

Chapter 1

Introduction to Vectors

1.1 Vectors and Linear Combinations

1.1.1 Definition of Vectors

\forall n —dimensional vector \mathbf{v} where $n \in \mathbb{N}^*$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = (v_1, v_2, \dots, v_n).$$

v_1, v_2, \dots, v_n are the 1st, 2nd, \dots , n -th component of \mathbf{v} . Every vector is written as a **column**.

1.1.2 Operations of Vectors

1. Addition

\forall n —dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ where $m, n \in \mathbb{N}^*$:

$$\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{n1} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{n2} \end{bmatrix}, \quad \dots, \quad \mathbf{v}_m = \begin{bmatrix} v_{1m} \\ v_{2m} \\ \vdots \\ v_{mn} \end{bmatrix}.$$

The vector addition of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ is

$$\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_m = \begin{bmatrix} v_{11} + v_{12} + \dots + v_{1m} \\ v_{21} + v_{22} + \dots + v_{2m} \\ \vdots \\ v_{m1} + v_{m2} + \dots + v_{mn} \end{bmatrix}.$$

2. Scalar Multiplication

\forall n —dimensional vector \mathbf{v} where $n \in \mathbb{N}^*$ and \forall number c :

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

The scalar multiplication of c and \mathbf{v} is

$$c\mathbf{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}.$$

The number c is called a “scalar”.

1.1.3 Definition of Linear Combination

Combine addition with scalar multiplication to produce a “linear combination”. $\forall n$ —dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ where $m, n \in \mathbb{N}^*$ and \forall number $\alpha_1, \alpha_2, \dots, \alpha_m$. The sum of $\alpha_1\mathbf{v}_1, \alpha_2\mathbf{v}_2, \dots, \alpha_m\mathbf{v}_m$ is a linear combination

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \dots + \alpha_m\mathbf{v}_m.$$

1.1.4 Geometrical Significance of Linear Combination

Parallelogram Law

The parallelogram law gives the rule for vector addition of vectors \mathbf{u} and \mathbf{v} . The sum $\mathbf{u} + \mathbf{v}$ of the vectors is obtained by placing them head to tail and drawing the vector from the free tail to the free head.

Line, Plane, Space and Three-dimensional Vectors

$\forall \alpha, \beta, \gamma \in \mathbb{R}$ and \forall nonzero three-dimensional vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$:

- All combinations $\alpha\mathbf{u}$ fill a line through $(0, 0, 0)$;
- If \mathbf{u} and \mathbf{v} are not on the same line, all combinations $\alpha\mathbf{u} + \beta\mathbf{v}$ fill a plane through $(0, 0, 0)$;
- If \mathbf{w} is not on the same plane formed by $\alpha\mathbf{u} + \beta\mathbf{v}$, all combinations $\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$ fill three-dimensional space.

1.2 Lengths and Dot Products

1.2.1 Definition of Dot Products

$\forall n$ —dimensional vectors \mathbf{v}, \mathbf{w} where $n \in \mathbb{N}^*$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}.$$

The dot product or inner product of \mathbf{v}, \mathbf{w} is the number $\mathbf{v} \cdot \mathbf{w}$ or $\mathbf{w} \cdot \mathbf{v}$:

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} = v_1w_1 + v_2w_2 + \dots + v_nw_n.$$

1.2.2 Definition of Lengths

\forall n —dimensional vectors \mathbf{v} where $n \in \mathbb{N}^*$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

The length $\|\mathbf{v}\|$ of a vector \mathbf{v} is the square root of $\mathbf{v} \cdot \mathbf{v}$:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

1.2.3 Definition of Unit Vectors

A unit vector \mathbf{u} is a vector whose length equals one. Then $\mathbf{u} \cdot \mathbf{u} = 1$. \forall n —dimensional vectors \mathbf{v} where $n \in \mathbb{N}^*$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector in the same direction as \mathbf{v} .

1.2.4 The Angle Between Vectors

Two-dimensional Vectors

\forall 2—dimensional nonzero vectors \mathbf{v}, \mathbf{w} :

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (v_1^2 + v_2^2 \neq 0), \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad (w_1^2 + w_2^2 \neq 0).$$

Let the angle between \mathbf{v} and \mathbf{w} is θ ($0 \leq \theta < \pi$), then

$$\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} = \frac{v_1 w_1 + v_2 w_2}{\sqrt{v_1^2 + v_2^2} \sqrt{w_1^2 + w_2^2}} = \cos \theta.$$

Three-dimensional Vectors

\forall 3—dimensional nonzero vectors \mathbf{v}, \mathbf{w} :

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad (v_1^2 + v_2^2 + v_3^2 \neq 0), \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \quad (w_1^2 + w_2^2 + w_3^2 \neq 0).$$

Let the angle between \mathbf{v} and \mathbf{w} is θ ($0 \leq \theta < \pi$), then

$$\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{\sqrt{v_1^2 + v_2^2 + v_3^2} \sqrt{w_1^2 + w_2^2 + w_3^2}} = \cos \theta.$$

Schwarz inequality and Triangle inequality

Schwarz inequality

$$|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\|.$$

Triangle inequality

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|.$$