$$\boldsymbol{\mu_{ML}^{N} = \mu_{ML}^{N-1} + \frac{1}{N}(x_N - \mu_{ML}^{N-1})}$$

$$\boldsymbol{\Sigma_{ML}^{N-1} = \frac{1}{N-1}\sum_{i=1}^{N-1}(x_i - \mu_{ML}^{N-1})(x_i - \mu_{ML}^{N-1})^T}$$

GOAL : express $\boldsymbol{\Sigma_{ML}^{N}}$ by $\boldsymbol{\Sigma_{ML}^{N-1}}$, $\boldsymbol{\mu_{ML}^{N}}$, $\boldsymbol{\mu_{ML}^{N-1}}$, $\boldsymbol{x_N}$, and N

$$\Sigma_{ML}^{N} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu_{ML}^{N})(x_i - \mu_{ML}^{N})^T$$

(Take out the last iteration)

$$= \frac{1}{N}\sum_{i=1}^{N-1}(x_i - \mu_{ML}^{N})(x_i - \mu_{ML}^{N})^T + \frac{1}{N}(x_N - \mu_{ML}^{N})(x_N - \mu_{ML}^{N})^T$$

(Replace $\boldsymbol{\mu_{ML}^{N}}$ with $\boldsymbol{\mu_{ML}^{N-1} + \frac{1}{N}(x_N - \mu_{ML}^{N-1})}$)

$$= \frac{1}{N}\sum_{i=1}^{N-1}((x_i - \mu_{ML}^{N-1}) - \frac{1}{N}(x_N - \mu_{ML}^{N-1}))((x_i - \mu_{ML}^{N-1}) - \frac{1}{N}(x_N - \mu_{ML}^{N-1}))^T +$$

$$\frac{1}{N}(x_N - \mu_{ML}^{N})(x_N - \mu_{ML}^{N})^T$$

$$= \frac{1}{N}\sum_{i=1}^{N-1}(x_i - \mu_{ML}^{N-1})(x_i - \mu_{ML}^{N-1})^T - \frac{1}{N^2}\sum_{i=1}^{N-1}(x_i - \mu_{ML}^{N-1})(x_N - \mu_{ML}^{N-1})^T$$

$$-\frac{1}{N^2}\sum_{i=1}^{N-1}(x_i - \mu_{ML}^{N-1})^T(x_N - \mu_{ML}^{N-1})$$

$$+\frac{N-1}{N^3}(x_N - \mu_{ML}^{N-1})(x_N - \mu_{ML}^{N-1})^T + \frac{1}{N}(x_N - \mu_{ML}^{N})(x_N - \mu_{ML}^{N})^T$$

$$= \frac{\mathrm{N}-1}{\mathrm{N}}\Sigma_{ML}^{N-1} - \frac{1}{N^2}\sum_{i=1}^{N-1}(x_i - \mu_{ML}^{N-1})(x_N - \mu_{ML}^{N-1})^T$$

$$-\frac{1}{N^2}\sum_{i=1}^{N-1}(x_i - \mu_{ML}^{N-1})^T(x_N - \mu_{ML}^{N-1})$$

$$+\frac{N-1}{N^3}(x_N - \mu_{ML}^{N-1})(x_N - \mu_{ML}^{N-1})^T + \frac{1}{N}(x_N - \mu_{ML}^{N})(x_N - \mu_{ML}^{N})^T$$

(We prove that the second and third term equals 0)

(Consider the second term of the equation)

$$\frac{1}{N^2}\sum_{i=1}^{N-1}(x_i - \mu_{ML}^{N-1})(x_N - \mu_{ML}^{N-1})^T$$

$$= \frac{1}{N^2}\sum_{i=1}^{N-1}x_i(x_N - \mu_{ML}^{N-1})^T - \frac{1}{N^2}\sum_{i=1}^{N-1}\mu_{ML}^{N-1}(x_N - \mu_{ML}^{N-1})^T$$

$$= \left(\frac{N-1}{N^2}\mu_{ML}^{N-1}\right)(x_N - \mu_{ML}^{N-1})^T - \frac{N-1}{N^2}\mu_{ML}^{N-1}(x_N - \mu_{ML}^{N-1})^T = 0$$

(Now consider the third term of the equation)

$$\frac{1}{N^2}\sum_{i=1}^{N-1}(x_i - \mu_{ML}^{N-1})^T(x_N - \mu_{ML}^{N-1})$$

$$= \frac{1}{N^2}\sum_{i=1}^{N-1}x_N(x_i - \mu_{ML}^{N-1})^T - \frac{1}{N^2}\sum_{i=1}^{N-1}\mu_{ML}^{N-1}(x_i - \mu_{ML}^{N-1})^T$$

$$= \frac{N-1}{N^2}\mu_{ML}^{N-1}(x_i - \mu_{ML}^{N-1})^T - \frac{1}{N^2}\sum_{i=1}^{N-1}\mu_{ML}^{N-1}(x_i - \mu_{ML}^{N-1})^T = 0$$

Thus the equation becomes

$$\frac{\mathbf{N-1}}{\mathbf{N}}\boldsymbol{\Sigma}_{\boldsymbol{ML}}^{\boldsymbol{N-1}} + \frac{\boldsymbol{N-1}}{\boldsymbol{N^3}}\left(\boldsymbol{x_N} - \boldsymbol{\mu}_{\boldsymbol{ML}}^{\boldsymbol{N-1}}\right)\left(\boldsymbol{x_N} - \boldsymbol{\mu}_{\boldsymbol{ML}}^{\boldsymbol{N-1}}\right)^{\boldsymbol{T}} + \frac{\mathbf{1}}{\boldsymbol{N}}\left(\boldsymbol{x_N} - \boldsymbol{\mu}_{\boldsymbol{ML}}^{\boldsymbol{N}}\right)\left(\boldsymbol{x_N} - \boldsymbol{\mu}_{\boldsymbol{ML}}^{\boldsymbol{N}}\right)^{\boldsymbol{T}}$$