Representation Theory Notes

MIT PRIMES 2022 Reading Group with Nathan Xiong and Ilaria Seidel, mentored by Sasha Utiralova

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1 Introduction

Representation theory involves studying algebra structures by representing their elements as linear maps between vector spaces. It has applications ranging from number theory and combinatorics to geometry, probability theory, quantum mechanics, and quantum field theory.

An important sub-problem is the representation theory of groups—since linear algebra is nicer and easier to work with than abstract algebra, it is nice to be able to take groups and represent them as sets of matrices instead.

As prerequisites to reading these notes, I suggest backgrounds in both abstract and linear algebra.

2 Basic Notions of Representation Theory

§2.1 What is representation theory?

In this section, the book gives a very quick summary of the topics covered in representation theory. However, to a beginner in the subject, it makes absolutely no sense, so we'll skip it.

§2.2 Algebras

Let k be a field. In general, we will consider k to be algebraically closed (assume so unless stated otherwise). The main examples of such fields are $\mathbb{C}$ and $\overline{\mathbb{F}_p}$.

Definition 2.2.1. An **associative algebra** over k is a vector space A over k together with a bilinear map $A \times A \to A$, where we write $(a, b) \mapsto ab$, such that $(ab)c = a(bc)$.

So, an associative algebra is a vector space together with some associative "multiplication" operation. This operation is also bilinear, so it respects the distributive property: $a(b + c) = ab + ac$, for example. We will present some examples of algebras soon.

Definition 2.2.2. A **unit** in an associative algebra is an element $1 \in A$ such that $1a = a1 = a$. If an associative algebra has a unit, we call it **unital**.

Note that if an algebra is unital, that unit is unique: given two units 1 and $1'$, $1 = 11' = 1'$.

From now on, whenever we say algebra, we mean a unital associative algebra, unless stated otherwise.

Example 2.2.3 (Algebras)

Here are some examples of algebras over k:

(a) $A = k$. The multiplication is just given by normal multiplication in k.

(b) The polynomial ring $A = k[x_1, \ldots, x_n]$. Again, the multiplication is just given by normal multiplication of polynomials.

(c) $A = \operatorname{End} V$, the algebra of endomorphisms of a vector space V over k, which is the set of linear maps from V to itself. Multiplication is given by composition of maps. If we specify a basis, we can also interpret these as the set of $n \times n$ matrices with entries in k.

Before we move on, we should note that there's another way to think of these algebras:

An algebra (unital and associative) A over k is a possibly noncommutative ring with a copy of k inside of it.

Note that the copy of k inside of A implies that it is a k-vector space (since you can scale by elements of k).

Why is this essentially the same definition? Well, the definition before involved two operations in A: an addition operation given by the k-vector space, and a multiplication operation given by the bilinear map; these become the two operations of the noncommutative ring. For a more formal definition:

Definition 2.2.4. An **algebra** A over k is a *possibly noncommutative* ring, equipped with an injective ring homomorphism $k \hookrightarrow A$ (the image of this map is the "copy of k inside of" A). When k is viewed as a subset of A, we additionally require $\lambda \cdot a = a \cdot \lambda$ for $\lambda \in k$ and $a \in A$.

For example, when $A = \operatorname{End} V$, the image of the injective ring homomorphism $k \hookrightarrow \operatorname{End} V$ is the set $kI = \{\lambda I : \lambda \in k\}$ where I is the identity map. In general, the image of this injective ring homomorphism is always $k1$ where 1 is the unit in A.

Definition 2.2.5. An algebra A is **commutative** if $ab = ba$ for all $a, b \in A$.

Example 2.2.6 (More algebras)

(a) The **free algebra** $A = k\langle x_1, \ldots, x_n \rangle$, with a basis given by words with letters $x_1, \ldots, x_n$, and multiplication given by concatenation of words. For example, $(x_1x_2x_1 + x_3x_4) \cdot x_1^2x_2 = x_1x_2x_1^3x_2 + x_3x_4x_1^2x_2$.

(b) The **group algebra** $A = k[G]$ of a group G, which is a k-vector space formally spanned by elements of G, and the product of two basis elements is given by the group operation. In particular, its basis is $\{a_g : g \in G\}$, with $a_ga_h = a_{gh}$.

Question 2.2.7. When are each of the five examples of algebras mentioned above commutative?

Definition 2.2.8. A **homomorphism of algebras** $\varphi : A \to B$ is a linear map such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in A$, and $\varphi(1) = 1$.

In other words, this map respects both operations in the algebras: addition and multiplication; it also sends 0 to 0 and 1 to 1. It is therefore a homomorphism in both senses: as a ring and as a vector space.