SOLUTIONS

UNIVERSITY OF TORONTO

Faculty of Applied Science and Engineering

Final Examination

First Year — Program 5

MAT185S — Linear Algebra

Examiners: A D Rennet & G M T D'Eleuterio

21 April 2014

Student Name: Answers Selected

Last Name First Names

Student Number:

Instructions:

1. Attempt *all* questions.
2. The value of each question is indicated at the end of the space provided for its solution; a summary is given in the table opposite.
3. Write the final answers *only* in the boxed space provided for each question.
4. *No* aid is permitted.
5. There are 16 pages and 6 questions in this examination paper.

For Examiners Only		
Question	**Value**	**Mark**
A		
1	10	
B		
2	20	
C		
3	15	
4	15	
5	20	
6	20	
Total	100	

[illegible]

A. Definitions and Statements

Fill in the blanks.

1(a). The *commutativity property* of a vector space states

____

____

/2

1(b). State the definition of a *subspace* $\mathcal{U}$ of a vector space $\mathcal{V}$.

____

____

/2

1(c). State the *dimension formula* for $\mathbf{A} \in {}^m\mathbb{R}^n$.

____

____

/2

1(d). State the definition of *diagonalizability* for $\mathbf{A} \in {}^n\mathbb{R}^n$.

____

____

/2

1(e). State the definition of the *geometric multiplicity* of an eigenvalue.

____

____

/2

B. True or False

Determine if the following statements are true *or* false *and indicate by "$\mathcal{T}$" (for true) and "$\mathcal{F}$" (for false) in the box beside the question. The value of each question is 2 marks.*

2(a). The equation

$$\det \begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0$$

describes a straight line passing through (x_1, y_1) and (x_2, y_2). T

2(b). If $\mathbf{A} \in {}^n\mathbb{R}^n$ is not invertible, then $\operatorname{col} \operatorname{adj} \mathbf{A} \subset \operatorname{null} \mathbf{A}$ and $\operatorname{col} \mathbf{A} \subset \operatorname{null} \operatorname{adj} \mathbf{A}$. F

2(c). The sets $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \subset \mathcal{V}$ and $\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m\} \subset \mathcal{V}$ are both bases for the vector space $\mathcal{V}$ if and only if $m = n$. F

2(d). Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}\} \subset {}^n\mathbb{R}$ such that any subset of n vectors is linearly independent. If

$$\sum_{i=1}^{n+1} \lambda_i \mathbf{x}_i = \mathbf{0}$$

then either all $\lambda_i = 0$ or all $\lambda_i \neq 0$. T

2(e). If

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{O} & \mathbf{C} \end{bmatrix}$$

where $\mathbf{A}$ and $\mathbf{C}$ are square matrices, then $\det \mathbf{M} = \det \mathbf{AC}$. F

2(f). If $\mathbf{A} \in {}^n\mathbb{R}^n$ is upper triangular and $\mathbf{A}^k = \mathbf{O}$ for some integer $k > 1$ but $\mathbf{A} \neq \mathbf{O}$, then $\mathbf{A}$ is not diagonalizable. T

2(g). If $\mathbf{A}^2 = \mathbf{1} \in {}^n\mathbb{R}^n$, then $\mathbf{A}$ has an eigenvalue equal to 1. F

2(h). If $\mathbf{A} \in {}^n\mathbb{R}^n$ is diagonalizable with only one distinct eigenvalue, then $\mathbf{A}$ is diagonal. T

2(i). If $\mathbf{A} \in {}^n\mathbb{R}^n$ is diagonalizable, then $\mathbf{A}$ is invertible. F

2(j). Let $\mathbf{A} \in {}^4\mathbb{R}^4$ have two distinct eigenvalues λ_1, λ_2 each repeated twice. If $\dim \mathcal{E}_{\lambda_1} = 3$ and $\dim \mathcal{E}_{\lambda_2} = 1$, then $\mathbf{A}$ is diagonalizable. F

C. Problems

3. Show that the set of all determinant functions for $n \times n$ real matrices is a vector space over the field $\mathbb{R}$ and under the usual vector addition and scalar multiplication for functions. (Note that the set of functions $f : {}^n\mathbb{R}^n \to \mathbb{R}$ forms a vector space over the field $\mathbb{R}$ and under the usual vector addition and scalar multiplication.)

Conditions for det fcn:

2

1

$$\left.\begin{aligned} &\text{DI.} \quad \Delta_{\times}[\underline{E}(1;i;j)\underline{A}] = \Delta_{\times}(\underline{A}) \\ &\text{DII.} \quad \Delta_{\times}[\underline{E}(\lambda;i)\underline{A}] = \lambda\Delta_{\times}(\underline{A}) \end{aligned}\right\} \Rightarrow \Delta \in D.$$

5

2

Subspace test:

SI. $\Delta_0 = 0$, i.e., $\Delta(\underline{A}) = 0 \quad \forall \underline{A}$.

SII. $\Delta_1 + \Delta_2 \in D$ -2 - 1st error

SIII. $\lambda\Delta \in D$. -1 - 2nd error

5

Apply test:

$$\left.\begin{aligned} \text{SI.} \quad &\Delta_0[\underline{E}(1;i;j)\underline{A}] = \Delta_0(\underline{A}) = 0 \\ &\Delta_0[\underline{E}(\lambda;i)\underline{A}] = \lambda\Delta_0(\underline{A}) = \lambda\cdot 0 = 0 \end{aligned}\right\} \Delta_0 \in D.$$

1

SII $(\Delta_1+\Delta_2)(\underline{A}) = \Delta_1(\underline{A}) + \Delta_2(\underline{A})$

$$\Rightarrow \Delta_1[\underline{E}\underline{A}] = \Delta_1[\underline{E}\underline{A}] + \Delta_2[\underline{E}\underline{A}] = \Delta_1(\underline{A}) + \Delta_2(\underline{A}) = (\Delta_1+\Delta_2)(\underline{A})$$ ✓

$\cancel{(\lambda\Delta)(\underline{A}) =}$

...cont'd

3. ... cont'd

$$(\Delta_1 \oplus \Delta_2)[\underline{E}\underline{A}] = \Delta_1[\underline{E}\underline{A}] + \Delta_2[\underline{E}\underline{A}] = \lambda\Delta_1(\underline{A}) + \lambda\Delta_2(\underline{A})$$

$$= \lambda[\Delta_1(\underline{A}) + \Delta_2(\underline{A})] = \lambda\Delta(\underline{A}) \checkmark$$

2

S III . $(\alpha\Delta)(\underline{A}) = \alpha\Delta(\underline{A})$

$$(\alpha\Delta)[\underline{E}\underline{A}] = \alpha\Delta[\underline{E}\underline{A}] = \alpha\Delta(\underline{A}) = (\alpha\Delta)(\underline{A}) \checkmark$$

$$(\alpha\Delta)[\underline{E}\underline{A}] = \alpha\Delta[\underline{E}\underline{A}] = \alpha\lambda\Delta(\underline{A})$$

$$= \lambda[\alpha\Delta(\underline{A})] = \lambda(\alpha\Delta)(\underline{A}) \checkmark$$

2

5

$\therefore \underline{\Delta}$ is a U.S.

/15

4. Consider the set of polynomials $\mathbb{P}_2$ over the field $\mathbb{R}$ and under vector addition and scalar multiplication defined as

$$(p \oplus q)(x) = p(x) + q(x) - 1, \qquad \forall p, q \in \mathbb{P}_2$$
$$(\alpha \odot p)(x) = \alpha p(x) - \alpha + 1, \qquad \forall \alpha \in \mathbb{R}, p \in \mathbb{P}_2$$

This *is* a vector space.

(a) What is the zero of the vector space?

(b) Determine a basis for the vector space and prove that it is a basis.

(c) Determine the coordinates of $r(x) = 1 + 3x - 2x^2$ in your chosen basis.

4(a). What is the zero of the vector space?

$p(x) = 1$

/3

4(b). Determine a basis for the vector space and prove that it is a basis.

$\{ a, x, x^2 \}$

/8

4(c). Determine the coordinates of $r(x) = 1 + 3x - 2x^2$ in your chosen basis.

$$\begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

/4

5. Let $\mathbf{A} \in {}^m\mathbb{R}^n$, $\mathbf{B} \in {}^n\mathbb{R}^m$ and $m < n$.

(a) Show that if $\mathbf{x} \in {}^m\mathbb{R}$ is an eigenvector of $\mathbf{AB}$ then $\mathbf{Bx}$ is an eigenvector of $\mathbf{BA}$.

(b) Show that if λ is an eigenvalue of $\mathbf{AB}$ then λ is an eigenvalue of $\mathbf{BA}$.

(c) Show furthermore that $\mathbf{BA}$ has an eigenvalue of zero with algebraic multiplicity of at least $(n - m)$.

5(a). Show that if $\mathbf{x} \in {}^m\mathbb{R}$ is an eigenvector of $\mathbf{AB}$ then $\mathbf{Bx}$ is an eigenvector of $\mathbf{BA}$.

/6

5(b). Show that if λ is an eigenvalue of $\mathbf{AB}$ then λ is an eigenvalue of $\mathbf{BA}$.

/6

5(c). Show furthermore that **BA** has an eigenvalue of zero with algebraic multiplicity of at least $(n - m)$.

/8

6. The population dynamics of three neighboring countries evolve according to the following equations:

$$\begin{aligned}\dot{x}_1 &= \ 4x_2 + 4x_3 \\ \dot{x}_2 &= \ \ x_1 + \ \ x_2 - 2x_3 \\ \dot{x}_3 &= -x_1 + \ \ x_2 + 4x_3\end{aligned}$$

where $x_1(t), x_2(t), x_3(t)$ are the populations of the three countries in tens of millions.

(a) Cast the differential equations in the matrix form, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

(b) Determine the eigenvalues of the system.

(c) Determine a basis for each eigenspace of the system.

(d) Explain why $\mathbf{A}$ is diagonalizable.

(e) Determine the unique solution to the system given that the initial values at $t = 0$ are $x_1(0) = 1$ and $x_2(0) = x_3(0) = 2$.

6(a). Cast the differential equations in the matrix form, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

/2

6(b). Determine the eigenvalues of the system.

/6

6(c). Determine a basis for each eigenspace of the system.

...cont'd

6(c). ... cont'd

/6

6(d). Explain why **A** is diagonalizable.

/2

6(e). Determine the unique solution to the system given that the initial values at $t = 0$ are $x_1(0) = 1$ and $x_2(0) = x_3(0) = 2$.

/4

$$\binom{4}{1}\frac{8}{81} + \binom{4}{2}\frac{4}{81} + \binom{4}{3}\frac{2}{81} + \frac{1}{81}$$

$$\frac{1}{81}(32 + 24 + 8 + 1) = \frac{65}{81} \sim \frac{8}{9} \sim .888 \sim \frac{8.9}{10} \sim \left(1 - \frac{1.1}{10}\right)^{56}$$

$$10^{\log \frac{8.9}{10}}$$

$$e^{\ln \frac{8.9}{10}}$$

$$\ln\left(1 - \frac{1.1}{10}\right) = -\frac{1.1}{10}$$

$$= \cancel{0.89}$$

$$-0.11$$

$$10 = e^{2.2} \qquad \sim 10^{-0.28}$$

$$10^{-14}$$

$$\frac{1}{2^{10}} = 10^{4}$$