UNIVERSITY OF TORONTO

Faculty of Applied Science and Engineering

Final Examination

First Year — Program 5

MAT185S — Linear Algebra

Examiners: G S Scott & G M T D'Eleuterio

18 April 2016

Student Name:

Last Name First Names

Student Number:

Instructions:

1. Attempt *all* questions.
2. The value of each question is indicated at the end of the space provided for its solution; a summary is given in the table opposite.
3. Write the final answers *only* in the boxed space provided for each question.
4. *No* aid is permitted.
5. There are 16 pages and 6 questions in this examination paper.

For Examiners Only		
Question	**Value**	**Mark**
A		
1	10	
B		
2	20	
C		
3	15	
4	15	
5	20	
6	20	
Total	100	

A. Definitions and Statements

Fill in the blanks.

1(a). State the *additive inverse* axiom for a vector space.

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/2

1(b). A *subspace* of a vector space is defined as

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/2

1(c). The *row space* of a matrix **A** is defined as

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/2

1(d). The (i, j) *minor* of a matrix $\mathbf{A} \in {}^n\mathbb{R}^n$ is defined as

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/2

1(e). State the *Diagonalization Test* (*i.e.*, necessary and sufficient conditions for the diagonalizability of a matrix).

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/2

B. True or False

Determine if the following statements are true *or* false *and indicate by "$\mathcal{T}$" (for true) and "$\mathcal{F}$" (for false) in the box beside the question. The value of each question is 2 marks.*

2(a). The set $\mathcal{V} = \mathbb{R}^2$ is a vector space over $\mathbb{R}$ with vector addition defined as $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and scalar multiplication as $\alpha(x, y) = (2\alpha x, 2\alpha y)$.

2(b). If $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq \mathbb{R}^3$ such that $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_3$, then B cannot be a basis for $\mathbb{R}^3$.

2(c). The columns of a matrix $\mathbf{A}$ corresponding to the nonzero columns of the row-reduced echelon form of $\mathbf{A}$ form a basis for the column space of $\mathbf{A}$.

2(d). Let $\mathbf{A} \in {}^m\mathbb{R}^n$. Then $\operatorname{null} \mathbf{A} = \{\mathbf{0}\}$ if and only if $\mathbf{Ax} = \mathbf{Ay}$ implies that $\mathbf{x} = \mathbf{y}$.

2(e). Let $\mathbf{A}_1, \mathbf{A}_2 \in {}^2\mathbb{R}^7$. If $\mathbf{v}, \mathbf{w} \in {}^7\mathbb{R}$ such that

$$\mathbf{Av} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \qquad \mathbf{Aw} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

then there is a nonzero linear combination of $\{\mathbf{v}, \mathbf{w}\}$ in $\operatorname{null} \mathbf{A}$.

2(f). The set of functions $\{e^t, te^t, t^2e^t\}$ is linearly independent in the function space $\mathcal{F}(-\infty, \infty)$ over $\mathbb{R}$ with the usual pointwise definition of function (vector) addition and scalar multiplication.

2(g). If $\mathbf{A} \in {}^n\mathbb{R}^n$ is diagonalizable and has all real nonnegative eigenvalues, then there exists $\mathbf{B} \in {}^n\mathbb{R}^n$ such that $\mathbf{B}^2 = \mathbf{A}$.

2(h). Let $\mathbf{A} \in {}^3\mathbb{R}^3$ be diagonalizable. If $\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$ is the eigenequation of $\mathbf{A}$, then $\mathbf{A}^3 - 2\mathbf{A}^2 - \mathbf{A} + 2\mathbf{1} = \mathbf{O}$.

2(i). The matrix

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

is diagonalizable for any choice of $a, b, c \in \mathbb{R}$.

2(j). Let $\mathbf{A} \in {}^n\mathbb{R}^n$ be a matrix with n distinct eigenvalues $\lambda_\alpha, \alpha = 1 \cdots n$. Then $\sum_{\alpha=1}^{n} \operatorname{rank}(\lambda_\alpha \mathbf{1} - \mathbf{A}) = n^2 - n$.

C. Problems

3. Define

$$\mathcal{T}_n = \{\mathbf{A} \in {}^n\mathbb{R}^n \mid \operatorname{tr}\mathbf{A} = 0\}$$

(a) Show that $\mathcal{T}_n$ is a subspace of ${}^n\mathbb{R}^n$.

(b) Determine a basis for $\mathcal{T}_2$.

3(a). Show that $\mathcal{T}_n$ is a subspace of ${}^n\mathbb{R}^n$.

/10

3(b). Determine a basis for $\mathcal{T}_2$.

/5

4. Let $\mathbf{A} \in {}^n\mathbb{R}^n$ and $r_i = \operatorname{rank} \mathbf{A}^i$ for all integers $i \geq 1$.

(a) Prove that $r_i \geq r_{i+1}$ for all i.

(b) Prove that $\mathbf{A}$ is invertible if and only if $r_i = n$ for all i.

4(a). Prove that $r_i \geq r_{i+1}$ for all i.

/7

4(b). Prove that $\mathbf{A}$ is invertible if and only if $r_i = n$ for all i.

/8

5. Let $\mathbf{A} = \mathbf{A}^T \in {}^n\mathbb{R}^n$ with distinct eigenvalues $\lambda_\alpha, \alpha = 1 \cdots n$, and corresponding (nonzero) eigenvectors $\mathbf{p}_\alpha, \alpha = 1 \cdots n$, which satisfy

$$\mathbf{p}_\alpha^T \mathbf{p}_\alpha = 1, \qquad \mathbf{p}_\alpha^T \mathbf{p}_\beta = 0, \ \alpha \neq \beta$$

(a) Let

$$\mathbf{H} = \lambda_1 \mathbf{p}_1 \mathbf{p}_1^T + \cdots + \lambda_n \mathbf{p}_n \mathbf{p}_n^T$$

Show that

$$(\mathbf{A} - \mathbf{H})\mathbf{p}_\alpha = \mathbf{0}$$

for all the eigenvectors $\mathbf{p}_\alpha$.

(b) Show that $\mathbf{A} = \mathbf{H}$.

5(a). Let

$$\mathbf{H} = \lambda_1 \mathbf{p}_1 \mathbf{p}_1^T + \cdots + \lambda_n \mathbf{p}_n \mathbf{p}_n^T$$

Show that

$$(\mathbf{A} - \mathbf{H})\mathbf{p}_\alpha = \mathbf{0}$$

for all the eigenvectors $\mathbf{p}_\alpha$.

. . . cont'd

5. . . . cont'd

/10

5(b). Show that $\mathbf{A} = \mathbf{H}$.

/10

6. A butterfly starts its life as a caterpillar, then transforms into a butterfly and then dies. Let x_1, x_2, x_3 be the number of caterpillars, live butterflies and dead butterflies at any given time t. Suppose that

i. the rate at which caterpillars are born is $r_1 x_2$

ii. the rate at which caterpillars are transformed into butterflies is $r_2 x_1$

iii. the rate at which butterflies die is $r_3 x_2$

Then...

(a) What is the rate of change of caterpillars $\dot{x}_1$? Of live butterflies $\dot{x}_2$? Of dead butterflies $\dot{x}_3$? (Don't forget, for example, that the population of caterpillars depends on their birth rate as well as their transformation rate?)

(b) Let $\mathbf{x}(t) = [\ x_1 \quad x_2 \quad x_3\ ]^T$. Expressing in mathematical terms the ecology described above as $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, find the matrix $\mathbf{A}$.

(c) What is the characteristic polynomial of $\mathbf{A}$?

(d) If $r_1 = \frac{2}{3}, r_2 = 3$ and $r_3 = 2$, determine the eigenvalues of $\mathbf{A}$.

(e) Given these values of r_1, r_2, r_3, find the general solution for $\mathbf{x}(t)$.

(f) If you start with 6000 caterpillars, how many dead butterflies will you have after 10 units of time? (You do not have to evaluate exponentials!)

6(a). What is the rate of change of caterpillars $\dot{x}_1$? Of live butterflies $\dot{x}_2$? Of dead butterflies $\dot{x}_3$? (Don't forget, for example, that the population of caterpillars depends on their birth rate as well as their transformation rate?)

/3

6(b). Let $\mathbf{x}(t) = [\ x_1 \quad x_2 \quad x_3\ ]^T$. Expressing in mathematical terms the ecology described above as $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, find the matrix $\mathbf{A}$.

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6(c). What is the characteristic polynomial of $\mathbf{A}$?

/3

6(d). If $r_1 = \frac{2}{3}, r_2 = 3$ and $r_3 = 2$, determine the eigenvalues of $\mathbf{A}$.

/3

6(e). Given these values of r_1, r_2, r_3, find the general solution for $\mathbf{x}(t)$.

. . . cont'd

6. ... cont'd

/7

6(f). If you start with 6000 caterpillars, how many dead butterflies will you have after 10 units of time? (You do not have to evaluate exponentials!)

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