Descriptive Units of Heterogeneity: An Axiomatic Approach to Measuring Heterogeneity

(Latest Version)

Willy Chen*

October 2, 2024†

Abstract

I address the challenge of measuring heterogeneity in a system, because existing measures are unsatisfactory for providing cardinal interpretation and comparability across systems. Using an axiomatic approach, I highlight the strengths and limitations of existing measures and generalize the properties that shall be satisfied by alternatives. Using these axioms, I propose a class of measures I term the *descriptive units of heterogeneity* (DUH), which overcome prior limitations without limiting the applicable contexts. DUH achieves the generalized comparability of concentration units while remaining able to reflect changes in the distribution of small groups in the population. I provide several empirical examples to demonstrate that DUH is a valuable tool for researchers studying heterogeneity in systems in various contexts, such as racial composition in a city, revenue shares by product of a firm, and trade flows between countries.

Keywords: Diversity, Concentration, Heterogeneity
JEL: B41, D30, D63, J15, L11

*Michigan State University: willyc@msu.edu

†I thank Hanzhe Zhang, Jon Eguia, and Ce Liu for suggestions and feedback.

1 Introduction

I address the challenge of measuring heterogeneity in a system, because existing measures are unsatisfactory for providing cardinal interpretation and comparability across systems (Nunes et al. 2020; Kvålseth 2022). My objective is to measure the degree of heterogeneity present in a system. Take Apple's revenue stream as a system. Heterogeneity in a system means that there is variation in the outcome of interest, and homogeneity is the lack of variation. For example, all else equal, if iPhone sales and iPad sales are 85% and 5% of Apple's revenue, respectively, then the revenue streams of Apple are less heterogeneous than if iPhone sales and iPad sales were each 45% of Apple's revenue. Such examples are easy to understand, but how would we rank two systems with $(55\%, 35\%, 10\%)$ and $(60\%, 20\%, 20\%)$?

Measuring heterogeneity is, at its heart, a dimension-reduction problem. When studying a complex population, one hopes to simplify the complexity without losing sight of the big picture. Choosing the essential elements to *describe* a system is the key to defining a measure that can tractably identify changes in a system. I define perfect heterogeneity as when all groups in the system have the same number of elements and perfect homogeneity as when all but one group in the system have zero elements. The next tasks are to (1) balance the influence of large groups and the influence of the distribution of elements across groups and (2) define the role of the number of groups in determining heterogeneity.[1]

The heterogeneity of a system is commonly measured in one of two ways: dispersion units or concentration units (James and Taeuber 1985). Dispersion units measure the distance between the observed population distribution and a benchmark distribution and yield concise interpretation at the cost of comparability between some systems. Atkinson (1970) showed that comparing any two systems using the distance between distributions requires significant restrictions on the domain of comparable systems; otherwise, different measures can be made to rank any two systems in opposing ways. Within the bounds of such restrictions, the interpretation of any dispersion unit is simply "the higher the number, the greater the heterogeneity." One well-known example of a dispersion unit is the *Gini coefficient*. While its simple interpretation contributes to its popularity, Schwartz and Winship (1980) point out that many empirical researchers fail to account for these restrictions when using the Gini coefficient to rank income inequality (pp. 2, 8) and such failure can lead to obscured inferences (pp. 9-13).

Concentration units measure the richness of information from select subgroups of the population. These units differ from dispersion units because of the generalized compatibility between systems, but they understate the information provided by small groups. Both the *Herfindahl-Hirschman index* (HHI) and *Shannon's entropy* (SE) are popular examples of concentration units. By emphasizing the influence of large groups, concentration units well reflect changes in heterogeneity in systems when a large group shrinks. However, this feature can cause concentration units to report negligible changes in heterogeneity when there are drastic changes between small groups. Also, because the objective of these units is to measure concentration, they omit any information provided by the presence of zero-groups—i.e., groups with zero elements. These features are not desirable for a measure of

[1]For example, it may be intuitive to say that $\left(\frac{1}{2}, \frac{1}{2}\right)$ is less heterogeneous than $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, but formal reasoning must be defined for such inference.

heterogeneity.

Learning from the strengths and weaknesses of these units, I aim to create an index that (1) yields cardinal interpretation, (2) shares basic properties with existing indices, and (3) accounts for the presence of zero-groups. In this paper, I propose a new yet intuitive way to think about the makeup of heterogeneity by separating it into *contribution from the relative size of the largest group* and *contribution from the evenness of the rest of the groups* (minority) and show that this approach builds on the existing paradigm. I divide the desirable properties of measures of heterogeneity into two groups: *fundamental axioms* and *characterization axioms*.

Fundamental axioms are axioms that induce partial ordering. They are *group symmetry* (SYM), *scale invariance* (INV), and the *principle of diminishing transfers* (PT). The first two axioms state that two systems with the same number of groups are equally heterogeneous if one is a permutation or rescaling of the other. PT pins down how two marginally different systems should be ordered.

Characterization axioms induce total ordering and uniquely characterize measures. I define *independence* (IND), the *principle of proportional Transfers* (PPT), and *contractibility* (CON). IND ensures that the contribution of the largest group and the evenness of minority groups are orthogonal; PPT refines PT and enables my index to yield cardinal interpretation; and CON ensures that adding groups with zero elements (zero-groups) decreases heterogeneity. My new axioms, along with the fundamental axioms, characterize a class of indices that focus on rendering the comparisons between systems *descriptive*: generally comparable, cardinally interpretable, and reflective of changes to small groups. This class of indices is termed the *descriptive units of heterogeneity* (DUH).

Along with the axioms, I put forth the idea of a *reasonable collection of groups*—the set of grouping labels the researcher deems reasonable and comparable—for practical uses of any measure of heterogeneity. To demonstrate DUH use cases, I employ three sets of examples. First, using changes in racial heterogeneity in San Francisco from 1900 to 1990, I examine the case in which a measure's ability to reflect changes in the evenness in minority groups is pertinent, and how DUH satisfies this need better than HHI and SE. Second, using changes in Apple's revenue source from 2012 to 2023, I show that DUH is sensitive to changes in the distribution of minority groups and can reflect the growth of a small group better than existing measures. And third, using changes in international trade flow, I show that concentration units and DUH are complementary in their strengths and should optimally be used with different objectives in mind.

The rest of the paper proceeds as follows. Section 2 discusses existing measures, such as the Gini coefficient, HHI, and SE, and motivates the need for a new measure. Section 3 defines fundamental axioms and characterization axioms and presents my proposed paradigm. Section 4 defines descriptive units of heterogeneity and compares their behavior with that of existing concentration units. Section 5 elucidates best practices for application and presents empirical examples to highlight the strength of the descriptive units of heterogeneity in various settings. Proofs that establish the uniqueness of my index under the proposed axioms are contained in the appendix.

2 Related Literature

This paper is not the first attempt in this strand of literature at an axiomatic characterization of a measure. Rothschild and Stiglitz (1971, 1973) and Schwartz and Winship (1980) axiomatized the Gini coefficient; Chakravarty and Eichhorn (1991) and Kvålseth (2022) the Herfindahl–Hirschman index (HHI); and Nambiar et al. (1992), Suyari (2004), and Chakrabarti et al. (2005) Shannon's entropy (SE). While the Gini coefficient is the simplest form of a unit of heterogeneity in that it operates with the fewest prior assumptions; the HHI and SE are more versatile and are uniquely characterized by their own sets of characterization axioms.

Let Θ be the universe of distinct groups labels. Given some collection of $G \in \mathbb{N}$ groups $\vartheta \subseteq \Theta$, a system $S(Population, \vartheta)$ is the image of a mapping that takes a population and an element from $\{\vartheta \subset \Theta^G \mid |\vartheta| = G\}$ to $\mathbb{Z}_+^G$ such that $S(Population, \vartheta) = (n_1, \ldots, n_g \ldots, n_G)$ where n_g is the positive integer that represents the number of elements in the group $\theta_g \in \vartheta$ in the system. For example, the faculty-student-staff composition of Michigan State University can be thought of as $S(MSU, \{faculty, student, staff\}) = (number\,of\,faculty,\ number\,of\,staff,\ number\,of\,students)$. For simplicity, I use the subscripts to denote the population with an implied ϑ so the system here can be written as S_{MSU}.

The measure of heterogeneity is then a mapping $\Phi : \mathbb{Z}_+^G \to \mathbb{R}$ such that for any two systems S_1 and S_2,

$$\Phi(S_2) \geq \Phi(S_1) \iff S_2 \text{ is weakly more heterogeneous than } S_1.$$

Heterogeneity of a system is the presence of mixture—e.g. the presence of a mix of faculty, staff, and students. Homogeneity is the lack of mixture, meaning that only one or two of these groups are present in the system. I can thus define mathematically what it means for a system to be maximally/minimally heterogeneous.

Definition 1: A system S_{max} of G groups is said to achieve **maximum heterogeneity** if it can be represented as a scalar multiple of the identity vector of size $G \in \mathbb{N}$:

$$S_{max} = (\underbrace{n, n, \ldots, n}_{\substack{G \text{ groups each} \\ \text{with } n \text{ elements}}}) = n \cdot (1, 1, \ldots, 1).$$

Definition 2: A system S_{min} of G groups is said to achieve **minimum heterogeneity/perfect homogeneity** if it can be represented as a $1 \times G$ vector where all but one entries are 0:

$$S_{min} = (0, 0, \ldots, 0, n, 0, \ldots, 0) = n \cdot (0, 0, \ldots, 0, 1, 0, \ldots, 0).$$

The one-dimensional (presence of mixture) nature of this definition makes it convenient for any measure to be bounded between $\Phi(S_{min}) = 0$ and $\Phi(S_{max}) \in \mathbb{R}_{++}$.[2] These units can generally be separated into two categories—dispersion units and concentration units (James and Taeuber 1985).

[2]Typical measures of heterogeneity are bounded between 0 and 1.

Dispersion units.—This class of units compares the observed distribution to a benchmark distribution. The most commonly used dispersion unit is the Gini coefficient. It compares the distribution of an outcome to the uniform distribution and measures heterogeneity based on, e.g., "the top x% of households in the US earns the top y% of income." In a uniformly distributed world (maximum heterogeneity), x and y should be equal, and if it is not, there is inequality. This simple interpretation comes at the cost of a restrictive assumption—the *Lorenz criterion*—in which the order of the Gini coefficients of two systems implies the order of heterogeneity between them if and only if their Lorenz curves do not cross (Atkinson 1970; Marshall et al. 1979; Schwartz and Winship 1980; James and Taeuber 1985).[3] However, the common limitations of dispersion units render them less practical in modern empirical research, for which we may need a general comparison between systems.

Concentration Units.—Both HHI and SE are derived from the *Hannah-Kay* class of concentration units (Hannah and Kay 1977), with perception α and $G \in \mathbb{N}$ firms in industry $S \in D^G$, defined as

$$H_\alpha^G(S) = \begin{cases} \left[\sum_{g=1}^{G} P_g^\alpha\right]^{\frac{1}{\alpha-1}} & \text{if } \alpha > 0,\ \alpha \neq 1 \\ \prod_{g=1}^{G} P_g^{P_g} & \text{if } \alpha = 1 \end{cases}, \quad P_g = \frac{n_g}{n_S}.$$

Consider a system S with G groups. HHI and its complement *Gini-Simpson index* (GSI) of system S are defined as

$$HHI(S) = \sum_{g=1}^{G} \left(\frac{n_g}{n_S}\right)^2 = H_2^G(S), \quad GSI(S) = 1 - HHI(S).$$

SE is defined as

$$SE(S) = -\sum_{g=1}^{G} \left[\frac{n_g}{n_S} \cdot ln\left(\frac{n_g}{n_S}\right)\right] = -ln\left(H_1^G(S)\right).$$

HHI is comparable across systems and yields cardinal interpretation—the probability that 2 random draws with replacement, from the system S, will be from the same group. Nevertheless, it disproportionately accounts for changes in large groups, which is desirable in assessing the market power of firms (Chakravarty and Eichhorn 1991) or the power of political parties (Laakso and Taagepera 1979). Consequently, the simple interpretation of HHI/GSI can obscure obvious differences between systems: Two systems, $(48\%, 47\%, 5\%)$ and $(60\%, 29\%, 11\%)$, will yield 0.5462 and 0.5438 GSI. Even though they have similar deconcentration values, the first system comprises of two large groups and one small group while the second only has one large group. In comparison, the DUHs of these two systems are 0.2865 and 0.3170, which indicates that the second system is much more heterogeneous. This necessitates contextual limitations of HHI/GSI's application to discussions concerning market shares and concentrations (Kvålseth 2022), rather than employing it as a measure of heterogeneity.

[3] Nunes et al. (2020) discuss several other dispersion units in the context of ecology, biology, and medicine.

SE is a popular measure of uncertainty/informativeness in the information theory and rational inattention literature (Sims 2010, 2003; Pomatto et al. 2023). Notice that SE is a negative logarithmic transformation of H_1^G, and hence it (1) loses the probabilistic interpretation of HHI and (2) behaves differently when groups are broken up. A key thing to notice is that in both of these units, zero-groups—i.e., groups with zero elements—do not affect the measure at all. This property is intuitive when used to capture market shares or uncertainty, but it should not be salient when used to measure heterogeneity.

Comparability between systems hinges on what the comparison is intended to capture and whether the two systems are similar enough.[4] To effectively compare heterogeneity between two systems, it is crucial to establish the qualifiers that make them comparable. Including information from all groups, even those with zero elements, provides a baseline for measurement. Thus, comparing systems with varying group counts requires viewing the system with fewer groups as encompassing additional groups with zero elements, which ensures the accurate evaluation of heterogeneity.

Adapting an index to the inclusion of zero-groups is hardly revolutionary. A normalized version of HHI attempts to solve this issue by revising the formula to

$$NHHI(S,G) = \frac{HHI(S) - \frac{1}{G}}{1 - \frac{1}{G}} \in [0,1].$$

This index improves system comparability by accounting for zero-groups via normalization, but it is done at the cost of HHI's probabilistic interpretation. [5]

3 Axioms for Units of Heterogeneity

Fundamental Axioms.—I start with the set of axioms that induce partial ordering. These axioms pin down when two systems are equally heterogeneous and how two systems should be ordered when one is a simple elementary transfer of the other.

[SYM] Group Symmetry. For any permutation $\pi(S)$ of S, $\Phi(S) = \Phi(\pi(S))$.

For example, take $n_a, n_b, n_c \in \mathbb{N}$,

[4]Comparing apples with oranges might not necessarily be nonsensical, as the popular idiom suggests. In the proper context, I can compare an apple to an orange in its density, brightness of color, amount of sugar per milliliter of water, etc. I can even compare an apple to an orange based on how good each fruit is at being citrus (clearly, the apple will lose). What I cannot do is say that an apple is x times denser than an ice cube and an orange is y times rounder than a bowling ball.

[5]Consider the following two systems:

$$S_1 = (0.4, 0.4, 0.2) \text{ and } S_2 = (0.5, 0.3, 0.1, 0.1).$$

These two systems have the same HHI (0.36), but they have different NHHIs ($NHHI(S, G = 3) = 0.04$ and $NHHI(S', G = 4) \approx 0.15$). By observation, it may not be clear whether S_1 and S_2 are equally homogeneous, but a comparison of these two NHHIs is unlikely to be convincing. Once we account for zero-groups and make $S_1 = (0.4, 0.4, 0.2, 0)$, the NHHIs of the two systems are the same (0.15)—just like their HHIs—but the level of heterogeneity can no longer be intuitively interpreted.

S. / G.	S_1		S_2		S_3
A	n_a		n_b		n_c
B	n_b	$\sim$	n_a	$\sim$	n_b
C	n_c		n_c		n_a

SYM is an intuitive axiom, because it enables the index to focus on the distribution of groups in a system rather than the labels of individual groups. Satisfying SYM means that any two systems with the same number of groups can be compared. By focusing on distribution over the same number of groups, SYM enables comparisons between systems that map from different collections of groups, so long as the two collections have the same number of groups.

[INV] Scale Invariance. For any system S and a scalar $\lambda \in \mathbb{R}_{++}$, $\Phi(S) = \Phi(\lambda \cdot S)$.

INV allows for a further generalization of the systems to allow groups with sizes of any nonnegative real number. This axiom ensures that the index reflects only the distribution of sizes in a system rather than the absolute sizes of the groups.

[PT] Principle of transfers. Take $S = (n_1, \ldots, n_{g-1}, n_g, n_{g+1}, \ldots, n_G)$ such that $\forall k \in \{1, \ldots, G\}$, $n_k \geq n_{k+1}$ and let e_i^G be a $1 \times G$ vector where the ith element is 1 and the rest are 0. Then $\Phi(S) < \Phi(S - e_i + e_k)$ if $i > k$, $n_i - 1 \geq n_{i+1}$, and $n_{k-1} \geq n_k + 1$.

In plain words, *holding the order of groups constant*, a transfer from a larger group to a smaller group increases heterogeneity. For example, take $n_a, n_b, n_c \in \mathbb{N}$ such that $n_a > n_b > n_c$ and $\varepsilon < \min\left\{n_b - n_c, \dfrac{n_a - n_b}{2}\right\}$.

S. / G.	S_1		S_2		S_3
A	n_a		$n_a - \varepsilon$		$n_a - \varepsilon$
B	n_b	$\prec$	$n_b + \varepsilon$	$\prec$	n_b
C	n_c		n_c		$n_c + \varepsilon$

This axiom originated as the *principle of transfers*, first formulated by Dalton (1920): "If there are only two income-receivers, and a transfer of income takes place from the richer to the poorer, inequality is diminished" (p. 351). For a measure of heterogeneity, I believe that the decrease in inequality should be increasing in the difference in the proportions of the two groups, adding the "diminishing" to the axiom formulated by Dalton (1920).[6]

A New Way to Think About Heterogeneity.—Gini, HHI, and SE treat each group identically, which ensures SYM. However, SYM can also be satisfied by treating the same types of groups identically rather than each group. Notice that for any system, there is always the largest group and the remaining groups—so an index that treats the largest group differently than the rest of the groups in the system can still satisfy SYM. The advantage of this new

[6] The richer in Dalton's case translates to a group that has more elements.

perspective on heterogeneity is that if the largest group's influence on the index is orthogonal to the rest, changes in the one-dimensional index can be equated to changes in either group while keeping the other constant.

For convenience, I refer to groups that are not the largest group the *minority groups*. Under this new paradigm, an index of heterogeneity is then $\Phi = \Phi(\varphi, \psi)$, where $\varphi(P_1)$ is the influence of the relative size of the largest group and $\psi(P_2, \ldots, P_G)$ the influence of the relative sizes of the minority groups.

Characterization Axioms.—My next step is to build on the fundamental axioms to gain total ordering and uniquely characterize my measure. In prior literature, this is done by imposing cardinal interpretation and prescribing how the measure changes when groups are broken up into smaller groups.

[IND] Independence. The influence of the relative size of the largest group should be independent of the relative sizes of the minority groups, and vice versa.

$$\varphi(n_1, n_2, \ldots, n_G) = \varphi\left(\frac{n_1}{n_1 + \cdots + n_G}\right) = \varphi(P_1).$$
$$\psi(n_1, n_2, \ldots, n_G) = \psi(n_2, n_3, \ldots, n_G) = \psi\left(P_2, \ldots, P_G\right).$$

IND enables us to determine what an equivalent change in heterogeneity would look like if it resulted solely from changes in either the size of the largest group or the evenness of the minority groups; in reality, however, it often lies between these extremes. To accommodate an arbitrary number of groups, ψ needs to capture features of the minority groups' distribution. I define ψ to be a function of the distance between the observed distribution in the minority groups and the ideal uniform distribution in the minority groups.

Definition 3: A function $\psi_p : \mathbb{R}_+^{G-1} \to \mathbb{R}_+$ is a measure of evenness in minority groups' distribution if it is of the following form:

$$p \in \mathbb{R}_+,\ \psi_p(S) = \psi\left(\tilde{P}_2, \ldots, \tilde{P}_G\right) = 1 - \left(\sum_{g=2}^{G} \left|\tilde{P}_g - \frac{1}{G-1}\right|^p\right)^{\frac{1}{p}}.$$

Proposition 1: Consider an index $\Phi_p = \Phi(\varphi, \psi)$ that satisfies SYM, INV, and IND. Holding P_1 constant, if Φ is strictly increasing in ψ_p, then $\Phi(\varphi(P_1), \psi)$ satisfies PT if and only if $p > 1$.[7]

Having formally defined ψ_p, I want to pin down φ with an axiom that fully describes changes in heterogeneity when ψ is fixed. In pursuit of an index whose changes are easy to interpret, I propose a minimalist refinement of PT that builds on the notion of IND: the principle of proportional transfers.

[7]This proposition suggests that we need to be careful regarding the functional form of $\varphi(P_1)$. To satisfy PT, it must be that when there is a transfer from the largest group to the minority group, the increase in $\varphi(P_1)$ must dominate the decrease in ψ_p in the case in which evenness decreases. Also, if $G = 3$, this proposition holds with $p \geq 1$.

[PPT] Principle of proportional transfers. Take $S = (n_1, \ldots, n_G)$ s.t. $\forall k \in \{1, \ldots, G\}$, $n_k \geq n_{k+1}$. Let $n_S = n_1 + \cdots + n_G$ and $\tilde{P}_g = \frac{n_g}{n_S - n_1}$, and let e_i^G be a $1 \times G$ vector where the ith element is 1 and the rest are 0. Then

$$\frac{n_1 - c}{n_S} = \left(\frac{n_1}{n_S}\right)^{\alpha} \text{ and } n_1 - c \geq n_2 + \tilde{P}_2 \cdot c$$
$$\Rightarrow \Phi\left(S - c \cdot e_1^G + \sum_{g=2}^{G} \tilde{P}_g c \cdot e_g^G\right) = \alpha \cdot \Phi(S)$$

In plain words, *holding the order of groups constant*, a transfer from the largest group proportionally to the minority groups that reduces P_1 to $(P_1)^{\alpha}$ increases heterogeneity by a factor of α. For example,

S. / G.	S_1		S_2
θ_1	P_1		P_1^{α}
θ_2	P_2	$\prec$	$P_2 + \frac{P_2}{P_2+P_3}(P_1 - P_1^{\alpha})$
θ_3	P_3		$P_3 + \frac{P_3}{P_2+P_3}(P_1 - P_1^{\alpha})$

Then

$$\Phi(S_1) < \alpha\Phi(S_1) = \Phi(S_2).$$

PPT gives changes in the index a simple interpretation. If $2\Phi(S_1) = \Phi(S_2)$, we can say that S_2 *is twice as diverse as* S_1 because it has the same heterogeneity it would if the largest group proportion in S_1 shrunk to the power of 2 while still being the largest group, holding the evenness in the minority groups' distribution equal.

[CON] Contractibility. Φ satisfies contractibility if adding one zero-group to a system of G groups decreases the heterogeneity of the system.

$$\Phi(n_2, \ldots, n_G, 0) < \Phi(n_2, \ldots, n_G).$$

A practical implication of CON is that the comparison between systems with a unit assumes that the two systems have the same number of groups, even if some groups have 0 elements. For example, $\left(\frac{1}{2}, \frac{1}{2}, 0\right) \prec \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Axioms used to characterize HHI and SE are presented in Appendix B.

4 Descriptive Units of Heterogeneity

Definition: Let $n_1 \geq n_2 > 0$, $P_1 = \frac{n_1}{n_1+n_2+\cdots+n_G}$, and $\tilde{P}_g = \frac{n_g}{n_2+\cdots+n_G}$, $g > 1$. The descriptive units of heterogeneity (DUH) of the system S with $G \geq 2$ groups are:

$$DUH(S) = \frac{ln(P_1)}{ln(G)} \cdot \left[\left(\sum_{g=2}^{G} \left| \tilde{P}_g - \frac{1}{G-1} \right|^p \right)^{\frac{1}{p}} - 1 \right].$$

Theorem: Descriptive units of heterogeneity are a unique class of units, up to positive scalar multiplication, that satisfy group symmetry, scale invariance, independence, the principle of diminishing transfers, the principle of proportional transfers, and contractibility and uses ψ_p to incorporate the measure of evenness.

Proof sketch: (1) I show that any index using an ordered vector of group proportions as inputs satisfies SYM and INV. (2) I then show that any $\Phi(\varphi, \psi)$ that satisfies SYM, INV, and PPT must be a positive monotonic transformation of $\frac{1}{P_1}$. (3) I show if an $\Phi(\varphi, \psi)$ satisfies INV, PPT, and IND, then $\Phi = \varphi \cdot \psi$. (4) I show that if $\Phi(\varphi, \psi)$ satisfies INV, IND, and PPT, then $\varphi = -cln(P_1)$, $c \in \mathbb{R}_{++}$. (5) I show that the theorem is true by taking the derivative of the extreme case in which P_1 is close to 1 and $\psi = 1$ with respect to a transfer from the largest group to the second-largest group.

Notice that the parameter p controls how much evenness is reflected in DUH through ψ_p. As p increases, minority groups that are farther away from $\frac{1}{G-1}$ take more weight. As $p \to \infty$,

$$d_p \to d_\infty = \sup_{g \in \{1,\ldots,G\}} \left\{ \left| P_g - \frac{1}{G-1} \right| \right\}.$$

Figure 1 shows how the progression of DUH changes with different p's. As p increases, the contribution of the evenness in the minority takes less weight. When $p = 2,000$, the effects of transfers between minority groups become negligible, which causes it to look like P_1 dominates evenness in calculating DUH.

Figure 1: DUH with Different p

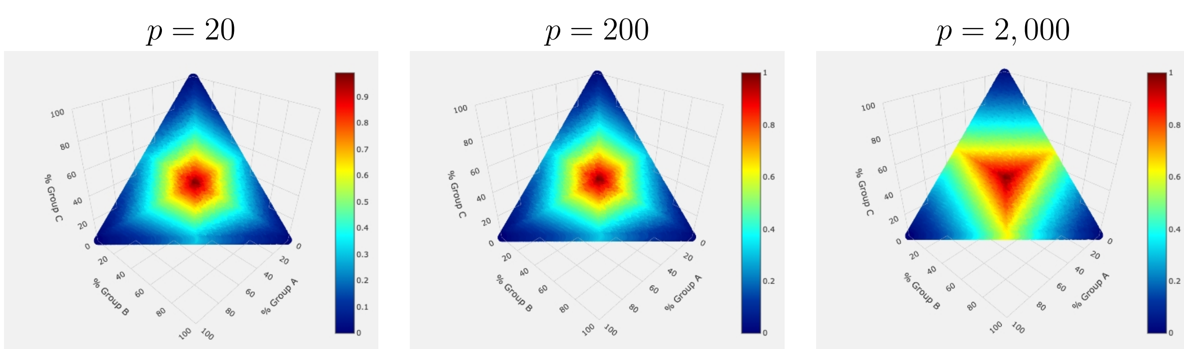


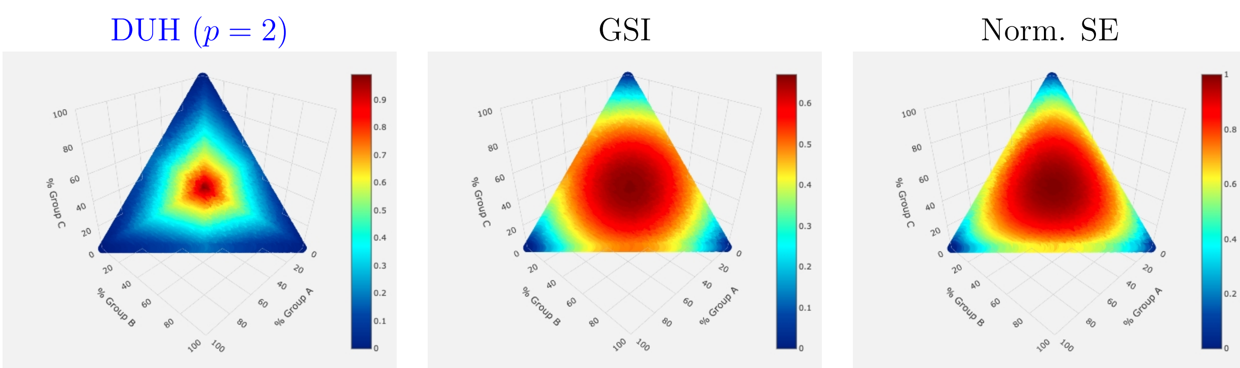
Table 1 outlines the axioms discussed and whether Gini, DUH, HHI, or SE satisfy them. Notice that DUH can be considered a refined Gini coefficient with generalized comparability across systems consisting of discrete and unordered groups. DUH builds on the partial ordering of the Lorenz criterion to induce a total order that enables comparisons across systems by further refining PT and incorporating ψ_p. DUH is characterized differently than concentration units, because it focuses on the overall distribution without losing comparability. Nevertheless, these indices are complementary, because they satisfy different objectives. If you want an index that is invariant to the presence of zero-groups, then you should use either HHI or SE, depending on whether you want to satisfy the replication principle or Shannon's additivity.[8] If you want an index that is subject to the presence of zero-groups and yields cardinal interpretation, then DUH should be used.

Table 1: Measures and Axioms

Type	Axiom	Gini	DUH	HHI	SE
Fundamental	Type symmetry	✓	✓	✓	✓
	Scale invariance	✓	✓	✓	✓
	Principle of transfers	✓	✓	✓	✓
Characterization	Independence	×	✓	✓	✓
	Principle of proportional transfers	×	✓	×	×
	Contractibility	✓	✓	×	×
	Expandability[8]	×	×	✓	✓
	Replication principle[8]	×	×	✓	×
	Shannon's additivity[8]	×	×	×	✓

Figure 2 compares DUH with the concentration units where $G = 3$. The tetrahedrons of each measure below show how each measure changes as the distribution of groups becomes more heterogeneous.[9] The centers of the triangles represent a perfectly heterogeneous system, and the vertices of the triangles represent perfectly homogeneous systems.

Figure 2: Differences between DUH, GSI, and SE



[8]See Appendix B for details on these axioms and characterizations.
[9]SE is normalized to be between 0 and 1 by dividing it by $ln(3)$.

5 Practical Uses of DUH

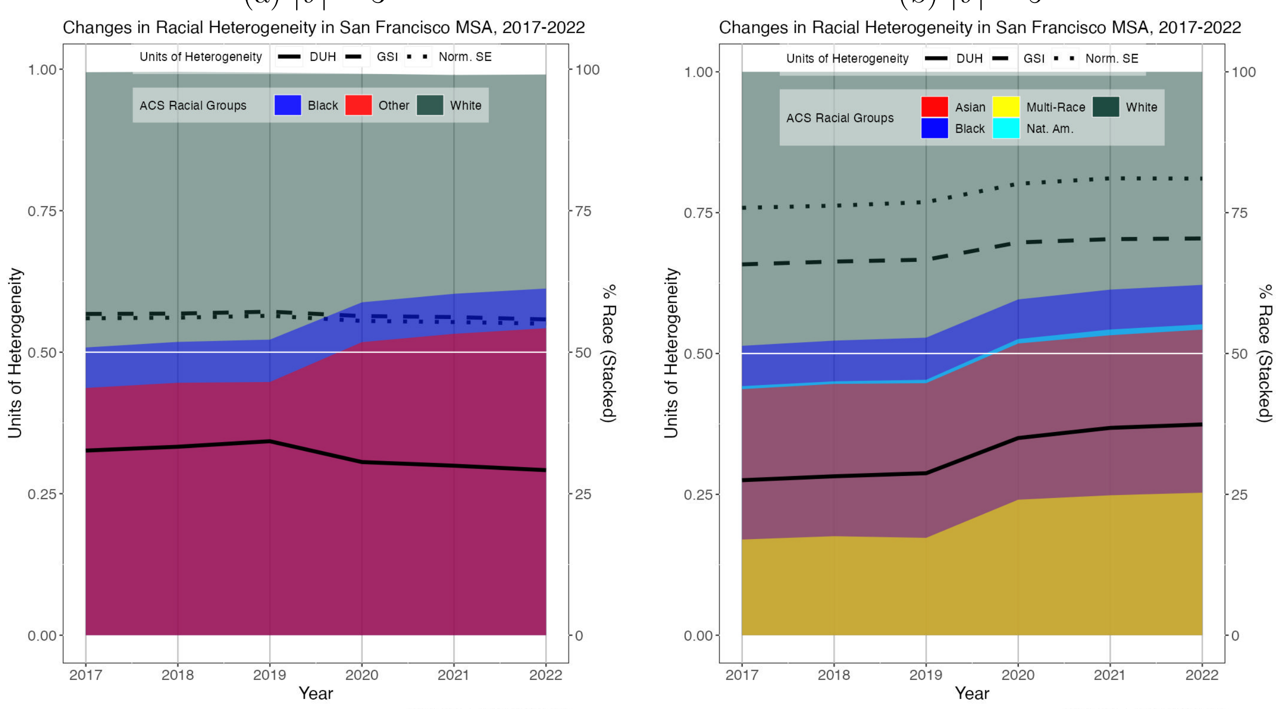
DUH, as a simple description of heterogeneity in systems, can be used in various contexts with discrete distributions over unordered groups in a system. This section presents several empirical examples in which the strength of DUH is demonstrated. Since HHI is a measure of concentration/homogeneity between 0 and 1, I use GSI (=1-HHI) here to simplify the comparisons. Similarly, SE is normalized to be between 0 and 1 to simplify the comparisons.

5.1 Reasonable Collection of Groups ϑ

Recall that a system is the image of a mapping from the population and a collection of group labels $\vartheta \subseteq \Theta$ to a vector whose length is $|\vartheta|$. The set ϑ must be handled with care because each $\theta \in \vartheta$ must be similar/comparable to each other. Figure 3 illustrate this idea with a practical example. These two figures present the racial composition of San Francisco metropolitan statistical area (MSA) from 2017 to 2022 using the American community survey (ACS) 1-year data (Ruggles et al. 2024). In part (a), ϑ is selected to be {White, Black, Other} and in part (b) the *Other* group is split into 3 subgroups, yielding ϑ = {White, Black, Asian, Native America, and Multi-Race}.

Figure 3: Example for the Importance of Reasonable Collection of Groups ϑ

(a) $|\vartheta| = 3$ (b) $|\vartheta| = 5$



In (a), the heterogeneity of this system is somewhat stable due to the influence of the shrinkage in the White population and the increase in the Other population. The heterogeneity started to decrease post-2019 when the White population became a minority group and the Other population became the largest group. This change shows the importance of SYM, which allows researchers to study heterogeneity as a distributional property free of labels. However, the story is different once ϑ is redefined to further capture distributional changes in subgroups. Figure 3 shows that when the Asian population and multi-race population are considered separately, heterogeneity increases post-2019, because the groups, at a glance, are proportionally growing. Such distributional changes are what PPT is designed to reflect.

When measuring heterogeneity in a system, we must understand the implications of choosing ϑ. Determining the elements of ϑ is a framing problem and also a judgment call by the researcher. Just as the use of Gini coefficient requires the Lorenz criterion, the use of any units of heterogeneity requires justifying the reasonable groupings. In the examples here, the simple split of a subgroup changed the inference, and serves as an excellent reason why the reasonable collection ϑ must be chosen with care.

Keeping this in mind, let us consider examples regarding when DUH can be used and why they should be used. For simplicity, I will use my preferred version of DUH, where $p = 2$ so that ψ uses the Euclidean distance.

5.2 Examples

The examples here demonstrate how DUH can be useful for interpreting heterogeneity in different environments. The examples are arraigned to progress in the size of the reasonable collection of groups to show that DUH is sensitive only to the two components— the size of the largest group and the evenness of minority groups—and not to the number of groups.

5.2.1 Using DUH for Racial Heterogeneity

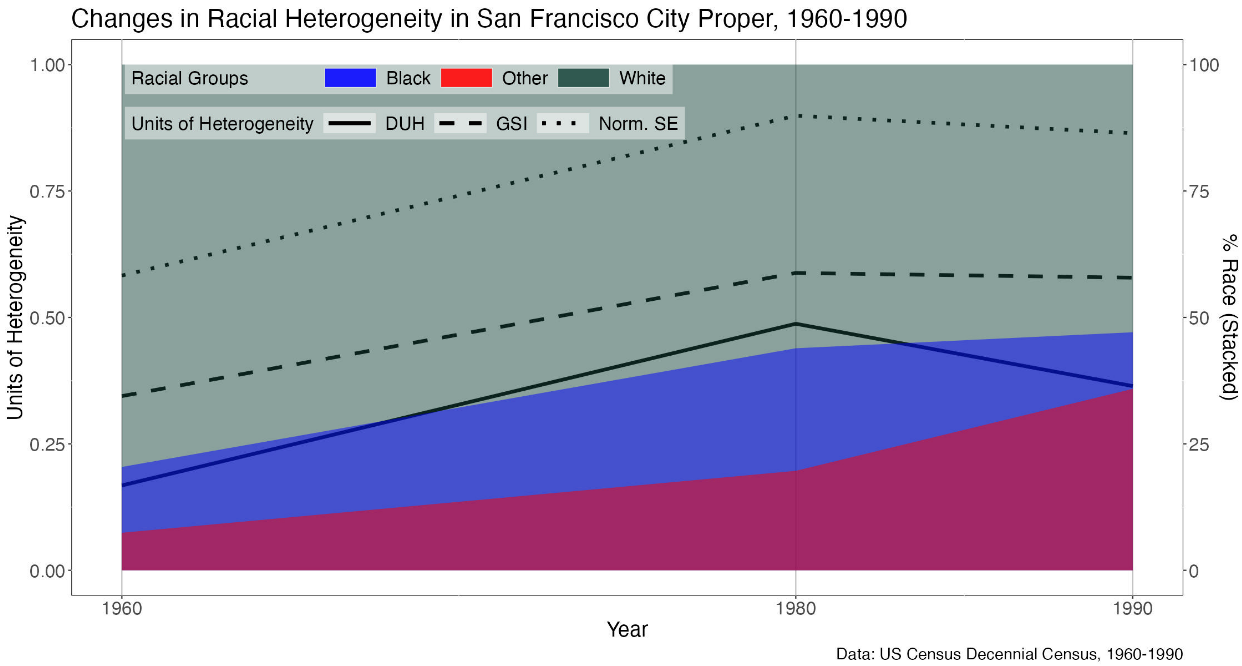
The first example uses DUH to measure racial heterogeneity when there are only 3 groups—White, Black, and Other—in the reasonable collection ϑ. Figure 4 shows the progression of racial heterogeneity in San Francisco city proper from 1960 to 1990 using decennial Census data from IPUMS USA (Ruggles et al. 2024).[10]

From 1960 to 1980, the population of the largest group (White) decreased by about 20 percentage points and the population of Black and Other grew approximately proportionally. From 1980 to 1990, the White population decreased slightly, but the other population (mostly Asian) grew so much that it rendered the minority groups' distribution much less even. In this case, GSI indicated only a slight decrease in heterogeneity, while the larger decrease in SE reflects more of this change in the minority groups' distribution. DUH, on the other hand, generally follows the same trends as GSI and SE, yet can reflect much more of the decrease in evenness of the minority groups' distribution.

Recall that the main weakness in GSI and SE is that the size of the influence from changes in a group positively correlates with the size of the group. This example shows that DUH can dial back the correlation and reflect changes in the evenness of minority groups.

[10]Due to Census coding of the inner-city variable, data are missing for 1970.

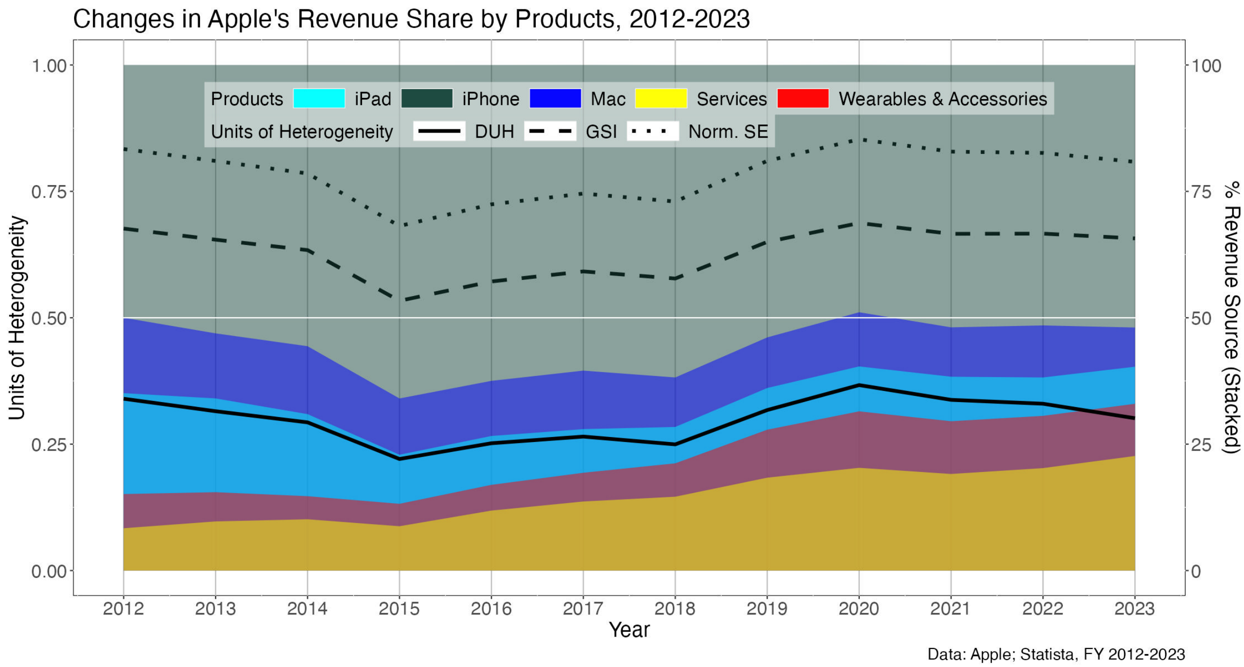
Figure 4: Comparisons between Different Units for Racial Heterogeneity



5.2.2 Using DUH for Revenue Heterogeneity

The second example uses DUH to proxy how much a firm relies on specific products for its revenue. In this example, there are 5 groups—iPhone, iPad, Mac, Wearables & Accessories, and Services—in the reasonable collection for Apple's revenue. Figure 5 illustrates how DUH compares with GSI and SE in a space that often uses units of concentration to examine Apple's revenue sources by product (Apple and Statista 2024). For the most part, the three units move in the same way. However, notice that from 2020 to 2023, Apple's revenue share for Services, as well as Wearables (like Apple watch) & Accessories (like AirPods), grew without diminishing the revenue share of iPhones. This decrease in the evenness of minority groups is captured by a continuous and sizable decrease in DUH, while decreases in GSI in this period are limited.

Figure 5: Comparisons between Different Units for Revenue Heterogeneity



5.2.3 Using DUH for Trade Share Heterogeneity

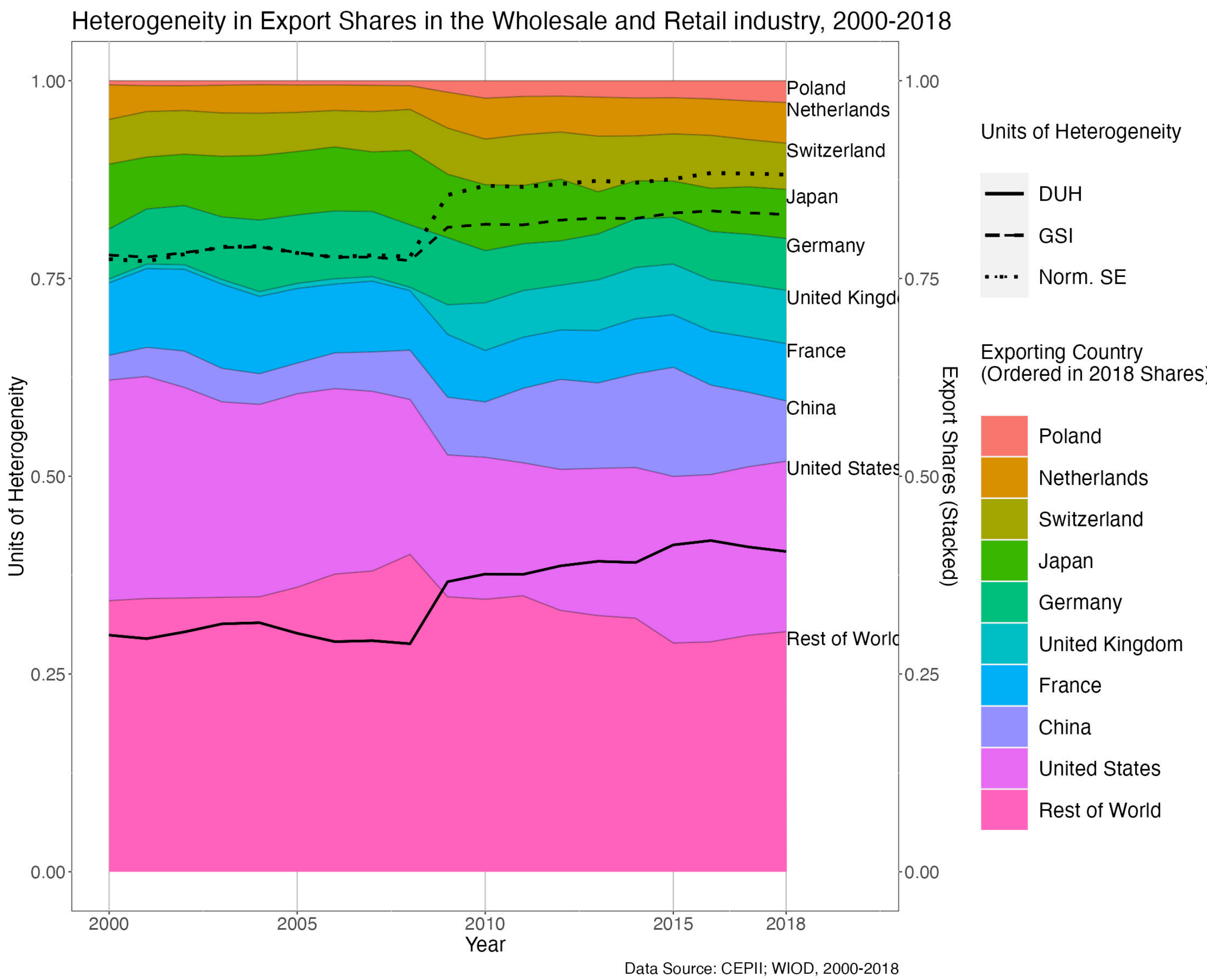
The last set of examples uses DUH in the environment in which HHI and SE are typically used: trade. In this setting, it is reasonable to focus on the concentration and not the heterogeneity in the system. As I progress through the three examples, readers should notice why a measure of heterogeneity does not necessarily need to agree with a measure of concentration. Using trade flow data from CEPII-BACI (Gaulier and Zignago 2010) and the World Input-Output Database (WIOD) (Timmer et al. 2012), I calculated the export share of each of the 29+1 countries,[11] import shares from each of the 28+1 countries to the US, and export shares to each of the 28+1 countries from the US.[12]

[11] I use 29 of the 164 member countries of the World Trade Organization as well as non-WTO countries, combined as the rest of the world (RoW).

[12] Special thanks to Erin Eidschun, an economics PhD student at Boston University, who graciously provided a cleaned and organized version of this data set. Since these are only illustrative examples, I am less concerned about any inaccuracies.

Within-Industry Export Shares of Countries.[13]—Figure 6 graphs the global export shares of the top 9+1 countries (in 2018) for *Wholesale and Retail*. In 2009, there was a drastic increase in all three measures, coinciding with both the significant growth in export shares of the United Kingdom and Poland and shrinkage in the share of the rest of the world (RoW). From 2010 to 2015, export shares for the US and China grew as the shares of RoW decreased. This growth is well reflected in DUH, but changes in GSI and SE are minimal. Similarly, from 2016 to 2018, China's export share shrunk as the export share of the RoW grew—resulting in a decrease in DUH—while changes in GSI and SE remained minimal.

Figure 6: Comparisons between Different Units for International Export Heterogeneity

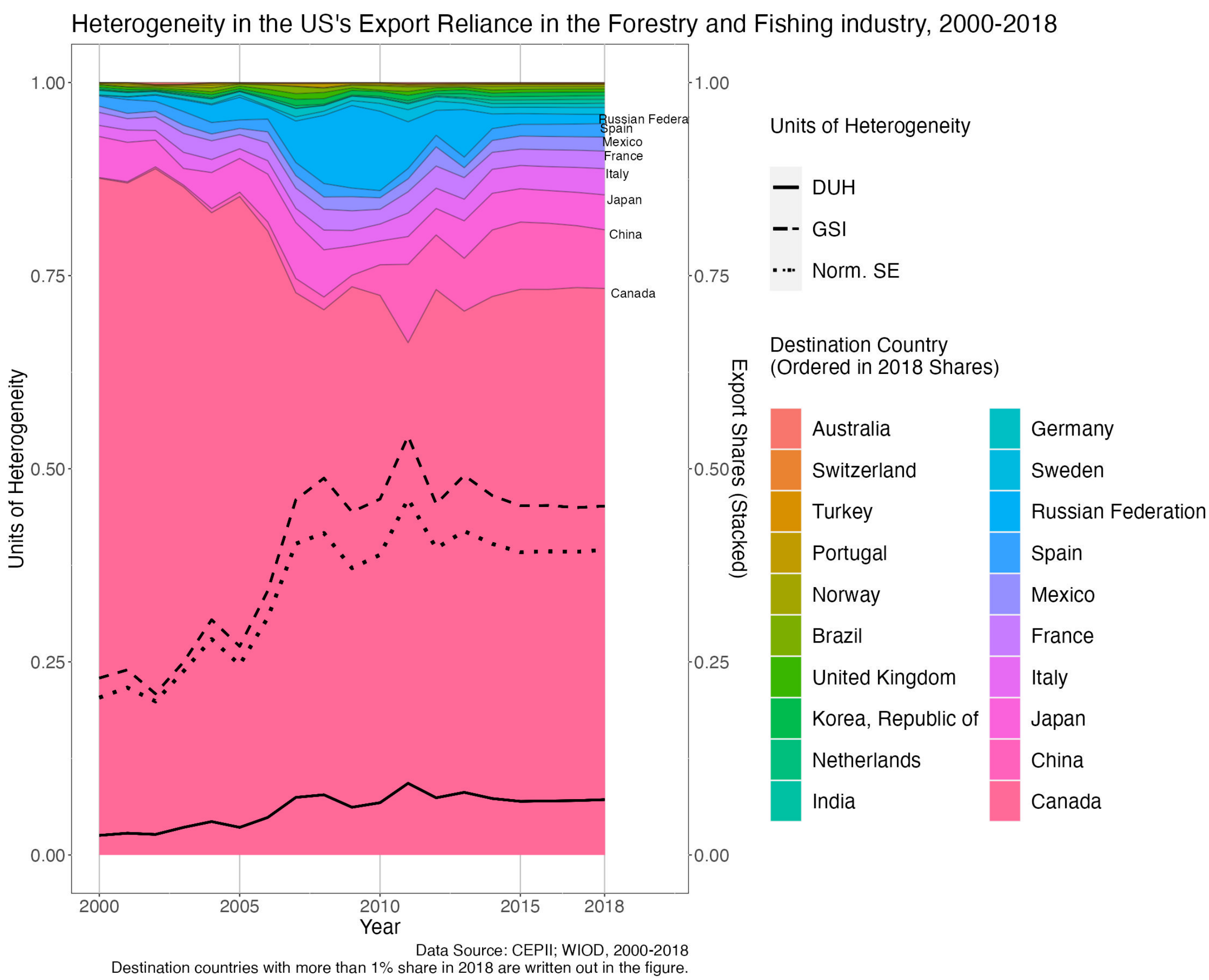


This example shows how DUH reflect changes in small groups better than HHI and SE. The next two examples demonstrate why DUH are not substitutes for concentration units.

[13]The share of all exported goods in the industry from the country.

Export Reliance.—Export reliance is similar to the revenue stream. If a country's export in an industry heavily relies on one or a few countries, then we can argue that said country has little power in determining trade policy regarding that industry. Figure 7 graphs the share of the US's exports in *Forestry and Fishing* to the top 20 destination countries.[14] Changes in DUH are relatively minor compared with changes in GSI and SE. Due to PT, the range of the influence of evenness in the minority decreases in the size of the largest group. In Figure 7, without looking at the shares directly, DUH suggests that the US's exports of forestry and fishing were heavily reliant on one or a few countries in 2000-2018; both GSI and SE suggest an initial reliance on a few countries, followed by a decrease in export reliance in 2007. Comparing that with the shares, whether DUH or concentration units are more descriptive of export reliance is at the researcher's discretion.

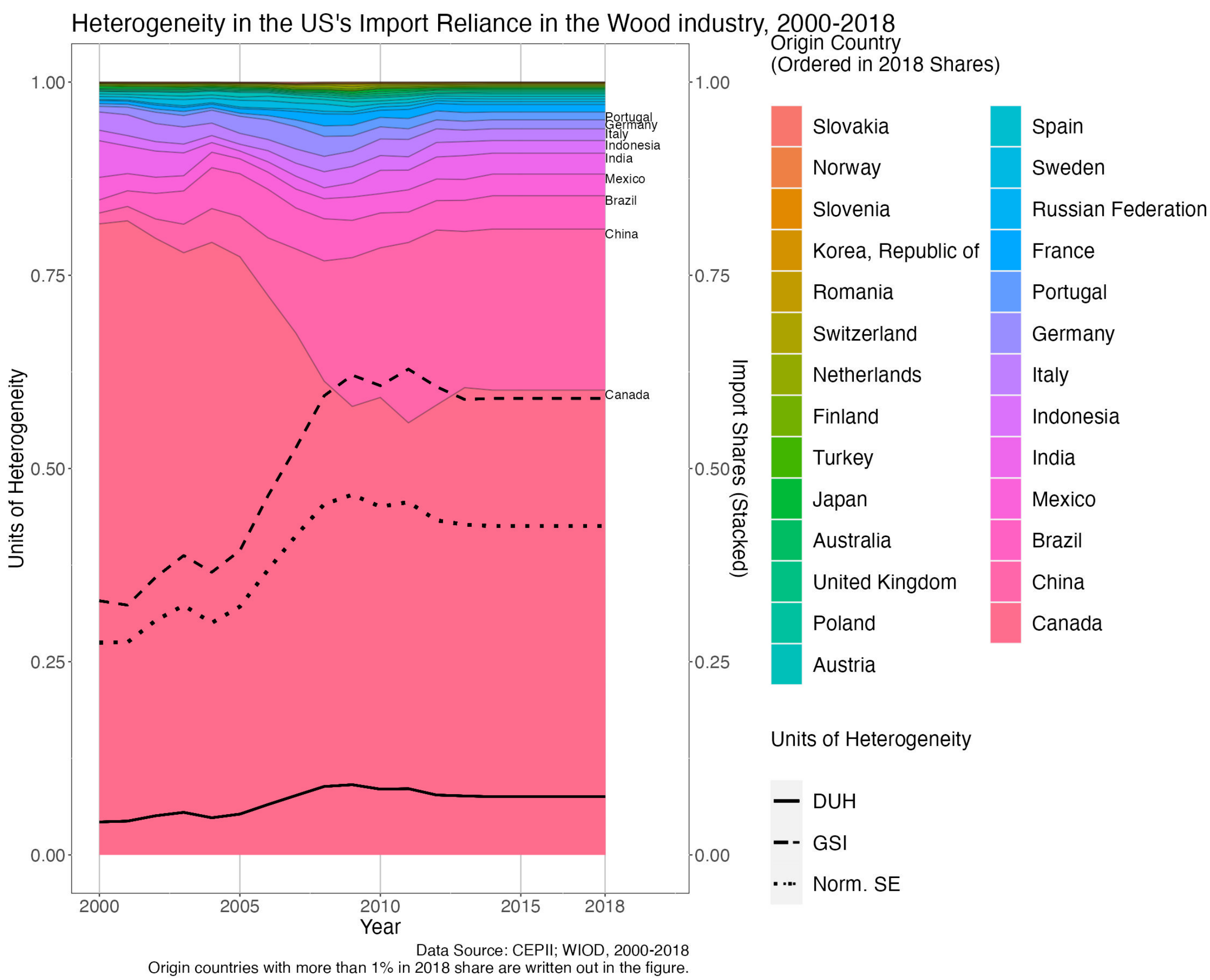
Figure 7: Comparisons between Different Units for Export Heterogeneity of the US



[14] Destination countries are the destinations of US exports. Countries that import little, as well as the RoW, are excluded because this example aims to measure export reliance. This means that the distinct grouping of countries in the reasonable collection matters. The cutoff of the top 20 is an arbitrary choice for this illustrative example.

Import Reliance.—Import reliance is not quite diametrically opposed to export reliance. If a country's import in an industry heavily relies on one country, we can argue that said country's economic activity in that industry can be easily influenced by exogenous shocks in the origin country. Figure 8 graphs the share of the US's imports of *Wood* from 28 origin countries.[15] Figure 8 demonstrates quite well why concentration units are suitable for measuring concentration but not heterogeneity. In this example, the US's imports of wood heavily rely on Canada. This reliance is partially substituted by Chinese imports starting in 2007, but the reliance is still high. In terms of market concentration, GSI and SE reflect that the US's imports of wood became less concentrated in 2007. In terms of heterogeneity, DUH reflect that the distribution in this system is still highly homogeneous.

Figure 8: Comparisons between Different Units for Import Heterogeneity of the US



This last example brings my discussion back to the definition of heterogeneity and the

[15] Origin countries are defined as the destination of US exports. Countries belonging to RoW are excluded as this example aims to measure import reliance, meaning the distinct grouping of countries in the reasonable collection ϑ matters. The 28 countries are what is available to me in this data.

place in the literature for a measure of it. As shown here, DUH are not meant to replace or serve as an improvement over concentration units; rather, they are meant to complement concentration units when heterogeneity is the outcome of interest. Heterogeneity concerns the presence of mixture in the system, while concentration concerns the presence of large groups. I agree with the sentiment in Chakravarty and Eichhorn (1991), whereby concentration units differ fundamentally from units meant to measure inequality, although I believe that the difference stems from more than simply the axiom of expandability.[9] The examples here show the difference between heterogeneity and concentration, as well as how units made to measure them can behave differently in the same environment.

6 Conclusion

Building on axiomatizations of the Gini coefficient, Herfindahl–Hirschman index, and Shannon's entropy, I uniquely characterize the class of units/indices/measures called descriptive units of heterogeneity. This set of units is simple to use, similar to existing units, and can fairly reflect changes in the evenness of minority groups. DUH provide a specific meaning for the sentence "S_2 is x times more heterogeneous than S_1" via the principle of proportional transfers.

DUH is not meant to be a replacement for either HHI or SE. Improvement in the ability to reflect the addition of zero-groups and changes in minority groups may not be a desirable property in cases in which only information about large groups should be considered. Future researchers, empirical or otherwise, can use this measure—or others similar to it—to reveal more insights into the evolution of heterogeneity in systems.

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Appendix A Proofs

Definition 3: A function $\psi : \mathbb{R}_+^{G-1} \to \mathbb{R}_+$ is a measure of evenness in minority groups' distribution if it is of the following form:

$$p \in \mathbb{R}_+,\ \psi_p(S) = \psi\left(\tilde{P}_2, \ldots, \tilde{P}_G\right) = 1 - \left(\sum_{g=2}^{G} \left|\tilde{P}_g - \frac{1}{G-1}\right|^p\right)^{\frac{1}{p}}.$$

Proposition 1: Consider an index $\Phi_p = \Phi(\varphi, \psi)$ that satisfies *SYM*, *INV*, and *IND*. Holding P_1 constant, if Φ is strictly increasing in ψ_p, then $\Phi(\varphi(P_1), \psi)$ satisfies the principle of diminishing transfers (PDT) if and only if $p > 1$.

Proof of Proposition 1:

Consider two ordered systems $S = (P_1, \ldots, P_g, P_{g+1}, \ldots, P_G)$ and $S' = (P_1, \ldots, P_g - c, P_{g+1} + c, \ldots, P_G)$ where $c < \frac{P_g - P_{g+1}}{2}$. Define $\tilde{c} = \frac{c}{P_2 + \cdots + P_G}$. I want to show that $\psi(S) < \psi(S')$ and $\Phi(S) < \Phi(S')$, thus satisfying *PDT*.

Given S and S', we have

$$\psi_p(S) = 1 - \left(\left|\tilde{P}_2 - \frac{1}{G-1}\right|^p + \cdots + \left|\tilde{P}_g - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} - \frac{1}{G-1}\right|^p + \cdots + \left|\tilde{P}_G - \frac{1}{G-1}\right|^p\right)^{\frac{1}{p}}$$
$$\psi_p(S') = 1 - \left(\left|\tilde{P}_2 - \frac{1}{G-1}\right|^p + \cdots + \left|\tilde{P}_g - \tilde{c} - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} + \tilde{c} - \frac{1}{G-1}\right|^p + \cdots + \left|\tilde{P}_G - \frac{1}{G-1}\right|^p\right)^{\frac{1}{p}}.$$

Observe that

$$\begin{aligned}
&\psi_p(S) < \psi_p(S') \\
\iff &\left(\tilde{P}_g - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} - \frac{1}{G-1}\right|^p + C\Bigg)^{\frac{1}{p}} > \left(\tilde{P}_g - \tilde{c} - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} + \tilde{c} - \frac{1}{G-1}\right|^p + C\Bigg)^{\frac{1}{p}} \\
\iff &\left|\tilde{P}_g - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} - \frac{1}{G-1}\right|^p > \left|\tilde{P}_g - \tilde{c} - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} + \tilde{c} - \frac{1}{G-1}\right|^p.
\end{aligned}$$

Case 1: $\frac{1}{G-1} < \tilde{P}_{g+1} < \tilde{P}_g$, then

$$\begin{aligned}
&\left|\tilde{P}_g - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} - \frac{1}{G-1}\right|^p > \left|\tilde{P}_g - \tilde{c} - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} + \tilde{c} - \frac{1}{G-1}\right|^p \\
\iff &\left(\tilde{P}_g - \frac{1}{G-1}\right)^p + \left(\tilde{P}_{g+1} - \frac{1}{G-1}\right)^p > \left(\tilde{P}_g - \tilde{c} - \frac{1}{G-1}\right)^p + \left(\tilde{P}_{g+1} + \tilde{c} - \frac{1}{G-1}\right)^p \\
\iff &\frac{\left(\tilde{P}_g - \frac{1}{G-1}\right)^p + \left(\tilde{P}_{g+1} - \frac{1}{G-1}\right)^p}{2} > \frac{\left(\tilde{P}_g - \tilde{c} - \frac{1}{G-1}\right)^p + \left(\tilde{P}_{g+1} + \tilde{c} - \frac{1}{G-1}\right)^p}{2}. \\
\iff &p > 1 (\text{ making the function } x^p \text{ convex}).
\end{aligned}$$

Case 2: $\tilde{P}_{g+1} < \frac{1}{G-1} < \tilde{P}_g$, then

$$\begin{aligned}
&\left|\tilde{P}_g - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} - \frac{1}{G-1}\right|^p > \left|\tilde{P}_g - c - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} + c - \frac{1}{G-1}\right|^p \\
\iff &\left(\tilde{P}_g - \frac{1}{G-1}\right)^p + \left(\frac{1}{G-1} - \tilde{P}_{g+1}\right)^p > \left(\tilde{P}_g - c - \frac{1}{G-1}\right)^p + \left(\frac{1}{G-1} - \tilde{P}_{g+1} - c\right)^p \\
\iff &\underbrace{\left(\tilde{P}_g - \frac{1}{G-1}\right)^p - \left(\tilde{P}_g - c - \frac{1}{G-1}\right)^p}_{>0} + \underbrace{\left(\frac{1}{G-1} - \tilde{P}_{g+1}\right)^p - \left(\frac{1}{G-1} - \tilde{P}_{g+1} - c\right)^p}_{>0} > 0.
\end{aligned}$$

Case 3: $\tilde{P}_{g+1} < \tilde{P}_g < \frac{1}{G-1}$, then

$$\begin{aligned}
&\left|\tilde{P}_g - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} - \frac{1}{G-1}\right|^p > \left|\tilde{P}_g - c - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} + c - \frac{1}{G-1}\right|^p \\
\iff &\left(\frac{1}{G-1} - \tilde{P}_g\right)^p + \left(\frac{1}{G-1} - \tilde{P}_{g+1}\right)^p > \left(\frac{1}{G-1} - \tilde{P}_g + c\right)^p + \left(\frac{1}{G-1} - \tilde{P}_{g+1} - c\right)^p \\
\iff &\frac{\left(\frac{1}{G-1} - \tilde{P}_g\right)^p + \left(\frac{1}{G-1} - \tilde{P}_{g+1}\right)^p}{2} > \frac{\left(\frac{1}{G-1} - \tilde{P}_g + c\right)^p + \left(\frac{1}{G-1} - \tilde{P}_{g+1} - c\right)^p}{2}. \\
\iff &p > 1(\text{ making the function } x^p \text{ convex}).
\end{aligned}$$

Last, if $G = 3$, then case 2 is the only case, meaning that the RHS of the statement can be expanded to $p \geq 1$. □

Lemma 1: Any measure $\Phi(n_1, \ldots, n_G)$ of system $S = (n_1, \ldots, n_G)$ satisfies SYM if $(n_1, \ldots, n_G)$ is a vector ordered such that $n_1 \geq n_2 \geq \cdots \geq n_G$.

Proof of Lemma 1:

The proof is trivial, given that the groups are ordered by size and not the label of the groups. This is a convenient consequence of defining systems as mappings from the universe of groups to a vector of numbers.

Lemma 2: If an index $\Phi(\varphi, \psi)$ of heterogeneity satisfies SYM, INV, and PPT, it is monotonically decreasing in P_1, and therefore a positive monotonic transformation of $\frac{1}{P_1}$.

Proof of Lemma 2:

Take any 2 systems of G groups $S = (n_1, n_2, \ldots, n_G)$ and $S' = (n'_1, n'_2, \ldots, n'_G)$ such that $\Phi(S) > \Phi(S')$ and that the $(n_2, \ldots, n_G) = \tilde{S} = \lambda \cdot \tilde{S}' = \lambda \cdot (n'_2, \ldots, n'_G)$, $\lambda \in \mathbb{R}_{++}$, then by scale invariance:

$$\Phi(n_1, n_2, \ldots, n_G) > \Phi(n'_1, n'_2, \ldots, n'_G) = \Phi\left(n'_1 \cdot \frac{n_S}{n'_S}, n'_2 \cdot \frac{n_S}{n'_S}, \ldots, n'_G \frac{n_S}{n'_S}\right).$$

By the principle of proportional transfers, since $n_1 + n_2 + \cdots + n_G = n'_1 \frac{n_S}{n'_S} + n'_2 \frac{n_S}{n'_S} + \cdots + n'_G \frac{n_S}{n'_S}$,

$$\Phi(S) > \Psi(S') \iff n_1 < n'_1 \cdot \frac{n_S}{n'_S} \iff P_1 < P'_1.$$

$\square$

Lemma 3: If an index $\Phi(\varphi, \psi)$ of heterogeneity satisfies SYM, INV, PPT, and IND, then $\Phi = \varphi \cdot \psi$.

Proof of Lemma 3:

Notice first that independence trivially implies that φ and ψ must be separable. Take any system S with G groups. By the principle of proportional transfers, it must be that $\forall \alpha \in \left[1, \frac{n_1 + \tilde{P}_2 \cdot n_1}{n-2+\tilde{P}_2 \cdot n_1}\right]$

$$\alpha \cdot \Phi(P_1, P_2, \ldots, P_G) = \Phi\left(P_1^{\alpha}, P_2 + \tilde{P}_2\left(P_1 - P_1^{\alpha}\right), \ldots, P_G + \tilde{P}_G\left(P_1 - P_1^{\alpha}\right)\right)$$

$$\iff \alpha\Phi(P_1, P_2, \ldots) = \Phi\left(P_1^{\alpha}, P_2', \ldots, P_G'\right) \iff \alpha = \frac{\varphi\left(P_1^{\alpha}\right)}{\varphi(P_1)}.$$

where $\exists \lambda \in \mathbb{R}_{++}\, s.t.\, \lambda P_g = P_g'$, $\forall g \in \{2, \ldots, G\}$.

$\square$

Proposition 2: If an index $\Phi(\varphi, \psi)$ of heterogeneity satisfies INV, IND, and PPT, then it must be $\Phi = \varphi(P_1) \cdot \psi(\tilde{P}_2, \ldots, \tilde{P}_G)$ where $\varphi(P_1) = -c \cdot log_q(P_1)$, $c \in \mathbb{R}_{++}$.

Proof of Proposition 2:

From the previous two lemmas, we know that $\varphi(P_1)$ must be a positive monotonic transformation of $\frac{1}{P_1}$ and that for α such that $P_1^{\alpha} > P_2 + \tilde{P}_2(P_1 - P_1^{\alpha})$, we must have

$$\alpha = \frac{\varphi(P_1^{\alpha})}{\varphi(P_1)}.$$

Notice that the only positive monotonic transformation that would satisfy this is $log_q\left(\frac{1}{P_1}\right)$, up to a positive scalar multiplication. Further, notice that any $log_q\left(\frac{1}{P_1}\right)$ can be rewritten as $\frac{ln\left(\frac{1}{P_1}\right)}{ln(q)}$, so it is equivalent to write $c \cdot ln\left(\frac{1}{P_1}\right)$. As such, $\varphi(P_1) = c \cdot ln\left(\frac{1}{P_1}\right)$, $c \in \mathbb{R}_{++}$ is the unique function, up to positive scalar multiplication, of majority proportions that can lead to $\Phi(\varphi, \psi)$ satisfying independence, scale invariance, and the principle of proportional transfers. $\square$

Theorem: Descriptive units of heterogeneity, Φ defined as

$$\Phi_p(n_1, \ldots, n_G) = -\frac{ln\left(P_1\right)}{ln(G)}\left[1 - \left(\sum_{g=2}^{G}\left(\tilde{P}_g - \frac{1}{G-1}\right)^p\right)^{\frac{1}{p}}\right]$$

$$= \frac{ln\left(P_1\right)}{ln(G)}\left[\left(\sum_{g=2}^{G}\left(\tilde{P}_g - \frac{1}{G-1}\right)^p\right)^{\frac{1}{p}} - 1\right],$$

where $P_1 = \dfrac{n_1}{n_1 + \cdots + n_G}$, $\tilde{P}_g = \dfrac{n_g}{n_2 + \cdots + n_g}$, $p \in (1, \infty)$. =, is a unique class of units that satisfy scale invariance, group symmetry, independence, the principle of diminishing transfers, the principle of proportional transfers, and contractibility and uses ψ_p to account for evenness in minority groups.

Proof of Theorem 1:

Propositions 1 and 2 combined implies that DUH satisfy SYM, INV, PPT, IND, and CON, but not necessarily PDT. I have only shown that DUH satisfy PDT when either φ or ψ_p is held constant, which means that I need to be careful with the functional form of $\varphi(P_1)$. To satisfy *PDT* overall, it must be that when there is a transfer from the majority group to the minority group, the increase in Φ through $\varphi(P_1)$ dominates the decrease in Φ through ψ_p in the case in which evenness decreases as a result of the transfer.

To show that Φ satisfies PDT, we only need to look at the extreme case in which P_1 is close to 1 and $\psi_p = 1$. In this case, a simple transfer from n_1 to n_2 will decrease ψ_p the most. For simplicity, we will consider the case in which $p = 2$, so that Φ is simply

$$\Phi(n_1, \ldots, n_G) = \frac{ln\,(P_1)}{ln(G)} \left[\sqrt{\sum_{g=2}^{G} \left(\tilde{P}_g - \frac{1}{G-1} \right)^2} - 1 \right].$$

Denote $n_2 + \cdots + n_G$ as $\tilde{n}_S$, a transfer of x from n_1 to n_2 when $\psi = 1$ can be written as

$$\Phi_2 = ln \left(\frac{n_1 - x}{n_S} \right) \left[\sqrt{ \left(\frac{\frac{\tilde{n}_S}{G-1} + x}{\tilde{n}_S + x} \right) + (G-2) \left(\frac{\frac{\tilde{n}_S}{G-1}}{\tilde{n}_S + x} - \frac{1}{G-1} \right)^2 } - 1 \right].$$

Taking the derivative of this expression with respect to x, we have $\forall x \in \left[0, \frac{(G-1)n_1 - \tilde{n}_S}{G}\right]$:

$$\frac{d}{dx}\Phi_2(n_1 - x, n_2 + x, n_3, \ldots, n_G) = \frac{\sqrt{\frac{G-2}{G-1}} \left[\tilde{n}_S(x - n_1) ln \left(\frac{n_1 - x}{n_S} \right) + x(b + x) \right]}{(x - n_1)(x + \tilde{n}_S)^2} + \frac{1}{n_1 - x} > 0.$$

□

Appendix B Characterization Axioms of HHI and SE

[EXP] Expandability. $\Phi(n_1,\ldots,n_G)$ satisfies expandability if

$$\Phi(n_1,\ldots,n_G,0)=\Phi(n_1,\ldots,n_G).$$

It should be clear that neither EXP nor CON attempts to pin down the functional form of a unit. Rather, these two opposing axioms serve as the divide between a unit for concentration and a unit for heterogeneity.

As discussed extensively by Atkinson (1970), the fact that PDT only induces partial ordering implies that specific functional forms can always be chosen to induce different total orders when neither system's distribution second-order stochastically dominates the other.[16] The functional forms of both HHI and SE were able to be uniquely characterized because of this feature.[17] HHI uses EXP and the replication principle (REP), and SE uses EXP and Shannon's additivity (SADD).

[REP] Replication Principle. $\Phi(n_1,\ldots,n_G)$ satisfies the replication principle for concentration if replicating a system k times divides the system concentration by k.

For example, take $k\in\mathbb{N}$,

$$\frac{1}{k}\Phi(n_1,\ldots,n_G)=\Phi\left(\underbrace{\frac{n_1}{k},\frac{n_1}{k},\ldots,\frac{n_1}{k}}_{\text{Sum to } n_1},\frac{n_2}{k},\frac{n_2}{k},\ldots,\underbrace{\frac{n_G}{k},\ldots,\frac{n_G}{k}}_{\text{Sum to } n_G}\right).$$

REP pins down the cardinal meaning of the unit by linking the multiplication of the unit to how many times a system is divided/replicated into a system with more groups. Chakravarty and Eichhorn (1991) show that if a concentration unit C can be represented as a self-weighted quasilinear mean, then C is the Hannah-Kay index of concentration if and only if C satisfies the fundamental axioms and REP.[18] Since HHI is $H^n_{\alpha=2}(S)$, it is the unique self-weighted quasilinear concentration unit that satisfies the fundamental axioms, EXP, and REP. Notice that REP applies only when *every group* is broken into multiple groups. To prescribe how the index behaves when only one group is broken up, SE forgoes REP and restricts the measure based on Shannon's additivity instead.

[16] Newbery (1970) shows that no additive functional forms can be chosen to induce the same order as the Gini coefficient. This impossibility theorem stems from and reaffirms the point made by Atkinson (1970).

[17] Similarly, I use this feature to uniquely characterize the descriptive units of heterogeneity.

[18] A relative concentration index $C:D\to\mathbb{R}$ is called a self-weighted quasilinear mean if for all $n\in\mathbb{N}$, $x\in D^n$, $C^n(x)$ is of the form

$$C^n(x)=\phi^{-1}\left[\sum_{g=1}^{G}P_g\phi\left(P_g\right)\right],$$

where $\phi:(0,1]\to\mathbb{R}$ is strictly monotonic.

[SADD] Shannon's additivity. Define $n_{gj} \geq 0$ such that $n_g = \sum_{j=1}^{m_g} n_{gj}$, $\forall g \in \{1, \ldots, G\}$ and $\forall j \in \{1, \ldots, m_g\}$,

$\Phi(n_1, \ldots, n_G)$ satisfies Shannon's additivity if

$$\Phi(n_{11}, \ldots, n_{Gm_G}) = \Phi(n_1, \ldots, n_G) + \sum_{g=1}^{G} \frac{n_g}{n_S} \cdot \Phi\left(\frac{n_{g1}}{n_g}, \ldots, \frac{n_{gm_g}}{n_g}\right).$$

which implies (by setting $m_g = 1$, $\forall g \in \{1, \ldots, G-1\}$ and $n_{G'} = n_G + n_{G+1}$),

$$\Phi(n_1, \ldots, n_G, n_{G+1}) = \Phi(n_1, \ldots, n_{G'}) + \frac{n_{G'}}{n_1 + \ldots + n_{G-1} + n_{G'}} \cdot \Phi\left(\frac{n_G}{n_{G'}}, \frac{n_{G+1}}{n_{G'}}\right).$$

SADD pins down how the decomposition of group(s) in a system should influence the unit.[19]

For detailed proofs of the unique characterization of SE and explanations of SADD, readers should refer to Suyari (2004) and Chakrabarti et al. (2005).

[19] An example of decomposing a group is to split the sales of Macs into Mac desktops and Mac laptops, for the purpose of measuring the heterogeneity of Apple's revenue streams.