Ejercicio 6 clase 5: para $\mathbb{R}^3$ si $\{e_i\}$ define un sistema de coordenadas (dextrógiro), demuestre

a) $$e^i = \frac{e_j \times e_k}{e_i \cdot (e_j \times e_k)} \; ; \; i = j = k = 1, 2, 3$$ y sus permutaciones cíclicas

Dados e_i, e_j y e_k $\Rightarrow$ Se puede tener $e^i = \alpha (e_j \times e_k)$

ya que $e_i \cdot e^j = \delta_i^j$ además, $e_1 \cdot e^1 = 1$

$$\Rightarrow \quad e_i \cdot \alpha (e_j \times e_k) = 1 \quad \Rightarrow \quad \alpha = \frac{1}{e_i (e_j \times e_k)}$$

$$\Rightarrow \quad (e_j \times e_k)\alpha = \frac{e_j \times e_k}{e_i (e_j \times e_k)} \quad \Rightarrow \quad e^i = \alpha (e_j \times e_k) = \frac{e_j \times e_k}{e_i (e_j \times e_k)}$$

$$\Rightarrow \quad \boxed{e^i = \frac{e_j \times e_k}{e_i (e_j \times e_k)}}$$

b) Si $V = e_1 \cdot (e_2 \times e_3)$ y $\tilde{V} = e^1 \cdot (e^2 \times e^3)$ $\rightarrow$ $V\tilde{V} = 1$

$\Rightarrow$ $V\tilde{V} = e_1 \cdot (e_2 \times e_3) \cdot e^1 \cdot (e^2 \times e^3)$ como $e_2 \times e_3 = e_1$

y $e^2 \times e^3 = e^1$

$\Rightarrow$ $V\tilde{V} = e_1 \cdot e_1 \cdot e^1 \cdot e^1 = e_1 \cdot e^1$

$\Rightarrow$ $\boxed{V\tilde{V} = 1}$

c) $a \cdot e^i = 1$ como $V = \{e^i, e^j, e^k\}$

$V^* = \{e_i, e_j, e_k\}$

como $e^i \perp e_j, e_k$; $e^j \perp e_i, e_k$; $e^k \perp e_i, e_j$

$e_i \perp e^j, e^k$; $e_j \perp e^i, e^k$ y $e_k \perp e^i, e^j$

$\Rightarrow$ la única forma de que $a \cdot e^i = 1$, es que

$\boxed{a = e^i}$ siendo este único

d) Sea $W_i = \begin{cases} W_1 = 4\hat{\imath} + 2\hat{\jmath} + \hat{k} \\ W_2 = 3\hat{\imath} + 3\hat{\jmath} \\ W_3 = 2\hat{k} \end{cases}$ como ya tenemos $e^i = \dfrac{e_j \times e_k}{e_i \cdot (e_j \times e_k)}$

$\Rightarrow$ $W^1 = \dfrac{W_2 \times W_3}{W_1 \cdot (W_2 \times W_3)} = \dfrac{6\hat{\imath} - 6\hat{\jmath}}{12}$ $\Rightarrow$ $\boxed{W^1 = \frac{1}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath}}$

$\boxed{W^2 = -\frac{1}{3}\hat{\imath} - \frac{2}{3}\hat{\jmath}}$ y $\boxed{W^3 = -\frac{1}{4}\hat{\imath} + \frac{1}{4}\hat{\jmath} + \frac{1}{2}\hat{k}}$

2) $a = \hat{i} + 2\hat{j} + 3\hat{k}$

$\Rightarrow |a\rangle = a^i |w_i\rangle = a^1 w_1 + a^2 w_2 + a^3 w_3$

$$\Rightarrow \begin{aligned} 4a^1 + 3a^2 &= 1 \\ 2a^1 + 3a^2 &= 2 \\ a^1 + 2a^3 &= 3 \end{aligned} \Rightarrow \left.\begin{aligned} a^1 &= -1/2 \\ a^2 &= 1 \\ a^3 &= 7/4 \end{aligned}\right\} \text{contravariantes}$$

$\langle a| = a_i \langle w^i|$

$$\begin{aligned} \frac{1}{2}a_1 - \frac{1}{3}a_2 - \frac{1}{4}a_3 &= 1 \\ -\frac{1}{2}a_1 - \frac{2}{3}a_2 + \frac{1}{4}a_3 &= 2 \\ \frac{1}{2}a_3 &= 3 \end{aligned} \Rightarrow \left.\begin{aligned} a_1 &= 3 \\ a_2 &= -3 \\ a_3 &= 6 \end{aligned}\right\} \text{covariantes}$$

Ejercicio 7 : con $\langle a|b\rangle \rightleftarrows Tr(A^{+}B)$

y $\sigma_i = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ encontrar $\sigma^i = \{\sigma^0, \sigma^1, \sigma^2, \sigma^3\}$

$$Tr(\sigma_0^T \cdot \sigma^0) = Tr\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}a_{11}^0 & a_{12}^0\\ a_{21}^0 & a_{22}^0\end{bmatrix}\right) = Tr\begin{pmatrix}a_{11}^0 & a_{12}^0\\ a_{21}^0 & a_{22}^0\end{pmatrix} = 1$$

$$\Rightarrow a_{11}^0 + a_{22}^0 = 1$$

$$Tr(\sigma_1^T \cdot \sigma^0) = Tr\begin{pmatrix}a_{21}^0 & a_{22}^0\\ a_{11}^0 & a_{12}^0\end{pmatrix} = a_{21}^0 + a_{12}^0 = 0$$

$$Tr(\sigma_2^T \cdot \sigma^0) = Tr\begin{pmatrix}-a_{21}^0 \hat{\jmath} & -a_{22}^0 \hat{\jmath}\\ a_{11}^0 \hat{\jmath} & a_{12}^0 \hat{\jmath}\end{pmatrix} = -a_{21}^0 + a_{12}^0 = 0$$

$$\Rightarrow a_{12}^0 = 0 \qquad a_{21}^0 = 0$$

$$Tr(\sigma_3^T \cdot \sigma^0) = Tr\begin{pmatrix}a_{11}^0 & a_{12}^0\\ -a_{21}^0 & -a_{22}^0\end{pmatrix} = a_{11}^0 - a_{22}^0 = 0 \quad \Rightarrow a_{11}^0 = \frac{1}{2}$$

$$\Rightarrow a_{22}^0 = 1/2$$

$$\Rightarrow \quad \sigma^0 = \begin{pmatrix}1/2 & 0\\ 0 & 1/2\end{pmatrix} \quad \Rightarrow \quad \sigma^1 = \begin{pmatrix}0 & 1/2\\ 1/2 & 0\end{pmatrix}$$

$$\sigma^2 = \begin{pmatrix}0 & -\hat{\imath}/2\\ \hat{\imath}/2 & 0\end{pmatrix} \quad ; \quad \sigma^3 = \begin{pmatrix}1/2 & 0\\ 0 & -1/2\end{pmatrix}$$