Chapter 1 Section A Exercises

1. Suppose a and b are real numbers, not both 0. Find real numbers c and d such that

$$\frac{1}{a+bi} = c + di.$$

Solution. Because $a+bi$ is the denominator, a and b can not be both 0, $a^2+b^2 \neq 0$. Then

$$\frac{1}{a+bi} = c+di \quad \Rightarrow \quad (a+bi)(c+di) = (ac-bd)+(ad+bc)i = 1 = 1+0i.$$

Therefore, we can get an equations set

$$\begin{cases} ac - bd = 1, \\ ad + bc = 0. \end{cases}$$

We will get the solution after solving this equations set:

$$\begin{cases} c = \dfrac{a}{a^2+b^2}, \\ d = \dfrac{-b}{a^2+b^2}. \end{cases}$$

Thus, the values of real numbers c and d are $c = a/(a^2+b^2), d = -b/(a^2+b^2)$.

2. Show that

$$\frac{-1+\sqrt{3}i}{2}.$$

is a cube root of 1 (meaning that its cube equals 1).

Proof. We have

$$\begin{aligned} \left(\frac{-1+\sqrt{3}i}{2}\right)^3 &= \left(\frac{-1+\sqrt{3}i}{2}\right)\left(\frac{-1+\sqrt{3}i}{2}\right)\left(\frac{-1+\sqrt{3}i}{2}\right) \\ &= \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) \\ &= \left[\left(-\frac{1}{2}\right)+\left(-\frac{\sqrt{3}}{2}\right)i\right]\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) \\ &= 1+0i \\ &= 1. \end{aligned}$$

■

3. Find two distinct square roots of i.

Solution. Suppose that the square root of i is $x \in \mathbf{C}$. Let $x = a + bi$, where $a, b \in \mathbf{R}$. Then

$$x^2 = (a + bi)(a + bi) = (a^2 - b^2) + 2abi = i = 0 + 1i.$$

Therefore, we can get an equations set

$$\begin{cases} a^2 - b^2 = 0, \\ 2ab \qquad = 1. \end{cases}$$

We will get the solution after solving this equations set:

$$\begin{cases} a_1 = b_1 = \ -\dfrac{\sqrt{2}}{2}, \\ a_2 = b_2 = \dfrac{\sqrt{2}}{2}. \end{cases}$$

Thus, two distinct square roots of i is

$$x_1 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \qquad x_2 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i.$$

4. Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbf{C}$.

Proof. For all $\alpha, \beta \in \mathbf{C}$, let $\alpha = a + bi, \beta = c + di$, where $a, b, c, d \in \mathbf{R}$.

$$\begin{aligned} \alpha + \beta &= (a + bi) + (c + di) \\ &= (a + c) + (b + d)i \\ &= (c + a) + (d + b)i \\ &= (c + di) + (a + bi) \\ &= \beta + \alpha. \end{aligned}$$

■

5. Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

Proof. For all $\alpha, \beta, \lambda \in \mathbf{C}$, let $\alpha = a + bi, \beta = c + di, \lambda = e + fi$, where $a, b, c, d, e, f \in \mathbf{R}$.

$$\begin{aligned} (\alpha + \beta) + \lambda &= [(a + bi) + (c + di)] + (e + fi) \\ &= [(a + c) + (b + d)i] + (e + fi) \\ &= (a + c + e) + (b + d + f)i \end{aligned}$$

$$
\begin{aligned}
&= (a+bi) + [(c+e) + (d+f)i] \\
&= (a+bi) + [(c+di) + (e+fi)] \\
&= \alpha + (\beta + \lambda).
\end{aligned}
$$

■

6. Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

Proof. For all $\alpha, \beta, \lambda \in \mathbf{C}$, let $\alpha = a + bi, \beta = c + di, \lambda = e + fi$, where $a, b, c, d, e, f \in \mathbf{R}$.

$$
\begin{aligned}
(\alpha\beta)\lambda &= [(a+bi)(c+di)](e+fi) \\
&= [(ac-bd) + (ad+bc)i](e+fi) \\
&= (ace - bde - adf - bcf) + (acf - bdf + ade + bce)i \\
&= (a+bi)[(ce-df) + (cf+de)i] \\
&= (a+bi)[(c+di)(e+fi)] \\
&= \alpha(\beta\lambda).
\end{aligned}
$$

■

7. Show that for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$.

Proof. For all $\alpha \in \mathbf{C}$, let $\alpha = a + bi$, where $a, b \in \mathbf{R}$. Let $c = -a, d = -b, \beta \in \mathbf{C}$ and $\beta = c + di$, then

$$\alpha + \beta = (a+bi) + (c+di) = (a+c) + (b+d)i = (a-a) + (b-b)i = 0 + 0i = 0.$$

Thus, for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$.

■

8. Show that for every $\alpha \in \mathbf{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha\beta = 1$.

Proof. For all $\alpha \in \mathbf{C}$ with $\alpha \neq 0$, let $\alpha = a + bi$, where $a, b \in \mathbf{R}$. Because $\alpha \neq 0$, $a^2 + b^2 \neq 0$. Let $c = a/(a^2+b^2), d = -b/(a^2+b^2), \beta \in \mathbf{C}$ and $\beta = c + di$, then

$$\alpha\beta = (a+bi)\left(\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i\right) = \left(\frac{a^2+b^2}{a^2+b^2} + \frac{-ab+ab}{a^2+b^2}i\right) = (1+0i) = 1.$$

Thus, for every $\alpha \in \mathbf{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha\beta = 1$.

$\blacksquare$

9. Show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbf{C}$.

Proof. For all $\lambda, \alpha, \beta \in \mathbf{C}$, we have

$$\begin{aligned}
\lambda(\alpha + \beta) &= (e + fi)[(a + bi) + (c + di)] \\
&= (e + fi)[(a + c) + (b + d)i] \\
&= (ea + ec - fb - fd) + (eb + ed + fa + fc)i \\
&= [(ea - fb) + (eb + fa)i] + [(ec - fd) + (ed + fc)i] \\
&= (e + fi)(a + bi) + (e + fi)(c + di) \\
&= \lambda\alpha + \lambda\beta.
\end{aligned}$$

$\blacksquare$

10. Find $x \in \mathbf{R}^4$ such that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8).$$

Solution. We have

$$\begin{aligned}
2x &= (5, 9, -6, 8) - (4, -3, 1, 7) \\
&= (5 - 4, 9 + 3, -6 - 1, 8 - 7) \\
&= (1, 12, -7, 1) \\
x &= \frac{1}{2}(1, 12, -7, 1) \\
&= \left(\frac{1}{2}, 6, -\frac{7}{2}, \frac{1}{2}\right).
\end{aligned}$$

11. Explain why there does not exist $\lambda \in \mathbf{C}$ such that

$$\lambda(2 - i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i).$$

Proof. Suppose that there exists $\lambda \in \mathbf{C}$ that satisfies this requirement. Let $\lambda = a + bi$, where $a, b \in \mathbf{R}$. Then

$$\lambda(2 - i, 5 + 4i, -6 + 7i)$$

$$
\begin{aligned}
&= (a+bi)(2-i, 5+4i, -6+7i) \\
&= [(a+bi)(2-i), (a+bi)(5+4i), (a+bi)(-6+7i)] \\
&= [(2a+b)+(-a+2b)i, (5a-4b)+(4a+5b)i, (-6a-7b)+(7a-6b)i] \\
&= (12-5i, 7+22i, -32-9i).
\end{aligned}
$$

Therefore, we can get an equations set

$$
\begin{cases}
2a \quad + b \ = 12, \\
-a \quad + 2b = -5, \\
5a \quad - 4b = 7, \\
4a \quad + 5b = 22, \\
-6a - 7b = -32, \\
7a \quad - 6b = -9.
\end{cases}
$$

However, we will know that there is no solution for this equations set after we solve it. Thus, there does not exist $\lambda \in \mathbf{C}$ such that

$$
\lambda(2-i, 5+4i, -6+7i) = (12-5i, 7+22i, -32-9i).
$$

■

12. Show that $(x+y)+z = x+(y+z)$ for all $x, y, z \in \mathbf{F}^n$.

Proof. Let $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n), z = (z_1, \cdots, z_n)$, where $x_j, y_j, z_j \in \mathbf{F}$ for $j = 1, \cdots, n$. Then

$$
\begin{aligned}
(x+y)+z &= [(x_1, \cdots, x_n) + (y_1, \cdots, y_n)] + (z_1, \cdots, z_n) \\
&= (x_1+y_1, \cdots, x_n+y_n) + (z_1, \cdots, z_n) \\
&= (x_1+y_1+z_1, \cdots, x_n+y_n+z_n) \\
&= (x_1, \cdots, x_n) + (y_1+z_1, \cdots, y_n+z_n) \\
&= (x_1, \cdots, x_n) + [(y_1, \cdots, y_n) + (z_1, \cdots, z_n)] \\
&= x+(y+z).
\end{aligned}
$$

■

13. Show that $(ab)x = a(bx)$ for all $x \in \mathbf{F}^n$ and all $a, b \in \mathbf{F}$.

Proof. Let $x = (x_1, \cdots, x_n)$, where $a, b, x_j \in \mathbf{F}$ for $j = 1, \cdots, n$. Then

$$\begin{aligned}
(ab)x &= (ab)(x_1, \cdots, x_n) \\
&= (abx_1, \cdots, abx_n) \\
&= a(bx_1, \cdots, bx_n) \\
&= a[b(x_1, \cdots, x_n)] \\
&= a(bx).
\end{aligned}$$

■

14. Show that $1 \cdot x = x$ for all $x \in \mathbf{F}^n$.

Proof. Let $x = (x_1, \cdots, x_n)$, where $x_j \in \mathbf{F}$ for $j = 1, \cdots, n$. Then

$$\begin{aligned}
1 \cdot x &= 1 \cdot (x_1, \cdots, x_n) \\
&= (1 \cdot x_1, \cdots, 1 \cdot x_n) \\
&= (x_1, \cdots, x_n) \\
&= x.
\end{aligned}$$

■

15. Show that $\lambda(x + y) = \lambda x + \lambda y$ for all $\lambda \in \mathbf{F}$ and all $x, y \in \mathbf{F}^n$.

Proof. Let $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n)$, where $\lambda, x_j, y_j \in \mathbf{F}$ for $j = 1, \cdots, n$. Then

$$\begin{aligned}
\lambda(x + y) &= \lambda[(x_1, \cdots, x_n) + (y_1, \cdots, y_n)] \\
&= \lambda(x_1 + y_1, \cdots, x_n + y_n) \\
&= [\lambda(x_1 + y_1), \cdots, \lambda(x_n + y_n)] \\
&= (\lambda x_1 + \lambda y_1, \cdots, \lambda x_n + \lambda y_n) \\
&= (\lambda x_1, \cdots, \lambda x_n) + (\lambda y_1, \cdots, \lambda y_n) \\
&= \lambda(x_1, \cdots, x_n) + \lambda(y_1, \cdots, y_n) \\
&= \lambda x + \lambda y.
\end{aligned}$$

■

16. Show that $(a+b)x = ax + bx$ for all $a, b \in \mathbf{F}$ and all $x \in \mathbf{F}^n$.

Proof. Let $x = (x_1, \cdots, x_n)$, where $a, b, x_j \in \mathbf{F}$ for $j = 1, \cdots, n$. Then

$$\begin{aligned}(a+b)x &= (a+b)(x_1, \cdots, x_n) \\ &= [(a+b)x_1, \cdots, (a+b)x_n] \\ &= (ax_1 + bx_1, \cdots, ax_n + bx_n) \\ &= (ax_1, \cdots, ax_n) + (bx_1, \cdots, bx_n) \\ &= a(x_1, \cdots, x_n) + b(x_1, \cdots, x_n) \\ &= ax + bx.\end{aligned}$$

■