Chapter 2

Solving Linear Equations

2.1 Linear Equations

2.1.1 Definition of Linear Equations

$\forall$ linear equations

$$\begin{cases} a_{11}x_1 & + a_{12}x_2 & + & \cdots + a_{1n}x_n & = b_1, \\ a_{21}x_1 & + a_{22}x_2 & + & \cdots + a_{2n}x_n & = b_2, \\ & & \vdots & & \\ a_{m1}x_1 & + a_{m2}x_2 & + & \cdots + a_{mn}x_n & = b_m. \end{cases} \tag{2.1}$$

where $m, n \in \mathbb{N}^+$. The equations are linear, which means that the unknowns are only multiplied by numbers.

2.1.2 The Matrix Form of the Equations

The matrix form of linear equations 2.1 is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

This linear equations could be expressed as

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \\ | & | & \cdots & | \end{bmatrix} = \begin{bmatrix} — & \mathbf{r}_1 & — \\ — & \mathbf{r}_2 & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{r}_m & — \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

A is called the <u>coefficient matrix</u>. This linear equations could also be expressed as

$$A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{x} \\ \mathbf{r}_2 \cdot \mathbf{x} \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{x} \end{bmatrix} = \mathbf{b}.$$

2.2 Gauss-Jordan Elimination

2.2.1 Objective Matrix

$\forall\ m \times n$ matrix A without zero columns

$$A = \begin{bmatrix} A(1,1) & A(1,2) & \cdots & A(1,n) \\ A(2,1) & A(2,2) & \cdots & A(2,n) \\ \vdots & \vdots & \ddots & \vdots \\ A(m,1) & A(m,2) & \cdots & A(m,n) \end{bmatrix}$$

where $m, n \in \mathbb{N}^+$. $\forall\ 1 \le i \le m, 1 \le j \le n$ and $i, j \in \mathbb{N}^+$, $A(i,j)$ represents the component at the i-th row, j-th column of A, so its value will not necessarily remain the same in the following steps of Gauss-Jordan Elimination.

2.2.2 Operations and Shorthands

1. $P_{(i,j)}A$

   This notation represents a matrix multiplication, which means that switching the i-th row and j-th row of the matrix A. The matrix $P_{(i,j)}$ is shown as follows:

$$P_{(i,j)} = P = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}.$$

   where P is a $m \times m$ matrix. For $1 \le l \le m, l \ne i, l \ne j$ and $l \in \mathbb{N}^+$, $P(i,j) = P(j,i) = P(l,l) = 1$ and other components of P is 0.

2. $D^{-1}_{(i,k)}A$

   This notation represents a matrix multiplication, which means that multiplying the i-th row of the matrix A by a number k. The matrix $D_{(i,k)}^{-1}$ is shown as follows:

$$D_{(i,k)}^{-1} = D^{-1} = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}.$$

where D^{-1} is a $m \times m$ matrix. For $k \in \mathbb{R}, 1 \leq l \leq m, l \neq i$ and $l \in \mathbb{N}^{+}$, $D^{-1}(i,i) = k, D^{-1}(l,l) = 1$ and other components of D^{-1} is 0.

3. $E_{(j,i,k)}A$

   This notation represents a matrix multiplication, which means that adding the multiple of the i-th row of A by a number k into the j-th row of A. The matrix $E_{(j,i,k)}$ is shown as follows:

$$E_{(j,i,k)} = E = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}.$$

   where E is a $m \times m$ matrix. For $k \in \mathbb{R}, 1 \leq l \leq m$ and $l \in \mathbb{N}^{+}$, $E(j,i) = k, E(l,l) = 1$ and other components of E is 0.

4. **eliminationBelow**(i,j)

   The prerequisite of this operation is $A(i,j) \neq 0$. This notation represents a series of operations: For $l = i+1, i+2, \cdots, m$, adding the multiple of the i-th row of A by a number $-\frac{A(l,j)}{A(i,j)}$ into the l-th row of A. This operation is implemented as follows:

$$E_{\left[m,i,-\frac{A(m,j)}{A(i,j)}\right]} E_{\left[m-1,i,-\frac{A(m-1,j)}{A(i,j)}\right]} \cdots E_{\left[i+1,i,-\frac{A(i+1,j)}{A(i,j)}\right]} A.$$

5. **eliminationAbove**(i,j)

The prerequisite of this operation is $A(i,j) \neq 0$. This notation represents a series of operations: For $l = 1, 2 \cdots, i-1$, adding the multiple of the i-th row of A by a number $-\frac{A(l,j)}{A(i,j)}$ into the l-th row of A. This operation is implemented as follows:

$$E_{\left[i-1,i,-\frac{A(i-1,j)}{A(i,j)}\right]} \cdots E_{\left[2,i,-\frac{A(2,j)}{A(i,j)}\right]} E_{\left[1,i,-\frac{A(1,j)}{A(i,j)}\right]} A.$$

6. **searchPivot**(i,j)

   This notation represents a series of operations to search the pivot at the i-th row and no less than j-th column of the matrix A:

   (1) If $A(i,j) \neq 0$, $A(i,j)$ is the pivot at the i-th row of the matrix A.

   (2) If $A(i,j) = 0$, we check $A(i+1,j), A(i+2,j), \cdots, A(m,j)$ from the small row number to the large row number.

      a. If $A(l,j) \neq 0$ where $i < l \leq m$ and $l \in \mathbb{N}^+$, $A(l,j)$ is the pivot. Then we take the operation $P_{(i,l)}A$. Now $A(i,j)$ is the pivot at the i-th row of the matrix A.

      b. If $A(i,j) = A(i+1,j) = \cdots = A(m,j) = 0$, we take the operation **searchPivot**$(i,j+1)$, **searchPivot**$(i,j+2)$, $\cdots$, **searchPivot**(i,n) from the small column number to the large column number.

         (a) If we can search the pivot in the operation **searchPivot**(i,l) where $j < l \leq n$ and $l \in \mathbb{N}^+$, after the exchange of rows (if there is), $A(i,l)$ is the pivot at the i-th row of the matrix A.

         (b) If we have reached **searchPivot**(i,n) but still can not find the pivot. There is no pivot at the i-th row of the matrix A.

2.2.3 The Flow of Gauss-Jordan Elimination

Gauss-Jordan Elimination which is applied to the matrix A is shown as follows:

1. **searchPivot**$(1,1)$, and we find the pivot $A(1,1)$;

2. **eliminationBelow**$(1,1)$;

3. **searchPivot**$(2,2)$, and we find the pivot $A(2,c_1)$ where $2 \leq c_1 \leq n$ and $c_1 \in \mathbb{N}^+$;

4. **eliminationBelow**$(2,c_1)$;

5. **searchPivot**$(3, c_1 + 1)$, and we find the pivot $A(3, c_2)$ where $c_1 + 1 \leq c_2 \leq n$ and $c_2 \in \mathbb{N}^+$;
6. **eliminationBelow**$(3, c_2)$;
7. **searchPivot**$(4, c_2 + 1)$, and we find the pivot $A(4, c_3)$ where $c_2 + 1 \leq c_3 \leq n$ and $c_3 \in \mathbb{N}^+$;
8. **eliminationBelow**$(4, c_3)$;

   $\vdots$

We teriminate this process until we meet one of these three situations:

1. We have found the pivot at the m-th row of the matrix A;
2. We have found the pivot at the n-th column of the matrix A;
3. We have found that a row of the matrix A does not have a pivot.

Now, the matrix A has become the Row Echelon Form (REF). There are l pivots in this matrix where $1 \leq l \leq \min\{m, n\}$ and $l \in \mathbb{N}^+$. The REF of the matrix A is shown as follows:

$$\begin{bmatrix} B_1 & B_2 & \cdots & B_l \end{bmatrix}$$

where

$$B_j = \begin{bmatrix} e_{1\left(j+\sum_{i=0}^{j-1} k_i\right)} & b_{1\left(j+1+\sum_{i=0}^{j-1} k_i\right)} & \cdots & b_{1\left(j+\sum_{i=0}^{j} k_i\right)} \\ e_{2\left(j+\sum_{i=0}^{j-1} k_i\right)} & b_{2\left(j+1+\sum_{i=0}^{j-1} k_i\right)} & \cdots & b_{2\left(j+\sum_{i=0}^{j} k_i\right)} \\ \vdots & \vdots & \ddots & \vdots \\ e_{(j-1)\left(j+\sum_{i=0}^{j-1} k_i\right)} & b_{(j-1)\left(j+1+\sum_{i=0}^{j-1} k_i\right)} & \cdots & b_{(j-1)\left(j+\sum_{i=0}^{j} k_i\right)} \\ a_j & b_{j\left(j+1+\sum_{i=0}^{j-1} k_i\right)} & \cdots & b_{j\left(j+\sum_{i=0}^{j} k_i\right)} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

and $1 \leq j \leq l$, $k_0 = 0$, $j, k_1, k_2, \cdots, k_l \in \mathbb{N}^+$. The implicit constraint is

$$l + \sum_{i=0}^{l} k_i = n.$$

The Row Echelon Form (REF) of the matrix A could be transformed into the Reduced Row Echelon Form (RREF). The process is shown as follows:

1. $D^{-1}_{\left(l,\frac{1}{a_l}\right)} \cdots D^{-1}_{\left(2,\frac{1}{a_2}\right)} D^{-1}_{\left(1,\frac{1}{a_1}\right)} A$;
2. **eliminationAbove**$(1, 1)$;
3. **eliminationAbove**$(2, 2 + k_1)$;

   $\vdots$

$l+1$. **eliminationAbove**$\left(l, l - 2 + \sum_{i=1}^{l-1} k_i\right)$.

Now, the matrix A has become the Reduced Row Echelon Form (RREF). There are stll l pivots in this matrix. The RREF of the matrix A is shown as follows:

$$\begin{bmatrix} RB_1 & RB_2 & \cdots & RB_l \end{bmatrix}$$

where

$$RB_j = \begin{bmatrix} 0 & c_{1\left(j+1+\sum_{i=0}^{j-1} k_i\right)} & \cdots & c_{1\left(j+\sum_{i=0}^{j} k_i\right)} \\ 0 & c_{2\left(j+1+\sum_{i=0}^{j-1} k_i\right)} & \cdots & c_{2\left(j+\sum_{i=0}^{j} k_i\right)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c_{(j-1)\left(j+1+\sum_{i=0}^{j-1} k_i\right)} & \cdots & c_{(j-1)\left(j+\sum_{i=0}^{j} k_i\right)} \\ 1 & c_{j\left(j+1+\sum_{i=0}^{j-1} k_i\right)} & \cdots & c_{j\left(j+\sum_{i=0}^{j} k_i\right)} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

and $1 \leq j \leq l$, $k_0 = 0$, $j, k_1, k_2, \cdots, k_l \in \mathbb{N}^+$. The implicit constraint is

$$l + \sum_{i=0}^{l} k_i = n.$$

2.2.4 Gauss-Jordan Elimination and Linear Equations

$\forall$ linear equations, we eliminate the unknowns with only 0 coefficients to get the coefficient matrix A and vector **b**

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

where $m, n \in \mathbb{N}^+$ and $\forall\, 1 \leq i \leq n, i \in \mathbb{N}^+$

$$a_{1i}^2 + a_{2i}^2 + \cdots + a_{mi}^2 \neq 0.$$

The augmented matrix of the linear equations is

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}.$$

In order to find the solution of these linear equations, we apply the Gauss-Jordan Elimination into this augmented matrix to get its RREF matrix

$$\begin{bmatrix} RB_1 & RB_2 & \cdots & RB_l & \mathbf{d} \end{bmatrix}$$

where

$$RB_j = \begin{bmatrix} 0 & c_{1\left(j+1+\sum\limits_{i=0}^{j-1} k_i\right)} & \cdots & c_{1\left(j+\sum\limits_{i=0}^{j} k_i\right)} \\ 0 & c_{2\left(j+1+\sum\limits_{i=0}^{j-1} k_i\right)} & \cdots & c_{2\left(j+\sum\limits_{i=0}^{j} k_i\right)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c_{(j-1)\left(j+1+\sum\limits_{i=0}^{j-1} k_i\right)} & \cdots & c_{(j-1)\left(j+\sum\limits_{i=0}^{j} k_i\right)} \\ 1 & c_{j\left(j+1+\sum\limits_{i=0}^{j-1} k_i\right)} & \cdots & c_{j\left(j+\sum\limits_{i=0}^{j} k_i\right)} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{l-1} \\ d_l \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and $1 \leq j \leq l$, $k_0 = 0, j, k_1, k_2, \cdots, k_l \in \mathbb{N}^+$.

The solution of the linear equations is divided into two parts:

1. Independent Unknowns:

$$x_2, x_3, \cdots, x_{1+k_1} \in \mathbb{R};$$
$$x_{3+k_1}, x_{4+k_1}, \cdots, x_{2+k_1+k_2} \in \mathbb{R};$$
$$x_{4+k_1+k_2}, x_{5+k_1+k_2}, \cdots, x_{3+k_1+k_2+k_3} \in \mathbb{R};$$
$$\vdots$$
$$x_{l+1+\sum\limits_{i=1}^{l-1} k_i}, x_{l+2+\sum\limits_{i=1}^{l-1} k_i}, \cdots, x_{l+\sum\limits_{i=1}^{l} k_i} \in \mathbb{R}.$$

2. Dependent Unknowns:

$$\begin{aligned}
x_1 &= d_1 - \sum_{j=0}^{l-1} \sum_{u=1}^{k_{j+1}} c_{1\left(j+u+1+\sum\limits_{i=0}^{j} k_i\right)} x_{\left(j+u+1+\sum\limits_{i=0}^{j} k_i\right)}, \\
x_{2+k_1} &= d_2 - \sum_{j=1}^{l-1} \sum_{u=1}^{k_{j+1}} c_{2\left(j+u+1+\sum\limits_{i=1}^{j} k_i\right)} x_{\left(j+u+1+\sum\limits_{i=1}^{j} k_i\right)}, \\
x_{3+k_1+k_2} &= d_3 - \sum_{j=2}^{l-1} \sum_{u=1}^{k_{j+1}} c_{3\left(j+u+1+\sum\limits_{i=2}^{j} k_i\right)} x_{\left(j+u+1+\sum\limits_{i=2}^{j} k_i\right)}, \\
&\vdots \\
x_{l+\sum\limits_{i=1}^{l-1} k_i} &= d_l - \sum_{j=l-1}^{l-1} \sum_{u=1}^{k_{j+1}} c_{l\left(j+u+1+\sum\limits_{i=l-1}^{j} k_i\right)} x_{\left(j+u+1+\sum\limits_{i=l-1}^{j} k_i\right)}.
\end{aligned}$$

2.3 Inverse Matrices

2.3.1 Definition of Inverse Matrix

$\forall$ $n \times n$ square matrix A where $n \in \mathbb{N}^+$, the matrix A is invertible if there exists a matrix A^{-1} such that

$$A^{-1}A = AA^{-1} = I.$$

2.3.2 The Properties of Inverse Matrices

1. $\forall$ $n \times n$ invertible square matrix A where $n \in \mathbb{N}^+$, its inverse A^{-1} is unique.

2. For $n \times n$ square matrix $A_1, A_2, \cdots, A_k$ where $n, k \in \mathbb{N}^+$, if $A_1, A_2, \cdots, A_k$ are separately invertible, then

$$(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_1^{-1}.$$

2.3.3 Calculating Inverse Matrices by Gauss-Jordan Elimination

$\forall$ $n \times n$ invertible square matrix A where $n \in \mathbb{N}^+$, apply Gauss-Jordan Elimination into the matrix

$$\begin{bmatrix} A & I \end{bmatrix}$$

to get the matrix

$$\begin{bmatrix} I & A^{-1} \end{bmatrix}.$$

2.3.4 Invertible Matrix Theorem

$\forall$ $n \times n$ square matrix A where $n \in \mathbb{N}^+$, A is invertible if and only if any of the following hold:

1. The RREF of A has n pivots;

2. The equation $A\mathbf{x} = \mathbf{b}$ has only one solution.

3. The equation $A\mathbf{x} = \mathbf{0}$ has no nonzero solutions.

4. The determinant of A is not zero.

2.4 Transposes and Permutations

2.4.1 Definition of Transpose

$\forall\ m \times n$ matrix A where $m, n \in \mathbb{N}^+$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

The transpose of the matrix A is

$$A^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.$$

The properties of transposes are shown as follows:

1. $\forall\ m \times n$ matrices A and B where $m, n \in \mathbb{N}^+$

$$(A + B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}}.$$

2. $\forall\ p \times q$ matrix A and $\forall\ q \times r$ matrix B where $p, q, r \in \mathbb{N}^+$

$$(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}}.$$

3. $\forall\ m \times n$ matrix A where $m, n \in \mathbb{N}^+$

$$(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1}.$$

2.4.2 Symmetric Matrix

$\forall\ n \times n$ symmetric matrix S has

$$S^T = S.$$

If a symmetric matrix S has its inverse, then

$$(S^{-1})^T = S^{-1}.$$

2.4.3 Permutation Matrix

A permutation matrix P has the rows of the identity I in any order. The properties of permutation matrices are shown as follows:

1. $\forall\, n \in \mathbb{N}^{+}$, there are $n!$ permutation matrices of order n;
2. P^{-1} is also a permutation matrix;
3. $P^{-1} = P^{\mathrm{T}}$.

2.5 Elimination = Factorization: $A = LU$

2.5.1 Elimination and $A = LU$

Suppose that A is a $n \times n$ square matrix where $n \in \mathbb{N}^+$. If we apply Gauss-Jordan Elimination into the matrix A to get its REF, and there are no any row switches in this process, then the matrix A becomes the product of two special matrices:

$$A = LU,$$

where L is a lower triangular matrix and U is a upper triangular matrix. In addition, U is the REF of the matrix A. Suppose that

$$E_k \cdots E_2 E_1 A = U,$$

where $k \in \mathbb{N}^+$ and $E_1, E_2, \cdots, E_k$ is a series of **eliminationBelow** operations in Gauss-Jordan Elimination. Therefore,

$$A = (E_k \cdots E_2 E_1)^{-1} U = (E_1^{-1} E_2^{-1} \cdots E_k^{-1}) U = LU.$$

2.5.2 $A = LU$ and $A = LDU$

Suppose that A is a $n \times n$ square matrix where $n \in \mathbb{N}^+$ and A can be factored into $A = LU$. The matrix U can be splited as the product of two special matrices. Suppose that the matrix U is

$$U = \begin{bmatrix} p_1 & b_{12} & \cdots & b_{1k_1} & b_{1(1+k_1)} & b_{1(2+k_1)} & \cdots & b_{1k_2} & \cdots & b_{1k_{l-1}} & b_{1(1+k_{l-1})} & \cdots & b_{1k_l} \\ 0 & 0 & \cdots & 0 & p_2 & b_{2(2+k_1)} & \cdots & b_{2k_2} & \cdots & b_{2k_{l-1}} & b_{2(1+k_{l-1})} & \cdots & b_{2k_l} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & p_l & b_{l(1+k_{l-1})} & \cdots & b_{lk_l} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

where $1 < k_1 < k_2 < \cdots < k_l = n$ and $k_1, k_2, \cdots, k_l \in \mathbb{N}^+$. The matrix U can be splited as the product of two special matrices: D and U. Both the matrix D and the latter matrix U are $n \times n$ matrices. The matrix D is shown as follows:

$$D = \begin{bmatrix} p_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_l & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The latter matrix U is shown as follows:

$$U = \begin{bmatrix} 1 & \frac{b_{12}}{p_1} & \cdots & \frac{b_{1k_1}}{p_1} & \frac{b_{1(1+k_1)}}{p_1} & \frac{b_{1(2+k_1)}}{p_1} & \cdots & \frac{b_{1k_2}}{p_1} & \cdots & \frac{b_{1k_{l-1}}}{p_1} & \frac{b_{1(1+k_{l-1})}}{p_1} & \cdots & \frac{b_{1k_l}}{p_1} \\ 0 & 0 & \cdots & 0 & 1 & \frac{b_{2(2+k_1)}}{p_2} & \cdots & \frac{b_{2k_2}}{p_2} & \cdots & \frac{b_{2k_{l-1}}}{p_2} & \frac{b_{2(1+k_{l-1})}}{p_2} & \cdots & \frac{b_{2k_l}}{p_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & \frac{b_{l(1+k_{l-1})}}{p_l} & \cdots & \frac{b_{lk_l}}{p_l} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Thus, the matrix A becomes the product of three special matrices:

$$A = LDU.$$

If $S = S^T$ is factored into LDU with no row switches, then U is exactly L^T:

$$S = LDL^T.$$

2.5.3 $PA = LU$

Suppose that A is a $n \times n$ square matrix where $n \in \mathbb{N}^+$. If we apply Gauss-Jordan Elimination into the matrix A to get its REF, and there are row switches in this process, then the row switches can be done in advance. Their product P puts the rows of A in the right order, so that no exchanges are needed for PA. Then

$$PA = LU.$$