Chapter 2

Solving Linear Equations

2.1 Matrix Operations

2.1.1 Matrix Addition

Matrices can be added if their shapes are the same. $\forall\ m \times n$ matrices $A_1, A_2, \cdots A_k$ where $k, m, n \in \mathbb{N}^+$

$$A_1 = \begin{bmatrix} a_{111} & a_{112} & \cdots & a_{11n} \\ a_{121} & a_{122} & \cdots & a_{12n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m1} & a_{1m2} & \cdots & a_{1mn} \end{bmatrix}, \qquad A_2 = \begin{bmatrix} a_{211} & a_{212} & \cdots & a_{21n} \\ a_{221} & a_{222} & \cdots & a_{22n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{2m1} & a_{2m2} & \cdots & a_{2mn} \end{bmatrix}, \qquad \cdots,$$
$$A_i = \begin{bmatrix} a_{i11} & a_{i12} & \cdots & a_{i1n} \\ a_{i21} & a_{i22} & \cdots & a_{i2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{im1} & a_{im2} & \cdots & a_{imn} \end{bmatrix}, \qquad \cdots, \qquad A_k = \begin{bmatrix} a_{k11} & a_{k12} & \cdots & a_{k1n} \\ a_{k21} & a_{k22} & \cdots & a_{k2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{km1} & a_{km2} & \cdots & a_{kmn} \end{bmatrix}.$$

Then the addition of $A_1, A_2, \cdots A_k$ is

$$\sum_{i=1}^{k} A_i = \begin{bmatrix} \sum\limits_{i=1}^{k} a_{i11} & \sum\limits_{i=1}^{k} a_{i12} & \cdots & \sum\limits_{i=1}^{k} a_{i1n} \\ \sum\limits_{i=1}^{k} a_{i21} & \sum\limits_{i=1}^{k} a_{i22} & \cdots & \sum\limits_{i=1}^{k} a_{i2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sum\limits_{i=1}^{k} a_{im1} & \sum\limits_{i=1}^{k} a_{im2} & \cdots & \sum\limits_{i=1}^{k} a_{imn} \end{bmatrix}.$$

2.1.2 Matrix Scalar Multiplication

$\forall$ number $k \in \mathbb{R}$, $\forall$ $m \times n$ matrix A where $k \in \mathbb{R}$ and $m, n \in \mathbb{N}^+$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Then the scalar multiplication of k and A is

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}.$$

2.1.3 Matrix Multiplication

For two matrices A and B, in order to multiply AB, if A has q columns, B must have q rows, where $q \in \mathbb{N}^+$. $\forall$ $p \times q$ matrix A and $\forall$ $q \times r$ matrix B where $p, q, r \in \mathbb{N}^+$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ b_{q1} & b_{q2} & \cdots & b_{qr} \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} \sum\limits_{i=1}^{q} a_{1i}b_{i1} & \sum\limits_{i=1}^{q} a_{1i}b_{i2} & \cdots & \sum\limits_{i=1}^{q} a_{1i}b_{ir} \\ \sum\limits_{i=1}^{q} a_{2i}b_{i1} & \sum\limits_{i=1}^{q} a_{2i}b_{i2} & \cdots & \sum\limits_{i=1}^{q} a_{2i}b_{ir} \\ \vdots & \vdots & \ddots & \vdots \\ \sum\limits_{i=1}^{q} a_{pi}b_{i1} & \sum\limits_{i=1}^{q} a_{2i}b_{i2} & \cdots & \sum\limits_{i=1}^{q} a_{pi}b_{ir} \end{bmatrix}.$$

The computation of AB uses pqr separate multiplications.

2.1.4 The Laws of Matrix Operations

1. **Laws of Matrix Addition**

   $\forall$ number $k \in \mathbb{R}$, $\forall$ three $m \times n$ matrix A, B and C where $m, n \in \mathbb{N}^+$

- **Commutative Law**

$$A + B = B + A.$$

- **Distributive Law**

$$k(A + B) = kA + kB.$$

- **Associative Law**

$$A + (B + C) = (A + B) + C.$$

2. **Laws of Matrix Multiplication**

- **Distributive Law from The Left**

  $\forall\ p \times q$ matrix A, $\forall\ q \times r$ matrices B and C where $p, q, r \in \mathbb{N}^+$

$$A(B + C) = AB + AC.$$

- **Distributive Law from The Right**

  $\forall\ p \times q$ matrix A and B, $\forall\ q \times r$ matrix C where $p, q, r \in \mathbb{N}^+$

$$(A + B)C = AC + BC.$$

- **Associative Law**

  $\forall\ p \times q$ matrix A, $\forall\ q \times r$ matrix B and $\forall\ r \times s$ matrix C where $p, q, r, s \in \mathbb{N}^+$

$$A(BC) = (AB)C.$$

3. **Laws of Matrix Powers**

$\forall$ number p, q and $\forall\ n \times n$ matrix A where $p, q, n \in \mathbb{N}^+$

$$A^p = AA \cdots A \text{ (p factors)},$$

$$(A^p)(A^q) = A^{p+q},$$

$$(A^p)^q = A^{pq}.$$

2.2 The Extension of Matrix Multiplication

2.2.1 Variants of Matrix Multiplication

$\forall\ p \times q$ matrix A and $\forall\ q \times r$ matrix B where $p, q, r \in \mathbb{N}^+$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{bmatrix} = \begin{bmatrix} — & \mathbf{a}_{\mathbf{r}_1} & — \\ — & \mathbf{a}_{\mathbf{r}_2} & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{a}_{\mathbf{r}_p} & — \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{a}_{\mathbf{c}_1} & \mathbf{a}_{\mathbf{c}_2} & \cdots & \mathbf{a}_{\mathbf{c}_q} \\ | & | & \cdots & | \end{bmatrix},$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ b_{q1} & b_{q2} & \cdots & b_{qr} \end{bmatrix} = \begin{bmatrix} — & \mathbf{b}_{\mathbf{r}_1} & — \\ — & \mathbf{b}_{\mathbf{r}_2} & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{b}_{\mathbf{r}_q} & — \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{b}_{\mathbf{c}_1} & \mathbf{b}_{\mathbf{c}_2} & \cdots & \mathbf{b}_{\mathbf{c}_r} \\ | & | & \cdots & | \end{bmatrix}.$$

There are three variants of matrix multiplication AB

1. **Matrix A times every column of matrix B**

$$AB = A \begin{bmatrix} | & | & \cdots & | \\ \mathbf{b}_{\mathbf{c}_1} & \mathbf{b}_{\mathbf{c}_2} & \cdots & \mathbf{b}_{\mathbf{c}_r} \\ | & | & \cdots & | \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ A\mathbf{b}_{\mathbf{c}_1} & A\mathbf{b}_{\mathbf{c}_2} & \cdots & A\mathbf{b}_{\mathbf{c}_r} \\ | & | & \cdots & | \end{bmatrix}.$$

2. **Every row of matrix A times matrix B**

$$AB = \begin{bmatrix} — & \mathbf{a}_{\mathbf{r}_1} & — \\ — & \mathbf{a}_{\mathbf{r}_2} & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{a}_{\mathbf{r}_p} & — \end{bmatrix} B = \begin{bmatrix} — & \mathbf{a}_{\mathbf{r}_1}B & — \\ — & \mathbf{a}_{\mathbf{r}_2}B & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{a}_{\mathbf{r}_p}B & — \end{bmatrix}.$$

3. **The sum of column i of A times row i of B from 1 to q**

$$AB = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{a}_{\mathbf{c}_1} & \mathbf{a}_{\mathbf{c}_2} & \cdots & \mathbf{a}_{\mathbf{c}_q} \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} — & \mathbf{b}_{\mathbf{r}_1} & — \\ — & \mathbf{b}_{\mathbf{r}_2} & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{b}_{\mathbf{r}_q} & — \end{bmatrix} = \mathbf{a}_{\mathbf{c}_1}\mathbf{b}_{\mathbf{r}_1} + \mathbf{a}_{\mathbf{c}_2}\mathbf{b}_{\mathbf{r}_2} + \cdots + \mathbf{a}_{\mathbf{c}_q}\mathbf{b}_{\mathbf{r}_q}.$$

2.2.2 Row Operations of Matrix

$\forall$ $m \times n$ matrix A where $m \geq 2$ and $m, n \in \mathbb{N}^+$, let $1 \leq i < j \leq m$ where $i, j \in \mathbb{N}^+$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

1. **Switching Rows**

   For $1 \leq l \leq m, l \neq i, l \neq j$ and $l \in \mathbb{N}^+$, let P be a $m \times m$ matrix

$$P = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}.$$

   where $P(i,j) = P(j,i) = P(l,l) = 1$ and other components of P is 0. In order to switch the i-th row and j-th row of A,

$$PA = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

2. **Multiplying a Row by a Number**

For $k \in \mathbb{R}, 1 \leq l \leq m, l \neq i$ and $l \in \mathbb{N}^+$, let D^{-1} be a $m \times m$ matrix

$$D^{-1} = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}.$$

where $D^{-1}(i,i) = k, D^{-1}(l,l) = 1$ and other components of D^{-1} is 0. In order to multiply the i-th row of A by k,

$$D^{-1}A = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ ka_{i1} & \cdots & ka_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

If i is replace by j, this row operation still works.

3. **Adding Multiples of Rows**

For $k \in \mathbb{R}, 1 \leq l \leq m$ and $l \in \mathbb{N}^+$, let E be a $m \times m$ matrix

$$E = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}.$$

where $E(i,j) = k, E(l,l) = 1$ and other components of E is 0. In order to add the multiple of the i-th row of A by k into the j-th row of A

$$EA = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ ka_{i1} + a_{j1} & \cdots & ka_{in} + a_{jn} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

If the order of i and j is switched, this row operation still works.

2.2.3 Identity Matrix

$\forall\, k \in \mathbb{N}^+$, the $k \times k$ identity matrix is

$$I = I_k = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

The diagonal components of I is 1, and other components of I is 0. $\forall\, m \times n$ matrix A where $m, n \in \mathbb{N}^+$

$$I_m A = A I_n = A.$$

In particular, $\forall\, n \times n$ square matrix A where $n \in \mathbb{N}^+$

$$IA = AI = A.$$

2.3 Linear Equations and Elimination

2.3.1 Definition of Linear Equations

$\forall$ linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{cases} \tag{2.1}$$

where $m, n \in \mathbb{N}^+$. The equations are linear, which means that the unknowns are only multiplied by numbers.

2.3.2 The Matrix Form of the Equations

The matrix form of linear equations 2.3.4 is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \\ | & | & \cdots & | \end{bmatrix} = \begin{bmatrix} — & \mathbf{r}_1 & — \\ — & \mathbf{r}_2 & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{r}_m & — \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

This linear equations could be expressed as

$$A\mathbf{x} = \mathbf{b}$$

where A is called the coefficient matrix. This linear equations could also be expressed as

$$A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{b} \\ \mathbf{r}_2 \cdot \mathbf{b} \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{b} \end{bmatrix} = \mathbf{b}.$$

2.3.3 Gauss-Jordan Elimination

$\forall\ m \times n$ matrix A without zero columns

$$A = \begin{bmatrix} A(1,1) & A(1,2) & \cdots & A(1,n) \\ A(2,1) & A(2,2) & \cdots & A(2,n) \\ \vdots & \vdots & \ddots & \vdots \\ A(m,1) & A(m,2) & \cdots & A(m,n) \end{bmatrix}$$

where $m, n \in \mathbb{N}^+$. $\forall\ 1 \leq i \leq m, 1 \leq j \leq n$ and $i, j \in \mathbb{N}^+$, $A(i,j)$ represents the component at the i-th row, j-th column of A, so its value will not necessarily remain the same in the following steps. Gauss-Jordan Elimination which is applied to the matrix A is shown as follows:

1. Define a method F_1 with two parameters i and j where $1 \leq i \leq m, 1 \leq j \leq n$ where $i, j \in \mathbb{N}^+$ to find the pivot at the i-th row and not less than the j-th column of A:

   (1) If $A(i,j)$ is nonzero, then $A(i,j)$ is the pivot at the i-th row of A.

   (2) If $A(i,j)$ is zero and if $i+1 < m$, check if $A(i+1,j)$ is nonzero:

      a. If $A(i+1,j)$ is zero and if $i+2 < m$, then check if $A(i+2,j)$ is nonzero. If $A(i+2,j)$ is zero and if $i+3 < m$, then check if $A(i+3,j)$ is nonzero. Repeat this step until we find $A(k,j)$ where $i < k \leq m, k \in \mathbb{N}^+$ is nonzero at the k-th row, j-th column of A. Then switch the i-th row and k-th row of A, now $A(i,j)$ is the pivot at the i-th row of A.

      b. In the step a., if $\forall\ i \leq l \leq m$ and $l \in \mathbb{N}^+$, $A(l,j)$ are all zero, then the j-th column of A does not have the pivot. If $j+1 < n$, then check if $A(i,j+1)$ is nonzero and do the same thing as the step (1) and (2). If the $(j+1)$-th column of A does not have the pivot either and if $j+2 < n$, then check if $A(i,j+2)$ is nonzero and do the same thing as the step (1) and (2).

c. In the step b., if $\forall\ j \leq l \leq n$ and $l \in \mathbb{N}^+$, the l-th column of A does not have the pivot. There is no pivot at the i-th row of A.

2. Suppose that there is a pivot at the i-th row and j-th column of A where $1 \leq i \leq m, 1 \leq j \leq n$ and $i, j \in \mathbb{N}^+$. Define a method F_2 with two parameters i and j to eliminate some components of A.

   (a) If $A(i, j)$ is not equal to 1, multiply the i-th row of A by the number of $\dfrac{1}{A(i,j)}$.

   (b) $\forall\ 1 \leq k \leq m, k \neq i$ and $k \in \mathbb{N}^+$, if $A(k, j)$ is nonzero, then add the multiple of the i-th row by -1 into the k-th row of A.

3. For the matrix A, apply the method F_1 with the parameter 1 and 1, then get the first pivot at $A(1, 1)$. After that, apply the method F_2 with the parameter 1 and 1 to eliminate some components of A.

   If $2 \leq m$ and $2 \leq n$, apply the method F_1 with the parameter 2 and 2, then get the next pivot at $A(2, p)$ where $2 \leq p \leq n$ and $p \in \mathbb{N}^+$. After that, apply the method F_2 with the parameter 2 and p to eliminate some components of A.

   If $3 \leq m$ and $p + 1 \leq n$, apply the method F_1 with the parameter 3 and $p + 1$, then get the next pivot at $A(3, q)$ where $p + 1 \leq q \leq n$ and $q \in \mathbb{N}^+$. After that, apply the method F_2 with the parameter 3 and q to eliminate some components of A.

   Repeat this step until we meet these three cases:

   (1) The present pivot is at the last row of A;

   (2) The present pivot is at the last column of A;

   (3) The present row does not have the pivot.

After these three steps, Gauss-Jordan Elimination has been finished.

2.3.4 Gauss-Jordan Elimination and Linear Equations

$\forall$ linear equations, we eliminate the unknowns with only 0 coefficients to get the coefficient matrix A and vector **b**

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

where $m, n \in \mathbb{N}^+$ and $\forall\ 1 \le i \le n,\ i \in \mathbb{N}^+$

$$a_{1i}^2 + a_{2i}^2 + \cdots + a_{mi}^2 \neq 0.$$

The augmented matrix of the linear equations is

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}.$$

Apply the Gauss-Jordan Elimination into this augmented matrix to get the result matrix

$$\begin{bmatrix} 1 & c_{11} & \cdots & c_{1k_1} & 0 & c_{1(k_1+1)} & \cdots & c_{1(k_1+k_2)} & \cdots & 0 & c_{1(\sum\limits_{i=2}^{l-1} k_i+1)} & \cdots & c_{1(\sum\limits_{i=1}^{l} k_i)} & d_1 \\ 0 & 0 & \cdots & 0 & 1 & c_{21} & \cdots & c_{2k_2} & \cdots & 0 & c_{2(\sum\limits_{i=2}^{l-1} k_i+1)} & \cdots & c_{2(\sum\limits_{i=2}^{l} k_i)} & d_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & c_{l1} & \cdots & c_{lk_l} & d_l \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

This matrix is called the Reduced Row-Echelon Form (RREF) of the linear equations. There are l pivots in this matrix where $1 \le l \le \min\{m, n\}$ and $l \in \mathbb{N}^+$. The solution of the linear equations is shown as follows:

$$
\begin{cases}
x_1 & = d_1 - \sum\limits_{p=1}^{l} \sum\limits_{q=1}^{k_p} c_{1(\sum\limits_{r=1}^{p-1} k_r + q)} x_{(\sum\limits_{r=1}^{p-1} k_r + p + q)}, \\
x_2 & \in \mathbb{R}, \\
 & \vdots \\
x_{k_1} & \in \mathbb{R}, \\
x_{k_1+1} & = d_2 - \sum\limits_{p=2}^{l} \sum\limits_{q=1}^{k_p} c_{1(\sum\limits_{r=1}^{p-1} k_r + q)} x_{(\sum\limits_{r=1}^{p-1} k_r + p + q)}, \\
x_{k_1+2} & \in \mathbb{R}, \\
 & \vdots \\
x_{k_1+k_2+1} & \in \mathbb{R}, \\
x_{k_1+k_2+2} & = d_3 - \sum\limits_{p=3}^{l} \sum\limits_{q=1}^{k_p} c_{1(\sum\limits_{r=1}^{p-1} k_r + q)} x_{(\sum\limits_{r=1}^{p-1} k_r + p + q)}, \\
x_{k_1+k_2+3} & \in \mathbb{R}, \\
 & \vdots \\
x_{(\sum\limits_{p=1}^{l-1} k_p + l - 2)} & \in \mathbb{R}, \\
x_{(\sum\limits_{p=1}^{l-1} k_p + l - 1)} & = d_l - \sum\limits_{p=l}^{l} \sum\limits_{q=1}^{k_p} c_{1(\sum\limits_{r=1}^{p-1} k_r + q)} x_{(\sum\limits_{r=1}^{p-1} k_r + p + q)}, \\
x_{(\sum\limits_{p=1}^{l-1} k_p + l)} & \in \mathbb{R}, \\
 & \vdots \\
x_n & \in \mathbb{R}.
\end{cases}
$$

2.4 Inverse Matrices

2.4.1 Definition of Inverse Matrix

$\forall$ $n \times n$ square matrix A where $n \in \mathbb{N}^+$, the matrix A is invertible if there exists a matrix A^{-1} such that

$$A^{-1}A = AA^{-1} = I.$$

2.4.2 The Properties of Inverse Matrices

1. $\forall\ n \times n$ invertible square matrix A where $n \in \mathbb{N}^+$, its inverse A^{-1} is unique.

2. For $n \times n$ square matrix $A_1, A_2, \cdots, A_k$ where $n, k \in \mathbb{N}^+$, if $A_1, A_2, \cdots, A_k$ are separately invertible, then

$$(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_1^{-1}.$$

2.4.3 Calculating Inverse Matrices by Gauss-Jordan Elimination

$\forall\ n \times n$ invertible square matrix A where $n \in \mathbb{N}^+$, apply Gauss-Jordan Elimination into the matrix

$$\begin{bmatrix} A & I \end{bmatrix}$$

to get the matrix

$$\begin{bmatrix} I & A^{-1} \end{bmatrix}.$$

2.4.4 Invertible Matrix Theorem

$\forall\ n \times n$ square matrix A where $n \in \mathbb{N}^+$, A is invertible if and only if any of the following hold:

1. The RREF of A has n pivots;

2. The equation $A\mathbf{x} = \mathbf{b}$ has only one solution.

3. The equation $A\mathbf{x} = \mathbf{0}$ has no nonzero solutions.

4. The determinant of A is not zero.