Price Theory III
Spring 2019

(Lars Stole)

Solutions to
Assignment 3
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Contents

For typos/comments, email me at takumahabu@uchicago.edu.

v1.0 Initial version

v1.1 Fixed a typos. Q1(a) c_e not c_θ in the FOC. Q4 9(a): $w(x_1)$ changed to -215 from -225, also changed IC from $w(x_2) \geq 40$ to $w(x_2) \geq 20$. Added a remark about agent's IR constraint in Q6 (a).

1 Problem 1

(Variation on MWG, Exercise 13.C.2). Reconsider the two-type signalling model with $r(\theta_\ell) = r(\theta_h) = 0$, assuming that a worker's productivity is $\theta(1+e)$. As before, $\theta_h > \theta_\ell > 0$, the probability of θ_h is $\phi \in (0,1)$, and the worker's cost of education is $c(e,\theta)$, where $c(e,\theta)$ is strictly increasing and strictly convex in e, decreasing in θ, $c(0,\theta) = 0$ and $c_{e\theta}(e,\theta) < 0$ for $e > 0$.

1.1 Part (a)

Assume that the worker's productivity is observable and contractible by the labour market. Characterise the competitive equilibrium wages and education levels for each type of worker in this complete information game.

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We start off with the following lemma.

Lemma 1.1 (Zero profit lemma)**.** *Let $\mu(e) \in [0,1]$ denote the firms' (common) belief that the worker is type h after observing education level e. Then, in any equilibrium (competitive or perfect Bayesian), wages must equal the expected productivity of the worker with education level e.*

Proof. Given belief $\mu(e)$, if wages do not equal the expected productivity of the worker with education level e, then the firm must expect either strictly positive or strictly negative profits. If strictly positive, then at least one firm would have the incentive to increase wages to obtain the worker for sure (we are assuming some break rule); if strictly negative, then at least one firm would have the incentive to not hire the worker to earn zero profits. In either case, not all firms could not have been maximising their profits. ∎

Proposition 1.1. *Suppose that worker productivity is observable. Then, the competitive equilibrium, $(e_i^*, w_i^*)_{i\in\{\ell,h\}}$, satisfies, for each $i \in \{\ell, h\}$,*

$$w_i^* = \theta_i (1 + e_i^*), \ \theta_i = c_\theta(e_i^*, \theta_i).$$

Proof. By the lemma above, wages must equal the worker's productivity in equilibrium. Letting w_i^* and e_i^* denote the equilibrium wage and education levels, respectively, for each $i \in \{\ell, h\}$, we have that

$$w_i^* = \theta_i (1 + e_i^*), \ \forall i \in \{\ell, h\}.$$

Each type of workers maximises utility taking as given the wage:

$$\max_{e_i} \ \theta_i (1 + e_i) - c(e_i, \theta_i).$$

Since the objective is strictly concave, the first-order condition characterises the optimal education level:

$$\theta_i = c_e\left(e_i^*, \theta_i\right). \tag{1.1}$$

■

To interpret (1.1), let's give a graphical analysis of this question. We first think about how the indifference curves look like now in (e, w)-space for the workers. First, the indifference curves for each type remains unchanged as in the case when workers were not productive so that the indifference for each type $i \in \{\ell, h\}$ is still characterised by

$$u_i = w(e) - c(e, \theta_i) \Leftrightarrow w(e) = u_i + c(e, \theta_i),$$

where we write $w(e)$ to emphasise the fact that we are in (e, w) space. Since $c(0, \theta) = 0$, it follows that $w(0) = u_i$. Moreover, since $\frac{\mathrm{d}w(e)}{\mathrm{d}e} = c_e(e, \theta_i) > 0$, the indifference curves are upward sloping and since $\frac{\mathrm{d}^2 w(e)}{\mathrm{d}e\mathrm{d}\theta} = c_{e\theta}(e, \theta_i) < 0$ and $\theta_h > \theta_\ell$, the slope of the indifference curve for type-ℓ is everywhere steeper. Finally, as higher wages for a given level of education, and a lower level of education for a given level of wages is preferred by the worker, the indifference curves are increasing in the Northwesterly direction.

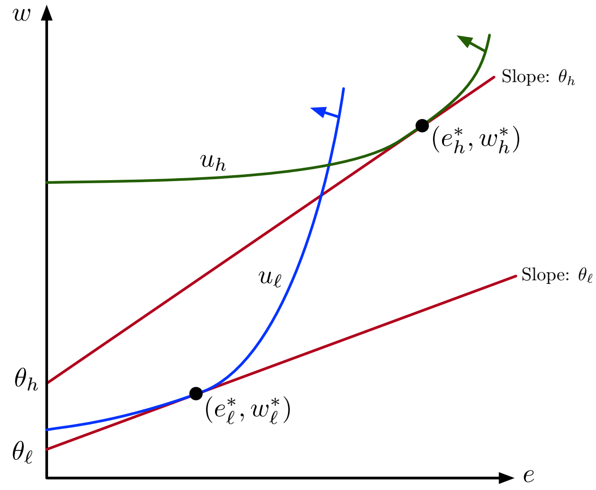
From each type of worker $i \in \{\ell, h\}$, the firm's isoprofit curve is characterised by

$$\pi_i = \theta_i(1 + e) - w(e) \Leftrightarrow w(e) = \pi_i + \theta_i(1 + e),$$

where, again, we write $w(e)$ to emphasise that we are in (e, w) space. Observe that the isoprofit curves are: (i) linear; (ii) has slope θ_i; and (iii) intersects the y-axis at $w(0) = \pi_i + \theta_i$. Note that, as lower wages given a level of information, and a higher level of education for a given wage level is preferred by the firm, the isoprofit curve is increasing in the Southeasterly direction.

As argued before, in any competitive equilibrium with complete information, wages must equal worker's productivity. This pins down the firm's isoprofit curves at $w(0) = \theta_i$ for each $i \in \{\ell, h\}$. Each type of worker then maximises utility by choosing a level of education, e_i^*, such that the indifference curve, u_i, is tangent to the isoprofit curves. We can therefore interpret (1.1) as equating marginal revenue and marginal cost for of education for the worker as well as equating marginal revenue and cost for the firm (in equilibrium). See figure below.

Figure 1.1: Competitive equilibrium with complete information (no envy case).



1.2 Part (b)

Demonstrate that, for some preferences, the outcome in (a) *may* arise in a separating equilibrium in the incomplete information game.

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The figure above is an example in which the separating equilibrium in the incomplete information game coincides with the competitive equilibrium in the complete information case. To see this, recall that the worker's utilities are increasing in the Northwesterly direction. Then, taking wages as given, neither type would want to deviate to the other type's wage-education level pair.

To argue this more formally, recall that a perfect Bayesian equilibrium in this case is a set of strategies, $\sigma^* : \{\theta_\ell, \theta_h\} \to \Delta(\mathbb{R}_+)$ and a belief function $\mu^*(e) : \mathbb{R}_+ \to [0,1]$ that describes the firm's (common) probability assessment that the worker is of high ability after observing education level e such that:

(i) worker's strategy is optimal given the firm's strategies;

(ii) $\mu^*(e)$ obeys Bayes' rule whenever possible;

(iii) wage offers following each choice e constitutes a Nash equilibrium of the simultaneous-move wage offer game in which the probability that the worker is of high ability is $\mu^*(e)$.

We already argued that (i) holds true and

$$\sigma^*(\cdot|\theta_h) = \delta_{e_h^*},\ \sigma^*(\cdot|\theta_\ell) = \delta_{e_\ell^*},$$

where δ_e denotes a degenerate distribution with all the probability mass at e. It remains to specify $\mu(e)$ and verify (iii). Here's one possible belief function:

$$\mu^*(e) = \begin{cases} 1 & \text{if } e = e_h^*, \\ 0 & \text{if } e \neq e_h^*. \end{cases}$$

We only need to check that $\mu^*(e)$ obeys Bayes' rule on the equilibrium paths; i.e. $e \in \{e_h^*, e_\ell^*\}$, which is clearly satisfied:

$$\mu^*(e_h^*) = 1 = \frac{\phi_h \sigma^*(e_h^*|\theta_h)}{\phi_h \sigma^*(e_h^*|\theta_h) + (1-\phi_h)\sigma^*(e_h^*|\theta_\ell)},$$
$$\mu^*(e_\ell^*) = 0 = \frac{\phi_h \sigma^*(e_\ell^*|\theta_h)}{\phi_h \sigma^*(e_\ell^*|\theta_h) + (1-\phi_h)\sigma^*(e_\ell^*|\theta_\ell)}.$$

For any e that differ from e_h^* or e_ℓ^*, the firm believes that the worker is of low productivity type and so the firm offers $w_\ell^* = \theta_\ell(1+e_\ell^*)$, which is a Nash equilibrium of the simultaneous-move wage offer game in which the probability that the worker is of high ability is $\mu^*(e)$. To be complete, an equilibrium wage function is

$$w^*(e) = \begin{cases} \theta_h(1+e_h^*) & \text{if } e = e_h^* \\ \theta_\ell(1+e_\ell^*) & \text{if } e \neq e_h^* \end{cases}.$$

1.3 Part (c)

Assume that the underlying preferences are such that full-information competitive allocation in (a) does not arise in any separating equilibrium to the incomplete information game. Indicate the range of education levels that can be supported in a PBE separating equilibrium and the range of outputs that can be supported in a PBE pooling equilibrium.

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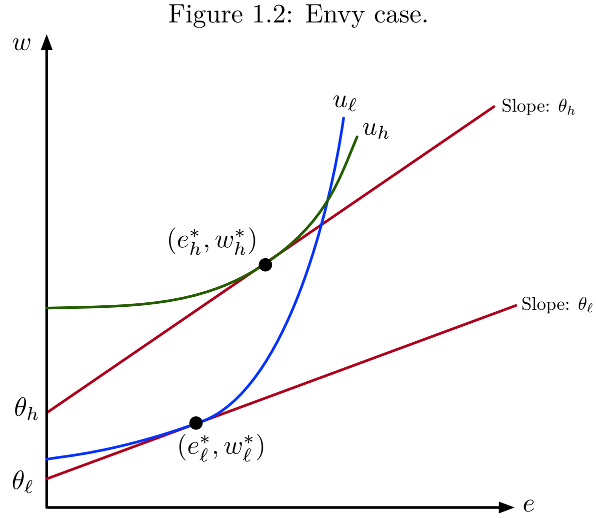
If (e_h^*, w_h^*) lies to the left of the intersection between u_h and u_ℓ, then we would not observe the competitive equilibrium with complete information as a separating equilibrium in the incomplete information game. This is sometimes called the *envy* case (and the previous case the *no-envy* case).

Figure 1.2: Envy case.



1.3.1 Separating equilibria

Let $(e_i^s, w_i^s)_{i\in\{\ell,\}h}$ denote the wage-effort offers and $\mu^s(e)$ the belief function in a separating equilibrium. We will characterise the separating equilibria below.

Lemma 1.2. *In any separating PBE, $w^s(e_h^s) = \theta_h$ and $w^s(e_\ell^s) = \theta_\ell$.*

Proof. In any separating PBE, since the firm is able to infer the type from the observed level of education, and e_h^* and e_ℓ^* are "on path". Then, the zero-profit lemma yields the result. ■

Above lemma, in particular, means that (e_i^s, w_i^s) must be related as

$$w_i^s = \theta_i\left(1+e_i^s\right),\ \forall i\in\{\ell,h\}.$$

Lemma 1.3. *In any PBE, type ℓ attains utility at least $w_\ell^* - c(e_\ell^*, \theta_\ell)$.*

Proof. Given $\mu^s(e)$, the zero-profit lemma implies that

$$w^s(e) = \left[\mu^s(e)\,\theta_h + (1-\mu^s(e))\,\theta_\ell\right](1+e).$$

Since $\theta_h > \theta_\ell$, it follows that

$$w^s(e) \geq \theta_\ell(1+e),\ \forall\mu^s(e)\in[0,1].$$

Thus, by choosing e_ℓ^*, type-ℓ can assure himself of a wage of at least w_ℓ^* so that his utility must be at least $w_\ell^* - c(e_\ell^*, \theta_\ell)$. ■

Lemma 1.4. *In any separating PBE, $(e_\ell^s, w_\ell^s) = (e_\ell^*, w_\ell^*)$.*

Proof. In a separating PBE, if e is chosen by the type-ℓ worker, then $w^s(e) = \theta_\ell(1+e)$. But, by construction,

$$\max_e\ \theta_\ell(1+e) - c(e,\theta_\ell) = \theta_\ell(1+e_\ell^*) - c(e_\ell^*,\theta_\ell) \geq \theta_\ell(1+e) - c(e,\theta_\ell).$$

Since type-ℓ worker can attain utility equal to the right-hand side by the previous lemma, we must have $(w_\ell^s, e_\ell^s) = (w_\ell^*, e_\ell^*)$. ■

Proposition 1.2. *There exist separating equilibria such that $e_h^s \in [e', e'']$, where e' and e'' are such that*

$$\begin{aligned}
w_\ell^* - c(e_\ell^*,\theta_\ell) + c(e',\theta_\ell) &= \theta_h(1+e'),\\
w_\ell^* - c(e_\ell^*,\theta_h) + c(e'',\theta_h) &= \theta_h(1+e'').
\end{aligned}$$

That is, e' is the intersection of the ℓ-type's indifference curve through (w_ℓ^*, e_ℓ^*) and the zero-profit curve for the h type; and e'' is the intersection of the h-type's indifference curve through (e_ℓ^*, w_ℓ^*) and the zero-profit curve for the h type.

Proof. We will argue this graphically. Consider the figure below.

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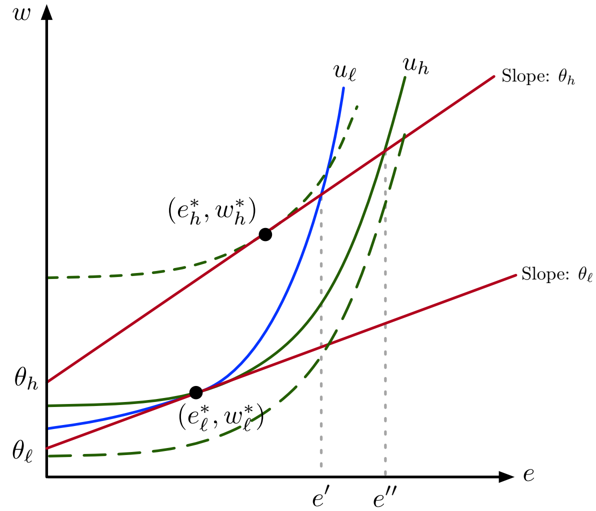
Figure 1.3: Characterising equilibrium education levels in separating PBE.

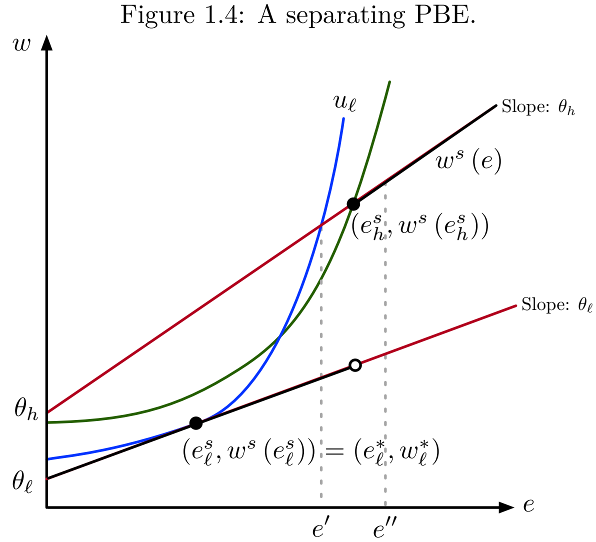


We already argued that if firms chose (e_h^*, w_h^*), then the type ℓwould want to mimic type h so this cannot be an equilibrium. Such incentives exist unless $e_h^s \geq e'$. If firms offers $e > e''$, then the h type would want to mimic type ℓ. Such incentives exist unless $e_h^s \leq e''$. Thus, any separating PBE must be involve h type choosing education level in $[e', e'']$. To show that any education level in this level is sustainable as a separating PBE, notice that we can set beliefs and wage functions to be:

$$\mu^s(e) = \begin{cases} 1 & \text{if } e \geq e_h^s \\ 0 & \text{if } e < e_h^s \end{cases}, \quad w^s(e) = \begin{cases} \theta_h(1+e) & \text{if } e \geq e_h^s \\ \theta_\ell(1+e) & \text{if } e < e_h^s \end{cases}. \qquad \blacksquare$$

This equilibrium wage function is shown in the figure below.

Figure 1.4: A separating PBE.



So to answer the question formally, the range of education levels that can be supported in a separating PBE are:

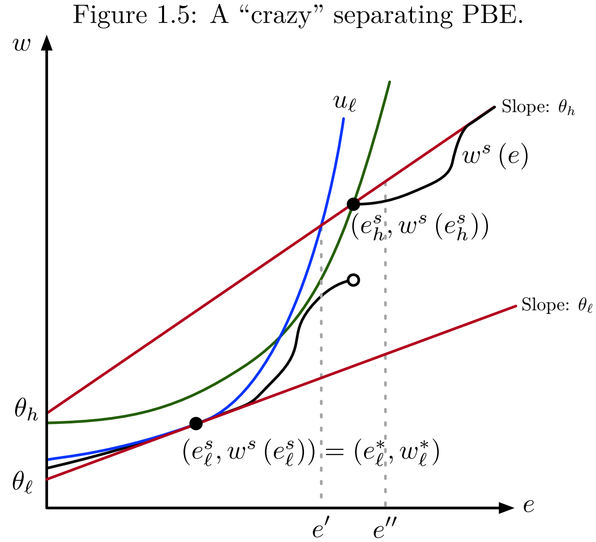
$$\{e_\ell^*\} \cup [e', e''] .$$

Remark 1.1. Just as in the case where e was not productive, there can be many wage functions that can be supported as separating PBE so long as it satisfies the zero-profit condition

$$w^e(e) = [\mu(e)\theta_h + (1-\mu(e))\theta_\ell](1-e)$$

and it lies below the lower envelope of the indifference curves. So something "crazy" like the figure below is a separating PBE.

Figure 1.5: A "crazy" separating PBE.



1.3.2 Pooling equilibria

Lemma 1.5. *In any pooling equilibrium,* (e^p, w^p) *satisfies*

$$w^p = (\phi\theta_h + (1-\phi)\theta_\ell)(1+e^p) .$$

Proof. By definition, in a pooling equilibrium, both types of the worker must choose the same level of education; i.e. $\sigma(e^p|\theta_h) = \sigma(e^p|\theta_\ell) = 1$. Thus, Bayes' rule tells us that

$$\mu(e^p) = \frac{\phi(1)}{\phi(1) + (1-\phi)(1)} = \phi.$$

By the zero-profit lemma, it follows that $w^p = (\phi\theta_h + (1-\phi)\theta_\ell)(1+e^p)$.

It remains to pin down e^p, off-equilibrium belief and the wage functions. ∎

Proposition 1.3. *There exist pooling equilibria such that*

$$(e^p, (\phi\theta_h + (1-\phi)\theta_\ell)(1+e^p)) ,$$

for all $e_p \in [0, \overline{e}]$, *where* $\overline{e}$ *satisfies*

$$(\phi\theta_h + (1-\phi)\theta_\ell)(1+\overline{e}) = w_\ell^* - c(e_\ell^*, \theta_\ell) + c(\overline{e}, \theta_h);$$

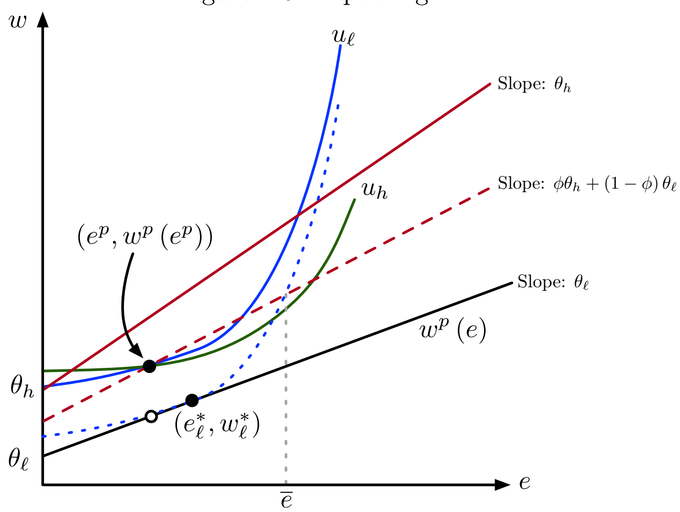
i.e. $\overline{e}$ *is the intersection of the pooling zero-profit line and* ℓ*-type's indifference curve through* (e_ℓ^*, w_ℓ^*).

Proof. Lemma 1.3 tells us that, in any pooling equilibrium, type ℓ must attain utility at least $w_\ell^* - c(e_\ell^*, \theta_\ell)$. This, together with the previous lemma implies that $e^p \in [0, \overline{e}]$. To show that any such e^p can be supported as a pooling equilibria, let

$$\mu^p(e) = \begin{cases} \phi & \text{if } e = e^p \\ 0 & \text{if } e \neq e^p \end{cases}, \quad w^p(e) = \begin{cases} (\phi\theta_h + (1-\phi)\theta_\ell)(1+e^p) & \text{if } e = e^p \\ \theta_\ell(1+e) & \text{if } e \neq e^p \end{cases}. \qquad \blacksquare$$

Notice that beliefs and wage functions are such that if the h-type chooses any other education level other than e^p, he is paid the low-type wage which is lower than the pooling equilibrium; i.e. h-type has no incentive to deviate. ℓ type does not deviate since deviating gives a strictly lower payoff given that $w_\ell^* \leq w^p(e)$. This equilibrium wage function is shown in the figure below.

Figure 1.6: A pooling PBE.

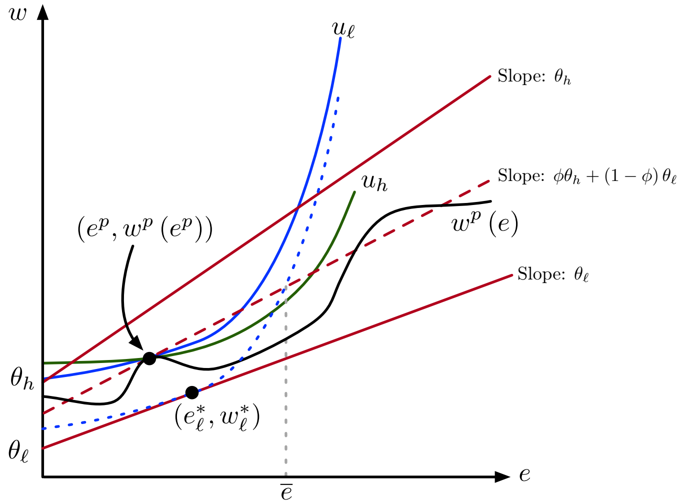


Since output equals wages in equilibrium, the range of outputs that can be supported in a PBE pooling equilibrium is:

$$\{(\phi\theta_h + (1-\phi)\theta_\ell)(1+e^p) : e^p \in [0, \overline{e}]\}.$$

Remark 1.2. Once again, there are multiplicity of pooling equilibria as we can specify $\mu^p(e)$ and $w^p(e)$ off equilibrium (i.e. for $e \neq e^p$) arbitrary so long as $w^p(e)$ lie below the lower envelope of the h- and ℓ-types indifference curves through (e^p, w^p) as shown in the figure below.

Figure 1.7: A "crazy" pooling PBE.



Exercise 1.1. There are multiplicity of hybrid equilibrium in which one type plays a pure strategy but another type mixes. Can you think give an example?

Exercise 1.2. Apply the intuitive criterion to this case to see what equilibrium we would end up with.

2 Problem 2

(Variation on MWG, Exercise 13.D.1). Extend the screening model of MWG, Chapter 13.D, to the case in which tasks are productive. Assume that a type-θ worker produces $\theta(1+t)$ units of output when her task level is t. As before, $\theta_h > \theta_\ell > 0$, the probability of θ_h is $\phi \in (0,1)$ and the worker's cost of task is $c(t,\theta)$, where $c(t,\theta)$ is increasing and convex in t, decreasing in θ, $c(0,\theta) = 0$ and $c_{t\theta}(t,v) < 0$ for $t > 0$.

. . . . . .

This is the competitive screening version of the previous signalling question. We maintain the assumption that there are more than two firms and that firms offer a menu of contracts and the appropriate equilibrium concept is subgame perfect Nash equilibrium. We first establish that, in any equilibrium, firms must earn zero profits.

Lemma 2.1 (Zero profit lemma)**.** *In any subgame perfect equilibrium, firms earn zero profits.*

Proof. Same proof as in the case with unproductive tasks—if not, at least one firm can deviate to make themselves strictly better off. ■

As in class/MWG, we first consider the complete information case.

Complete information

Proposition 2.1. *Suppose firms can observe worker productivity. Then, equilibrium contracts,* $(t_i^*, w_i^*)_{i\in\{\ell,h\}}$*, satisfies, for each* $i \in \{\ell, h\}$,

$$w_i^* = \theta_i(1+t_i^*), \ \theta_i = c_t(t_i^*, \theta_i).$$

Proof. By the zero-profit lemma, we know that $w_i^* = \theta_i(1+t_i^*)$. To ensure that each type $i \in \{\ell, h\}$ has no incentive to deviate given w_i^*, it must satisfy, for each $i \in \{\ell, h\}$,

$$t_i^* \in \arg\max_{t_i} \ \theta_i(1+t_i) - c(t_i, \theta_i),$$

Since the first-order condition is necessary and sufficient, it follows that

$$\theta_i = c_t(t_i^*, \theta_i), \ \forall i \in \{\ell, h\}.$$

■

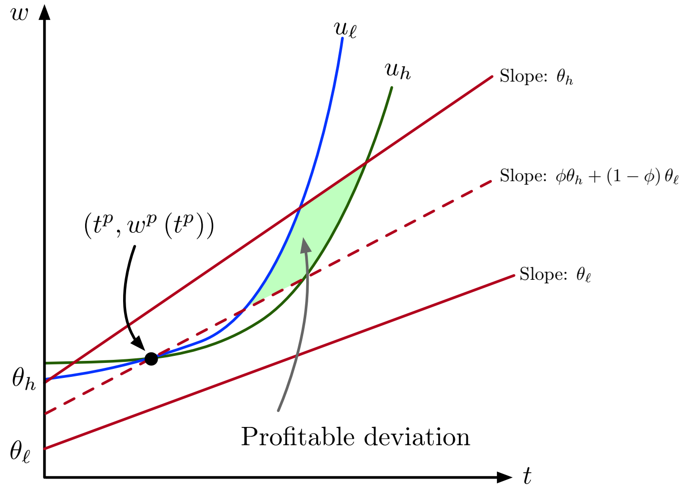
The relevant figure is exactly the same as in the signalling model.

Incomplete information

Lemma 2.2. *No pooling equilibria exist.*

Proof. As in the signalling case, in any pooling equilibrium, $w(p) = (\theta\phi_h + (1-\phi)\theta_\ell)(1+e^p)$. Suppose that a pooling equilibria exist as show in the figure below.

Figure 2.1: No pooling equilibrium exists.



Suppose a firm offers a contract (e', w') in the shaded region. Since indifference curve for the worker is increasing in Northwesterly direction, this will attract only the h type. Moreover, these points lie below the zero profit line associated with h type (recall that firm's isoprofit lines are increasing in the Southeasterly direction) so such deviations would earn the deviating firm strictly positive profit. Observe that the highlighted regions will always exist. ■

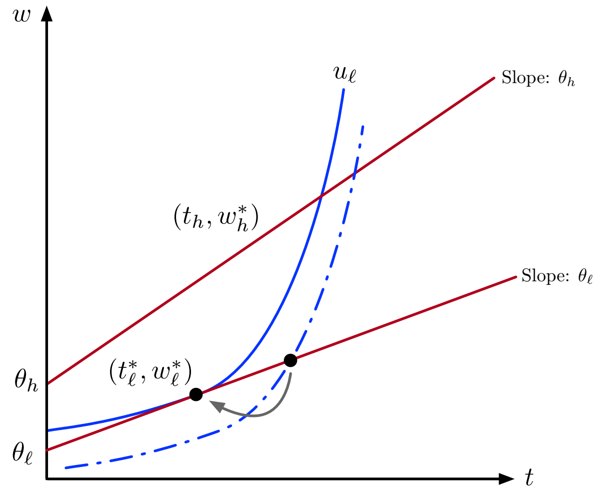
Lemma 2.3. *In any separating equilibria,* $(t_\ell^s, w_\ell^s) = (t_\ell^*, w_\ell^*)$.

Proof. In any separating equilibria, (t_ℓ^s, w_ℓ^s) attracts only the ℓ-type workers. The zero profit lemma implies that

$$w_\ell^s = \theta_\ell\,(1 + t_\ell^s)\,.$$

To pin down t_ℓ^s, consider first the case when $t > t_\ell^s$ (with $w = \theta_\ell\,(1+t)$). But then, one firm could attract all the ℓ type workers by deviating to (t_ℓ^*, w_ℓ^*) since the workers obtain higher utility from such a contract. This incentive exists unless (t_ℓ^*, w_ℓ^*). Note that firms would not choose $e < e_\ell^*$ since this results in strictly negative profits (recall that firm's isoprofit is increasing in the Southeasterly direction).

Figure 2.2: $(t_\ell^s, w_\ell^s) = (t_\ell^*, w_\ell^*)$.



Given w_ℓ^*, we showed, in the complete information case, that t_ℓ^* maximises type ℓ's utility; i.e. (t_ℓ^*, w_ℓ^*) is incentive compatible from the buyer's perspective. By the zero-profit lemma, we know that $w_\ell^* = \theta_\ell (1 + t_\ell^*)$. ■

Lemma 2.4. *In any separating equilibria, (t_h^s, w_h^s) satisfies*

$$w_\ell^* - c(t_\ell^*, \theta_\ell) - c(t_h^s, \theta_\ell) = \theta_h (1 + t_h^s) = w_h^s$$

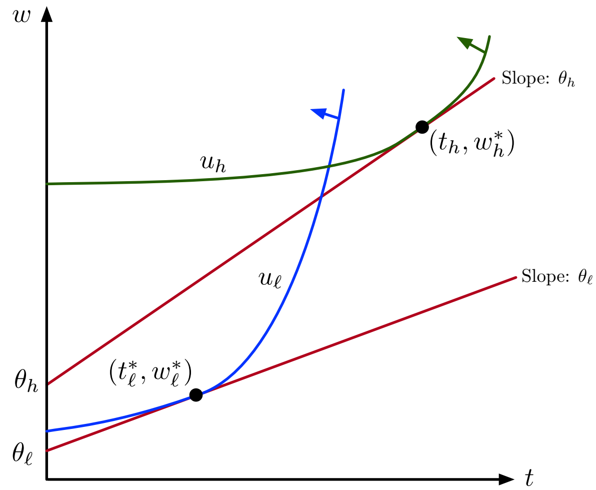
so that (t_h^s, w_h^s) is the intersection of type ℓ's indifference curve through (t_ℓ^s, w_ℓ^s) and the zero-profit line associated with type h workers.

Proof. s ■

Proposition 2.2 (No envy case)**.** *There could exist a separating equilibrium with the same outcome as in the complete information case.*

Proof. Consider the no envy case from the signalling model (where we replace e with t). Observe that the pooling deviation (the intersection of u_h and u_ℓ) results in lower profit for both firms (recall that isoprofit is increasing in the southeasterly direction). If firms offer contracts below the respective zero-profit lines, this would necessarily lower the utility for the workers and so such contracts would not attract any workers. Finally, observe that each type of worker's utilities are maximised in this case.

Figure 2.3: No-envy case.



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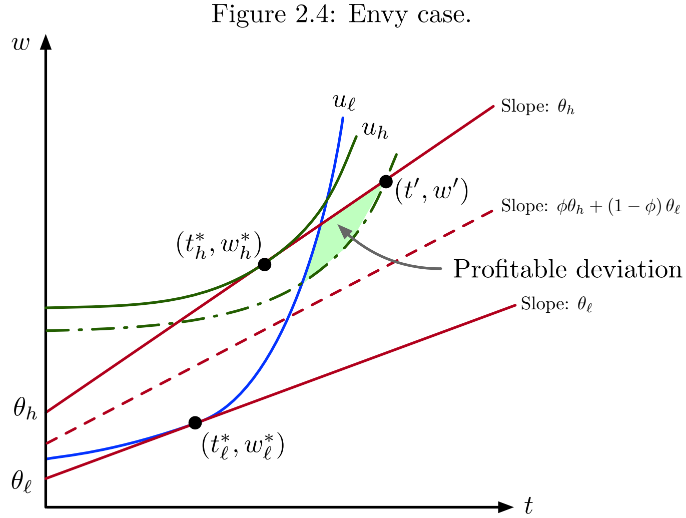
Proposition 2.3. *There may or may not exist a separating equilibrium with the same outcome as in the complete information case depending on ϕ.*

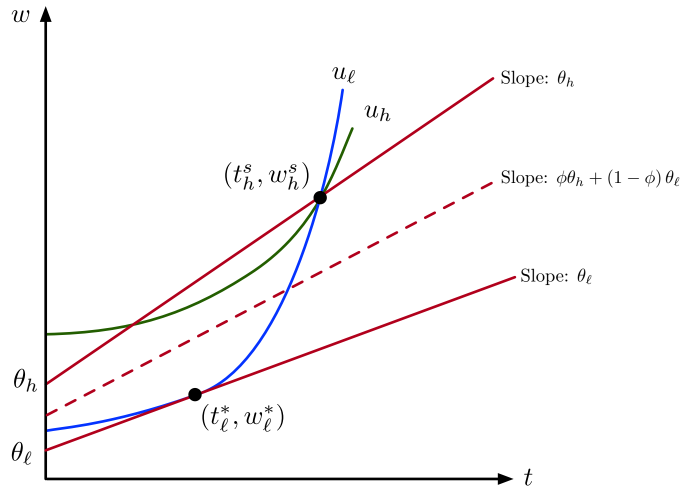
Proof. Consider the envy case as below. Note that, the zero-profit lemma implies that (t_h^s, w_h^s) must lie on the zero-profit line associated with the h type workers; i.e. $w_h^s = \theta_h\left(1 + t_h^s\right)$.

Figure 2.4: Envy case.



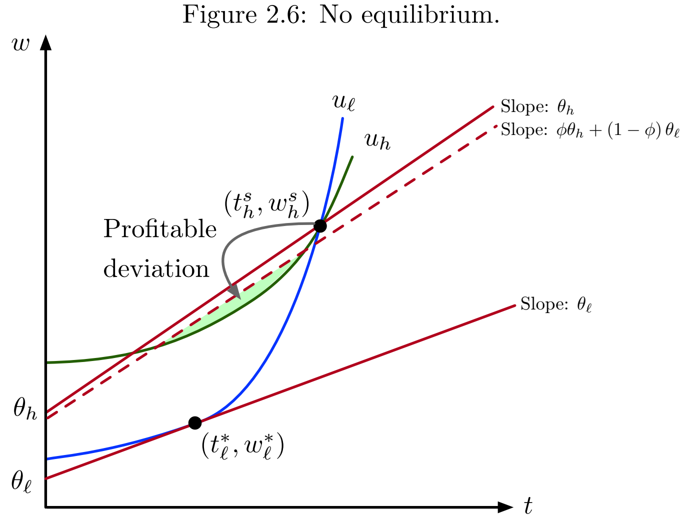
Observe that if firms offer (t_h^*, w_h^*) and (t_ℓ^*, w_ℓ^*), both types would choose the contract (t_h, w_h^*) such that the firms would make strictly negative profits (since the point lies above the pooling zero-profit line). Hence, firms can do strictly better by not offering the contract. Now consider contract such as (t', w'). Then, a firm can offer contract in the shaded region to attract all of the type-h workers yield a strictly greater profits for the deviating firm. These incentives to deviate exists unless (t_h^s, w_h^s) is at the intersection of the type ℓ's indifference curve through (t_ℓ^*, w_ℓ^*) and the zero-profit line associated with type h. As drawn below, ϕ is sufficiently low such that there firms can do no better than this—in particular, there does not exist a profitable pooling deviation.

Figure 2.5: The separating equilibrium.



Now suppose that ϕ is sufficiently high so that the pooling zero-profit line cuts is as shown below. In this case, there exists a profitable pooling deviation—both types would be attracted to a contract in the shaded region but since the deviating contract lies below the pooling zero-profit line, firms would obtain strictly positive profits.. But we already concluded that there cannot be any pooling equilibrium. Hence, in this case, there is no equilibrium to the screening model.

Figure 2.6: No equilibrium.



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3 Problem 3

Consider a principle-agent model with moral hazard in which the principal is risk neutral and the agent is risk averse with $u(w) = \sqrt{100 + w}$. There are two effort levels, $e_h > e_\ell$, with personal cost to the agent of $\psi(e_h) = 1$ and $\psi(e_\ell) = 0$. The agent's expected utility from wage contract $w(\cdot)$ and effort e is $\mathbb{E}\left[\sqrt{100 + w(x)}|e\right] - \psi(e)$.

There are two outcomes. The high-output outcome is $x_2 = 200$ and the low-outcome output is $x_1 = 100$. When the agent exerts low effort, each output is equally likely. When the agent exerts high effort, the probability of high output is 3/4 (and the probability of low output is 1/4). The agent's reservation utility is $\underline{U} = u(0) = 10$.

3.1 Part (a)

Does this distribution satisfy MLRP?

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Let $f(\cdot|e)$ denote the probability density of the outcome conditional on e level of effort exerted by the agent. If f were differentiable in e,

$$f \text{ satisfies MLRP} \Leftrightarrow \frac{f_e(x|e)}{f(x|e)} \text{ is increasing in } x.$$

Since the effort levels are discrete, f is not differentiable in e. Recall the definition of $f_e(x|e)$:

$$\lim_{\Delta \to 0} \frac{f(x|e+\Delta) - f(x|e)}{\Delta} = f_e(x|e).$$

Hence, the discrete version of the MLRP condition is therefore

$$f \text{ satisfies MLRP} \Leftrightarrow \frac{1}{f(x|e)} \frac{f(x|e+\Delta) - f(x|e)}{\Delta} \text{ is increasing in } x.$$

Let $e_H = e + \Delta$ and $e_L = e$, above becomes

$$\begin{aligned} f \text{ satisfies MLRP} &\Leftrightarrow \frac{1}{f(x|e_H)} \frac{f(x|e_H) - f(x|e_L)}{e_H - e_L} \text{ is increasing in } x. \\ &\Leftrightarrow \frac{f(x|e_H) - f(x|e_L)}{f(x|e)} \text{ is increasing in } x. \end{aligned}$$

So this is how we can get the condition in the class notes. Substituting the values into the expression yields that

$$\frac{f(x_2|e_h) - f(x_2|e_\ell)}{f(x_2|e_h)} = \frac{3/4 - 1/2}{3/4} = \frac{1}{3} > -1 = \frac{1/4 - 1/2}{1/4} = \frac{f(x_1|e_h) - f(x_1|e_\ell)}{f(x_1|e_h)}.$$

Since $x_2 > x_1$, we may conclude that the distribution in this question satisfies MLRP.

3.2 Part (b)

Solve for the optimal output-contingent wage contract, $(w^*(x_1), w^*(x_2))$.

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We will assume that w and x (i.e. wages and outputs) are in the same unit. Since the principal is risk neutral, we may, without loss of generality, assume that his utility is given by $u(m) = m$ (where m is in units of output). The principal's problem is the following:

$$\begin{aligned}
\max_{e \in \{e_\ell, e_h\}} \max_{w(x_1), w(x_2)} \quad & \mathbb{E}_x \left[x - w(x) | e \right] \\
\text{s.t.} \quad & \mathbb{E}_x \left[\sqrt{100 + w(x)} | e \right] - \psi(e) \geq \underline{U}, \\
& e \in \arg\max_{\tilde{e} \in \{e_\ell, e_h\}} \mathbb{E}_x \left[\sqrt{100 + w(x)} | \tilde{e} \right] - \psi(\tilde{e}),
\end{aligned}$$

where the first constraint is the participation constraint and the second constraint is the incentive compatibility constraint. This tells us that we ought to solve the problem separately assuming that the principal wishes to induce effort level e_h and e_ℓ.

Low level of effort Suppose the principal wishes to induce level of effort e_ℓ. Clearly, $w(x) = 0$ for all x is the best outcome for the principal in this case. Observe that

$$\mathbb{E}_x \left[\sqrt{100 + w(x)} | e_\ell \right] - \psi(e_\ell) = \frac{1}{4} \sqrt{100 + 0} + \frac{3}{4} \sqrt{100 + 0} - 0 = 10 = \underline{U}$$

so that this satisfies the IR constraint. For the IC constraint, note that

$$\begin{aligned}
\mathbb{E}_x \left[\sqrt{100 + w(x)} | e_h \right] &= \frac{1}{2} \sqrt{100 + 0} + \frac{1}{2} \sqrt{100 + 0} - 1 \\
&< \frac{1}{4} \sqrt{100 + 0} + \frac{3}{4} \sqrt{100 + 0} = \mathbb{E}_x \left[\sqrt{100 + w(x)} | e_\ell \right]
\end{aligned}$$

so that the IC constraint is satisfied too. In this case, the principal's expected profit is

$$\mathbb{E}_x [x | e_\ell] = \frac{1}{2} 100 + \frac{1}{2} 200 = 150.$$

High level of effort The principal's problem to induce e_h is

$$\begin{aligned}
\max_{w(x_1), w(x_2)} \quad & f(x_1 | e_h)(x_1 - w(x_1)) + f(x_2 | e_h)(x_2 - w(x_2)) \\
\text{s.t.} \quad & f(x_1 | e_h) \sqrt{100 + w(x_1)} + f(x_2 | e_h) \sqrt{100 + w(x_2)} - \psi(e_h) \geq \overline{U}, \\
& f(x_1 | e_h) \sqrt{100 + w(x_1)} + f(x_2 | e_h) \sqrt{100 + w(x_2)} - \psi(e_h) \\
& \quad \geq f(x_1 | e_\ell) \sqrt{100 + w(x_1)} + f(x_2 | e_\ell) \sqrt{100 + w(x_2)} - \psi(e_\ell).
\end{aligned}$$

Substituting the values and simplifying the constraint yields

$$\begin{aligned}\max_{w(x_1),w(x_2)} \quad & \frac{1}{4}\left(100 - w\left(x_1\right)\right) + \frac{3}{4}\left(200 - w\left(x_2\right)\right) \\ \text{s.t.} \quad & \frac{1}{4}\sqrt{100 + w\left(x_1\right)} + \frac{3}{4}\sqrt{100 + w\left(x_2\right)} \geq 11, \\ & \frac{1}{4}\left[\sqrt{100 + w\left(x_2\right)} - \sqrt{100 + w\left(x_1\right)}\right] \geq 1.\end{aligned}$$

Now, suppose at the optimum, the IR constraint does not bind. But then, the principal can reduce $w\left(x_1\right)$ and $w\left(x_2\right)$ together in such a way as to leave the IC constraint unchanged.[1] This contradicts that the initial wage contracts were optimal so we conclude that the IR constraint must bind at the optimum. That is,

$$\begin{aligned}\frac{1}{4}\sqrt{100 + w\left(x_1\right)} &= 11 - \frac{3}{4}\sqrt{100 + w\left(x_2\right)} \\ \Leftrightarrow w\left(x_1\right) &= \left(44 - 3\sqrt{100 + w\left(x_2\right)}\right)^2 - 100.\end{aligned}$$

We may rewrite the IC constraint as

$$\begin{aligned}\frac{1}{4}\left[\sqrt{100 + w\left(x_2\right)} - \sqrt{100 + w\left(x_1\right)}\right] &\geq 1 \\ \frac{1}{4}\sqrt{100 + w\left(x_2\right)} - \left(11 - \frac{3}{4}\sqrt{100 + w\left(x_2\right)}\right) &\geq 1 \\ \Leftrightarrow \sqrt{100 + w\left(x_2\right)} &\geq 12 \\ \Leftrightarrow w\left(x_2\right) &\geq 44.\end{aligned}$$

We can write the maximisation problem as

$$\begin{aligned}\max_{w(x_2)} \quad & \frac{1}{4}\left(200 - \left(44 - 3\sqrt{100 + w\left(x_2\right)}\right)^2\right) + \frac{3}{4}\left(200 - w\left(x_2\right)\right) \\ \text{s.t.} \quad & w\left(x_2\right) \geq 44.\end{aligned}$$

Note that the objective function is concave:

$$\begin{aligned}\frac{\partial \Pi}{\partial w\left(x_2\right)} &= \frac{3}{4}\frac{44 - 3\sqrt{100 + w\left(x_2\right)}}{\sqrt{100 + w\left(x_2\right)}} - \frac{3}{4} = 3\left[\frac{11 - \sqrt{100 + w\left(x_2\right)}}{\sqrt{100 + w\left(x_2\right)}}\right] \\ \frac{\partial^2 \Pi}{\partial w\left(x_2\right)\partial w\left(x_2\right)} &= \frac{\partial}{\partial w\left(x_2\right)} 3\left[11\left(100 + w\left(x_2\right)\right)^{-1/2} - 1\right] \\ &= -\frac{33}{2}\left(100 + w\left(x_2\right)\right)^{-3/2} < 0.\end{aligned}$$

[1] To be precise, suppose at the optimum, $\sqrt{100 + w\left(x_2\right)} - \sqrt{100 + w\left(x_1\right)} = c$ for some constant c. Then, we may reduce $\left(w\left(x_1\right), w\left(x_2\right)\right)$ to $\left(\hat{w}\left(x_1\right), \hat{w}\left(x_2\right)\right)$ such that

$$\begin{aligned}\hat{w}\left(x_2\right) &= \left[c + \sqrt{100 + \hat{w}\left(x_1\right)}\right]^2 - 100 \\ \frac{1}{4}\sqrt{100 + w\left(x_1\right)} + \frac{3}{4}\sqrt{100 + w\left(x_2\right)} &= 11.\end{aligned}$$

(For this argument to work, we do need that $100 + w\left(x_i\right) \geq 0$.)

The unconstrained, global maximum is at

$$11 = \sqrt{100 + w\left(x_2\right)} \Leftrightarrow w\left(x_2\right) = 21,$$

which violates the IC constraint. So it follows that the IC constraint must bind at the optimum; i.e.

$$w\left(x_2\right) = 44,$$

which, in turn, implies that

$$w\left(x_1\right) = \left(4\left[11 - \frac{3}{4}\sqrt{100 + w\left(x_2\right)}\right]\right)^2 - 100 = -36.$$

The expected profit for the principle in this case is

$$\mathbb{E}_x\left[x - w\left(x\right)|e_h\right] = \frac{1}{4}\left(100 - \left(-36\right)\right) + \frac{3}{4}\left(200 - 44\right) = 151.$$

Since this is greater than the low effort case, we conclude that the optimal contract is for the principal to induce high level of effort with

$$w^*\left(x_1\right) = -36,\ w^*\left(x_2\right) = 44.$$

Remark 3.1. You can invoke the general result from the paper/class notes that the IC constraint must be binding. But you really should check that this problem here is a special case of the model from the paper.

4 Problem 4

Consider a principal-agent model with moral hazard in which the principal is risk neutral and the agent is also risk neutral. There are two levels of effort, $e_h > e_\ell$, with personal cost to the agent of $\psi(e_h) = 10$ and $\psi(e_\ell) = 0$. The agent's expected utility is $\mathbb{E}[w|e] - \psi(e)$.

There are two outcomes. The high-output payoff is $x_2 = 200$, but the low-outcome payoff represents a loss, $x_1 = -100$. When the agent exerts low effort, each output is equally likely. When the agent exerts high effort, the probability of high output is $3/4$ (and the probability of low output is $1/4$). The agent's reservation utility is $\underline{U} = 0$.

4.1 Part (a)

Solve for an optimal output-contingent wage contract, $(w^*(x_1), w^*(x_2))$. Does the worker earn any surplus above his reservation value?

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In class, we mentioned that, if both parties are risk neutral, then the principal should "sell" the firm to the agent. But before thinking about what this means, let us first try and solve it the "usual" way. As in the previous question, we first write down the principal's problem:

$$\begin{aligned} \max_{e\in\{e_\ell,e_h\}} \max_{w(x_1),w(x_2)} \quad & \mathbb{E}_x[x - w(x)|e] \\ \text{s.t.} \quad & \mathbb{E}_x[w(x)|e] - \psi(e) \geq \underline{U}, \\ & e \in \arg\max_{\tilde{e}\in\{e_\ell,e_h\}} \mathbb{E}_x[w(x)|\tilde{e}] - \psi(\tilde{e}). \end{aligned}$$

Let's suppose that the principal wishes to induce e_h. Then, the problem reduces to

$$\begin{aligned} \max_{w(x_1),w(x_2)} \quad & \frac{1}{4}(-100 - w(x_1)) + \frac{3}{4}(200 - w(x_2)) \\ \text{s.t.} \quad & 3w(x_2) + w(x_1) \geq 40, \\ & w(x_2) - w(x_1) \geq 40. \end{aligned}$$

Notice that it is still the case that lower w's are preferred by the principal—in particular, w's are allowed to be negative here. As before, suppose that, at the optimum, the IR constraint does not bind; i.e. $3w(x_2) + w(x_1) > 40$. Then, reducing $w(x_2)$ and $w(x_1)$ (in proportion of 1:3) would leave the IC constraint unaffected but make the IR constraint bind. This tells us that, at any optimal, it must be that

$$w(x_1) = 40 - 3w(x_2).$$

Substituting this into the objective function gives that

$$\begin{aligned} & \frac{1}{4}(-100 - (40 - 3w(x_2))) + \frac{3}{4}(200 - w(x_2)) \\ & = \frac{1}{4}[-100 - 40 + 3w(x_2) + 600 - 3w(x_2)] = 115. \end{aligned}$$

In particular, observe that the objective function does not depend on w in this case. The IC constraint, however, rearranges to

$$w\left(x_2\right)-\left[40-3w\left(x_2\right)\right]\geq 40 \Leftrightarrow w\left(x_2\right)\geq 20.$$

So we may conclude that any

$$\left(w^*\left(x_1\right),w^*\left(x_2\right)\right)\in\left\{\left(w_1,w_2\right):w_1=40-3w_2,w_2\geq 20\right\}$$

is optimal. It remains to verify that inducing e_h is preferred over inducing e_ℓ. In this case, the principal's problem is

$$\begin{aligned}\max_{w(x_1),w(x_2)}\quad & \frac{1}{2}\left(-100-w\left(x_1\right)\right)+\frac{1}{2}\left(200-w\left(x_2\right)\right)\\ \text{s.t.}\quad & w\left(x_2\right)+w\left(x_1\right)\geq 0,\\ & w\left(x_2\right)-w\left(x_1\right)\leq 40.\end{aligned}$$

Once again, the IR constraint must bind so that the objective can be written as

$$50-\frac{1}{2}\left(w\left(x_1\right)+w\left(x_2\right)\right)=50.$$

Since $115>50$, it follows that the principal indeed wishes to induce e_h.

To answer the question, because the IR constraint binds, the worker does not earn any surplus above his reservation value.

Now, let us go back to this idea of the principal selling the firm to the agent. The problem from moral hazard arises because the principal is unable to observe the agent's effort; in other words, it is a problem that arises from delegation. However, if the agent owns the firm and manages it, no such problem could arise. In other words, the agent can realise the first best outcome, which in this case is given by exerting high level of effort to realise expected profit of

$$\frac{1}{4}\left(-100\right)+\frac{3}{4}\left(200\right)-10=115.$$

(from exerting low effort, the agent would realise a profit of 50 as the owner of the firm). But these numbers look familiar. So, in the complete information case, the firm is worth 115. The principal can then sell the firm to the agent for its expected value, which is achieved by setting

$$w\left(x\right)=x-115=\begin{cases}-215 & \text{if } x=x_1\\ 85 & \text{if } x=x_2\end{cases}.$$

Since $w\left(x_2\right)=95>20$, we know that this is incentive compatible and

$$40-3w\left(x_2\right)=40-3\times 85=-215=w\left(x_1\right)$$

so that it also satisfies IR. That is, selling the firm to the agent for its expected value constitutes *an* optimal wage schedule.

4.2 Part (b)

Now suppose that the agent is still risk neutral, but legally, the principal is not allowed to pay a wage that is negative. [Aside: This is sometimes called the *limited-liability* assumption. It is also equivalent to a situation in which an agent is infinitely risk averse at $w < 0$; i.e., $u(w) = w$ for $w \geq 0$ and $u(w) = -\infty$ for $w < 0$.] Solve for an optimal output-contingent wage contract, $(w^*(x_1), w^*(x_2))$.

What is the expected cost to the principal of the limited-liability constraint?

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Let us suppose that the principal wishes to induce e_h. Then, the problem is now given by

$$\begin{aligned}
\max_{w(x_1),w(x_2)} \quad & \frac{1}{4}(-100 - w(x_1)) + \frac{3}{4}(200 - w(x_2)) \\
\text{s.t.} \quad & 3w(x_2) + w(x_1) \geq 40, \\
& w(x_2) - w(x_1) \geq 40 \\
& w(x_1), w(x_2) \geq 0
\end{aligned}$$

Consider the constraint $w(x_1) \geq 0$. When we did not have the limited-liability constraint, we found that $w(x_1) = 40 - 3w(x_2)$ (this came from the fact that the IR constraint binds). And so

$$\begin{aligned}
w(x_1) \geq 0 &\Leftrightarrow 40 - 3w(x_2) \geq 0 \\
&\Leftrightarrow w(x_2) \leq \frac{40}{3}.
\end{aligned}$$

Clearly, above inequality cannot hold at the same time as the IC constraint, which simplifies to $w(x_2) \geq 0$ when the IR constraint binds. It follows then that, at any optimal with limited-liability constraints in which effort e_h is induced, the IR constraint cannot bind. With this in mind, suppose now that $w(x_1) > 0$ at the optimum. The principal can then reduce $w(x_1)$ by $\epsilon > 0$ small which leads to an increase of $\epsilon/4$ in expected profit. The IC constraint is relaxed b such a change. Moreover, since we know that the IR constraint does not bind, we can always make such small changes. Therefore, we conclude that, at the optimum,

$$w^*(x_1) = 0.$$

This allows us to simplify the problem to:

$$\begin{aligned}
\max_{w(x_2)} \quad & \frac{1}{4}(100) + \frac{3}{4}(200 - w(x_2)) \\
\text{s.t.} \quad & 3w(x_2) \geq 40, \\
& w(x_2) \geq 40.
\end{aligned}$$

The solution to this problem is to set $w^*(x_2) = 40$. This leads to an expected profit of:

$$\mathbb{E}_x[x|e_h] - \left(\frac{1}{4}w(x_1) + \frac{3}{4}w(x_2)\right) = 125 - \left(\frac{1}{4}(0) + \frac{3}{4}(40)\right) = 95.$$

It remains to verify that this leads to greater expected profit that inducing e_ℓ. Since there is no IC constraint to worry about in this case, inducing e_ℓ leads to expected profit of 50 as before (we just set $w(x_1) = w(x_2) = 0$ in this case).

Finally, observe that the firm is worse off by $115 - 95 = 20$ because of the limited-liability constraint.

5 Problem 5

Consider a simple moral hazard where the principal and the agent are risk neutral (i.e. $v'(\cdot) = u'(\cdot) = 1$), there is no uncertainty over output, but now output depends upon the actions of the agent and the principal. Specifically, the agent chooses $e_1 \in [0, 2]$, the principal chooses $e_2 \in [0, 2]$ and output is deterministic:

$$x = e_1 + e_2 \in \mathcal{X} = [0, 4].$$

Assume that the cost of effort for each player is $\frac{1}{2}e_i^2$, so that first-best production requires $e_i^{fb} = 1$ and $x^{fb} = 2$.

5.1 Part (a)

Assume that the principal can offer a contract to the agent which promises $s(x)$ in payment for the outcome x, and the residual profit, $x - s(x)$, is kept by the principal. After the agent accepts the contract, both the principal and the agent simultaneously choose their individual efforts, e_i. Once x is revealed, payments are shared as promised with $s(x)$ going to the agent and $x - s(x)$ going to the principal.

If $s(x)$ is required to be continuously differentiable on $\mathcal{X}$, show that the first best cannot be implemented by the principal. [Hint: remember that the principal must also be given incentives. Another way to think of this problem is that there is a team of two players that need to devise a sharing rule $(s(x), x - s(x))$ to give each incentives.]

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Let us try backward induction. Suppose that the agent has accepted the contract $(s(x), x - s(x))$. The agent's problem is

$$\max_{e_1 \in [0,2]} \; s(e_1 + e_2) - \frac{1}{2}e_1^2,$$

and the principle's problem is

$$\max_{e_2 \in [0,2]} \; (e_1 + e_2) - s(e_1 + e_2) - \frac{1}{2}e_2^2.$$

Since we are allowed to assume that $s(x)$ is continuously differentiable, the first-order conditions tell us that

$$s'(e_1^* + e_2^*) = e_1^*, \; 1 = s'(e_1^* + e_2^*) + e_2^* \Rightarrow e_1^* + e_2^* = 1$$

in any Nash equilibrium of the second stage. But, as given in the question, the first best is for $e_i^{fb} = 1$. Hence, we conclude that outputs are below first best.

Remark 5.1. That s is continuously differentiable is not important for this result. See Holmström (1982) (appendix) for a more general result that applies to discontinuous sharing rules also.

5.2 Part (b)

Suppose that instead of designing a sharing rule, $(s(x), x - s(x))$, the principal can commit to giving some of the output to a third party for some values of x; i.e., the principal can offer $s(x)$ to the agent, and can commit to giving $z(x)$ to a third party, keeping $x - s(x) - z(x)$ for herself. Show that the first best can now be implemented as a Nash equilibrium between the principal and agent and that nothing is giving to the third party *along the equilibrium path.* For this part, you may construct the implementing functions using discontinuous s and z functions. [Hint: think about simple contracts which split the output between principal and the agent for some x, and give the entire output to a third party for other x.]

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Consider the following schedule:

$$s(x) := \mathbb{1}_{\{x=2\}}, \; z(x) := 2\mathbb{1}_{\{x \neq 2\}}.$$

With these sharing rules, both the principal and the agent get 1 if $x = 2$ and 0 otherwise. Assuming that $e_j^* = 1$ in equilibrium, then player i will choose $e_i = 1$ because the payoff is 1/2. Anything else can generate a payoff of zero, at best. The idea here is a general one—if there exists someone who is known not to have shirked (i.e. a third party), the other players can commit to give the output to that party off the equilibrium path, in order to provide incentives for first-best efforts.

5.3 Part (c)

Assume we are back to setting (a) without a third party, and thus the share contracts are limited to $(s(x), x - s(x))$. Now, however, assume that the sets of feasible actions are discrete $e_1 \in \{0.5, 1, 1.5\}$ and $e_2 \in \{0.67, 1, 1.33\}$. Can the first best $e_1^* = e_2^* = 1$ be implemented?

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This is based on Legros and Matthews (1993) who show, among other things, that, if the output function is discrete and generic, the outputs will identify some party that did not shirk, allowing that person to obtain the entire output. This effectively allows the budget breaking that is used in (b).

To see how we can do this here, notice that, if player 1 chooses $e_1^* = 1$, then only three outputs can arise

$$\{1.67, 2, 2.33\}.$$

Furthermore, these outputs cannot arise if player 1 chooses either $e_1 = 0.5$ or $e_1 = 1.5$. Thus, we can assign

$$s(2) := 1, \; s(1.67) := 1.67, \; s(2.33) := 2.33.$$

This effectively punishes player 2 with a payoff of zero for not choosing $e_2 = 2$. Similarly, if player 2 chooses $e_2 = 1$, then outputs can only be

$$\{1.5, 2, 2.5\}.$$

For $x = 2$, we already assigned $s(x) = 1$ and so player 2 also gets 1. For $x = 1.5$ or 2.5, player 1 must have shirked, so we give the output to player 2:

$$s(1.5) \coloneqq 0, \ \ s(2.5) \coloneqq 0.$$

It remains to assign a sharing rule for outputs where both parties have shirked. In these cases, it is enough to split the output 50-50. We now have a Nash equilibrium in which $e_1^* = e_2^* = 1$. If either party chooses an effort other than 1 (while the other party plays the equilibrium effort), they are assured a payoff of zero (at best). Hence, it is optimal to choose $e_i^* = 1$ and obtain a payoff of 1/2.

6 Problem 6

Consider a risk-averse individual with utility function of money $u(\cdot)$ with initial wealth y who faces the risk of having an accident and losing an amount x of her wealth. She has access to a perfectly competitive market of risk-neutral insurers who can offer coverage schedule $b(x)$ (i.e., in the event of loss x, the insurance company pays out $b(x)$) in exchange for an insurance premium, p. Assume that the distribution of x, which depends on accident-prevention effort e, has an atom at $x = 0$:

$$f(x|e) = \begin{cases} 1 - \phi(e) & \text{if } x = 0 \\ \phi(e)\, g(x) & \text{if } x < 0 \end{cases},$$

where $g(\cdot)$ is a probability density. Assume $\phi'(e) < 0 < \phi''(e)$. Also assume that the individual's (increasing and convex) cost of effort, separable from her utility of money, is $\psi(e)$.

6.1 Part (a)

What is the full information insurance contract when e is contractible? Specifically, what is e and $b(x)$?

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This is a problem of hidden action, where the agent's action, e, *only* affects the likelihood of an "accident" occurring—in particular, the extent of the loss, x, is not affected by e. There are two ways to get to the solution, and, by popular request, I will provide both methods.

- ▷ First way, which is consistent with Holmström (1979) is to maximise the expected profit of the insurer subject to a constraint that the expected utility for the buyer is equal to some constant. To reflect the fact that insurers are in perfect competition, we then change the constant such that insurers make zero profits.
- ▷ A second way is to maximise the expected utility of the buyer subject to a nonnegativity constraint for the insurer(s).

Either way, since insurers are risk neutral, we may, without loss of generality, assume that their utility function is linear in money. Notice the particular way in which the question defines $b(x)$—'in the event of loss x, the insurance company pays out $b(x)$'. This means that $b(x)$ is defined for all $x > 0$; i.e. $b : \mathbb{R}_{++} \to \mathbb{R}$; but not for $x = 0$. But since the individual has to pay the premium p no matter the value of p, we can define, $\tilde{b} : \mathbb{R}_{+} \to \mathbb{R}$ as

$$\tilde{b}(x) := \begin{cases} -p + b(x) & \text{if } x > 0 \\ -p & \text{if } x = 0 \end{cases}.$$

We will only deal with the function $\tilde{b}(x)$ in what follows.

Since we are dealing with the full information case when e is contractible, our maximisation problem does not feature the IC constraint. To answer the question from the TA session, you can think of the agent as accepting a contract $(b(\cdot), e)$ from the insurers—the agent will only be paid $b(\cdot)$ only if he exerts level of effort e (and must pay $-\infty$ otherwise, say).

6.1.1 Maximising insurer's expected profit

Each insurer's problem is given by

$$\begin{aligned} \max_{\tilde{b}(x),e} \quad & -\mathbb{E}_x\left[\tilde{b}\left(x\right)|e\right] \\ \text{s.t.} \quad & \mathbb{E}_x\left[u\left(y-x+\tilde{b}\left(x\right)\right)|e\right]-\psi\left(e\right)\geq c, \end{aligned}$$

where we assume that the individual's outside option has util value c. We can think about this as the consumer's value from buying an alternative contract from another insurer (thanks Aleksei for this interpretation!).

The Lagrangian for this problem is given by

$$\begin{aligned} \mathcal{L} &= -\mathbb{E}_x\left[\tilde{b}\left(x\right)|e\right]+\lambda\left(\mathbb{E}_x\left[u\left(y-x+\tilde{b}\left(x\right)\right)|e\right]-\psi\left(e\right)-c\right) \\ &= -\int\tilde{b}\left(x\right)f\left(x|e\right)\mathrm{d}x+\lambda\left[\int_{x\geq0}u\left(y-x+\tilde{b}\left(x\right)\right)f\left(x|e\right)\mathrm{d}x-\psi\left(e\right)-c\right] \\ &= \int_{x\geq0}\left(-\tilde{b}\left(x\right)+\lambda u\left(y-x+\tilde{b}\left(x\right)\right)\right)f\left(x|e\right)\mathrm{d}x-\lambda\psi\left(e\right)-\lambda c. \end{aligned}$$

Observe that $\tilde{b}\left(x\right)$ and $\tilde{b}\left(x'\right)$ for $x\neq x'$ are unrelated—this means that we can maximise the Lagrangian pointwise (i.e. for each value of x). So fix some $x\geq0$, then the first-order condition with respect to $\tilde{b}\left(x\right)$ is

$$\begin{aligned} 0=\frac{\partial\mathcal{L}}{\partial\tilde{b}\left(x\right)} &= -1+\lambda u'\left(y-x+\tilde{b}^{*}\left(x\right)\right) \\ \Leftrightarrow\lambda &= \frac{1}{u'\left(y-x+\tilde{b}^{*}\left(x\right)\right)}. \end{aligned}$$

Observe that the IR constraint has to bind in this case (for any given e, if the IR constraint did not bind, then the insurer can lower $\tilde{b}\left(x\right)$ which increases expected profits; thanks Scott for point this out). So since the that the right-hand side must be a strictly positive constant (with respect to x), the first-order condition holds if and only if

$$\tilde{b}^{*}\left(x\right)=x-\delta,\ \forall x\geq0,$$

for some constant δ. Let us compute the optimal level of effort now. Substituting for $f\left(x|e\right)$ yields

$$\begin{aligned} \mathcal{L}=-&\left(\left[1-\phi\left(e\right)\right]\tilde{b}\left(0\right)+\phi\left(e\right)\int_{x>0}\tilde{b}\left(x\right)g\left(x\right)\mathrm{d}x\right)+ \\ &+\lambda\left(\left[1-\phi\left(e\right)\right]u\left(y+\tilde{b}\left(0\right)\right)+\phi\left(e\right)\int_{x>0}u\left(y-x+\tilde{b}\left(x\right)\right)g\left(x\right)\mathrm{d}x-\psi\left(e\right)\right) \end{aligned}$$

The first-order condition with respect to e is:

$$0 = \frac{\partial \mathcal{L}}{\partial e} = -\phi'\left(e^*\right)\left(-\tilde{b}\left(0\right) + \int_{x>0} \tilde{b}\left(x\right) g\left(x\right) \mathrm{d}x\right) \\ + \lambda\left(\phi'\left(e^*\right)\left[-u\left(y + \tilde{b}\left(0\right)\right) + \int_{x>0} u\left(y - x + \tilde{b}\left(x\right)\right) g\left(x\right) \mathrm{d}x\right] - \psi'\left(e^*\right)\right).$$

Substituting in that $\tilde{b}\left(x\right) = x - \delta$ yield

$$\begin{aligned} 0 =& -\phi'\left(e^*\right) \int_{x>0} xg\left(x\right) \mathrm{d}x \\ &+ \lambda\left(\phi'\left(e^*\right)\left[-u\left(y-\delta\right) + u\left(y-\delta\right)\right] - \psi'\left(e^*\right)\right) \\ =& -\phi'\left(e^*\right) \mathbb{E}_x\left[x|x>0\right] - \lambda\psi'\left(e^*\right) \\ \Leftrightarrow \lambda\psi'\left(e^*\right) =& -\phi'\left(e^*\right) \mathbb{E}_x\left[x|x>0\right]. \end{aligned}$$

Substituting the expression for λ yields

$$\frac{\psi'\left(e^*\right)}{u'\left(y-\delta\right)} = -\phi'\left(e^*\right) \mathbb{E}_x\left[x|x>0\right].$$

To pin down e^* and δ, we use the fact that, since there are many insurers competing, expected profit in equilibrium must be zero; i.e.

$$\begin{aligned} 0 = \mathbb{E}_x\left[\tilde{b}^*\left(x\right)|e^*\right] &= \left[1 - \phi\left(e^*\right)\right]\left(-\delta\right) + \phi\left(e^*\right) \mathbb{E}_x\left[x - \delta|x>0\right] \\ \Leftrightarrow \delta &= \phi\left(e^*\right) \mathbb{E}_x\left[x|x>0\right]. \end{aligned}$$

So e^* is pinned down as

$$\frac{\psi'\left(e^*\right)}{u'\left(y - \phi\left(e^*\right) \mathbb{E}_x\left[x|x>0\right]\right)} = -\phi'\left(e^*\right) \mathbb{E}_x\left[x|x>0\right].$$

Since ϕ is a decreasing and concave function, the right-hand side is decreasing in e^*. ψ is convex so that ψ' is increasing; u is concave so that u' is decreasing $-\phi$, and since $-\phi$ is increasing in e, it follows that the denominator is decreasing in e so that the left-hand side is increasing in e. So we conclude that there is at most one unique solution for e^*.

Remark 6.1. Note that the consumer always has the option to not buy any contract and be fully exposed to the loss. Thus, it must be that

$$c \geq \mathbb{E}_x\left[u\left(y-x\right)|e^*\right] - \psi\left(e^*\right).$$

Since u is concave, by Jensen's inequality,

$$u\left(y - \mathbb{E}_x\left[x|e^*\right]\right) \geq \mathbb{E}_x\left[u\left(y-x\right)|e^*\right],$$

where $\mathbb{E}_x[x|e^*] = \phi(e^*)\mathbb{E}_x[x|x>0]$. With the contract that we found above, the agent's utility is

$$\begin{aligned}\mathbb{E}_x[u(y-\delta)|e^*] - \psi(e^*) &= \mathbb{E}_x[u(y-\phi(e^*)\mathbb{E}_x[x|x>0])|e^*] - \psi(e^*) \\ &= u\left(y - \underbrace{\phi(e^*)\mathbb{E}_x[x|x>0]}_{=\mathbb{E}_x[x|e^*]}\right) - \psi(e^*) \\ &\geq \mathbb{E}_x[u(y-x)|e^*] - \psi(e^*).\end{aligned}$$

Hence, the agent indeed has the incentive to buy this contract! You probably expected this to be true since we have a risk averse agent being offered full insurance. Note that the inequality is, in general, strict. That is, buyers earn a positive surplus in general.

6.1.2 Maximising buyer's expected utility

The buyer's problem is

$$\begin{aligned}\max_{\tilde{b}(x),e} \quad & \mathbb{E}_x\left[u\left(y-x+\tilde{b}(x)\right)|e\right] - \psi(e) \\ \text{s.t.} \quad & -\mathbb{E}_x\left[\tilde{b}(x)|e\right] \geq 0.\end{aligned}$$

The Lagrangian is given by

$$\mathcal{L} = \mathbb{E}_x\left[u\left(y-x+\tilde{b}(x)\right)|e\right] - \psi(e) + \tau\mathbb{E}_x\left[-\tilde{b}(x)|e\right].$$

We know that the constraint must bind so that $\tau > 0$. This means that, whenever we obtain a first-order condition using the Lagrangian above, we can divide both sides by $1/\tau \equiv \lambda$ and obtain the same expression as above. That is, we'll get the same outcome as above! The trick works because, when we solve the the maximisation problem, the zero-profit condition is automatically imposed by the fact that the constraint must bind at the optimum.

6.2 Part (b)

If e is not observable, prove that the optimal contract consists of a premium and a deductible; i.e. $b(x) = x - \delta$. You may assume that a solution exists and the first-order approach is valid.

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6.2.1 Maximising insurer's expected profit

Now that e is not observable/contractible, we need to add the IC constraint to the problem we defined above.

$$\begin{aligned}\max_{\tilde{b}(x),e} \quad & -\mathbb{E}_x\left[\tilde{b}(x)|e\right] \\ \text{s.t.} \quad & \mathbb{E}_x\left[u\left(y-x+\tilde{b}(x)\right)|e\right] - \psi(e) \geq c, \\ & e \in \arg\max_{\tilde{e}} \mathbb{E}_x\left[u\left(y-x+\tilde{b}(x)\right)|\tilde{e}\right] - \psi(\tilde{e}).\end{aligned}$$

We will assume that the first-order approach is valid and will therefore replace the IC constraint with the first-order condition. Since

$$\begin{aligned}
&\mathbb{E}_x\left[u\left(y-x+\tilde{b}(x)\right)|e\right]-\psi(e)\\
&=(1-\phi(e))\,u\left(y+\tilde{b}(0)\right)+\phi(e)\int_{x>0}u\left(y-x+\tilde{b}(x)\right)g(x)\,\mathrm{d}x-\psi(e),
\end{aligned}$$

the first-order condition is

$$0=-\phi'(e)\,u\left(y+\tilde{b}(0)\right)+\phi'(e)\int_{x>0}u\left(y-x+\tilde{b}(x)\right)g(x)\,\mathrm{d}x-\psi'(e)$$

We can write the Lagrangian for this problem as

$$\begin{aligned}
\mathcal{L}=-&\left([1-\phi(e)]\,\tilde{b}(0)+\phi(e)\int_{x>0}\tilde{b}(x)\,g(x)\,\mathrm{d}x\right)\\
&+\lambda\left([1-\phi(e)]\,u\left(y+\tilde{b}(0)\right)+\phi(e)\int_{x>0}u\left(y-x+\tilde{b}(x)\right)g(x)\,\mathrm{d}x-\psi(e)\right)\\
&+\mu\left[-\phi'(e)\,u\left(y+\tilde{b}(0)\right)+\phi'(e)\int_{x>0}u\left(y-x+\tilde{b}(x)\right)g(x)\,\mathrm{d}x-\psi'(e)\right].
\end{aligned}$$

Once again, we can maximise the benefit function, $\tilde{b}(\cdot)$, pointwise over $x>0$, by taking the derivative with respect to $\tilde{b}(\cdot)$ for any given $x>0$. Note

$$\left.\frac{\partial\mathcal{L}}{\partial\tilde{b}(x)}\right|_{x>0}=\phi(e)\left[-1+\lambda u'\left(y-x+\tilde{b}(x)\right)\right]g(x)+\mu\phi'(e)\,u'\left(y-x+\tilde{b}(x)\right)g(x),$$

and setting this equal to zero yields that (assuming that $g(x)>0$ for all $x>0$)

$$\frac{1}{u'\left(y-x+\tilde{b}^*(x)\right)}=\lambda+\mu\frac{\phi'(e)}{\phi(e)}.$$

Assuming that $\lambda,\mu>0$ (i.e. IC and IR constraints are binding), since the right-hand side is constant with respect to x, it must be that $y-x+\tilde{b}^*(x)$ must be constant for all $x>0$. That is,

$$\tilde{b}^*(x)=x-\delta,\ \forall x>0,$$

where δ is some constant (may not be the same as in part (a)).

We also need to think about $\tilde{b}(0)$. The first-order condition with respect to $\tilde{b}(\cdot)$, fixing $x=0$, is

$$0=\frac{\partial\mathcal{L}}{\partial\tilde{b}(0)}=(1-\phi(e))\left(-1+\lambda u'\left(y+\tilde{b}^*(0)\right)\right)-\mu\phi'(e)\,u'\left(y+\tilde{b}^*(0)\right).$$

Setting this equal to zero yields that

$$\frac{1}{u'\left(y+\tilde{b}^*(0)\right)}=\lambda+\mu\frac{-\phi'(e)}{1-\phi(e)}>\lambda,$$

where the inequality follows since $\phi'<0$ by assumption.

Claim 6.1. $\tilde{b}^*(0) > \lim_{x\downarrow 0} \tilde{b}^*(x) =: \tilde{b}_0 \equiv -\delta$.

Proof. By way of contradiction, suppose instead that $\tilde{b}(0) \leq \tilde{b}_0$. By definition, we have

$$\lim_{x\downarrow 0} \frac{1}{u'\left(y - x + \tilde{b}^*(x)\right)} = \frac{1}{u'\left(y + \tilde{b}_0\right)}.$$

Since $u' > 0$ and $u'' < 0$,

$$\begin{aligned}
\tilde{b}^*(0) \leq \tilde{b}_0 &\Leftrightarrow y + \tilde{b}^*(0) \leq y + \tilde{b}_0 \\
&\Rightarrow u'\left(y + \tilde{b}^*(0)\right) \geq u'\left(y + \tilde{b}_0\right) \\
&\Leftrightarrow \frac{1}{u'\left(y + \tilde{b}_0\right)} \geq \frac{1}{u'\left(y + \tilde{b}^*(0)\right)} > \lambda.
\end{aligned}$$

But, since, for all $x < 0$, and $\phi'(e) < 0$,

$$\begin{aligned}
\frac{1}{u'\left(y - x + \tilde{b}^*(x)\right)} &= \lambda + \mu \frac{\phi'(e)}{\phi(e)} < \lambda \\
\Rightarrow \lim_{x\downarrow 0} \frac{1}{u'\left(y - x + \tilde{b}^*(x)\right)} = \frac{1}{u'\left(y + \tilde{b}^*(0)\right)} &< \lambda,
\end{aligned}$$

which is a contradiction. Hence, we must have $\tilde{b}^*(0) > \tilde{b}_0$. ∎

Let $b_0^* > 0$ be such that $\tilde{b}^*(0) = b_0^* - \delta$. Then, the zero-profit condition gives us that

$$\begin{aligned}
0 &= \mathbb{E}_x\left[b(x)\,|e^*\right] = \left[1 - \phi(e^*)\right]\left(b_0^* - \delta\right) + \phi(e^*)\,\mathbb{E}_x\left[x - \delta | x > 0\right] \\
\Leftrightarrow \delta &= \left[1 - \phi(e^*)\right] b_0^* + \phi(e^*)\,\mathbb{E}_x\left[x | x > 0\right].
\end{aligned}$$

To pin things down, we need the first-order condition with respect to e:

$$\begin{aligned}
0 = \frac{\partial \mathcal{L}}{\partial e} = &-\phi'(e^*)\left(-\tilde{b}(0) + \int_{x>0} \tilde{b}(x)\,g(x)\,\mathrm{d}x\right) \\
&+ \lambda\left[-\phi'(e^*)\,u\left(y + \tilde{b}(0)\right) + \phi'(e^*)\int_{x>0} u\left(y - x + \tilde{b}(x)\right)g(x)\,\mathrm{d}x - \psi'(e^*)\right] \\
&+ \mu\left[-\phi''(e^*)\,u\left(y + \tilde{b}(0)\right) + \phi''(e^*)\int_{x>0} u\left(y - x + \tilde{b}(x)\right)g(x)\,\mathrm{d}x - \psi''(e^*)\right].
\end{aligned}$$

Observe that that the coefficient on λ is zero from the first-order condition of the IC constraint. Substituting in for $\tilde{b}(x)$ and $\tilde{b}(0)$ yields that

$$\begin{aligned}
0 = &-\phi'(e^*)\left(-b_0^* + \int_{x>0} x g(x)\,\mathrm{d}x\right) \\
&+ \mu\left[-\phi''(e^*)\,u\left(y + b_0^* - \delta\right) + \phi''(e^*)\,u(y - \delta) - \psi''(e^*)\right] \\
\Leftrightarrow -\phi'(e^*)\left(b_0^* - \mathbb{E}_x\left[x | x > 0\right]\right) &= \mu\left[\phi''(e^*)\left[u(y - \delta) - u\left(y + b_0^* - \delta\right)\right] - \psi''(e^*)\right].
\end{aligned}$$

We also have that e^* satisfies the first-oder condition from the IC constraint; i.e.[2]

$$\begin{aligned} 0 &= -\phi'(e^*)\, u\left(y + \tilde{b}^*(0)\right) + \phi'(e^*)\, u(y-\delta) - \psi'(e^*) \\ \Leftrightarrow \frac{\psi'(e^*)}{\phi'(e^*)} &= u(y-\delta) - u(y + b_0^* - \delta). \end{aligned}$$

Combining the two first-order conditions with respect to $\tilde{b}(\cdot)$ to eliminate λ gives

$$\begin{aligned} \frac{1}{u'\left(y + \tilde{b}^*(0)\right)} - \mu \frac{-\phi'(e)}{1-\phi(e)} &= \frac{1}{u'\left(y - x + \tilde{b}^*(x)\right)} - \mu \frac{\phi'(e)}{\phi(e)} \\ \Leftrightarrow \frac{1}{u'(y + b_0^* - \delta)} - \frac{1}{u'(y-\delta)} &= \mu \frac{-\phi'(e)}{[1-\phi(e)]\,\phi(e)}. \end{aligned}$$

So we have the following system of equations:

$$\begin{aligned} \delta &= [1-\phi(e^*)]\, b_0^* + \phi(e^*)\, \mathbb{E}_x[x|x>0], \\ -\phi'(e^*)\left(b_0^* - \mathbb{E}_x[x|x>0]\right) &= \mu\left[\phi''(e^*)\left[u(y-\delta) - u(y+b_0^*-\delta)\right] - \psi''(e^*)\right], \\ \frac{\psi'(e^*)}{\phi'(e^*)} &= u(y-\delta) - u(y+b_0^*-\delta). \\ \frac{1}{u'(y+b_0^*-\delta)} - \frac{1}{u'(y-\delta)} &= \mu \frac{-\phi'(e)}{[1-\phi(e)]\,\phi(e)}. \end{aligned}$$

This is a system of four equation in four unknowns (δ, b_0^*, e^* and μ), which we can (hopefully) solve.

Let's go over what we found. The payout function we found is as follows

$$\tilde{b}(x) = \begin{cases} -\delta + x & \text{if } x > 0 \\ -\delta + b_0^* & \text{if } x = 0 \end{cases},$$

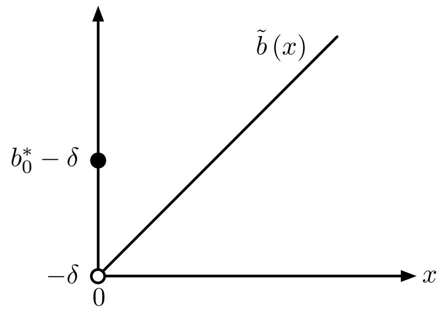
where $b_0^* > 0$. This is similar to what we found in the complete information case (although δ is going to be different between the two) in that there is full insurance conditional on loss. However, the payout function differs when there is no loss (i.e. $x = 0$). The reason this is the case is that, effort exerted by the agent does not affect the extent of the loss; i.e. $g(x)$ does not depend on e. This means that $x > 0$ is not informative about e (conditional on the accident occurring) so that marginal utilities should not be equated across all accident states. The deductible is used as the single lever for incentive.

[2] Note that the coefficient on μ is the second derivative of the IC constraint:

$$\phi''(e^*)\underbrace{\left(u(y-\delta) - u(y+b_0^*-\delta)\right)}_{<0} - \psi''(e^*) < 0,$$

since $\phi''(e) > 0$ and $\psi''(e) > 0$, so the inequality tells us that e^* is indeed a maximum.

Figure 6.1: Optimal insurance contract.



Remark 6.2. To check that the first-order approach is valid, we should verify the second-order condition with respect to $b(\cdot)$ for $x > 0$. Note

$$\left.\frac{\partial^2 \mathcal{L}}{\partial \tilde{b}(x)\,\partial \tilde{b}(x)}\right|_{x>0} = \left[\phi(e)\lambda + \mu\phi'(e)\right] u''\left(y - x + \tilde{b}(x)\right) g(x).$$

To ensure that this is negative, since $u'' < 0$, we need that

$$\phi(e)\lambda + \mu\phi'(e) > 0.$$

But this isn't immediately obvious since $\phi'(\cdot) < 0$. But recall the first-order condition:

$$\begin{aligned} 0 &= \phi(e)\left[-1 + \lambda u'\left(y - x + \tilde{b}(x)\right)\right] g(x) + \mu\phi'(e)\, u'\left(y - x + \tilde{b}(x)\right) g(x) \\ \Leftrightarrow \phi(e) &= \left(\phi(e)\lambda + \mu\phi'(e)\right) u'\left(y - x + \tilde{b}(x)\right) g(x). \end{aligned}$$

Since $\phi(\cdot), u', g(\cdot) > 0$, it follows that $\phi(e)\lambda + \mu\phi'(e) > 0$ so that the second-order condition holds at the optimum. Moreover, note the right-hand side in the first-order condition is monotone in $\tilde{b}(x)$ (for fixed x). Hence, there exists a unique $\tilde{b}(x)$. In other words, $\tilde{b}(x)$ we found must, in fact, be a global maximum.

6.2.2 Maximising buyer's expected utility

Just as in part (a), we could have solve the following problem instead: Alternatively, given that there is perfect competition, we can think of the consumer maximising their expected utility subject to incentive compatibility and a requirement that insurer makes nonnegative profits in expectation. Then, the problem is

$$\begin{aligned} \max_{\tilde{b}(x), e} \quad & \mathbb{E}_x\left[u\left(y - x + \tilde{b}(x)\right)|e\right] - \psi(e) \\ \text{s.t.} \quad & -\mathbb{E}_x\left[\tilde{b}(x)\,|e\right] \geq 0 \\ & e \in \arg\max_{\tilde{e}}\ \mathbb{E}_x\left[u\left(y - x + \tilde{b}(x)\right)|\tilde{e}\right] - \psi(\tilde{e}). \end{aligned}$$

But the exact same argument as above as we used in part (a) means that the answer from solving this will be the same as as above.