

Action-based Milky Way Disk Modelling with *RoadMapping* and our imperfect Knowledge of the “Real World”

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0.1. Effect of measurement errors on recovery of potential?

[TO DO]

Collection tests and plots (tests are still running on the cluster)

- *Plot 1*: number of MC samples needed for the error convolution vs. maximum velocity error inside the observed volume, such that a given accuracy in potential and qDF parameters is reached. Similar to what I had on the poster. However, we still haven’t tested, if this plot depends on: hotness of stars and or umber of stars.
- *Plot 2*: 2 columns of panels (one row for each parameter), bias vs. standard error. First column: only proper motion and vlos errors → shows that our error convolution works and should be bias free, plus, when knowing the errors perfectly we can get a perfect deconvolution and tight constraints. Second column: proper motion, vlos and distance modulus errors → shows that for too large proper motion and distance errors our approximation for the error convolution does not work anymore.

Convergence of the error integral. In §?? we introduced how we convolve the model probability with the measurement errors. In the absence of distance errors the accuracy of the parameter recovery is limited by an insufficient MC sampling of the convolution integral in Equation (??). Test ⑥ in Table 3 and Figure 1 investigate how many MC samples are needed, given the size of the velocity error, for the integral to be accurate within certain

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limits:

For each $\delta\mu \in [2, 3, 4, 5]\text{mas yr}^{-1}$ we set up four mock data sets and evaluated the likelihood for different N_{error} . We used $N_{\text{conv}} := 800$ and 1200 MC samples to calculate the numerically converged likelihood for proper motion errors $\delta\mu \leq 3\text{mas yr}^{-1}$ and $\delta\mu > 3\text{mas yr}^{-1}$, respectively. We determined the mean bias

$$\text{BIAS}(N_{\text{error}}, \delta\mu) \equiv \frac{1}{4} \sum_{j=1}^4 [\langle p_i \rangle(N_{\text{error}}, \delta\mu)]_j - [\langle p_i \rangle(N_{\text{conv}}, \delta\mu)]_j,$$

where $[\langle p_i \rangle(N_{\text{error}}, \delta\mu)]_j$ is the best estimate for the i -th model parameter $p_i \in p_M$ from the analysis of the j -th mock data realisation with $\delta\mu$ using N_{error} MC samples. From this we then generated the curves $N_{\text{error},i}(\delta v_{\text{max}}, \text{BIAS})$ in Figure 1 by linear interpolation, that show how many MC samples are needed for parameter p_i given a velocity error and a systematic bias in units of the standard error (SE) of the estimate. The proper motion error $\delta\mu$ translates to a velocity error according to

$$\delta v_{\text{max}}[\text{km s}^{-1}] \equiv 4.74047 \cdot r_{\text{max}}[\text{kpc}] \cdot \delta\mu[\text{mas yr}^{-1}], \quad (1)$$

where r_{max} is the maximum distance of stars. We find in Figure 1 the relation

$$N_{\text{error},i}(\delta v_{\text{max}}, \text{BIAS}) \propto (\delta v_{\text{max}})^2.$$

Figure 1 also demonstrates that different model parameters do not have the same sensitivity to the numerical inaccuracies introduced by insufficient sampling.

Underestimation of the proper motion error. We found that in case we perfectly knew the measurement errors (and the distance error is negligible), the convolution of the model probability with the measurement errors gives precise and accurate constraints on the model parameters - even if the error itself is quite large. Now we investigate what would happen if the quoted measurement errors, e.g. the proper motion errors, were actually smaller than the true errors. Figure 3 shows the case for two different stellar populations and an error underestimation of 10% and 50%.

Overall the parameter recovery gets worse the larger the proper motion error and the stronger the underestimation. The relation between the bias due to error misjudgment and the size of the proper motion error seems to be linear.

For the recovery of the isochrone potential scale length b the hotness of the population does not matter (see lower left panel in Figure 3). The circular velocity $v_{\text{circ}}(R_{\odot})$ is, as always, better measured by cooler than by hotter populations (see upper left panel in Figure 3).

We find that the recovery of the qDF parameters on the other hand is more strongly affected

by the misjudgment of the velocity error for *cooler* stellar populations. The measured velocity dispersion is the convolution of the intrinsic dispersion with the measurement errors. If the proper motion error is underestimated, the deconvolved velocity dispersion is larger than the intrinsic velocity dispersion and the relative difference is bigger for a cooler population (see upper right panel for σ_z in Figure 3). The intrinsic velocity dispersion is also cooler at larger radii than at smaller radii, therefore the deconvolved dispersion is overestimated more strongly at large R and the velocity dispersion scale length will be overestimated as well (see lower left panel for h_{σ_z} in Figure 3). We get analogous results for the qDF parameters σ_R and h_{σ_R} . The recovery of the tracer density scale length h_R is not affected by the misjudgment of velocity errors.

The most important and encouraging result from Figure 3 is, that for an underestimation of 10% the bias is still $\lesssim 2\sigma$ - even for proper motion errors of 3 mas/yr.

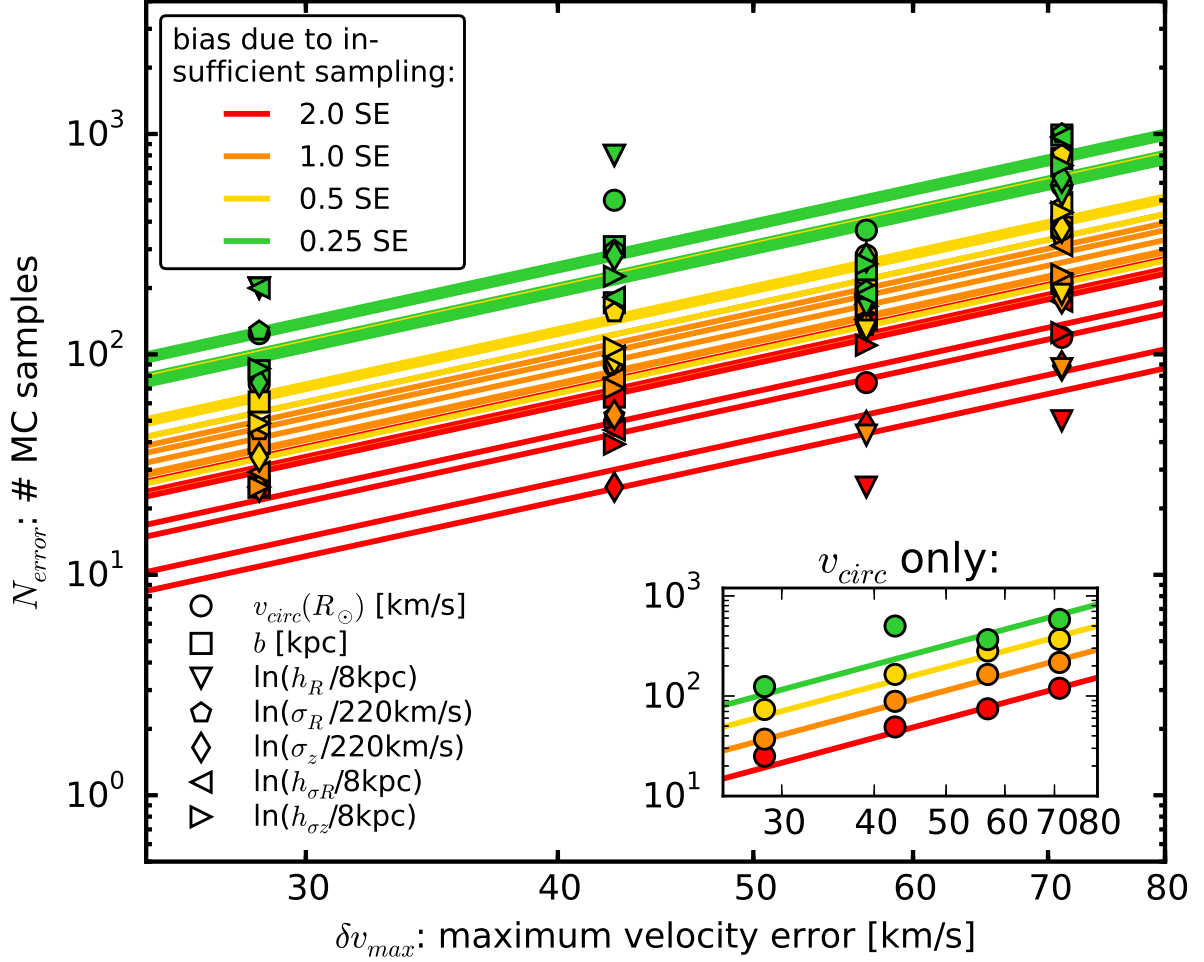


Fig. 1.— Number of Monte Carlo (MC) samples N_{error} needed for the numerical error convolution in Equation (??), given the maximum velocity error δv_{max} in the sample to reach a given accuracy. An insufficient sampling of the convolution integral leads to systematic biases in the reconstruction of the true model parameters. The size of the bias is color coded as indicated in the legend and is given in units of the standard error (SE). The model parameters, marked by different symbols, have different sensitivities to the numerical inaccuracy of the error convolution, therefore the range in N_{error} for the same given bias. Here we assume that the distance error is zero and the proper motion error $\delta\mu$ translates to a velocity error according to Equation (1) and $\delta v_{\text{los}} \ll \delta v_{\text{max}}$. All model parameters are listed in Table 3 as Test ⑥. The number of MC samples needed increases with the velocity error as $N_{\text{error}} \propto (\delta v_{\text{max}})^2$, as can be seen especially well in the inset figure for the potential parameter $v_{\text{circ}}(R_{\odot})$. All lines are fits of this functional form to each four points derived for a given model parameter (symbol) and bias (color). The large scatter in the points comes from low number statistics and errors introduced by linear interpolation of the bias vs. N_{error} relation found from the analyses.

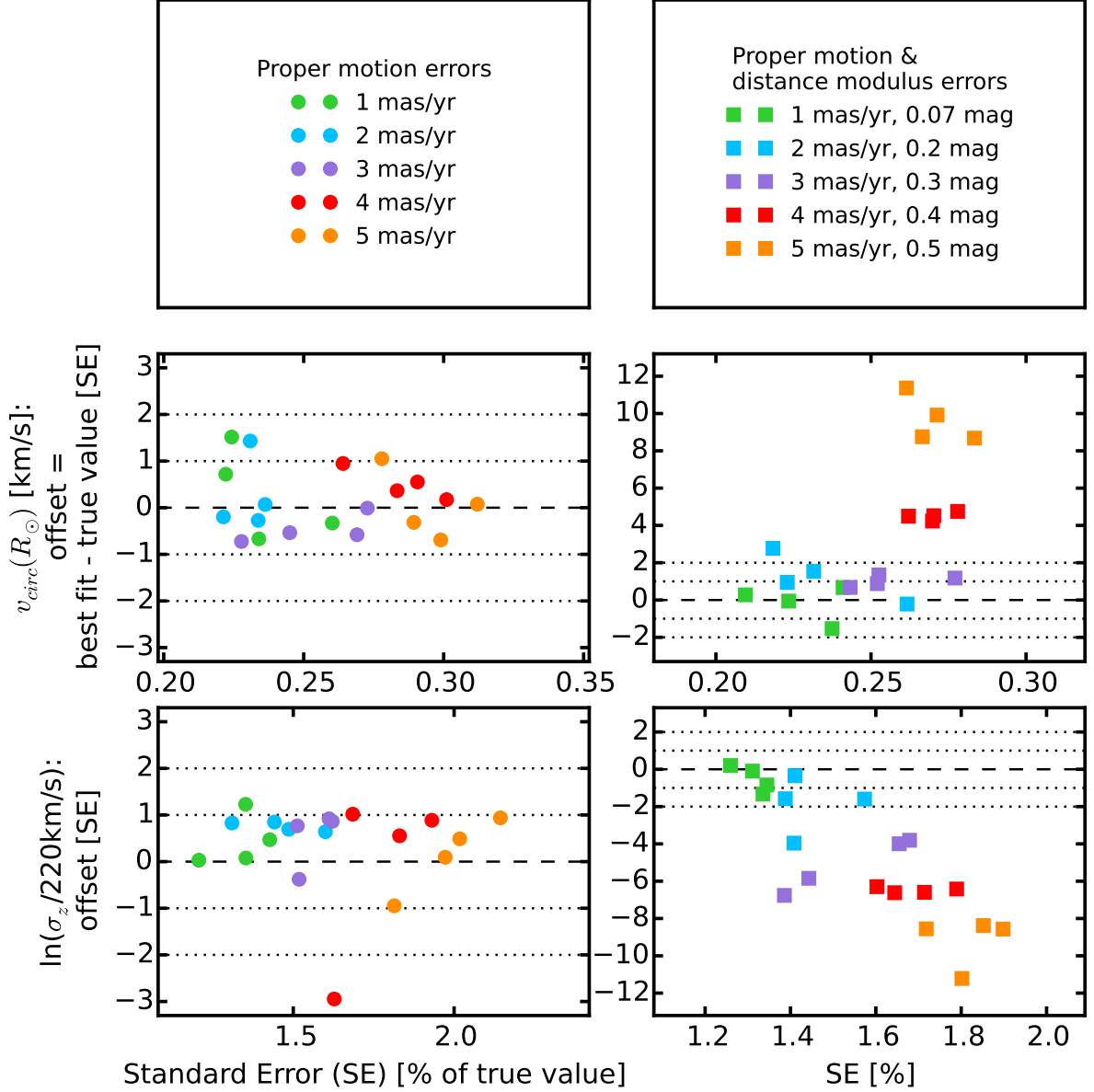


Fig. 2.— Parameter recovery using the approximation for the measurement error convolved likelihood in Equation (??). All model parameters used to create the mock data sets analyzed for this figure are given as Test 12 in Table 3. The mock data sets in the left panels have only errors in line-of-sight velocity and proper motions, while the data sets in the right panels also have distance modulus errors, as indicated in the legends in the first row. The size of the error is color coded. The other panels plot the offset of the recovered model parameter to the true parameter vs. the relative standard error for two of the seven model parameters, the potential parameter $v_{\text{circ}}(R_{\odot})$ and qDF parameter σ_z . For data sets with proper motion error errors $\delta(m - M) \leq 3\text{mas yr}^{-1}$ Equation (??) was evaluated with $N_{\text{error}} = 800$, for $\delta(m - M) > 3\text{mas yr}^{-1}$ we used $N_{\text{error}} = 1200$. In the absence of distance errors Equation (??) gives unbiased results, for $\delta(m - M) \geq 3\text{mas yr}^{-1}$ (which corresponds to $\delta v_{\text{max}} = [\text{TO DO: CHECK}]$, see Equation (1)) however biases of several sigma [TO DO: CAN I SAY THAT] are introduced as Equation (??) is only an approximation for the true likelihood in

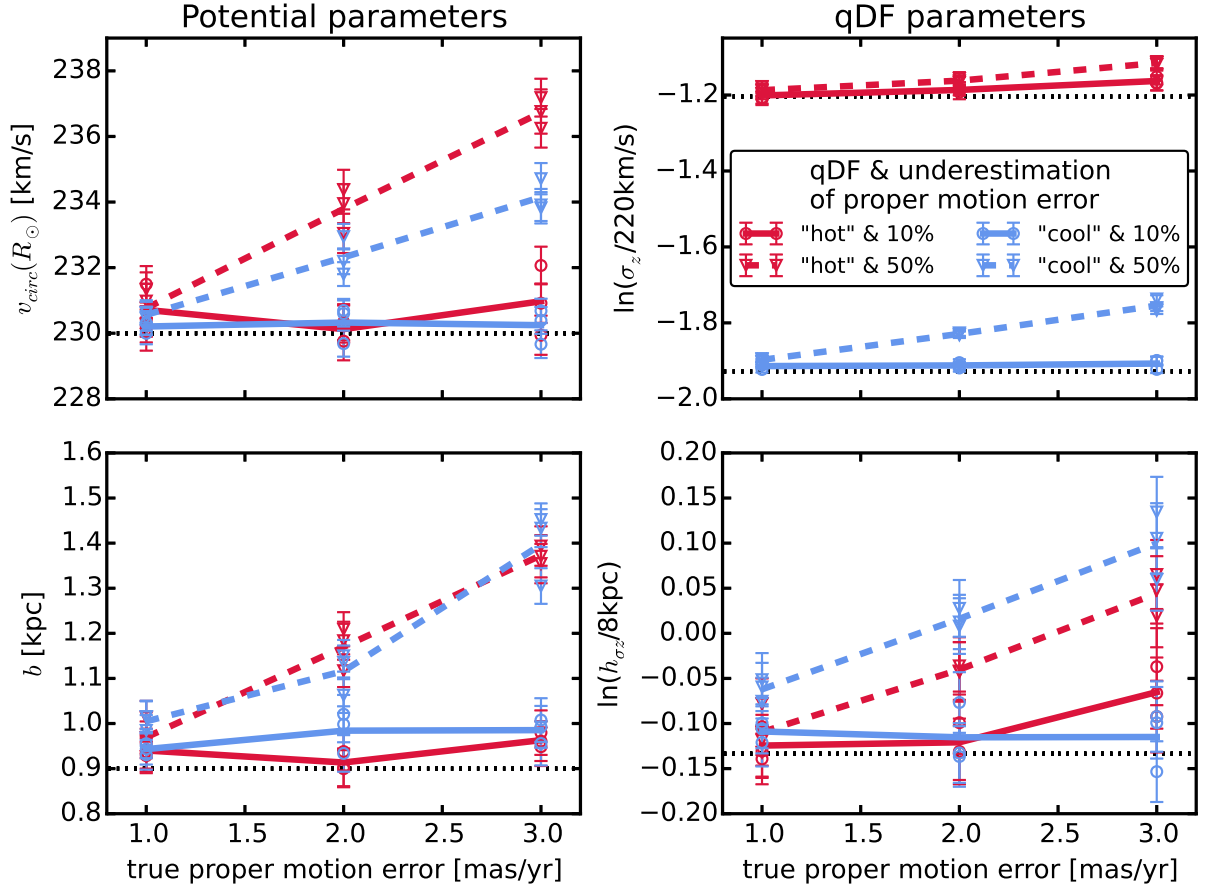


Fig. 3.— Effect of an systematic underestimation of proper motion errors in the recovery of the model parameters. The true model parameters used to create the mock data are summarized as Test ① in Table 3, four of them are given on the y -axes and the true values are indicated as black dashed lines. The velocities of the mock data were perturbed according to Gaussian errors in the α and δ proper motions as indicated on the x -axis. The circles and triangles are the best fit parameters of several mock data set assuming the proper motion error, with which the model probability was convolved, was underestimated in the analysis by 10% or 50%, respectively. The error bars correspond to 1σ confidence. The lines connect the mean of each two data realisations and are just guides to the eyes.

Table 1. Gravitational potentials of the reference galaxies used throughout this work and the respective ways to calculate actions in these potentials. All four potentials are axisymmetric. The potential parameters are fixed for the mock data creation at the values given in this table. In the subsequent analyses we aim to recover these potential parameters again. The parameters of "MW13-Pot" and "KKS-Pot" were found as direct fits to the "MW14-Pot".

name	potential type	potential parameters p_Φ		action calculation	reference for potential type
"Iso-Pot"	isochrone potential	circular velocity at the sun isochrone scale length	$v_{\text{circ}} = 230 \text{ km s}^{-1}$ $b = 0.9 \text{ kpc}$	<i>analytical and exact</i> J_r, J_ϑ, L_z ; $?$ use $J_r \rightarrow J_R, J_\vartheta \rightarrow J_z$ in Equation (??)	
"KKS-Pot"	2-component Kuzmin-Kutuzov- Stäckel potential (disk + halo) (analytic potential)	circular velocity at the sun focal distance of coordinate system ^a axis ratio of the coordinate surfaces ^aof the disk component ...of the halo component relative contribution of the disk mass to the total mass	$v_{\text{circ}} = 230 \text{ km s}^{-1}$ $\Delta = 0.3$ $\left(\frac{a}{c}\right)_{\text{Disk}} = 20$ $\left(\frac{a}{c}\right)_{\text{Halo}} = 1.07$ $k = 0.28$	<i>exact</i> J_R, J_z, L_z using "Stäckel Fudge" (?) and interpolation on action grid ^b (?)	$?$
"MW13-Pot"	MW-like potential with Hernquist bulge, 2 exponential disks (stars + gas), spherical power-law halo (interpolated potential)	circular velocity at the sun stellar disk scale length stellar disk scale height relative halo contribution to $v_{\text{circ}}^2(R_\odot)$ "flatness" of rotation curve	$v_{\text{circ}} = 230 \text{ km s}^{-1}$ $R_d = 3 \text{ kpc}$ $z_h = 0.4 \text{ kpc}$ $f_h = 0.5$ $\frac{d \ln(v_{\text{circ}}(R_\odot))}{d \ln(R)} = 0$	<i>approximate</i> J_R, J_z, L_z using "Stäckel Fudge" (?) and interpolation on action grid ^a (?)	$?$
"MW14-Pot"	MW-like potential with cut-off power-law bulge, Miyamoto-Nagai stellar disk, NFW halo	-	-	<i>approximate</i> J_R, J_z, L_z (see "MW13-Pot")	$?$

^aThe coordinate system of each of the two Stäckel-potential components is $\frac{R^2}{\tau_{i,p} + \alpha_p} + \frac{z^2}{\tau_{i,p} + \gamma_p} = 1$ with $p \in \{\text{Disk}, / \text{Halo}\}$ and $\tau_{i,p} \in \{\lambda_p, \nu_p\}$. Both components have the same focal distance $\Delta = \sqrt{\gamma_p - \alpha_p}$, to make sure that the superposition of the two components itself is still a Stäckel potential. The axis ratio of the coordinate surfaces $\left(\frac{a}{c}\right)_p := \sqrt{\frac{\alpha_p}{\gamma_p}}$ describes the flatness of the corresponding Stäckel component.

^bWe use a finely spaced action interpolation grid with $R_{\text{max}} = 10$ [TO DO: What's that??? units???] and 50 grid points in E and ψ [TO DO: Find out what's that???], and 60 grid points in L_z . [TO DO: more details?]

Table 2. Reference distribution function parameters for the qDF in Equations (??)-(??). These qDFs describe the phase-space distribution of stellar *MAPs* for which mock data is created and analysed throughout this work for testing purposes. The parameters of the "cooler" & "colder" ("hotter" & "warmer") *MAPs* were chosen such, that the they have the same σ_R/σ_z ratio as the "hot" ("cool") *MAP* . The "colder" and "warmer" *MAPs* have a free parameter X that governs how much colder/warmer they are then the reference "hot" and "cool" qDFs. Hotter populations have shorter tracer scale lengths (?) and the velocity dispersion scale lengths were fixed according to ?.

name of <i>MAP</i>	qDF parameters p_{DF}				
	h_R [kpc]	σ_R [km s ⁻¹]	σ_z [km s ⁻¹]	h_{σ_R} [kpc]	h_{σ_z} [kpc]
"hot"	2	55	66	8	7
"cool"	3.5	42	32	8	7
"cooler"	2 +50%	55-50%	66-50%	8	7
"hotter"	3.5-50%	42+50%	32+50%	8	7
"colder"	2 +X%	55-X%	66-X%	8	7
"warmer"	3.5-X%	42+X%	32+X%	8	7

Table 3. Summary of test suites in this work: The first column indicates the test suite, the second column the potential, DF and selection function model etc. used for the mock data creation, the third model the corresponding model assumed in the analysis, and the last column lists the figures belonging to the test suite. Parameters that are not left free in the analysis, are always fixed to their true value. Unless otherwise stated we calculate the likelihood by the nested-grid and MCMC approach outlined in §?? and use $N_{\text{spatial}} = 16$, $N_{\text{velocity}} = 24$, $N_{\text{sigma}} = 5$ as numerical accuracy for the likelihood normalisation in Equations (??) and (??). [TO DO: Change encircled numbers to proper order. Make sure the plot references are the right ones.]

Test	Model for Mock Data		Model in Analysis	Figures
① Influence of survey volume on mock data distribution, also in action space	<i>Potential:</i> <i>MAP :</i> <i>Survey volume:</i> <i># stars per data set:</i> <i># data sets:</i>	"KKS-Pot" 2 MAPs "hot" or "cold" qDF a) $R \in [4, 12]$ kpc, $z \in [-4, 4]$ kpc, $\phi \in [-20^\circ, 20^\circ]$. b) $R \in [6, 10]$ kpc, $z \in [1, 5]$ kpc, $\phi \in [-20^\circ, 20^\circ]$. 20,000 4 (= 2×2 models)	-	Mock data: Figure ??
⑨ Numerical accuracy in calculation of the likelihood normalisation	<i>Potential:</i> <i>MAP :</i> <i>Survey volume:</i> <i>Numerical accuracy:</i>	"Iso-Pot", "MW13-Pot" & "KKS-Pot" "hot" qDF sphere around sun, $r_{\text{max}} = 0.2, 1, 2, 3$ or 4 kpc $N_{\text{spatial}} \in [5, 20]$, $N_{\text{velocity}} \in [6, 40]$, $N_{\text{sigma}} \in [3.5, 7]$	-	Convergence of normalisation: Figure ?? 6
⑩ <i>pdf</i> is a multivariate Gaussian for large data sets.	<i>Potential:</i> <i>MAP :</i> <i>Survey Volume:</i> <i># stars per data set:</i> <i># data sets:</i> <i>Numerical accuracy:</i>	"Iso-Pot" "hot" qDF sphere around sun, $r_{\text{max}} = 2$ kpc 20,000 5 (only one is shown)	"Iso-Pot", all parameters free qDF, all parameters free (fixed & known) $N_{\text{velocity}} = 20$ and $N_{\text{sigma}} = 4$	Figure ??
② Width of the likelihood scales with number of stars by $\propto 1/\sqrt{N}$.	<i>Potential:</i> <i>MAP :</i> <i>Survey volume:</i> <i># stars per data set:</i> <i># data sets:</i> <i>Analysis method:</i> <i>Numerical accuracy:</i>	"Iso-Pot" "hot" qDF sphere around sun, $r_{\text{max}} = 3$ kpc between 100 and 40,000 132 likelihood on grid $N_{\text{velocity}} = 20$ and $N_{\text{sigma}} = 4$ (for speed)	"Iso-Pot", free parameter: b "hot" qDF, free parameters: $\ln\left(\frac{h_R}{8\text{kpc}}\right), \ln\left(\frac{\sigma_R}{230\text{km s}^{-1}}\right), \ln\left(\frac{h_{\sigma,R}}{8\text{kpc}}\right)$ (fixed & known)	Figure ??
③ Parameter estimates are unbiased.	<i>Potential:</i> <i>MAP :</i> <i>Survey volume:</i> <i># stars per data set:</i> <i># data sets:</i> <i>Analysis method:</i> <i>Numerical accuracy:</i>	2 "Iso-Pot" with $b = 0.8$ kpc or $b = 1.5$ kpc 2 MAPs, "hot" or "cool" qDF 5 spheres around sun, $r_{\text{max}} = 0.2, 1, 2, 3$ or 4 kpc 20,000 640 (= $2 \times 2 \times 5$ models \times 32 realisations) likelihood on grid $N_{\text{velocity}} = 20$ and $N_{\text{sigma}} = 4$ (for speed)	"Iso-Pot", free parameter: b "hot"/"cool" qDF, free parameters: $\ln\left(\frac{h_R}{8\text{kpc}}\right), \ln\left(\frac{\sigma_R}{230\text{km s}^{-1}}\right), \ln\left(\frac{h_{\sigma,R}}{8\text{kpc}}\right)$ (fixed & known)	Figure ??
④	<i>Potential:</i>	i) "Iso-Pot", ii) "MW13-Pot" or iii) "KKS-Pot"	i) "Iso-Pot", all parameters free	Figure ??

Table 3—Continued



Test	Model for Mock Data		Model in Analysis	Figures
Influence of position & shape of survey volume on parameter recovery	<i>MAP :</i> <i>Survey volume:</i> <i># of stars per data set:</i> <i># data sets:</i> <i>Analysis method:</i> <i>Action calculation:</i>	"hot" qDF 4 different wedges, see Figure ??, upper right panel 20,000 48 ($= 4 \times 3$ models \times 4 realisations) ii) & iii) low accuracy "Stäckel Fudge" grid (?) for speed (# grid points: 25 in each E and ψ , 30 in L_z , $R_{\max} = 5$ [TO DO: What is psi and Rmax (units)?])	ii) "MW13-Pot", R_d and f_h free iii) "KKS-Pot", all free except $v_{\text{circ}}(R_{\odot})$ i) & iii) qDF, all parameters free ii) qDF, only h_R , $\sigma_{z,0}$ and h_{σ_R} free (fixed & known) i) & ii) MCMC, iii) likelihood on grid (same as mock data creation)	
⑤ Influence of wrong assumptions about the data set (in-)completeness on parameter recovery	<i>Potential:</i> <i>MAP :</i> <i>Survey volume:</i> <i>Completeness:</i> <i># stars per data set:</i> <i># data sets:</i>	"Iso-Pot" 2 MAPs , a) "hot" or b) "cool" qDF sphere around sun, $r_{\max} = 3$ kpc <i>Example 1:</i> radial incompleteness, $\text{completeness}(r) = 1 - \epsilon_r \frac{r}{r_{\max}}$, twenty $\epsilon_r \in [0, 0.7]$ $r \equiv$ distance from sun, <i>Example 2:</i> planar incompleteness, $\text{completeness}(z) = 1 - \epsilon_z \frac{ z }{r_{\max}}$, $\epsilon_r \in [0, 0.7]$, $z \equiv$ distance from Gal. plane. 20,000 40 ($= 2 \times 2 \times 20$)	"Iso-Pot", all parameters free qDF, all parameters free (fixed & known) data set complete, $\text{completeness}(r) = 1$, $\epsilon_r = 0$ data set complete, $\text{completeness}(r) = 1$, twenty $\epsilon_z = 0$	Illustration & mock data: Figures ?? & ?? Analysis results:  Figures ?? & ?? Analysis results:  when not using v_T data: Figure ??
⑥ Numerical convergence of convolution with measurement errors	<i>Potential:</i> <i>MAP :</i> <i>Survey Volume:</i> <i>Errors:</i> <i>Numerical Accuracy:</i> <i># stars per data set:</i> <i># data sets:</i>	"Iso-Pot" "hot" qDF sphere around sun, $r_{\max} = 3$ kpc $\delta\text{RA} = \delta\text{DEC} = \delta(m - M) = 0$ $\delta v_{\text{los}} = 2$ km/s $\delta\mu_{\text{RA}} = \delta\mu_{\text{DEC}} = 2, 3, 4$ or 5 mas/yr	"Iso-Pot, all parameters free" qDF, all parameters free (fixed & known) Convolution with perfectly known errors convolution using MC integration with between 25 and 1200 MC samples	Figure 1
⑫ Testing the	<i>Potential:</i>	"Iso-Pot"	"Iso-Pot", all parameters free	Figure 2

Table 3—Continued

Test	Model for Mock Data		Model in Analysis	Figures
convolution with measurement errors with & without distance errors	<i>MAP :</i> <i>Survey Volume:</i> <i>Errors:</i> <i>Numerical Accuracy:</i> <i># stars per data set:</i> <i># data sets:</i>	"hot" qDF sphere around sun, $r_{\max} = 3$ kpc $\delta\text{RA} = \delta\text{DEC} = 0$ $\delta v_{\text{los}} = 2$ km/s $\delta\mu_{\text{RA}} = \delta\mu_{\text{DEC}} = 1, 2, 3, 4$ or 5 mas/yr a) $\delta(m - M) = 0$, b) $\delta(m - M) \neq 0$ (see Figure 2)	qDF, all parameters free (fixed & known) Convolution with errors, ignoring distance errors in position (see §[TO DO: CHECK??]) 800 or 1200 MC samples	
① Underestimation of proper motion errors	<i>Potential:</i> <i>MAP :</i> <i>Survey volume:</i> <i>Errors:</i> <i># stars per data set:</i> <i># data sets:</i>	"Iso-Pot" "hot" or "cool" qDF sphere around sun, $r_{\max} = 3$ kpc [TO DO: CHECK] only proper motion errors 1, 2 or 3 mas/yr	"Iso-Pot", all parameters free qDF, all parameters free (fixed & known) Convolution with proper motion errors 10% or 50% underestimated	Figure 3
⑦ Deviations in the assumed DF from the star's true DF	<i>Potential:</i> <i>MAP :</i> <i>Survey volume:</i> <i># stars per data set:</i> <i># data sets:</i>	"Iso-Pot" mix of two qDFs <i>Example 1:</i> with fixed qDF parameters, but 20 different mixing rates: a) "hot" & "cooler" qDF or b) "cool" & "hotter" qDF <i>Example 2:</i> 20 fixed 50/50 mixtures, with varying qDF parameters (by $X\%$): a) "hot" & "colder" qDF or b) "cool" & "warmer" qDF	"Iso-Pot", all parameters free single qDF, all parameters free (fixed & known)	mock data: Figure ?? Analysis results: ?? & Figure ??
⑧ Deviations of the assumed potential model from the star's true potential	<i>Potential:</i> <i>MAP :</i> <i>Survey volume:</i> <i># stars per data set:</i> <i># data sets:</i>	"MW14-Pot" "hot" or "cool" qDF sphere around sun, $r_{\max} = 4$ kpc 20,000 40 ($= 2 \times 2 \times 20$)	"KKS-Pot", all parameters free, only $v_{\text{circ}}(R_{\odot}) = 230 \text{ km s}^{-1}$ fixed qDF, all parameters free (fixed & known)	potential contours: Figure ?? qDF recovery: Figure ??

