

Ejercicio 6 clase 5: para \mathbb{R}^3 si $\{e_i\}$ define un sistema de coordenadas (dextrogiro), demuestre

a)
$$e^i = \frac{e_j \times e_k}{e_i \cdot (e_j \times e_k)} ; i = j = k = 1, 2, 3 \text{ y sus permutaciones cíclicas}$$

Dados e_i, e_j y $e_k \Rightarrow$ Se puede tener $e^i = \alpha (e_j \times e_k)$
ya que $e_i \cdot e^j = \delta_i^j$ además, $e_i \cdot e^i = 1$

$$\Rightarrow e_i \cdot \alpha (e_j \times e_k) = 1 \Rightarrow \alpha = \frac{1}{e_i (e_j \times e_k)}$$

$$\Rightarrow (e_j \times e_k) \alpha = \frac{e_j \times e_k}{e_i (e_j \times e_k)} \Rightarrow e^i = \alpha (e_j \times e_k) = \frac{e_j \times e_k}{e_i (e_j \times e_k)}$$

$$\Rightarrow \boxed{e^i = \frac{e_j \times e_k}{e_i (e_j \times e_k)}}$$

b) Si $V = e_1 \cdot (e_2 \times e_3)$ y $\tilde{V} = e^1 \cdot (e^2 \times e^3) \Rightarrow V \tilde{V} = 1$

$$e^1 = \frac{e_2 \times e_3}{e_1(e_2 \times e_3)} \quad ; \quad e_1 = \frac{e^2 \times e^3}{e^1(e^2 \times e^3)}$$

$$e_2 \times e_3 = \alpha e_1$$

$$\alpha(e_2 \times e_3) = e^1$$

$$\alpha(e^2 \times e^3) = e^1$$

$$(e^2 \times e^3) = \alpha e_1$$

$$V \tilde{V} = e_1 \cdot \frac{e^1}{\alpha} \cdot e^1 \cdot \alpha e_1 = 1 \cdot 1 = 1$$

c) $a \cdot e^i = 1$ como $V = \{e^i, e^j, e^k\}$
 $V^* = \{e_i, e_j, e_k\}$

como $e^i \perp e_j, e_k$; $e^j \perp e_i, e_k$; $e^k \perp e_i, e_j$

$e_i \perp e^j, e^k$; $e_j \perp e^i, e^k$ y $e_k \perp e^i, e^j$

\Rightarrow la única forma de que $a \cdot e^i = 1$, es que

$a = e_i$ siendo este único

d) Sea $W_i = \begin{cases} W_1 = 4\hat{i} + 2\hat{j} + \hat{k} \\ W_2 = 3\hat{i} + 3\hat{j} \\ W_3 = 2\hat{k} \end{cases}$ como ya tenemos $e^i = \frac{e_j \times e_k}{e_i(e_j \times e_k)}$

$$\Rightarrow W^1 = \frac{W_2 \times W_3}{W_1 \cdot (W_2 \times W_3)} = \frac{6\hat{i} - 6\hat{j}}{12} \Rightarrow W^1 = \frac{1}{2}\hat{i} - \frac{1}{2}\hat{j}$$

$$W^2 = -\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j}$$

$$y \quad W^3 = -\frac{1}{4}\hat{i} + \frac{1}{4}\hat{j} + \frac{1}{2}\hat{k}$$

$$2) \quad a = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\Rightarrow |a\rangle = a^i |w_i\rangle = a^1 w_1 + a^2 w_2 + a^3 w_3$$

$$\Rightarrow 4a^1 + 3a^2 = 1$$

$$2a^1 + 3a^2 = 2$$

$$a^1 + 2a^3 = 3$$

$$\Rightarrow \begin{cases} a^1 = -\frac{1}{2} \\ a^2 = 1 \\ a^3 = \frac{7}{4} \end{cases}$$

} contravariantes

$$y \quad \langle a| = a_i \langle w^i|$$

$$\frac{1}{2}a_1 - \frac{1}{3}a_2 - \frac{1}{4}a_3 = 1$$

$$-\frac{1}{2}a_1 - \frac{2}{3}a_2 + \frac{1}{4}a_3 = 2$$

$$\frac{1}{2}a_3 = 3$$

$$\Rightarrow \begin{cases} a_1 = 3 \\ a_2 = -3 \\ a_3 = 6 \end{cases} \text{ covariantes}$$

Ejercicio 1 : con $\langle a|b \rangle \equiv \text{Tr}(A^\dagger B)$

y $\sigma_i = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ encontrar $\sigma^i = \{\sigma^0, \sigma^1, \sigma^2, \sigma^3\}$

$$\text{Tr}(\sigma_0^T \cdot \sigma^0) = \text{Tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11}^0 & a_{12}^0 \\ a_{21}^0 & a_{22}^0 \end{bmatrix}\right) = \text{Tr}\begin{pmatrix} a_{11}^0 & a_{12}^0 \\ a_{21}^0 & a_{22}^0 \end{pmatrix} = 1$$

$$\Rightarrow a_{11}^0 + a_{22}^0 = 1$$

$$\left. \begin{aligned} \text{Tr}(\sigma_1^T \cdot \sigma^0) &= \text{Tr}\begin{pmatrix} a_{21}^0 & a_{22}^0 \\ a_{11}^0 & a_{12}^0 \end{pmatrix} = a_{21}^0 + a_{12}^0 = 0 \\ \text{Tr}(\sigma_2^T \cdot \sigma^0) &= \text{Tr}\begin{pmatrix} -a_{21}^0 & -a_{22}^0 \\ a_{11}^0 & a_{12}^0 \end{pmatrix} = -a_{21}^0 + a_{12}^0 = 0 \end{aligned} \right\} \Rightarrow \begin{aligned} a_{12}^0 &= 0 \\ a_{21}^0 &= 0 \end{aligned}$$

$$\text{Tr}(\sigma_3^T \cdot \sigma^0) = \text{Tr}\begin{pmatrix} a_{11}^0 & a_{12}^0 \\ -a_{21}^0 & -a_{22}^0 \end{pmatrix} = a_{11}^0 - a_{22}^0 = 0 \Rightarrow a_{11}^0 = \frac{1}{2}$$

$$\Rightarrow a_{22}^0 = \frac{1}{2}$$

para σ^1 :

$$\text{Tr}(\sigma_0^T \cdot \sigma^1) = \text{Tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11}^1 & a_{12}^1 \\ a_{21}^1 & a_{22}^1 \end{bmatrix}\right) = a_{11}^1 + a_{22}^1 = 0$$

$$\text{Tr}(\sigma_1^T \cdot \sigma^1) = \text{Tr}\begin{pmatrix} a_{21}^1 & a_{22}^1 \\ a_{11}^1 & a_{12}^1 \end{pmatrix} = a_{21}^1 + a_{12}^1 = 1$$

$$\text{Tr}(\sigma_2^T \cdot \sigma^1) = -a_{21}^1 + a_{12}^1 = 0 \quad ; \quad \text{Tr}(\sigma_3^T \cdot \sigma^1) = a_{11}^1 - a_{22}^1 = 0$$

$$\Rightarrow 2a_{12}^1 = 1 \Rightarrow a_{12}^1 = \frac{1}{2} \quad ; \quad a_{21}^1 = \frac{1}{2} \quad ; \quad a_{11}^1 = a_{22}^1 = 0$$

$$\Rightarrow \boxed{\begin{aligned} \sigma^0 &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad ; \quad \sigma^1 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad ; \quad \sigma^2 = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \\ \sigma^3 &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \end{aligned}}$$