Chapter 1

Introduction to Vectors

1.1 Vectors

1.1.1 Definition of Vector

$\forall$ $n-$dimensional vector **v** where $n \in \mathbb{N}^+$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = (v_1, v_2, \cdots, v_n).$$

$v_1, v_2, \cdots, v_n$ are the 1st, 2nd, $\cdots$, n-th component of **v**. Every vector is written as a column.

1.1.2 Operations of Vectors

1. **Vector Addition**

   $\forall$ $m-$dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m$ where $m, n \in \mathbb{N}^+$:

$$\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix}, \qquad \cdots, \qquad \mathbf{v}_n = \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

The vector addition of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ is

$$\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n = \begin{bmatrix} v_{11} + v_{12} + \cdots + v_{1n} \\ v_{21} + v_{22} + \cdots + v_{2n} \\ \vdots \\ v_{m1} + v_{m2} + \cdots + v_{mn} \end{bmatrix}.$$

2. **Vector Scalar Multiplication**

   $\forall$ number $k \in \mathbb{R}$ and $\forall$ $n-$dimensional vector $\mathbf{v}$ where $n \in \mathbb{N}^+$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

   The scalar multiplication of k and $\mathbf{v}$ is

$$k\mathbf{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}.$$

   The number k is called a "scalar".

1.1.3 Linear Combination

1. **Definition of Linear Combination**

   Combine addition with scalar multiplication to produce a "linear combination". $\forall$ $m-$dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ where $m, n \in \mathbb{N}^+$ and $\forall$ number $\alpha_1, \alpha_2, \cdots,$ $\alpha_n \in \mathbb{R}$. The sum of $\alpha_1\mathbf{v}_1, \alpha_2\mathbf{v}_2, \cdots, \alpha_n\mathbf{v}_n$ is a linear combination

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n.$$

2. **Geometrical Significance of Linear Combination (Parallelogram Law)**

   The parallelogram law gives the rule for vector addition of vectors $\mathbf{u}$ and $\mathbf{v}$. The sum $\mathbf{u} + \mathbf{v}$ of the vectors is obtained by placing them head to tail and drawing the vector from the free tail to the free head.

3. **Line, Plane, Space and Three-dimensional Vectors**

   $\forall$ number $\alpha, \beta, \gamma \in \mathbb{R}$ and $\forall$ nonzero three-dimensional vectors **u**, **v**, **w**:

   - All combinations $\alpha\mathbf{u}$ fill a line through $(0, 0, 0)$;
   - If **u** and **v** are not on the same line, all combinations $\alpha\mathbf{u} + \beta\mathbf{v}$ fill a plane through $(0, 0, 0)$;
   - If **w** is not on the same plane formed by $\alpha\mathbf{u}+\beta\mathbf{v}$, all combinations $\alpha\mathbf{u}+\beta\mathbf{v}+\gamma bfw$ fill three-diemnsional space.

1.1.4 Vector Dot Product

$\forall$ two $n-$dimensional vectors **v**, **w** where $n \in \mathbb{N}^+$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}.$$

The dot product or inner product of **v**, **w** is the number $\mathbf{v} \cdot \mathbf{w}$ or $\mathbf{w} \cdot \mathbf{v}$:

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

1.1.5 The Length of Vector

$\forall$ $n-$dimensional vectors **v** where $n \in \mathbb{N}^+$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

The length $||\mathbf{v}||$ of a vector **v** is the square root of $\mathbf{v} \cdot \mathbf{v}$:

$$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

1.1.6 Unit Vector

A unit vector **u** is a vector whose length equals one. Then $\mathbf{u} \cdot \mathbf{u} = 1$. $\forall$ $n-$dimensional vectors **v** where $n \in \mathbb{N}^{+}$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

$\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector in the same direction as **v**.

1.1.7 The Angle Between Vectors

1. **Two-dimensional Vectors**

   $\forall$ two $2-$dimensional nonzero vectors **v**, **w**:

   $$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (v_1^2 + v_2^2 \neq 0), \qquad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad (w_1^2 + w_2^2 \neq 0).$$

   Let the angle between **v** and **w** is θ $(0 \leq \theta < \pi)$, then

   $$\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \, \|\mathbf{w}\|} = \frac{v_1 w_1 + v_2 w_2}{\sqrt{v_1^2 + v_2^2}\sqrt{w_1^2 + w_2^2}} = \cos\theta.$$

2. **Three-dimensional Vectors**

   $\forall$ two $2-$dimensional nonzero vectors **v**, **w**:

   $$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad (v_1^2 + v_2^2 + v_3^2 \neq 0), \qquad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \quad (w_1^2 + w_2^2 + w_3^2 \neq 0).$$

   Let the angle between **v** and **w** is θ $(0 \leq \theta < \pi)$, then

   $$\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \, \|\mathbf{w}\|} = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{\sqrt{v_1^2 + v_2^2 + v_3^2}\sqrt{w_1^2 + w_2^2 + w_3^2}} = \cos\theta.$$

3. **Schwarz inequality and Triangle inequality**

   Schwarz inequality

   $$|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \, \|\mathbf{w}\|.$$

   Triangle inequality

   $$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|.$$

1.2 Matrix

1.2.1 Definition of Matrix

$\forall$ $m-$dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ where $m, n \in \mathbb{N}^+$:

$$\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix}, \qquad \cdots, \qquad \mathbf{v}_n = \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

and $\forall$ number $x_1, x_2, \cdots, x_n \in \mathbb{R}$. The linear combination of

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = x_1 \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix} + x_2 \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

can be expressed as

$$A\mathbf{x} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where A is a matrix and $\mathbf{x}$ is a vector:

$$A = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & \cdots & | \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

1.2.2 The Format of Matrix

Let $m-$dimensional vectors $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n$ and

$$\mathbf{c}_1 = \mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix}, \qquad \mathbf{c}_2 = \mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix}, \qquad \cdots, \qquad \mathbf{c}_n = \mathbf{v}_n = \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

Let $n-$dimensional vectors $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_m$ and

$$\begin{aligned}
\mathbf{r}_1 &= \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \end{bmatrix}, \\
\mathbf{r}_2 &= \begin{bmatrix} v_{21} & v_{22} & \cdots & v_{2n} \end{bmatrix}, \\
&\vdots \\
\mathbf{r}_m &= \begin{bmatrix} v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix}.
\end{aligned}$$

Then the matrix A can be expressed as

$$A = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \\ | & | & \cdots & | \end{bmatrix} = \begin{bmatrix} — & \mathbf{r}_1 & — \\ — & \mathbf{r}_2 & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{r}_m & — \end{bmatrix}.$$

1.2.3 The Form of $A\mathbf{x} = \mathbf{b}$

Let a m-dimensional vector $\mathbf{b}$

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

and

$$A\mathbf{x} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Then

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{b} \\ \mathbf{r}_2 \cdot \mathbf{b} \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{b} \end{bmatrix}.$$

1.2.4 Difference Matrix and Cyclic Difference Matrix

Given a n-dimensional vector **x** and a n-dimensional vector **b**

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

its difference matrix A such that A**x** = **b** is

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}.$$

Then we will get

$$\mathbf{x} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ \vdots \\ b_1 + b_2 + \cdots + b_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \mathbf{A}^{-1}\mathbf{b}.$$

The matrix A is invertible. From **b** we can recover **x**. We write **x** as A^{-1}**b**. In particular, $A\mathbf{x} = \mathbf{0}$ has one solution

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Its cyclic difference matrix C such that $C\mathbf{x} = \mathbf{b}$ is

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}.$$

$C\mathbf{x} = \mathbf{b}$ has many solutions or else no solution. In particular, if $C\mathbf{x} = \mathbf{0}$ has many solutions, C is a singular matrix.

1.3 Matrix Operations

1.3.1 Matrix Addition

Matrices can be added if their shapes are the same. $\forall$ $m \times n$ matrices A and B where $k, m, n \in \mathbb{N}^+$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}.$$

Then the addition of A and B is

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}.$$

1.3.2 Matrix Scalar Multiplication

$\forall$ number $k \in \mathbb{R}$, $\forall$ $m \times n$ matrix A where $k \in \mathbb{R}$ and $m, n \in \mathbb{N}^+$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Then the scalar multiplication of k and A is

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}.$$

1.3.3 Matrix Multiplication

For two matrices A and B, in order to multiply AB, if A has q columns, B must have q rows, where $q \in \mathbb{N}^+$. $\forall$ $p \times q$ matrix A and $\forall$ $q \times r$ matrix B where $p, q, r \in \mathbb{N}^+$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ b_{q1} & b_{q2} & \cdots & b_{qr} \end{bmatrix}.$$

Then the multiplication of A and B is

$$AB = \begin{bmatrix} \sum_{i=1}^{q} a_{1i}b_{i1} & \sum_{i=1}^{q} a_{1i}b_{i2} & \cdots & \sum_{i=1}^{q} a_{1i}b_{ir} \\ \sum_{i=1}^{q} a_{2i}b_{i1} & \sum_{i=1}^{q} a_{2i}b_{i2} & \cdots & \sum_{i=1}^{q} a_{2i}b_{ir} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{q} a_{pi}b_{i1} & \sum_{i=1}^{q} a_{2i}b_{i2} & \cdots & \sum_{i=1}^{q} a_{pi}b_{ir} \end{bmatrix}.$$

The computation of AB uses pqr separate multiplications.

1.3.4 The Laws of Matrix Operations

1. **Laws of Matrix Addition**

   $\forall$ number $k \in \mathbb{R}$, $\forall$ three $m \times n$ matrix A, B and C where $m, n \in \mathbb{N}^+$

   - **Commutative Law**

     $$A + B = B + A.$$

   - **Distributive Law**

     $$k(A + B) = kA + kB.$$

   - **Associative Law**

     $$A + (B + C) = (A + B) + C.$$

2. **Laws of Matrix Multiplication**

   - **Distributive Law from The Left**

     $\forall$ $p \times q$ matrix A, $\forall$ $q \times r$ matrices B and C where $p, q, r \in \mathbb{N}^+$

     $$A(B + C) = AB + AC.$$

- **Distributive Law from The Right**

  $\forall\ p \times q$ matrix A and B, $\forall\ q \times r$ matrix C where $p, q, r \in \mathbb{N}^+$

  $$(A + B)C = AC + BC.$$

- **Associative Law**

  $\forall\ p \times q$ matrix A, $\forall\ q \times r$ matrix B and $\forall\ r \times s$ matrix C where $p, q, r, s \in \mathbb{N}^+$

  $$A(BC) = (AB)C.$$

3. **Laws of Matrix Powers**

   $\forall$ number p, q and $\forall\ n \times n$ matrix A where $p, q, n \in \mathbb{N}^+$

$$A^p = AA \cdots A \text{ (p factors)},$$
$$(A^p)(A^q) = A^{p+q},$$
$$(A^p)^q = A^{pq}.$$

1.4 The Extension of Matrix Multiplication

1.4.1 Variants of Matrix Multiplication

$\forall\ p \times q$ matrix A and $\forall\ q \times r$ matrix B where $p, q, r \in \mathbb{N}^+$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{bmatrix} = \begin{bmatrix} — & \mathbf{a_{r_1}} & — \\ — & \mathbf{a_{r_2}} & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{a_{r_p}} & — \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{a_{c_1}} & \mathbf{a_{c_2}} & \cdots & \mathbf{a_{c_q}} \\ | & | & \cdots & | \end{bmatrix},$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ b_{q1} & b_{q2} & \cdots & b_{qr} \end{bmatrix} = \begin{bmatrix} — & \mathbf{b_{r_1}} & — \\ — & \mathbf{b_{r_2}} & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{b_{r_q}} & — \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{b_{c_1}} & \mathbf{b_{c_2}} & \cdots & \mathbf{b_{c_r}} \\ | & | & \cdots & | \end{bmatrix}.$$

There are three variants of matrix multiplication AB

1. **Matrix A times every column of matrix B**

$$AB = A \begin{bmatrix} | & | & \cdots & | \\ \mathbf{b_{c_1}} & \mathbf{b_{c_2}} & \cdots & \mathbf{b_{c_r}} \\ | & | & \cdots & | \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ A\mathbf{b_{c_1}} & A\mathbf{b_{c_2}} & \cdots & A\mathbf{b_{c_r}} \\ | & | & \cdots & | \end{bmatrix}.$$

2. **Every row of matrix A times matrix B**

$$AB = \begin{bmatrix} — & \mathbf{a_{r_1}} & — \\ — & \mathbf{a_{r_2}} & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{a_{r_p}} & — \end{bmatrix} B = \begin{bmatrix} — & \mathbf{a_{r_1}}B & — \\ — & \mathbf{a_{r_2}}B & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{a_{r_p}}B & — \end{bmatrix}.$$

3. **The sum of column i of A times row i of B from 1 to q**

$$AB = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{a_{c_1}} & \mathbf{a_{c_2}} & \cdots & \mathbf{a_{c_q}} \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} — & \mathbf{b_{r_1}} & — \\ — & \mathbf{b_{r_2}} & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{b_{r_q}} & — \end{bmatrix} = \mathbf{a_{c_1}}\mathbf{b_{r_1}} + \mathbf{a_{c_2}}\mathbf{b_{r_2}} + \cdots + \mathbf{a_{c_q}}\mathbf{b_{r_q}}.$$

1.4.2 Row Operations of Matrix

$\forall$ $m \times n$ matrix A where $m \geq 2$ and $m, n \in \mathbb{N}^+$, let $1 \leq i < j \leq m$ where $i, j \in \mathbb{N}^+$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

1. **Switching Rows**

   For $1 \leq l \leq m, l \neq i, l \neq j$ and $l \in \mathbb{N}^+$, let P be a $m \times m$ matrix

$$P = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}.$$

   where $P(i, j) = P(j, i) = P(l, l) = 1$ and other components of P is 0. In order to switch the i-th row and j-th row of A,

$$PA = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

2. **Multiplying a Row by a Number**

For $k \in \mathbb{R}, 1 \leq l \leq m, l \neq i$ and $l \in \mathbb{N}^+$, let D^{-1} be a $m \times m$ matrix

$$D^{-1} = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}.$$

where $D^{-1}(i,i) = k, D^{-1}(l,l) = 1$ and other components of D^{-1} is 0. In order to multiply the i-th row of A by k,

$$D^{-1}A = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ ka_{i1} & \cdots & ka_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

If i is replace by j, this row operation still works.

3. **Adding Multiples of Rows**

For $k \in \mathbb{R}, 1 \leq l \leq m$ and $l \in \mathbb{N}^+$, let E be a $m \times m$ matrix

$$E = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}.$$

where $E(j,i) = k, E(l,l) = 1$ and other components of E is 0. In order to add the multiple of the i-th row of A by k into the j-th row of A

$$EA = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ ka_{i1} + a_{j1} & \cdots & ka_{in} + a_{jn} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

If the order of i and j is switched, this row operation still works.

1.4.3 Identity Matrix

$\forall\, k \in \mathbb{N}^+$, the $k \times k$ identity matrix is

$$I = I_k = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

The diagonal components of I is 1, and other components of I is 0. $\forall\, m \times n$ matrix A where $m, n \in \mathbb{N}^+$

$$I_m A = A I_n = A.$$

In particular, $\forall\, n \times n$ square matrix A where $n \in \mathbb{N}^+$

$$IA = AI = A.$$