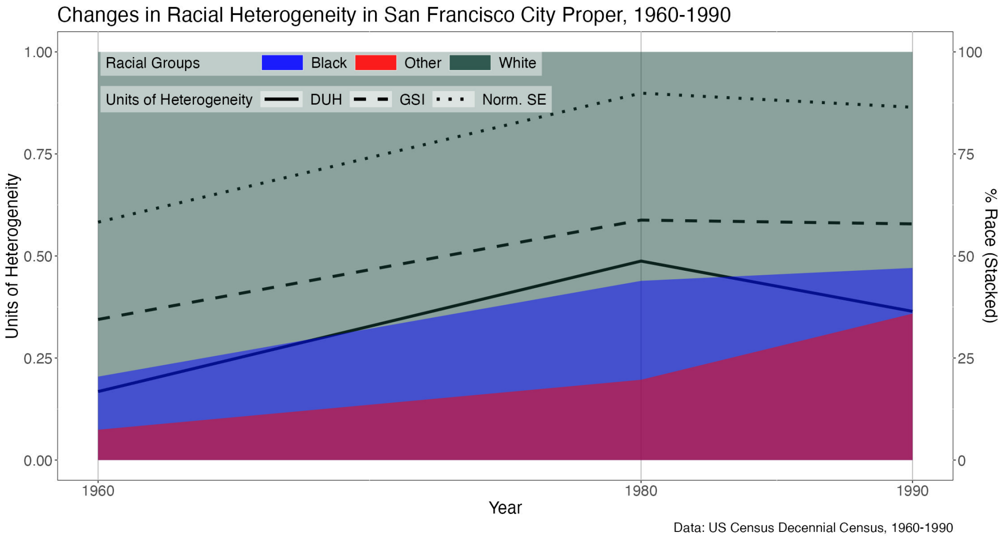
Descriptive Units of Heterogeneity: An Axiomatic Approach to Measuring Heterogeneity

Willy Chen

AFRE Brown Bag

MICHIGAN STATE UNIVERSITY

Empirical Motivation



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$\forall s \in \mathcal{S}$, denote

- The length of s as $|s|$.
- The "total population" of s as $||s||_1 = \sum_{g=1}^{|s|} s_g$.
- The mean group size of s as $\mu(s) = \frac{||s||_1}{|s|}$.

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A function $\Phi : \mathscr{S} \to \mathbb{R}$ is a measure of heterogeneity if, $\forall s, s' \in \mathscr{S}$,

$$\Phi(s) \geq \Phi(s') \iff s \text{ is weakly more heterogeneous } (\succsim) \text{ than } s'.$$

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A system $\underline{s}$ is minimally heterogeneous (or perfectly homogeneous) if all but one element of $\underline{s}$ are 0.

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Problem: If we treat heterogeneity as a distributional property without value-judgments, the there are cases when dispersion and deconcentration are sufficient.

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- Require arbitrary normative judgments (Atkinson and Genralized Gini).
- Require implicit order of outcomes (Atkinson, Generalized Gini, and Gini).
- Lacks compressibility and the sensitivity to transfers (Hoover).

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Measures of concentration quantify heterogeneity by the non-dominance of one or a few groups in a system (deconcentration).

- Not sensitive to redistribution between small groups when one group is sufficiently large ((HHI and SE) .
- Discard information provided by the presence of groups with 0 population (HHI and SE).

Descriptive Units of Heterogeneity

Let σ of s be the permutation of s such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{|s|}$. σ is called the *ordered system* of s.

Define

$$\hat{\sigma}_1 = \frac{\sigma_1}{||s||_1} \text{ and } \tilde{\sigma}_g = \frac{\sigma_g}{||s||_1 - \sigma_1}$$

The Descriptive Units of Heterogeneity (DUH) is defined as:

$$DUH_p(s) = \frac{ln(\hat{\sigma}_1)}{ln(|s|)} \cdot \left[\left(\sum_{g=2}^{|s|} \left| \tilde{\sigma}_g - \frac{1}{|s| - 1} \right|^p \right)^{\frac{1}{p}} - 1 \right]$$

Fundamental Axioms

Axioms that induce partial ordering by pinning down

1. When two systems of $|s|$ groups are equally heterogeneous
2. How to order the heterogeneity of two systems of $|s|$ groups that are marginally different

[GSYM] Group Symmetry

For any permutation $\pi(s)$ of s, $\Phi(s) = \Phi(\pi(s))$.

For example, take $s, s', s'' \in \mathcal{S}$,

$$\begin{array}{c} s \\ \begin{array}{|c|}\hline s_1 \\ \hline s_2 \\ \hline s_3 \\ \hline \end{array} \end{array} \sim \begin{array}{c} s' \\ \begin{array}{|c|}\hline s_2 \\ \hline s_1 \\ \hline s_3 \\ \hline \end{array} \end{array} \sim \begin{array}{c} s'' \\ \begin{array}{|c|}\hline s_3 \\ \hline s_2 \\ \hline s_1 \\ \hline \end{array} \end{array}$$

[INV] Scale Invariance

A measure of heterogeneity Φ satisfies the property of Scale Invariance if for any system s and a scalar $\lambda \in \mathbb{R}_{++}$, $\Phi(s) = \Phi\left(\lambda \cdot s\right)$.

For example, take $s, s' \in \mathcal{S}$, $\lambda \in \mathbb{R}_{++}$,

$$\begin{array}{ccc} s & & s' \\ \begin{array}{|c|}\hline s_1 \\ \hline s_2 \\ \hline s_3 \\ \hline \end{array} & \sim & \begin{array}{|c|}\hline \lambda s_1 \\ \hline \lambda s_2 \\ \hline \lambda s_3 \\ \hline \end{array} \end{array}$$

[PT] Principle of Transfers

Let σ be the ordered system of s. Let $e_i^{|s|}$ be an ordered tuple of length $|s|$ such that its i^{th} element is some ε and the rest are 0. A measure of heterogeneity Φ satisfies the Principle of Transfers if $\forall i < j \leq |\sigma|$ and $\varepsilon \in \mathbb{R}_+$,

$$\begin{cases} \sigma_i - \sigma_j \geq 2\varepsilon \\ \sigma_i - \sigma_{i+1} \geq \varepsilon \\ \sigma_{j-1} - \sigma_j \geq \varepsilon \end{cases} \quad \text{together imply } \Phi(\sigma) < \Phi\left(\sigma - e_i^{|s|} + e_j^{|s|}\right).$$

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$$\begin{array}{c} \sigma \\ \begin{array}{|c|} \hline \sigma_1 \\ \hline \sigma_2 \\ \hline \sigma_3 \\ \hline \end{array} \end{array} \prec \begin{array}{c} \sigma' \\ \begin{array}{|c|} \hline \sigma_1 - \varepsilon \\ \hline \sigma_2 + \varepsilon \\ \hline \sigma_3 \\ \hline \end{array} \end{array} \prec \begin{array}{c} \sigma'' \\ \begin{array}{|c|} \hline \sigma_1 - \varepsilon \\ \hline \sigma_2 \\ \hline \sigma_3 + \varepsilon \\ \hline \end{array} \end{array}, \quad \text{i.e. } \Phi(\sigma) < \Phi(\sigma') < \Phi(\sigma''),$$

A Different Sufficient Condition for GSYM

- Existing units (satisfying INV) has the same arithmetic treatment for every group in the system, partially to satisfy GSYM

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  - For example, HHI is the sum of squares of every group's proportion, instead of sum of squares of some groups' proportions and cubes of other groups' proportions.
- This uniform treatment can be relaxed by treating the same type of groups the same.

What I propose is to treat the largest group separate from the rest of the groups.

What Matters in Measuring Heterogeneity?

The heterogeneity of system S is measured by $\Phi : S \to \mathbb{R}$ such that

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$$\Phi(s) \geq \Phi(s') \iff s \text{ is weakly more heterogeneous than } s'.$$

- Relative size of the largest group: $\hat{\sigma}_1 = \frac{\sigma_1}{||s||_1}$
- Relative size(s) of the minority group(s): $\tilde{\sigma}_g = \frac{\sigma_g}{||s||_1 - \sigma_1}$, $g \in \{2, \ldots, |s|\}$

A unit of heterogeneity can be thought of as $\Phi = \Phi(\phi, \psi)$ where
$\phi(\hat{\sigma}_1)$ is the influence of the relative size of the largest group
$\psi(\tilde{\sigma}_2, \ldots, \tilde{\sigma}_G)$ the influence of the relative sizes of the minority group.

Characterization Axioms

Axioms that can be combined to uniquely characterize measures with total ordering and cardinal interpretation.

[IND] Independence

Let σ be the ordered system of s. A measure of heterogeneity $\Phi(s)$ satisfies Independence if it is a composite function of $\phi : \mathcal{S} \to \mathbb{R}$ and $\psi : \mathcal{S} \to \mathbb{R}$ such that $\psi\left(s\right) = \psi\left(c, \sigma_2, \ldots, \sigma_{|s|}\right), \forall c \in \mathbb{R}_{++}$.

Using Evenness for ψ

I use the L^p-norm between the minority distribution and the uniform distribution to quantify evenness in minority distribution.

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Definition: Take $p \in [1, \infty)$. The function $\psi_p : \mathcal{S} \to \mathbb{R}$ defined as:

$$\psi_p(s) = 1 - \left(\sum_{g=2}^{|s|} \left| \tilde{\sigma}_g - \frac{1}{|s| - 1} \right|^p \right)^{\frac{1}{p}},$$

is a measure of evenness in the distribution of minority groups.

Proposition

Let s be an arbitrary system such that $|s| \geq 4$. Let Φ be a measure of heterogeneity that satisfies GSYM, INV, and IND. Holding $\hat{\sigma}_1$ constant, if Φ is strictly increasing in ψ_p, then Φ satisfies PT if and only if $p > 1$.

[PPT] Principle of Proportional Transfers

Let σ be the ordered system of s. Let $e_i^{|s|}$ be an ordered tuple of length $|s|$ such that its i^{th} element is some ε and the rest are 0. A measure of heterogeneity $\Phi(s)$ satisfies the Principle of Proportional Transfers if $\forall \varepsilon \in \mathbb{R}_+, \exists \alpha \in \mathbb{R}_{++}$

$$\begin{cases} \frac{\sigma_1 - \varepsilon}{||s||_1} = \left(\frac{\sigma_1}{||s||_1}\right)^\alpha \\ \sigma_1 - \varepsilon \geq \sigma_2 + \tilde{\sigma}_2 \cdot \varepsilon \end{cases} \quad \text{together imply } \Phi\left(\sigma - e_1^{|s|} + \sum_{g=2}^{|s|} \tilde{\sigma}_g \cdot e_g^{|s|}\right) = \alpha \cdot \Phi\left(\sigma\right).$$

In other words, *holding the order of groups constant*, a transfer from the largest group proportionally to the minority groups that reduces $\hat{\sigma}_1$ to $(\hat{\sigma}_1)^\alpha$ increases heterogeneity by a factor of α.

PPT Example

$$
\begin{array}{ccc}
\sigma & & \sigma' \\
\boxed{\begin{array}{c} \tilde{\sigma}_1 \\ \hline \hat{\sigma}_2 \\ \hline \hat{\sigma}_3 \end{array}} & \prec & \boxed{\begin{array}{c} \hat{\sigma}_1^{\alpha} \\ \hline \hat{\sigma}_2 + \frac{\hat{\sigma}_2}{\hat{\sigma}_2+\hat{\sigma}_3}\left(\hat{\sigma}_1 - \hat{\sigma}_1^{\alpha}\right) \\ \hline \hat{\sigma}_3 + \frac{\hat{\sigma}_3}{\hat{\sigma}_2+\hat{\sigma}_3}\left(\hat{\sigma}_1 - \hat{\sigma}_1^{\alpha}\right) \end{array}}
\end{array}.
$$

A measure Φ satisfying GSYM, INV, and PPT would yield:

$$\Phi\left(\sigma\right) < \alpha\Phi\left(\sigma\right) = \Phi\left(\sigma'\right).$$

[CON] Contractibility

Let s be an arbitrary system. Let s' be the concatenation of s and the tuple (0) such that $s' = (s, 0)$. Let σ and σ' denote the ordered systems of s and s'. A measure of heterogeneity Φ satisfies Contractibility if

$$\sigma_2 > 0 \ \Rightarrow \ \Phi\left(\sigma'\right) < \Phi\left(\sigma\right).$$

[UNI] Unity

Let σ be the ordered system of s. A measure of heterogeneity Φ satisfies Unity if

$$\Phi\left(s\right) = 0 \iff \hat{\sigma}_1 = 1$$

and

$$\Phi\left(s\right) = 1 \iff \hat{\sigma}_1 = \hat{\sigma}_2 = \cdots = \hat{\sigma}_{|s|} = \frac{1}{|s|}$$

Descriptive Units of Heterogeneity

Let σ be the ordered system of s. Denote $\hat{\sigma}_1 = \frac{\sigma_1}{||s||_1}$ and $\tilde{\sigma}_g = \frac{\sigma_g}{||s||_1 - \sigma_1}$, where $g \in \{2, \ldots, |s|\}$ and $p \in (1, \infty)$. The family of descriptive units of heterogeneity (DUH) of the system s with $|s| \geq 2$ is:

$$DUH_p\left(s\right) = \frac{ln\left(\hat{\sigma}_1\right)}{ln(|s|)} \cdot \left[\left(\sum_{g=2}^{|s|} \left|\tilde{\sigma}_g - \frac{1}{|s| - 1}\right|^p\right)^{\frac{1}{p}} - 1\right].$$

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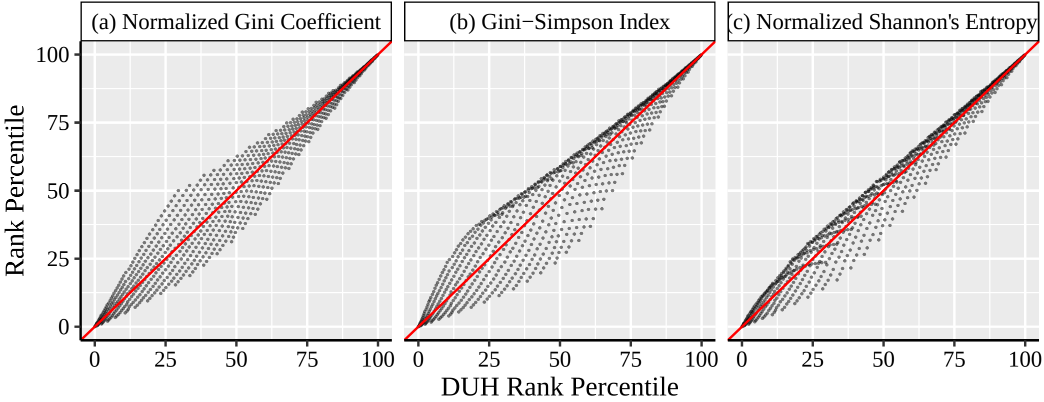
Theorem:
The descriptive units of heterogeneity constitute a uniquely determined family of measures—up to positive scalar multiplication—that incorporate evenness through the function ψ_p, and satisfy GSYM, INV, PT, IND, PPT, CON, and UNI.

Existing Measures Satisfy Some Axioms

Type	Axiom	Gini	DUH	HHI	SE
Fundamental	Type Symmetry	✓	✓	✓	✓
	Scale Invariance	✓	✓	✓	✓
	Principle of Transfers	✓	✓	✓	✓
Characterization	Independence	×	✓	✓	✓
	Principle of Proportional Transfers	×	✓	×	×
	Contractibility	✓	✓	×	×
	Unity	×	✓	×	×
	Expandability	×	×	✓	✓
	Replication Principle	×	×	✓	×
	Shannon's Additivity	×	×	×	✓

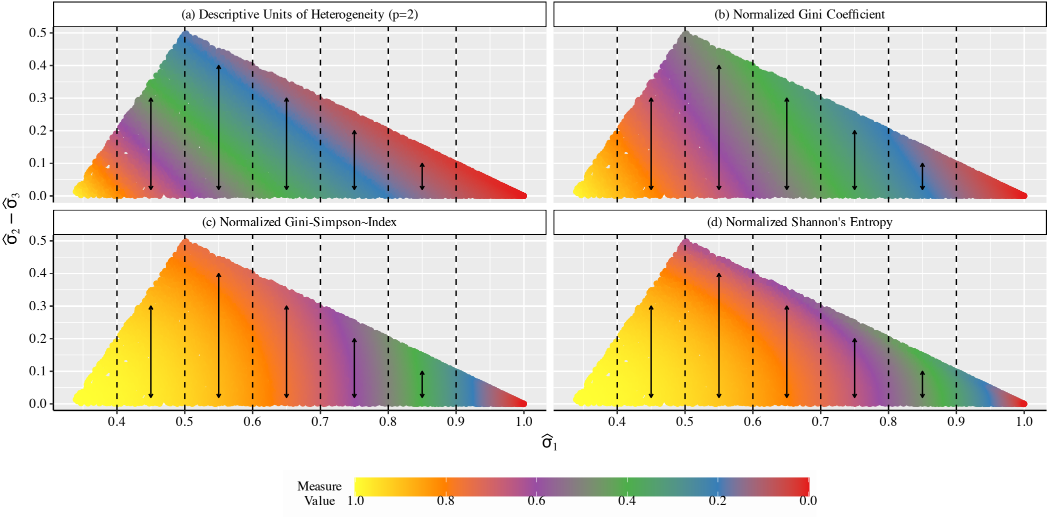
Comparing Measures

Figure 1: Rank Correlation between Measures over Systems with $|\sigma| = 3$ and $||\sigma||_1 = 100$



Comparing Measures

Figure 2: Comparison between DUH, Gini, GSI, and SE



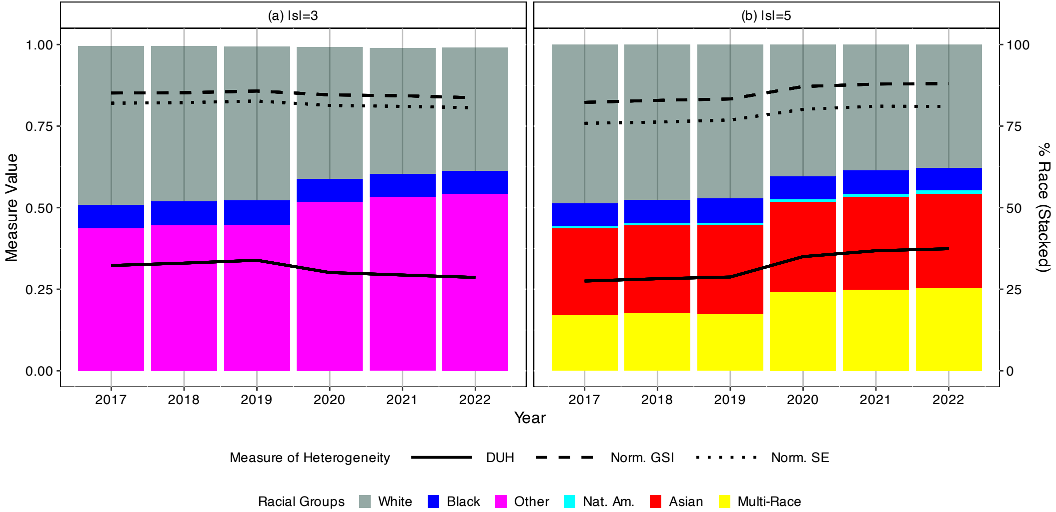
Reasonable Group Labels

While the satisfaction of GSYM implies that exact group labels are independent of the heterogeneity of a system, the satisfaction of CON implies that two systems are only directly comparable if the labeling of elements reflect a reasonable and interpretable grouping scheme.

Figure 3 illustrates this caveat with a practical example: the evolution of the racial composition of the San Francisco Metropolitan Statistical Area from 2017 to 2022, based on the American Community Survey (ACS) 1-year data (Ruggles et al., 2024).

Reasonable Group Labels

Figure 3: Different Group Labels Can Yield Opposite Inferences



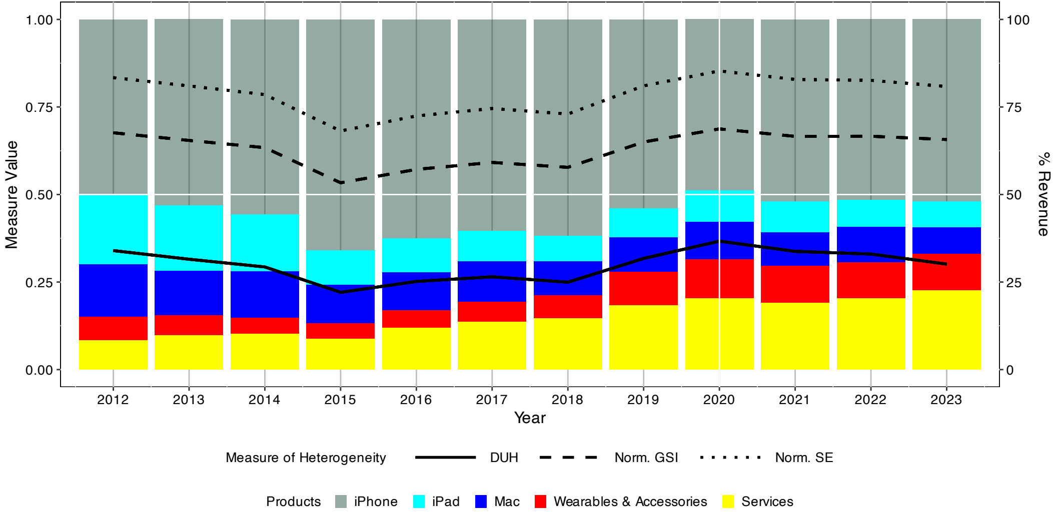
Empirical Example: Racial Heterogeneity

Table 1: The Progression of Racial Composition and Racial Heterogeneity of a Hypothetical City

Share (%)	Decade			
	1	2	3	4
White	60%	65%	66%	69%
Black	34%	26%	22%	17%
Other	6%	9%	11%	14%
DUH	0.235	0.257	0.284	0.315
Gini	0.460	0.440	0.450	0.450
GSI	0.521	0.502	0.499	0.475
Norm. GSI	0.781	0.753	0.749	0.713
Norm. SE	0.767	0.771	0.778	0.758

Empirical Example: Racial Heterogeneity

Figure 4: Revenue Heterogeneity using DUH, GSI, and SE



Thank You!

Bibliography I

Ruggles, Steven et al. (2024). *IPUMS USA: Version 15.0*. dataset. Minneapolis, MN. DOI: 10.18128/D010.V15.0. URL: https://doi.org/10.18128/D010.V15.0.

Characterization of DUH

Proof sketch

1. I show that any measure mapping an ordered system of group shares to the real numbers satisfies GSYM and INV.
2. I show that any Φ that satisfies GSYM, INV, and PPT must be a positive monotonic transformation of $\frac{1}{\hat{\sigma}_1}$.
3. I show that if $\Phi(\phi, \psi_p)$ satisfies INV, IND, and PPT, then ϕ and ψ_p must be multiplicatively separable. In other words, $\Phi = \phi \cdot \psi_p$.
4. I show that, holding $\hat{\sigma}_1$ constant and assuming GSYM, INV, and IND, the measure Φ, using the measure of evenness ψ_p as defined, satisfies PT if and only if $p > 1$ when $|s| > 3$ and $p \geq 1$ when $|s| = 3$.

Characterization of DUH

5. I show that if $\Phi(\phi, \psi_p)$ satisfies GSYM, INV, IND, and PPT, then $\phi = -c \cdot log_q\left(\hat{\sigma}_1\right), \, c, q \in \mathbb{R}_{++}$.
6. For the case of $p = 2$, using the Euclidean distance for ψ_p, I show that the DUH family satisfies PT, in addition to GSYM, INV, IND, and PPT, by taking the derivative of the extreme case in which $\hat{\sigma}_1$ is close to 1 and $\psi_2 = 1$ with respect to a transfer from the largest group to the second-largest group.
7. I show that the DUH family satisfies CON and UNI, in addition to GSYM, INV, IND, PT, and PPT.

[EXP] Expandability

Adding any arbitrary number of zero-groups does not affect the measure Φ.

$\Phi(n_1, \ldots, n_G)$ satisfies *Expandability* if

$$\Phi(n_1, \ldots, n_G) = \Phi(n_1, \ldots, n_G, 0)$$

[REP] Replication Principle

$\Phi(n_1, \ldots, n_G)$ satisfies *Replication Principle* (for concentration) if $\forall k \in \mathbb{N}$

$$\frac{1}{k}\Phi(n_1, \ldots, n_G) = \Phi\left(\underbrace{\frac{n_1}{k}, \frac{n_1}{k}, \ldots, \frac{n_1}{k}}_{\text{Sum to } n_1}, \frac{n_2}{k}, \frac{n_2}{k}, \ldots, \underbrace{\frac{n_G}{k}, \ldots, \frac{n_G}{k}}_{\text{Sum to } n_G}\right)$$

[SADD] Shannon's Additivity

Define $n_{gj} \geq 0$ such that $n_g = \sum_{j=1}^{m_g} n_{gj}$, $\forall g \in \{1, \ldots, G\}$, $\forall j \in \{1, \ldots, m_g\}$

$\Phi(n_1, \ldots, n_G)$ satisfies *Shannon's Additivity* if

$$\Phi(n_{11}, \ldots, n_{Gm_G}) = \Phi(n_1, \ldots, n_G) + \sum_{g=1}^{G} \frac{n_g}{n_S} \cdot \Phi\left(\frac{n_{g1}}{n_g}, \ldots, \frac{n_{gm_g}}{n_g}\right)$$

which implies (by setting $m_g = 1$, $\forall g \in \{1, \ldots, G-1\}$ and $n_{G'} = n_G + n_{G+1}$),

$$\begin{aligned}\Phi(n_1, \ldots, n_G, n_{G+1}) =& \Phi(n_1, \ldots, n_{G'}) \\ &+ \frac{n_{G'}}{n_1 + \ldots + n_{G-1} + n_{G'}} \cdot \Phi\left(\frac{n_G}{n_{G'}}, \frac{n_{G+1}}{n_{G'}}\right)\end{aligned}$$