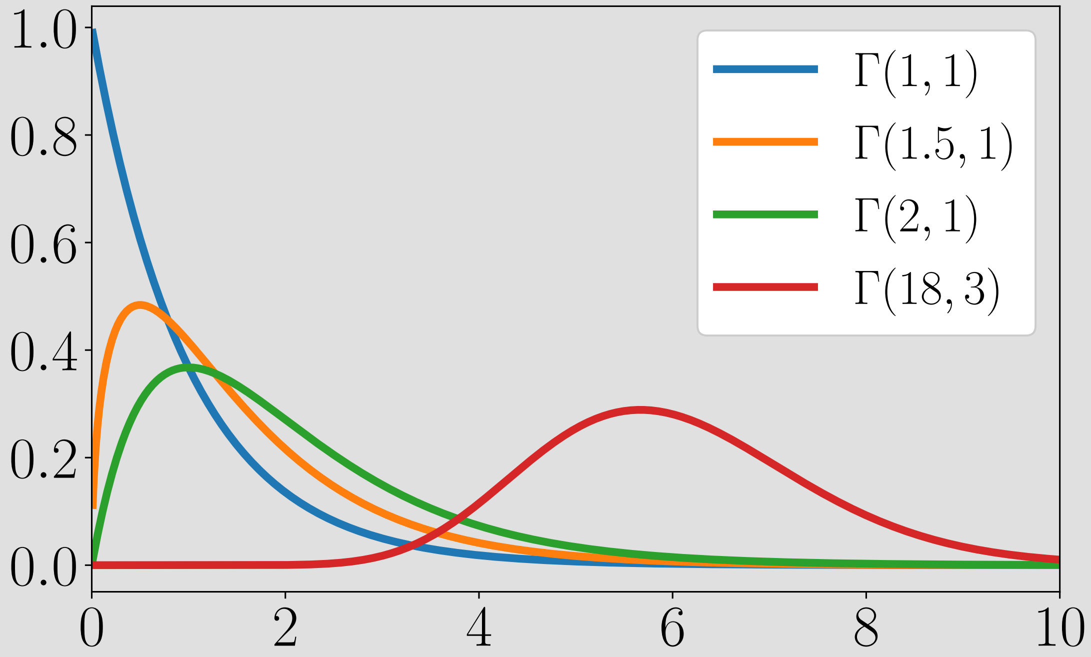
Distributions: Gamma

Gamma distribution

$$\Gamma(\gamma|a, b) = \frac{b^a}{\Gamma(a)} \gamma^{a-1} e^{-b\gamma}$$

$$\gamma, a, b > 0$$



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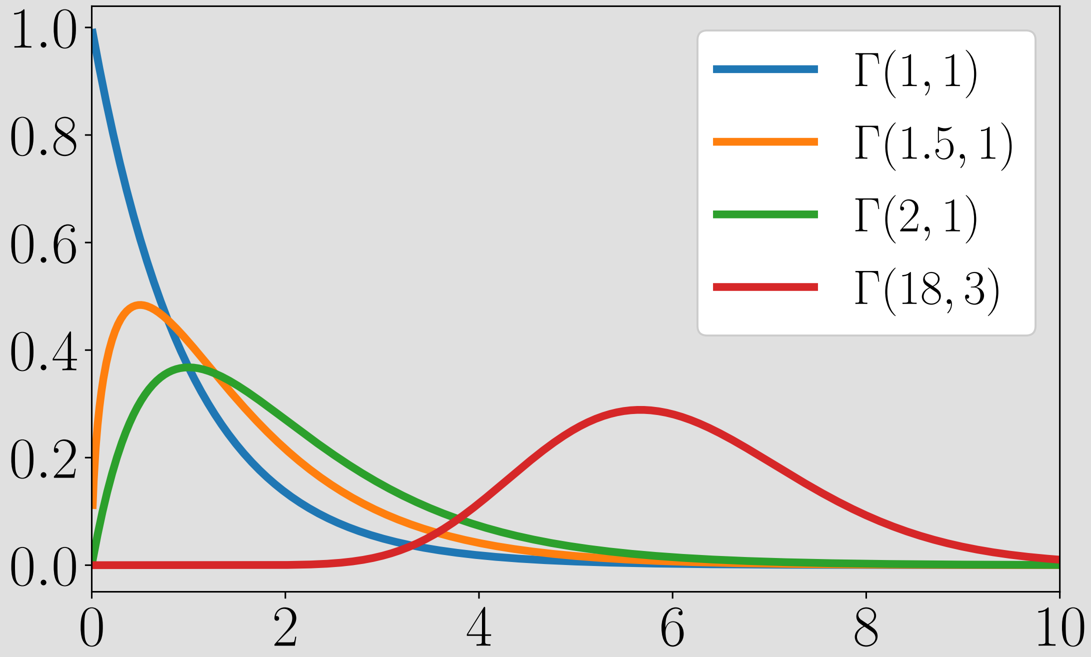
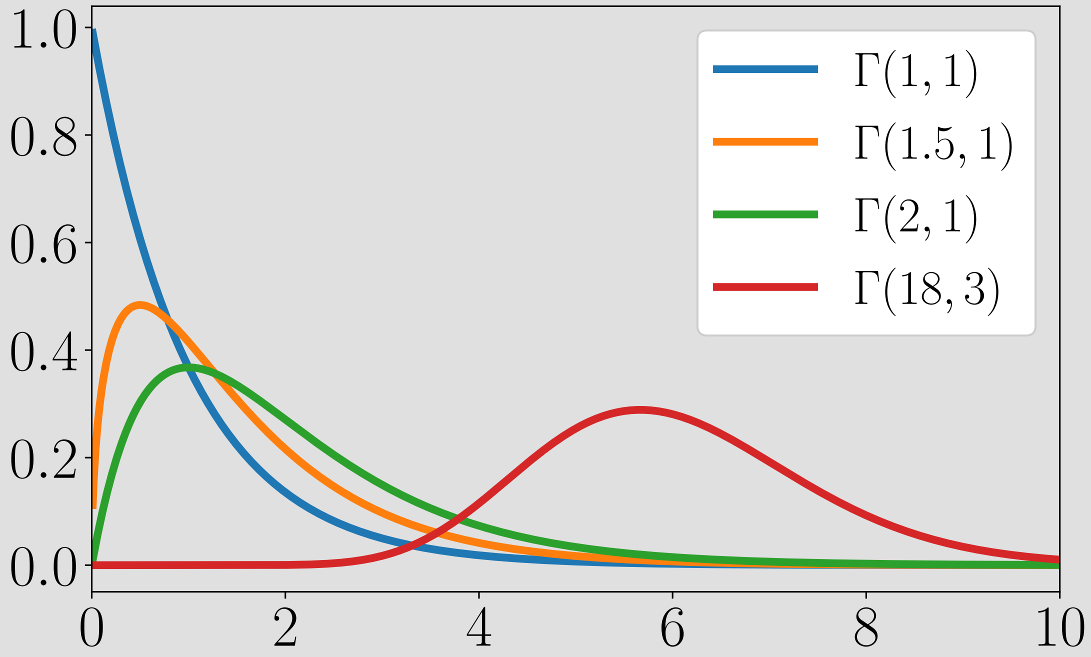


Gamma distribution

$$\Gamma(\gamma|a,b) = \frac{b^a}{\Gamma(a)}\gamma^{a-1}e^{-b\gamma}$$

$$\Gamma(n) = (n-1)!$$



Statistics

$$\Gamma(\gamma|a,b) = \frac{b^a}{\Gamma(a)}\gamma^{a-1}e^{-b\gamma}$$

$$\mathbb{E}[\gamma] = a/b$$

$$\mathrm{Mode}[\gamma] = \frac{a-1}{b}$$

$$\mathrm{Var}[\gamma] = {}^{a}\!/_{b^2}$$

Example

Std.

You run $5\text{km} \pm 100\text{m}$ a day

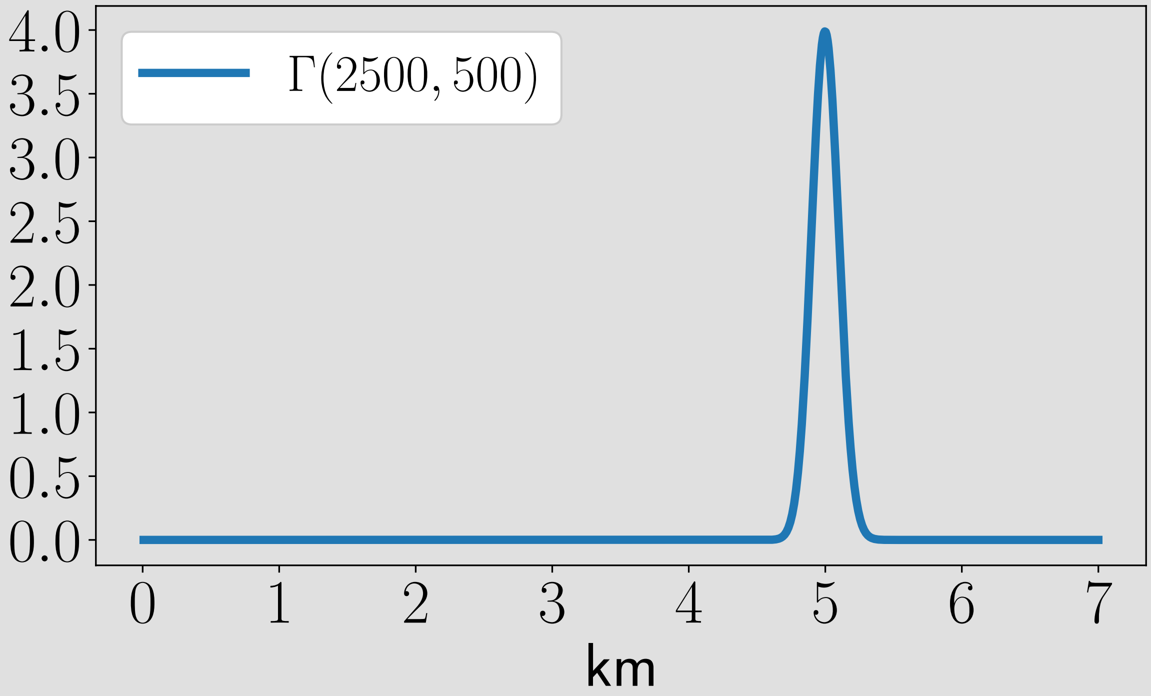
Expectation

Example

You run 5km ± 100m a day

$$\mathbb{E}[x] = {}^{a}/_{b} = 5,\ Var[x] = {}^{a}/_{b^2} = 0.1^2$$

$$\Rightarrow a = 2500, \mathrm{b} = 500$$



Example: Normal, precision

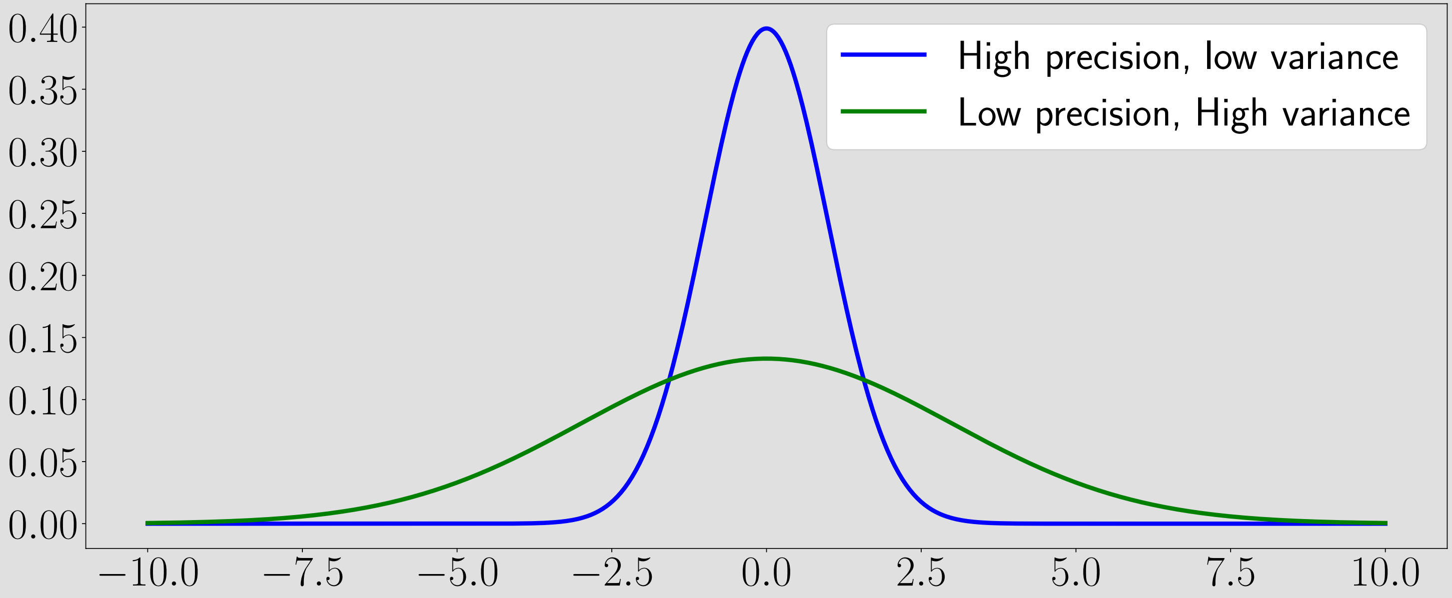
Precision

The gamma distribution is conjugate to the normal w.r.t the precision.

Precision $\rightarrow$ $$\gamma = \frac{1}{\sigma^2}$$ $\leftarrow$ Variance

Precision is simply inverse of the variance.

Higher precision, the better you can predict the whereabouts of the sample.



Precision

Now. Let's do an example to demonstrate precision.

Let us use the normal distribution. Replace variance with the reciprocal of precision.

$$\mathcal{N}(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\mathcal{N}(x|\mu,\gamma^{-1}) = \frac{\sqrt{\gamma}}{\sqrt{2\pi}} e^{-\gamma\frac{(x-\mu)^2}{2}}$$

We want to ask: What is the conjugate prior w.r.t to the precision?
i.e. we want to find p(gamma).

in:
p(gamma |x) = p(x|gamma) p(gamma) / p(x)

Functional form

$$\mathcal{N}(x|\mu,\gamma^{-1}) = \frac{\sqrt{\gamma}}{\sqrt{2\pi}} e^{-\gamma\frac{(x-\mu)^2}{2}}$$

Now, let us drop all constants that do not depend on gamma (e.g. x , mu).

$$\mathcal{N}(x|\mu,\gamma^{-1}) \propto \gamma^{\frac{1}{2}} e^{-b\gamma}$$

$$p(\gamma) \propto \gamma^{\frac{1}{2}} e^{-b\gamma}?$$

Find the conjugate distribution p(gamma).
Suppose that the prior uses the same distribution as the likelihood

Functional form

$$\mathcal{N}(x|\mu,\gamma^{-1}) = \frac{\sqrt{\gamma}}{\sqrt{2\pi}} e^{-\gamma\frac{(x-\mu)^2}{2}}$$

$$\mathcal{N}(x|\mu,\gamma^{-1}) \propto \gamma^{\frac{1}{2}} e^{-b\gamma}$$

$$p(\gamma) \propto \gamma^{\frac{1}{2}} e^{-b\gamma}?$$

Now, lets test if the posterior has the same distribution as the prior.

$$p(\gamma|x) = \frac{p(x|\gamma)p(\gamma)}{p(x)} \propto \gamma e^{-\gamma(b+\frac{(x-\mu)^2}{2})}$$

Functional form

$$\mathcal{N}(x|\mu,\gamma^{-1}) = \frac{\sqrt{\gamma}}{\sqrt{2\pi}} e^{-\gamma\frac{(x-\mu)^2}{2}}$$

$$\mathcal{N}(x|\mu,\gamma^{-1}) \propto \gamma^{\frac{1}{2}} e^{-b\gamma}$$

$$p(\gamma) \propto \gamma^{\frac{1}{2}} e^{-b\gamma}?$$

$$p(\gamma|x) = \frac{p(x|\gamma)p(\gamma)}{p(x)} \propto \gamma e^{-\gamma(b+\frac{(x-\mu)^2}{2})}$$

IT DOES NOT! the gamma here is not a square rooted one, like the prior

Functional form

So we try another approach:

$$\mathcal{N}(x|\mu,\gamma^{-1}) = \frac{\sqrt{\gamma}}{\sqrt{2\pi}} e^{-\gamma\frac{(x-\mu)^2}{2}}$$

$$\mathcal{N}(x|\mu,\gamma^{-1}) \propto \gamma^{\frac{1}{2}} e^{-b\gamma}$$

$$p(\gamma) \propto \gamma^{a-1} e^{-b\gamma}$$

$$p(\gamma) = \Gamma(\gamma|a,b)$$

Suppose we try to set the prior to be a gamma distribution.

Gamma prior

This is our prior.

$$p(\gamma) = \Gamma(\gamma|a,b) \propto \gamma^{a-1}e^{-b\gamma}$$

$$p(\gamma|x) \propto p(x|\gamma)p(\gamma)$$

$$p(\gamma|x) \propto \left(\gamma^{\frac{1}{2}}e^{-\gamma\frac{(x-\mu)^2}{2}}\right)\cdot\left(\gamma^{a-1}e^{-b\gamma}\right)$$

$$p(\gamma|x) \propto \gamma^{\frac{1}{2}+a-1}e^{-\gamma(b+\frac{(x-\mu)^2}{2})}$$

to acct for the 1/2 + a - 1

$$p(\gamma|x) = \Gamma(\boxed{a+\frac{1}{2}}, b+\frac{(x-\mu)^2}{2})$$

As we can see, the posterior is also a gamma distribution!
This is great. Prior is conjugate to the liklihood.

By choosing a sensible conjugate prior, we avoided computing the evidence
because they lie in the same distribution.