1. 1 The density of $\vec{u}$, is $f_{\vec{u}} = f_{\vec{x}}(g^{-1}(\vec{u}))||J||$

First, we find the partials:

$$\frac{\partial x_i}{\partial u_j} = \frac{1}{u_{d+1}}$$
$$\frac{\partial x_i}{\partial j_i} = \frac{1}{u_i} + \frac{1}{u_{d+1}}$$

Which gives us the Jacobian:

$$J = \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \cdots & \frac{\partial x_1}{\partial u_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_d}{\partial u_1} & \cdots & \frac{\partial x_n}{\partial u_d} \end{bmatrix} = \begin{bmatrix} \frac{1}{u_1} + \frac{1}{u_{d+1}} & & \frac{1}{u_{d+1}} \\ & \ddots & \\ \frac{1}{u_{d+1}} & & \frac{1}{u_d} + \frac{1}{u_{d+1}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{u_1} \\ \vdots \\ \frac{1}{u_d} \end{bmatrix} I_d + \begin{bmatrix} \vdots \\ \frac{1}{u_{d+1}} \\ \vdots \end{bmatrix}$$

Letting $X = \begin{bmatrix} \frac{1}{u_1} \\ \vdots \\ \frac{1}{u_d} \end{bmatrix}$ and $C = \begin{bmatrix} \vdots \\ \frac{1}{u_{d+1}} \\ \vdots \end{bmatrix}$, Sylvester's Formula gives us that:

$$\begin{aligned} \det(J) &= \det(X \times I_d) \cdot (1 + CX^{-1} \begin{bmatrix} \ldots & 1 & \ldots \end{bmatrix}) \\ &= \left(\prod_{i=1}^{d} \frac{1}{u_i}\right) \left(1 + \left(\frac{\frac{1}{u_{d+1}}}{\frac{1}{u_1}} + \ldots + \frac{\frac{1}{u_{d+1}}}{\frac{1}{u_d}}\right)\right) \\ &= \left(\prod_{i=1}^{d} \frac{1}{u_i}\right) \left(\frac{u_1 + \ldots + u_d}{u_{d+1}} + \frac{u_{d+1}}{u_{d+1}}\right) \\ &= \left(\prod_{i=1}^{d} \frac{1}{u_i}\right) \left(\frac{1 - u_{d+1}}{u_{d+1}} + \frac{u_{d+1}}{u_{d+1}}\right) \\ &= \left(\prod_{i=1}^{d+1} u_i\right)^{-1} \end{aligned}$$

Thus, $||J|| = \left(\prod_{i=1}^{d+1} u_i\right)^{-1}$. We need not worry about the absolute value as each element $u_i \in [0, 1]$.

So we know the density of $\vec{u}$ to be:

$$f_{\vec{u}}(\vec{u}) = |2\pi\Sigma|^{-1/2}\left(\prod_{i=1}^{d+1} u_i\right)^{-1} \exp\left[-\frac{1}{2}\left\{log\left(\frac{u}{u_{d+1}}\right)-\mu\right\}^T \Sigma^{-1}\left\{log\left(\frac{u}{u_{d+1}}\right)-\mu\right\}\right]$$