

# Chapter 1 Exercises

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercises are understood to be real.

1. If  $r$  is rational ( $r \neq 0$ ) and  $x$  is irrational, prove that  $r + x$  and  $rx$  are irrational.

**Proof.** Let  $r = p_1/q_1$ , where  $p_1, q_1$  are integers and  $p_1 \neq 0, q_1 \neq 0$ . Suppose that  $r + x$  and  $rx$  are rational. Let  $r + x = p_2/q_2$ , where  $p_2, q_2$  are integers and  $q_2 \neq 0$ . We have

$$x = (r + x) - r = \frac{p_2}{q_2} - \frac{p_1}{q_1} = \frac{p_2q_1 - p_1q_2}{q_1q_2},$$

so  $x$  is rational, which is contrary to the condition that  $x$  is irrational. Therefore,  $r + x$  is irrational. Suppose that  $rx = p_3/q_3$ , where  $p_3, q_3$  are integers and  $q_3 \neq 0$ . With  $r \neq 0$ , we have

$$x = \frac{rx}{r} = \left(\frac{p_3}{q_3}\right) / \left(\frac{p_1}{q_1}\right) = \frac{p_3q_1}{p_1q_3},$$

so  $x$  is rational, which is contrary to the condition that  $x$  is irrational. Therefore,  $rx$  is irrational. ■

2. Prove that there is no rational number whose square is 12.

**Proof.** Suppose that there exists a rational number  $r$  whose square is 12. Let  $r = p/q$ , where  $p, q$  are integers with no common factors which are greater than 1 and  $q \neq 0$ . Then we have

$$r^2 = \frac{p^2}{q^2} = 12 \quad \Rightarrow \quad p^2 = 12q^2,$$

so  $p$  is even and has a factor 2. Let  $p = 2m$ , where  $m$  is an integer. We have

$$p^2 = (2m)^2 = 4m^2 = 12q^2 \quad \Rightarrow \quad m^2 = 3q^2,$$

so  $m$  has a factor 3. Let  $m = 3n$ , where  $n$  is an integer. We have

$$m^2 = (3n)^2 = 9n^2 = 3q^2 \quad \Rightarrow \quad 3n^2 = q^2,$$

so  $q$  has also a factor 3. However,  $m$  is a factor of  $p$  and  $m$  has a factor 3, so  $p$  has a factor 3. Therefore, 3 is a common factor which is greater than 1 of  $p$  and  $q$ , which is contrary to the condition that  $p, q$  are integers with no common factors which are greater than 1. Thus, there is no rational number whose square is 12.

■

3. Prove the following propositions:

(a) If  $x \neq 0$  and  $xy = xz$  then  $y = z$ .

(b) If  $x \neq 0$  and  $xy = x$  then  $y = 1$ .

(c) If  $x \neq 0$  and  $xy = 1$  then  $y = 1/x$ .

(d) If  $x \neq 0$  then  $1/(1/x) = x$ .

**Proof.**

(a)

$$y \xrightarrow{(M4)} 1 \cdot y \xrightarrow{(M5)} (x \cdot \frac{1}{x}) \cdot y \xrightarrow{(M2)} xy \cdot \frac{1}{x} = xz \cdot \frac{1}{x} \xrightarrow{(M2)} (x \cdot \frac{1}{x}) \cdot z \xrightarrow{(M5)} 1 \cdot z \xrightarrow{(M4)} z.$$

(b) With the (a) statement, let  $z = 1$ ,

$$xy = xz = x \cdot 1 \xrightarrow{(M2)} 1 \cdot x \xrightarrow{(M4)} x \xrightarrow{(a)} y = z = 1.$$

(c) With the (a) statement, let  $z = 1/x$ ,

$$xy = xz = x \cdot \frac{1}{x} \xrightarrow{(M5)} 1 \xrightarrow{(a)} = z = \frac{1}{x}.$$

(d)

$$x \cdot \frac{1}{x} \left\{ \begin{array}{c} \xrightarrow{(M2)} \frac{1}{x} \cdot x \\ \xrightarrow{(M5)} 1 \end{array} \right\} \Rightarrow \frac{1}{x} \cdot x = 1 \xrightarrow{(M5)} \frac{1}{\frac{1}{x}} = x.$$

■

4. Let  $E$  be a nonempty subset of an ordered set; suppose  $\alpha$  is a lower bound of  $E$  and  $\beta$  is an upper bound of  $E$ . Prove that  $\alpha \leq \beta$ .

**Proof.** Suppose that  $\alpha > \beta$ , because  $E$  is a nonempty set and  $\alpha$  is a lower bound of  $E$ , there exists an element  $a \in E$  such that  $a \geq \alpha$ . Therefore, we have  $a \geq \alpha > \beta$ . However,  $\beta$  is an upper bound of  $E$ , so every element of  $E$  should be no more than  $\beta$ , which means that  $a \leq \beta$ . This is contrary to the condition that  $a \geq \alpha > \beta$ . Therefore,  $\alpha \leq \beta$ .

■

5. Let  $A$  be a nonempty set of real numbers which is bounded below. Let  $-A$  be the set of all numbers  $-x$ , where  $x \in A$ . Prove that

$$\inf A = -\sup(-A).$$

**Proof.** Because  $A$  is a nonempty set of real numbers which is bounded below,  $A$  has the greatest lower bound  $\alpha = \inf A$ .

- For every element  $x \in A$ , there is  $x \geq \alpha$ . Therefore, we have  $-x \leq -\alpha$ . Because  $-x \in -A$ ,  $-\alpha$  is a upper bound of  $-A$ .
- For any real  $\beta > \alpha$ , there exists at least an element  $y \in A$  such that  $y < \beta$ . Therefore, we have  $-\beta < -\alpha$  and  $-y > -\beta$ . Let  $-\beta = \gamma$  and  $-y = z$ , then for any real  $\gamma < -\alpha$ , there exists at least an element  $z \in -A$  such that  $z > \gamma$ . Thus,  $-\alpha$  is the least upper bound of  $-A$ .

To sum up, we can conclude that

$$-\alpha = -\inf A = \sup(-A) \quad \Rightarrow \quad \inf A = -\sup(-A).$$

■

6. Fix  $b > 1$ .

(a) If  $m, n, p, q$  are integers,  $n > 0$ ,  $q > 0$ , and  $r = m/n = p/q$ , prove that

$$(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}.$$

Hence it makes sense to define  $b^r = (b^m)^{1/n}$ .

(b) Prove that  $b^{r+s} = b^r b^s$  if  $r$  and  $s$  are rational.

(c) If  $x$  is real, define  $B(x)$  to be the set of all numbers  $b^t$ , where  $t$  is rational and  $t \leq x$ .

Prove that

$$b^x = \sup B(x)$$

when  $r$  is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

for every real  $x$ .

(d) Prove that  $b^{x+y} = b^x b^y$  for all real  $x$  and  $y$ .

***Proof.***

(a) We have

$$\begin{aligned} (b^{\frac{m}{n}})^n &= (\sqrt[n]{b^m})^n = b^m \quad \Rightarrow \quad b^{\frac{m}{n}} = \sqrt[n]{b^m} = (b^m)^{\frac{1}{n}}, \\ (b^{\frac{p}{q}})^q &= (\sqrt[q]{b^p})^q = b^p \quad \Rightarrow \quad b^{\frac{p}{q}} = \sqrt[q]{b^p} = (b^p)^{\frac{1}{q}}. \end{aligned}$$

Because  $m/n = p/q$ , we have  $b^{m/n} = b^{p/q}$ , so  $(b^m)^{1/n} = (b^p)^{1/q}$ .

■