

# Chapter 1

## Introduction to Vectors

### 1.1 Vectors and Linear Combinations

#### 1.1.1 Definition of Vectors

$\forall$   $n$ —dimensional vector  $\mathbf{v}$  where  $n \in \mathbb{N}^*$ :

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = (v_1, v_2, \dots, v_n).$$

$v_1, v_2, \dots, v_n$  are the 1st, 2nd,  $\dots$ ,  $n$ -th component of  $\mathbf{v}$ . Every vector is written as a column.

#### 1.1.2 Operations of Vectors

##### 1. Addition

$\forall$   $m$ —dimensional vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$  where  $m, n \in \mathbb{N}^*$ :

$$\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix}, \quad \dots, \quad \mathbf{v}_n = \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

The vector addition of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  is

$$\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_n = \begin{bmatrix} v_{11} + v_{12} + \dots + v_{1n} \\ v_{21} + v_{22} + \dots + v_{2n} \\ \vdots \\ v_{m1} + v_{m2} + \dots + v_{mn} \end{bmatrix}.$$

##### 2. Scalar Multiplication

$\forall$   $n$ —dimensional vector  $\mathbf{v}$  where  $n \in \mathbb{N}^*$  and  $\forall$  number  $c$ :

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

The scalar multiplication of  $c$  and  $\mathbf{v}$  is

$$c\mathbf{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}.$$

The number  $c$  is called a “scalar”.

### 1.1.3 Definition of Linear Combination

Combine addition with scalar multiplication to produce a “linear combination”.  $\forall m$ —dimensional vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  where  $m, n \in \mathbb{N}^*$  and  $\forall$  number  $\alpha_1, \alpha_2, \dots, \alpha_n$ . The sum of  $\alpha_1\mathbf{v}_1, \alpha_2\mathbf{v}_2, \dots, \alpha_n\mathbf{v}_n$  is a linear combination

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \dots + \alpha_n\mathbf{v}_n.$$

### 1.1.4 Geometrical Significance of Linear Combination

#### Parallelogram Law

The parallelogram law gives the rule for vector addition of vectors  $\mathbf{u}$  and  $\mathbf{v}$ . The sum  $\mathbf{u} + \mathbf{v}$  of the vectors is obtained by placing them head to tail and drawing the vector from the free tail to the free head.

#### Line, Plane, Space and Three-dimensional Vectors

$\forall \alpha, \beta, \gamma \in \mathbb{R}$  and  $\forall$  nonzero three-dimensional vectors  $\mathbf{u}, \mathbf{v}, \mathbf{w}$ :

- All combinations  $\alpha\mathbf{u}$  fill a line through  $(0, 0, 0)$ ;
- If  $\mathbf{u}$  and  $\mathbf{v}$  are not on the same line, all combinations  $\alpha\mathbf{u} + \beta\mathbf{v}$  fill a plane through  $(0, 0, 0)$ ;
- If  $\mathbf{w}$  is not on the same plane formed by  $\alpha\mathbf{u} + \beta\mathbf{v}$ , all combinations  $\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$  fill three-dimensional space.

## 1.2 Lengths and Dot Products

### 1.2.1 Definition of Dot Products

$\forall n$ —dimensional vectors  $\mathbf{v}, \mathbf{w}$  where  $n \in \mathbb{N}^*$ :

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}.$$

The dot product or inner product of  $\mathbf{v}, \mathbf{w}$  is the number  $\mathbf{v} \cdot \mathbf{w}$  or  $\mathbf{w} \cdot \mathbf{v}$ :

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} = v_1w_1 + v_2w_2 + \dots + v_nw_n.$$

### 1.2.2 Definition of Lengths

$\forall$   $n$ -dimensional vectors  $\mathbf{v}$  where  $n \in \mathbb{N}^*$ :

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

The length  $\|\mathbf{v}\|$  of a vector  $\mathbf{v}$  is the square root of  $\mathbf{v} \cdot \mathbf{v}$ :

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

### 1.2.3 Definition of Unit Vectors

A unit vector  $\mathbf{u}$  is a vector whose length equals one. Then  $\mathbf{u} \cdot \mathbf{u} = 1$ .  $\forall$   $n$ -dimensional vectors  $\mathbf{v}$  where  $n \in \mathbb{N}^*$ :

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$  is a unit vector in the same direction as  $\mathbf{v}$ .

### 1.2.4 The Angle Between Vectors

#### Two-dimensional Vectors

$\forall$  2-dimensional nonzero vectors  $\mathbf{v}, \mathbf{w}$ :

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (v_1^2 + v_2^2 \neq 0), \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad (w_1^2 + w_2^2 \neq 0).$$

Let the angle between  $\mathbf{v}$  and  $\mathbf{w}$  is  $\theta$  ( $0 \leq \theta < \pi$ ), then

$$\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} = \frac{v_1 w_1 + v_2 w_2}{\sqrt{v_1^2 + v_2^2} \sqrt{w_1^2 + w_2^2}} = \cos \theta.$$

#### Three-dimensional Vectors

$\forall$  3-dimensional nonzero vectors  $\mathbf{v}, \mathbf{w}$ :

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad (v_1^2 + v_2^2 + v_3^2 \neq 0), \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \quad (w_1^2 + w_2^2 + w_3^2 \neq 0).$$

Let the angle between  $\mathbf{v}$  and  $\mathbf{w}$  is  $\theta$  ( $0 \leq \theta < \pi$ ), then

$$\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{\sqrt{v_1^2 + v_2^2 + v_3^2} \sqrt{w_1^2 + w_2^2 + w_3^2}} = \cos \theta.$$

## Schwarz inequality and Triangle inequality

Schwarz inequality

$$|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\|.$$

Triangle inequality

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|.$$

## 1.3 Matrices

### 1.3.1 Matrix and Linear Combination of Vectors

#### Definition of Matrix

$\forall$   $m$ —dimensional vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  where  $m, n \in \mathbb{N}^*$ :

$$\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix}, \quad \dots, \quad \mathbf{v}_n = \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

and  $\forall$  number  $x_1, x_2, \dots, x_n$ . The linear combination of

$$x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \dots + x_n \mathbf{v}_n = x_1 \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix} + x_2 \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

can be expressed as

$$\mathbf{A} \mathbf{x} = \begin{bmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \dots & v_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where  $\mathbf{A}$  is a matrix and  $\mathbf{x}$  is a vector:

$$\mathbf{A} = \begin{bmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \dots & v_{mn} \end{bmatrix} = \begin{bmatrix} | & | & \dots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & \dots & | \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

#### The Format of Matrix

Let  $n$   $m$ —dimensional vectors  $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$  and

$$\mathbf{c}_1 = \mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix}, \quad \mathbf{c}_2 = \mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix}, \quad \dots, \quad \mathbf{c}_n = \mathbf{v}_n = \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

Let  $m$   $n$ -dimensional vectors  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m$  and

$$\mathbf{r}_1 = \begin{bmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} v_{21} \\ v_{22} \\ \vdots \\ v_{2n} \end{bmatrix}, \quad \dots, \quad \mathbf{r}_m = \begin{bmatrix} v_{m1} \\ v_{m2} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

Then the matrix  $\mathbf{A}$  can be expressed as

$$\mathbf{A} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \\ | & | & \cdots & | \end{bmatrix} = \begin{bmatrix} \text{---} & \mathbf{r}_1 & \text{---} \\ \text{---} & \mathbf{r}_2 & \text{---} \\ \vdots & \vdots & \vdots \\ \text{---} & \mathbf{r}_m & \text{---} \end{bmatrix}.$$

### The Form of $\mathbf{Ax} = \mathbf{b}$

Let a  $m$ -dimensional vector  $\mathbf{b}$

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

and

$$\mathbf{Ax} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Then

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = x_1 \mathbf{c}_1 + x_2 \mathbf{c}_2 + \cdots + x_n \mathbf{c}_n = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{b} \\ \mathbf{r}_2 \cdot \mathbf{b} \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{b} \end{bmatrix}.$$

### 1.3.2 Difference Matrix and Cyclic Difference Matrix

Given a  $n$ -dimensional vector  $\mathbf{x}$  and a  $n$ -dimensional vector  $\mathbf{b}$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

its difference matrix  $\mathbf{A}$  such that  $\mathbf{Ax} = \mathbf{b}$  is

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}.$$

Then we will get

$$\mathbf{x} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ \vdots \\ b_1 + b_2 + \cdots + b_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \mathbf{A}^{-1}\mathbf{b}.$$

The matrix  $\mathbf{A}$  is invertible. From  $\mathbf{b}$  we can recover  $\mathbf{x}$ . We write  $\mathbf{x}$  as  $\mathbf{A}^{-1}\mathbf{b}$ . In particular,  $\mathbf{Ax} = \mathbf{0}$  has one solution

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Its cyclic difference matrix  $\mathbf{C}$  such that  $\mathbf{Cx} = \mathbf{b}$  is

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}.$$

$\mathbf{Cx} = \mathbf{b}$  has many solutions or else no solution. In particular, if  $\mathbf{Cx} = \mathbf{0}$  has many solutions,  $\mathbf{C}$  is a singular matrix.