Chapter 1

Introduction to Vectors

1.1 Vectors and Linear Combinations

1.1.1 Definition of Vectors

$\forall\ n-$dimensional vector **v** where $n \in \mathbb{N}^*$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = (v_1, v_2, \cdots, v_n).$$

$v_1, v_2, \cdots, v_n$ are the 1st, 2nd, $\cdots$, n-th component of **v**. Every vector is written as a column.

1.1.2 Operations of Vectors

1. Addition

   $\forall\ m-$dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m$ where $m, n \in \mathbb{N}^*$:

$$\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix}, \qquad \cdots, \qquad \mathbf{v}_n = \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

The vector addition of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ is

$$\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n = \begin{bmatrix} v_{11} + v_{12} + \cdots + v_{1n} \\ v_{21} + v_{22} + \cdots + v_{2n} \\ \vdots \\ v_{m1} + v_{m2} + \cdots + v_{mn} \end{bmatrix}.$$

2. Scalar Multiplication

   $\forall\, n-$dimensional vector $\mathbf{v}$ where $n \in \mathbb{N}^*$ and $\forall$ number c:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

   The scalar multiplication of c and $\mathbf{v}$ is

$$c\mathbf{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}.$$

   The number c is called a "scalar".

1.1.3 Definition of Linear Combination

Combine addition with scalar multiplication to produce a "linear combination". $\forall\, m-$dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ where $m, n \in \mathbb{N}^*$ and $\forall$ number $\alpha_1, \alpha_2, \cdots, \alpha_n$. The sum of $\alpha_1\mathbf{v}_1, \alpha_2\mathbf{v}_2, \cdots, \alpha_n\mathbf{v}_n$ is a linear combination

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n.$$

1.1.4 Geometrical Significance of Linear Combination

Parallelogram Law

The parallelogram law gives the rule for vector addition of vectors $\mathbf{u}$ and $\mathbf{v}$. The sum $\mathbf{u} + \mathbf{v}$ of the vectors is obtained by placing them head to tail and drawing the vector from the free tail to the free head.

Line, Plane, Space and Three-dimensional Vectors

$\forall\, \alpha, \beta, \gamma \in \mathbb{R}$ and $\forall$ nonzero three-dimensional vectors **u**, **v**, **w**:

- All combinations α**u** fill a line through $(0, 0, 0)$;
- If **u** and **v** are not on the same line, all combinations α**u** $+$ β**v** fill a plane through $(0, 0, 0)$;
- If **w** is not on the same plane formed by α**u** $+$ β**v**, all combinations α**u** $+$ β**v** $+ \gamma bfw$ fill three-diemnsional space.

1.2 Lengths and Dot Products

1.2.1 Definition of Dot Products

$\forall\, n-$dimensional vectors **v**, **w** where $n \in \mathbb{N}^*$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}.$$

The dot product or inner product of **v**, **w** is the number $\mathbf{v} \cdot \mathbf{w}$ or $\mathbf{w} \cdot \mathbf{v}$:

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

1.2.2 Definition of Lengths

$\forall\, n-$dimensional vectors **v** where $n \in \mathbb{N}^*$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

The length $||\mathbf{v}||$ of a vector **v** is the square root of $\mathbf{v} \cdot \mathbf{v}$:

$$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

1.2.3 Definition of Unit Vectors

A unit vector **u** is a vector whose length equals one. Then $\mathbf{u} \cdot \mathbf{u} = 1$. $\forall\ n-$dimensional vectors **v** where $n \in \mathbb{N}^*$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

$\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector in the same direction as **v**.

1.2.4 The Angle Between Vectors

Two-dimensional Vectors

$\forall\ 2-$dimensional nonzero vectors **v**, **w**:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (v_1^2 + v_2^2 \neq 0), \qquad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad (w_1^2 + w_2^2 \neq 0).$$

Let the angle between **v** and **w** is θ $(0 \leqslant \theta < \pi)$, then

$$\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\ \|\mathbf{w}\|} = \frac{v_1 w_1 + v_2 w_2}{\sqrt{v_1^2 + v_2^2}\sqrt{w_1^2 + w_2^2}} = \cos\theta.$$

Three-dimensional Vectors

$\forall\ 2-$dimensional nonzero vectors **v**, **w**:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad (v_1^2 + v_2^2 + v_3^2 \neq 0), \qquad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \quad (w_1^2 + w_2^2 + w_3^2 \neq 0).$$

Let the angle between **v** and **w** is θ $(0 \leqslant \theta < \pi)$, then

$$\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\ \|\mathbf{w}\|} = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{\sqrt{v_1^2 + v_2^2 + v_3^2}\sqrt{w_1^2 + w_2^2 + w_3^2}} = \cos\theta.$$

Schwarz inequality and Triangle inequality

Schwarz inequality

$$|\mathbf{v} \cdot \mathbf{w}| \leqslant \|\mathbf{v}\|\ \|\mathbf{w}\|.$$

Triangle inequality

$$\|\mathbf{v}+\mathbf{w}\| \leqslant \|\mathbf{v}\| + \|\mathbf{w}\|.$$

1.3 Matrices

1.3.1 Matrix and Linear Combination of Vectors

Definition of Matrix

$\forall$ $m-$dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ where $m, n \in \mathbb{N}^*$:

$$\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix}, \qquad \cdots, \qquad \mathbf{v}_n = \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

and $\forall$ number $x_1, x_2, \cdots, x_n$. The linear combination of

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = x_1 \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix} + x_2 \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

can be expressed as

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where **A** is a matrix and **x** is a vector:

$$\mathbf{A} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & \cdots & | \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The Format of Matrix

Let n $m-$dimensional vectors $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n$ and

$$\mathbf{c}_1 = \mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix}, \quad \mathbf{c}_2 = \mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix}, \quad \cdots, \quad \mathbf{c}_n = \mathbf{v}_n = \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

Let m $n-$dimensional vectors $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_m$ and

$$\mathbf{r}_1 = \begin{bmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} v_{21} \\ v_{22} \\ \vdots \\ v_{2n} \end{bmatrix}, \quad \cdots, \quad \mathbf{r}_m = \begin{bmatrix} v_{m1} \\ v_{m2} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

Then the matrix **A** can be expressed as

$$\mathbf{A} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \\ | & | & \cdots & | \end{bmatrix} = \begin{bmatrix} — & \mathbf{r}_1 & — \\ — & \mathbf{r}_2 & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{r}_m & — \end{bmatrix}.$$

The Form of Ax = b

Let a m-dimensional vector **b**

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

and

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Then

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{b} \\ \mathbf{r}_2 \cdot \mathbf{b} \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{b} \end{bmatrix}.$$

1.3.2 Difference Matrix and Cyclic Difference Matrix

Given a n-dimensional vector **x** and a n-dimensional vector **b**

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

its difference matrix **A** such that **Ax** = **b** is

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}.$$

Then we will get

$$\mathbf{x} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ \vdots \\ b_1 + b_2 + \cdots + b_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \mathbf{A}^{-1}\mathbf{b}.$$

The matrix $\mathbf{A}$ is invertible. From $\mathbf{b}$ we can recover $\mathbf{x}$. We write $\mathbf{x}$ as $\mathbf{A}^{-1}\mathbf{b}$. In particular, $\mathbf{Ax} = \mathbf{0}$ has one solution

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Its cyclic difference matrix $\mathbf{C}$ such that $\mathbf{Cx} = \mathbf{b}$ is

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}.$$

$\mathbf{Cx} = \mathbf{b}$ has many solutions or else no solution. In particular, if $\mathbf{Cx} = \mathbf{0}$ has many solutions, $\mathbf{C}$ is a singular matrix.