Chapter 1 Exercises

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercises are understood to be real.

1. If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

Proof. Let $r = p_1/q_1$, where p_1, q_1 are integers and $p_1 \neq 0, q_1 \neq 0$. Suppose that $r + x$ and rx are rational. Let $r + x = p_2/q_2$, where p_2, q_2 are integers and $q_2 \neq 0$. We have

$$x = (r + x) - r = \frac{p_2}{q_2} - \frac{p_1}{q_1} = \frac{p_2 q_1 - p_1 q_2}{q_1 q_2},$$

so x is rational, which is contrary to the condition that x is irrational. Therefore, $r + x$ is irrational. Suppose that $rx = p_3/q_3$, where p_3, q_3 are integers and $q_3 \neq 0$. With $r \neq 0$, we have

$$x = \frac{rx}{r} = \left(\frac{p_3}{q_3}\right) / \left(\frac{p_1}{q_1}\right) = \frac{p_3 q_1}{p_1 q_3},$$

so x is rational, which is contrary to the condition that x is irrational. Therefore, rx is irrational.

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2. Prove that there is no rational number whose square is 12.

Proof. Suppose that there exists a rational number r whose square is 12. Let $r = p/q$, where p, q are integers with no common factors which are greater than 1 and $q \neq 0$. Then we have

$$r^2 = \frac{p^2}{q^2} = 12 \quad \Rightarrow \quad p^2 = 12q^2,$$

so p is even and has a factor 2. Let $p = 2m$, where m is an integer. We have

$$p^2 = (2m)^2 = 4m^2 = 12q^2 \quad \Rightarrow \quad m^2 = 3q^2,$$

so m has a factor 3. Let $m = 3n$, where n is an integer. We have

$$m^2 = (3n)^2 = 9n^2 = 3q^2 \quad \Rightarrow \quad 3n^2 = q^2,$$

so q has also a factor 3. However, m is a factor of p and m has a factor 3, so p has a factor 3. Therefore, 3 is a common factor which is greater than 1 of p and q, which is contray to the condition that p, q are integers with no common factors which are greater than 1. Thus, there is no rational number whose square is 12.

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