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STAT 221
Pset 3
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question 1. Show:

(a)

$$\begin{aligned} E\left[\nabla l(\theta; y)\right] &= 0 \text{ at } \theta^* \\ \nabla l(\theta; y) &= \frac{(y - b'(x^T\theta^*)) * x}{\phi} \\ E\left[\frac{y_n - b'(x^T\theta^*)}{\phi}\right] &= \frac{E(Y) - b'(x^T\theta^*)}{\phi} = 0 \\ E(Y) &= b'(x^T\theta^*) \end{aligned}$$

(b)

$$\begin{aligned} \text{We know that } -E\left[\nabla\nabla l(\theta; y)\right] &= E\left[\nabla l(\theta; y)\right]^2 \\ \nabla\nabla l(\theta; y) &= \frac{-b''(x^T\theta^*)}{\phi} \\ E\left[\frac{Y - b'(\lambda)}{\phi}\right]^2 &= \frac{E(Y - E(Y))^2}{\phi^2} = \frac{Var(Y)}{\phi^2} \\ \phi b''(\lambda_n) = \phi h'(\lambda_n) &= var(y_n|\lambda_n) \end{aligned}$$

(c)

$$\begin{aligned} l(\theta; y) &= \frac{\lambda_n y_n - b(\lambda_n)}{\phi} + \log y_n, \phi \\ \nabla l(\theta, y) &= \frac{1}{\phi} * (y_n - b'(x^T\theta))x_n \\ &= \frac{1}{\phi} * (y_n - h(x^T\theta))x_n \end{aligned}$$

(d) This is the fisher's information:

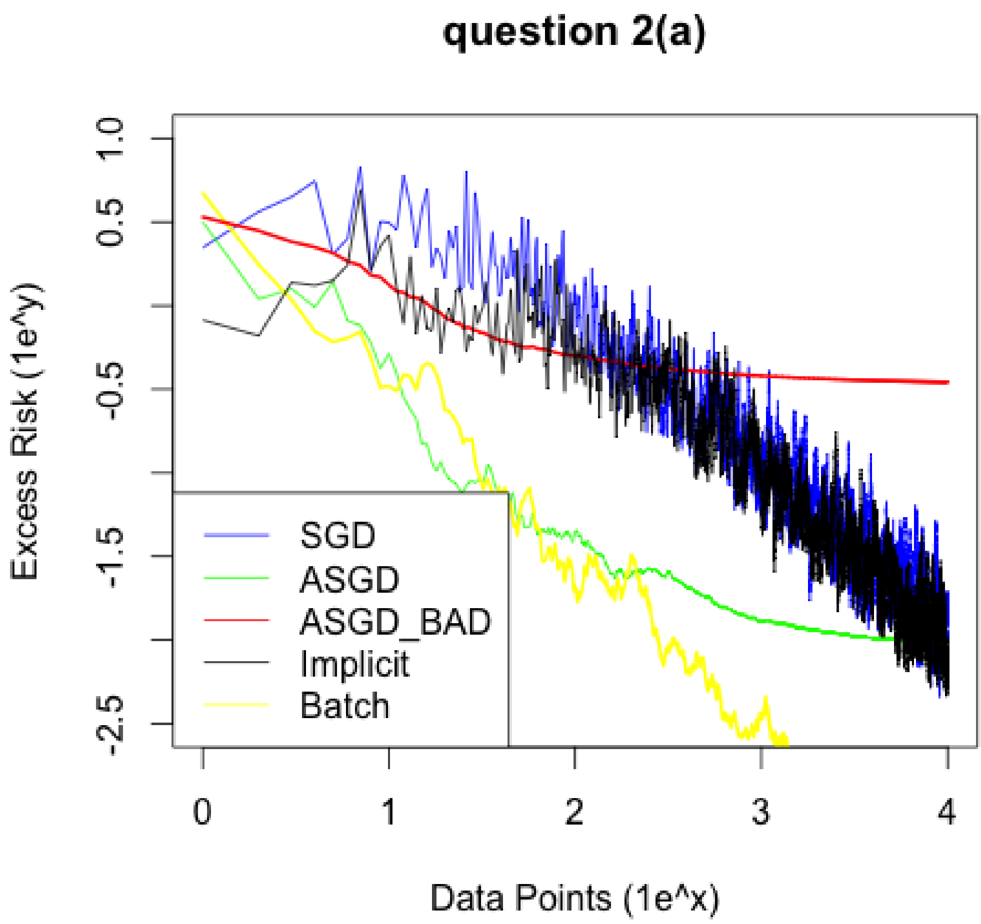
$$\begin{aligned} -\nabla\frac{1}{\phi} * (y_n - h(x^T\theta))x_n &= \nabla\frac{1}{\phi} * h(x^T\theta)x_n \\ I(\theta) &= \frac{1}{\phi}E(h'(x^T\theta)x_n x_n^T) \end{aligned}$$

(e) Because we know the fisher's information must be positive (it's a variance) and we know that $x_n x_n^T$ is positive-definite, then that means $h'(x^T\theta)$ must also be positive. If this is true then it means $h(\cdot)$ is non-decreasing.

question 2. Experiment:

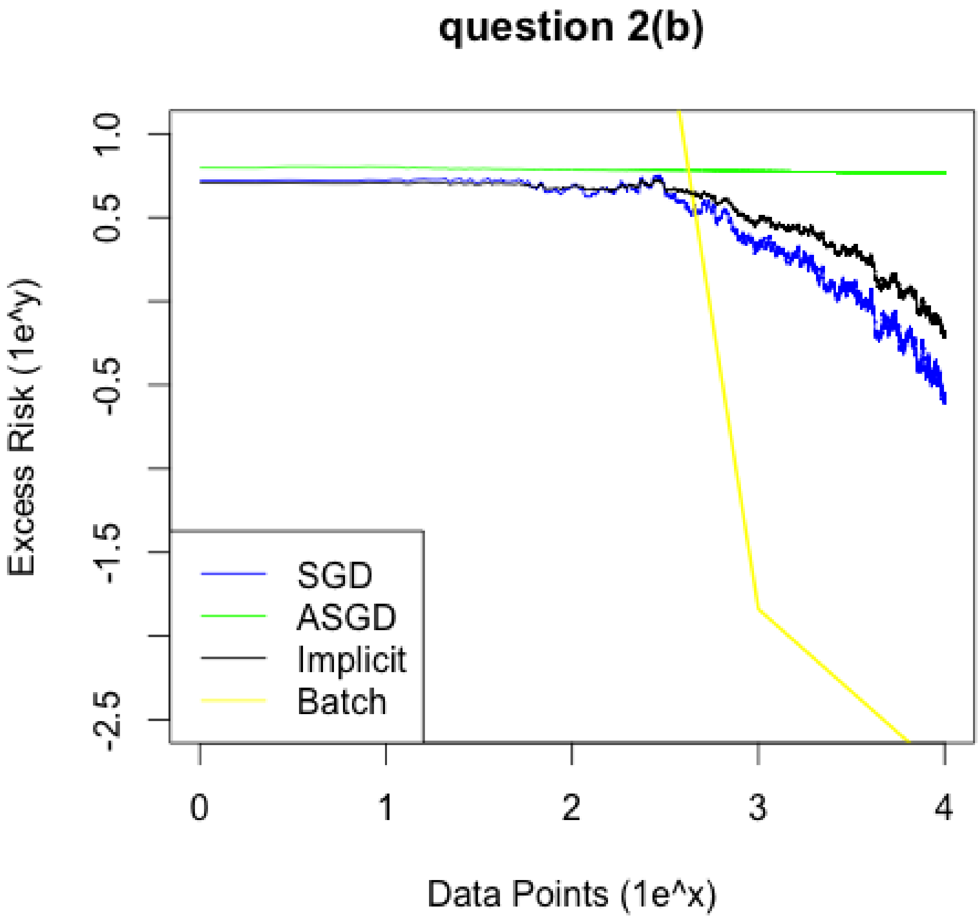
(a)

$$\text{SGD: } \theta_{t+1} = \theta_t + (1 + .02 * t)^{-1} * (\Sigma^{-1} * (\vec{x_t} - \vec{\theta_t}))$$
$$\text{ASGD: } \overline{\theta}_{t+1} = (1 - 1/t) * \overline{\theta}_t + (1/t) * \theta_{t+1}$$
$$\text{where } \theta_{t+1} = \theta_t + (1 + .02 * t)^{-2/3} * (\Sigma^{-1} * (\vec{x_t} - \vec{\theta_t}))$$
$$\text{ASGD\_BAD: Same as ASGD, but with learning rate } (1 + t)^{-1}$$
$$\text{Implicit: } \theta_{t+1} = (1 + \gamma_t)^{-1} * (\theta_t + \gamma_t * \vec{x_t}), \text{ where } \gamma_t = (1 + .02 * t)^{-1}$$



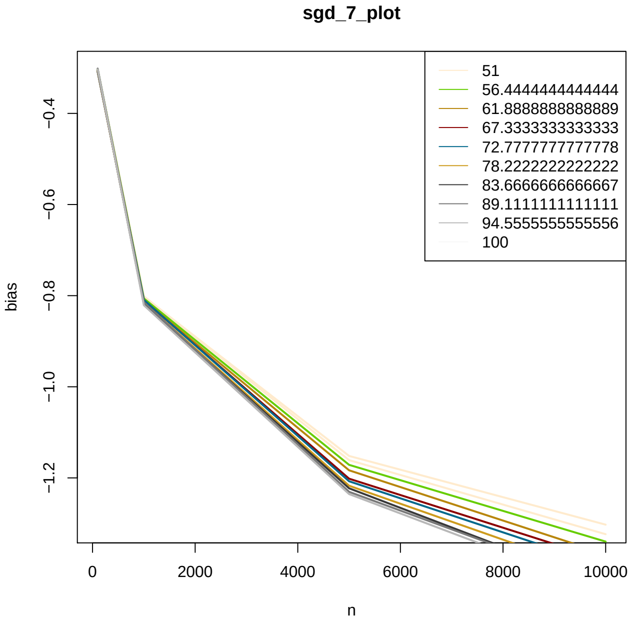
(b) For SGD and implicit, we use a learning rate a_t of $\gamma_0 * (1 + \gamma_0 \lambda_0 t)^{-1}$ and for ASGD we use $\gamma_0 * (1 + \gamma_0 \lambda_0 t)^{-2/3}$. Where $\gamma_0 = tr(A)$ and $\lambda_0 = .01$.

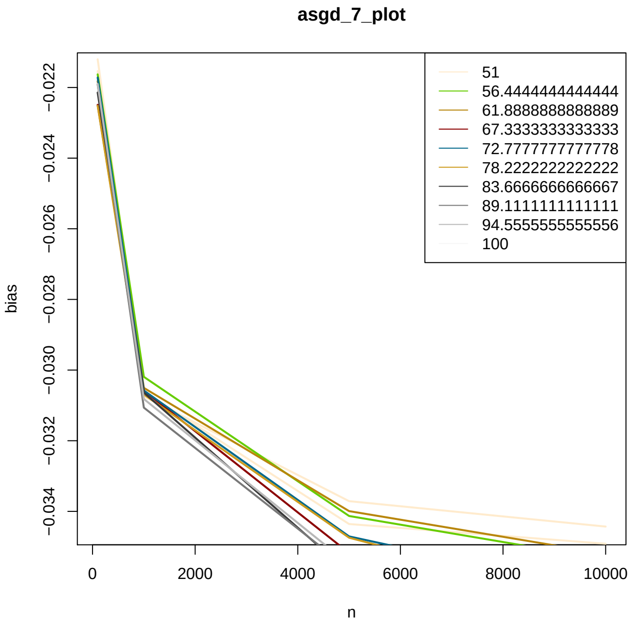
$$
\begin{aligned}
\text{SGD: } \theta_{t+1} &= \theta_t - a_t * x^T \theta_t x + a_t * y * x \\
\text{ASGD: } \overline{\theta}_{t+1} &= (1 - 1/t) * \overline{\theta}_t + (1/t) * \theta_{t+1} \\
\text{Implicit: } \theta_{t+1} &= \theta_t - a_t f_t x^T \theta_t x + a_t y x - a_t^2 f_t y \Sigma(x^2) x \\
& \text{where: } f_t = 1/(1 + a_t \Sigma(x^2)) \\
& \text{Note: this is taken from Panos' distro code} \\
\text{Batch: } & \text{We ran the linear regression using R's lm(y } \sim \text{ x + 0)}
\end{aligned}
$$

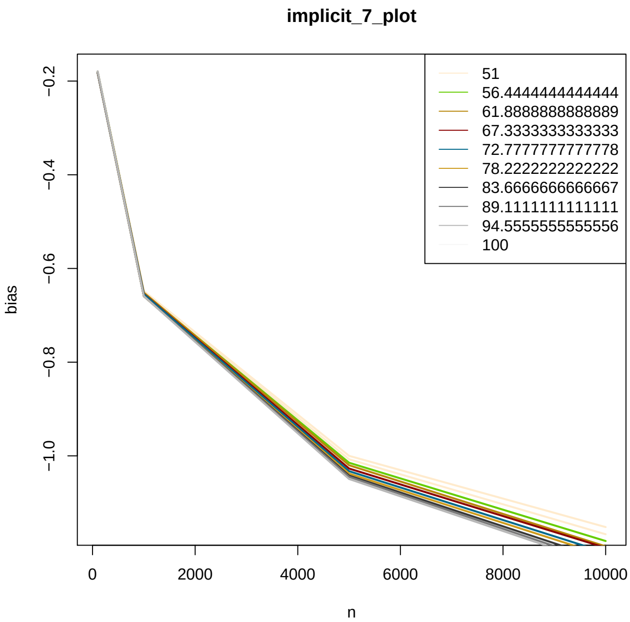


Note: After trying multiple learning rates and averaging rates for ASGD, for some reason we were still unable to produce the correct convergence rate. It may be some misunderstanding on our part or simply a bug in the code, but this will affect also 2(c).

(c) We ran our script task2c.slurm. We ran it for $n = 100, 1000, 5000, 10000$ each at 400 reps. Then we chose amin to be 51 and amax to be 100. All code for this is in task2c_runner.R. Plots below:



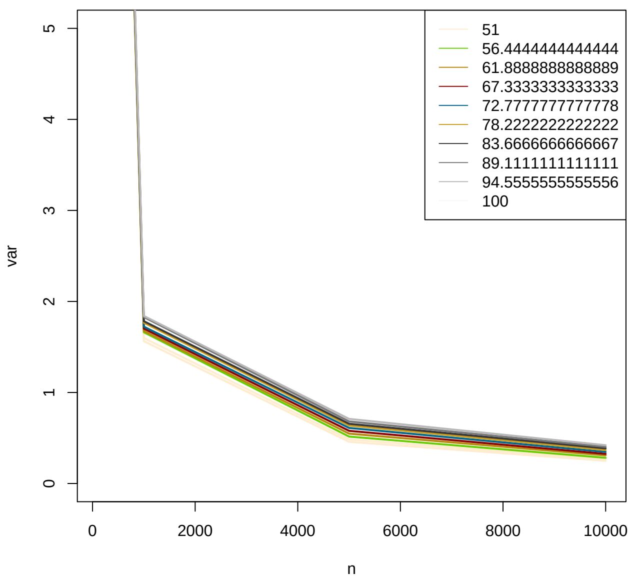




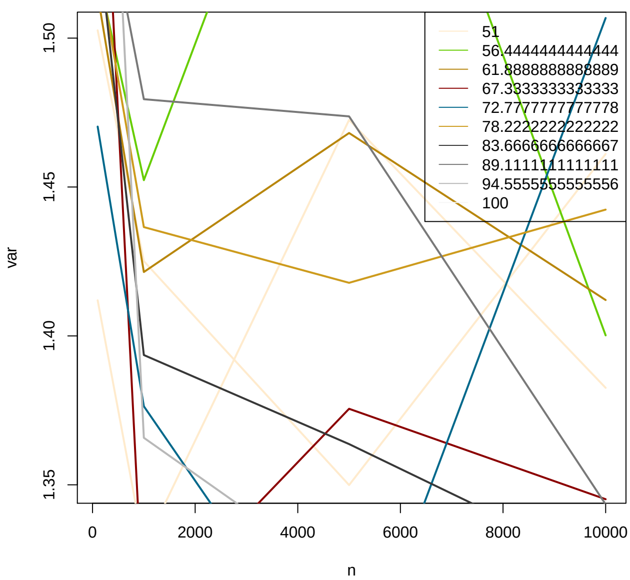
Discussion: Do note that ASGD might be a little off relatively to the other ones because of the learning rate problem we had with 2(b). Regardless, of that looking at the plots of bias over the different α values, we see that it seems that as alpha gets bigger, the bias is relatively smaller. This makes sense because as we have bigger α's we will converge faster because each jump is slightly more powerful.

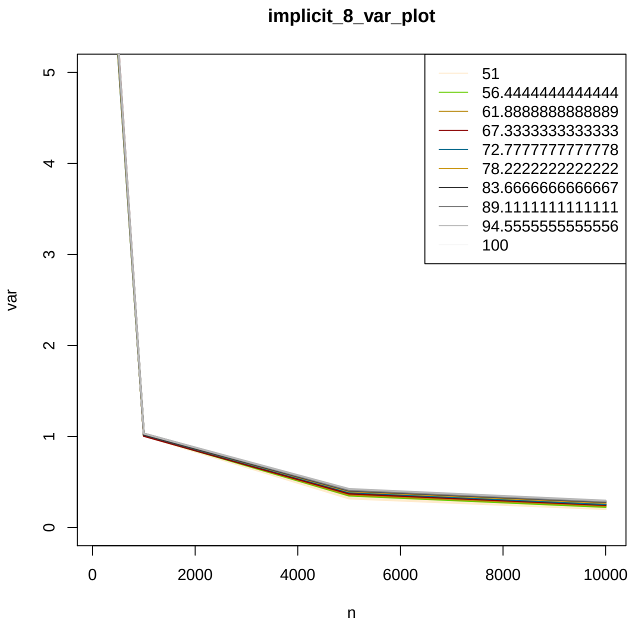
(d) Trace of the empirical variance plotted on a log scale, from same data as part (c):

sgd_8_var_plot



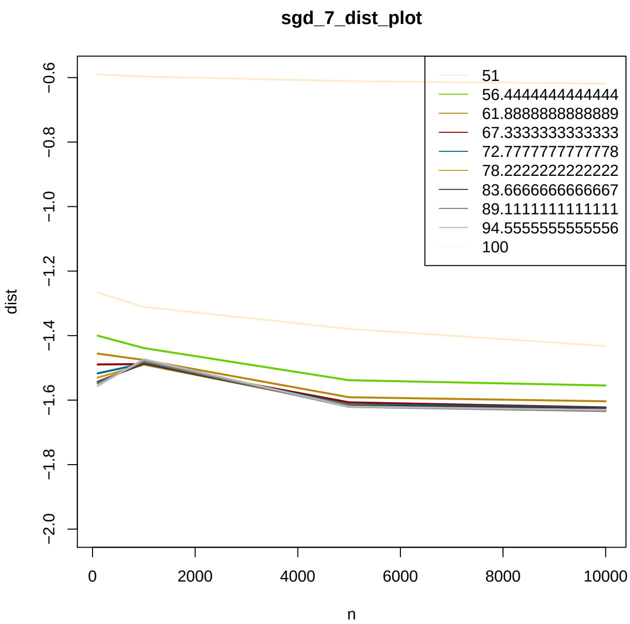
asgd_8_var_plot

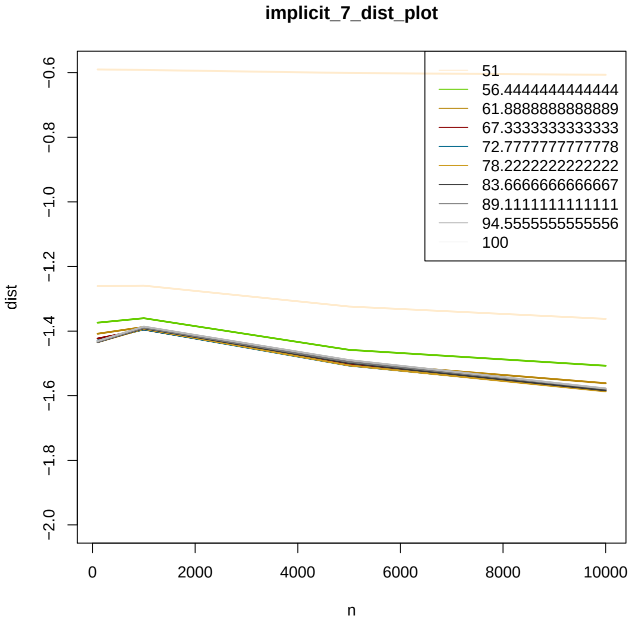




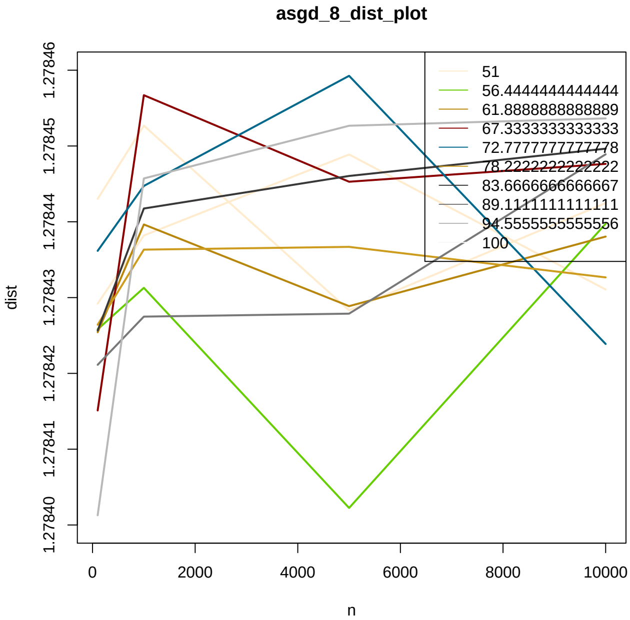
Discussion: Again, ASGD doesn't seem to follow the same trend as the other charts. It's clear that while all the variances decrease, the learning rates with greater α's exhibit more variance. This makes sense as it trades off with the bias from before.

(e) First investigating the asymptotic distances for ASGD and implicit (log-scale):





It does appear that the limit is correct for SGD and Implicit, as $n \to \infty$ the distance decreases. Now looking at the distance from $(1/n) * I(\theta^*)^{-1}$ for ASGD:



Note: This graph is messed up as a result of our ASGD algorithm's mistake from 2(b).
Calculating the ideal optimal value. Because we know that $I(\theta^*)$ is the

optimal variance, we simply want to minimize the following: $\alpha^2(2\alpha I(\theta^*) - I)^{-1} - I)I(\theta^*)$. Empirically for our example, we see that we want to find the minimal of $tr(2\alpha I(\theta^*) - I)^{-1} = -|\Sigma_{i=1}^{p} 1/(2 * \alpha + (\lambda_i - 1)) - 1|$. Running this through R's optim function we get: 50.238. Looking at the sgd and implicit plots, we see that our lowest variance is indeed 51 which is the closest α to 50.23. It does make us question, however, the distances plot. While the distances plot was calculated assuming a rate of just α/n, we actually used a more complicated rate as explained above in 2(b).

question 3. Part 3

(a.) The elastic net penalty is a compromise between ridge-regression penalty and lasso penalty, behaving more like the former for alphas close to 0 and more like the latter for alphas close to 1. The quadratic term in the regularized problem is an approximation of our log-likelihood which we get by doing a Taylor expansion about our current estimates. To take advantage of cases where the feature matrix X is sparse (like in a bag of words model), only the non-zero entries of the matrix (and their coordinates) are stored. Since it uses coordinate descent, the inner-product operations can sum over only the non-zero entries (since the other calculations would just yield 0 as the result).

(b.) Tables 1 through 6 below reproduce the Table 1 in the paper.
Note: We did the lars part before Panos said it wasn't necessary on Piazza. Additionally, Tables 11 through 16 have the corresponding MSEs from the glmnet runs so we can make adequete comparisons with SGD.

(c.) Tables 7 through 10 correspond to tables 1-4 (and their respective MSE tables 11-14) but give results for SGD (normal and implicit) rather than glmnet. In general we see that SGD (both normal and implicit) is much slower than glmnet. Additionally it appears that SGD had a lot higher MSE than glmnet on the first three cases, though they were pretty similar on the 4th (N=100, p = 5000), and I'm guessing it would be close for the last two as well. This is pretty surprising, as I'd expect SGD to have a lower MSE. Also it should be noted that implicit SGD generally had lower MSEs than normal SGD.

(d.) Tables 17 and 18 show runtime and MSE running the glmnet part over a larger sample size and more covariates. We see that while it took much longer to run, MSE is a lot smaller than when we had smaller sample sizes (with a similar number of covariates). We also see that with a large sample size, the difference in MSE between glmnet type = "naive" and type = "cov" dissapears, though "naive" runs a lot faster for low rho values and "cov" runs faster when we have high rho values. Ideally we could have compared these with results from SGD with the same number of covariates, but the code for that seems to take too long to run (which is why it is commented out). If we had more time we could potentially try running that part on Odyssey to speed things up.

Table 1: N = 1000, p = 100

	0	0.1	0.2	0.5	0.9	0.95
glmnet (type = "naive")	0.035	0.037	0.041	0.058	0.144	0.249
glmnet (type = "cov")	0.010	0.011	0.011	0.011	0.017	0.021
lars	0.210	0.207	0.216	0.214	0.210	0.213

Table 2: N = 5000, p = 100

	0	0.1	0.2	0.5	0.9	0.95
glmnet (type = "naive")	0.129	0.129	0.148	0.192	0.480	0.804
glmnet (type = "cov")	0.033	0.034	0.032	0.035	0.038	0.039
lars	1.023	1.053	1.036	1.047	1.038	1.029

Table 3: N = 100, p = 1000

	0	0.1	0.2	0.5	0.9	0.95
glmnet (type = "naive")	0.026	0.024	0.025	0.030	0.048	0.051
glmnet (type = "cov")	0.047	0.045	0.051	0.070	0.151	0.136
lars	0.319	0.296	0.303	0.288	0.306	0.327

Table 4: N = 100, p = 5000

	0	0.1	0.2	0.5	0.9	0.95
glmnet (type = "naive")	0.073	0.075	0.073	0.086	0.110	0.194
glmnet (type = "cov")	0.237	0.257	0.256	0.312	0.655	0.679
lars	1.893	1.658	1.619	1.694	1.758	1.578

Table 5: N = 100, p = 20000

	0	0.1	0.2	0.5	0.9	0.95
glmnet (type = ”naive”)	0.264	0.268	0.269	0.288	0.347	0.528
glmnet (type = ”cov”)	0.981	1.034	1.069	1.267	2.348	2.777
lars	6.854	7.487	7.398	7.351	8.118	7.584

Table 6: N = 100, p = 50000

	0	0.1	0.2	0.5	0.9	0.95
glmnet (type = ”naive”)	0.657	0.714	0.731	0.722	0.839	1.598
glmnet (type = ”cov”)	2.566	2.666	2.536	3.227	6.326	8.347
lars	22.491	22.561	25.666	25.018	23.115	24.605

Table 7: N = 1000, p = 100

	0	0.1	0.2	0.5	0.9	0.95
SGD Time	0.261	0.263	0.258	0.276	0.266	0.268
Implicit SGD Time	0.246	0.246	0.245	0.248	0.251	0.250
SGD MSE	0.147	0.157	0.148	0.168	0.163	0.154
Implicit SGD MSE	0.129	0.132	0.133	0.133	0.142	0.139

Table 8: N = 5000, p = 100

	0	0.1	0.2	0.5	0.9	0.95
SGD Time	5.486	5.540	5.665	5.529	5.539	5.620
Implicit SGD Time	5.476	5.516	5.605	5.563	5.617	5.577
SGD MSE	4.421	4.342	4.275	4.133	4.150	4.187
Implicit SGD MSE	4.399	4.274	4.162	4.136	4.151	4.192

Table 9: N = 100, p = 1000

	0	0.1	0.2	0.5	0.9	0.95
SGD Time	1.643	1.607	1.623	1.617	1.617	1.631
Implicit SGD Time	1.602	1.623	1.629	1.622	1.631	1.659
SGD MSE	0.037	0.036	0.040	0.039	0.039	0.035
Implicit SGD MSE	0.035	0.037	0.043	0.040	0.040	0.035

Table 10: N = 100, p = 5000

	0	0.1	0.2	0.5	0.9	0.95
SGD Time	172.810	172.320	172.874	172.882	172.961	174.794
Implicit SGD Time	172.752	175.411	175.180	177.150	175.977	177.233
SGD MSE	0.820	0.827	0.881	0.865	0.881	0.831
Implicit SGD MSE	0.809	0.867	0.938	0.993	1.004	0.910

Table 11: MSEs: N = 1000, p = 100

	0	0.1	0.2	0.5	0.9	0.95
glmnet (type = "naive")	0.045	0.045	0.045	0.048	0.057	0.080
glmnet (type = "cov")	0.039	0.044	0.040	0.044	0.062	0.072

Table 12: MSEs: N = 5000, p = 100

	0	0.1	0.2	0.5	0.9	0.95
glmnet (type = "naive")	0.028	0.027	0.028	0.030	0.050	0.070
glmnet (type = "cov")	0.027	0.028	0.028	0.029	0.051	0.068

Table 13: MSEs: N = 100, p = 1000

	0	0.1	0.2	0.5	0.9	0.95
glmnet (type = "naive")	0.064	0.060	0.063	0.060	0.067	0.071
glmnet (type = "cov")	0.062	0.064	0.064	0.057	0.064	0.069

Table 14: MSEs: N = 100, p = 5000

	0	0.1	0.2	0.5	0.9	0.95
glmnet (type = "naive")	0.033	0.031	0.032	0.031	0.030	0.032
glmnet (type = "cov")	0.030	0.033	0.029	0.030	0.032	0.031

Table 15: MSEs: N = 100, p = 20000

	0	0.1	0.2	0.5	0.9	0.95
glmnet (type = "naive")	0.016	0.017	0.015	0.016	0.016	0.017
glmnet (type = "cov")	0.017	0.017	0.016	0.017	0.017	0.017

Table 16: MSEs: N = 100, p = 50000

	0	0.1	0.2	0.5	0.9	0.95
glmnet (type = "naive")	0.010	0.011	0.011	0.011	0.011	0.010
glmnet (type = "cov")	0.010	0.010	0.010	0.011	0.011	0.010

Table 17: Times: N = 100000, p = 1000

	0	0.1	0.2	0.5	0.9	0.95
glmnet (type = "naive")	14.419	14.793	15.274	16.445	41.139	60.373
glmnet (type = "cov")	50.527	50.504	50.647	50.268	50.955	49.562

Table 18: MSEs: N = 100000, p = 1000

	0	0.1	0.2	0.5	0.9	0.95
glmnet (type = "naive")	0.005	0.005	0.005	0.006	0.011	0.018
glmnet (type = "cov")	0.005	0.005	0.005	0.006	0.011	0.018