

Chapter 1

Introduction to Vectors

1.1 Vectors

1.1.1 Definition of Vector

\forall n —dimensional vector \mathbf{v} where $n \in \mathbb{N}^+$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = (v_1, v_2, \dots, v_n).$$

v_1, v_2, \dots, v_n are the 1st, 2nd, \dots , n -th component of \mathbf{v} . Every vector is written as a column.

1.1.2 Operations of Vectors

1. Vector Addition

\forall m —dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ where $m, n \in \mathbb{N}^+$:

$$\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix}, \quad \dots, \quad \mathbf{v}_n = \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

The vector addition of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is

$$\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_n = \begin{bmatrix} v_{11} + v_{12} + \dots + v_{1n} \\ v_{21} + v_{22} + \dots + v_{2n} \\ \vdots \\ v_{m1} + v_{m2} + \dots + v_{mn} \end{bmatrix}.$$

2. Vector Scalar Multiplication

\forall number $k \in \mathbb{R}$ and \forall n -dimensional vector \mathbf{v} where $n \in \mathbb{N}^+$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

The scalar multiplication of k and \mathbf{v} is

$$k\mathbf{v} = \begin{bmatrix} kv_1 \\ kv_2 \\ \vdots \\ kv_n \end{bmatrix}.$$

The number k is called a “scalar”.

3. Vector Dot Product (Inner Product)

\forall two n -dimensional vectors \mathbf{v}, \mathbf{w} where $n \in \mathbb{N}^+$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}.$$

The dot product or inner product of \mathbf{v}, \mathbf{w} is the number $\mathbf{v} \cdot \mathbf{w}$ or $\mathbf{w} \cdot \mathbf{v}$:

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w} = \mathbf{w} \cdot \mathbf{v} = \mathbf{w}^T \mathbf{v} = v_1 w_1 + v_2 w_2 + \dots + v_n w_n.$$

1.1.3 The Length of Vector

$\forall n$ —dimensional vectors \mathbf{v} where $n \in \mathbb{N}^+$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

The length $\|\mathbf{v}\|$ of a vector \mathbf{v} is the square root of $\mathbf{v} \cdot \mathbf{v}$:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

1.1.4 Unit Vector

A unit vector \mathbf{u} is a vector whose length equals one. Then $\mathbf{u} \cdot \mathbf{u} = 1$. $\forall n$ —dimensional vectors \mathbf{v} where $n \in \mathbb{N}^+$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector in the same direction as \mathbf{v} .

1.1.5 The Angle Between Vectors

1. Two-dimensional Vectors

\forall two 2—dimensional nonzero vectors \mathbf{v}, \mathbf{w} :

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (v_1^2 + v_2^2 \neq 0), \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad (w_1^2 + w_2^2 \neq 0).$$

Let the angle between \mathbf{v} and \mathbf{w} is θ ($0 \leq \theta < \pi$), then

$$\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} = \frac{v_1 w_1 + v_2 w_2}{\sqrt{v_1^2 + v_2^2} \sqrt{w_1^2 + w_2^2}} = \cos \theta.$$

2. Three-dimensional Vectors

\forall two 2-dimensional nonzero vectors \mathbf{v}, \mathbf{w} :

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad (v_1^2 + v_2^2 + v_3^2 \neq 0), \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \quad (w_1^2 + w_2^2 + w_3^2 \neq 0).$$

Let the angle between \mathbf{v} and \mathbf{w} is θ ($0 \leq \theta < \pi$), then

$$\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{\sqrt{v_1^2 + v_2^2 + v_3^2} \sqrt{w_1^2 + w_2^2 + w_3^2}} = \cos \theta.$$

3. Schwarz inequality and Triangle inequality

Schwarz inequality \square

$$|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\|.$$

Triangle inequality \square

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|.$$

1.1.6 Linear Combination

1. Definition of Linear Combination

Combine addition with scalar multiplication to produce a “linear combination”. $\forall m$ -dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ where $m, n \in \mathbb{N}^+$ and \forall number $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$. The sum of $\alpha_1 \mathbf{v}_1, \alpha_2 \mathbf{v}_2, \dots, \alpha_n \mathbf{v}_n$ is a linear combination

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n.$$

2. Geometrical Significance of Linear Combination (Parallelogram Law)

The parallelogram law gives the rule for vector addition of vectors \mathbf{u} and \mathbf{v} . The sum $\mathbf{u} + \mathbf{v}$ of the vectors is obtained by placing them head to tail and drawing the vector from the free tail to the free head.

3. Line, Plane, Space and Three-dimensional Vectors

\forall number $\alpha, \beta, \gamma \in \mathbb{R}$ and \forall nonzero three-dimensional vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$:

- All combinations $\alpha \mathbf{u}$ fill a line through $(0, 0, 0)$;
- If \mathbf{u} and \mathbf{v} are not on the same line, all combinations $\alpha \mathbf{u} + \beta \mathbf{v}$ fill a plane through $(0, 0, 0)$;

- If \mathbf{w} is not on the same plane formed by $\alpha\mathbf{u} + \beta\mathbf{v}$, all combinations $\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$ fill three-dimensional space.

1.2 Matrix

1.2.1 Definition of Matrix

$\forall m$ -dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ where $m, n \in \mathbb{N}^+$:

$$\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix}, \quad \dots, \quad \mathbf{v}_n = \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

and \forall number $x_1, x_2, \dots, x_n \in \mathbb{R}$. The linear combination of

$$x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \dots + x_n \mathbf{v}_n = x_1 \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix} + x_2 \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

can be expressed as

$$A\mathbf{x} = \begin{bmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \dots & v_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where A is a matrix and \mathbf{x} is a vector:

$$A = \begin{bmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \dots & v_{mn} \end{bmatrix} = \begin{bmatrix} | & | & \dots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & \dots & | \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

1.2.2 The Format of Matrix, $A\mathbf{x}_c = \mathbf{b}_c$ and $\mathbf{x}_r^T A = \mathbf{b}_r^T$

For m -dimensional vectors $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ such that

$$\mathbf{c}_1 = \mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix}, \quad \mathbf{c}_2 = \mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix}, \quad \dots, \quad \mathbf{c}_n = \mathbf{v}_n = \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

For n -dimensional vectors $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m$ such that

$$\mathbf{r}_1 = \begin{bmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} v_{21} \\ v_{22} \\ \vdots \\ v_{2n} \end{bmatrix}, \quad \dots, \quad \mathbf{r}_m = \begin{bmatrix} v_{m1} \\ v_{m2} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

Then the matrix A can be expressed as

$$A = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \\ | & | & \cdots & | \end{bmatrix} = \begin{bmatrix} - & \mathbf{r}_1^T & - \\ - & \mathbf{r}_2^T & - \\ \vdots & \vdots & \vdots \\ - & \mathbf{r}_m^T & - \end{bmatrix}.$$

For a m -dimensional vector \mathbf{b}_c and a m -dimensional vector \mathbf{b}_r such that

$$\mathbf{b}_c = \begin{bmatrix} b_{c1} \\ b_{c2} \\ \vdots \\ b_{cm} \end{bmatrix}, \quad \mathbf{b}_r = \begin{bmatrix} b_{r1} \\ b_{r2} \\ \vdots \\ b_{rn} \end{bmatrix},$$

there are

$$A\mathbf{x}_c = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} \begin{bmatrix} x_{c1} \\ x_{c2} \\ \vdots \\ x_{cn} \end{bmatrix} = \mathbf{b}_c = \begin{bmatrix} b_{c1} \\ b_{c2} \\ \vdots \\ b_{cm} \end{bmatrix},$$

and

$$\mathbf{x}_r^T A = \begin{bmatrix} x_{r1} & x_{r2} & \cdots & x_{rn} \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} = \mathbf{b}_r^T = \begin{bmatrix} b_{r1} & b_{r2} & \cdots & b_{rn} \end{bmatrix}.$$

Then

$$\mathbf{b}_c = \begin{bmatrix} b_{c1} \\ b_{c2} \\ \vdots \\ b_{cm} \end{bmatrix} = x_{c1}\mathbf{c}_1 + x_{c2}\mathbf{c}_2 + \cdots + x_{cn}\mathbf{c}_n = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{b}_c \\ \mathbf{r}_2 \cdot \mathbf{b}_c \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{b}_c \end{bmatrix},$$

and

$$\mathbf{b}_r^T = \begin{bmatrix} b_{r1} & b_{r2} & \cdots & b_{rn} \end{bmatrix} = x_{r1}\mathbf{r}_1^T + x_{r2}\mathbf{r}_2^T + \cdots + x_{rn}\mathbf{r}_n^T = \begin{bmatrix} \mathbf{c}_1 \cdot \mathbf{b}_r & \mathbf{c}_2 \cdot \mathbf{b}_r & \cdots & \mathbf{c}_m \cdot \mathbf{b}_r \end{bmatrix}.$$

1.2.3 Matrix Operations

1. Matrix Addition

Matrices can be added if their shapes are the same. $\forall m \times n$ matrices A and B where $k, m, n \in \mathbb{N}^+$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}.$$

Then the addition of A and B is

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}.$$

2. Matrix Scalar Multiplication

\forall number $k \in \mathbb{R}$, $\forall m \times n$ matrix A where $k \in \mathbb{R}$ and $m, n \in \mathbb{N}^+$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Then the scalar multiplication of k and A is

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}.$$

3. Matrix Multiplication

For two matrices A and B , in order to multiply AB , if A has q columns, B must have q

rows, where $q \in \mathbb{N}^+$. $\forall p \times q$ matrix A and $\forall q \times r$ matrix B where $p, q, r \in \mathbb{N}^+$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{bmatrix} = \begin{bmatrix} \text{---} & \mathbf{a}_{\mathbf{r}_1}^T & \text{---} \\ \text{---} & \mathbf{a}_{\mathbf{r}_2}^T & \text{---} \\ \vdots & \vdots & \vdots \\ \text{---} & \mathbf{a}_{\mathbf{r}_p}^T & \text{---} \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{a}_{\mathbf{c}_1} & \mathbf{a}_{\mathbf{c}_2} & \cdots & \mathbf{a}_{\mathbf{c}_q} \\ | & | & \cdots & | \end{bmatrix},$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ b_{q1} & b_{q2} & \cdots & b_{qr} \end{bmatrix} = \begin{bmatrix} \text{---} & \mathbf{b}_{\mathbf{r}_1}^T & \text{---} \\ \text{---} & \mathbf{b}_{\mathbf{r}_2}^T & \text{---} \\ \vdots & \vdots & \vdots \\ \text{---} & \mathbf{b}_{\mathbf{r}_q}^T & \text{---} \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{b}_{\mathbf{c}_1} & \mathbf{b}_{\mathbf{c}_2} & \cdots & \mathbf{b}_{\mathbf{c}_r} \\ | & | & \cdots & | \end{bmatrix}.$$

Then the multiplication of A and B is

$$AB = \begin{bmatrix} \sum_{i=1}^q a_{1i}b_{i1} & \sum_{i=1}^q a_{1i}b_{i2} & \cdots & \sum_{i=1}^q a_{1i}b_{ir} \\ \sum_{i=1}^q a_{2i}b_{i1} & \sum_{i=1}^q a_{2i}b_{i2} & \cdots & \sum_{i=1}^q a_{2i}b_{ir} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^q a_{pi}b_{i1} & \sum_{i=1}^q a_{pi}b_{i2} & \cdots & \sum_{i=1}^q a_{pi}b_{ir} \end{bmatrix}.$$

The computation of AB uses pqr separate multiplications. There are three variants of matrix multiplication AB

(a) **Matrix A times every column of matrix B** \square

$$AB = A \begin{bmatrix} | & | & \cdots & | \\ \mathbf{b}_{\mathbf{c}_1} & \mathbf{b}_{\mathbf{c}_2} & \cdots & \mathbf{b}_{\mathbf{c}_r} \\ | & | & \cdots & | \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ A\mathbf{b}_{\mathbf{c}_1} & A\mathbf{b}_{\mathbf{c}_2} & \cdots & A\mathbf{b}_{\mathbf{c}_r} \\ | & | & \cdots & | \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}\mathbf{a}_{\mathbf{c}_1} & b_{12}\mathbf{a}_{\mathbf{c}_1} & \cdots & b_{1r}\mathbf{a}_{\mathbf{c}_1} \\ + & + & \cdots & + \\ b_{21}\mathbf{a}_{\mathbf{c}_2} & b_{22}\mathbf{a}_{\mathbf{c}_2} & \cdots & b_{2r}\mathbf{a}_{\mathbf{c}_2} \\ + & + & \cdots & + \\ \vdots & \vdots & \cdots & \vdots \\ + & + & \cdots & + \\ b_{q1}\mathbf{a}_{\mathbf{c}_q} & b_{q2}\mathbf{a}_{\mathbf{c}_q} & \cdots & b_{qr}\mathbf{a}_{\mathbf{c}_q} \end{bmatrix}.$$

(b) Every row of matrix A times matrix B \square

$$\begin{aligned}
 AB &= \begin{bmatrix} \text{---} & \mathbf{a}_{r_1}^T & \text{---} \\ \text{---} & \mathbf{a}_{r_2}^T & \text{---} \\ \vdots & \vdots & \vdots \\ \text{---} & \mathbf{a}_{r_p}^T & \text{---} \end{bmatrix} B = \begin{bmatrix} \text{---} & \mathbf{a}_{r_1}^T B & \text{---} \\ \text{---} & \mathbf{a}_{r_2}^T B & \text{---} \\ \vdots & \vdots & \vdots \\ \text{---} & \mathbf{a}_{r_p}^T B & \text{---} \end{bmatrix} \\
 &= \begin{bmatrix} a_{11}\mathbf{b}_{r_1}^T + a_{12}\mathbf{b}_{r_2}^T + \cdots + a_{1q}\mathbf{b}_{r_q}^T \\ a_{21}\mathbf{b}_{r_1}^T + a_{22}\mathbf{b}_{r_2}^T + \cdots + a_{2q}\mathbf{b}_{r_q}^T \\ \vdots \\ a_{p1}\mathbf{b}_{r_1}^T + a_{p2}\mathbf{b}_{r_2}^T + \cdots + a_{pq}\mathbf{b}_{r_q}^T \end{bmatrix}.
 \end{aligned}$$

(c) The sum of column i of A times row i of B from 1 to q \square

$$AB = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{a}_{c_1} & \mathbf{a}_{c_2} & \cdots & \mathbf{a}_{c_q} \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} \text{---} & \mathbf{b}_{r_1}^T & \text{---} \\ \text{---} & \mathbf{b}_{r_2}^T & \text{---} \\ \vdots & \vdots & \vdots \\ \text{---} & \mathbf{b}_{r_q}^T & \text{---} \end{bmatrix} = \mathbf{a}_{c_1}\mathbf{b}_{r_1}^T + \mathbf{a}_{c_2}\mathbf{b}_{r_2}^T + \cdots + \mathbf{a}_{c_q}\mathbf{b}_{r_q}^T.$$

4. The Laws of Matrix Operations \square

(a) **Laws of Matrix Addition**

\forall number $k \in \mathbb{R}$, \forall three $m \times n$ matrices A , B and C where $m, n \in \mathbb{N}^+$

- **Commutative Law**

$$A + B = B + A.$$

- **Distributive Law**

$$k(A + B) = kA + kB.$$

- **Associative Law**

$$A + (B + C) = (A + B) + C.$$

(b) **Laws of Matrix Multiplication**

- **Distributive Law from The Left**

$\forall p \times q$ matrix A , $\forall q \times r$ matrices B and C where $p, q, r \in \mathbb{N}^+$

$$A(B + C) = AB + AC.$$

- **Distributive Law from The Right**

$\forall p \times q$ matrices A and B , $\forall q \times r$ matrix C where $p, q, r \in \mathbb{N}^+$

$$(A + B)C = AC + BC.$$

- **Associative Law**

$\forall p \times q$ matrix A , $\forall q \times r$ matrix B and $\forall r \times s$ matrix C where $p, q, r, s \in \mathbb{N}^+$

$$A(BC) = (AB)C.$$

(c) Laws of Matrix Powers

\forall number p, q and $\forall n \times n$ matrix A where $p, q, n \in \mathbb{N}^+$

$$A^p = AA \cdots A \text{ (} p \text{ factors),}$$

$$(A^p)(A^q) = A^{p+q},$$

$$(A^p)^q = A^{pq}.$$

1.2.4 Row Operations of Matrix

$\forall m \times n$ matrix A where $m \geq 2$ and $m, n \in \mathbb{N}^+$, let $1 \leq i < j \leq m$ where $i, j \in \mathbb{N}^+$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

1. Switching Rows

For $1 \leq l \leq m, l \neq i, l \neq j$ and $l \in \mathbb{N}^+$, let P be a $m \times m$ matrix

$$P = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}.$$

where $P(i, j) = P(j, i) = P(l, l) = 1$ and other components of P is 0. In order to switch the i -th row and j -th row of A ,

$$PA = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

If the order of i and j is switched, this row operation still works.

2. Multiplying a Row by a Number

For $k \in \mathbb{R}, 1 \leq l \leq m, l \neq i$ and $l \in \mathbb{N}^+$, let D^{-1} be a $m \times m$ matrix

$$D^{-1} = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}.$$

where $D^{-1}(i, i) = k, D^{-1}(l, l) = 1$ and other components of D^{-1} is 0. In order to multiply the i -th row of A by k ,

$$D^{-1}A = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ ka_{i1} & \cdots & ka_{in} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

3. Adding Multiples of Rows

For $k \in \mathbb{R}, 1 \leq l \leq m$ and $l \in \mathbb{N}^+$, let E be a $m \times m$ matrix

$$E = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}.$$

where $E(j, i) = k, E(l, l) = 1$ and other components of E is 0. In order to add the multiple of the i -th row of A by k into the j -th row of A

$$\begin{aligned} EA &= \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ ka_{i1} + a_{j1} & \cdots & ka_{in} + a_{jn} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}. \end{aligned}$$

If the order of i and j is switched, this row operation still works.

1.2.5 Inverse Matrices

1. Definition of Inverse Matrix

$\forall n \times n$ square matrix A where $n \in \mathbb{N}^+$, the matrix A is invertible if there exists a matrix A^{-1} such that

$$A^{-1}A = AA^{-1} = I.$$

2. The Properties of Inverse Matrices \square

- (a) $\forall n \times n$ invertible square matrix A where $n \in \mathbb{N}^+$, its inverse A^{-1} is unique.
- (b) For $n \times n$ square matrices A_1, A_2, \dots, A_k where $n, k \in \mathbb{N}^+$, if A_1, A_2, \dots, A_k are separately invertible, then

$$(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_1^{-1}.$$

3. Calculating Inverse Matrices by Gauss-Jordan Elimination \square

$\forall n \times n$ invertible square matrix A where $n \in \mathbb{N}^+$, apply Gauss-Jordan Elimination into the matrix

$$\begin{bmatrix} A & I \end{bmatrix}$$

to get the matrix

$$\begin{bmatrix} I & A^{-1} \end{bmatrix}.$$

4. Invertible Matrix Theorem

$\forall n \times n$ square matrix A where $n \in \mathbb{N}^+$, A is invertible if and only if any of the following hold:

- (a) The RREF of A has n pivots;
- (b) The equation $A\mathbf{x} = \mathbf{b}$ has only one solution.
- (c) The equation $A\mathbf{x} = \mathbf{0}$ has no nonzero solutions.
- (d) The determinant of A is not zero.

1.2.6 Transposes and Permutations

1. Definition of Transpose

$\forall m \times n$ matrix A where $m, n \in \mathbb{N}^+$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

The transpose of the matrix A is

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.$$

2. The Properties of Transposes \square

(a) For $m \times n$ matrices A_1, A_2, \dots, A_k where $m, n, k \in \mathbb{N}^+$

$$(A_1 + A_2 + \cdots + A_k)^T = A_1^T + A_2^T + \cdots + A_k^T.$$

(b) $\forall n_1 \times n_2$ matrix $A_1, \forall n_2 \times n_3$ matrix $A_2, \dots, \forall n_{k-1} \times n_k$ matrix $A_{k-1}, \forall n_k \times n_{k+1}$ matrix A_k where $k, n_1, n_2, \dots, n_k, n_{k+1} \in \mathbb{N}^+$

$$(A_1 A_2 \cdots A_{k-1} A_k)^T = A_k^T A_{k-1}^T \cdots A_2^T A_1^T.$$

(c) $\forall n \times n$ invertible matrix A where $n \in \mathbb{N}^+$

$$(A^{-1})^T = (A^T)^{-1}.$$

3. Symmetric Matrix

$\forall n \times n$ symmetric matrix S has

$$S^T = S.$$

The inverse of a symmetric matrix is also symmetric. \square

4. Permutation Matrix

A permutation matrix P has the rows of the identity I in any order. The properties of permutation matrices are shown as follows:

(a) $\forall n \in \mathbb{N}^+$, there are $n!$ permutation matrices of order n ;

(b) P^{-1} is also a permutation matrix;

(c) $P^{-1} = P^T$.

1.2.7 Special Matrices

1. Difference Matrix and Cyclic Difference Matrix

Given a n -dimensional vector \mathbf{x} and a n -dimensional vector \mathbf{b}

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

its difference matrix A such that $A\mathbf{x} = \mathbf{b}$ is

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}.$$

Then we will get

$$\mathbf{x} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ \vdots \\ b_1 + b_2 + \cdots + b_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \mathbf{A}^{-1}\mathbf{b}.$$

The matrix A is invertible. From \mathbf{b} we can recover \mathbf{x} . We write \mathbf{x} as $A^{-1}\mathbf{b}$. In particular, $A\mathbf{x} = \mathbf{0}$ has one solution

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Its cyclic difference matrix C such that $C\mathbf{x} = \mathbf{b}$ is

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}.$$

$C\mathbf{x} = \mathbf{b}$ has many solutions or else no solution. In particular, if $C\mathbf{x} = \mathbf{0}$ has many solutions, C is a singular matrix.

2. Identity Matrix

$\forall k \in \mathbb{N}^+$, the $k \times k$ identity matrix is

$$I = I_k = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

The diagonal components of I is 1, and other components of I is 0. $\forall m \times n$ matrix A where $m, n \in \mathbb{N}^+$

$$I_m A = A I_n = A.$$

In particular, $\forall n \times n$ square matrix A where $n \in \mathbb{N}^+$

$$I A = A I = A.$$