

Chapter 1 Exercises

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercises are understood to be real.

1. If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

Proof. Let $r = p_1/q_1$, where p_1, q_1 are integers and $p_1 \neq 0, q_1 \neq 0$. Suppose that $r + x$ and rx are rational. Let $r + x = p_2/q_2$, where p_2, q_2 are integers and $q_2 \neq 0$. We have

$$x = (r + x) - r = \frac{p_2}{q_2} - \frac{p_1}{q_1} = \frac{p_2q_1 - p_1q_2}{q_1q_2},$$

so x is rational, which is contrary to the condition that x is irrational. Therefore, $r + x$ is irrational. Suppose that $rx = p_3/q_3$, where p_3, q_3 are integers and $q_3 \neq 0$. With $r \neq 0$, we have

$$x = \frac{rx}{r} = \left(\frac{p_3}{q_3}\right) / \left(\frac{p_1}{q_1}\right) = \frac{p_3q_1}{p_1q_3},$$

so x is rational, which is contrary to the condition that x is irrational. Therefore, rx is irrational. ■

2. Prove that there is no rational number whose square is 12.

Proof. Suppose that there exists a rational number r whose square is 12. Let $r = p/q$, where p, q are integers with no common factors which are greater than 1 and $q \neq 0$. Then we have

$$r^2 = \frac{p^2}{q^2} = 12 \quad \Rightarrow \quad p^2 = 12q^2,$$

so p is even and has a factor 2. Let $p = 2m$, where m is an integer. We have

$$p^2 = (2m)^2 = 4m^2 = 12q^2 \quad \Rightarrow \quad m^2 = 3q^2,$$

so m has a factor 3. Let $m = 3n$, where n is an integer. We have

$$m^2 = (3n)^2 = 9n^2 = 3q^2 \quad \Rightarrow \quad 3n^2 = q^2,$$

so q has also a factor 3. However, m is a factor of p and m has a factor 3, so p has a factor 3. Therefore, 3 is a common factor which is greater than 1 of p and q , which is contrary to the condition that p, q are integers with no common factors which are greater than 1. Thus, there is no rational number whose square is 12.

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3. Prove the following propositions:

(a) If $x \neq 0$ and $xy = xz$ then $y = z$.

(b) If $x \neq 0$ and $xy = x$ then $y = 1$.

(c) If $x \neq 0$ and $xy = 1$ then $y = 1/x$.

(d) If $x \neq 0$ then $1/(1/x) = x$.

Proof.

(a)

$$y \stackrel{(M4)}{=} 1 \cdot y \stackrel{(M5)}{=} (x \cdot \frac{1}{x}) \cdot y \stackrel{(M2)}{=} xy \cdot \frac{1}{x} = xz \cdot \frac{1}{x} \stackrel{(M2)}{=} (x \cdot \frac{1}{x}) \cdot z \stackrel{(M5)}{=} 1 \cdot z \stackrel{(M4)}{=} z.$$

(b) With the (a) statement, let $z = 1$,

$$xy = xz = x \cdot 1 \stackrel{(M2)}{=} 1 \cdot x \stackrel{(M4)}{=} x \stackrel{(a)}{\implies} y = z = 1.$$

(c) With the (a) statement, let $z = 1/x$,

$$xy = xz = x \cdot \frac{1}{x} \stackrel{(M5)}{=} 1 \stackrel{(a)}{\implies} y = z = \frac{1}{x}.$$

(d)

$$x \cdot \frac{1}{x} \left\{ \begin{array}{c} \stackrel{(M2)}{=} \frac{1}{x} \cdot x \\ \stackrel{(M5)}{=} 1 \end{array} \right\} \implies \frac{1}{x} \cdot x = 1 \stackrel{(M5)}{\implies} \frac{1}{\frac{1}{x}} = x.$$

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4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E . Prove that $\alpha \leq \beta$.

Proof. Suppose that $\alpha > \beta$, because E is a nonempty set and α is a lower bound of E , there exists an element $a \in E$ such that $a \geq \alpha$. Therefore, we have $a \geq \alpha > \beta$. However, β is an upper bound of E , so every element of E should be no more than β , which means that $a \leq \beta$. This is contrary to the condition that $a \geq \alpha > \beta$. Therefore, $\alpha \leq \beta$.

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