

Chapter 1

Introduction to Vectors

1.1 Vectors

1.1.1 Definition of Vectors

\forall n —dimensional vector \mathbf{v} where $n \in \mathbb{N}^+$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = (v_1, v_2, \dots, v_n).$$

v_1, v_2, \dots, v_n are the 1st, 2nd, \dots , n -th component of \mathbf{v} . Every vector is written as a column.

1.1.2 Operations of Vectors

1. Addition

\forall m —dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ where $m, n \in \mathbb{N}^+$:

$$\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix}, \quad \dots, \quad \mathbf{v}_n = \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

The vector addition of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is

$$\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_n = \begin{bmatrix} v_{11} + v_{12} + \dots + v_{1n} \\ v_{21} + v_{22} + \dots + v_{2n} \\ \vdots \\ v_{m1} + v_{m2} + \dots + v_{mn} \end{bmatrix}.$$

2. Scalar Multiplication

\forall n -dimensional vector \mathbf{v} where $n \in \mathbb{N}^+$ and \forall number c :

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

The scalar multiplication of c and \mathbf{v} is

$$c\mathbf{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}.$$

The number c is called a “scalar”.

1.1.3 Linear Combination

1. Definition of Linear Combination

Combine addition with scalar multiplication to produce a “linear combination”. \forall m -dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ where $m, n \in \mathbb{N}^+$ and \forall number $\alpha_1, \alpha_2, \dots, \alpha_n$. The sum of $\alpha_1\mathbf{v}_1, \alpha_2\mathbf{v}_2, \dots, \alpha_n\mathbf{v}_n$ is a linear combination

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \dots + \alpha_n\mathbf{v}_n.$$

2. Geometrical Significance of Linear Combination (Parallelogram Law)

The parallelogram law gives the rule for vector addition of vectors \mathbf{u} and \mathbf{v} . The sum $\mathbf{u} + \mathbf{v}$ of the vectors is obtained by placing them head to tail and drawing the vector from the free tail to the free head.

3. Line, Plane, Space and Three-dimensional Vectors

$\forall \alpha, \beta, \gamma \in \mathbb{R}$ and \forall nonzero three-dimensional vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$:

- All combinations $\alpha\mathbf{u}$ fill a line through $(0, 0, 0)$;
- If \mathbf{u} and \mathbf{v} are not on the same line, all combinations $\alpha\mathbf{u} + \beta\mathbf{v}$ fill a plane through $(0, 0, 0)$;
- If \mathbf{w} is not on the same plane formed by $\alpha\mathbf{u} + \beta\mathbf{v}$, all combinations $\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$ fill three-dimensional space.

1.1.4 Definition of Dot Products

$\forall n$ -dimensional vectors \mathbf{v}, \mathbf{w} where $n \in \mathbb{N}^+$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}.$$

The dot product or inner product of \mathbf{v}, \mathbf{w} is the number $\mathbf{v} \cdot \mathbf{w}$ or $\mathbf{w} \cdot \mathbf{v}$:

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

1.1.5 Definition of Lengths

$\forall n$ -dimensional vectors \mathbf{v} where $n \in \mathbb{N}^+$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

The length $\|\mathbf{v}\|$ of a vector \mathbf{v} is the square root of $\mathbf{v} \cdot \mathbf{v}$:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

1.1.6 Definition of Unit Vectors

A unit vector \mathbf{u} is a vector whose length equals one. Then $\mathbf{u} \cdot \mathbf{u} = 1$. $\forall n$ -dimensional vectors \mathbf{v} where $n \in \mathbb{N}^+$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector in the same direction as \mathbf{v} .

1.1.7 The Angle Between Vectors

1. Two-dimensional Vectors

\forall 2-dimensional nonzero vectors \mathbf{v}, \mathbf{w} :

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (v_1^2 + v_2^2 \neq 0), \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad (w_1^2 + w_2^2 \neq 0).$$

Let the angle between \mathbf{v} and \mathbf{w} is θ ($0 \leq \theta < \pi$), then

$$\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} = \frac{v_1 w_1 + v_2 w_2}{\sqrt{v_1^2 + v_2^2} \sqrt{w_1^2 + w_2^2}} = \cos \theta.$$

2. Three-dimensional Vectors

\forall 3-dimensional nonzero vectors \mathbf{v}, \mathbf{w} :

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad (v_1^2 + v_2^2 + v_3^2 \neq 0), \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \quad (w_1^2 + w_2^2 + w_3^2 \neq 0).$$

Let the angle between \mathbf{v} and \mathbf{w} is θ ($0 \leq \theta < \pi$), then

$$\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{\sqrt{v_1^2 + v_2^2 + v_3^2} \sqrt{w_1^2 + w_2^2 + w_3^2}} = \cos \theta.$$

3. Schwarz inequality and Triangle inequality

Schwarz inequality

$$|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\|.$$

Triangle inequality

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|.$$

1.2 Matrices

1.2.1 Definition of Matrix

$\forall m$ —dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ where $m, n \in \mathbb{N}^+$:

$$\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix}, \quad \dots, \quad \mathbf{v}_n = \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

and \forall number x_1, x_2, \dots, x_n . The linear combination of

$$x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \dots + x_n \mathbf{v}_n = x_1 \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix} + x_2 \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

can be expressed as

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \dots & v_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where \mathbf{A} is a matrix and \mathbf{x} is a vector:

$$\mathbf{A} = \begin{bmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \dots & v_{mn} \end{bmatrix} = \begin{bmatrix} | & | & \dots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & \dots & | \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

1.2.2 The Format of Matrix

Let n m —dimensional vectors $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ and

$$\mathbf{c}_1 = \mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{bmatrix}, \quad \mathbf{c}_2 = \mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{bmatrix}, \quad \dots, \quad \mathbf{c}_n = \mathbf{v}_n = \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

Let m n -dimensional vectors $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m$ and

$$\begin{aligned}\mathbf{r}_1 &= \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \end{bmatrix}, \\ \mathbf{r}_2 &= \begin{bmatrix} v_{21} & v_{22} & \cdots & v_{2n} \end{bmatrix}, \\ &\vdots \\ \mathbf{r}_m &= \begin{bmatrix} v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix}.\end{aligned}$$

Then the matrix \mathbf{A} can be expressed as

$$\mathbf{A} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \\ | & | & \cdots & | \end{bmatrix} = \begin{bmatrix} \text{---} & \mathbf{r}_1 & \text{---} \\ \text{---} & \mathbf{r}_2 & \text{---} \\ \vdots & \vdots & \vdots \\ \text{---} & \mathbf{r}_m & \text{---} \end{bmatrix}.$$

1.2.3 The Form of $\mathbf{Ax} = \mathbf{b}$

Let a m -dimensional vector \mathbf{b}

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

and

$$\mathbf{Ax} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Then

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = x_1 \mathbf{c}_1 + x_2 \mathbf{c}_2 + \cdots + x_n \mathbf{c}_n = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{b} \\ \mathbf{r}_2 \cdot \mathbf{b} \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{b} \end{bmatrix}.$$

1.2.4 Difference Matrix and Cyclic Difference Matrix

Given a n -dimensional vector \mathbf{x} and a n -dimensional vector \mathbf{b}

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

its difference matrix \mathbf{A} such that $\mathbf{Ax} = \mathbf{b}$ is

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}.$$

Then we will get

$$\mathbf{x} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ \vdots \\ b_1 + b_2 + \cdots + b_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \mathbf{A}^{-1}\mathbf{b}.$$

The matrix \mathbf{A} is invertible. From \mathbf{b} we can recover \mathbf{x} . We write \mathbf{x} as $\mathbf{A}^{-1}\mathbf{b}$. In particular, $\mathbf{Ax} = \mathbf{0}$ has one solution

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Its cyclic difference matrix \mathbf{C} such that $\mathbf{Cx} = \mathbf{b}$ is

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}.$$

$\mathbf{Cx} = \mathbf{b}$ has many solutions or else no solution. In particular, if $\mathbf{Cx} = \mathbf{0}$ has many solutions, \mathbf{C} is a singular matrix.