Chapter 1

The Real and Complex Number Systems

1.1 Sets

Definition 1.1 (Set, Member, Empty Set, Nonempty Set, Subset, Proper Subset, Equal Sets)

- If A is any **set** (whose elements may be numbers or any other objects), we write $x \in A$ to indicate that x is a member (or an element) of A. If x is not a member of A, we write: $x \notin A$.
- The set which contains no element will be called the **empty set**. If a set has at least one element, it is called **nonempty**.
- If A and B are sets, and if every element of A is an element of B, we say that A is a subset of B, and write $A \subseteq B$, or $B \supseteq A$. If, in addition, there is an element of B which is not in A, then A is said to be a **proper subset** of B.
- If $A \subseteq B$ and $B \subseteq A$, we write $A = B$. Otherwise $A \neq B$.

Definition 1.2 The set of all rational numbers will be denoted by Q.

1.2 Ordered Sets

Definition 1.3 Let S be a set. An **order** on S is a realtion, denoted by $<$, with the following two properties:

(i) If $x \in S$ and $y \in S$ then only one of the statements

$$x < y, \qquad x = y, \qquad y < x$$

is true.

(ii) If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

Definition 1.4 An **ordered set** is a set S in which an order is defined.

Definition 1.5 Suoopse S is an ordered set, and $E \subseteq S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is **bounded above**, and call β an **upper bound** of E. If there exists a $\gamma \in S$ such that $x \geq \gamma$ for every $x \in E$, we say that E is **bounded below**, and call γ an **lower bound** of E.

Definition 1.6 Suppose S is an ordered set, $E \subseteq S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

(i) α is an upper bound of E.

(ii) If $\gamma < \alpha$ then γ is not an upper bound of E.

Then α is called the **least upper bound** of E or the **supremum** of E, and we write

$$\alpha = \sup E.$$

Suppose there exists an $\beta \in S$ with the following properties:

(i) β is an lower bound of E.

(ii) If $\gamma > \beta$ then γ is not an lower bound of E.

Then α is called the **greatest lower bound** of E or the **infimum** of E, and we write

$$\beta = \inf E.$$

Definition 1.7 An **ordered set** S is said to have the **least-upper-bound property** if the following is true: If $E \subseteq S$, E is not empty, and E is bounded above, then $\sup E$ exists in S.

Theorem 1.1 Suppose S is an ordered set with the least-upper-bound property, $B \subseteq S$, B is not empty, and B is bounded below. Let L be the set of all lower bounds of B. Then

$$\alpha = \sup L$$

exists in S, and $\alpha = \inf B$. In particular, $\inf B$ exists in S.

1.3 Fields

Definition 1.8 A **field** is a set F with two operations, called **addition** and **multiplication**, which satisfy the following so-called "field axioms" (A), (M) and (D):

(A) Axioms for addition

- (A1) If $x \in F$ and $y \in F$, then their sum $x + y$ is in F.
- (A2) Addition is commutative: $x + y = y + x$ for all $x, y \in F$.
- (A3) Addition is associative: $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.
- (A4) F contains an element 0 such that $0 + x = x$ for every $x \in F$.
- (A5) To every $x \in F$ corresponds an element $-x \in F$ such that

$$x + (-x) = 0.$$

(M) Axioms for multiplication

- (M1) If $x \in F$ and $y \in F$, then their product xy is in F.
- (M2) Multiplication is commutative: $xy = yx$ for all $x, y \in F$.
- (M3) Multiplication is associative: $(xy)z = x(yz)$ for all $x, y, z \in F$.
- (M4) F contains an element $1 \neq 0$ such that $1x = x$ for every $x \in F$.
- (M5) If $x \in F$ and $x \neq 0$ then there exists an element $1/x \in F$ such that

$$x \cdot \frac{1}{x} = 1.$$

(D) The distributive law

$$x(y + z) = xy + xz$$

holds for all $x, y, z \in F$.

Proposition 1.1 The axioms for addition imply the following statements:

(a) If $x + y = x + z$ then $y = z$.

(b) If $x + y = x$ then $y = 0$.

(c) If $x + y = 0$ then $y = -x$.

(d) $-(-x) = x$.

Proposition 1.2 The axioms for multiplication imply the following statements:

(a) If $x \neq 0$ and $xy = xz$ then $y = z$.

(b) If $x \neq 0$ and $xy = x$ then $y = 1$.

(c) If $x \neq 0$ and $xy = 1$ then $y = 1/x$.

(d) If $x \neq 0$ then $1/(1/x) = x$.

Proposition 1.3 The field axioms imply the following statements for any $x, y, z \in F$:

(a) $0x = 0$.

(b) If $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.

(c) $(-x)y = -(xy) = x(-y)$.

(d) $(-x)(-y) = xy$.

Definition 1.9 An **ordered field** is a field F which is also an **ordered set**, such that

(i) $x + y < x + z$ if $x, y, z \in F$ and $y < z$.

(ii) $xy > 0$ if $x \in F, y \in F, x > 0$, and $y > 0$.

If $x > 0$, we call x **positive**; if $x < 0$, x is **negative**.

Proposition 1.4 The following statements are true in every ordered field:

(a) If $x > 0$ then $-x < 0$, and vice versa.

(b) If $x > 0$ and $y < z$ then $xy < xz$.

(c) If $x < 0$ and $y < z$ then $xy > xz$.

(d) If $x \neq 0$ then $x^2 > 0$. In particular, $1 > 0$.

(e) If $0 < x < y$ then $0 < 1/y < 1/x$.

1.4 The Real Field

Theorem 1.2 There exists an ordered field R which has the least-upper-bound property. Moreover, R contains Q as a subfield.

Theorem 1.3 (Archimedean Property and Density of $\boldsymbol{Q}$)

(a) If $x \in R, y \in R$, and $x > 0$, then there is a positive integer n such that

$$nx > y.$$

(b) If $x \in R, y \in R$, and $x < y$, then there is a $p \in Q$ such that

$$x < p < y.$$

Theorem 1.4 For every real $x > 0$ and every integer $n > 0$ there is one and only one positive real y such that $y^n = x$. This number y is written $\sqrt[n]{x}$ or $x^{1/n}$.

Corollary 1.1 If a and b are positive real numbers and n is a positive integer, then

$$(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}.$$

1.5 The Extended Real Number System

Definition 1.10 The extended real number system consists of the real field R and two symbols, $+\infty$ and $-\infty$. We preserve order in R, and define

$$-\infty < x < +\infty$$

for every $x \in R$. $+\infty$ is an upper bound of every subset of the extended real number system, and that every nonempty subset has a least upper bound. There are three conventions:

(a) If x is real then

$$x + \infty = +\infty, \qquad x - \infty = -\infty, \qquad \frac{x}{+\infty} = \frac{x}{-\infty} = 0.$$

(b) If $x > 0$ then $x \cdot (+\infty) = +\infty$, $x \cdot (-\infty) = -\infty$.

(c) If $x < 0$ then $x \cdot (+\infty) = -\infty$, $x \cdot (-\infty) = +\infty$.

1.6 The Complex Field

Definition 1.11 A **complex number** is an ordered pair (a, b) of real numbers. "Ordered" means that (a, b) and (b, a) are regarded as distinct if $a \neq b$. Let $x = (a, b), y = (c, d)$ be two complex numbers. We write $x = y$ if and only if $a = c$ and $b = d$. We define

$$x + y = (a + c, b + d), \qquad xy = (ac - bd, ad + bc).$$

Theorem 1.5 These definitions of addition and multiplication turn the set of all complex numbers into a field, with $(0, 0)$ and $(1, 0)$ in the role of 0 and 1.

Theorem 1.6 For any real numbers a and b we have

$$(a, 0) + (b, 0) = (a + b, 0), \qquad (a, 0)(b, 0) = (ab, 0).$$

Definition 1.12 $i = (0, 1)$.

Theorem 1.7 $i^2 = -1$.

Theorem 1.8 If a and b are real, then $(a, b) = a + bi$.

Definition 1.13 If a, b are real and $z = a + bi$, then the complex number $\overline{z} = a - bi$ is called the **conjugate** of z. The numbers a and b are **real part** and **imaginary part** of z, respectively. We shall occassionally write

$$a = \mathrm{Re}(z), \qquad b = \mathrm{Im}(z).$$

Theorem 1.9 If z and w are complex, then

(a) $\overline{z + w} = \overline{z} + \overline{w}$,

(b) $\overline{zw} = \overline{z} \cdot \overline{w}$,

(c) $z + \overline{z} = 2\ \mathrm{Re}(z),\ z - \overline{z} = 2i\ \mathrm{Im}(z)$,

(d) $z\overline{z}$ is real and positive (except when $z = 0$).

Definition 1.14 If z is a complex number, its **absolute value** $|z|$ is the non-negative square root of $z\overline{z}$; that is $|z| = (z\overline{z})^{1/2}$.

Theorem 1.10 Let z and w be complex numbers. Then

(a) $|z| > 0$ unless $z = 0$, $|0| = 0$,

(b) $|\overline{z}| = |z|$,

(c) $|zw| = |z|\ |w|$,

(d) $|\mathrm{Re}(z)| \leq |z|$,

(e) $|z + w| \leq |z| + |w|$.

Theorem 1.11 (Cauchy–Schwarz Inequality) If $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ are complex numbers, then

$$\left|\sum_{j=1}^{n} a_j \overline{b_j}\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2.$$

1.7 Euclidean Spaces

Definition 1.15 For each positive integer k, let R^k be the set of all ordered k-tuples

$$\mathbf{x} = (x_1, x_2, \cdots, x_k),$$

where $x_1, x_2, \cdots, x_k$ are real numbers, called the **coordinates** of $\mathbf{x}$. The elements of R^k are called points, or vectors, especially when $k > 1$.

- Addition of vectors

  If $\mathbf{y} = (y_1, y_2, \cdots, y_k)$ and if α is a real number, put

  $$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_1 + y_1, \cdots, x_k + y_k)$$

  so that $\mathbf{x} + \mathbf{y} \in R^k$.

- Multiplication of a vector by a real number (a scalar)

  $$\alpha\mathbf{x} = (\alpha x_1, \alpha x_2, \cdots, \alpha x_k)$$

  so that $\alpha\mathbf{x} \in R^k$.

These two operations make R^k into a **vector space over the real field**.

- Inner product (or scalar product) of vectors

  $$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{k} x_i y_i.$$

- Norm of a vector

  $$|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{\sum_{i=1}^{k} x_i^2}.$$

The structure now defined (the vector space R^k with the above inner product and norm) is called Eulidean k-space.

Theorem 1.12 Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in R^k$, and α is real. Then

(a) $|\mathbf{x}| \geq 0$;

(b) $|\mathbf{x}| = 0$ if and only if $\mathbf{x} = \mathbf{0}$;

(c) $|\alpha\mathbf{x}| = |\alpha|\ |\mathbf{x}|$;

(d) $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}|\ |\mathbf{y}|$;

(e) $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$;

(f) $|\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|$.