Outline of the Presentation on Convex Body Approximation

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1 John's Ellipsoid

Outline:
In this part I want to start with introducing the John's ellipsoid theorem, which is one of the famous results in this field, by showing some samples in both asymmetric and symmetric cases in 2 dimension. Then I would like to talk in details about theorem 1.2 and 1.3, and give the proof on blackboard. I would skip the proof of lemma 1.1 and also the explanation of tensor product in the proof. Later on I will introduce the important concept of John's position and show a simple illustration. At last I will briefly talk about the definition of Banach-Mazur distance and the theorem 1.4 and its proof, just to give an idea behind the John's ellipsoid theorem that mentioned at the beginning.

Lemma 1.1. *$\mathcal{D} = \{A \in \mathcal{P} : det(A) \geq 1\}$ is a closed, smooth and strictly convex set in $\mathcal{P}$ with non-empty interior.*

Proof of Lemma 1.1. Let $A = I \in \mathcal{D} \Rightarrow \mathcal{D} \neq \emptyset$, hence $\mathcal{D}$ has non-empty interior.

It is clear that $\mathcal{D}$ is closed and smooth.

For convexity, it suffices to prove that the function $f(A) = \log\det A$ is concave. Then:

$$\log\det(\lambda A + (1-\lambda)B) \geq \lambda \log\det A + (1-\lambda)\log\det B$$

$$\Rightarrow \det(\lambda A + (1-\lambda)B) \geq (\det A)^{\lambda}(\det B)^{1-\lambda}$$

Thus, if $A, B \in \mathcal{D}$, then $\det(\lambda A + (1-\lambda)B) \geq 1$, so $\lambda A + (1-\lambda)B \in \mathcal{D}$, and convexity holds.

To show that $f(A) = \log\det A$ is concave, compute its second derivative.

(1) Define $f(t) := \log\det(A + tH), \quad t \in (-\varepsilon, \varepsilon),\ A \in \mathcal{P},\ H \in \mathrm{Sym}_n$.

(2) Compute the first derivative:

$$f'(t) = \frac{d}{dt}\log\det(A+tH) = \mathrm{tr}\left((A+tH)^{-1}H\right)$$

(3) Compute the second derivative:

$$f''(t) = \frac{d}{dt}\mathrm{tr}\left((A+tH)^{-1}H\right) = \mathrm{tr}\left(-(A+tH)^{-1}\left(\frac{d}{dt}(A+tH)\right)(A+tH)^{-1}H\right)$$

$$= -\mathrm{tr}\left(\left[(A+tH)^{-1}H\right]^2\right) < 0$$

with equality iff $H = 0$.

Therefore, f is strictly concave, hence $\mathcal{D}$ is strictly convex. $\square$

Theorem 1.2. *If $C \subset \mathbb{R}^d$ be a convex body, then the inner ellipsoid of maximal volume is unique.*

Proof of Theorem 1.2. There exists at least one ellipsoid in C of maximum volume since C is compact.

We prove it by contradiction, assume that there are two distinct maximum volume ellipsoids in C. Without loss of generality, we may assume that their volumes are equal to that of B^d. These ellipsoids can be represented in the form AB^d, BB^d, with suitable $d \times d$ matrices $A, B \in \mathcal{P}$, where $A \neq B$ and $\det A = \det B = 1$.

Then

$$\frac{1}{2}(A+B)B^d \subseteq C$$

by the convexity of C. Since

$$\det\left(\frac{1}{2}(A+B)\right) > \frac{1}{2}detA + \frac{1}{2}detB = 1$$

by lemma 1.1, the ellipsoid

$$\frac{1}{2}(A+B)B^d$$

has greater volume than the maximum volume ellipsoids AB^d, BB^d, a contradiction. ☐

Theorem 1.3 (John's Characterization)**.** *Let $C \in \mathcal{C}_p$ be symmetric in origin and $B^d \subseteq C$. Then the following statements are equivalent:*

(i) B^d is the unique John's Ellipsoid in C

(ii) There are $u_i \in B^d \cap bdC$ and $\lambda_i > 0$, $i = 1, ..., m$, where $d \leq m \leq \frac{1}{2}d(d+1)$, such that

$$I = \sum \lambda_i u_i \otimes u_i \ , \sum \lambda_i = d.$$

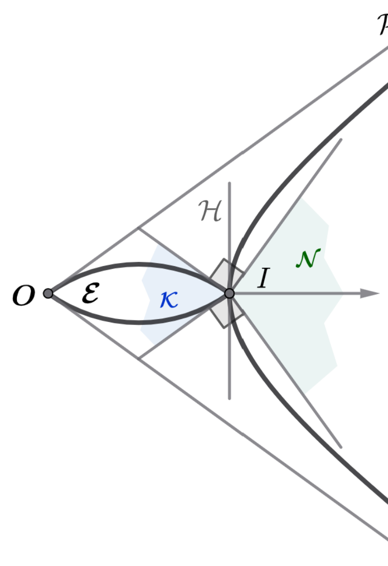

Figure 1: Proof sketch of uniqueness of John's ellipsoid

Proof of Theorem 1.3. $(i) \Rightarrow (ii)$:

The family of all ellipsoids in C is represented by the following set: $\mathcal{E} = \{A \in \mathcal{P} : Au \cdot v \leq h_C(v)$ for all $u, v \in bd\ B^d\}$, the first key step is to rewrite $Au \cdot v$ as $A \cdot (u \otimes v)$, since

$$\langle Au, v\rangle = (Au)^T v = \sum_{i=1}^{d}\left(\sum_{j=1}^{d} A_{ij}u_j\right) v_i = \sum_{i,j} A_{ij}u_j v_i$$

and

$$\langle A, u \otimes v\rangle = \sum_{i,j} A_{ij} u_i v_j$$

and by A is symmetric we know they are the same. This step is to express the "quadratic form" into "linear constraints in matrix space", thus $\mathcal{E}$ is the intersection of family of the closed halfspaces

$$\{A \in \mathbb{E}^{\frac{1}{2}d(d+1)} : A \cdot (u \otimes v) \leq h_C(v)\} : u, v \in bd\ B^d, \tag{1}$$

by assumption of (ii) we know: $detA < 1$ for all $A \in \mathcal{E}/\{I\}$; $\mathcal{E}$is convex, $\mathcal{D} \cap \mathcal{E} = \{I\} and \mathcal{E}$ is separated from $\mathcal{D}$ by the unique support hyperplane $\mathcal{H}$ of $\mathcal{D}$ at I.
Now let's look at the support cone $\mathcal{K}$ of $\mathcal{E}$ at I, which will lead us to construct the normal cone and then prove the statement. Recall that in (1) we write $\mathcal{E}$ as intersection of those halfspaces, also in $\mathcal{P}$, then by definition $\mathcal{K}$ is the intersection of those halfspaces in (1) which contains I on their boundaries, i.e. $I \cdot (u \otimes v) = u \cdot v = h_C(v)$. Notice that:

$$u \cdot v \leq 1,\ B^d \subset C \Rightarrow 1 = h_{B^d}(v) \leq h_C(v) \quad for\ u, v \in bd\ B^d. \tag{2}$$

which means the equality $I \cdot (u \otimes v) = h_C(v)$ only holds when $u = v$, thus $\mathcal{K}$ is the intersection of the following halfspaces:

$$\{A \in \mathbb{E}^{\frac{1}{2}d(d+1)} : A \cdot (u \otimes v) \leq 1\} : u, v \in B^d \cap bd\ C. \tag{3}$$

The normal cone $\mathcal{N}$ of $\mathcal{E}$ at I is just the polar cone of $\mathcal{K} - I$ so it is:

$$\mathcal{N} = pos\{u \otimes u : u \in B^d \cap bd\ C\}. \tag{4}$$

Thus $I \in \mathcal{N}$. Caratheodory's theorem implies that there are $u_i \in B^d \cap bd\, C, \lambda_i > 0, i = 1, ..., m, m \leq \frac{1}{2}d(d+1)$, such that

$$I = \sum_i \lambda_i u_i \otimes u_i.$$

and

$$d = tr(I) = \sum_i \lambda_i tr(u_i \otimes u_i) = \sum_i \lambda_i$$

For the $m \geq d$ is obvious, since at least we need d linear independent vectors to span $\mathbb{E}^d$.
$(ii) \Rightarrow (i)$:
Following the same definition of $\mathcal{N}, \mathcal{K}, \mathcal{E}$. $(ii) \Rightarrow I \in \mathcal{N}$, thus the hyperplane $\mathcal{H}$ through I and orthogonal to it separates $\mathcal{K}$ and $\mathcal{D}$, by convexity $\mathcal{D} \cap \mathcal{E} = \{I\}$, which leads to $detA < 1$ for all $A \in \mathcal{E}/\{I\}$, which is the same as statement(i). $\square$

Corollary 1.4. *$\delta^{BM}(\|\cdot\|, |\cdot|) \leq \sqrt{d}$, where $\|\cdot\|$ is the Euclidean norm and $|\cdot| \in \mathcal{N}(\mathbb{E}^d)$*

Proof of Corollary 1.4. Assume that T is chosen such that $B^d \subseteq TB_{|\cdot|}$ is the ellipsoid of maximum volume in $C = TB_{|\cdot|}$. Which establishes the left-hand inclusion in the definition of the Banach–Mazur distance. Now we need to show the right-hand inclusion, which is: $C = TB_{|\cdot|} \subseteq \sqrt{d}\, B^d$.
Choose u_i, λ_i according to theorem 1.3. i.e.

$$I = \sum_i \lambda_i u_i \otimes u_i, \qquad \sum_i \lambda_i = d. \tag{5}$$

Note that $u_i \in B^d \cap \mathrm{bd}\, C$. Hence C is contained in the support halfspace of B^d at u_i, and thus $u_i \cdot x \leq 1$ for $x \in C = TB_{|\cdot|}$. Represent x by (5) in the form:

$$x = Ix = \sum_i \lambda_i (u_i \otimes u_i) x = \sum_i \lambda_i (u_i \cdot x) u_i.$$

Then

$$x^2 = x \cdot x = \sum_i \lambda_i (u_i \cdot x)^2 \leq \sum_i \lambda_i = d \quad \text{for } x \in TB_{|\cdot|},$$

again by (5). Thus $TB_{|\cdot|} \subseteq \sqrt{d}\, B^d$. $\square$

2 Reverse Isoperimetric Inequality

Outline:
In this section I would like to talk about an important application of John's Characterization Theorem: the reverse isoperimetric inequality. I will start by introducing the isoperimetric inequality and the definition of isoperimetric quotient. Then I will talk about the theorem 2.1 and show the curve of the volume, surface area and isoperimetric quotient of the ball and cube, in terms of dimension n. At last I will briefly prove the theorem 2.1.

Theorem 2.1. *Let $C \in \mathcal{C}_p$ be symmetric in origin, and T is the transformation makes C into John's position, then:*

$$\frac{S(TC)}{V(TC)^{\frac{d-1}{d}}} \leq 2d.$$

In particular, the equality holds iff C is a parallelotope.

Proof of Theorem 2.1. Choose a linear transformation T such that B^d is the ellipsoid of maximum volume in TC. We will show that

$$V(TC) \leq 2^d. \tag{6}$$

Take u_i, λ_i as in Theorem 1.3 and consider the convex body

$$D = \{x : |u_i \cdot x| \leq 1,\ i = 1, \ldots, m\}.$$

For each i, let f_i be the characteristic function of the interval $[-1, 1]$. i.e.

$$f_i(t) := \chi_{[-1,1]}(t) = \begin{cases} 1, & \text{if } |t| \leq 1 \\ 0, & \text{if } |t| > 1 \end{cases}$$

Then the function

$$x \mapsto \prod_i f_i(u_i \cdot x)^{\lambda_i}$$

is the characteristic function of D. Now integrate and use Brascamp-Lieb inequality to see that $V(D) \leq 2^d$.

$$V(D) = \int_{\mathbb{R}^d} \chi_D(x)dx = \int\limits_{\mathbb{E}^d} \prod_i f_i(u_i \cdot x)^{\lambda_i} dx$$

And by Brascamp-Lieb inequality we get

$$\int\limits_{\mathbb{E}^d} \prod_i f_i(u_i \cdot x)^{\lambda_i} dx \leq \prod_i \left(\int\limits_{\mathbb{R}} f_i(t)dt \right)^{\lambda_i}$$

And $f_i(t) = \chi_{[-1,1]}(t) \Rightarrow \int_{\mathbb{R}} f_i(t)dt = 2$, thus

$$\prod_i \left(\int\limits_{\mathbb{R}} f_i(t)dt \right)^{\lambda_i} = \prod_i 2^{\lambda_i} = 2^d$$

Since $TC \subseteq D$, this implies (6).

Using the definition of Minkowski surface area, the inclusion $B^d \subseteq TC$, and (6) together yield the following:

$$S(TC) = \lim_{\varepsilon \to +0} \frac{V(TC + \varepsilon B^d) - V(TC)}{\varepsilon} \leq \lim_{\varepsilon \to +0} \frac{V(TC + \varepsilon TC) - V(TC)}{\varepsilon} = dV(TC) \leq 2d\, V(TC)^{\frac{d-1}{d}}.$$

$\square$

3 Asymptotic best Approximation by polytopes

Outline:
In this section I will talk about how the difference between the convex body and the approximated n-facets polytopes behaves when the dimension goes to infinity. First I want to start with the example of approximating the ellipsoid or unit ball with n-facets polytopes. Although In dimension $d = 2$ it's easy, for $d > 2$ it's already difficult to find the precise one, and list some known results related to this problem refer to several literature without further explanation. Then I will talk about the theorem Asymptotic formula and explain the result it gets. Since it's a very complicated proof I will only show the strategy and idea of it to give the audience a basic knowledge of this. After that I will show the former isoperimetric problem for convex polytopes is an application of this theorem. In the end I will give an inscribed version of the main theorem, which will be useful in the last section.

4 Heuristic Principle

Outline:
In this last section I will talk about the famous heuristic principle. I would like to introduce the concept of random polytope approximation, and show a result from other literature that indicates the error of the random approximation. This will lead to our observation of how random approximation is as good as the precise one in high dimensions.