Chapter 1

Introduction to Vectors

1.1 Vectors

1.1.1 Definition of Vector

$\forall$ $n-$dimensional vector $\mathbf{v}$ where $n \in \mathbb{N}^+$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = (v_1, v_2, \cdots, v_n).$$

$v_1, v_2, \cdots, v_n$ are the 1st, 2nd, $\cdots$, n-th component of $\mathbf{v}$. Every vector is written as a column.

1.1.2 Operations of Vectors

1. **Vector Addition**

   $\forall$ $m-$dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m$ where $m, n \in \mathbb{N}^+$:

$$\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix}, \qquad \cdots, \qquad \mathbf{v}_n = \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

The vector addition of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ is

$$\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n = \begin{bmatrix} v_{11} + v_{12} + \cdots + v_{1n} \\ v_{21} + v_{22} + \cdots + v_{2n} \\ \vdots \\ v_{m1} + v_{m2} + \cdots + v_{mn} \end{bmatrix}.$$

2. **Vector Scalar Multiplication**

   $\forall$ number $k \in \mathbb{R}$ and $\forall$ $n-$dimensional vector $\mathbf{v}$ where $n \in \mathbb{N}^+$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

   The scalar multiplication of k and $\mathbf{v}$ is

$$k\mathbf{v} = \begin{bmatrix} kv_1 \\ kv_2 \\ \vdots \\ kv_n \end{bmatrix}.$$

   The number k is called a "scalar".

3. **Vector Dot Product (Inner Product)**

   $\forall$ two $n-$dimensional vectors $\mathbf{v}$, $\mathbf{w}$ where $n \in \mathbb{N}^+$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}.$$

   The dot product or inner product of $\mathbf{v}$, $\mathbf{w}$ is the number $\mathbf{v} \cdot \mathbf{w}$ or $\mathbf{w} \cdot \mathbf{v}$:

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^{\mathrm{T}}\mathbf{w} = \mathbf{w} \cdot \mathbf{v} = \mathbf{w}^{\mathrm{T}}\mathbf{v} = v_1w_1 + v_2w_2 + \cdots + v_nw_n.$$

1.1.3 The Length of Vector

$\forall$ $n-$dimensional vectors $\mathbf{v}$ where $n \in \mathbb{N}^+$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

The length $\|\mathbf{v}\|$ of a vector $\mathbf{v}$ is the square root of $\mathbf{v} \cdot \mathbf{v}$:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

1.1.4 Unit Vector

A unit vector $\mathbf{u}$ is a vector whose length equals one. Then $\mathbf{u} \cdot \mathbf{u} = 1$. $\forall$ $n-$dimensional vectors $\mathbf{v}$ where $n \in \mathbb{N}^+$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

$\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector in the same direction as $\mathbf{v}$.

1.1.5 The Angle Between Vectors

1. **Two-dimensional Vectors**

   $\forall$ two $2-$dimensional nonzero vectors $\mathbf{v}$, $\mathbf{w}$:

   $$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (v_1^2 + v_2^2 \neq 0), \qquad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad (w_1^2 + w_2^2 \neq 0).$$

   Let the angle between $\mathbf{v}$ and $\mathbf{w}$ is θ $(0 \leq \theta < \pi)$, then

   $$\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \ \|\mathbf{w}\|} = \frac{v_1 w_1 + v_2 w_2}{\sqrt{v_1^2 + v_2^2}\sqrt{w_1^2 + w_2^2}} = \cos\theta.$$

2. **Three-dimensional Vectors**

$\forall$ two 2−dimensional nonzero vectors $\mathbf{v}$, $\mathbf{w}$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad (v_1^2 + v_2^2 + v_3^2 \neq 0), \qquad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \quad (w_1^2 + w_2^2 + w_3^2 \neq 0).$$

Let the angle between $\mathbf{v}$ and $\mathbf{w}$ is θ ($0 \leq \theta < \pi$), then

$$\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \ \|\mathbf{w}\|} = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{\sqrt{v_1^2 + v_2^2 + v_3^2}\sqrt{w_1^2 + w_2^2 + w_3^2}} = \cos\theta.$$

3. **Schwarz inequality and Triangle inequality**

Schwarz inequality $\square$

$$|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \ \|\mathbf{w}\|.$$

Triangle inequality $\square$

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|.$$

1.1.6 Linear Combination

1. **Definition of Linear Combination**

Combine addition with scalar multiplication to produce a "linear combination". $\forall$ $m-$ dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ where $m, n \in \mathbb{N}^+$ and $\forall$ number $\alpha_1, \alpha_2, \cdots,$ $\alpha_n \in \mathbb{R}$. The sum of $\alpha_1\mathbf{v}_1, \alpha_2\mathbf{v}_2, \cdots, \alpha_n\mathbf{v}_n$ is a linear combination

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n.$$

2. **Geometrical Significance of Linear Combination (Parallelogram Law)**

The parallelogram law gives the rule for vector addition of vectors $\mathbf{u}$ and $\mathbf{v}$. The sum $\mathbf{u} + \mathbf{v}$ of the vectors is obtained by placing them head to tail and drawing the vector from the free tail to the free head.

3. **Line, Plane, Space and Three-dimensional Vectors**

$\forall$ number $\alpha, \beta, \gamma \in \mathbb{R}$ and $\forall$ nonzero three-dimensional vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$:

- All combinations $\alpha\mathbf{u}$ fill a line through $(0, 0, 0)$;
- If $\mathbf{u}$ and $\mathbf{v}$ are not on the same line, all combinations $\alpha\mathbf{u} + \beta\mathbf{v}$ fill a plane through $(0, 0, 0)$;

- If $\mathbf{w}$ is not on the same plane formed by $\alpha\mathbf{u}+\beta\mathbf{v}$, all combinations $\alpha\mathbf{u}+\beta\mathbf{v}+\gamma\mathbf{w}$ fill three-diemnsional space.

1.2 Matrix

1.2.1 Definition of Matrix

$\forall$ $m-$dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ where $m, n \in \mathbb{N}^+$:

$$\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix}, \qquad \cdots, \qquad \mathbf{v}_n = \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

and $\forall$ number $x_1, x_2, \cdots, x_n \in \mathbb{R}$. The linear combination of

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = x_1 \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix} + x_2 \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

can be expressed as

$$A\mathbf{x} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where A is a matrix and $\mathbf{x}$ is a vector:

$$A = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & \cdots & | \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

1.2.2 The Format of Matrix, $A\mathbf{x}_c = \mathbf{b}_c$ and $\mathbf{x}_r^{\mathrm{T}} A = \mathbf{b}_r^{\mathrm{T}}$

For $m-$dimensional vectors $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n$ such that

$$\mathbf{c}_1 = \mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix}, \qquad \mathbf{c}_2 = \mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix}, \qquad \cdots, \qquad \mathbf{c}_n = \mathbf{v}_n = \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

For $n-$dimensional vectors $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_m$ such that

$$\mathbf{r}_1 = \begin{bmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \end{bmatrix}, \qquad \mathbf{r}_2 = \begin{bmatrix} v_{21} \\ v_{22} \\ \vdots \\ v_{2n} \end{bmatrix}, \qquad \cdots, \qquad \mathbf{r}_m = \begin{bmatrix} v_{m1} \\ v_{m2} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

Then the matrix A can be expressed as

$$A = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \\ | & | & \cdots & | \end{bmatrix} = \begin{bmatrix} — & \mathbf{r}_1^{\mathrm{T}} & — \\ — & \mathbf{r}_2^{\mathrm{T}} & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{r}_m^{\mathrm{T}} & — \end{bmatrix}.$$

For a m-dimensional vector $\mathbf{b}_c$ and a m-dimensional vector $\mathbf{b}_r$ such that

$$\mathbf{b}_c = \begin{bmatrix} b_{c1} \\ b_{c2} \\ \vdots \\ b_{cm} \end{bmatrix}, \qquad \mathbf{b}_r = \begin{bmatrix} b_{r1} \\ b_{r2} \\ \vdots \\ b_{rn} \end{bmatrix},$$

there are

$$A\mathbf{x}_c = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} \begin{bmatrix} x_{c1} \\ x_{c2} \\ \vdots \\ x_{cn} \end{bmatrix} = \mathbf{b}_c = \begin{bmatrix} b_{c1} \\ b_{c2} \\ \vdots \\ b_{cm} \end{bmatrix},$$

and

$$\mathbf{x}_r^{\mathrm{T}} A = \begin{bmatrix} x_{r1} & x_{r2} & \cdots & x_{rn} \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} = \mathbf{b}_r^{\mathrm{T}} = \begin{bmatrix} b_{r1} & b_{r2} & \cdots & b_{rn} \end{bmatrix}.$$

Then

$$\mathbf{b}_c = x_{c1}\mathbf{c}_1 + x_{c2}\mathbf{c}_2 + \cdots + x_{cn}\mathbf{c}_n = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{b}_c \\ \mathbf{r}_2 \cdot \mathbf{b}_c \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{b}_c \end{bmatrix},$$

and

$$\mathbf{b}_r^{\mathrm{T}} = x_{r1}\mathbf{r}_1^{\mathrm{T}} + x_{r2}\mathbf{r}_2^{\mathrm{T}} + \cdots + x_{rn}\mathbf{r}_n^{\mathrm{T}} = \begin{bmatrix} \mathbf{c}_1 \cdot \mathbf{b}_r & \mathbf{c}_2 \cdot \mathbf{b}_r & \cdots & \mathbf{c}_m \cdot \mathbf{b}_r \end{bmatrix}.$$

1.2.3 Matrix Operations

1. **Matrix Addition**

   Matrices can be added if their shapes are the same. $\forall$ $m \times n$ matrices A and B where $k, m, n \in \mathbb{N}^+$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}.$$

   Then the addition of A and B is

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}.$$

2. **Matrix Scalar Multiplication**

   $\forall$ number $k \in \mathbb{R}$, $\forall$ $m \times n$ matrix A where $k \in \mathbb{R}$ and $m, n \in \mathbb{N}^+$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

   Then the scalar multiplication of k and A is

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}.$$

3. **Matrix Multiplication**

   For two matrices A and B, in order to multiply AB, if A has q columns, B must have q rows, where $q \in \mathbb{N}^+$. $\forall\ p \times q$ matrix A and $\forall\ q \times r$ matrix B where $p, q, r \in \mathbb{N}^+$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{bmatrix} = \begin{bmatrix} — & \mathbf{a}_{\mathbf{r}_1}^{\mathrm{T}} & — \\ — & \mathbf{a}_{\mathbf{r}_2}^{\mathrm{T}} & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{a}_{\mathbf{r}_p}^{\mathrm{T}} & — \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{a}_{\mathbf{c}_1} & \mathbf{a}_{\mathbf{c}_2} & \cdots & \mathbf{a}_{\mathbf{c}_q} \\ | & | & \cdots & | \end{bmatrix},$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ b_{q1} & b_{q2} & \cdots & b_{qr} \end{bmatrix} = \begin{bmatrix} — & \mathbf{b}_{\mathbf{r}_1}^{\mathrm{T}} & — \\ — & \mathbf{b}_{\mathbf{r}_2}^{\mathrm{T}} & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{b}_{\mathbf{r}_q}^{\mathrm{T}} & — \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{b}_{\mathbf{c}_1} & \mathbf{b}_{\mathbf{c}_2} & \cdots & \mathbf{b}_{\mathbf{c}_r} \\ | & | & \cdots & | \end{bmatrix}.$$

Then the multiplication of A and B is

$$AB = \begin{bmatrix} \sum_{i=1}^{q} a_{1i}b_{i1} & \sum_{i=1}^{q} a_{1i}b_{i2} & \cdots & \sum_{i=1}^{q} a_{1i}b_{ir} \\ \sum_{i=1}^{q} a_{2i}b_{i1} & \sum_{i=1}^{q} a_{2i}b_{i2} & \cdots & \sum_{i=1}^{q} a_{2i}b_{ir} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{q} a_{pi}b_{i1} & \sum_{i=1}^{q} a_{2i}b_{i2} & \cdots & \sum_{i=1}^{q} a_{pi}b_{ir} \end{bmatrix}.$$

The computation of AB uses pqr separate multiplications. There are three variants of matrix multiplication AB

(a) **Matrix A times every column of matrix B** $\square$

$$AB = A \begin{bmatrix} | & | & \cdots & | \\ \mathbf{b}_{\mathbf{c}_1} & \mathbf{b}_{\mathbf{c}_2} & \cdots & \mathbf{b}_{\mathbf{c}_r} \\ | & | & \cdots & | \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ A\mathbf{b}_{\mathbf{c}_1} & A\mathbf{b}_{\mathbf{c}_2} & \cdots & A\mathbf{b}_{\mathbf{c}_r} \\ | & | & \cdots & | \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}\mathbf{a}_{\mathbf{c}_1} & b_{12}\mathbf{a}_{\mathbf{c}_1} & & b_{1r}\mathbf{a}_{\mathbf{c}_1} \\ + & + & & + \\ b_{21}\mathbf{a}_{\mathbf{c}_2} & b_{22}\mathbf{a}_{\mathbf{c}_2} & & b_{2r}\mathbf{a}_{\mathbf{c}_2} \\ + & + & & + \\ \vdots & \vdots & \cdots & \vdots \\ + & + & & + \\ b_{q1}\mathbf{a}_{\mathbf{c}_q} & b_{q2}\mathbf{a}_{\mathbf{c}_q} & & b_{qr}\mathbf{a}_{\mathbf{c}_q} \end{bmatrix}.$$

(b) **Every row of matrix A times matrix B** $\square$

$$AB = \begin{bmatrix} — & \mathbf{a}_{\mathbf{r}_1}^{\mathrm{T}} & — \\ — & \mathbf{a}_{\mathbf{r}_2}^{\mathrm{T}} & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{a}_{\mathbf{r}_p}^{\mathrm{T}} & — \end{bmatrix} B = \begin{bmatrix} — & \mathbf{a}_{\mathbf{r}_1}^{\mathrm{T}} B & — \\ — & \mathbf{a}_{\mathbf{r}_2}^{\mathrm{T}} B & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{a}_{\mathbf{r}_p}^{\mathrm{T}} B & — \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}\mathbf{b}_{\mathbf{r}_1}^{\mathrm{T}} + a_{12}\mathbf{b}_{\mathbf{r}_2}^{\mathrm{T}} + \cdots + a_{1q}\mathbf{b}_{\mathbf{r}_p}^{\mathrm{T}} \\ a_{21}\mathbf{b}_{\mathbf{r}_1}^{\mathrm{T}} + a_{22}\mathbf{b}_{\mathbf{r}_2}^{\mathrm{T}} + \cdots + a_{2q}\mathbf{b}_{\mathbf{r}_p}^{\mathrm{T}} \\ \vdots \\ a_{p1}\mathbf{b}_{\mathbf{r}_1}^{\mathrm{T}} + a_{p2}\mathbf{b}_{\mathbf{r}_2}^{\mathrm{T}} + \cdots + a_{pq}\mathbf{b}_{\mathbf{r}_p}^{\mathrm{T}} \end{bmatrix}.$$

(c) **The sum of column i of A times row i of B from 1 to q** $\square$

$$AB = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{a}_{\mathbf{c}_1} & \mathbf{a}_{\mathbf{c}_2} & \cdots & \mathbf{a}_{\mathbf{c}_q} \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} — & \mathbf{b}_{\mathbf{r}_1}^{\mathrm{T}} & — \\ — & \mathbf{b}_{\mathbf{r}_2}^{\mathrm{T}} & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{b}_{\mathbf{r}_q}^{\mathrm{T}} & — \end{bmatrix} = \mathbf{a}_{\mathbf{c}_1}\mathbf{b}_{\mathbf{r}_1}^{\mathrm{T}} + \mathbf{a}_{\mathbf{c}_2}\mathbf{b}_{\mathbf{r}_2}^{\mathrm{T}} + \cdots + \mathbf{a}_{\mathbf{c}_q}\mathbf{b}_{\mathbf{r}_q}^{\mathrm{T}}.$$

4. **The Laws of Matrix Operations** $\square$

(a) **Laws of Matrix Addition**

$\forall$ number $k \in \mathbb{R}$, $\forall$ three $m \times n$ matrices A, B and C where $m, n \in \mathbb{N}^+$

- **Commutative Law**

$$A + B = B + A.$$

- **Distributive Law**

$$k(A + B) = kA + kB.$$

- **Associative Law**

$$A + (B + C) = (A + B) + C.$$

(b) **Laws of Matrix Multiplication**

- **Distributive Law from The Left**

$\forall$ $p \times q$ matrix A, $\forall$ $q \times r$ matrices B and C where $p, q, r \in \mathbb{N}^+$

$$A(B + C) = AB + AC.$$

- **Distributive Law from The Right**

  $\forall$ $p \times q$ matrices A and B, $\forall$ $q \times r$ matrix C where $p, q, r \in \mathbb{N}^+$

  $$(A + B)C = AC + BC.$$

- **Associative Law**

  $\forall$ $p \times q$ matrix A, $\forall$ $q \times r$ matrix B and $\forall$ $r \times s$ matrix C where $p, q, r, s \in \mathbb{N}^+$

  $$A(BC) = (AB)C.$$

(c) **Laws of Matrix Powers**

$\forall$ number p, q and $\forall$ $n \times n$ matrix A where $p, q, n \in \mathbb{N}^+$

$$A^p = AA \cdots A \text{ (p factors)},$$
$$(A^p)(A^q) = A^{p+q},$$
$$(A^p)^q = A^{pq}.$$

1.2.4 Row Operations of Matrix

$\forall$ $m \times n$ matrix A where $m \geq 2$ and $m, n \in \mathbb{N}^+$, let $1 \leq i < j \leq m$ where $i, j \in \mathbb{N}^+$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

1. **Switching Rows**

   For $1 \leq l \leq m, l \neq i, l \neq j$ and $l \in \mathbb{N}^+$, let P be a $m \times m$ matrix

   $$P = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}.$$

where $P(i,j) = P(j,i) = P(l,l) = 1$ and other components of P is 0. In order to switch the i-th row and j-th row of A,

$$PA = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

If the order of i and j is switched, this row operation still works.

2. **Multiplying a Row by a Number**

For $k \in \mathbb{R}, 1 \leq l \leq m, l \neq i$ and $l \in \mathbb{N}^+$, let D^{-1} be a $m \times m$ matrix

$$D^{-1} = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}.$$

where $D^{-1}(i,i) = k, D^{-1}(l,l) = 1$ and other components of D^{-1} is 0. In order to multiply the i-th row of A by k,

$$D^{-1}A = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ ka_{i1} & \cdots & ka_{in} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

3. **Adding Multiples of Rows**

For $k \in \mathbb{R}, 1 \leq l \leq m$ and $l \in \mathbb{N}^+$, let E be a $m \times m$ matrix

$$E = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}.$$

where $E(j,i) = k, E(l,l) = 1$ and other components of E is 0. In order to add the multiple of the i-th row of A by k into the j-th row of A

$$\begin{aligned} EA &= \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ ka_{i1} + a_{j1} & \cdots & ka_{in} + a_{jn} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}. \end{aligned}$$

If the order of i and j is switched, this row operation still works.

1.2.5 Inverse Matrices

1. **Definition of Inverse Matrix**

   $\forall$ $n \times n$ square matrix A where $n \in \mathbb{N}^+$, the matrix A is invertible if there exists a matrix A^{-1} such that

$$A^{-1}A = AA^{-1} = I.$$

2. **The Properties of Inverse Matrices** $\square$

   (a) $\forall\ n \times n$ invertible square matrix A where $n \in \mathbb{N}^+$, its inverse A^{-1} is unique.

   (b) For $n \times n$ square matrices $A_1, A_2, \cdots, A_k$ where $n, k \in \mathbb{N}^+$, if $A_1, A_2, \cdots, A_k$ are separately invertible, then

$$(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_1^{-1}.$$

3. **Calculating Inverse Matrices by Gauss-Jordan Elimination** $\square$

   $\forall\ n \times n$ invertible square matrix A where $n \in \mathbb{N}^+$, apply Gauss-Jordan Elimination into the matrix

$$\begin{bmatrix} A & I \end{bmatrix}$$

   to get the matrix

$$\begin{bmatrix} I & A^{-1} \end{bmatrix}.$$

4. **Invertible Matrix Theorem**

   $\forall\ n \times n$ square matrix A where $n \in \mathbb{N}^+$, A is invertible if and only if any of the following hold:

   (a) The RREF of A has n pivots;

   (b) The equation $A\mathbf{x} = \mathbf{b}$ has only one solution.

   (c) The equation $A\mathbf{x} = \mathbf{0}$ has no nonzero solutions.

   (d) The determinant of A is not zero.

1.2.6 Transposes and Permutations

1. **Definition of Transpose**

   $\forall\ m \times n$ matrix A where $m, n \in \mathbb{N}^+$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

The transpose of the matrix A is

$$A^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.$$

2. **The Properties of Transposes** $\square$

   (a) For $m \times n$ matrices $A_1, A_2, \cdots, A_k$ where $m, n, k \in \mathbb{N}^+$

   $$(A_1 + A_2 + \cdots + A_k)^{\mathrm{T}} = A_1^{\mathrm{T}} + A_2^{\mathrm{T}} + \cdots + A_k^{\mathrm{T}}.$$

   (b) $\forall$ $n_1 \times n_2$ matrix A_1, $\forall$ $n_2 \times n_3$ matrix A_2, $\cdots$, $\forall$ $n_{k-1} \times n_k$ matrix A_{k-1}, $\forall$ $n_k \times n_{k+1}$ matrix A_k where $k, n_1, n_2, \cdots, n_k, n_{k+1} \in \mathbb{N}^+$

   $$(A_1 A_2 \cdots A_{k-1} A_k)^{\mathrm{T}} = A_k^{\mathrm{T}} A_{k-1}^{\mathrm{T}} \cdots A_2^{\mathrm{T}} A_1^{\mathrm{T}}.$$

   (c) $\forall$ $n \times n$ invertible matrix A where $n \in \mathbb{N}^+$

   $$(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1}.$$

3. **Symmetric Matrix**

   $\forall$ $n \times n$ symmetric matrix S has

   $$S^{\mathrm{T}} = S.$$

   The inverse of a symmetric matrix is also symmetric. $\square$

4. **Permutation Matrix**

   A permutation matrix P has the rows of the identity I in any order. The properties of permutation matrices are shown as follows:

   (a) $\forall$ $n \in \mathbb{N}^+$, there are $n!$ permutation matrices of order n;

   (b) P^{-1} is also a permutation matrix;

   (c) $P^{-1} = P^{\mathrm{T}}$.

1.2.7 Special Matrices

1. **Difference Matrix and Cyclic Difference Matrix**

Given a n-dimensional vector $\mathbf{x}$ and a n-dimensional vector $\mathbf{b}$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

its difference matrix A such that $A\mathbf{x} = \mathbf{b}$ is

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}.$$

Then we will get

$$\mathbf{x} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ \vdots \\ b_1 + b_2 + \cdots + b_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \mathbf{A}^{-1}\mathbf{b}.$$

The matrix A is invertible. From $\mathbf{b}$ we can recover $\mathbf{x}$. We write $\mathbf{x}$ as $A^{-1}\mathbf{b}$. In particular, $A\mathbf{x} = \mathbf{0}$ has one solution

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Its cyclic difference matrix C such that $C\mathbf{x} = \mathbf{b}$ is

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}.$$

$C\mathbf{x} = \mathbf{b}$ has many solutions or else no solution. In particular, if $C\mathbf{x} = \mathbf{0}$ has many solutions, C is a singular matrix.

2. **Identity Matrix**

$\forall\ k \in \mathbb{N}^+$, the $k \times k$ identity matrix is

$$I = I_k = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

The diagonal components of I is 1, and other components of I is 0. $\forall\ m \times n$ matrix A where $m, n \in \mathbb{N}^+$

$$I_m A = A I_n = A.$$

In particular, $\forall\ n \times n$ square matrix A where $n \in \mathbb{N}^+$

$$IA = AI = A.$$