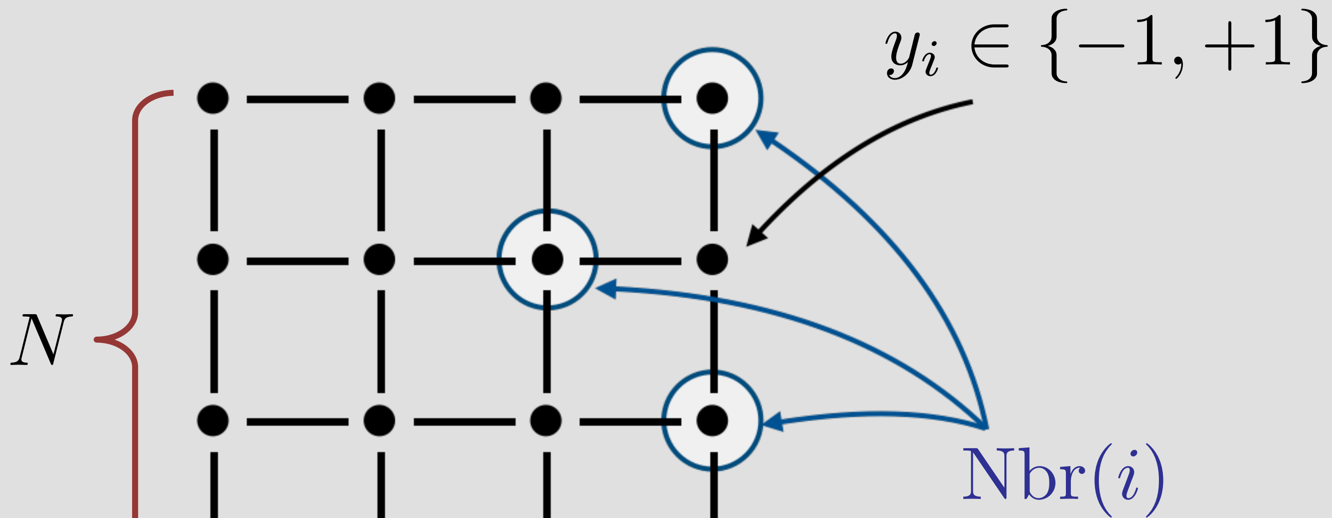
Example: Ising model

Ising model

The model is a 2D lattice. Each dot y_i can take in either -1 or 1.



There is a function Nbr(i) which returns the neighbours for some dot y_i.

sum of all edges

for each edge, sum the adjacent vals.

$$p(y) \propto \underbrace{\exp(\frac{1}{2} J \sum_i \sum_{j \in \mathrm{Nbr}(i)} y_i y_j + \sum_i b_i y_i)}_{\phi(y)}$$

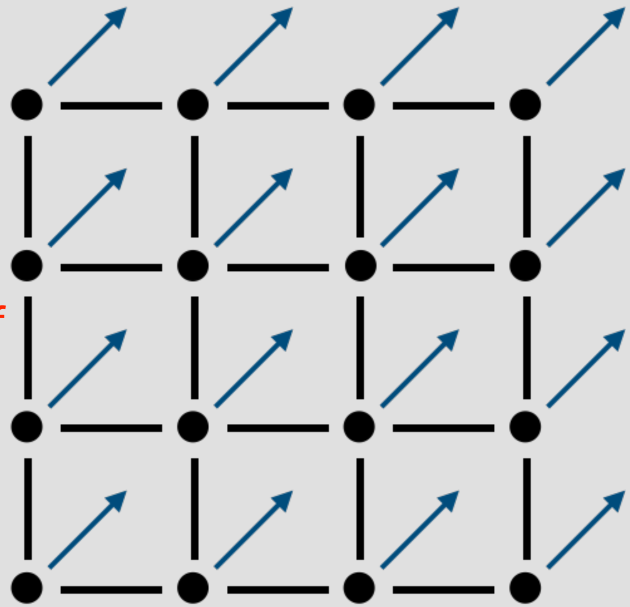
bi is the 'external field'

J is the parameter of the model.

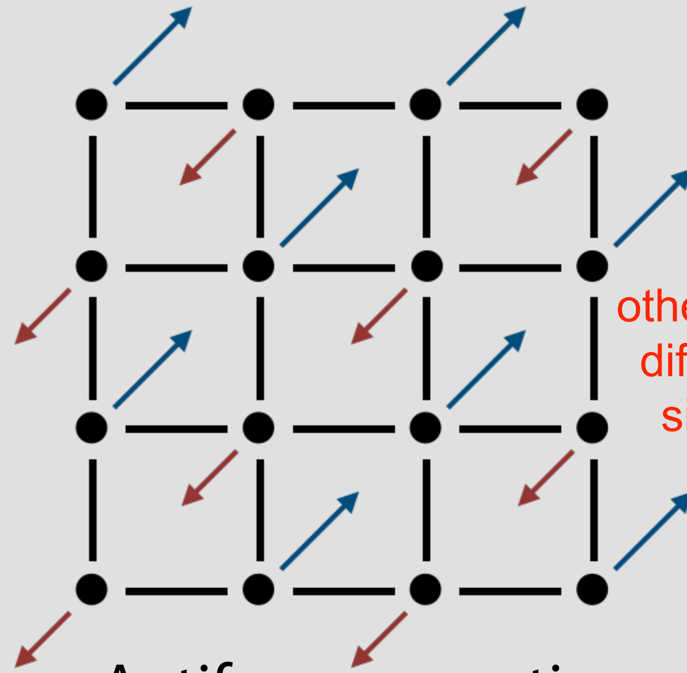
if neighbouring values have same sign, then 1. Else, -1.

Ising model

y_i will tend to have the same sign if ferro magnetic.



Ferromagnetic
$J > 0$



otherwise, differing signs.

Antiferromagnetic
$J < 0$

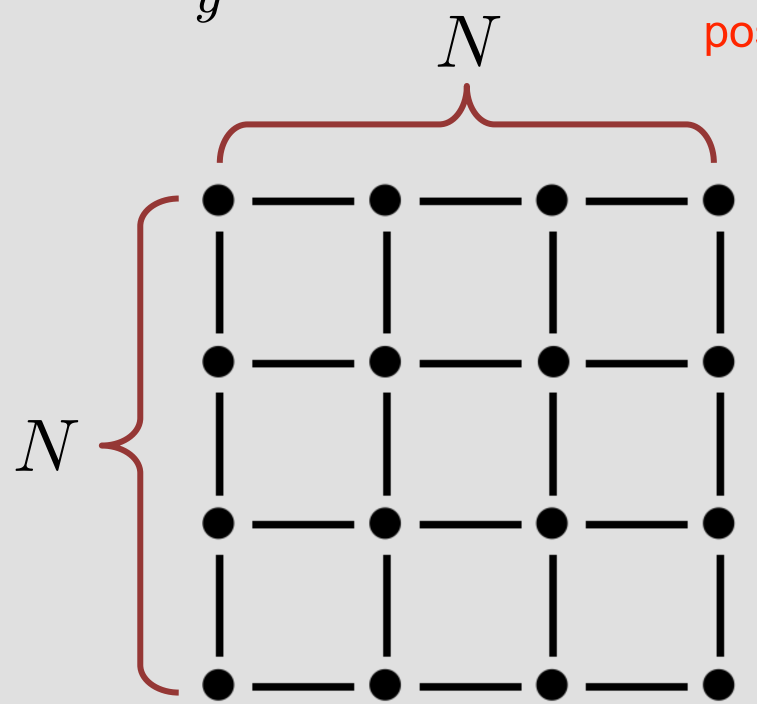
$$p(y) \propto \exp\Big(\underbrace{\frac{1}{2} J \sum_i \sum_{j \in \mathrm{Nbr}(i)} y_i y_j + \sum_i b_i y_i}_{\phi(y)}\Big)$$

Normalization constant

$$p(y) = \frac{1}{Z}\phi(y)$$

$$Z = \sum_y \phi(y) \longleftarrow 2^{N^2} \text{ terms}$$

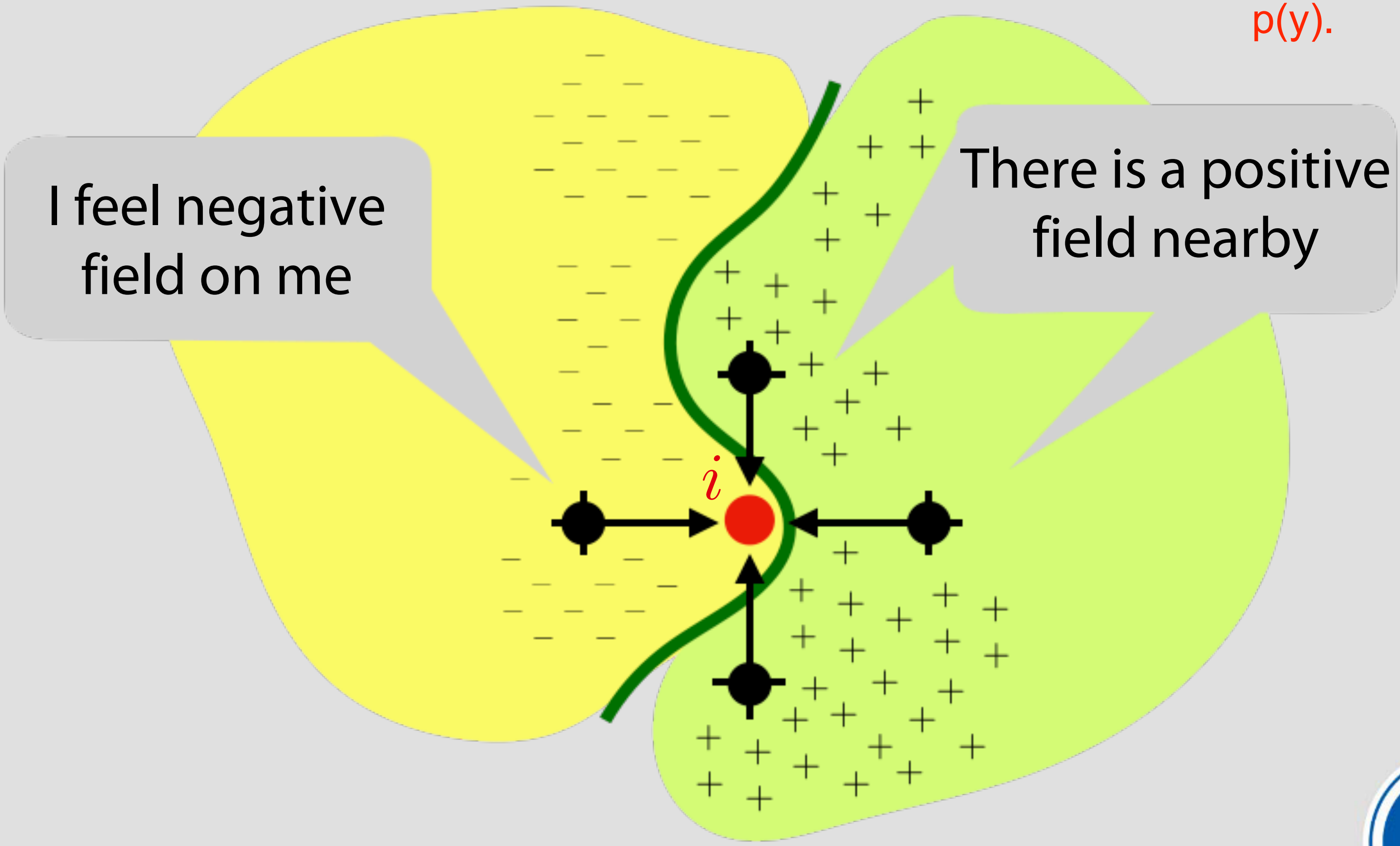
^ because we want to get all the possible states. This sounds infeasible.



Mean field

$$p(y) \approx q(y) = \prod_i q_i(y_i)$$

We want to use an MFA to approximate p(y).



Технический слайд (5 минут на доску)

$$\log q_i(y_i) = \mathbb{E}_{y \setminus y_i} p(y) + \text{const}$$

$$= \mathbb{E}_{y \setminus y_i} J \sum_{j \in \text{Nbr}(i)} y_i y_j + b_i y_i + \text{const}$$

$$= J \sum_{j \in \text{Nbr}(i)} y_i \mathbb{E} y_j + b_i y_i + \text{const}$$

$$= J \sum_{j \in \text{Nbr}(i)} y_i \mu_j + b_i y_i + \text{const}$$

$$= y_i \overbrace{\left(J \sum_{j \in \text{Nbr}(i)} \mu_j + b_i\right)}^{M} + \text{const}$$

$$= M y_i + \text{const}$$

Технический слайд

$$q_i(y_i) = \text{const} \cdot e^{My_i}$$

$$q_i(+1) + q_i(-1) = \text{const}(e^{M} + e^{-M}) = 1$$

$$q_i(+1) = \frac{e^{M}}{e^{M} + e^{-M}} = \sigma(2M)$$

$$q_i(-1) = \frac{e^{-M}}{e^{M} + e^{-M}} = 1 - \sigma(2M)$$

$$M = (J \sum_{j \in \mathrm{Nbr}(i)} {\color{red}\mu_j} + b_i)$$

Технический слайд

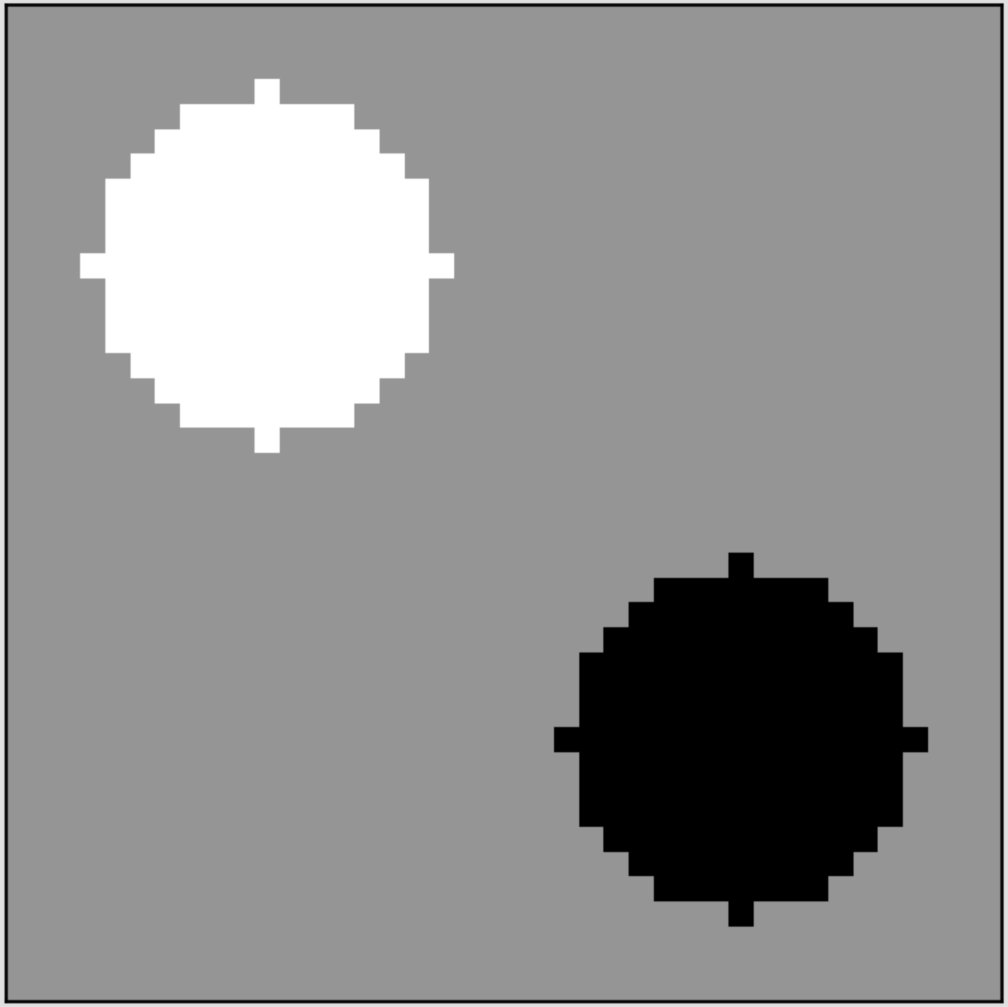
$$q_k(y_k) \propto \exp\Big(J y_k \sum_{j \in Nb(k)} \mu_j\Big) = \exp(y_k M), \quad M = J \sum_{j \in Nb(k)} \mu_j$$

$$q_k(+1) = \frac{e^M}{e^M + e^{-M}} = \frac{1}{1 + e^{-2M}} = \sigma(2M)$$

$$\mu_k = q_k(+1) - q_k(-1) = \frac{1}{1+e^{-2M}} - \frac{e^{-2M}}{1+e^{-2M}} = \frac{1-e^{-2M}}{1+e^{-2M}} = \frac{e^M - e^{-M}}{e^M + e^{-M}} = \tanh(M)$$

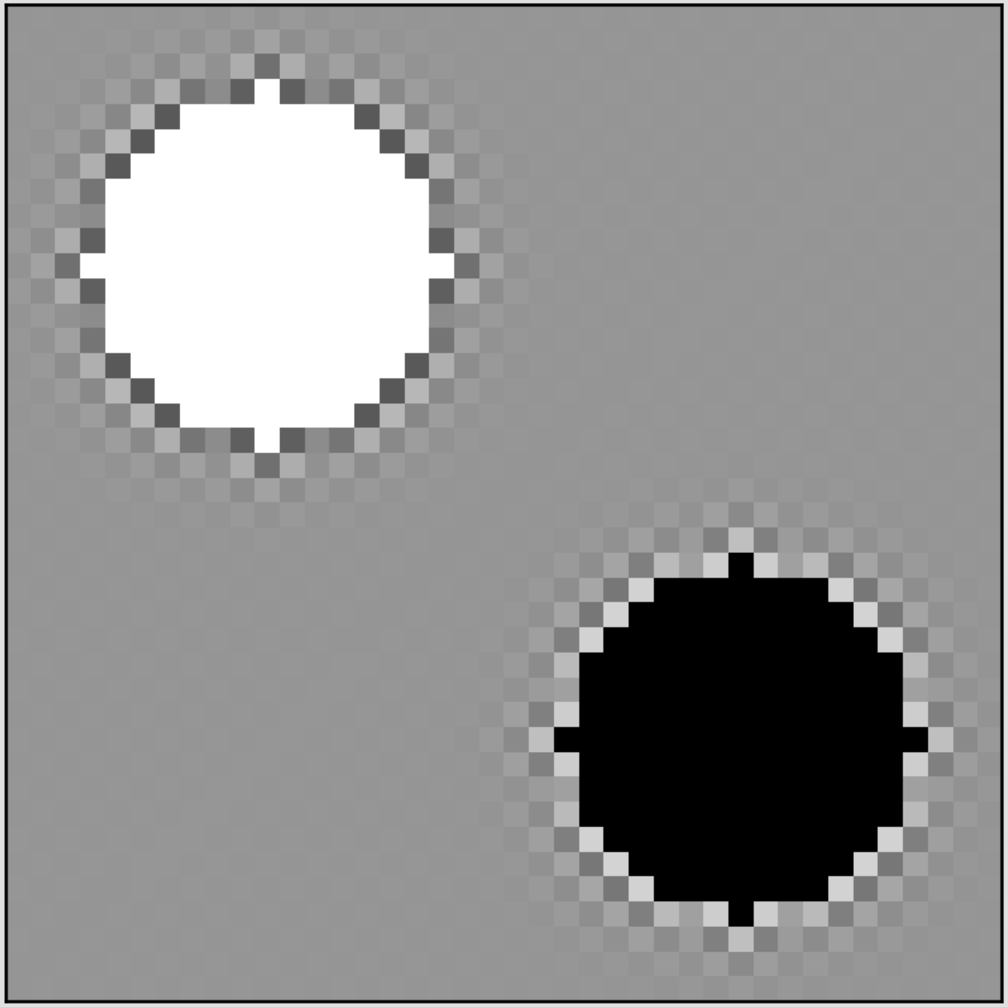
Example

$$J = 0$$



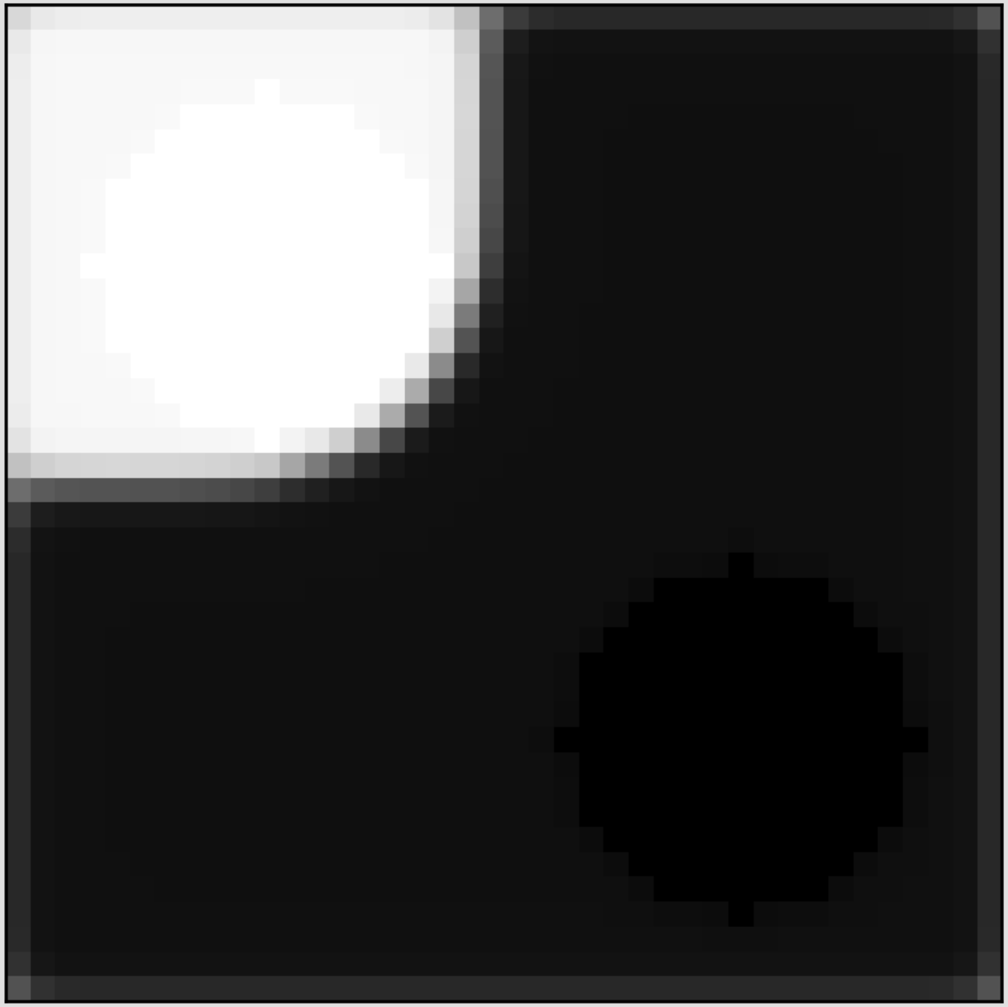
Example

$$J = -0.05$$

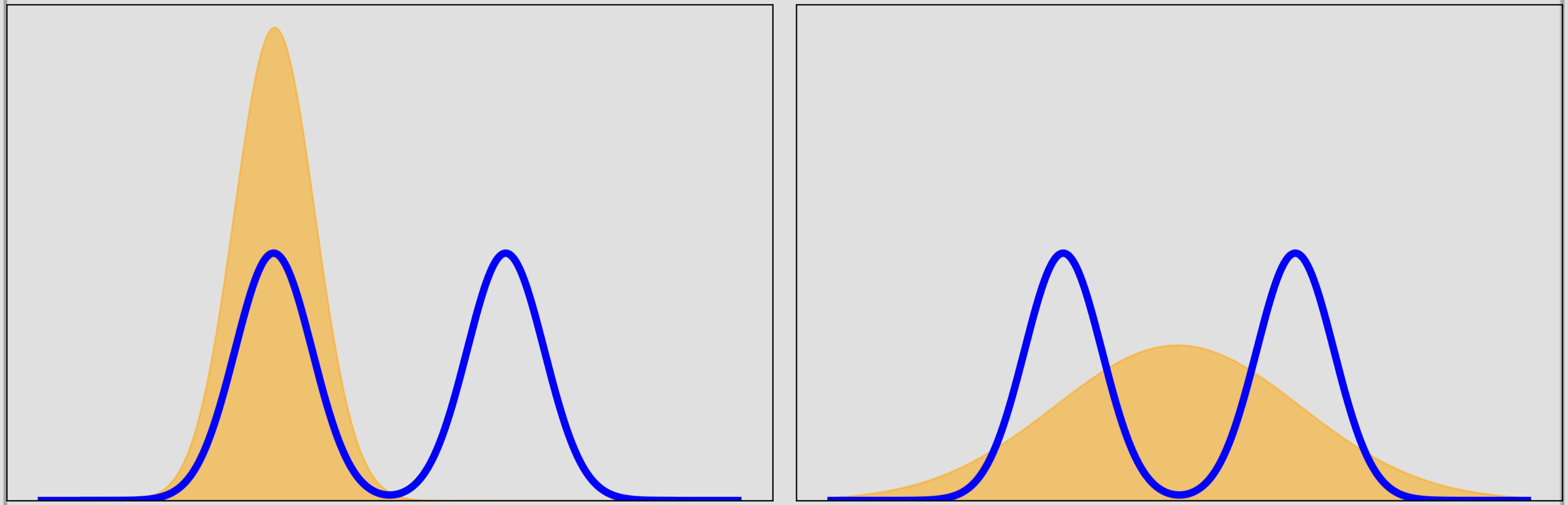


Example

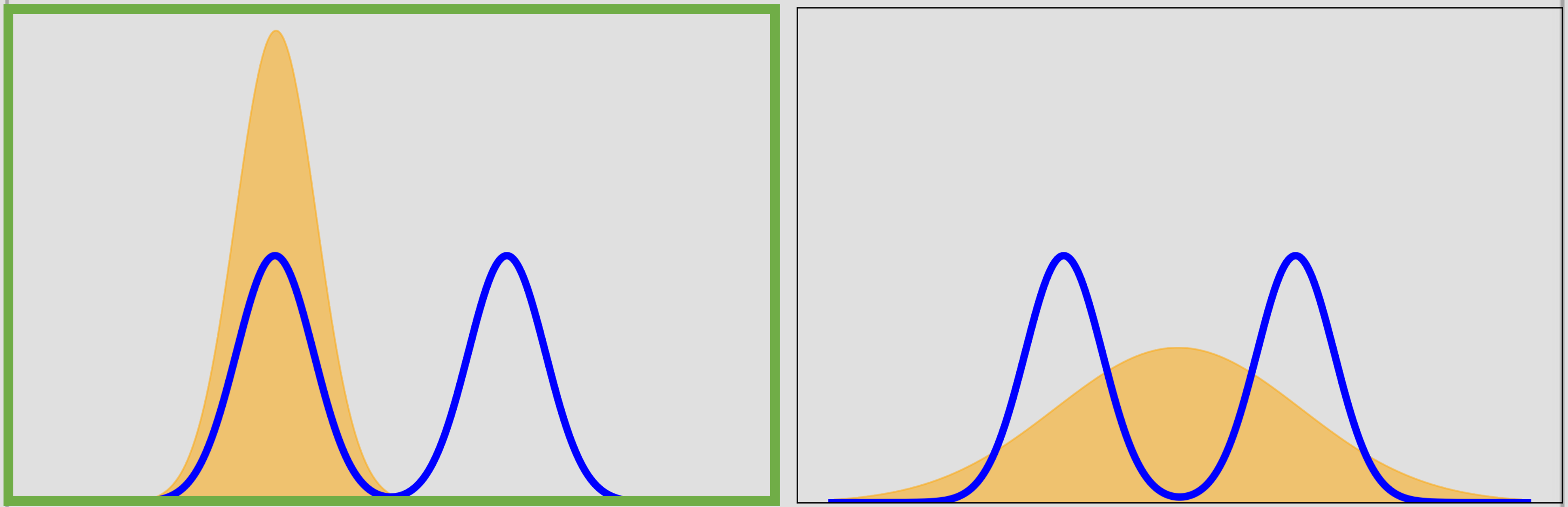
$$J = 0.1$$



Optimization solutions



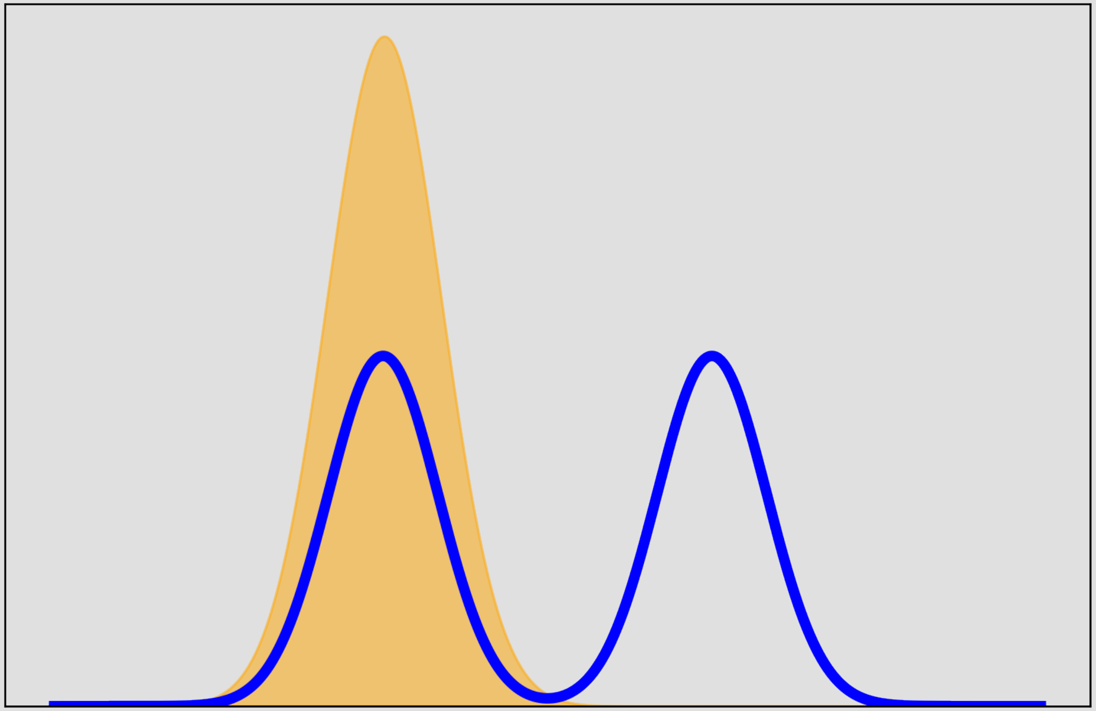
Optimization solutions



Captures statistics

Mode has high probability

Optimization solutions



$0 \neq$

$$\mathcal{KL}(q \,\|\, p^*) = \int q(z) \log \frac{q(z)}{p^*(z)} dz = +\infty$$

$0 =$