Chapter 1

Vector Spaces

1.A $\mathbf{R}^n$ and $\mathbf{C}^n$

Definition 1.1 (Complex numbers)

- A **complex number** is an ordered pait (a, b), where $a, b \in \mathbf{R}$, but we will write this as $a + bi$.
- The set of all complex numbers is denoted by **C**:

$$\mathbf{C} = \{a + bi : a, b \in \mathbf{R}\}.$$

- **Addition and multiplication** on **C** are defined by

$$\begin{aligned}(a + bi) + (c + di) &= (a + c) + (b + d)i,\\ (a + bi)(c + di) &= (ac - bd) + (ad + bc)i;\end{aligned}$$

  here $a, b, c, d \in \mathbf{R}$.

If $a \in \mathbf{R}$, we identity $a + 0i$ with the real number a. Thus we can think of **R** as a subset of **C**. We also usually write $0 + bi$ as just bi, and we usually write $0 + 1i$ as just i.

Theorem 1.1 (Properties of complex arithmetic)

- **Commutativity**

  $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbf{C}$.

- **Associativity**

  $(\alpha+\beta)+\lambda=\alpha+(\beta+\lambda)$ and $(\alpha\beta)\lambda=\alpha(\beta\lambda)$ for all $\alpha,\beta,\lambda\in\mathbf{C}$.

- **Identities**

  $\lambda+0=\lambda$ and $\lambda\cdot 1=\lambda$ for all $\lambda\in\mathbf{C}$.

- **Additive inverse**

  For every $\alpha\in\mathbf{C}$, there exists a unique $\beta\in\mathbf{C}$ such that $\alpha+\beta=0$.

- **Multiplicative inverse**

  For every $\alpha\in\mathbf{C}$ with $\alpha\neq 0$, there exists a unique $\beta\in\mathbf{C}$ such that $\alpha\beta=1$.

- **Distributive property**

  $\lambda(\alpha+\beta)=\lambda\alpha+\lambda\beta$ for all $\lambda,\alpha,\beta\in\mathbf{C}$.

Definition 1.2 ($-\alpha$, subtraction, $1/\alpha$, division) Let $\alpha,\beta\in\mathbf{C}$.

- Let $-\alpha$ denote the additive inverse of α. Thus $-\alpha$ is the unique complex number such that
$$\alpha+(-\alpha)=0.$$

- **Subtraction** on **C** is defined by
$$\beta-\alpha=\beta+(-\alpha).$$

- For $\alpha\neq 0$, let $1/\alpha$ denote the multiplicative inverse of α. Thus $1/\alpha$ is the unique complex number such that
$$\alpha\cdot\frac{1}{\alpha}=1.$$

- **Division** on **C** is defined by
$$\frac{\beta}{\alpha}=\beta\cdot\frac{1}{\alpha}.$$

Notation 1.1 (F) **F** stands for either **R** or **C**. Elements of **F** are called **scalars**. For $\alpha\in\mathbf{F}$ and m a positive integer, we define α^m to denote the product of α with itself m times.

Corollary 1.1 For all $\alpha,\beta\in\mathbf{F}$ and all positive integers m,n, there is $(\alpha^m)^n=\alpha^{mn}$ and $(\alpha\beta)^m=\alpha^m\beta^m$.

Definition 1.3 (List, length) Suppose n is a nonnegative integer. A **list** of **length** n is an ordered collection of n elements (which might be numbers, other lists, or more abstract entities) separated by commas and surrounded by parentheses. A list of length n looks like this:

$$(x_1, \cdots, x_n).$$

Two lists are equal if and only if they have the same length and the same elements in the same order. A list of length 0 looks like this: (). A list of length n can also be called as n-**tuple**.

Definition 1.4 ($\mathbf{F}^n$) $\mathbf{F}^n$ is the set of all lists of length n of elements of $\mathbf{F}$:

$$\mathbf{F}^n = \{(x_1, \cdots, x_n) : x_j \in \mathbf{F} \text{ for } j = 1, \cdots, n\}.$$

For $(x_1, \cdots, x_n) \in \mathbf{F}^n$ and $j \in \{1, \cdots, n\}$, we say that x_j is thej^{th} **coordinate** of $(x_1, \cdots, x_n)$.

Definition 1.5 (Addition in $\mathbf{F}^n$) **Addition** in $\mathbf{F}^n$ is defined by adding corresponding coordinates:

$$(x_1, \cdots, x_n) + (y_1, \cdots, y_n) = (x_1 + y_1, \cdots, x_n + y_n).$$

Theorem 1.2 (Commutativity of addition in $\mathbf{F}^n$) If $x, y \in \mathbf{F}^n$, then $x + y = y + x$.

Definition 1.6 (0) Let 0 denote the list of length n whose coordinates are all 0:

$$0 = (0, \cdots, 0).$$

Definition 1.7 (Additive inverse in $\mathbf{F}^n$) For $x \in \mathbf{F}^n$, the **additive inverse** of x, denoted $-x$, is the vector $-x \in \mathbf{F}^n$ such that

$$x + (-x) = 0.$$

In other words, if $x = (x_1, \cdots, x_n)$, then $-x = (-x_1, \cdots, -x_n)$.

Definition 1.8 (Scalar multiplication in $\mathbf{F}^n$) The **product** of a number λ and a vector in $\mathbf{F}^n$ is computed by multiplying each coordinate of the vector by λ:

$$\lambda(x_1, \cdots, x_n) = (\lambda x_1, \cdots, \lambda x_n);$$

here $\lambda \in \mathbf{F}^n$ and $(x_1, \cdots, x_n) \in \mathbf{F}^n$.

1.B Definition of Vector Space

Definition 1.9 (Addition, scalar multiplication)

- An **addition** on a set V is a function that assigns an element $u + v \in V$ to eaach pair of elements $u, v \in V$.
- A **scalar multiplication** on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in V$ to each $\lambda \in$ **F** and each $v \in V$.

Definition 1.10 (Vector space) A **vector space** is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

- **Commutativity**

  $u + v = v + u$ for all $u, v \in V$.

- **Associativity**

  $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and all $a, b \in$ **F**.

- **Additive identity**

  There exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$.

- **Additive inverse**

  For every $v \in V$, there exists $w \in V$ such that $v + w = 0$.

- **Multiplicative identity**

  $1 \cdot v = v$ for all $v \in V$.

- **Distributive identity**

  $a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in$ **F** and all $u, v \in V$.

Definition 1.11 (Vector, point) Elements of a vector space are called **vectors** or **points**.

Definition 1.12 (Real vector space, complex vector space)

- A vector space over **R** is called a **real vector space**.
- A vector space over **C** is called a **complex vector space**.

Notation 1.2 ($\mathbf{F}^S$)

- If S is a set, then $\mathbf{F}^S$ denotes the set of functions from S to $\mathbf{F}$.
- For $f, g \in \mathbf{F}^S$, the **sum** $f + g \in \mathbf{F}^S$ is the function defined by

  $$(f + g)(x) = f(x) + g(x)$$

  for all $x \in S$.
- For $\lambda \in \mathbf{F}$ and $f \in \mathbf{F}^S$, the **product** $\lambda f \in \mathbf{F}^S$ is the function defined by

  $$(\lambda f)(x) = \lambda f(x)$$

  for all $x \in S$.

Theorem 1.3 (Unique additive identity) A vector space has a unique additive indentity.

Theorem 1.4 (Unique additive inverse) Every element in a vector space has a unique additive inverse.

Notation 1.3 ($-v, w - v$) Let $v, w \in V$. Then

- $-v$ denotes the additive inverse of v.
- $w - v$ is defined to be $w + (-v)$.

Notation 1.4 (V) V denotes a vector space over $\mathbf{F}$.

Theorem 1.5 (The number 0 times a vector) $0 \cdot v = 0$ for every $v \in V$.

Theorem 1.6 (A number times the vector 0) $a \cdot 0 = 0$ for every $a \in \mathbf{F}$.

Theorem 1.7 (The number -1 times a vector) $(-1) \cdot v = -v$ for every $v \in V$.

1.C Subspace

Definition 1.13 (Subspace) A subset U of V is called a **subspace** of V if U is also a vector space (using the same addition and scalar multiplication as on V).

Theorem 1.8 (Conditions for a subspace) A subset U of V is a subspace of V if and only if U satisfies the following three conditions:

- **Additive identity**

$$0 \in U.$$

- **Closed under addition**

$$u, w \in U \text{ implies } u + w \in U.$$

- **Closed under scalar multiplication**

$$a \in \mathbf{F} \text{ and } u \in U \text{ implies } au \in U.$$

Definition 1.14 (Sum of subsets) Suppose $U_1, \cdots, U_m$ are subsets of V. The **sum** of $U_1, \cdots, U_m$ denoted $U_1 + \cdots + U_m$, is the set of all possible sums of elements of $U_1, \cdots, U_m$. More precisely,

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_1 \in U_1, \cdots, u_m \in U_m\}.$$

Theorem 1.9 (Sum of subspaces is the smallest containing subspace) Suppose $U_1, \cdots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \cdots, U_m$.

Definition 1.15 (Direct sum) Suppose $U_1, \cdots, U_m$ are subspaces of V.

- The sum $U_1 + \cdots + U_m$ is called a **direct sum** if each element of $U_1 + \cdots + U_m$ can be written in only one way as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j.
- If $U_1 + \cdots + U_m$ is a direct sum, then $U_1 \oplus \cdots \oplus U_m$ denotes $U_1 + \cdots + U_m$, with the $\oplus$ notation serving as an indication that this is a direct sum.

Theorem 1.10 (Condition for a direct sum) Suppose $U_1, \cdots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j, is by taking each u_j equal to 0.

Theorem 1.11 (Direct sum of two subspaces) Suppose U and W are subspaces of V. Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.