_{jt} = k, \mathbf{t}_{-(ji)}, \boldsymbol{\pi}_0 \propto \begin{cases} n_{jt}^{-(ji)} & t \leq T \\ \alpha_0\pi_{0,k} & t = T+1 \end{cases}$$

and as in Fox [2009], sample $I(t_{ji} = t_{new}|k_{jt_{new}} = k)$ directly from:

$$Bern\Big(\frac{\alpha_0\pi_{0,k}}{n_{jk} + \alpha_0\pi_{0,k}}\Big)$$

3. Sample $\boldsymbol{\pi}_0$ from distribution:

$$\boldsymbol{\pi}_0 \sim Dir(m_1, \dots, m_K, \gamma)$$

[1] so we can integrate out the mixture component parameters

Algorithm 1 HDP, Gibbs Sampler through Direct Assignment

procedure `hdp_gibbs_ds`($\mathbf{K}$, $\mathbf{y}$, (τ, μ, σ))
  $\boldsymbol{\alpha}^0 = \mathbf{0}$
  for $p = 1$ to `MAX_ITER` **do**
    $\boldsymbol{\alpha}_0^p = (1 - \frac{\mu}{\sigma})\boldsymbol{\alpha}^{p-1} - \frac{1}{\sigma n}(\mathbf{K}\boldsymbol{\alpha}^{p-1} - \mathbf{y})$
    $\boldsymbol{\alpha}^p = \mathbf{S}_{\frac{\tau}{\sigma}}(K, \boldsymbol{\alpha}_0^p)$
  end for
  return $f^{\texttt{MAX\_ITER}} = (\boldsymbol{\alpha}^{\texttt{MAX\_ITER}})^T\mathbf{k}$
end procedure

A.3 Application: Clustering Hierarchical Gaussian Data

Consider mixture of Gaussian data $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_K\}$ with $\mathbf{x}_k \overset{iid}{\sim} MVN(\boldsymbol{\theta}_{k,2\times 1}, \mathbf{I}_{2\times 2})$ with unknown mean $\boldsymbol{\theta}$. Assuming diffused Gaussian prior $\boldsymbol{\theta} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$, the form of likelihood F and base measure H are:

$$f(x_{ji}|\boldsymbol{\theta}_k) \propto exp(-\frac{1}{2\sigma^2}(x_{ji} - \boldsymbol{\theta}_k)^T(x_{ji} - \boldsymbol{\theta}_k))$$
$$h(\boldsymbol{\theta}_k) \propto exp(-\frac{1}{2\sigma_0^2}\boldsymbol{\theta}_k^T\boldsymbol{\theta}_k)$$

Then $f_k^{-x_{ji}}(x_{ji})$ should be:

$$f_k^{-x_{ji}}(x_{ji}) \sim N(\frac{n_k^{-(ji)}\sigma_0^2}{n_k^{-(ji)}\sigma_0^2 + \sigma^2}\bar{\mathbf{x}}_k^{-(ji)}, (1 + \frac{\sigma_0^2}{n_k^{-(ji)}\sigma_0^2 + \sigma^2})\mathbf{I})$$

B HDP for Hidden Markov Model

B.1 Hidden Markov Model

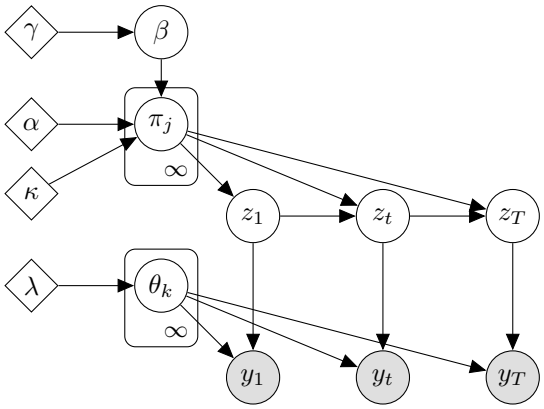


Figure 3: Hidden Markov Model

$$
\begin{aligned}
\beta|\gamma &\sim GEM(\gamma) \\
\pi_j|\beta,\alpha &\sim DP(\alpha,\beta) \\
\theta_k|H,\lambda &\sim H(\lambda)
\end{aligned}
$$

$$
\begin{aligned}
z_t|z_{t-1},\boldsymbol{\pi} &\sim \pi_{z_{t-1}} \\
y_t|z_t,\boldsymbol{\theta} &\sim F(\theta_{z_t})
\end{aligned}
$$

$$f_k(y_t) = p(y_t|\boldsymbol{\theta}_{z_t})p(z_t|z_{t-1})$$

B.2 Sticky HDP

Though flexible, the fact that HDP-HMM is deploying $\pi_k \sim DP(\alpha,\beta)$ leads to:

1. large posterior probability for unrealistically transition dynamics
2. once instantiated, the unrealistically transition dynamics will be reinforced by CRF

Sticky HDP address above issues by encouraging self-transition. More specifically, the base measure for π_k is augmented *a priori* from β to:

$$\pi_j \sim DP(\alpha+\kappa, \frac{\alpha\beta+\kappa\delta_j}{\alpha+\kappa})$$

B.3 Inference

Inference for HMM with Sticky HDP prior follows the sticky extension of CRF. For a observation y_t at time t, "restaurant" corresponds to the state z_t that y_t is at, and dishes at restaurant z_t indicates the potential states that y_{t+1} can transit to. To improve mixing rate of state sequence $\mathbf{z}$, we deploy the blocked sampler which uses a weak limit approximation of the infinite-dimension DP prior. More specifically, we assume there are L states, and β and π follows:

$$
\begin{aligned}
\beta|\gamma &\sim Dir(\frac{\gamma}{L},\ldots,\frac{\gamma}{L}) \\
\pi_j|\alpha,\beta,\kappa &\sim Dir(\alpha\beta_1,\ldots,\alpha\beta_j+\kappa,\ldots,\alpha\beta_L)
\end{aligned}
$$

Define $\boldsymbol{\theta}_k$ as emission parameter for state k, we sample $(\mathbf{z},\mathbf{m},\boldsymbol{\pi}_0,\boldsymbol{\theta})$ as follows:

1. Sample z_t from the distribution:

$$z_t|\mathbf{z}_{-(ji)}, \mathbf{m}, \boldsymbol{\pi}_0, \boldsymbol{\theta} \sim f(z_t = k|\mathbf{y}, \mathbf{m}, \boldsymbol{\pi}_0, \boldsymbol{\theta})$$

   where $f(z_t = k|\mathbf{y})$ is calculated using the forward-backward message passing algorithm in B.3.1).

2. Sample m_{jk} through override correction:
   (a) Sample $m'_{jk} = \sum_i I(t_{ji} = t_{new}|k_{jt_{new}} = k)$, where:

$$I(t_{ji} = t_{new}|k_{jt_{new}} = k) \sim Bern\Big(\frac{\alpha\pi_{0,k} + \kappa\delta_j(k)}{n_{jk} + \alpha\pi_{0,k} + \kappa\delta_j(k)}\Big)$$

   (b) Sample override variable:

$$w_j \sim Binom\Big(m'_{jj}, \frac{\kappa}{\kappa + \alpha\boldsymbol{\pi}_{0,j}}\Big)$$

   (c) Finally calculate m_{jk} as:

$$m_{jk} = \begin{cases} m'_{ij} & j \neq k \\ m'_{jj} - w_j & j = k \end{cases}$$

3. Sample $\boldsymbol{\pi}_0$ from distribution:

$$\boldsymbol{\pi}_0 \sim Dir(\frac{\gamma}{L} + m_1, \ldots, \frac{\gamma}{L} + m_K)$$

4. Sample $\boldsymbol{\theta}$ from distribution:

$$\boldsymbol{\theta} \sim p(\boldsymbol{\theta}|\lambda, \mathbf{y})$$

B.3.1 Forward-backward Message Passing

The forward-backward algorithm provide an efficient method for computing node marginals $p(y_t)$. Define:

$$\begin{aligned} \text{Backward Message}: &\quad \beta_t(z_t) = p(\mathbf{y}_{T>t}|z_t) \\ \text{Forward Message}: &\quad \alpha_t(z_t) = p(\mathbf{y}_{T\leq t}, z_t) \\ \text{Joint Message}: &\quad \alpha_t(z_t)\beta_t(z_t) = p(\mathbf{y}, z_t) \end{aligned}$$

which can be alternatively defined using message m_{t_1,t_2}

$$\begin{aligned} \text{Backward Message}: &\quad \beta_t(z_t) = p(\mathbf{y}_{T>t}|z_t) = m_{t+1,t}(z_t) \\ \text{Forward Message}: &\quad \alpha_t(z_t) = p(y_t|z_t)p(\mathbf{y}_{T<t}, z_t) = p(y_t|z_t)m_{t-1,t}(z_t) \end{aligned}$$

.

These two types of messages can be computed β_t backward and α_t forward in time as:

$$\beta_{t-1} = \sum_{z_t} p(y_t|z_t) \qquad p(z_t|z_{t-1})\beta_t(z_t) \qquad \text{with} \quad \beta_T(z_T) = 1$$

$$\alpha_{t+1} = \sum_{z_t} p(y_{t+1}|z_{t+1})p(z_{t+1}|z_t)\alpha_t(z_t) \qquad \text{with} \quad \alpha_1(z_1) = p(y_1, z_1) = p(y_1|z_1)\pi^0(z_1)$$

Using the forward and backward messages, we can compute state assignment posterior as:

$$p(z_t|\mathbf{y}) = \frac{p(z_t, \mathbf{y})}{\sum_{z_t} p(z_t, \mathbf{y})} = \frac{\alpha_t(z_t)\beta_t(z_t) = p(\mathbf{y}, z_t)}{\sum_{z_t} \alpha_t(z_t)\beta_t(z_t) = p(\mathbf{y}, z_t)}$$

C Chinese Restaurant Franchise

A hierarchical analogy of Chinese Restaurant Process, the Chinese Restaurant Franchise offers a convenient scheme to sample from the posterior of cluster-specific θ's in HDP. This process draw below analogy:

- H as the dish distribution for all possible dishes in the world, with the types of possible dishes being $(\theta_k)_{k=1}^{\infty}$.
- $G_0 \sim DP(\gamma, H)$ as the dish distribution for the franchise
- $G_j \sim DP(\alpha_0, G_0)$ as the dish distribution for restaurant j in the franchise
- $\psi_{jt} \sim G_0$ as the dish served at table t in restaurant j.
  $k_{jt} \sim \pi_0$ as the index of dish choice for this table.
- $\theta_{ji} \sim G_j$ as the dish will be enjoyed by customer i in restaurant j.
  $t_{ji} \sim \pi_j$ as the index of table choice for this customer.

Integrating over G_j, the sampling scheme for subject-specific dish $\theta_{ji} \sim G_j$ is:

$$\theta_{ji}|\boldsymbol{\theta}_{j(-i)}, \alpha_0, G_0 \sim \sum_{k=1}^{K} \frac{n_{jt.}}{\alpha_0 + n_{j..}} \delta_{\psi_{jt}} + \frac{\gamma}{n_{j..} + \gamma} G_0$$

Integrating over G_0, the sampling scheme for table-specific dish $\psi_{jt} \sim G_0$ is:

$$\psi_{jk}|\Psi_{j(-k)}, \gamma, H \sim \sum_{k=1}^{K} \frac{m_{.k}}{\gamma + m_{..}} \delta_{\theta_k} + \frac{\gamma}{m_{..} + \gamma} H$$

D References

References

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