1 Auctions

Below are the four main types of auctions:

- ▷ First-price auction: Bidders submit sealed bids, higher bidder wins and pays his bid.
- ▷ Second-price auction: Bidders submit sealed bids, higher bidder wins and pays the second highest bid.
- ▷ Ascending (English) auction: Bidders raise their hands when they are willing to buy the good. Prices increase from zero until only one is left and the remaining bidder pays the price at which the last bidder dropped out.
- ▷ Descending (Dutch) auction: Bidders Price decrease from some high number. First bidder to raise their hand wins the auction and pays the price at that time.

Throughout, we will not worry about "tie breaks" (which will be a measure zero event in the set up).

1.1 Independent private values model

1.1.1 First-price auction

A risk neutral seller wishes to sell a single indivisible good. There are N risk-neutral bidders indexed by $i \in \{1, 2, \ldots, N\}$. Each bidder has a valuation of the good, denoted v_i. We assume that v_i is drawn from $[0, 1]$ according to the density function $f_i : [0, 1] \to \mathbb{R}_{++}$.[1] We assume that all values are independent. The CDF function is given by

$$F_i(v_i) = \int_0^{v_i} f_i(x)\, dx.$$

If v_i wins and pays p, his v.N-M utility is $v_i - p$. If he does not win and pays p, his v.N-M utility is simply $-p$.

A strategy for bidder i is a mapping $b_i : [0, 1] \to [0, 1]$ (from the set of types to the set of actions).

Definition 1.1. *(Bayesian Nash equilibrium)* A strategy profile $(b_1, b_2, \ldots, b_N)$ is a Bayesian Nash equilibrium (BNE) if and only if, for all $i \in \{1, 2, \ldots, N\}$, and for all $v_i \in [0, 1]$, $b_i(v_i)$ maximises v_i's expected utility given that the other bidder use strategy the equilibrium strategy b_{-i}.

We refer to this model as independent private values model. Independent refers to the fact that each buyer's private information, v_i, is independent of every other player's private information. Private value refers to the fact that once a buyer employs his own private information to assess the value of the objective, this assessment would be unaffected were he subsequently to learn any other buyer's private information; i.e. each buyer's private information is sufficient for determining his value.

For simplicity, we assume symmetry; i.e.

$$f_i \equiv f, \forall i$$

[1] We can think of each v_i as different types of players.

and look for a BNE in symmetric, increasing functions; i.e.

$$b_i = b, \ \forall i \in \{1, 2, \ldots, N\},$$
$$b(v') > b(v), \ \forall v' > v.$$

Denote $\hat{b} : [0,1] \rightarrow [0,1]$ as the increasing function that represent the symmetric equilibrium strategy. What must it be?

Deriving the bidding function Let $u_i(r_i, v_i)$ denote the bidder i with value v_i (from now on, player v_i)'s expected utility when he bids $\hat{b}(r_i)$ and all others bid according to $\hat{b}(\cdot)$ and their true values. We want to check that, if everyone else is playing $\hat{b}(v_i)$, then you will also want to play $\hat{b}(v_i)$ and not $\hat{b}(r_i)$. If player v_i bids $\hat{b}(r_i)$, then his payoff is given by

$$\begin{aligned} u_i(r_i, v_i) = {} & \left(v_i - \hat{b}(r_i)\right) \mathbb{P}\left(\text{win} | \hat{b}(r_i) \text{ vs } N-1 \text{ others bid } \hat{b}(\cdot)\right) \\ & + (0)\, \mathbb{P}\left(\text{lose} | \hat{b}(r_I) \text{ vs } N-1 \text{ others bid } \hat{b}(\cdot)\right). \end{aligned}$$

Note that player v_i wins by bidding $\hat{b}(r_i)$ if and only if

$$\hat{b}(r_i) > \hat{b}(v_j), \ \forall j \neq i.$$

But we assume $\hat{b}$ is (strictly) increasing so that

$$\hat{b}(r_i) > \hat{b}(v_j), \ \forall j \neq i \Leftrightarrow r_i > v_j, \ \forall j \neq i.$$

Since we assume symmetry and independence,

$$\begin{aligned} \mathbb{P}(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_N < r_i) &= \mathbb{P}(v_1 < r_i) \cdots \mathbb{P}(v_{i-1} < r_i)\, \mathbb{P}(v_{i+1} < r_i) \cdots \mathbb{P}(v_N < r_i) \\ &= (F(r_i))^{N-1}. \end{aligned}$$

Hence,

$$u_i(r_i, v_i) = \left(v_i - \hat{b}(r_i)\right) F^{N-1}(r_i).$$

For $\hat{b}$ to be an equilibrium strategy, $u(r_i, v_i)$ must be maximised in r_i when evaluated at $r_i = v_i$.

$$\begin{aligned} \left.\frac{\partial u_i(r_i, v_i)}{\partial r_i}\right|_{r_i = v_i} &= (N-1) f(r_i) \left(v_i - \hat{b}(r_i)\right) F^{N-2}(r_i) - \hat{b}'(r_i) F^{N-1}(r_i)\Big|_{r_i = v_i} \\ &= (N-1) f(v_i) \left(v_i - \hat{b}(v_i)\right) F^{N-2}(v_i) - \hat{b}'(v_i) F^{N-1}(v_i). \end{aligned}$$

Let us, for now, assume that the first-order condition is sufficient for a maximum, then

$$\begin{aligned} \left.\frac{\partial u(r_i, v_i)}{\partial r_i}\right|_{r_i = v_i} &= 0 \\ \Rightarrow (N-1) f(v_i)\, \hat{b}(v_i)\, F^{N-2}(v_i) + \hat{b}'(v_i)\, F^{N-1}(v_i) &= (N-1) f(v_i)\, F^{N-2}(v_i)\, v_i, \end{aligned}$$

Observe that the left-hand side is the derivative of $\hat{b}(v_i) F^{N-1}(v_i)$; i.e.

$$\frac{d}{dv_i}\left[\hat{b}(v_i) F^{N-1}(v_i)\right] = (N-1) f(v_i) F^{N-2}(v_i) v_i.$$

Since the first-order condition holds for all $v_i \in [0,1]$, we can integrating both sides with respect to v_i between $[0, v_i]$ to obtain

$$\begin{aligned}
\hat{b}(v_i) F^{N-1}(v_i) - \underbrace{\hat{b}(0) F^{N-1}(0)}_{=0} &= \int_0^{v_i} (N-1) f(x) F^{N-2}(x) x dx \\
\Leftrightarrow \hat{b}(v_i) &= \frac{1}{F^{N-1}(v_i)} \int_0^{v_i} (N-1) f(x) F^{N-2}(x) x dx.
\end{aligned}$$

Notice that

$$(N-1) f(x) F^{N-2}(x) = \frac{dF^{N-1}(x)}{dx},$$

where $F^{N-1}(x)$ is the CDF of $\max\{v_j\}_{j=1}^{N-1}$. Hence, we can write the bidding function as

$$\begin{aligned}
\hat{b}(v_i) &= \frac{1}{F^{N-1}(v_i)} \int_0^{v_i} x \frac{dF^{N-1}(x)}{dx} dx \\
&= \frac{1}{F^{N-1}(v_i)} \int_0^{v_i} x dF^{N-1}(x).
\end{aligned}$$

We can verify that $\hat{b}(v_i)$ is strictly increasing in v_i:

$$\begin{aligned}
\hat{b}'(v_i) &= \left(\frac{\partial}{\partial v_i} (F(v_i))^{1-N}\right) \int_0^{v_i} x dF^{N-1}(x) + \frac{1}{F^{N-1}(v_i)} \left((N-1) f(v_i) F^{N-2}(v_i) v_i\right) \\
&= (1-N) f(v_i) F(v_i)^{1-N-1} \int_0^{v_i} x dF^{N-1}(x) + (N-1) \frac{f(v_i)}{F(v_i)} v_i \\
&= -(N-1) \frac{f(v_i)}{F(v_i)} \underbrace{\frac{1}{F^{N-1}(v_i)} \int_0^{v_i} x dF^{N-1}(x)}_{=\hat{b}(v_i)} + (N-1) \frac{f(v_i)}{F(v_i)} v_i \\
&= (N-1) \frac{f(v_i)}{F(v_i)} \left(v_i - \hat{b}(v_i)\right).
\end{aligned}$$

We need to determine the sign of $v_i - \hat{b}(v_i)$. To do so, we can use integration by parts on $\hat{b}(v_i)$:

$$\begin{aligned}
\hat{b}(v_i) &= \frac{1}{F^{N-1}(v_i)} \int_0^{v_i} x \frac{dF^{N-1}(x)}{dx} dx \\
&= \frac{1}{F^{N-1}(v_i)} \left(\left[F^{N-1}(x) x\right]_0^{v_i} - \int_0^{v_i} F^{N-1}(x) dx\right) \\
&= \frac{1}{F^{N-1}(v_i)} \left(F^{N-1}(v_i) v_i - \int_0^{v_i} F^{N-1}(x) dx\right) \\
&= v_i - \frac{1}{F^{N-1}(v_i)} \int_0^{v_i} F^{N-1}(x) dx. \qquad (1.1)
\end{aligned}$$

Since $F(v_i) > 0$, it follows that

$$v_i > \hat{b}(v_i) \Leftrightarrow v_i - \hat{b}(v_i) > 0.$$

Given this, as well as the fact that $f(v_i) > 0$ for all v_i by assumption, $N > 1$, $F(v_i) > 0$, it follows that $\hat{b}'(v_i) > 0$ so that $\hat{b}(v_i)$ is strictly increasing.

To show that this is a maximum, observe that we can write, using the expression we found for $\hat{b}'(v_i)$:

$$\begin{aligned}\frac{\partial u_i(r_i, v_i)}{\partial r_i} &= (N-1) f(r_i) \left(v_i - \hat{b}(r_i)\right) F^{N-2}(r_i) - \hat{b}'(r_i) F^{N-1}(r_i) \\ &= (N-1) f(r_i) \left(v_i - \hat{b}(r_i)\right) F^{N-2}(r_i) - (N-1) \frac{f(r_i)}{F(r_i)} \left(r_i - \hat{b}(r_i)\right) F^{N-1}(r_i) \\ &= (N-1) f(r_i) F^{N-2}(r_i)(v_i - r_i).\end{aligned}$$

Then, $u_i(r_i, v_i)$ is upward sloping in r_i for all $r_i < v_i$ and downward sloping in r_i for all $r_i > v_i$. Thus, it attains a unique global maximum at $r_i = v_i$. In other words, the first-order condition gives us the maximum as we wanted.

Interpretation To interpret $\hat{b}$, recall that $F^{N-1}(x)$ is the CDF of $\upsilon_{-i} = \max\{v_j\}_{j \neq i}$. Hence,

$$\begin{aligned}\hat{b}(v_i) &= \frac{1}{F^{N-1}(v_i)} \int_0^{v_i} x dF^{N-1}(x) \\ &= \frac{\mathbb{E}\left[\mathbf{1}\{\upsilon_{-i} \leq v_i\} \upsilon_{-i}\right]}{\mathbb{P}(\upsilon_{-i} \leq v_i)} \\ &= \frac{\mathbb{E}\left[1 \cdot \upsilon_{-i} | \upsilon_{-i} \leq v_i\right] \mathbb{P}(\upsilon_{-i} \leq v_i) + \mathbb{E}\left[0 \cdot \upsilon_{-i} | \upsilon_{-i} > v_i\right](1 - \mathbb{P}(\upsilon_{-i} \leq v_i))}{\mathbb{P}(\upsilon_{-i} \leq v_i)} \\ &= \mathbb{E}\left[\upsilon_{-i} | \upsilon_{-i} \leq v_i\right] \\ &= \mathbb{E}\left[\text{highest value of others} | v_i \text{ is highest value}\right].\end{aligned}$$

That is, each bidder bids the expected value of the highest value held by other bidders conditional on his own value being the highest. Equivalently, letting υ_k denote the kth highest value among the N values, $\{\upsilon_1, \upsilon_2, \ldots, \upsilon_N\}$ (i.e. υ_k is the $N-k+1$th order statistic),

$$\hat{b}(v_i) = \mathbb{E}_{\upsilon_2}\left[\upsilon_2 | \upsilon_1 = v\right].$$

Alternatively, recall that we may write the bidding function as in (1.1):

$$\hat{b}(v_i) = v_i - \int_0^{v_i} \left(\frac{F(x)}{F(v_i)}\right)^{N-1} dx.$$

This form of the equilibrium bidding function highlights the amount by which bidders “shade” their bids (i.e. bid below their value). Notice also that since

$$F(x) \leq F(v_i), \ \forall x \in [0, v_i]$$

so that

$$\frac{F(x)}{F(v_i)} \leq 1, \ \forall x \in [0, v_i].$$

Hence,

$$\lim_{N \to \infty} \hat{b}(v_i) = v_i - \lim_{N \to \infty} \int_0^{v_i} \left(\frac{F(x)}{F(v_i)}\right)^{N-1} dx = v_i.$$

That is, as N tends to infinity, bidders bid their true value/more aggressively (and will not shade their bids).

Example 1.1. *(Uniform F)* Suppose that each bidder's value is uniformly distributed on $[0,1]$ so that $F(v) = v$ and $f(v) = 1$. With N bidders, the bidding function is given by

$$\begin{aligned}\hat{b}(v) &= \frac{1}{F^{N-1}(v)} \int_0^v (N-1) f(x) F^{N-2}(x) x dx \\ &= \frac{1}{v^{N-1}} \int_0^v (N-1) x^{N-2} x dx = \frac{1}{v^{N-1}} \int_0^v (N-1) x^{N-1} dx \\ &= \frac{1}{v^{N-1}} \left[\frac{N-1}{N} x^N\right]_0^v = \frac{1}{v^{N-1}} \left[\frac{N-1}{N} v^N\right] \\ &= v - \frac{1}{N} v.\end{aligned}$$

Thus, each bidders *shades* his bid, by bidding less than his value, and the bidders bid more aggressively as the number of bidders increase.

Remark 1.1. *(Dutch auction)* In a Dutch auction, every bidder must decide ex ante when to put their "hand" up as the price comes down. The decision that the bidder face is then exactly the same as in the first-price auction. That is, Dutch auction and the first-price auction are equivalent.

1.1.2 Second-price auction

Bidders will bid differently between first-price and second-price auctions. In a first-price auction, a bidder has an incentive to raise his bid to increase his chances of winning the auction, yet he has an incentive to reduce his bid to lower the price he pays when he does win. However, in a second-price auction, the second effect is absent because when a bidder wins, the amount he pays is independent of his bid. So we might expect bidders to bid more aggressively in a second-price auction than they would in a first-price auction. Let us first consider what an equilibrium in a second-price auction might look like, and then ask whether a second-price auction might generate higher expected revenues for the seller than a first-price auction.

In a second-price auction, bidding your own value is a weakly dominant strategy. To see this, recall that the rule is that when you win, you pay the highest bid among others, denoted $v_{-i} = \max\{v_j\}_{j \neq i}$. Suppose the bidder i bids $r_i < v_i$.

- ▷ If $r_i < v_i < v_{-i}$, then the bidder does not win and obtains zero utility;
- ▷ If $r_i < v_{-i} < v_i$, then reporting v_i gives the bidder utility of

  $$v_i - B_{-i} > 0$$

  and reporting r_i gives zero utility.
- ▷ If $v_{-i} < r_i < v_i$, then the bidder wins whether he reports r_i or v_i and the utility is given by

  $$v_i - v_{-i} > 0$$

  in both cases.

Hence, bidding v_i is a weakly dominant strategy.

Put differently, a bidder wants to win if $v_i > v_{-i}$ and wants to lose if $v_i < v_{-i}$. By bidding v_i, this is exactly what would happen—you will win when you would want to and lose when you would want to. Hence, every bidder bidding their own values is an equilibrium in the second-price auction.

Remark 1.2. *(English auction)* In an English auction, for each bidder, it is a weakly dominant strategy to drop out when the price increases to be above the bidder's valuation. Hence, this is equivalent to a second-price auction. But notice that the assumption of private value is crucial; if this did not hold, then as bidders "drop out" the remaining bidders may update their values.

1.1.3 Comparison of revenues between first and second-price auctions

In a first-price auction (FPA), because the bidder with the highest value submits the highest bid and wins, if v is the highest value among the N bidder values, then the seller's revenue is $\hat{b}(v)$. Thus, if the highest value is distributed according to the density $g(v)$, the seller's expected revenue is

$$R_{FPA} = \int_0^1 \hat{b}(v)\, g(v)\, dv.$$

Now, since all bidders are symmetric and the values are independent, the probability that v is the highest among all the bidders is given by $F^N(v)$. Then,

$$g(v) \equiv N f(v)\, F^{N-1}(v)$$

so that

$$R_{FPA} = N \int_0^1 \hat{b}(v)\, f(v)\, F^{N-1}(v)\, dv.$$

In a second-price auction (SPA), we found that each bidder bids their value, and the seller receives the second-highest bid/value among the B bids/values. Letting $h(v)$ denote the density of the second-highest value, the seller's expected revenue from a SPA is given by

$$R_{SPA} = \int_0^1 v h(v)\, dv.$$

We need to specify $h(v)$.

Fact 1.1. *(kth order statistic) Suppose we wish to know the kth order statistic (i.e. density of the kth smallest element) among $\{v_1, v_2, \ldots, v_N\}$ where v_i's are iid and have density $f(x)$. The density of the kth order statistic is defined as*

$$f_k(x) := \frac{N!}{(k-1)!\,(N-k)!} (F(x))^k (1 - F(x))^{N-k} f(x),$$

where $\frac{N!}{(k-1)!(N-k)!}$ is the number of different ways in which any draws of $v_1, \ldots, v_N$ could have been the kth order static, $F(x)^{k-1}$ is is the probability that $k-1$ draws are below x, $(1 - F(x))^{n-k}$ is the probability that $n-k$ draws are above x and $f(x)$ is the likelihood of drawing x.

In our case, we wish to know the density of the second highest value among the N possible values. We can follow the formula or think this through. The likelihood that some particular bidder's value v is $f(v)$ and the probability that exactly one of the remaining $N-1$ values are above v is given by $(N-1) F^{N-2}(v)(1-F(v))$.[2] Since there are N players, then

$$h(v) = N(N-1) F^{N-2}(v)(1-F(v)) f(v).$$

That is,

$$R_{SPA} = \int_0^1 vN(N-1) F^{N-2}(v)(1-F(v)) f(v)\, dv.$$

We now transform R_{FPA} into R_{SPA}:

$$\begin{aligned} R_{FPA} &= N \int_0^1 \hat{b}(v) f(v) F^{N-1}(v)\, dv \\ &= N \int_0^1 \left[\frac{1}{F^{N-1}(v)} \int_0^v (N-1) f(x) F^{N-2}(x)\, x dx \right] f(v) F^{N-1}(v)\, dv \\ &= N(N-1) \int_0^1 \int_0^v x F^{N-2}(x) f(x) f(v)\, dx dv. \end{aligned}$$

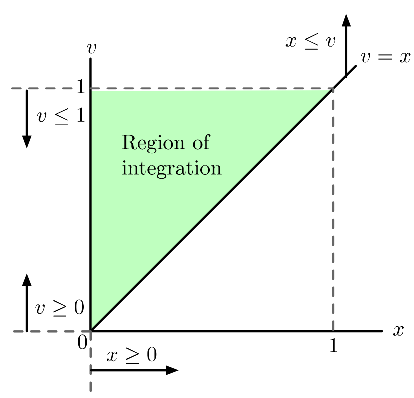
We then exchange the order of integration but we need to be careful with the limits. Consider the following double integration:

$$\int_0^1 \int_0^v (\cdot)\, dx dv.$$

The inner integration is with respect to x from zero to v. Hence, $0 \leq x \leq v$. The outer integration integrates with respect to v from 0 to 1, so $0 \leq v \leq 1$. Notice that

$$0 \leq x \leq v \leq 1.$$

We can plot this region in the (x, v) space as below.



In the figure, the inner integration takes the horizontal distance for each v up to from $x = 0$ to $x = v$, and the outer integration sums the horizontal distances for all different values of $v \in [0, 1]$. We wish now to integrate with respect to v first, and then x. In other words, we wish to integrate vertically for each x, and then sum the vertical distances for all different values of x Since the region of integration lies to the left of $v = x$; then it must be that $v \geq x$. Of course, upper bound for v is

[2]Suppose $N = 3$, then any one of bidder 1, 2 or 3 could have the highest valuation. Suppose 1 has value v, this is highest when: (i) 2 has higher value than v but 3 has lower value than v; or (ii) 3 has higher value than v but 2 has lower value than v. The probability of the two events is each $F(v)(1-F(v))$. So, the total probability that v is second highest is $2F(v)(1-F(v))$, which corresponds to $(N-1) F^{N-2}(v)(1-F(v))$ when $N = 3$.

one. On the other hand, we would integrate over the whole domain of x. So,

$$\int_0^1 \int_0^v (\cdot)\, dx dv = \int_0^1 \int_x^1 (\cdot)\, dv dx.$$

Observe that, on the right-hand side, we are integrating over

$$1 \geq v \geq x \geq 0,$$

which is the same as before. Using this, we can write

$$\begin{aligned} R_{FPA} &= N(N-1) \int_0^1 \int_x^1 xF^{N-2}(x) f(x) f(v)\, dv dx \\ &= N(N-1) \int_0^1 xF^{N-2}(x) f(x) \left(\int_x^1 f(v)\, dv \right) dx \\ &= N(N-1) \int_0^1 xF^{N-2}(x) f(x) (1 - F(x))\, dx \\ &= \int_0^1 xh(x)\, dx = R_{SPA}. \end{aligned}$$

We can think of this as an application of law of iterated expectations. We can write R_{SPA} as

$$R_{SPA} = \mathbb{E}[v_2].$$

The expected revenue from FPA, denoted R_{FPA}, is given by

$$\begin{aligned} R_{FPA} &= \mathbb{E}\left[\hat{b}(v_i)\right] = \mathbb{E}\left[\mathbb{E}_{v_2}[v_2 | v_1 = v_i]\right] \\ &= \mathbb{E}[v_2] = R_{SPA}. \end{aligned}$$

Remark 1.3. Ex post, the two auctions need not raise the same revenue. For example, when the highest value is quite high and the second-highest is quite low, running a first-price auction will yield more revenue than a second-price auction. However, ex ante, when the seller must decide on the selling mechanism, the result above shows that the expected revenue from the two types of auctions are the same.

Example 1.2. *(Uniform)* Consider again the case in which each bidder's value is uniform on $[0, 1]$

so that $F(v) = v$ and $f(v) = 1$. Then,

$$
\begin{aligned}
R_{FPA} &= N \int_0^1 \hat{b}(v) f(v) F^{N-1}(v)\, dv \\
&= N \int_0^1 \left(v - \frac{v}{N}\right) f(v) F^{N-1}(v)\, dv \\
&= N \int_0^1 \left(v - \frac{v}{N}\right) v^{N-1} dv \\
&= N \int_0^1 v^N \frac{N-1}{N} dv = (N-1) \int_0^1 v^N dv \\
&= (N-1) \left[\frac{1}{N+1} v^{N+1}\right]_0^1 \\
&= \frac{N-1}{N+1},
\end{aligned}
$$

and

$$
\begin{aligned}
R_{SPA} &= \int_0^1 vN(N-1) F^{N-2}(v)(1 - F(v)) f(v)\, dv \\
&= N(N-1) \int_0^1 v^{N-1}(1-v)\, dv \\
&= N(N-1) \left(\int_0^1 v^{N-1} dv - \int_0^1 v^N dv\right) \\
&= N(N-1) \left(\left[\frac{1}{N} v^N\right]_0^1 - \left[\frac{1}{N+1} v^{N+1}\right]_0^1\right) \\
&= N(N-1) \left(\frac{1}{N} - \frac{1}{N+1}\right) = N(N-1) \frac{1}{N(N+1)} \\
&= \frac{N-1}{N+1}.
\end{aligned}
$$

So, as expected, $R_{FPA} = R_{SPA}$.

1.2 A direct selling mechanism

We saw just now that first-price and second-price auctions yield the same expected revenue. What we want to show now is that, in fact, this *revenue equivalence result* is much more general. To do so, we introduce the concept of a direct selling mechanism.

Definition 1.2. *(Direct selling mechanism).* A direct selling mechanism (DSM) consists of N probability assignment functions,

$$\mathbf{p} = \{(p_1(v_1, v_2, \ldots, v_N), p_2(v_1, v_2, \ldots, v_N), \ldots, p_N(v_1, v_2, \ldots, v_N))\}$$

and N cost functions

$$\mathbf{c} = \{(c_1(v_1, v_2, \ldots, v_N), c_2(v_1, v_2, \ldots, v_N), \ldots, c_N(v_1, v_2, \ldots, v_N))\}$$

such that, for all $v_1, v_2, \ldots, v_N \in [0,1]$,

$$\begin{aligned} p_i(v_1, v_2, \ldots, v_N) &\in [0,1], \ \forall i, \\ \sum_{i=1}^{N} p_i(v_1, v_2, \ldots, v_N) &\leq 1, \\ c_i(v_1, v_2, \ldots, v_N) &\in \mathbb{R}, \ \forall i. \end{aligned}$$

Given a DSM, bidders submit their values to the seller, who would plug what he receives into $(\mathbf{p}, \mathbf{c})$, which is known to everyone, and the outcome is determined based on the output. Our ultimate goal is to check whether such a mechanism can induce each bidder to be truthful in reporting their values.

1.2.1 First-price auction

Let us translate the FPA equilibrium to a DSM. Let

$$\begin{aligned} p_i(v_1, v_2, \ldots, v_N) &= \begin{cases} 1 & \text{if } \hat{b}(v_i) > \hat{b}(v_j), \ \forall j \neq i \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } v_i > v_j, \ \forall j \neq i \\ 0 & \text{otherwise} \end{cases}, \end{aligned}$$

where the second line follows since $\hat{b}(\cdot)$ is strictly increasing. We also have

$$c_i(v_1, v_2, \ldots, v_N) = \begin{cases} \hat{b}(v_i) & \text{if } v_i > v_j, \ \forall j \neq i \\ 0 & \text{otherwise} \end{cases}.$$

We want to analyse whether there can be an equilibrium in which bidders report truthfully.

Suppose all but $j \neq i$ always report truthfully and let $u_i(r, v)$ denote bidder i's expected utility from reporting r when his true value is v. Given $(\mathbf{p}, \mathbf{c})$ as implied by above,

$$\begin{aligned} u_i(r, v) &= p_i(r, v_{-i})\, v - c_i(r, v_{-i}) \\ &= p_i(r, v_{-i})\, v - p_i(r, v_{-i})\, \hat{b}(r) \\ &= p_i(r, v_{-i}) \left(v - \hat{b}(r)\right) \\ &= F^{N-1}(r) \left(v - \hat{b}(r)\right), \end{aligned}$$

where we are using the fact that others are reporting truthfully when we write $F^{N-1}(r)$. Recall that, in deriving $\hat{b}$, this is exactly the function that we maximised when evaluating it at $r = v$. So we may conclude that truth telling is an equilibrium.

In effect, a DSM is a way for the seller to run the first-price auction on behalf of the bidders.

1.2.2 Second-price auction

For an SPA, an equivalent DSM is given by

$$p_i(v_1, v_2, \ldots, v_N) = \begin{cases} 1 & \text{if } v_i > v_j,\ \forall j \neq i \\ 0 & \text{otherwise} \end{cases},$$

$$c_i(v_1, v_2, \ldots, v_N) = \begin{cases} v_2 & \text{if } v_i > v_j,\ \forall j \neq i \\ 0 & \text{otherwise} \end{cases},$$

where v_2 denotes the second highest value reported.

1.2.3 Incentive compatibility and the revenue equivalence theorem

Define

$$\bar{p}_i(v_i) := \int_0^1 \cdots \int_0^1 p_i(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_N) \prod_{j \neq i} f_j(v_j)\, dj, \tag{1.2}$$

$$\bar{c}_i(v_i) := \int_0^1 \cdots \int_0^1 c_i(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_N) \prod_{j \neq i} f_j(v_j)\, dj,$$

$$u_i(r_i, v_i) := \bar{p}_i(r_i)\, v_i - \bar{c}_i(r_i).$$

$\bar{p}_i(v_i)$ is the probability of winning assuming that others report their value truthfully. Similarly, $\bar{c}_i(v_i)$ is the expected cost that i has to pay independent of whether he wins or not.

Definition 1.3. *(Incentive compatibility).* A DSM $(\mathbf{p}, \mathbf{c})$ is *incentive compatible (IC) if and only if, for all $i \in \{1, 2, \ldots, N\}$ and all $v_i \in [0, 1]$, $u_i(r_i, v_i)$ is maximised in r_i when $r_i = v_i$.*

Proposition 1.1. *A DSM $(\mathbf{p}, \mathbf{c})$ is incentive compatible if and only if, for all $i \in \{1, 2, \ldots, N\}$,*

(i) $\bar{p}_i(v_i)$ is nondecreasing in v_i;

(ii) $\bar{c}_i(v_i) = \bar{c}_i(0) + \bar{p}_i(v_i)\, v_i - \int_0^{v_i} \bar{p}_i(x)\, dx$, $\forall v_i \in [0, 1]$.

The definition of incentive compatibility does not say that truthful reporting is best for a bidder regardless of the others' reports—it only says that a bidder can do no better than to report truthfully so long as all other bidders report truthfully. Thus, although truthful reporting is a Bayesian Nash equilibrium in an incentive-compatible mechanism, it need not be a dominant strategy for any player.

There are many DSM that are not incentive compatible. For example, consider the example in which $p_i = 1/N$ for all i. Then, the optimal bid is $\hat{b}(v) = 0 \neq v$.

Proof. (Proposition 1.1) (Necessity: IC $\Rightarrow$ (i) and (ii)) That $(\mathbf{p}, \mathbf{c})$ is incentive compatible means that $u_i(r_i, v_i)$ *is maximised in* r_i *when* $r_i = v_i$ *for all* i *and* v_i ; i.e.

$$\begin{aligned}
0 = \left.\frac{\partial u_i(r_i, v_i)}{\partial r_i}\right|_{r_i = v_i} &= \left.\frac{\partial\left[\bar{p}_i(r_i) v_i - \bar{c}_i(r_i)\right]}{\partial r_i}\right|_{r_i = v_i} \\
&= \left.\bar{p}_i'(r_i) v_i - \bar{c}_i'(r_i)\right|_{r_i = v_i} \\
&= \bar{p}_i'(v_i) v_i - \bar{c}_i'(v_i) \\
\Leftrightarrow \bar{c}_i'(v_i) &= \bar{p}_i'(v_i) v_i, \ \forall v_i \in [0,1]. \qquad (1.3)
\end{aligned}$$

Since the equation holds for all v_i, we can integrate both sides with respect to v_i over the interval $[0, v_i]$:

$$\begin{aligned}
\bar{c}_i(v_i) - \bar{c}_i(0) &= \int_0^{v_i} \bar{p}_i'(x)\, x dx \\
\Leftrightarrow \bar{c}_i(v_i) &= \bar{c}_i(0) + \left[\bar{p}_i(x)\, x\right]_0^{v_i} - \int_0^{v_i} \bar{p}_i(x)\, dx \\
&= \bar{c}_i(0) + \bar{p}_i(v_i) v_i - \int_0^{v_i} \bar{p}_i(x)\, dx,
\end{aligned}$$

where we used integration by parts in the second line. This gives us condition (ii).

To see (i), consider

$$\begin{aligned}
\left.\frac{\partial^2 u_i(r_i, v_i)}{\partial r_i^2}\right|_{r_i = v_i} &= \left.\bar{p}_i''(r_i) v_i - \bar{c}_i''(r_i)\right|_{r_i = v_i} \\
&= \bar{p}_i''(v_i) v_i - \bar{c}_i''(v_i) \leq 0, \ \forall v_i \in [0,1], \qquad (1.4)
\end{aligned}$$

where the inequality follows from the second-order condition. Now since (1.3) holds for all v_i, we can differentiate both sides,

$$\begin{aligned}
\bar{c}_i''(v_i) &= \bar{p}_i''(v_i) v_i + \bar{p}_i'(v_i), \ \forall v_i \in [0,1] \\
\Leftrightarrow -\bar{p}_i'(v_i) &= \bar{p}_i''(v_i) v_i - \bar{c}_i''(v_i), \ \forall v_i \in [0,1].
\end{aligned}$$

Substituting this into (1.4) gives

$$\bar{p}_i'(v_i) \geq 0, \ \forall v_i \in [0,1].$$

So it must be that $\bar{p}_i'$ is nonnegative; i.e. $\bar{p}_i(v_i)$ is nondecreasing in v_i.

(Sufficiency: (i) and (ii) imply IC) Consider

$$\frac{\partial u_i(r_i, v_i)}{\partial r_i} = \bar{p}_i'(r_i) v_i - \bar{c}_i'(r_i). \qquad (1.5)$$

Since (ii) holds for all v_i, we can differentiate it with respect to v_i to obtain (and write v_i as r_i)

$$\begin{aligned}
\bar{c}_i'(r_i) &= \bar{p}_i'(r_i) r_i + \bar{p}_i(r_i) - \bar{p}_i(r_i) \\
\Leftrightarrow \bar{c}_i'(r_i) &= \bar{p}_i'(r_i) r_i.
\end{aligned}$$

Substituting this into (1.5),

$$\frac{\partial u_i(r_i, v_i)}{\partial r_i} = \bar{p}_i'(r_i)\, v_i - \bar{p}_i'(r_i)\, r_i = \bar{p}_i'(r_i)(v_i - r_i).$$

From (i), we know that $p_i'(r_i) \geq 0$ for all r_i.

- ▷ Suppose first that $p_i'(r_i) > 0$ for all $r_i \in [0,1]$, then the slope is positive while if $r_i > v_i$, and negative if $r_i < v_i$. Hence, there exists a unique global maximum at $r_i = v_i$, which implies that truth telling is a dominant strategy for player i.
- ▷ If, instead, $p_i'(r_i) = 0$ for all $r_i \in [0,1]$, then the player is indifferent between reporting $r_i = v_i$ and any other value (expected utility is constant with respect to r_i in this case). Hence, truth telling is a weakly dominant strategy.
- ▷ Finally, suppose that $p_i(r_i) > 0$ for some values of $r_i \in [0,1]$ but $p_i(\tilde{r}_i) = 0$ some other values of $\tilde{r}_i \in [0,1]$. In this case, truth telling is a weakly dominant strategy.

Hence, we realise that $r_i = v_i$ is a weakly dominant strategy for player i given (i) and (ii) so that we have our desired result. ■

Remark 1.4. Condition (i) of Theorem 1.1 ensures that the second-order condition of the maximisation of $u_i(r_i, v_i)$ with respect to r_i (evaluated at $r_i = v_i$) is satisfied, while condition (ii) ensures that the first-order condition is satisfied. That $\bar{p}_i(v_i)$ is nondecreasing means that there is incentive for bidder i to report a higher r_i than his true valuation since he can increase his probability of winning by reporting $r_i > v_i$. We can then interpret the integral term in condition (ii) as putting a limit on the bidder's incentive to report higher r_i.

Theorem 1.1. *(Revenue Equivalence Theorem). If two incentive compatible direct selling mechanisms have the same probability assignment functions and that each bidder is indifferent between the two mechanisms when her value is zero, then the two mechanisms raise the same expected revenue.*

Proof. Let R_1 denote the expected revenue from the first DSM, given by

$$R_1 = \sum_{i=1}^{N} \int_0^1 \bar{c}_i(v_i)\, f_i(v_i)\, dv_i.$$

By Theorem 1.1, since the DSM is IC, we can write $\bar{c}_i(v_i)$ as

$$R_1 = \sum_{i=1}^{N} \int_0^1 \left[\bar{c}_i(0) + \bar{p}_i(v_i)\, v_i - \int_0^{v_i} \bar{p}_i(x)\, dx\right] f_i(v_i)\, dv_i.$$

For the second DSM, we are told that the probability assignment functions are the same so that $\bar{p}_i(v_i)$ are the same between the two mechanisms. Now, the utility when the bidder has zero value is given by

$$u_i(0,0) = -\bar{c}_i(0).$$

Hence, if these are the same between the two DSM, then it follows that

$$R_1 = R_2.$$ ■

1.3 Designing a revenue maximising mechanism

1.3.1 Revelation principle

Theorem 1.2. *(Revelation Principle). If some extensive-form game has a Bayesian Nash equilibrium in which the seller's expected revenue is R_0, then there exists an incentive compatible direct selling mechanism (IC DSM) in which the seller's expected revenue is R_0 in the truth-telling Bayesian Nash equilibrium.*

Corollary 1.1. *If R^* is the maximum expected revenue that can be achieved among the class of all IC DSMs, then there is no Bayesian Nash equilibrium of any extensive-form game that can yield a higher expected revenue than R^*.*

Proof. *(Revelation Principle)* (Sketch) Let $\{\beta_1^*(v_1), \beta_2^*(v_2), \ldots, \beta_N^*(v_N)\}$ be a Bayesian Nash equilibrium of the extensive-form game, where $\beta_i(v_i) \in S_i$ for all $i \in \{1, 2, \ldots, N\}$. Then, we generate an IC DSM as follows:

- ▷ bidders report $\{v_1, v_2, \ldots, v_N\}$ to the seller;
- ▷ the sellers "plays" the strategies $\{\beta_i^*\}_{i=1}^N$ on behalf of the bidders. ■

The corollary implies that, to find a revenue-maximising (selling) mechanism, we can restrict our attention to the set of IC DSMs.

1.3.2 Individual rationality

We assume that participation by the bidders is voluntary. This means that no bidder's expected payoff can be negative—if it were, then the bidder will simply choose not to participate in the selling mechanism. Thus, we restrict attention to IC DSMs that are individually rational; i.e. that yield each bidder, regardless of his value, a non-negative expected payoff in the truth-telling equilibrium.

Recall that, in an IC DSM, bidder i with value v_i receives expected payoffs of $u_i(v_i, v_i)$ in the truth-telling equilibrium.

Definition 1.4. *(Individual rationality).* An IC DSM $(\mathbf{p}, \mathbf{c})$ satisfies *individually rationally if and only if*

$$u_i(v_i, v_i) = \bar{p}_i(v_i)\, v_i - \bar{c}_i(v_i) \geq 0,\ \forall v_i \in [0,1],\ \forall i = 1, 2, \ldots, N.$$

Proposition 1.2. *An IC DSM is individual rational if and only if*

$$\bar{c}_i(0) \leq 0,\ \forall i.$$

Proof. Recall that an IC DSM satisfies condition (ii) of Proposition 1.1:

$$\bar{c}_i(v_i) = \bar{c}_i(0) + \bar{p}_i(v_i)\, v_i - \int_0^{v_i} \bar{p}_i(x)\, dx,\ \forall v_i \in [0,1].$$

Substituting for $\bar{c}_i(v_i)$ into the individual rationality constraint yields that, for all $v_i \in [0,1]$,

$$\begin{aligned} \bar{p}_i(v_i)\,v_i &\geq \bar{c}_i(0) + \bar{p}_i(v_i)\,v_i - \int_0^{v_i} \bar{p}_i(x)\,dx \\ \Leftrightarrow \int_0^{v_i} \bar{p}_i(x)\,dx &\geq \bar{c}_i(0). \end{aligned}$$

The left-hand side is smallest when $v_i = 0$, in which case, we require

$$\bar{c}_i(0) \leq 0.$$

Reversing the steps gives us the other direction. ■

1.3.3 An optimal selling mechanism

The seller receives the expected values of $\bar{c}_i(v_i)$ from each bidders. Then, the optimal IC DSM solves

$$\begin{aligned} \max_{(\mathbf{p},\mathbf{c})} \; & R := \sum_{i=1}^{N} \int_0^1 \bar{c}_i(v_i)\,f_i(v_i)\,dv_i \\ s.t. \quad & (\mathbf{p},\mathbf{c}) \text{ is incentive compatible and individually rational.} \end{aligned}$$

Using condition (ii) of Proposition 1.1, we can rewrite the objective function as

$$\begin{aligned} &\sum_{i=1}^{N} \int_0^1 \left(\bar{c}_i(0) + \bar{p}_i(v_i)\,v_i - \int_0^{v_i} \bar{p}_i(x)\,dx \right) f_i(v_i)\,dv_i \\ =& \sum_{i=1}^{N} \int_0^1 \left(\bar{p}_i(v_i)\,v_i - \int_0^{v_i} \bar{p}_i(x)\,dx \right) f_i(v_i)\,dv_i + \sum_{i=1}^{N} \bar{c}_i(0). \end{aligned}$$

Using Propositions 1.1 and 1.2, we can state the problem equivalently as

$$\begin{aligned} \max_{(\mathbf{p},\mathbf{c})} \quad & R = \sum_{i=1}^{N} \int_0^1 \left(\bar{p}_i(v_i)\,v_i - \int_0^{v_i} \bar{p}_i(x)\,dx \right) f_i(v_i)\,dv_i + \sum_{i=1}^{N} \bar{c}_i(0) \\ s.t. \quad & \bar{p}_i(v_i) \text{ is nondecreasing in } v_i; \\ & \bar{c}_i(v_i) = \bar{c}_i(0) + \bar{p}_i(v_i)\,v_i - \int_0^{v_i} \bar{p}_i(x)\,dx,\ \forall v_i \in [0,1], \\ & \bar{c}_i(0) \leq 0. \end{aligned}$$

We can immediately see that the objective function is strictly increasing in $\bar{c}_i$. Hence, given the individual rationality constraint, the optimal value of $\bar{c}_i(0)$ is given by

$$\bar{c}_i^*(0) = 0,\ \forall v_i \in [0,1]. \tag{1.6}$$

Our strategy is to find an upper bound for the maximised revenue by considering an unconstrained problem and then show that the upper bound is, in fact, feasible.

Let us first work with the objective function:

$$R=\sum_{i=1}^{N}\left(\int_0^1 \bar{p}_i\left(v_i\right) v_i f_i\left(v_i\right) dv_i-\int_0^1 \int_0^{v_i} \bar{p}_i\left(x\right) dx f_i\left(v_i\right) dv_i\right)+\sum_{i=1}^{N} \bar{c}_i\left(0\right).$$

Our first trick is to exchange the order of integrals to obtain

$$\begin{aligned}
R&=\sum_{i=1}^{N}\left(\int_0^1 \bar{p}_i\left(v_i\right) v_i f_i\left(v_i\right) dv_i-\int_0^1 \int_x^{1} f_i\left(v_i\right) dv_i \bar{p}_i\left(x\right) dx\right)+\sum_{i=1}^{N} \bar{c}_i\left(0\right)\\
&=\sum_{i=1}^{N}\left(\int_0^1 \bar{p}_i\left(v_i\right) v_i f_i\left(v_i\right) dv_i-\int_0^1 \left(1-F_i\left(x\right)\right)\bar{p}_i\left(x\right) dx\right)+\sum_{i=1}^{N} \bar{c}_i\left(0\right)\\
&=\sum_{i=1}^{N}\left(\int_0^1 \bar{p}_i\left(v_i\right) v_i f_i\left(v_i\right) dv_i-\int_0^1 \left(1-F_i\left(v_i\right)\right)\bar{p}_i\left(v_i\right) dv_i\right)+\sum_{i=1}^{N} \bar{c}_i\left(0\right)\\
&=\left(\sum_{i=1}^{N}\int_0^1 \bar{p}_i\left(v_i\right) f_i\left(v_i\right)\left[v_i-\frac{1-F_i\left(v_i\right)}{f_i\left(v_i\right)}\right] dv_i\right)+\sum_{i=1}^{N} \bar{c}_i\left(0\right).
\end{aligned}$$

Recall the definition of $\bar{p}_i$ from (1.2):

$$R=\sum_{i=1}^{N}\left(\int_0^1\cdots\int_0^1 p_i\left(v_1,v_2,\ldots,v_N\right)\left[v_i-\frac{1-F_i\left(v_i\right)}{f_i\left(v_i\right)}\right]\prod_{j=1}^{N} f_j\left(v_j\right) dv_j\right)+\sum_{i=1}^{N} \bar{c}_i\left(0\right)$$

Moving the summation inside gives

$$R=\int_0^1\cdots\int_0^1\left(\sum_{i=1}^{N} p_i\left(v_1,v_2,\ldots,v_N\right)\left[v_i-\frac{1-F_i\left(v_i\right)}{f_i\left(v_i\right)}\right]\right)\prod_{j=1}^{N} f_j\left(v_j\right) dv_j+\sum_{i=1}^{N} \bar{c}_i\left(0\right)$$

So the problem is now to maximise above subject to the three constraints. Note that $p_i\left(\cdot\right)\in\left(0,1\right)$ and $\sum_{i=1}^{N} p_i\left(\cdot\right)\leq 1$. Hence, we can write the terms inside the brackets as

$$\sum_{i=1}^{N} p_i\left(\cdot\right)\left(v_i-\frac{1-F_i\left(v_i\right)}{f_i\left(v_i\right)}\right)+\left(1-\sum_{i=1}^{N} p_i\left(\cdot\right)\right)0$$

and interpret it as a weighted average of $N+1$ numbers:

$$v_1-\frac{1-F_1\left(v_1\right)}{f_1\left(v_1\right)},\ v_2-\frac{1-F_2\left(v_2\right)}{f_2\left(v_2\right)},\ldots,\ v_N-\frac{1-F_N\left(v_N\right)}{f_N\left(v_N\right)},0.$$

The sum of these cannot be larger than the largest of these brackets terms if at least one of them is positive and no larger than zero if all of them are negative. Suppose, for now, that these terms are distinct (we will give an exact condition later). Then, we can define

$$p_i^*\left(v_1,v_2,\ldots,v_N\right)=\begin{cases}1 & \text{if } v_i-\frac{1-F_i\left(v_i\right)}{f_i\left(v_i\right)}>\max\left\{0,v_j-\frac{1-F_j\left(v_j\right)}{f_j\left(v_j\right)}\right\},\ \forall j\neq i\\ 0 & \text{otherwise.}\end{cases}. \tag{1.7}$$

That is p_i^* places all the weights on the largest term. Hence, it follows that

$$\sum_{i=1}^{N} p_i\left(v_1, v_2, \ldots, v_N\right)\left[v_i - \frac{1 - F_i\left(v_i\right)}{f_i\left(v_i\right)}\right] \leq \sum_{i=1}^{N} p_i^*\left(v_1, v_2, \ldots, v_N\right)\left[v_i - \frac{1 - F_i\left(v_i\right)}{f_i\left(v_i\right)}\right].$$

Therefore, if the bracketed terms are distinct with probability one,[3] we would have

$$R \leq \int_0^1 \cdots \int_0^1 \left(\sum_{i=1}^{N} p_i^*\left(v_1, v_2, \ldots, v_N\right)\left[v_i - \frac{1 - F_i\left(v_i\right)}{f_i\left(v_i\right)}\right]\right) \prod_{j=1}^{N} f_j\left(v_j\right) dv_j \tag{1.8}$$

for all IC DSMs $(\mathbf{p}, \mathbf{c})$.

We will now construct an IC DSM that achieves this upper bound. Consequently, this mechanism will maximise the seller's revenue, and so will be optimal for the seller.

We let the probability assignment functions in this IC DSM be as (1.7) (for all i). The cost function $\bar{c}_i$'s are then pinned down by the second constraint:

$$\bar{c}_i^*\left(v_i\right) = \bar{c}_i^*\left(0\right) + \bar{p}_i^*\left(v_i\right) v_i - \int_0^{v_i} \bar{p}_i^*\left(x\right) dx,\ \forall v_i \in [0, 1].$$

Since $\bar{c}_i^*$ and $\bar{p}_i^*$ are averages of c_i^* and p_i^*'s, the equation above holds if it holds for each and every vector of values $v_1, v_2, \ldots, v_N$. That is, the second constraints holds if we define

$$c_i^*\left(v_1, \ldots, v_N\right) = c_i^*\left(0, v_{-i}\right) + p_i^*\left(v_1, v_2, \ldots, v_N\right) v_i - \int_0^{v_i} p_i^*\left(x, v_{-i}\right) dx$$

for every possible $\{v_1, v_2, \ldots, v_N\} \in [0, 1]^N$.

It remains to show that the probability assignment function satisfies the first constraint; i.e. $\bar{p}_i^*\left(v_i\right)$ is non-decreasing in v_i. To ensure that this is satisfied, we assume that, for every $i = 1, 2, \ldots, N$, the "virtual valuation"

$$v_i - \frac{1 - F\left(v_i\right)}{f\left(v_i\right)}$$

is strictly increasing in v_i. With this assumption, consider some bidder i and some fixed vector of values, v_{-i}, for the other bidders. Now, suppose that $\bar{v}_i > \underline{v}_i$ and that $p_i^*\left(\underline{v}_i, v_{-i}\right) = 1$. Then, by the definition of p_i^*, it must be the case that

$$\underline{v}_i - \frac{1 - F\left(\underline{v}_i\right)}{f_i\left(\underline{v}_i\right)} > v_j - \frac{1 - F_j\left(v_j\right)}{f_j\left(v_j\right)} > 0,\ \forall j \neq i.$$

Consequently, because the virtual valuation is strictly increasing, it must also be the case that

$$\bar{v}_i - \frac{1 - F\left(\bar{v}_{ii}\right)}{f_i\left(\bar{v}_i\right)} \underline{v}_i > \frac{1 - F\left(\underline{v}_i\right)}{f_i\left(\underline{v}_i\right)} > v_j - \frac{1 - F_j\left(v_j\right)}{f_j\left(v_j\right)} > 0,\ \forall j \neq i,$$

which implies that $p_i^*\left(\bar{v}_i\right) = p^*\left(\underline{v}_i\right) = 1$. Hence, we have shown that $p_i^*\left(v_i, v_{-i}\right)$ is non-decreasing in v_i for every v_{-i} since p_i^* only takes values one or zero. It follows then that $\bar{p}_i^*\left(v_i\right)$ is non-decreasing in v_i so that constraint (i) is satisfied.

Finally, we show that p_i^* and c_i^* that we found achieve the upper bound. Substituting these expressions into R gives the upper bound we found in (1.8).

[3] When this is not true, then p_i^* is not optimal. However, it is still possible to construct an optimal mechanism.

We summarise above with the following theorem.

Theorem 1.3. *(Optimal selling mechanism). Suppose N bidders have independent private values with bidder i's value drawn from a continuous strictly positive density f such that*

$$v_i - \frac{1 - F(v_i)}{f_i(v)}$$

is strictly increasing in v_i. Then, the direct selling mechanism defined by

$$p_i^*(v_1, v_2, \ldots, v_N) = \begin{cases} 1 & \text{if } v_i - \frac{1-F_i(v_i)}{f_i(v_i)} > \max\left\{0, v_j - \frac{1-F_j(v_j)}{f_j(v_j)}\right\},\ \forall j \neq i \\ 0 & \text{otherwise} \end{cases},$$

$$c_i^*(v_1, \ldots, v_N) = c_i^*(0, v_{-i}) + p_i^*(v_1, v_2, \ldots, v_N)\, v_i - \int_0^{v_i} p_i^*(x, v_{-i})\, dx, \tag{1.9}$$

$$c_i^*(0, v_{-i}) = 0$$

yields the seller the largest possible expected revenue.

Remark 1.5. A sufficient condition for the $v_i - (1 - F(v_i))/f_i(v)$ to be strictly increasing is for $F(v_i)$ to be strictly convex since

$$\begin{aligned}\frac{\partial}{\partial v_i}\left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}\right] &= 1 - \frac{-f_i(v_i) - (1 - F_i(v_i))\, f_i'(v_i)}{(f_i(v_i))^2} \\ &= 1 + \frac{f_i(v_i) + (1 - F_i(v_i))\, f_i'(v_i)}{(f_i(v_i))^2}.\end{aligned}$$

So that $f_i'(v_i) \geq 0$ guarantees that the virtual valuation is strictly convex. An example of a common distribution that is not convex but still satisfies this condition is the exponential distribution.

$$F_i(v_i) = -\exp[-\lambda v_i],\ v_i \geq 0.$$

Then,

$$\begin{aligned}f_i(v_i) &= \lambda \exp[-\lambda v_i], \\ f_i'(v_i) &= -\lambda^2 \exp[-\lambda v_i] < 0.\end{aligned}$$

Since f_i' is negative, F is not convex. However,

$$v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} = v_i - \frac{\exp[-\lambda v_i]}{\lambda \exp[-\lambda v_i]} = v_i - \frac{1}{\lambda}$$

so that the virtual valuation is strictly increasing in v_i.

Allocation of goods under the optimal selling mechanism Recall the expression for virtual valuation:

$$v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

This, in fact, represents the marginal revenue, $MR_i(v_i)$, that the seller obtains from increasing the probability that the objective is assigned to bidder i when his value is v_i.

To see this, recall the IC constraint:

$$\bar{c}_i(v_i) = \bar{c}_i(0) + \bar{p}_i(v_i)\, v_i - \int_0^{v_i} \bar{p}_i(x)\, dx, \ \forall v_i \in [0,1]$$

as well as the cost function in the optimal selling mechanism:

$$c_i^*(v_1, \ldots, v_N) = c_i^*(0, v_{-i}) + \underbrace{p_i^*(v_1, v_2, \ldots, v_N)\, v_i}_{A} - \underbrace{\int_0^{v_i} p_i^*(x, v_{-i})\, dx}_{B}.$$

Now consider a marginal increase in p_i^*.

- ▷ From term A, we see that this allows the seller to increase the cost function for bidder i, c_i^*, at rate v_i. In turn, the expected revenue increases by $v_i f_i(v_i)$.
- ▷ But an increase in p_i^* also has another effect through the term B. In particular, a higher p_i^* affects bidder i's cost function at all values v_i' greater than v_i since B involves integrating p_i^* from zero to v_i' (and so would include $p_i^*(v_i, v_{-i})$). The B term tells us that, increasing p_i^* reduces $c_i^*(v_i', v_{-i})$ for all $v_i' > v_i$. Since mass $1 - F_i(v_i)$ represents the mass of $v_i' > v_i$, the seller's expected revenue is reduced by $1 - F_i(v_i)$.

The net increase in the seller's revenue is given by $v_i f_i(v_i) - (1 - F_i(v_i)) / f_i(v_i)$. But this is the total effect due to the density $f_i(v_i)$ of values equal to v_i. The marginal revenue associated with each v_i is

$$MR_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} = \text{virtual valuation.}$$

We can now interpret the allocation rule as

$$p_i^*(v_1, v_2, \ldots, v_N) = \begin{cases} 1 & \text{if } MR_i(v_i) > \max\{0, MR_j(v_j)\}_{j \neq i} \\ 0 & \text{otherwise} \end{cases}.$$

Thus, the optimal selling mechanism involves the seller assigning the object to the bidder with the highest marginal revenue with probability one (provided that the highest marginal revenue is greater than zero). If this were not the case, then the seller can increase revenue by reducing the probability that the object is assigned to bidder j and increasing the probability that it is assigned to bidder i.

Now, if all the marginal revenues are negative, then the seller does best by reducing all of the probabilities to zero and keeping the object.

Payments under the optimal selling mechanism Suppose that when values are reported truthfully, bidder i does not receive the object; i.e. $p_i^*(v_i, v_{-i}) = 0$. The cost function for i is then

$$c_i^*(v_i, v_{-i})\big|_{p_i^*(v_i, v_{-i}) = 0} = 0 v_i - \int_0^{v_i} p_i^*(x, v_{-i})\, dx = -\int_0^{v_i} p_i^*(x, v_{-i})\, dx.$$

What is the value of the integral? Recall that $p_i^*(x, v_{-i})$ is nondecreasing in x. So if

$$p_i^*(v_i, v_{-i}) = 0 \Rightarrow p_i^*(x, v_{-i}) = 0, \ \forall x \in [0, v_i].$$

Thus, when $p_i^*(v_i, v_{-i})$, bidder i pays

$$c_i^*(v_i, v_{-i})\big|_{p_i^*(v_i,v_{-i})=0} = 0.$$

Suppose instead that bidder i receives the object; i.e. $p_i^*(v_i, v_{-i}) = 1$. Then,

$$c_i^*(v_i, v_{-i})\big|_{p_i^*(v_i,v_{-i})=1} = v_i - \int_0^{v_i} p_i^*(x, v_{-i})\, dx.$$

Since $p_i^*(x, v_{-i})$ is nondecreasing in x, there exists some r_i^* such that

$$p_i^*(x, v_{-i}) = \begin{cases} 1 & \text{if } x \geq r_i^* \\ 0 & \text{if } x < r_i^* \end{cases}.$$

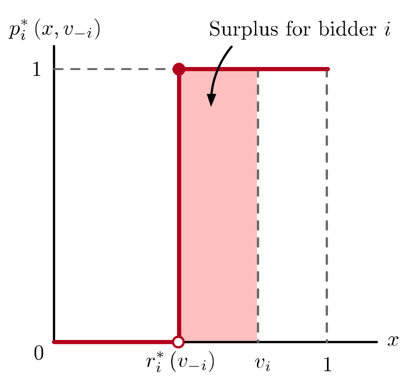
This also means that

$$r_i^* = \inf\left\{r_i : MR_i(r_i) > \max\{0, MR_j(v_j)\}_{j\neq i}\right\}$$

and so r_i^* is, in fact, independent of v_i. So we may write $r_i^*(v_{-i})$. We can now write

$$\begin{aligned} c_i^*(v_i, v_{-i})\big|_{p_i^*(v_i,v_{-i})=1} &= v_i - \left(\int_0^{r_i^*(v_{-i})} 0dx + \int_{r_i^*(v_{-i})}^{v_i} 1dx\right) \\ &= v_i - (v_i - r_i^*(v_{-i})) \\ &= r_i^*(v_{-i}). \end{aligned}$$

This tells us that the winning bidder i pays a price equal to $r_i^*(v_{-i})$ that is independent of his own reported value, and the price he pays is the maximum value he could have reported given the others' reported values and yet receiving the object. Note also that $v_i - r_i^*(v_{-i})$ represents the surplus earned by the winning bidder i as can be seen from the figure below.



We can now rephrase Theorem 1.3.

Theorem 1.4. *(The optimal selling mechanism, simplified). If N bidders have independent price values with bidder i's value drawn from the continuous positive density f_i and each $v_i - (1 - F_i(v_i)) / f_i(v_i)$ is strictly increasing, then the following direct selling mechanism yields the seller the largest possible expected revenue.*
For each reported vector of values, $\{v_1, v_2, \ldots, v_N\}$, the seller assigns the object to the bidder i whose virtual valuation, $v_i - (1 - F_i(v_i)) / f_i(v_i)$, is strictly largest and positive. If there is no such bidder, the seller keeps the object and no payments are made. If there is such a bidder, say i, then only this bidder makes a payment to the seller in the amount r_i^*, where

$$r_i^* = \max \left\{ \left\{ r_i : r_i - \frac{1 - F_i(r_i)}{f_i(r_i)} = 0 \right\}, \left\{ r_i : r_i - \frac{1 - F_i(r_i)}{f_i(r_i)} = \max \left\{ v_j - \frac{1 - F_j(v_j)}{f_j(v_j)} \right\}_{i \neq j} \right\} \right\}$$

This mechanism is incentive compatible so that truth telling is a Nash equilibrium (i.e. given that other report truthfully, it is optimal for the bidder to also report ruthfully). However, we can, in fact show that truth telling is a (weakly) dominant strategy; i.e. even if others do not report their values truthfully, bidder i can do no better than to report his value truthfully to the seller.

To see this, consider first possible deviations by the winning bidder. If he bids above his true value, then he still wins the object, and pays the same amount. Thus, this strategy does not strictly dominate the truth-telling strategy. The same logic applies if he bids below his true value and he still wins the objective. If he bids so low such hat he does not win the object, then he loses his surplus which is the difference between $v_i - r_i^* > 0$. Thus, we conclude that no strategy strictly dominates the truth-telling strategy for the winning bidder.

Now consider possible deviations by any losing bidder, say i. Let v_j^* denote the winner's bid. The only possibility to benefit is if i bids sufficiently high so that he wins the object such that

$$b_i - \frac{1 - F_i(b_i)}{f_i(b_i)} > v_j^* - \frac{1 - F_j(v^*)}{f_j(v^*)}.$$

But then i has to pay r_i^* as defined above. If we assume that i bids above his own value, then, r_i^* solves

$$r_i^* - \frac{1 - F_i(r_i^*)}{f_i(r_i^*)} = v_j^* - \frac{1 - F_j(v^*)}{f_j(v^*)}.$$

But since bidder i could not win if he had bid his own value, it follows that

$$r_i^* - \frac{1 - F_i(r_i^*)}{f_i(r_i^*)} > v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

Finally, since the virtual utilities are strictly increasing, above implies that

$$r_i^* > v_i \Leftrightarrow v_i - r_i^* < 0.$$

Thus, bidder i has no incentive to bid dishonestly.

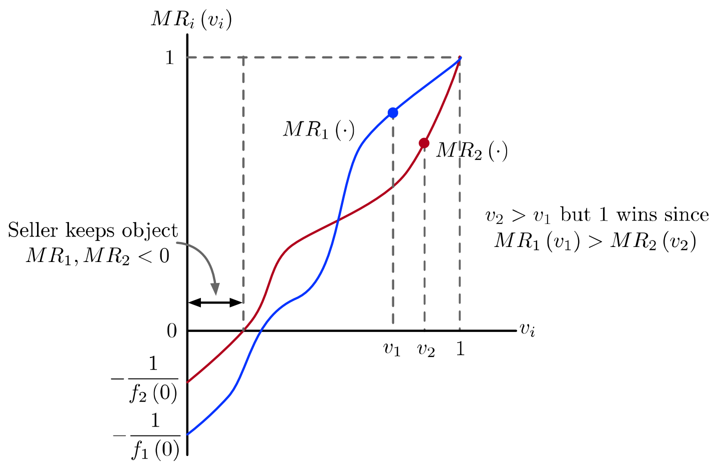
1.3.4 Efficiency

Let us think about allocative (in)efficiency in the context of a selling mechanism.

▷ We assume that the seller assigns zero value to the object, whereas bidders assign some positive value to the object. Hence, the seller should always sell the object to one of the bidders.

▷ In fact, the object should be assigned to the bidder with the highest valuation.

We already saw that the seller keeps the object if $MR_i(v_i)$ for each bidder is negative. To see how the second might arise, consider the following figure.



Note that: (i) when $v_i = 0$, $MR_i(0) = -1/f_i(0)$ since $F_i(0) = 0$; (ii) when $v_i = 1$, then $MR_i(1) = 1$; and (iii) MR_i is strictly increasing between these two points. The figure shows how there can be certain realisations of v_i's such that the seller will keep the object, as well as an example of realisation in which the bidder with the highest value does not get the object. We can view the fact that sometimes the seller keeps the object as the seller exercising his monopoly power over the good.

Now suppose that bidders are symmetric so that

$$f \equiv f_1 \equiv f_2, \; F \equiv F_1 \equiv F_2.$$

In this case, the two curves in the figure are overlapping so that the second type of inefficiency cannot arise. The optimal selling mechanism is now that the bidder with the highest reported value wins and pays the seller r_i^*, the largest value that he could have reported, given the other bidder's reported values, without winning the object. However, the first source of inefficiency still remains so that if there are no bidders with positive reported value of MR_i, then the seller keeps the object. This means that, in fact, the bidder with the highest value, say bidder i with value v_i, wins as so long as $v_i > \rho^* \in [0, 1]$, where ρ^* is implicitly defined as

$$\rho^* - \frac{1 - F(\rho^*)}{f(\rho^*)} = 0.$$

Bidder i does not get the object unless the reported value is strictly highest and strictly above ρ^*. So, the largest his report can be without receiving the object (i.e. the amount he has to pay) is the largest of the other bidders' values or ρ^*, whichever is larger. Thus, it is as if we've added another bidder whose value is ρ^* to the set up.

We can mimic this optimal direct selling mechanism by running a second-price auction with reserve price ρ^*.

Theorem 1.5. *(Optimal auction under symmetry). If N bidders have independent private values, each drawn from the same continuous positive density f, where $v - (1 - F(v))/f(v)$ is strictly increasing, then a second-price auction with reserve price ρ^* satisfying $\rho^* - (1 - F(\rho^*))/f(\rho^*) = 0$, maximises the seller's expected revenue.*

Example 1.3. *(JR3: Exercise 9.11).* Suppose values are distributed uniformly on $[0,1]$, then

$$F(v) = v,\ f(v) = 1.$$

So,

$$\mathcal{U}(v) := v - \frac{1 - F(v)}{f(v)} = v - \frac{1 - v}{1} = 2v - 1.$$

This is clearly strictly increasing in $v \in [0,1]$. Define

$$v^* = \frac{1}{2}$$

so that virtual valuation equals zero when $v = v^*$. A revenue maximising auction is therefore a second-price auction with reserve price $v^* = 1/2$.

Proposition 1.3. *(JR3: Exercise 9.21). Suppose that bidders are symmetric. Then, the first-price, Dutch and English auctions are all optimal for the seller once an appropriate reserve price is chosen. Moreover, the optimal reserve value is the same for all four of the standard auctions.*

Proof. By Theorem 1.5, we know that a second-price auction with reserve price $\rho^* - (1 - F(\rho^*))/f(\rho^*) = 0$ maximises the seller's expected revenue. We want to appeal to the Revenue Equivalence Theorem to argue that the first-price auction with the same reservation price is also revenue maximising for the seller.

With first-price and second-price auctions, bidders with zero valuations are indifferent between the two mechanisms since they will both bid zero and they have zero chances of winning. So it remains to show that the probability assignment functions in a second-price auction with reserve price ρ^* is equivalent to the probability assignment functions in a first-price auction with the same reserve price.

The probability assignment functions in a second-price auction with reserve price ρ^* is

$$p_i^{SA}(v) = \begin{cases} 1 & \text{if } v_i > \max\left\{\rho^*, \max\left\{v_j\right\}_{j \neq i}\right\} \\ 0 & \text{otherwise} \end{cases}.$$

Since bidders bid their value, reserve price and reserve value are the same.

Now, consider a first-price auction with reserve value ρ^*, i.e. with reserve price $\hat{b}(\rho^*)$. Then, the probability assignment functions in a first-price auction with reserve price π is

$$p_i^{FA}(v) = \begin{cases} 1 & \text{if } \hat{b}(v_i) > \max\left\{\hat{b}(\rho^*), \max\left\{\hat{b}(v_j)\right\}_{j\neq i}\right\} \\ 0 & \text{otherwise} \end{cases}.$$

Since $\hat{b}$ is strictly increasing, its inverse is well-defined and we can write

$$p_i^{FA}(v) = \begin{cases} 1 & \text{if } v_i > \max\left\{\rho^*, \max\left\{v_j\right\}_{j\neq i}\right\} \\ 0 & \text{otherwise} \end{cases}.$$

Hence, $p_i^{FA}(v) \equiv p_i^{SA}(v)$. Then, by the Revenue Equivalence Theorem, we conclude that the two auctions raise the same expected revenue for the seller.

Now consider a Dutch (i.e. descending) auction with reservation price $\hat{b}(\rho^*)$. Since the bidders must decide ex ante when to put their hand "up" as the price comes down, the decision that the bidder faces is exactly the same as in the first-price auction with reserve price $\hat{b}(\rho^*)$.

Consider an English (i.e. ascending) auction with reservation price/value ρ^*. It is a weakly dominant strategy for the bidder to drop out when the price increases to be above the bidder's valuation. The winner (to the extent that there is a winner) pays the maximum of the reserve price and the second highest value so this auction is equivalent to a second price auction with reserve price/value ρ^*. ■

1.4 All-pay auctions

In all-pay auctions, bidders are required to pay the seller independent of whether they win the object.

1.4.1 First-price all-pay auction (JR3: Exercise 9.8)

Consider first a *first-price all-pay* auction in which the highest bidder wins the object and all bidders (even the losers) must pay the auctioneer the amount of their bid. Let us consider the independent private values model with symmetric bidders whose values are each distributed according to the distribution function F with density f.

Equilibrium bidding function The derivation is isomorphic to the first-price auction.

Denote $\hat{b} : [0,1] \to [0,1]$ as an increasing function that represents the symmetric equilibrium strategy. The expected utility from bidding $r \in [0,1]$ given value v is now given by

$$u(r,v) = F^{N-1}(r)\, v - \hat{b}(r),$$

where $\hat{b}$ is no longer multiplied by the probability of winning to reflect the fact that bids must be paid no matter the outcome.

$$\begin{aligned}\left.\frac{\partial u(r,v)}{\partial r}\right|_{r=v} &= \left.(N-1)\,f(r)\,F^{N-2}(r)\,v-\hat{b}'(r)\right|_{r=v}\\ &= (N-1)\,f(v)\,F^{N-2}(r)\,v-\hat{b}'(v)\\ &= \frac{dF^{N-1}(v)}{dv}v-\hat{b}'(v)\,.\end{aligned}$$

Equating this derivative with zero, we obtain

$$\begin{aligned}\hat{b}'(v) &= \frac{dF^{N-1}(x)}{dv}v\\ \Rightarrow \int_0^v \hat{b}'(x)\,dx &= \int_0^v x\frac{dF^{N-1}(x)}{dx}dx = \int_0^v x dF^{N-1}(x)\\ \Rightarrow \hat{b}(v) &= \int_0^v x dF^{N-1}(x)\,,\end{aligned}$$

where we used $\hat{b}(0)=0$.

Clearly, $\hat{b}$ is strictly increasing given that $f(v)>0$. Moreover,

$$\begin{aligned}\frac{\partial u(r,v)}{\partial r} &= (N-1)\,f(r)\,F^{N-2}(r)\,v-\hat{b}'(r)\\ &= (N-1)\,f(r)\,F^{N-2}(r)\,v-(N-1)\,f(r)\,F^{N-2}(r)\,r\\ &= (N-1)\,f(r)\,F^{N-2}(r)\,(v-r)\,.\end{aligned}$$

By the same argument as before, we realise that $u(r,v)$ achieves a maximum at $r=v$.

Comparison of bidding function under first-price and first-price all-pay auctions In the standard first-price auction, the biding function was

$$\begin{aligned}\hat{b}_{FPA}(v) &= \frac{1}{F^{N-1}(v)}\int_0^v \frac{dF^{N-1}(x)}{dx}xdx\\ &= \int_0^v xd\frac{F^{N-1}(x)}{F^{N-1}(v)}.\end{aligned}$$

Hence,

$$\hat{b}_{FPA}(v) \equiv \frac{1}{F^{N-1}(v)}\hat{b}(v)\,.$$

Since $(F(v))^{N-1}\in(0,1)$, we realise that the bidders bid higher in the standard first-price auction; i.e.

$$\hat{b}_{FPA} \geq \hat{b}(v)\,.$$

In a first-price auction, a bidder has an incentive to raise his bid to increase his chances of winning the auction, yet he has an incentive to reduce his bid to lower the price he pays when he does win. In an all-pay first-price auction, the incentive to reduce the bid is even greater since the bidder must pay it no matter the outcome of the auction. Hence, bidders bid lower in the all-pay case, relative to the standard case.

Seller's expected revenue In a first-price all-pay auction, the probability assignment function is the same as in the standard first-price auction; i.e. whoever bids highest wins. Moreover, bidders whose value is zero would bid zero so $\bar{c}_i(0) = 0$—the same as in the standard first-price auction. Thus, by the Revenue Equivalence Theorem, the seller's expected revenue should be the same between the two auctions. Let us verify this explicitly.

Each bidder bids according to $\hat{b}(v_i)$ and the seller receives money from every bidder. Thus, the seller's expected revenue from the first-price all-pay auction (FPAPA) is

$$R_{FPAPA} = N\mathbb{E}\left[\hat{b}(v_i)\right].$$

Given symmetry,

$$R_{FPAPA} = N\int_0^1 \left(\int_0^v x dF^{N-1}(x)\right) f(v)\,dv.$$

Since

$$\begin{aligned} dF^{N-1}(x) &= \frac{dF^{N-1}(x)}{dx}dx \\ &= (N-1)\,f(x)\,F^{N-2}(x)\,dx, \end{aligned}$$

we can write

$$\begin{aligned} R_{FPAPA} &= N\int_0^1 \left(\int_0^v x(N-1)\,f(x)\,F^{N-2}(x)\,dx\right) f(v)\,dv \\ &= N(N-1)\int_0^1\int_0^v xF^{N-2}(x)\,f(x)\,f(v)\,dxdv. \end{aligned}$$

We exchange the integral, taking care about the bounds:

$$\begin{aligned} R_{FPAPA} &= N(N-1)\int_0^1 xF^{N-2}(x)\,f(x)\left(\int_x^1 f(v)\,dv\right)dx \\ &= N(N-1)\int_0^1 xF^{N-2}(x)\,f(x)\,(1-F(x))\,dx \\ &= R_{FPA}. \end{aligned}$$

1.4.2 Second-price all-pay auction (JR3: Exercise 9.9)

Now consider a second-price, all-pay auction: Each bidder submits a sealed bid. The highest bidder wins and *both* bidders pay the second highest bid. Assume that there are only two bidders.

Equilibrium bidding function Let F_j denote player j's CDF. If player i bids r_i given v_i, then there is probability $F_j(r_i)$ that i wins. In this case, he obtains the value v_i less the second highest bid; i.e. bid by player j, who we assume bids according to his true value, v_j. With probability $1 - F_j(r_i)$, player i loses, in which case, he has to pay $\hat{b}(r_i)$ (his own bid that is lower than j's).

Hence, we can write player i's expected utility as

$$\begin{aligned}
u_i\left(r_i, v_i\right) &= F\left(r_i\right) v_i - \int_0^1 \min\left\{\hat{b}\left(r_i\right), \hat{b}\left(v_j\right)\right\} f\left(v_j\right) dv_j \\
&= F\left(r_i\right) v_i - \left(\int_0^{r_i} \hat{b}\left(v_j\right) f\left(v_j\right) dv_j + \int_{r_i}^1 \hat{b}\left(r_i\right) f\left(v_j\right) dv_j\right) \\
&= F\left(r_i\right) v_i - \int_0^{r_i} \hat{b}\left(v_j\right) f\left(v_j\right) dv_j - \left(1 - F\left(r_i\right)\right) \hat{b}\left(r_i\right)
\end{aligned}$$

where we assumed symmetry; i.e. $F_i\left(r\right) = F\left(r\right)$ for all $i \in \{1, 2\}$, and assumed that $\hat{b}\left(\cdot\right)$ is strictly increasing. We want this function to be maximised when $r_i = v_i$:

$$\begin{aligned}
\left.\frac{\partial u_i\left(r_i, v_i\right)}{\partial r_i}\right|_{r_i = v_i} &= \left. f\left(r_i\right) v_i - \hat{b}\left(r_i\right) f\left(r_i\right) + f\left(r_i\right) \hat{b}\left(r_i\right) - \left(1 - F\left(r_i\right)\right) \hat{b}'\left(r_i\right)\right|_{r_i = v_i} \\
&= f\left(v_i\right) v_i - f\left(v_i\right) \hat{b}\left(v_i\right) - F\left(v_i\right) \hat{b}'\left(v_i\right) + \hat{b}\left(v_i\right) f\left(v_i\right) \\
&= f\left(v_i\right) v_i - \left(1 - F\left(v_i\right)\right) \hat{b}'\left(v_i\right)
\end{aligned}$$

Equating the derivative to be zero

$$\begin{aligned}
\hat{b}'\left(v_i\right) &= \frac{f\left(v_i\right)}{1 - F\left(v_i\right)} v_i \\
\Rightarrow \int_0^{v_i} \hat{b}'\left(x\right) dx &= \int_0^{v_i} \frac{f\left(x\right)}{1 - F\left(x\right)} x dx \\
\Rightarrow \hat{b}\left(v_i\right) &= \int_0^{v_i} \frac{x f\left(x\right)}{1 - F\left(x\right)} dx \\
&= \int_0^{v_i} \frac{1}{1 - F\left(x\right)} x \frac{dF\left(x\right)}{dx} dx \\
&= \int_0^{v_i} \frac{x}{1 - F\left(x\right)} dF\left(x\right).
\end{aligned}$$

Since $f\left(v_i\right) > 0$, it is immediate that $\hat{b}$ is strictly increasing. Moreover, since

$$\begin{aligned}
\frac{\partial u_i\left(r_i, v_i\right)}{\partial r_i} &= f\left(r_i\right) v_i - \left(1 - F\left(r_i\right)\right) \hat{b}'\left(r_i\right) \\
&= f\left(r_i\right) v_i - \left(1 - F\left(r_i\right)\right) \frac{f\left(r_i\right)}{\left(1 - F\left(r_i\right)\right)} r_i \\
&= f\left(r_i\right)\left(v_i - r_i\right).
\end{aligned}$$

and we have $f\left(r_i\right) > 0$, we conclude that $u_i\left(r_i, v_i\right)$ is maximised at $r_i = v_i$.

Comparison of equilibrium bidding function The bidding function in this case is

$$\hat{b}_{SPAPA}\left(v\right) = \int_0^v \frac{x f\left(x\right)}{1 - F\left(x\right)} dx.$$

The bidding function from standard first-price and second-price auctions with $N = 2$ are

$$\hat{b}_{FPA}(v) = \int_0^v \frac{xf(x)}{F(v)} dx,$$
$$\hat{b}_{SPA}(v) = v,$$
$$\hat{b}_{FPAPA}(v) = \int_0^v xf(x)\, dx.$$

So, we can see immediately that

$$\hat{b}_{SPAPA}(v) \geq \hat{b}_{FPAPA}(v).$$

That is, it remains the case that bidders bid more aggressively in the second-price auction as, conditional on winning the auction, the bids do not affect the amount the winning bidder pays.

Consider

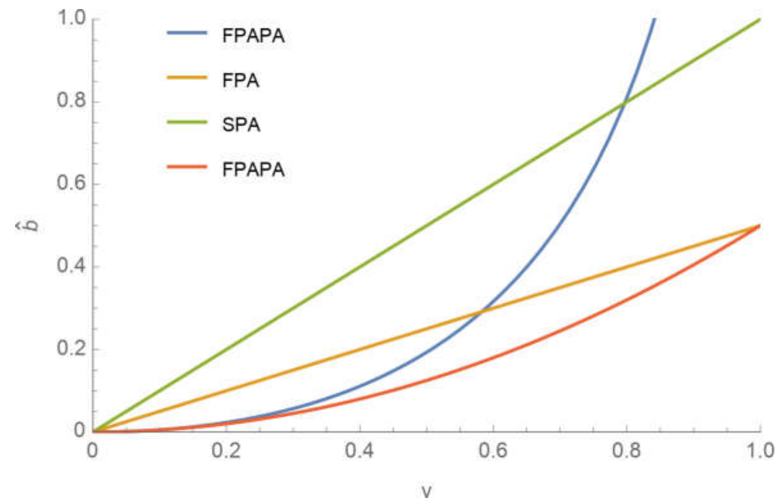
$$\begin{aligned}
\hat{b}_{SPAPA}(v) - \hat{b}_{FPA}(v) &= \int_0^v \frac{xf(x)}{1-F(x)} dx - \int_0^v \frac{xf(x)}{F(v)} dx \\
&= \int_0^v xf(x) \left(\frac{1}{1-F(x)} - \frac{1}{F(v)} \right) dx \\
&= \int_0^v xf(x) \left(\frac{F(v) - (1-F(x))}{(1-F(x))\, F(v)} \right) dx \\
&= \int_0^v \frac{xf(x)}{(1-F(x))\, F(v)} \left(F(v) - (1-F(x)) \right) dx.
\end{aligned}$$

The sign of the difference depends upon F.

Consider the case in which values are distributed uniformly in the interval $[0, 1]$; so $F(v) = v$ and $f(v) = 1$. Then,

$$\begin{aligned}
\hat{b}_{SPAPA}(v) &= \int_0^v \frac{x}{1-x} dx = \int_0^v \left(-1 + \frac{1}{1-x} \right) dx \\
&= \left[-x - \ln(1-x) \right]_0^v = -v - \ln(1-v), \\
\hat{b}_{FPA}(v) &= \int_0^v \frac{x}{v} dx = \frac{v}{2}, \\
\hat{b}_{SPA}(v) &= v, \\
\hat{b}_{FPAPA}(v) &= \int_0^v x dx = \frac{v^2}{2}.
\end{aligned}$$

See the figure below.



Seller's expected revenue In this case, both bidders pay the second highest bid. But, ex ante, neither v_1 nor v_2 is known, so the seller's expected revenue is given by

$$R_{SPAPA} = 2\int_0^1 \int_0^1 \min\left\{\hat{b}(v_1), \hat{b}(v_2)\right\} f(v_1) f(v_2)\, dv_1 dv_2.$$

Then,

$$\begin{aligned}
R_{SPAPA} &= 2\int_0^1 \int_0^1 \min\left\{\hat{b}(v_1), \hat{b}(v_2)\right\} f(v_1)\, dv_1 f(v_2)\, dv_2 \\
\Leftrightarrow \frac{1}{2} R_{SPAPA} &= \int_0^1 \left[\int_0^{v_2} \hat{b}(v_1) f(v_1)\, dv_1 + \int_{v_2}^1 \hat{b}(v_2) f(v_1)\, dv_1\right] f(v_2)\, dv_2 \\
&= \int_0^1 \left[\int_0^{v_2} \hat{b}(v_1) f(v_1)\, dv_1 + \hat{b}(v_2)\int_{v_2}^1 f(v_1)\, dv_1\right] f(v_2)\, dv_2 \\
&= \int_0^1 \int_0^{v_2} \hat{b}(v_1) f(v_1)\, dv_1 f(v_2)\, dv_2 + \int_0^1 \hat{b}(v_2)(1-F(v_2)) f(v_2)\, dv_2 \\
&= \int_0^1 \left(\int_{v_1}^1 f(v_2)\, dv_2\right) \hat{b}(v_1) f(v_1)\, dv_1 + \int_0^1 \hat{b}(v_2)(1-F(v_2)) f(v_2)\, dv_2 \\
&= \int_0^1 (1-F(v_1))\hat{b}(v_1) f(v_1)\, dv_1 + \int_0^1 \hat{b}(v_2)(1-F(v_2)) f(v_2)\, dv_2 \\
&= \int_0^1 (1-F(v))\hat{b}(v) f(v)\, dv + \int_0^1 \hat{b}(v)(1-F(v)) f(v)\, dv \\
\Leftrightarrow R_{SPAPA} &= 4\int_0^1 \hat{b}(v)(1-F(v)) f(v)\, dv.
\end{aligned}$$

Substituting the functional form for $\hat{b}(v)$ yields

$$\begin{aligned}
R_{SPAPA} &= 4\int_0^1 \int_0^v \frac{xf(x)}{1-F(x)} dx\,(1-F(v)) f(v)\, dv \\
&= 4\int_0^1 \left(\int_x^1 (1-F(v)) f(v)\, dv\right) \frac{xf(x)}{1-F(x)} dx.
\end{aligned}$$

Consider the inner integration first:

$$\begin{aligned}\int_x^1 (1-F(v))\,f(v)\,dv &= \int_x^1 f(v)\,dv - \int_x^1 F(v)\,f(v)\,dv \\ &= (1-F(x)) - \int_x^1 F(v)\,f(v)\,dv.\end{aligned}$$

Using integration by parts, we can write the the second term on the right-hand side as

$$\begin{aligned}\int_x^1 F(v)\,f(v)\,dv &= [F(v)\,F(v)]_x^1 - \int_x^1 F(v)\,f(v)\,dv \\ &= \left(1-F^2(x)\right) - \int_x^1 F(v)\,f(v)\,dv \\ \Leftrightarrow \int_x^1 F(v)\,f(v)\,dv &= \frac{1}{2}\left(1-F^2(x)\right) = \frac{1}{2}(1+F(x))(1-F(x)).\end{aligned}$$

Hence,

$$\begin{aligned}\int_x^1 (1-F(v))\,f(v)\,dv &= (1-F(x)) - \frac{1}{2}(1+F(x))(1-F(x)) \\ &= \frac{1}{2}(1-F(x))^2.\end{aligned}$$

Substituting this expression back into R_{SPAPA} gives

$$\begin{aligned}R_{SPAPA} &= 4\int_0^1 \left(\frac{1}{2}(1-F(x))^2\right)\frac{xf(x)}{1-F(x)}dx \\ &= 2\int_0^1 xf(x)(1-F(x))\,dx \\ &= R_{FPA}\end{aligned}$$

since, when $N=2$,

$$\begin{aligned}R_{FPA} &= 2\int_0^1\int_0^v xf(x)\,f(v)\,dxdv \\ &= 2\int_0^1 xf(x)\left(\int_x^1 f(v)\,dv\right)dx \\ &= 2\int_0^1 xf(x)(1-F(x))\,dx.\end{aligned}$$