Chapter 1

Introduction to Vectors

1.1 Vectors and Linear Combinations

1.1.1 Definition of Vectors

$\forall$ $n-$dimensional vector **v** where $n \in \mathbb{N}^*$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = (v_1, v_2, \cdots, v_n).$$

$v_1, v_2, \cdots, v_n$ are the 1st, 2nd, $\cdots$, n-th component of **v**. Every vector is written as a **column**.

1.1.2 Operations of Vectors

1. Addition

   $\forall$ $n-$dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m$ where $m, n \in \mathbb{N}^*$:

   $$\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{n1} \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{n2} \end{bmatrix}, \qquad \cdots, \qquad \mathbf{v}_m = \begin{bmatrix} v_{1m} \\ v_{2m} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

   The vector addition of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m$ is

   $$\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_m = \begin{bmatrix} v_{11} + v_{12} + \cdots + v_{1n} \\ v_{21} + v_{22} + \cdots + v_{2n} \\ \vdots \\ v_{m1} + v_{m2} + \cdots + v_{mn} \end{bmatrix}.$$

2. Scalar Multiplication

   $\forall$ $n-$dimensional vector **v** where $n \in \mathbb{N}^*$ and $\forall$ number c:

   $$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

The scalar multiplication of c and $\mathbf{v}$ is

$$c\mathbf{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}.$$

The number c is called a “scalar”.

1.1.3 Definition of Linear Combination

Combine addition with scalar multiplication to produce a “linear combination”. $\forall\, n-$dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m$ where $m, n \in \mathbb{N}^*$ and $\forall$ number $\alpha_1, \alpha_2, \cdots, \alpha_m$. The sum of $\alpha_1\mathbf{v}_1, \alpha_2\mathbf{v}_2, \cdots, \alpha_m\mathbf{v}_m$ is a linear combination

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_m\mathbf{v}_m.$$

1.1.4 Geometrical Significance of Linear Combination

Parallelogram Law

The parallelogram law gives the rule for vector addition of vectors **u** and **v**. The sum $\mathbf{u} + \mathbf{v}$ of the vectors is obtained by placing them head to tail and drawing the vector from the free tail to the free head.

Line, Plane, Space and Three-dimensional Vectors

$\forall\, \alpha, \beta, \gamma \in \mathbb{R}$ and $\forall$ nonzero three-dimensional vectors **u**, **v**, **w**:

- All combinations $\alpha\mathbf{u}$ fill a line through $(0, 0, 0)$;
- If **u** and **v** are not on the same line, all combinations $\alpha\mathbf{u} + \beta\mathbf{v}$ fill a plane through $(0, 0, 0)$;
- If **w** is not on the same plane formed by $\alpha\mathbf{u} + \beta\mathbf{v}$, all combinations $\alpha\mathbf{u} + \beta\mathbf{v} + \gamma bfw$ fill three-diemnsional space.

1.2 Lengths and Dot Products

1.2.1 Definition of Dot Products

$\forall\, n-$dimensional vectors **v**, **w** where $n \in \mathbb{N}^*$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}.$$

The dot product or inner product of **v**, **w** is the number $\mathbf{v} \cdot \mathbf{w}$ or $\mathbf{w} \cdot \mathbf{v}$:

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} = v_1w_1 + v_2w_2 + \cdots + v_nw_n.$$