# Chapter 1 Section B Exercises

**1.** Prove that $-(-v) = v$ for every $v \in V$.

***Proof.*** For every $v \in V$, we have

$$(-v) + [-(-v)] = 1 \cdot (-v) + (-1) \cdot (-v) = [1 + (-1)] \cdot (-v) = 0 \cdot (-v) = 0,$$

so $-(-v)$ is the additive inverse of $-v$. Because $(-v) + v = 0$, $v$ is also the additive inverse of $-v$. Since every element in a vector space has a unique additive inverse, we have $-(-v) = v$ for every $v \in V$.

$\blacksquare$

**2.** Suppose $a \in \mathbf{F}, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

***Proof.*** Suppose that $a \neq 0$ and $v \neq 0$, then we have

$$v + (a-1)v = v + av + (-1)v = v + 0 - v = v - v = 0.$$

Thus, $(a-1)v$ is the additive inverse of $v$. Because $v - v = 0$, $-v$ is also the additive inverse of $v$. However,

$$\left.\begin{array}{r} a \neq 0 \quad \Rightarrow \quad a - 1 \neq -1 \\ v \neq 0 \end{array}\right\} \quad \Rightarrow \quad (a-1)v \neq -v.$$

Therefore, there are two different additive inverses of $v$, which contradicts the statement that every element in a vector space has a unique additive inverse. Hence, $a = 0$ or $v = 0$.

$\blacksquare$

**3.** Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

***Proof.*** We have

$$\begin{aligned} v + 3x = w \quad &\Rightarrow \quad v + 3x - w = v - w + 3x = 0 \\ &\Rightarrow \quad \frac{1}{3}(v - w + 3x) = \frac{1}{3}v + \frac{1}{3}w + x = \frac{1}{3} \cdot 0 = 0. \end{aligned}$$

Therefore, $x$ is the additive inverse of $v/3 + w/3$. Since every element in a vector space has a unique additive inverse, there exists a unique $x \in V$ such that $v + 3x = w$.

■

**4.** The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in Definition 1.10. Which one?

***Solution.*** The third item: Additive identity because there does not exist any element in an empty set.

**5.** Show that in the definition of a vector space, the additive inverse condition can be replaced with the condition that

$$0 \cdot v = 0 \text{ for all } v \in V.$$

Here the $0$ on the left side is the number $0$, and the $0$ on the right side is the additive identity of $V$. (The phrase "a condition can be replaced" in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.)

***Proof.*** For every $v \in V$, we have $1 \cdot v = v$ and

$$v + (-1) \cdot v = 1 \cdot v + (-1) \cdot v = [1 + (-1)] \cdot v = 0 \cdot v = 0.$$

Therefore, for every $v \in V$, there exists $w = (-1) \cdot v \in V$ such that $v + w = 0$.

■

**6.** Let $\infty$ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t(\infty) = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0. \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0. \end{cases}$$

$$t + \infty = \infty + t = \infty, \qquad t + (-\infty) = (-\infty) + t = -\infty,$$

$$\infty + \infty = \infty, \qquad (-\infty) + (-\infty) = (-\infty) = -\infty, \qquad \infty + (-\infty) = 0.$$

Is $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbf{R}$? Explain.

***Solution.*** $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ is a vector space over $\mathbf{R}$. The proof is shown as follows:

***Proof.*** Let $V = \mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ and for all $u, v, w \in \mathbf{R}$. We have

- **Addition**

$$u + v \in V, \;\; u + \infty = \infty + u = \infty \in V, \;\; u + (-\infty) = (-\infty) + u = -\infty \in V,$$
$$\infty + \infty = \infty \in V, \;\; (-\infty) + (-\infty) = (-\infty) = -\infty \in V, \;\; \infty + (-\infty) = 0 \in V.$$

- **Scalar multiplication**

$$uv \in V, \quad u(\infty) = \begin{cases} -\infty \in V & \text{if } u < 0, \\ 0 \in V & \text{if } u = 0, \\ \infty \in V & \text{if } u > 0. \end{cases} \quad u(-\infty) = \begin{cases} \infty \in V & \text{if } u < 0, \\ 0 \in V & \text{if } u = 0, \\ -\infty \in V & \text{if } u > 0. \end{cases}$$

- **Commutativity**

$$u + v = v + u, \qquad u + \infty = \infty + u, \qquad u + (-\infty) = (-\infty) + u,$$
$$\infty + (-\infty) = (-\infty) + \infty = 0.$$

- **Associativity**

$$(u + v) + w = u + (v + w),$$
$$\left.\begin{aligned} (u+v) + \infty &= \infty \\ u + (v + \infty) = u + \infty &= \infty \end{aligned}\right\} \quad \Rightarrow \quad (u+v) + \infty = u + (v + \infty),$$
$$\left.\begin{aligned} (u+\infty) + v = \infty + v &= \infty \\ u + (\infty + v) = u + \infty &= \infty \end{aligned}\right\} \quad \Rightarrow \quad (u+\infty) + v = u + (\infty + v),$$
$$\left.\begin{aligned} (\infty+u) + v = \infty + v &= \infty \\ \infty + (u + v) &= \infty \end{aligned}\right\} \quad \Rightarrow \quad (\infty+u) + v = \infty + (u + v),$$
$$\left.\begin{aligned} (u+v) + (-\infty) &= -\infty \\ u + [v + (-\infty)] = u + (-\infty) &= -\infty \end{aligned}\right\} \Rightarrow (u+v) + (-\infty) = u + [v + (-\infty)],$$
$$\left.\begin{aligned} [u + (-\infty)] + v = -\infty + v &= -\infty \\ u + [(-\infty) + v] = u + (-\infty) &= -\infty \end{aligned}\right\} \Rightarrow [u + (-\infty)] + v = u + [(-\infty) + v],$$

$$\left.\begin{aligned}[(-\infty)+u]+v=-\infty+v=-\infty\\(-\infty)+(u+v)=-\infty\end{aligned}\right\}\ \Rightarrow\ [(-\infty)+u]+v=(-\infty)+(u+v),$$

$$\left.\begin{aligned}(u+\infty)+\infty=\infty+\infty=\infty\\u+(\infty+\infty)=u+\infty=\infty\end{aligned}\right\}\quad \Rightarrow\quad (u+\infty)+\infty=u+(\infty+\infty),$$

$$\left.\begin{aligned}(\infty+u)+\infty=\infty+\infty=\infty\\\infty+(u+\infty)=\infty+\infty=\infty\end{aligned}\right\}\quad \Rightarrow\quad (\infty+u)+\infty=\infty+(u+\infty),$$

$$\left.\begin{aligned}(\infty+\infty)+u=\infty+u=\infty\\\infty+(\infty+u)=\infty+\infty=\infty\end{aligned}\right\}\quad \Rightarrow\quad (\infty+\infty)+u=\infty+(\infty+u),$$

- **Additive identity**

$$u+0=u,\qquad \infty+0=\infty,\qquad -\infty+0=-\infty.$$

- **Additive inverse**

$$u+(-u)=0,\qquad \infty+(-\infty)=0,\qquad -\infty+\infty=0.$$

- **Multiplicative identity**

$$1\cdot u=u,\qquad 1\cdot\infty=\infty,\qquad 1\cdot(-\infty)=-\infty.$$

■