Chapter 1 Exercises

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercises are understood to be real.

1. If r is rational ($r \neq 0$) and x is irrational, prove that $r+x$ and rx are irrational.

Proof. Let $r = p_1/q_1$, where p_1, q_1 are integers and $p_1 \neq 0, q_1 \neq 0$. Suppose that $r+x$ and rx are rational. Let $r+x = p_2/q_2$, where p_2, q_2 are integers and $q_2 \neq 0$. We have

$$x = (r+x) - r = \frac{p_2}{q_2} - \frac{p_1}{q_1} = \frac{p_2 q_1 - p_1 q_2}{q_1 q_2},$$

so x is rational, which is contrary to the condition that x is irrational. Therefore, $r+x$ is irrational. Suppose that $rx = p_3/q_3$, where p_3, q_3 are integers and $q_3 \neq 0$. With $r \neq 0$, we have

$$x = \frac{rx}{r} = \left(\frac{p_3}{q_3}\right) / \left(\frac{p_1}{q_1}\right) = \frac{p_3 q_1}{p_1 q_3},$$

so x is rational, which is contrary to the condition that x is irrational. Therefore, rx is irrational.

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2. Prove that there is no rational number whose square is 12.

Proof. Suppose that there exists a rational number r whose square is 12. Let $r = p/q$, where p, q are integers with no common factors which are greater than 1 and $q \neq 0$. Then we have

$$r^2 = \frac{p^2}{q^2} = 12 \quad \Rightarrow \quad p^2 = 12q^2,$$

so p is even and has a factor 2. Let $p = 2m$, where m is an integer. We have

$$p^2 = (2m)^2 = 4m^2 = 12q^2 \quad \Rightarrow \quad m^2 = 3q^2,$$

so m has a factor 3. Let $m = 3n$, where n is an integer. We have

$$m^2 = (3n)^2 = 9n^2 = 3q^2 \quad \Rightarrow \quad 3n^2 = q^2,$$

so q has also a factor 3. However, m is a factor of p and m has a factor 3, so p has a factor 3. Therefore, 3 is a common factor which is greater than 1 of p and q, which is contray to the condition that p, q are integers with no common factors which are greater than 1. Thus, there is no rational number whose square is 12.

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3. Prove the following propositions:

(a) If $x \neq 0$ and $xy = xz$ then $y = z$.

(b) If $x \neq 0$ and $xy = x$ then $y = 1$.

(c) If $x \neq 0$ and $xy = 1$ then $y = 1/x$.

(d) If $x \neq 0$ then $1/(1/x) = x$.

Proof.

(a)

$$y \xlongequal{\text{(M4)}} 1 \cdot y \xlongequal{\text{(M5)}} (x \cdot \frac{1}{x}) \cdot y \xlongequal{\text{(M2)}} xy \cdot \frac{1}{x} = xz \cdot \frac{1}{x} \xlongequal{\text{(M2)}} (x \cdot \frac{1}{x}) \cdot z \xlongequal{\text{(M5)}} 1 \cdot z \xlongequal{\text{(M4)}} z.$$

(b) With the (a) statement, let $z = 1$,

$$xy = xz = x \cdot 1 \xlongequal{\text{(M2)}} 1 \cdot x \xlongequal{\text{(M4)}} x \quad \xRightarrow{\text{(a)}} \quad y = z = 1.$$

(c) With the (a) statement, let $z = 1/x$,

$$xy = xz = x \cdot \frac{1}{x} \xlongequal{\text{(M5)}} 1 \quad \xRightarrow{\text{(a)}} \quad = z = \frac{1}{x}.$$

(d)

$$x \cdot \frac{1}{x} \begin{Bmatrix} \xlongequal{\text{(M2)}} \frac{1}{x} \cdot x \\ \xlongequal{\text{(M5)}} 1 \end{Bmatrix} \quad \Rightarrow \quad \frac{1}{x} \cdot x = 1 \quad \xRightarrow{\text{(M5)}} \quad \frac{1}{\frac{1}{x}} = x.$$

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4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Proof. Suppose that $\alpha > \beta$, because E is a nonempty set and α is a lower bound of E, there exists an element $a \in E$ such that $a \geq \alpha$. Therefore, we have $a \geq \alpha > \beta$. However, β is an upper bound of E, so every element of E should be no more than β, which means that $a \leq \beta$. This is contrary to the condition that $a \geq \alpha > \beta$. Therefore, $\alpha \leq \beta$.

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5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Proof. Because A is a nonempty set of real numbers which is bounded below, A has the greatest lower bound $\alpha = \inf A$.

- For every element $x \in A$, there is $x \geq \alpha$. Therefore, we have $-x \leq -\alpha$. Because $-x \in -A$, $-\alpha$ is a upper bound of $-A$.
- For any real $\beta > \alpha$, there exists at least an element $y \in A$ such that $y < \beta$. Therefore, we have $-\beta < -\alpha$ and $-y > -\beta$. Let $-\beta = \gamma$ and $-y = z$, then for any real $\gamma < -\alpha$, there exists at least an element $z \in -A$ such that $z > \gamma$. Thus, $-\alpha$ is the least upper bound of $-A$.

To sum up, we can conclude that

$$-\alpha = -\inf A = \sup(-A) \quad \Rightarrow \quad \inf A = -\sup(-A).$$

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6. Fix $b > 1$.

(a) If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}.$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.

(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

(c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

when r is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

for every real x.

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

Proof.

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