105-1 Midterm, CSIE, NTPU

Advanced Algorithms (高等演算法)

Date: 2016/11/08 Class: 109 ID: 710583201 Name: Dmitrii Matveichev

1. (5%) Describe the **Master Theorem** for solving recurrences of the form $T(n) = a\, T(n/b) + f(n)$.
2. (15%) Use Master Theorem to solve the following **recurrences** for $T(n)$.
   (a) $T(n) = 4T(n/2) + n$ (5%)
   (b) $T(n) = 4T(n/2) + n^2$ (5%)
   (c) $T(n) = 4T(n/2) + n^3$ (5%)
3. (15%) The n-th Fibonacci number can be calculated with the following recursive function. Assume the time costs of all integer arithmetic (addition, subtraction, multiplication, division) operations are equal. First, find the time complexity of the following function. Then, rewrite the function for best (sub-linear) time efficiency and show your complexity analysis.

```
Fibo( n )
{
    if (n = 1) or (n = 2) then
        return 1
    else
        return Fibo(n – 1) + Fibo(n – 2)
}
```

4. (15%) Consider the following recursive function where the global variable *count* is initialized to 0 and input n is a positive integer.
   (a) Find out the asymptotic time complexity in Θ-notation. (5%)
   (b) Find out the asymptotic space complexity in Θ-notation. (5%)
   (c) Express the final value of *count* as a function of n. (5%)

```
Rec-x( n )
{
    if ( n = 1 ) or ( n = 2 ) then
        count ← count + 1
    else
    {
        Rec-x( n–2 )
        Rec-x( n–2 )
        Rec-x( n–1 )
        count ← count + 1
    }
}
```

5. (10%) A linear consecutive-k-out-of-n system ($k < n$) consists of n nodes arranged in a line, where the system fails if and only if some k consecutive nodes fail. Suppose the nodes are statistically independent and the reliability of node i is p_i for any $i \in \{1, 2, \ldots, n\}$ (node i functions with probability p_i and fails with probability $1 - p_i$). Let $R_L(i, j, k)$ denote the reliability of the linear consecutive-k-out-of-n subsystem consisting of nodes $i, i+1, i+2, \ldots, j$.

(a) Express $R_L(1, n, k)$ with $R_L(1, n-1, k)$ and $R_L(1, n-k-1, k)$.

(b) Design a $\Theta(n)$-time algorithm for $R_L(1, n, k)$ when p_i are given for $i \in \{1, 2, \ldots, n\}$.

6. (10%) Consider the following program segment.

```
i ← 2
x ← 3
while ( i < n ) do
{
    i ← i × i
    x ← x + 1
}
```

(a) Find the time complexity (in Θ notation) of this program segment. (5%)

(b) Express the final value of x as a function of n. (5%)

7. (15%) Given a subroutine Biased_Rand_Bit() that outputs 1 with probability p and 0 with probability $1 - p$, where $0 < p < 1$ and p is unknown, do the following tasks.

(a) Design an algorithm Unbiased_Rand_Bit() that returns 1 with probability 1/2 and 0 with probability 1/2. (5%)

(b) Prove that your algorithm is correct. (5%)

(c) What is the expected running time of your algorithm (as a function of p)? (5%)

8. (15%) Given 39 distinct numbers, please answer the following independent questions.

(a) In the worst case, how many comparisons at least do you need to find the largest number? Describe your algorithm with a brief proof. (5%)

(b) In the worst case, how many comparisons at least do you need to find the second largest number? Describe your algorithm with a brief proof. (5%)

(c) In the worst case, how many comparisons at least do you need to find both the maximum and minimum numbers? Describe your algorithm with a brief proof. (5%)