

Chapter 2

Solving Linear Equations

2.1 Vectors and Linear Equations

2.1.1 Definition of Linear Equations

\forall linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{cases} \quad (2.1)$$

where $m, n \in \mathbb{N}^*$. The equations are linear, which means that the unknowns are only multiplied by numbers.

2.1.2 The Matrix Form of the Equations

The matrix form of linear equations 2.1 is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \\ | & | & \cdots & | \end{bmatrix} = \begin{bmatrix} \text{---} & \mathbf{r}_1 & \text{---} \\ \text{---} & \mathbf{r}_2 & \text{---} \\ \vdots & \vdots & \vdots \\ \text{---} & \mathbf{r}_m & \text{---} \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

This linear equations could be expressed as $\mathbf{Ax} = \mathbf{b}$. \mathbf{A} is called the “coefficient matrix. This linear equations could also be expressed as

$$\mathbf{Ax} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{b} \\ \mathbf{r}_2 \cdot \mathbf{b} \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{b} \end{bmatrix} = \mathbf{b}.$$

2.1.3 Identity Matrix

\forall n —dimensional vector \mathbf{v} where $n \in \mathbb{N}^*$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

Its identity matrix is a $n \times n$ matrix \mathbf{I}

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The identity matrix always yields the multiplication $\mathbf{Iv} = \mathbf{v}$.