# Chapter 2

# Solving Linear Equations

## 2.1 Linear Equations

### 2.1.1 Definition of Linear Equations

$\forall$ linear equations

$$\begin{cases} a_{11}x_1 & + a_{12}x_2 & + & \cdots + a_{1n}x_n & = b_1, \\ a_{21}x_1 & + a_{22}x_2 & + & \cdots + a_{2n}x_n & = b_2, \\ & & \vdots & & \\ a_{m1}x_1 & + a_{m2}x_2 & + & \cdots + a_{mn}x_n & = b_m. \end{cases} \tag{2.1}$$

where $m, n \in \mathbb{N}^+$. The equations are linear, which means that the unknowns are only multiplied by numbers.

### 2.1.2 The Matrix Form of the Equations

The matrix form of linear equations 2.1 is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

This linear equations could be expressed as

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \\ | & | & \cdots & | \end{bmatrix} = \begin{bmatrix} — & \mathbf{r}_1^{\mathrm{T}} & — \\ — & \mathbf{r}_2^{\mathrm{T}} & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{r}_m^{\mathrm{T}} & — \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

$A$ is called the <u>coefficient matrix</u>. This linear equations could also be expressed as

$$A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{x} \\ \mathbf{r}_2 \cdot \mathbf{x} \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{x} \end{bmatrix} = \mathbf{b}.$$

## 2.2 Gauss-Jordan Elimination

### 2.2.1 Objective Matrix

$\forall$ $m \times n$ matrix $A$

$$A = \begin{bmatrix} A(1,1) & A(1,2) & \cdots & A(1,n) \\ A(2,1) & A(2,2) & \cdots & A(2,n) \\ \vdots & \vdots & \ddots & \vdots \\ A(m,1) & A(m,2) & \cdots & A(m,n) \end{bmatrix}$$

where $m, n \in \mathbb{N}^+$. $\forall\, 1 \leq i \leq m, 1 \leq j \leq n$ and $i, j \in \mathbb{N}^+$, $A(i,j)$ represents the component at the $i$-th row, $j$-th column of $A$, so its value will not necessarily remain the same in the following steps of Gauss-Jordan Elimination.

### 2.2.2 Operations and Shorthands

1. $P_{(i,j)}A$

   This notation represents a matrix multiplication, which means that switching the $i$-th row and $j$-th row of the matrix $A$. The matrix $P_{(i,j)}$ is shown as follows:

$$P_{(i,j)} = P = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}.$$

   where $P$ is a $m \times m$ matrix. For $1 \leq l \leq m, l \neq i, l \neq j$ and $l \in \mathbb{N}^+$, $P(i,j) = P(j,i) = P(l,l) = 1$ and other components of $P$ is 0.

2. $D^{-1}_{(i,k)}A$

   This notation represents a matrix multiplication, which means that multiplying the

$i$-th row of the matrix $A$ by a number $k$. The matrix $D^{-1}_{(i,k)}$ is shown as follows:

$$D^{-1}_{(i,k)} = D^{-1} = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}.$$

where $D^{-1}$ is a $m \times m$ matrix. For $k \in \mathbb{R}, 1 \leq l \leq m, l \neq i$ and $l \in \mathbb{N}^{+}$, $D^{-1}(i,i) = k, D^{-1}(l,l) = 1$ and other components of $D^{-1}$ is 0.

3. $E_{(j,i,k)}A$

   This notation represents a matrix multiplication, which means that adding the multiple of the $i$-th row of $A$ by a number $k$ into the $j$-th row of $A$. The matrix $E_{(j,i,k)}$ is shown as follows:

$$E_{(j,i,k)} = E = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}.$$

   where $E$ is a $m \times m$ matrix. For $k \in \mathbb{R}, 1 \leq l \leq m$ and $l \in \mathbb{N}^{+}$, $E(j,i) = k, E(l,l) = 1$ and other components of $E$ is 0.

4. **eliminationBelow**$(i,j)$

   The prerequisite of this operation is $A(i,j) \neq 0$. This notation represents a series of operations: For $l = i+1, i+2, \cdots, m$, adding the multiple of the $i$-th row of $A$ by a number $-\frac{A(l,j)}{A(i,j)}$ into the $l$-th row of $A$. This operation is implemented as follows:

$$E_{\left[m,i,-\frac{A(m,j)}{A(i,j)}\right]} E_{\left[m-1,i,-\frac{A(m-1,j)}{A(i,j)}\right]} \cdots E_{\left[i+1,i,-\frac{A(i+1,j)}{A(i,j)}\right]} A.$$

5. **eliminationAbove**$(i,j)$

The prerequisite of this operation is $A(i,j) \neq 0$. This notation represents a series of operations: For $l = 1, 2 \cdots, i-1$, adding the multiple of the $i$-th row of $A$ by a number $-\frac{A(l,j)}{A(i,j)}$ into the $l$-th row of $A$. This operation is implemented as follows:

$$E_{\left[i-1,i,-\frac{A(i-1,j)}{A(i,j)}\right]} \cdots E_{\left[2,i,-\frac{A(2,j)}{A(i,j)}\right]} E_{\left[1,i,-\frac{A(1,j)}{A(i,j)}\right]} A.$$

6. **searchPivot**$(i,j)$

   This notation represents a series of operations to search the pivot at the $i$-th row and no less than $j$-th column of the matrix $A$:

   (1) If $A(i,j) \neq 0$, $A(i,j)$ is the pivot at the $i$-th row of the matrix $A$.

   (2) If $A(i,j) = 0$, we check $A(i+1,j), A(i+2,j), \cdots, A(m,j)$ from the small row number to the large row number.

      a. If $A(l,j) \neq 0$ where $i < l \leq m$ and $l \in \mathbb{N}^+$, $A(l,j)$ is the pivot. Then we take the operation $P_{(i,l)}A$. Now $A(i,j)$ is the pivot at the $i$-th row of the matrix $A$.

      b. If $A(i,j) = A(i+1,j) = \cdots = A(m,j) = 0$, we take the operation **searchPivot**$(i,j+1)$, **searchPivot**$(i,j+2)$, $\cdots$, **searchPivot**$(i,n)$ from the small column number to the large column number.

         (a) If we can search the pivot in the operation **searchPivot**$(i,l)$ where $j < l \leq n$ and $l \in \mathbb{N}^+$, after the exchange of rows (if there is), $A(i,l)$ is the pivot at the $i$-th row of the matrix $A$.

         (b) If we have reached **searchPivot**$(i,n)$ but still can not find the pivot. There is no pivot at the $i$-th row of the matrix $A$.

### 2.2.3 The Flow of Gauss-Jordan Elimination

Gauss-Jordan Elimination which is applied to the matrix $A$ is shown as follows:

1. **searchPivot**$(1,1)$, and we find the pivot $A(1,c_1)$ where $1 \leq c_1 \leq n$ and $c_1 \in \mathbb{N}^+$;

2. **eliminationBelow**$(1,c_1)$;

3. **searchPivot**$(2,c_1+1)$, and we find the pivot $A(2,c_2)$ where $c_1 + 1 \leq c_2 \leq n$ and $c_2 \in \mathbb{N}^+$;

4. **eliminationBelow**$(2, c_2)$;

5. **searchPivot**$(3, c_2 + 1)$, and we find the pivot $A(3, c_3)$ where $c_2 + 1 \leq c_3 \leq n$ and $c_2 \in \mathbb{N}^+$;

6. **eliminationBelow**$(3, c_3)$;

7. **searchPivot**$(4, c_3 + 1)$, and we find the pivot $A(4, c_4)$ where $c_3 + 1 \leq c_4 \leq n$ and $c_3 \in \mathbb{N}^+$;

8. **eliminationBelow**$(4, c_4)$;

   $\vdots$

We teriminate this process until we meet one of these three situations:

1. We have found the pivot at the $m$-th row of the matrix $A$;

2. We have found the pivot at the $n$-th column of the matrix $A$;

3. We have found that a row of the matrix $A$ does not have a pivot.

Now, the matrix $A$ has become the Row Echelon Form (REF). There are $l$ pivots in this matrix where $1 \leq l \leq \min\{m, n\}$ and $l \in \mathbb{N}^+$. The REF of the matrix $A$ is shown as follows:

$$U = \begin{bmatrix} B_1 & B_2 & \cdots & B_l \end{bmatrix}$$

where

$$B_j = \begin{bmatrix} e_{1\left(j+\sum_{i=0}^{j-1} k_i\right)} & b_{1\left(j+1+\sum_{i=0}^{j-1} k_i\right)} & \cdots & b_{1\left(j+\sum_{i=0}^{j} k_i\right)} \\ e_{2\left(j+\sum_{i=0}^{j-1} k_i\right)} & b_{2\left(j+1+\sum_{i=0}^{j-1} k_i\right)} & \cdots & b_{2\left(j+\sum_{i=0}^{j} k_i\right)} \\ \vdots & \vdots & \ddots & \vdots \\ e_{(j-1)\left(j+\sum_{i=0}^{j-1} k_i\right)} & b_{(j-1)\left(j+1+\sum_{i=0}^{j-1} k_i\right)} & \cdots & b_{(j-1)\left(j+\sum_{i=0}^{j} k_i\right)} \\ a_j & b_{j\left(j+1+\sum_{i=0}^{j-1} k_i\right)} & \cdots & b_{j\left(j+\sum_{i=0}^{j} k_i\right)} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

and $1 \leq j \leq l$, $k_0 = 0$ and $j, k_1, k_2, \cdots, k_l \in \mathbb{N}^+$. The implicit constraint is

$$l + \sum_{i=0}^{l} k_i = n.$$

The Row Echelon Form (REF) of the matrix $A$ could be transformed into the Reduced Row Echelon Form (RREF). The process is shown as follows:

1. $D^{-1}_{\left(l,\frac{1}{a_l}\right)} \cdots D^{-1}_{\left(2,\frac{1}{a_2}\right)} D^{-1}_{\left(1,\frac{1}{a_1}\right)} A$;
2. **eliminationAbove**$(1, 1)$;
3. **eliminationAbove**$(2, 2 + k_1)$;

   $\vdots$

$l + 1$. **eliminationAbove**$\left(l, l - 2 + \sum\limits_{i=1}^{l-1} k_i\right)$.

Now, the matrix $A$ has become the Reduced Row Echelon Form (RREF). There are stll $l$ pivots in this matrix. The RREF of the matrix $A$ is shown as follows:

$$R = \begin{bmatrix} RB_1 & RB_2 & \cdots & RB_l \end{bmatrix}$$

where

$$RB_j = \begin{bmatrix} 0 & c_{1\left(j+1+\sum\limits_{i=0}^{j-1} k_i\right)} & \cdots & c_{1\left(j+\sum\limits_{i=0}^{j} k_i\right)} \\ 0 & c_{2\left(j+1+\sum\limits_{i=0}^{j-1} k_i\right)} & \cdots & c_{2\left(j+\sum\limits_{i=0}^{j} k_i\right)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c_{(j-1)\left(j+1+\sum\limits_{i=0}^{j-1} k_i\right)} & \cdots & c_{(j-1)\left(j+\sum\limits_{i=0}^{j} k_i\right)} \\ 1 & c_{j\left(j+1+\sum\limits_{i=0}^{j-1} k_i\right)} & \cdots & c_{j\left(j+\sum\limits_{i=0}^{j} k_i\right)} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

and $1 \leq j \leq l$, $k_0 = 0$, $j, k_1, k_2, \cdots, k_l \in \mathbb{N}^+$. The implicit constraint is

$$l + \sum_{i=0}^{l} k_i = n.$$

However, there may be zero columns in $U$ and $R$, but the conlusion will not change too much.

### 2.2.4 Gauss-Jordan Elimination and Linear Equations

$\forall$ linear equations, we eliminate the unknowns with only 0 coefficients to get the coefficient matrix $A$ and vector $\mathbf{b}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

where $m, n \in \mathbb{N}^+$ and $\forall\ 1 \le i \le n,\ i \in \mathbb{N}^+$

$$a_{1i}^2 + a_{2i}^2 + \cdots + a_{mi}^2 \neq 0.$$

The augmented matrix of the linear equations is

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}.$$

In order to find the solution of these linear equations, we apply the Gauss-Jordan Elimination into this augmented matrix to get its RREF matrix

$$\begin{bmatrix} RB_1 & RB_2 & \cdots & RB_l & \mathbf{d}. \end{bmatrix}$$

where (assume that there exists solutions in the linear equations)

$$RB_j = \begin{bmatrix} 0 & c_{1\left(j+1+\sum_{i=0}^{j-1} k_i\right)} & \cdots & c_{1\left(j+\sum_{i=0}^{j} k_i\right)} \\ 0 & c_{2\left(j+1+\sum_{i=0}^{j-1} k_i\right)} & \cdots & c_{2\left(j+\sum_{i=0}^{j} k_i\right)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c_{(j-1)\left(j+1+\sum_{i=0}^{j-1} k_i\right)} & \cdots & c_{(j-1)\left(j+\sum_{i=0}^{j} k_i\right)} \\ 1 & c_{j\left(j+1+\sum_{i=0}^{j-1} k_i\right)} & \cdots & c_{j\left(j+\sum_{i=0}^{j} k_i\right)} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{l-1} \\ d_l \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and $1 \le j \le l,\ k_0 = 0,\ j, k_1, k_2, \cdots, k_l \in \mathbb{N}^+$.

The solution of the linear equations is divided into two parts:

1. Free Variables:

$$\begin{aligned}
x_2, x_3, \cdots, x_{1+k_1} &\in \mathbb{R};\\
x_{3+k_1}, x_{4+k_1}, \cdots, x_{2+k_1+k_2} &\in \mathbb{R};\\
x_{4+k_1+k_2}, x_{5+k_1+k_2}, \cdots, x_{3+k_1+k_2+k_3} &\in \mathbb{R};\\
&\vdots\\
x_{l+1+\sum\limits_{i=1}^{l-1} k_i}, x_{l+2+\sum\limits_{i=1}^{l-1} k_i}, \cdots, x_{l+\sum\limits_{i=1}^{l} k_i} &\in \mathbb{R}.
\end{aligned}$$

2. Pivots:

$$\begin{aligned}
x_1 &= d_1 - \sum_{j=0}^{l-1} \sum_{u=1}^{k_{j+1}} c_{1\left(j+u+1+\sum\limits_{i=0}^{j} k_i\right)} x_{\left(j+u+1+\sum\limits_{i=0}^{j} k_i\right)},\\
x_{2+k_1} &= d_2 - \sum_{j=1}^{l-1} \sum_{u=1}^{k_{j+1}} c_{2\left(j+u+1+\sum\limits_{i=1}^{j} k_i\right)} x_{\left(j+u+1+\sum\limits_{i=1}^{j} k_i\right)},\\
x_{3+k_1+k_2} &= d_3 - \sum_{j=2}^{l-1} \sum_{u=1}^{k_{j+1}} c_{3\left(j+u+1+\sum\limits_{i=2}^{j} k_i\right)} x_{\left(j+u+1+\sum\limits_{i=2}^{j} k_i\right)},\\
&\vdots\\
x_{l+\sum\limits_{i=1}^{l-1} k_i} &= d_l - \sum_{j=l-1}^{l-1} \sum_{u=1}^{k_{j+1}} c_{l\left(j+u+1+\sum\limits_{i=l-1}^{j} k_i\right)} x_{\left(j+u+1+\sum\limits_{i=l-1}^{j} k_i\right)}.
\end{aligned}$$

## 2.3 Solutions to $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$

### 2.3.1 The Special Solution to $A\mathbf{x} = \mathbf{0}$

$\forall$ $m \times n$ matrix $A$ where $m, n \in \mathbb{N}^+$, for the equation $A\mathbf{x} = \mathbf{0}$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Suppose that $x_{p_1}, x_{p_2}, \cdots, x_{p_r}$ are pivots; $x_{q_1}, x_{q_2}, \cdots, x_{q_{n-r}}$ are free variables ($1 \leq p_1 < p_2 < \cdots < p_r \leq n$; $1 \leq q_1 < q_2 < \cdots < q_{n-r} \leq n$; $1 \leq r \leq n$; $p_1, p_2, \cdots, p_r, q_1, q_2, \cdots, q_{n-r}$, $r \in \mathbb{N}^+$). If we apply Gauss-Jordan Elimination to the matrix $A$, then the pivots can be expressed as a function of free variables.

$$\begin{aligned} x_{p_1} &= P_1(x_{q_1}, x_{q_2}, \cdots, x_{q_{n-r}}), \\ x_{p_2} &= P_2(x_{q_1}, x_{q_2}, \cdots, x_{q_{n-r}}), \\ &\vdots \\ x_{p_r} &= P_r(x_{q_1}, x_{q_2}, \cdots, x_{q_{n-r}}). \end{aligned}$$

where $P_1, P_2, \cdots, P_r$ are polynomial functions.

A series of $n$-dimensional vectors: $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{n-r}$ are constructed as the solutions to $A\mathbf{x} = \mathbf{0}$. $\forall$ $1 \leq i \leq n-r, 1 \leq j \leq n$ and $i, j \in \mathbb{N}^+$, the construction process of $\mathbf{x}_i$ is shown as following:

1. The $q_j$-th component of $\mathbf{x}_i$ is 1:
$$x_{q_j} = 1.$$
2. The $q_1, q_2, \cdots, q_{j-1}, q_{j+1}, \cdots, q_{n-r}$-th component of $\mathbf{x}_i$ are 0.
$$x_{q_1} = x_{q_2} = \cdots = x_{q_{j-1}} = x_{q_{j+1}} = \cdots = x_{q_{n-r}} = 0.$$
3. The $p_1, p_2, \cdots, p_r$-th component of $\mathbf{x}_i$ are determined as the polynomial functions above.

The special solution to $A\mathbf{x} = \mathbf{0}$ is a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{n-r}$:

$$\mathbf{x}_n = x_{q_1}\mathbf{x}_1 + x_{q_2}\mathbf{x}_2 + \cdots + x_{q_{n-r}}\mathbf{x}_{n-r},$$

where $x_{q_1}, x_{q_2}, \cdots, x_{q_{n-r}} \in \mathbb{R}$.

### 2.3.2 The Complete Solution to $A\mathbf{x} = \mathbf{b}$

$\forall$ $m \times n$ matrix $A$ where $m, n \in \mathbb{N}^+$ such that the equation $A\mathbf{x} = \boldsymbol{b}$ is solvable, let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Suppose that $x_{p_1}, x_{p_2}, \cdots, x_{p_r}$ are pivots; $x_{q_1}, x_{q_2}, \cdots, x_{q_{n-r}}$ are free variables ($1 \leq p_1 < p_2 < \cdots < p_r \leq n$; $1 \leq q_1 < q_2 < \cdots < q_{n-r} \leq n$; $1 \leq r \leq n$; $p_1, p_2, \cdots, p_r, q_1, q_2, \cdots, q_{n-r}$, $r \in \mathbb{N}^+$). If we apply Gauss-Jordan Elimination to the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$, then we can get the matrix $\begin{bmatrix} R & \mathbf{d} \end{bmatrix}$, where the matrix $R$ is the RREF (Reduced Row Echelon Form) of the matirx $A$. The pivots can be expressed as a function of free variables.

$$\begin{aligned} x_{p_1} &= d_1 + P_1(x_{q_1}, x_{q_2}, \cdots, x_{q_{n-r}}), \\ x_{p_2} &= d_2 + P_2(x_{q_1}, x_{q_2}, \cdots, x_{q_{n-r}}), \\ &\vdots \\ x_{p_r} &= d_r + P_r(x_{q_1}, x_{q_2}, \cdots, x_{q_{n-r}}). \end{aligned}$$

where $P_1, P_2, \cdots, P_r$ are polynomial functions.

The complete solution to $A\mathbf{x} = \mathbf{b}$ is divided into two parts: The particular solution $\mathbf{x}_p$ to $A\mathbf{x}_p = \mathbf{b}$ and the special soution $\mathbf{x}_n$ to $A\mathbf{x}_n = \mathbf{0}$.

1. **The particular solution $\mathbf{x}_p$ to $A\mathbf{x}_p = \mathbf{b}$**

   $\mathbf{x}_p$ is a $n-$dimensional vector. The construction process of $\mathbf{x}_p$ is shown as following:

   (a) The $q_1, q_2, \cdots, q_{n-r}$-th component of $\mathbf{x}_p$ are 0.

   $$x_{q_1} = x_{q_2} = \cdots = x_{q_{n-r}} = 0.$$

   (b) The $p_1, p_2, \cdots, p_r$-th component of $\mathbf{x}_p$ are determined as the polynomial functions above.

2. **The special soution $\mathbf{x}_n$ to $A\mathbf{x}_n = \mathbf{0}$.**

   $\mathbf{x}_n$ is a $n-$dimensional vector. The construction process of $\mathbf{x}_n$ is in the previous part.

Therefore, the complete solution to $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n.$$

### 2.3.3 Solutions Conditions of $A\mathbf{x} = \mathbf{b}$

Apply Gauss-Jordan Elimination into the argumented matrix $[A, \mathbf{b}]$ to get the matrix $[R, \mathbf{d}]$.

1. If there exists a row in the matirx $[R, \mathbf{d}]$ is

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & d \end{bmatrix}$$

   where $d \neq 0$, which means that all elements in this row are 0 except the element at the last column. Then the linear equations have no solutions.

2. If there are no such rows as above in the matirx $[R, \mathbf{d}]$, assume that the number of pivots in the matrix is $p$ and the number of unknowns is $n$

   - If $p < n$, then there are infinite solutions in the linear equations.
   - If $p = n$, then there is a unique solution in the linear equations.

In particular, if $\mathbf{b} = \mathbf{0}$, then the vector $\mathbf{0}$ is always a solution of the linear equations. Therefore, they can only have either a unique solution or infinite solutions.

## 2.4 Elimination = Factorization: $A = LU$

### 2.4.1 Elimination and $A = LU$

Suppose that $A$ is a $n \times n$ square matrix where $n \in \mathbb{N}^+$. If we apply Gauss-Jordan Elimination into the matrix $A$ to get its REF, and there are no any row switches in this process, then the matrix $A$ becomes the product of two special matrices:

$$A = LU,$$

where $L$ is a lower triangular matrix and $U$ is a upper triangular matrix. In addition, $U$ is the REF of the matrix $A$. Suppose that

$$E_k \cdots E_2 E_1 A = U,$$

where $k \in \mathbb{N}^+$ and $E_1, E_2, \cdots, E_k$ is a series of **eliminationBelow** operations in Gauss-Jordan Elimination. Therefore,

$$A = (E_k \cdots E_2 E_1)^{-1} U = (E_1^{-1} E_2^{-1} \cdots E_k^{-1}) U = LU.$$

### 2.4.2 $A = LU$ and $A = LDU$

Suppose that $A$ is a $n \times n$ square matrix where $n \in \mathbb{N}^+$ and $A$ can be factored into $A = LU$. The matrix $U$ can be splited as the product of two special matrices. Suppose that the matrix $U$ is

$$U = \begin{bmatrix} p_1 & b_{12} & \cdots & b_{1k_1} & b_{1(1+k_1)} & b_{1(2+k_1)} & \cdots & b_{1k_2} & \cdots & b_{1k_{l-1}} & b_{1(1+k_{l-1})} & \cdots & b_{1k_l} \\ 0 & 0 & \cdots & 0 & p_2 & b_{2(2+k_1)} & \cdots & b_{2k_2} & \cdots & b_{2k_{l-1}} & b_{2(1+k_{l-1})} & \cdots & b_{2k_l} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & p_l & b_{l(1+k_{l-1})} & \cdots & b_{lk_l} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

where $1 < k_1 < k_2 < \cdots < k_l = n$ and $k_1, k_2, \cdots, k_l \in \mathbb{N}^+$. The matrix $U$ can be splited as the product of two special matrices: $D$ and $U$. Both the matrix $D$ and the latter matrix $U$ are $n \times n$ matrices. The matrix $D$ is shown as follows:

$$D=\begin{bmatrix} p_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_l & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The latter matrix $U$ is shown as follows:

$$U=\begin{bmatrix} 1 & \frac{b_{12}}{p_1} & \cdots & \frac{b_{1k_1}}{p_1} & \frac{b_{1(1+k_1)}}{p_1} & \frac{b_{1(2+k_1)}}{p_1} & \cdots & \frac{b_{1k_2}}{p_1} & \cdots & \frac{b_{1k_{l-1}}}{p_1} & \frac{b_{1(1+k_{l-1})}}{p_1} & \cdots & \frac{b_{1k_l}}{p_1} \\ 0 & 0 & \cdots & 0 & 1 & \frac{b_{2(2+k_1)}}{p_2} & \cdots & \frac{b_{2k_2}}{p_2} & \cdots & \frac{b_{2k_{l-1}}}{p_2} & \frac{b_{2(1+k_{l-1})}}{p_2} & \cdots & \frac{b_{2k_l}}{p_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & \frac{b_{l(1+k_{l-1})}}{p_l} & \cdots & \frac{b_{lk_l}}{p_l} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Thus, the matrix $A$ becomes the product of three special matrices:

$$A = LDU.$$

If $S = S^{\mathrm{T}}$ is factored into $LDU$ with no row switches, then $U$ is exactly $L^{\mathrm{T}}$:

$$S = LDL^{\mathrm{T}}.$$

### 2.4.3 $PA = LU$

Suppose that $A$ is a $n \times n$ square matrix where $n \in \mathbb{N}^+$. If we apply Gauss-Jordan Elimination into the matrix $A$ to get its REF, and there are row switches in this process, then the row switches can be done in advance. Their product $P$ puts the rows of $A$ in the right order, so that no exchanges are needed for $PA$. Then

$$PA = LU.$$