Chapter 3

Vector Spaces and Subspaces

3.1 Vector Spaces

3.1.1 Definitions of Vector Space and Subspace

1. **Space**

   The space $\mathbf{R}^n$ consists of all vectors $\mathbf{v}$ with n components of real numbers. The space $\mathbf{C}^n$ consists of all vectors $\mathbf{v}$ with n components of complex numbers.

2. **Subspace**

   A subspace of a vector space is a set of vectors (including $\mathbf{0}$) that satisfies two requirements: If $\forall$ vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are in the subspace and $\forall\ k \in \mathbb{R}$, then

   (a) $\mathbf{v} + \mathbf{w}$ is in the subspace.

   (b) $k\mathbf{v}$ is in the subspace.

3.1.2 Properties of Vector Space and Subspace □

1. Every space and subspace contains the zero vector.

2. Lines through the origin are also subspaces.

3. **Closure Property**

   If $\forall$ vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are in a vector space or subspace and $\forall\ \alpha_1, \alpha_2 \in \mathbb{R}$, then the linear combination $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2$ must stay in this vector space or subspace.

3.1.3 Special Vector Spaces

1. **M**

   The vector space of all real 2 by 2 matrices.

2. **F**

   The vector space of all real functions $f(x)$.

3. **P** or $\mathbf{P}_n$

   The vector space that consists of all polynomials

   $$a_0 + a_1 x + \cdots + a_n x^n$$

   of degree n.

4. **Z**

   The vector space that consists only of a zero vector.

3.2 Four Subspaces of Matrices

$\forall\ m \times n$ matrix A where $m, n \in \mathbb{N}^+$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} — & \mathbf{r}_1^{\mathrm{T}} & — \\ — & \mathbf{r}_2^{\mathrm{T}} & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{r}_m^{\mathrm{T}} & — \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \\ | & | & \cdots & | \end{bmatrix}$$

and its transpose

$$A^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} — & \mathbf{c}_1^{\mathrm{T}} & — \\ — & \mathbf{c}_2^{\mathrm{T}} & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{c}_n^{\mathrm{T}} & — \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{r}_1 & \mathbf{r}_2 & \cdots & \mathbf{r}_m \\ | & | & \cdots & | \end{bmatrix}.$$

3.2.1 Definition of Four Subspaces of Matrices

1. **Row Space**

   The column space of A consists of all linear combinations of $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_m$, which is denoted by $\boldsymbol{C}(A^{\mathrm{T}})$. It is a subspace of $\mathbf{R}^n$.

2. **Column Space**

   The column space of A consists of all linear combinations of $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n$, which is denoted by $\boldsymbol{C}(A)$. It is a subspace of $\mathbf{R}^m$.

3. **Nullspace**

   The nullspace of A consists of solutions to $A\mathbf{x} = \mathbf{0}$, which is denoted by $\boldsymbol{N}(A)$. It is a subspace of $\mathbf{R}^n$.

4. **Left Nullspace**

   The nullspace of A consists of solutions to $A^{\mathrm{T}}\mathbf{x} = \mathbf{0}$, which is denoted by $\boldsymbol{N}(A^{\mathrm{T}})$. It is a subspace of $\mathbf{R}^m$.

3.3 Definitions and Properties about Vector Spaces

3.3.1 Definitions

1. **Linear Independence**

   The sequence of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ is linearly independent if the only combination that gives the zero vector is

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n.$$

2. **Maximal Linearly Independent Subset**

   For a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ where $n \in \mathbb{N}^+$, a subset $S' = \{\mathbf{v}_{k_1}, \mathbf{v}_{k_2}, \cdots, \mathbf{v}_{k_m}\}$ of S is a maximal linearly independent subset of S if and only if $\mathbf{v}_{k_1}, \mathbf{v}_{k_2}, \cdots, \mathbf{v}_{k_m}$ are linearly independent and every vector in S can be expressed as a linear combination of vectors in S', where $1 \leq m \leq n, 1 \leq k_1 \leq k_2 \leq \cdots \leq k_m \leq n$ and $m, k_1, k_2, \cdots, k_m \in \mathbb{N}^+$.

3. **Rank**

   (1) **The Rank of A Set of Vectors**

   For a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ where $n \in \mathbb{N}^+$, the rank of S is the number of vectors in its maximal linearly independent subset, which is denoted as r.

   (2) **The Rank of Matrices**

   $\forall$ matrix A, the rank of A is the number of pivots, which is denoted as r.

4. **Span**

   A set of vectors spans a space if their linear combinations fill the space.

5. **Basis**

   A basis for a vector space is a sequence of vectors with two properties:

   (1) The basis vectors are linearly independent;

   (2) The basis vectors span the space.

6. **Dimension**

   The dimension of a space is the number of vectors in every basis.

3.3.2 Properties of Linear Independence $\square$

1. **Zero Vector**

   (1) For vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ where $n \in \mathbb{N}^+$, if $\mathbf{v}_i = \mathbf{0}$ where $1 \le i \le n$ and $i \in \mathbb{N}^+$, then $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly dependent.

   (2) For vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ where $n \in \mathbb{N}^+$, if $\mathbf{v}_i = \mathbf{v}_j$ where $1 \le i < j \le n$ and $i, j \in \mathbb{N}^+$, then $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly dependent.

2. **Contraction and Extension**

   (1)
   - $\forall$ linearly dependent vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ and $\forall$ vectors $\mathbf{v}_{n+1}, \mathbf{v}_{n+2}, \cdots,$ $\mathbf{v}_m$ where $m > n$ and $m, n \in \mathbb{N}^+$, the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n, \mathbf{v}_{n+1}, \mathbf{v}_{n+2},$ $\cdots, \mathbf{v}_m$ are linearly dependent.
   - $\forall$ linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ where $n \in \mathbb{N}^+$, the vectors consisting of a subset are also linearly independent.

   (2)
   - $\forall$ linearly dependent vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n \in \mathbf{R}^m$ where $m, n \in \mathbb{N}^+$, if every vector adds a component at the same place and they become a series of new vectors $\mathbf{v}'_1, \mathbf{v}'_2, \cdots, \mathbf{v}'_n \in \mathbf{R}^{m+1}$, then the new vectors are also linearly dependent.
   - $\forall$ linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n \in \mathbf{R}^m$ where $m > 1$ and $m, n \in \mathbb{N}^+$, if every vector deletes a component at the same place and they become a series of new vectors $\mathbf{v}'_1, \mathbf{v}'_2, \cdots, \mathbf{v}'_n \in \mathbf{R}^{m-1}$, then the new vectors are also linearly independent.

   (3) $\forall$ linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$, if there exists a vector $\mathbf{u}$ such that $\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly dependent, then $\mathbf{u}$ can be represented as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ and the scalars are unique.

3. **Core Property of Linear Independence**

   For two sets of vectors $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m\}$ and $S_2 = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$, where $m, n \in \mathbb{N}^+$, and both S_1 and S_2 are linearly independent.

(1) If every vector in S_2 can be represented as a linear combination of the vectors in S_1, then $m \geq n$.

(2) If every vector in S_2 can be represented as a linear combination of the vectors in S_1 and every vector in S_1 can be represented as a linear combination of the vectors in S_2, then $m = n$.

4. **Maximal Linearly Independent Subset**

(1) There must exist a maximal linearly independent subset in every set of vectors.

(2) For a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ where $n \in \mathbb{N}^+$, assume that a subset $S' = \{\mathbf{v}_{k_1}, \mathbf{v}_{k_2}, \cdots, \mathbf{v}_{k_m}\}$ of S is a maximal linearly independent subset of S, where $1 \leq m \leq n, 1 \leq k_1 \leq k_2 \leq \cdots \leq k_m \leq n$ and $m, k_1, k_2, \cdots, k_m \in \mathbb{N}^+$. $\forall$ vector $\mathbf{v} \in S$, then $\mathbf{v}, \mathbf{v}_{k_1}, \mathbf{v}_{k_2}, \cdots, \mathbf{v}_{k_m}$ are linearly dependent.

(3) $\forall$ set of vectors, every maximal linearly independent subset of it has the same number of vectors.

5. **Number Restriction and Rank**

For two sets of vectors $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m\}$ and $S_2 = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ such that every vector in S_2 can be represented as a linear combination of the vectors in S_1, where $m, n \in \mathbb{N}^+$.

- If $m < n$, then the vectors in S_2 must be linearly dependent.
- If the vectors in S_2 are linearly independent, then $m \geq n$.
- Assume that the rank of S_1 is r_1 and the rank of S_2 is r_2, then $r_1 \geq r_2$.

6. **Basis and Dimension**

(1) Assume that $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ is a basis for a vector space,

- $\forall$ vector $\mathbf{u}$ in this vector space can be represented as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ and the scalars are unique.
- $\forall$ vector $\mathbf{u}$ in this vector space such that $\mathbf{u} \neq \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$, then $\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly dependent.

(2) $\forall$ $m, n \in \mathbb{N}^+$, if both $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m\}$ and $S_2 = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ are bases for the same vector space, then $m = n$.

(3)
- $\forall\, n \in \mathbb{N}^+$, the dimension of $\mathbf{R}^n$ is n.
- $\forall\, m, n \in \mathbb{N}^+$ such that $m < n$, $\forall$ vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n \in \mathbf{R}^m$ must be linearly dependent.
- $\forall\, n \in \mathbb{N}^+$, $\forall\, n$ independent vectors in $\mathbf{R}^n$ must span $\mathbf{R}^n$, which means that they are a basis.

(4) The row operations do not change the dimension of the row space of matrices.

3.3.3 Fundamental Theorem of Linear Algebra, Part 1 □

$\forall\, m \times n$ matrix A where $m, n \in \mathbb{N}^+$, suppose that the rank of A is r $(1 \leq r \leq \min\{m, n\})$. The column space $\boldsymbol{C}(A)$ and row space $\boldsymbol{C}(A^{\mathrm{T}})$ both have dimension r. The nullspace $\boldsymbol{N}(A)$ has dimension $n - r$. The left nullspace $\boldsymbol{N}(A^{\mathrm{T}})$ has dimension $m - r$.

3.3.4 Full Column Rank and Full Row Rank

1. Every $m \times n$ matrix A with full column rank $(r = n)$ has all these properties:

   (1) All columns of A are piovt columns;

   (2) There are no free variables or special solutions;

   (3) The nullspace $\boldsymbol{N}(A)$ contains only the zero vector $\mathbf{x} = \mathbf{0}$;

   (4) If $A\mathbf{x} = \mathbf{b}$ has a solution, then it has only one solution.

2. Every $m \times n$ matrix A with full row rank $(r = m)$ has all these properties:

   (1) All rows of A are piovot rows, and R has no zero rows;

   (2) The column space is the whole space $\mathbf{R}^m$;

   (3) The nullspace $\boldsymbol{N}(A)$ contains $n - r = n - m$ special solutions;

   (4) $A\mathbf{x} = \mathbf{b}$ has a solution for every right side $\mathbf{b}$.

3.4 Linear Equations and The Rank r

<table>
<tr><th rowspan="2">Size Relationship of r, m, n</th><th rowspan="2">A</th><th colspan="2">The number of solutions</th></tr>
<tr><th>$A\mathbf{x} = \mathbf{0}$</th><th>$A\mathbf{x} = \mathbf{b}$</th></tr>
<tr><td>$r = m = n$</td><td>Square and invertible</td><td>1</td><td>1</td></tr>
<tr><td>$r = m < n$</td><td>Short and wide</td><td>∞</td><td>∞</td></tr>
<tr><td>$r = n < m$</td><td>Tall and thin</td><td>1</td><td>0 (R has $\mathbf{0} = b$)
1 (R has no $\mathbf{0} = b$)</td></tr>
<tr><td>$r < m = n$</td><td>Square</td><td rowspan="3">∞</td><td rowspan="3">0 (R has $\mathbf{0} = b$)
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<tr><td>$r < n < m$</td><td>Tall and thin</td></tr>
</table>

For the matrix A, we move the pivots to the left side and move the free variables to the right side. Then we apply Gauss-Jordan Elimination to the matrix A to get its RREF (Reduced Row Echelon Form) R of the matirx A. The shapes of R related to the rank r are shown as follows:

1. $r = m = n$

$$R = \begin{bmatrix} I \end{bmatrix}.$$

2. $r = m < n$

$$R = \begin{bmatrix} I & F \end{bmatrix}.$$

3. $r = n < m$

$$R = \begin{bmatrix} I \\ \mathbf{0} \end{bmatrix}.$$

4. $r < m, r < n$

$$R = \begin{bmatrix} I & F \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$