2. Conjugate distributions

Bayes formula

Fixed by model

Our own choice!

$$P(\theta|X) = \frac{P(X|\theta)P(\theta)}{P(X)}$$

Fixed by data

Conjugate prior

$P(\theta)$ is **conjugate** to $P(X|\theta)$:

$\mathcal{A}(v)$

$$P(\theta|X) = \frac{P(X|\theta)P(\theta)}{P(X)}$$

$\mathcal{A}(v')$

The prior is conjugate to the likelihood if the prior P(theta) and the posterior P(theta | X) are in the same distribution.

Why is this important?
This is so that we can iterate it continuously to update our beliefs without changing
the 'model type' (i.e. distribution)

Example

$$P(X|\theta) = \mathcal{N}(X|\theta, \sigma^2)$$

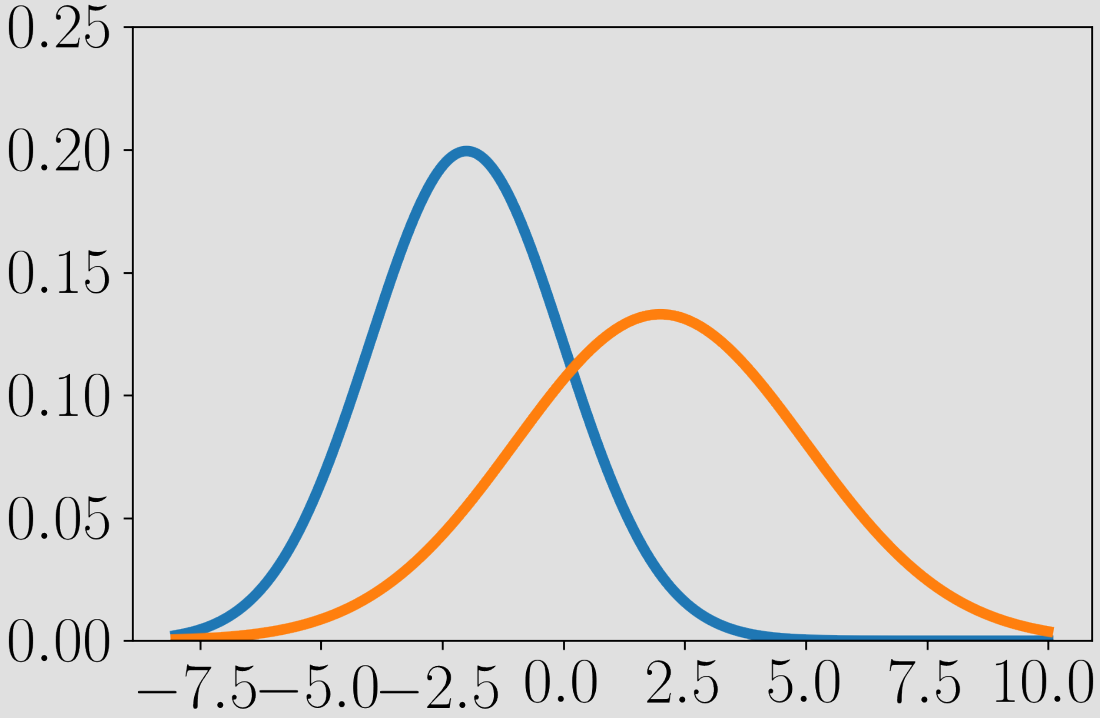
$$\mathcal{A}(v) = ?$$

$$\underbrace{P(\theta|X)}_{\mathcal{A}(v')} = \frac{\overbrace{P(X|\theta)}^{\mathcal{N}(X|\theta, \sigma^2)}\overbrace{P(\theta)}^{\mathcal{A}(v)}}{P(X)}$$

Two Gaussians

$$P(X_1) \sim \mathcal{N}(\mu_1, \sigma_1^2) \qquad P(X_2) \sim \mathcal{N}(\mu_2, \sigma_2^2)$$

$$\mathcal{N}(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \text{const} \cdot e^{-\text{parabola}}$$



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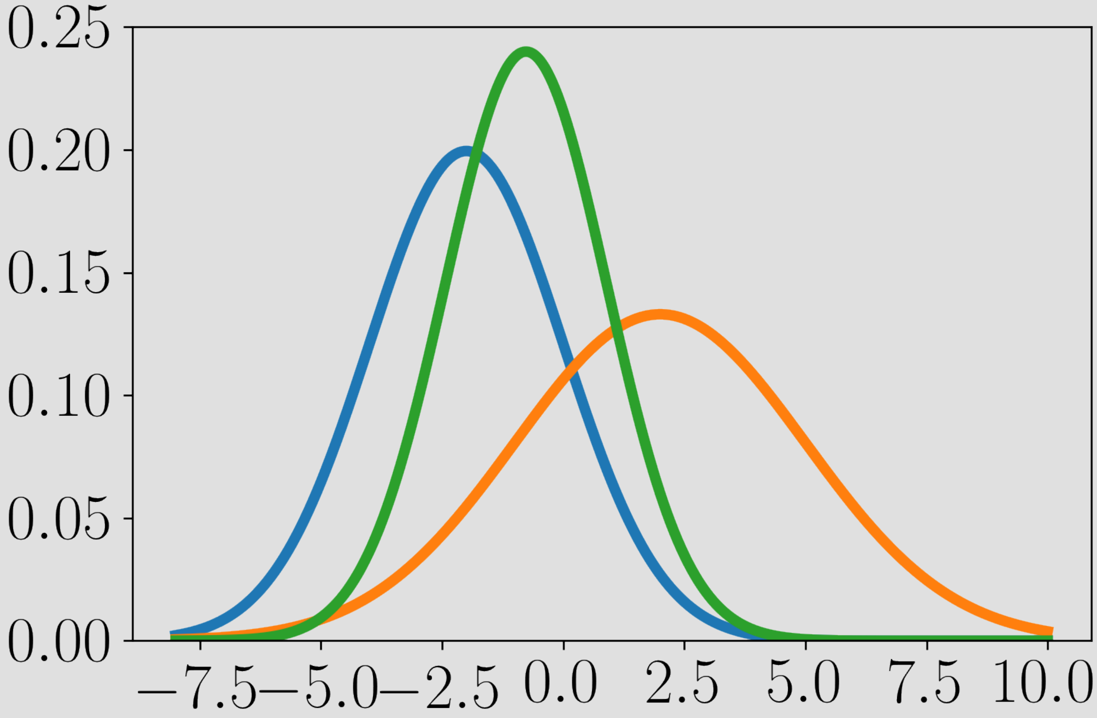

Solution

$$P(X|\theta) = \mathcal{N}(X|\theta, \sigma^2)$$

$$\mathcal{A}(v) = \mathcal{N}(\theta|a, b^2)$$

$$\mathcal{N}(X|\theta, \sigma^2) \qquad \mathcal{N}(\theta|m, s^2)$$

$$P(\theta|X) = \frac{P(X|\theta)P(\theta)}{P(X)}$$

$$\mathcal{N}(\theta|a, b^2)$$

Example

$$p(\theta|x) = \frac{p(x|\theta)p(\theta)}{p(x)} = \frac{\mathcal{N}(x|\theta,1)\mathcal{N}(\theta|0,1)}{p(x)}$$

$$p(\theta|x) \propto e^{-\frac{1}{2}(x-\theta)^2} e^{-\frac{1}{2}\theta^2}$$

$$p(\theta|x) \propto e^{-(\theta-\frac{x}{2})^2}$$

$$p(\theta|x) = \mathcal{N}(\theta|\frac{x}{2},\frac{1}{2})$$

So here, we see that the prior is conjugate to the likelihood.