Chapter 3

Vector Spaces and Subspaces

3.1 Spaces of Vectors

3.1.1 Definition of Vector Space and Subspace

1. **Space**

   The space $\mathbf{R}^n$ consists of all vectors $\mathbf{v}$ with n components of real numbers. The space $\mathbf{C}^n$ consists of all vectors $\mathbf{v}$ with n components of complex numbers.

2. **Subspace**

   A subspace of a vector space is a set of vectors (including $\mathbf{0}$) that satisfies two requirements: If $\forall$ vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are in the subspace and $\forall\ \alpha_1, \alpha_2 \in \mathbb{R}$, then the linear combination $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2$ is in the subspace.

3.1.2 Properties of Vector Space and Subspace

1. Every space and subspace contains the zero vector.

2. Lines through the origin are also subspaces.

3. **Closure Property**

   If $\forall$ vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are in a vector space or subspace and $\forall\ \alpha_1, \alpha_2 \in \mathbb{R}$, then the linear combination $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2$ must stay in this vector space or subspace.

3.1.3 Special Vector Spaces

1. **M**

   The vector space of all real 2 by 2 matrices.

2. **F**

   The vector space of all real functions $f(x)$.

3. **P** or $\mathbf{P}_n$

   The vector space that consists of all polynomials

   $$a_0 + a_1 x + \cdots + a_n x^n$$

   of degree n.

4. **Z**

   The vector space that consists only of a zero vector.

3.2 Independence, Basis, Rank and Dimension

3.2.1 Definition of Independence, Basis, Rank and Dimension

1. **Linear Independence**

   The sequence of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ is linearly independent if the only combination that gives the zero vector is

   $$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n.$$

2. **Basis**

   A set of vectors spans a space if their linear combinations fill the space. A basis for a vector space is a sequence of vectors with two properties:

   (1) The basis vectors are linearly independent;

   (2) The basis vectors span the space.

3. **Rank**

   $\forall$ matrix A, the rank of A is the number of pivots, which is denoted by the number r.

4. **Dimension**

   The dimension of a space is the number of vectors in every basis.

3.2.2 The Conclusion of Linear Independence and Dimension

1. $\forall$ matrix A, the columns of A are linearly independent when the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

2. $\forall\, m, n \in \mathbb{N}^+$, any set of n vectors in $\mathbf{R}^m$ must be linearly dependent if $n > m$.

3. $\forall\, m, n \in \mathbb{N}^+$, if $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m$ and $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n$ are both bases for the same vector space, then $m = n$.

3.3 Solutions to $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$

3.3.1 The Special Solution to $A\mathbf{x} = \mathbf{0}$

$\forall\, m \times n$ matrix A where $m, n \in \mathbb{N}^+$, for the equation $A\mathbf{x} = \mathbf{0}$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

suppose that $x_{p_1}, x_{p_2}, \cdots, x_{p_r}$ are pivots; $x_{q_1}, x_{q_2}, \cdots, x_{q_{n-r}}$ are free variables ($1 \le p_1 < p_2 < \cdots < p_r \le n$; $1 \le q_1 < q_2 < \cdots < q_{n-r} \le n$; $1 \le r \le n$; $p_1, p_2, \cdots, p_r, q_1, q_2, \cdots, q_{n-r}, r \in \mathbb{N}^+$). If we apply Gauss-Jordan Elimination to the matrix A, the pivots can be expressed as a function of free functions.

$$\begin{aligned} x_{p_1} &= f_1(x_{q_1}, x_{q_2}, \cdots, x_{q_{n-r}}), \\ x_{p_2} &= f_2(x_{q_1}, x_{q_2}, \cdots, x_{q_{n-r}}), \\ &\vdots \\ x_{p_r} &= f_r(x_{q_1}, x_{q_2}, \cdots, x_{q_{n-r}}). \end{aligned}$$

A series of n-dimensional vectors: $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{n-r}$ are the solutions to $A\mathbf{x} = \mathbf{0}$. $\forall\, 1 \le i \le n - r, 1 \le j \le n$ and $i, j \in \mathbb{N}^+$, the j-th component of $\mathbf{x}_i$ is the value of x_j given by assigning 1 to the x_i and assigning other free valuables to the 0. The special solution to $A\mathbf{x} = \mathbf{0}$ is a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{n-r}$:

$$\mathbf{x}_n = x_{q_1}\mathbf{x}_1 + x_{q_2}\mathbf{x}_2 + \cdots + x_{q_{n-r}}\mathbf{x}_{n-r},$$

where $x_{q_1}, x_{q_2}, \cdots, x_{q_{n-r}} \in \mathbb{R}$.

3.3.2 The Complete Solution to $A\mathbf{x} = \mathbf{b}$

$\forall\, m \times n$ matrix A where $m, n \in \mathbb{N}^+$, for the solvable equation $A\mathbf{x} = \mathbf{0}$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

suppose that $x_{p_1}, x_{p_2}, \cdots, x_{p_r}$ are pivots; $x_{q_1}, x_{q_2}, \cdots, x_{q_{n-r}}$ are free variables ($1 \leq p_1 < p_2 < \cdots < p_r \leq n$; $1 \leq q_1 < q_2 < \cdots < q_{n-r} \leq n$; $1 \leq r \leq n$; $p_1, p_2, \cdots, p_r, q_1, q_2, \cdots, q_{n-r}, r \in \mathbb{N}^+$). If we apply Gauss-Jordan Elimination to the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ to get $\begin{bmatrix} R & \mathbf{d} \end{bmatrix}$, where the matrix R is the RREF (Reduced Row Echelon Form) of the matirx A. The pivots can be expressed as a function of free functions.

$$
\begin{aligned}
x_{p_1} &= d_1 + f_1(x_{q_1}, x_{q_2}, \cdots, x_{q_{n-r}}), \\
x_{p_2} &= d_2 + f_2(x_{q_1}, x_{q_2}, \cdots, x_{q_{n-r}}), \\
&\vdots \\
x_{p_r} &= d_r + f_r(x_{q_1}, x_{q_2}, \cdots, x_{q_{n-r}}).
\end{aligned}
$$

The complete solution to $A\mathbf{x} = \mathbf{b}$ is divided into two parts: The particular solution $\mathbf{x}_p$ to $A\mathbf{x}_p = \mathbf{b}$ and the special soution $\mathbf{x}_n$ to $A\mathbf{x}_n = \mathbf{0}$.

1. $\mathbf{x}_p$ is a $n-$dimensional vector. $\forall\ 1 \leq i \leq n-r, 1 \leq j \leq n$ and $i, j \in \mathbb{N}^+$, the j-th component of $\mathbf{x}_p$ is the value of x_j given by assigning 0 to all free valuables.

2. $\mathbf{x}_n$ is a $n-$dimensional vector. A series of n-dimensional vectors: $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{n-r}$ are the solutions to $A\mathbf{x} = \mathbf{0}$. $\forall\ 1 \leq i \leq n-r, 1 \leq j \leq n$ and $i, j \in \mathbb{N}^+$, the j-th component of $\mathbf{x}_i$ is the value of x_j given by assigning 1 to the x_i and assigning other free valuables to the 0. The special solution to $A\mathbf{x} = \mathbf{0}$ is a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{n-r}$:

   $$\mathbf{x}_n = x_{q_1}\mathbf{x}_1 + x_{q_2}\mathbf{x}_2 + \cdots + x_{q_{n-r}}\mathbf{x}_{n-r},$$

   where $x_{q_1}, x_{q_2}, \cdots, x_{q_{n-r}} \in \mathbb{R}$.

Therefore, the complete solution to $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n.$$

3.4 Linear Equations and The Rank r

Size Relationship of r, m, n	A	The number of solutions	
		$A\mathbf{x} = \mathbf{0}$	$A\mathbf{x} = \mathbf{b}$
$r = m = n$	Square and invertible	1	1
$r = m < n$	Short and wide	∞	∞
$r = n < m$	Tall and thin	1	0 (R has $\mathbf{0} = b$) 1 (R has no $\mathbf{0} = b$)
$r < m = n$	Square	∞	0 (R has $\mathbf{0} = b$) ∞ (R has no $\mathbf{0} = b$)
$r < m < n$	Short and wide		
$r < n < m$	Tall and thin		

For the matrix A, we move the pivots to the left side and move the free variables to the right side. Then we apply Gauss-Jordan Elimination to the matrix A to get its RREF (Reduced Row Echelon Form) R of the matirx A. The shapes of R related to the rank r are shown as follows:

1. $r = m = n$

$$R = \begin{bmatrix} I \end{bmatrix}.$$

2. $r = m < n$

$$R = \begin{bmatrix} I & F \end{bmatrix}.$$

3. $r = n < m$

$$R = \begin{bmatrix} I \\ \mathbf{0} \end{bmatrix}.$$

4. $r < m, r < n$

$$R = \begin{bmatrix} I & F \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

3.5 Four Subspaces of Matrices

$\forall\ m \times n$ matrix A where $m, n \in \mathbb{N}^+$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} — & \mathbf{r}_1 & — \\ — & \mathbf{r}_2 & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{r}_m & — \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \\ | & | & \cdots & | \end{bmatrix}$$

and its transpose

$$A^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} — & \mathbf{c}_1 & — \\ — & \mathbf{c}_2 & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{c}_n & — \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{r}_1 & \mathbf{r}_2 & \cdots & \mathbf{r}_m \\ | & | & \cdots & | \end{bmatrix}.$$

3.5.1 Definition of Four Subspaces of A

1. **Row Space**

   The column space of A consists of all linear combinations of $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_m$, which is denoted by $\boldsymbol{C}(A^{\mathrm{T}})$. It is a subspace of $\mathbf{R}^n$.

2. **Column Space**

   The column space of A consists of all linear combinations of $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n$, which is denoted by $\boldsymbol{C}(A)$. It is a subspace of $\mathbf{R}^m$.

3. **Nullspace**

   The nullspace of A consists of solutions to $A\mathbf{x} = \mathbf{0}$, which is denoted by $\boldsymbol{N}(A)$. It is a subspace of $\mathbf{R}^n$.

4. **Left Nullspace**

   The nullspace of A consists of solutions to $A^{\mathrm{T}}\mathbf{x} = \mathbf{0}$, which is denoted by $\boldsymbol{N}(A^{\mathrm{T}})$. It is a subspace of $\mathbf{R}^m$.

3.5.2 Full Column Rank and Full Row Rank

1. Every $m \times n$ matrix A with full column rank ($r = n$) has all these properties:

(1) All columns of A are piovot columns;

(2) There are no free variables or special solutions;

(3) The nullspace $\boldsymbol{N}(A)$ contains only the zero vector $\mathbf{x} = \mathbf{0}$;

(4) If $A\mathbf{x} = \mathbf{b}$ has a solution, then it has only one solution.

2. Every $m \times n$ matrix A with full row rank ($r = m$) has all these properties:

(1) All rows of A are piovot rows, and R has no zero rows;

(2) The column space is the whole space $\mathbf{R}^m$;

(3) The nullspace $\boldsymbol{N}(A)$ contains $n - r = m - r$ special solutions;

(4) $A\mathbf{x} = \mathbf{b}$ has a solution for every right side $\mathbf{b}$.

3.5.3 Fundamental Theorem of Linear Algebra, Part 1

$\forall$ $m \times n$ matrix A where $m, n \in \mathbb{N}^+$, suppose that the rank of A is r ($1 \leq r \leq \min\{m, n\}$). The column space $\boldsymbol{C}(A)$ and row space $\boldsymbol{C}(A^{\mathrm{T}})$ both have dimension r. The nullspace $\boldsymbol{N}(A)$ has dimension $n - r$. The left nullspace $\boldsymbol{N}(A^{\mathrm{T}})$ has dimension $m - r$.