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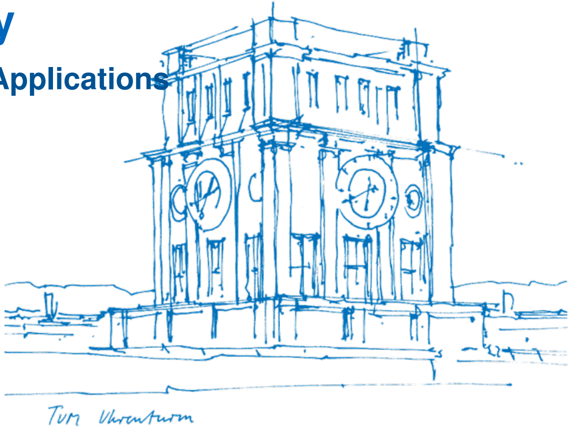
TUM School of Computation, Information and Technology
Technical University of Munich

Seminar: Convex Geometry

Approximation of Convex Bodies and Applications



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June 12th, 2025

Outline

Introduction to John's Ellipsoids

John's Theorem

Theorem 1 (John's Ellipsoid Theorem)**.** *Let $C \subset \mathbb{R}^d$ be a convex body. Then there exists an ellipsoid E of maximum volume in C (called the **John ellipsoid**) such that if c is the center of E, then the following inclusions hold:*

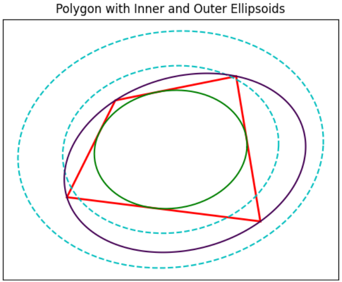
$$E \subseteq C \subseteq c + d(E - c),$$

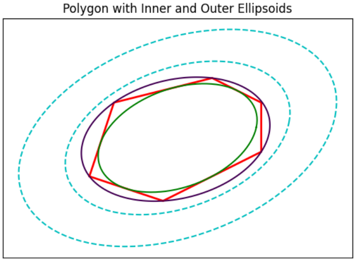
In particular, if C is symmetric in origin, then the tighter bound holds:

$$E \subseteq C \subseteq c + \sqrt{d}(E - c),$$

Introduction to John's Ellipsoids
2d samples

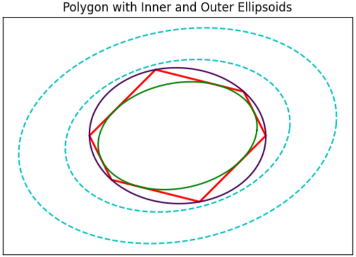
Asymmetric:

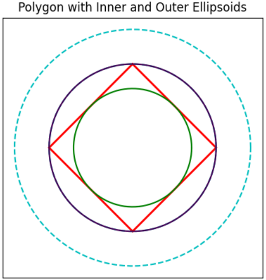




Introduction to John's Ellipsoids
2d samples

Symmetric:





Uniqueness and John's Characterization Theorem

Theorem

Theorem 2. *If $C \subset \mathbb{R}^d$ be a convex body, then the inner ellipsoid of maximal volume is unique.*

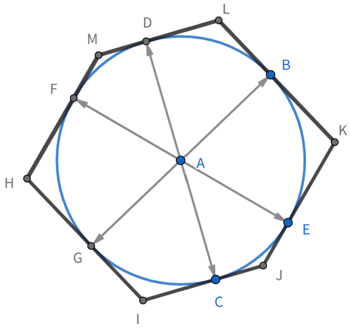
Theorem 3 (John's Characterization)**.** *Let $C \in \mathcal{C}_p$ be symmetric in origin and $B^d \subseteq C$. Then the following statements are equivalent:*

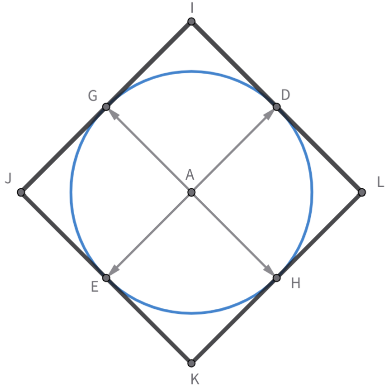
(i) B^d *is the unique John's Ellipsoid in* C

(ii) *There are* $u_i \in B^d \cap bdC$ *and* $\lambda_i > 0$, $i = 1, ..., m$, *where* $d \leq m \leq \frac{1}{2}d(d+1)$, *such that*

$$I = \sum \lambda_i u_i \otimes u_i \ , \sum \lambda_i = d.$$

John's Characterization Theorem
Illustration of the theorem





John's Characterization Theorem

Proof Sketch

$\mathcal{P}$: set of all symmetric positive definite $d \times d$ matrices, which is an open convex cone with apex at the origin;

$\mathcal{E} = \{A \in \mathcal{P} : Au \cdot v \leq h_C(v)$ for all $u, v \in bd\ B^d\}$;

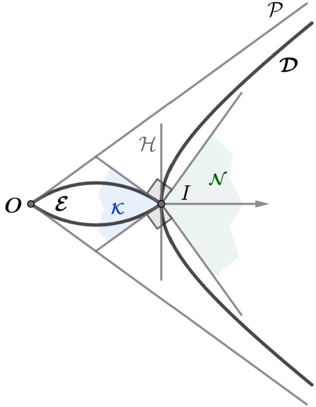
$\mathcal{D} = \{A \in \mathcal{P} : det(A) \geq 1\}$ is a closed, smooth and strictly convex set in $\mathcal{P}$ with non-empty interior;

$\mathcal{K}$: the support cone of $\mathcal{E}$ at I;

$\mathcal{N}$: the normal cone of $\mathcal{E}$ at I;

$\mathcal{H}$: the unique support hyperplane of $\mathcal{D}$ at I.

Proof Sketch

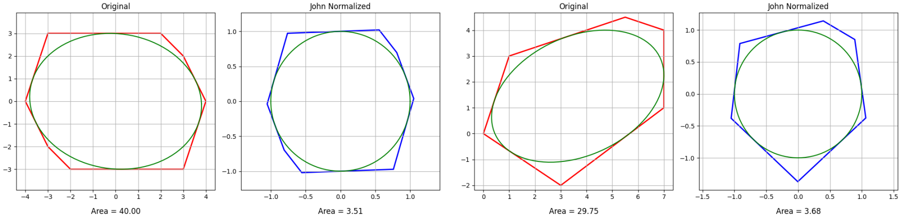


John's Characterization Theorem Applications(1)

John's position

Def: A convex body is in *John's position* when its John's Ellipsoid is B^d

Sample (2d):



John's Characterization Theorem Applications(2)

The Banach-Mazur distance

Def: $\delta^{BM}(\|\cdot\|_1, \|\cdot\|_2) = inf\{\lambda \geq 1 : \exists T \in GL(d), s.t. B_{\|\cdot\|_1} \subseteq TB_{\|\cdot\|_2} \subseteq \lambda B_{\|\cdot\|_1}\}$.

Other Def: $\delta^{BM}(X, Y) = inf\{\|T\|\|T^{-1}\| : T \in GL(X, Y)\}$, X, Y be two d-dim vector spaces.

Def: $\mathcal{N}(\mathbb{E}^d)$ is the set of all normed spaces on $\mathbb{E}^d$, endowed with δ^{BM}.

Corollary 1. $\delta^{BM}(\|\cdot\|, |\cdot|) \leq \sqrt{d}$, *where* $\|\cdot\|$ *is the Euclidean norm and* $|\cdot| \in \mathcal{N}(\mathbb{E}^d)$

Outline

Reverse Isoperimetric Inequality

Recall Isoperimetric Inequality

Recall: *Isoperimetric Inequality*:

Among all bodies of given volume in $\mathbb{R}^d$, the Euclidean ball minimizes surface area. Which is, given a convex body$K \subset \mathbb{R}^d$ and $S(K)$ and $V(K)$ being its surface area and volume, we have:

$$\frac{S(B^d)}{V(B^d)^{\frac{d-1}{d}}} \leq \frac{S(K)}{V(K)^{\frac{d-1}{d}}}$$

Def: *Isoperimetric quotient* : $\frac{S(K)}{V(K)^{\frac{d-1}{d}}}$

Reverse Isoperimetric Inequality

Introduction

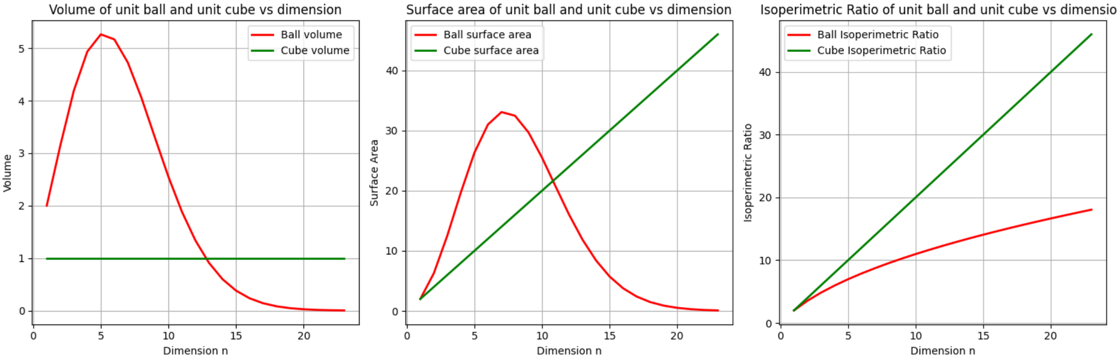
Theorem 4. *Let $C \in \mathcal{C}_p$ be symmetric in origin, and T is the transformation makes C into John's position, then:*

$$\frac{S(TC)}{V(TC)^{\frac{d-1}{d}}} \leq 2d.$$

In particular, the equality holds iff C is a parallelotope.

Reverse Isoperimetric Inequality

Ball vs. Cube by n-dims



Proof of Reverse Isoperimetric Inequality

Application of John's Characterization Theorem

Lemma 1 (Brascamp-Lieb inequality)**.** *Let* $u_i \in S^{d-1}$*,* $\lambda_i > 0$*, s.t.*

$$I = \sum \lambda_i u_i \otimes u_i \ , \sum \lambda_i = d.$$

and let f_i *be non-negative measurable functions on* $\mathbb{R}$*, then:*

$$\int_{\mathbb{E}^d} \prod_i f_i(u_i \cdot x)^{\lambda_i} dx \leq \prod_i \left(\int_{\mathbb{R}} f_i(t) dt \right)^{\lambda_i}.$$

Lemma 2 (Minkowski Surface Area)**.**

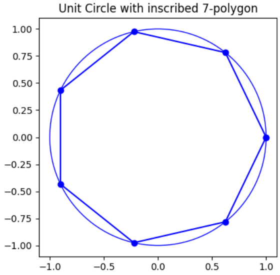
$$S(C) = \lim_{\epsilon \to +0} \frac{V(C + \epsilon B^d) - V(C)}{\epsilon}.$$

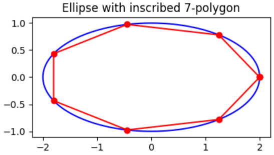
Outline

Polytope Approximation of Ellipsoid
d=2: result and example

Recall the fact: The best inscribed n-polygon in circle is the regular polygon.

Easy to see: The best inscribed n-polygon in an ellipse is the affine image of the regular n-polygon in the unit circle.





Polytope Approximation of Ellipsoid

d>2: hard

In dimension $d > 2$, finding the best n-facet approximation of the Euclidean ball is already difficult, so obtaining such approximations for general ellipsoids is equally challenging.

Some known results:

1. (*Lindelöf, 19th century*) Any optimal polyhedron circumscribed about a Euclidean ball touches each face at its centroid.
2. (*Ákos G.Horváth, Zsolt Lángi, 2014*) For the best polyhedron inscribed in the S^{d-1} with fixed $|V|$, the following cases are solved:
   a) $|V| = d + 2$, for any dimensions d;
   b) $|V| = d + 3$, for odd dimensions d find the maximum volume over the family of all polytopes; and for even dimensions d, over the family of not cyclic polytopes.

Asymptotic Formula

approximation by circumscribed convex polytopes

Def:

$\mathcal{P}^c_{(n)}(C)$: all polytopes circumscribed to C and have n facets;

$\delta(C, \mathcal{P}^c_{(n)}(C)) = min\{\delta(C, P) : P \in \mathcal{P}^c_{(n)}(C)\}$;

Symmetric difference metric: $\delta^V(C, D) = V(C \Delta D) = V\left((C/D) \cup (D/C)\right)$;

Affine surface area: $\kappa_C > 0$ is the Gauss curvature

$$A(C) = \int\limits_{bd\ C} \kappa_C(x)^{\frac{1}{d+1}} d\sigma(x)$$

Theorem 5. *Let $C \in \mathcal{C}_p$ be of class $\mathcal{C}^2$ with Gauss curvature $\kappa_C > 0$. Then there is a constant $\delta > 0$, depending only on d, such that*

$$\delta^V(C, \mathcal{P}^c_{(n)}) \sim \frac{\delta}{2} A(C)^{\frac{d+1}{d-1}} \frac{1}{n^{\frac{2}{d-1}}}\ as\ n \to \infty.$$

Asymptotic Formula

Strategy behind the Proof

Step 1: Prove the lower bound

$$\delta^V(C, \mathcal{P}^c_{(n)}) \geq \frac{\delta}{2\lambda^{f(d)}} \ldots$$

Use local quadratic approximation, apply Zador's theorem, and do global integration;

Step 2: Prove the upper bound

$$\delta^V(C, \mathcal{P}^c_{(n)}) \leq \frac{\lambda^{g(d)}\delta}{2} \ldots$$

To construct a specific polytope achieving this rate by doing curvature-weighted sampling, build circumscribed polytope and estimate volume excess.

Convex Polytopes with Minimum Isoperimetric quotient

An Application of asymptotic formula

Theorem 6. *Let $P_n \in \mathcal{P}_{(n)}$, $n = d+1, ...$, be polytopes with minimum isoperimetric quotient amongst all polytopes in $\mathcal{P}_{(n)}$. Then there is a constant $\delta > 0$, depending only on d, such that*

$$\frac{S(P_n)^d}{V(P_n)^{d-1}} \sim d^d V(B^d) + \frac{d^d \delta}{2} S(B^d)^{\frac{d+1}{d-1}} \frac{1}{n^{\frac{2}{d-1}}} \; as \; n \to \infty.$$

Asymptotic Formula

similar concequence for inscribed polytope

Theorem 7. *Let C be as defined before, then there is a constant $\gamma > 0$, depending only on d, such that*

$$\delta^V(C, \mathcal{P}^i_{(n)}) \sim \frac{\gamma}{2} A(C)^{\frac{d+1}{d-1}} \frac{1}{n^{\frac{2}{d-1}}} \; as \; n \to \infty.$$

Outline

Approximated by random polytopes

Def: Let $C \in \mathbb{R}^n$ be a convex body. A random polytope is the convex hull of finitely N points $[x_1, ..., x_N]$ in C that are chosen at random with respect to a probability measure $\mathbb{P}$. Denote the expected volume

$$\mathbb{E}(C, N) = \int\limits_{K^N} V([x_1, ..., x_N]) d\mathbb{P}$$

Theorem 8. *(J. Prochno, C. Schütt, E. M. Werne) Let $f : \partial K \to \mathbb{R}_{>0}$ be a density function, i.e. $\int_{\partial K} f(x) d\mu(x) = 1$, then we have*

$$\lim_{N \to \infty} \frac{V(C) - \mathbb{E}(f, N)}{N^{-\frac{2}{n-1}}} = c_n \int_{\partial K} \left(\frac{\kappa(x)}{f(x)^2} \right)^{\frac{1}{n-1}} d\mu_{\partial K}(x).$$

The minimum of right-hand integral is attained for the normalized affine surface area measure.

Heurisitic Observations

Precise approximation difference:

$$\delta^V(C, \mathcal{P}^i_{(n)}) \sim c_d A(C)^{\frac{d+1}{d-1}} n^{-\frac{2}{d-1}} \ as\ n \to \infty;$$

Random approximation difference(Q_k is convex hull of k random points):

$$\mathbb{E}(\delta^V(C, Q_k)) \sim \tilde{c}_d A(C) k^{-\frac{2}{d+1}} \ as\ k \to \infty;$$

$$n := \mathbb{E}(|V(Q_k)|) \sim \tilde{c}_d A(C) k^{\frac{d-1}{d+1}} \ as\ k \to \infty;$$

$$\Rightarrow \mathbb{E}(\delta^V(C, Q_k)) \sim \tilde{c}_d^{\frac{d+1}{d-1}} A(C)^{\frac{d+1}{d-1}} n^{-\frac{2}{d-1}} \ as\ n \to \infty.$$

$\Rightarrow$ In high dimensions, random approximation is **almost as good as** best approximation.

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Thank you!

Questions?