

Chapter 1 Proofs

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Theorem 1.1 Suppose S is an ordered set with the least-upper-bound property, $B \subseteq S$, B is not empty, and B is bounded below. Let L be the set of all lower bounds of B . Then

$$\alpha = \sup L$$

exists in S , and $\alpha = \inf B$. In particular, $\inf B$ exists in S .

Proof. First of all, we should prove that $\alpha = \sup L$ exists in S .

- (a) Because L is the set of all lower bounds of B and $B \subseteq S$, then $L \subseteq S$.
- (b) Because B is bounded below, there exists a $\beta \in S$ such that $x \geq \beta$ for every $x \in B$. Therefore, β is a lower bound of B , $\beta \in L$, so L is not empty.
- (c) Because L is the set of all lower bounds of B and B is not empty, there exists a $\gamma \in B$ such that $y \leq \gamma$ for every $y \in L$, then L is bounded above.

Because S is an ordered set with the least-upper-bound property, then $\alpha = \sup L$ exists in S . Now, we should prove that $\alpha = \inf B$.

- (a) Suppose that α is not the lower bound of B , then there exists $z \in B$ such that $z < \alpha$. However, α is the supremum of L . Because $z < (z + \alpha)/2 < \alpha$, $(z + \alpha)/2$ is not a upper bound of L , so there exists a $s \in L$ such that $s > (z + \alpha)/2$. Because L is the set of all lower bounds of B , s is a lower bounds of B . However, $z < s$ and $z \in B$, which is a cotradiction. Thus, α is the lower bound of B .

For all $\mu < \alpha$, μ is not an upper bound of L . Because L is the set of all lower bounds of B , $\mu \notin B$. It follows that $\alpha \leq x$ for every $x \in B$. Thus $\alpha \in L$.

- (b) For all $\lambda > \alpha$, there is $\lambda > \alpha \geq t$ for every $t \in L$, so λ is not an element of L . Because L is the set of all lower bounds of B and $B \subseteq S$, λ is not a lower bound of B .

To sum up, $\alpha = \inf B$ and $\inf B$ exists in S .

■

Proposition 1.1 The axioms for addition imply the following statements:

- (a) If $x + y = x + z$ then $y = z$.

(b) If $x + y = x$ then $y = 0$.

(c) If $x + y = 0$ then $y = -x$.

(d) $-(-x) = x$.

Proof.

(a)

$$\begin{aligned} y &\stackrel{(A4)}{=} 0 + y \stackrel{(A5)}{=} [x + (-x)] + y \stackrel{(A2)}{=} (x + y) + (-x) \\ &= (x + z) + (-x) \stackrel{(A2)}{=} [x + (-x)] + z \stackrel{(A5)}{=} 0 + z \stackrel{(A4)}{=} z. \end{aligned}$$

(b) With the (a) statement, let $z = 0$,

$$x + y = x + z = x + 0 \stackrel{(A2)}{=} 0 + x \stackrel{(A4)}{=} x \stackrel{(a)}{\implies} y = z = 0.$$

(c) With the (a) statement, let $z = -x$,

$$x + y = x + z = x + (-x) \stackrel{(A5)}{=} 0 \stackrel{(a)}{\implies} y = z = -x.$$

(d)

$$x + (-x) \left\{ \begin{array}{l} \stackrel{(A2)}{=} (-x) + x \\ \stackrel{(A5)}{=} 0 \end{array} \right\} \implies (-x) + x = 0 \stackrel{(A5)}{\implies} -(-x) = 0.$$

■

Proposition 1.2 The axioms for multiplication imply the following statements:

(a) If $x \neq 0$ and $xy = xz$ then $y = z$.

(b) If $x \neq 0$ and $xy = x$ then $y = 1$.

(c) If $x \neq 0$ and $xy = 1$ then $y = 1/x$.

(d) If $x \neq 0$ then $1/(1/x) = x$.

Proof.

(a)

$$y \stackrel{(M4)}{=} 1 \cdot y \stackrel{(M5)}{=} \left(x \cdot \frac{1}{x}\right) \cdot y \stackrel{(M2)}{=} xy \cdot \frac{1}{x} = xz \cdot \frac{1}{x} \stackrel{(M2)}{=} \left(x \cdot \frac{1}{x}\right) \cdot z \stackrel{(M5)}{=} 1 \cdot z \stackrel{(M4)}{=} z.$$

(b) With the (a) statement, let $z = 1$,

$$xy = xz = x \cdot 1 \xrightarrow{(M2)} 1 \cdot x \xrightarrow{(M4)} x \xrightarrow{(a)} y = z = 1.$$

(c) With the (a) statement, let $z = 1/x$,

$$xy = xz = x \cdot \frac{1}{x} \xrightarrow{(M5)} 1 \xrightarrow{(a)} = z = \frac{1}{x}.$$

(d)

$$x \cdot \frac{1}{x} \left\{ \begin{array}{c} \xrightarrow{(M2)} \frac{1}{x} \cdot x \\ \xrightarrow{(M5)} 1 \end{array} \right\} \Rightarrow \frac{1}{x} \cdot x = 1 \xrightarrow{(M5)} \frac{1}{\frac{1}{x}} = x.$$

■

Proposition 1.3 The field axioms imply the following statements, for any $x, y, z \in F$:

(a) $0 \cdot x = x$.

(b) If $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.

(c) $(-x)y = -(xy) = x(-y)$.

(d) $(-x)(-y) = xy$.

Proof.

(a)

$$0 \cdot x \xrightarrow{(A5)} [1 + (-1)]x \xrightarrow{(M2), (D)} 1 \cdot x - 1 \cdot x \xrightarrow{(M4)} x - x \xrightarrow{(A5)} 0.$$

(b) Suppose that $xy = 0$, then

$$xy \cdot \frac{1}{y} \cdot \frac{1}{x} \xrightarrow{(M3)} x \left(y \cdot \frac{1}{y} \right) \frac{1}{x} \xrightarrow{(M5)} x \cdot 1 \cdot \frac{1}{x} \xrightarrow{(M2), (M4)} x \cdot \frac{1}{x} \xrightarrow{(M5)} 1.$$

However, according to (a) statement,

$$xy \cdot \frac{1}{y} \cdot \frac{1}{x} = 0 \cdot \frac{1}{y} \cdot \frac{1}{x} \xrightarrow{(a)} 0 \cdot \frac{1}{x} \xrightarrow{(a)} 0.$$

This is a cotradiction, so if $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.

(c)

$$\begin{aligned} (-x)y + xy &\stackrel{(M2), (D)}{=} [(-x) + x]y \stackrel{(M5)}{=} 0 \cdot y \stackrel{(a)}{=} 0 \stackrel{(M5)}{\implies} (-x)y = -(xy), \\ xy + x(-y) &\stackrel{(D)}{=} x[y + (-y)] \stackrel{(A5)}{=} x \cdot 0 \stackrel{(M2)}{=} 0 \cdot x \stackrel{(a)}{=} 0 \stackrel{(M5)}{\implies} -(xy) = x(-y). \end{aligned}$$

To sum up, $(-x)y = -(xy) = x(-y)$.

(d)

$$\begin{aligned} (-x)(-y) + (-x)y &\left\{ \begin{array}{l} \stackrel{(D)}{=} (-x)(-y + y) \stackrel{(A2) (A5)}{=} (-x) \cdot 0 \stackrel{(M2) (a)}{=} 0 \\ \stackrel{(c)}{=} (-x)(-y) + (-xy) \end{array} \right\} \\ &\stackrel{(A5)}{\implies} (-x)(-y) = xy. \\ (-x)(-y) &\stackrel{(c)}{=} -[x(-y)] \stackrel{(c)}{=} -[-(xy)] \stackrel{\text{Proposition 1.4 (d)}}{=} xy. \end{aligned}$$

■

Proposition 1.4 The following statements are true in every ordered field:

- (a) If $x > 0$ then $-x < 0$, and vice versa.
- (b) If $x > 0$ and $y < z$ then $xy < xz$.
- (c) If $x < 0$ and $y < z$ then $xy > xz$.
- (d) If $x \neq 0$ then $x^2 > 0$. In particular, $1 > 0$.
- (e) If $0 < x < y$ then $0 < 1/y < 1/x$.

Proof.

(a)

$$\begin{aligned} x > 0 &\stackrel{\text{Definition 1.9 (i)}}{\implies} x - x \stackrel{(A5)}{=} 0 > 0 - x \stackrel{(A4)}{=} -x \implies -x < 0. \\ x < 0 &\stackrel{\text{Definition 1.9 (i)}}{\implies} x - x \stackrel{(A5)}{=} 0 < 0 - x \stackrel{(A4)}{=} -x \implies -x > 0. \end{aligned}$$

(b)

$$y < z \stackrel{\text{Definition 1.9 (i)}}{\implies} z - y > 0 \left. \begin{array}{l} x > 0 \\ z - y > 0 \end{array} \right\} \stackrel{\text{Definition 1.9 (ii)}}{\implies} x(z - y) > 0 \stackrel{(D)}{\implies} xy < xz.$$

(c)

$$\left. \begin{array}{l} x < 0 \xrightarrow{\text{Definition 1.9 (i)}} -x > 0 \\ y < z \xrightarrow{\text{Definition 1.9 (i)}} z - y > 0 \end{array} \right\} \xrightarrow{\text{Definition 1.9 (ii)}} -x(z - y) > 0$$

$$\xrightarrow{(D)} -xz + (-x)(-y) > 0 \xrightarrow{\text{Proposition 1.4 (d)}} xy > xz.$$

(d) • If $x < 0$, then

$$x < 0 \xrightarrow{(c)} x \cdot x = x^2 > x \cdot 0 \xrightarrow{(M2), \text{Proposition 1.3 (a)}} 0.$$

• If $x > 0$, then

$$x > 0 \xrightarrow{\text{Definition 1.9 (ii)}} x \cdot x = x^2 > 0.$$

(e) Suppose that $1/x < 0$, then

$$\left. \begin{array}{l} x > 0 \\ \frac{1}{x} < 0 \end{array} \right\} \xrightarrow{(b)} x \cdot \frac{1}{x} \xrightarrow{(M5)} 1 < 0 \cdot 0 \xrightarrow{\text{Proposition 1.3 (a)}} 0.$$

This is a contradiction, so $1/x > 0$. Likewise, $1/y > 0$. Suppose that $0 < 1/x < 1/y$, then

$$\left. \begin{array}{l} 0 < x < y \\ 0 < \frac{1}{x} < \frac{1}{y} \end{array} \right\} \xrightarrow{(b)} x \cdot \frac{1}{x} \xrightarrow{(M5)} 1 < x \cdot \frac{1}{y} < y \cdot \frac{1}{y} \xrightarrow{(M5)} 1.$$

This is a contradiction, so $0 < 1/y < 1/x$.

■

Theorem 1.2 There exists an ordered field R which has the least-upper-bound property. Moreover, R contains Q as a subfield.

Theorem 1.3 (Archimedean Property and Density of Q)

(a) If $x \in R, y \in R$, and $x > 0$, then there is a positive integer n such that

$$nx > y.$$

(b) If $x \in R, y \in R$, and $x < y$, then there is a $p \in Q$ such that

$$x < p < y.$$

Proof.

- (a) Let A be the set of all nx , where n runs through the positive integers. Suppose that this statement is false, then y would be an upper bound of A . But then A has a least upper bound in R . Put $\alpha = \sup A$. Since $x > 0$, $\alpha - x < \alpha$, and $\alpha - x$ is not an upper bound of A . Hence $\alpha - x < mx$ for some positive integer m . But then $\alpha < (m + 1)x \in A$, which is impossible, since α is an upper bound of A .

- (b) Because

$$x < y \quad \Rightarrow \quad y - x > 0.$$

According to Archimedean Property, there is a positive integer n such that

$$n(y - x) > 1.$$

With $1 > 0$, according to Archimedean Property, there is two positive integers m_1 and m_2 such that

$$m_1 \cdot 1 = m_1 > nx, \quad m_2 \cdot 1 = m_2 > -nx.$$

Therefore,

$$-m_2 < nx < m_1.$$

Find the integer m with $-m_2 \leq m \leq m_1$ such that

$$m - 1 \leq nx < m.$$

Now, we obtain

$$nx < m \leq 1 + nx < ny.$$

With $n > 0$, it follows that

$$x < \frac{m}{n} < y.$$

■

Theorem 1.4 For every real $x > 0$ and every integer $n > 0$ there is one and only one positive real y such that $y^n = x$. This number y is written $\sqrt[n]{x}$ or $x^{1/n}$.

Proof. Let x be any positive real and n be any positive integer, suppose that there is no positive real y such that $y^n = x$.

1. We will prove that for any positive real x and y such that $0 < x < y$, there is $0 < x^n < y^n$ for all positive integers n with mathematical induction.

- **Basis**

For $n = 1$, there is $0 < x < y$.

- **Inductive Assumption**

Assume that for $i = 1, 2, \dots, n$ where n is a positive integer, there is $0 < x^i < y^i$.

- **Inductive Step**

For $i = n + 1$, then

$$0 < x^{n+1} = x^n \cdot x < x^n \cdot y < y^n \cdot y = y^{n+1}.$$

To sum up, for any positive real x and y such that $0 < x < y$, there is $0 < x^n < y^n$ for all positive integers n .

2. For every real $y > 0$, if $y^n < x$, we put y into a set A ; if $y^n > x$, we put y into a set B . For any element a in A and any element b in B , if $t = x/(1+x)$ then $0 \leq t < 1$. Hence $t^n \leq t < x$. Thus $t \in A$, and A is not empty. With the conclusion in the part before, we have

$$0 < a^n < x < b^n \quad \Rightarrow \quad 0 < a < b.$$

Thus, every element in A is a lower bound of B ; every element in B is an upper bound of A . Let α be the least upper bound of A and let β be the greatest lower bound of A . We will prove that $\alpha = \beta$.

- If $\alpha < \beta$, then $(\alpha + \beta)/2$ is also a positive real. Let $\gamma = (\alpha + \beta)/2$. Because $\gamma > \alpha$, γ is not an element of A , so γ^n can not be less than x ; because $\gamma < \beta$, γ is not an element of B , γ^n can not be more than x ; thus, $\gamma^n = x$. This contradicts with our assumption that there is no positive real y such that $y^n = x$.

- If $\alpha > \beta$, then $(\alpha + \beta)/2$ is also a positive real. Let $\gamma = (\alpha + \beta)/2$. Because $\gamma < \alpha$, there is at least one element μ in A such that $\gamma < \mu$; because $\gamma > \beta$, there is at least one element ν in B such that $\gamma > \nu$. Thus, $0 < \nu < \gamma < \mu$. With the conclusion in the first part, we have $\nu^n < \mu^n$. However, we have $\mu^n < x < \nu^n$, so this is a contradiction.

To sum up, $\alpha = \beta$.

3. Because α is also a positive real, we will prove that α^n can not be less than x . Suppose that $\alpha^n < x$, then α is an element of A . Let ε be a positive real such that

$$\varepsilon = \min \left\{ \frac{x - \alpha^n}{2\alpha^{n-1}n!n}, \alpha \right\},$$

so

$$\frac{\varepsilon}{\alpha} \leq \frac{x - \alpha^n}{2\alpha^n n!n} < \frac{x - \alpha^n}{\alpha^n n!n} \quad \text{and} \quad 0 < \frac{\varepsilon}{\alpha} \leq 1.$$

Now, we have

$$\begin{aligned} (\alpha + \varepsilon)^n &= \alpha^n \left(1 + \frac{\varepsilon}{\alpha}\right)^n = \alpha^n \left[1 + \sum_{i=1}^n \binom{n}{i} \left(\frac{\varepsilon}{\alpha}\right)^i\right] \leq \alpha^n \left[1 + n! \sum_{i=1}^n \left(\frac{\varepsilon}{\alpha}\right)^i\right] \\ &\leq \alpha^n \left[1 + n!n \left(\frac{\varepsilon}{\alpha}\right)\right] < \alpha^n \left(1 + n!n \frac{x - \alpha^n}{\alpha^n n!n}\right) = \alpha^n \left(1 + \frac{x - \alpha^n}{\alpha^n}\right) = x. \end{aligned}$$

Therefore, $(\alpha + \varepsilon)$ is also an element of A . However, $\alpha + \varepsilon > \alpha$, α can not be the upper bound of A . This is a contradiction, so α^n can not be less than x .

4. Because α is also a positive real, we will prove that α^n can not be more than x . Suppose that $\alpha^n > x$, then α is an element of B . Let ε be a positive real such that

$$\varepsilon = \min \left\{ \frac{\alpha^n - x}{2\alpha^{n-1}n!n}, \alpha \right\},$$

so

$$\frac{\varepsilon}{\alpha} \leq \frac{\alpha^n - x}{2\alpha^n n!n} < \frac{\alpha^n - x}{\alpha^n n!n} \quad \text{and} \quad 0 < \frac{\varepsilon}{\alpha} \leq 1.$$

Then

$$-\frac{\varepsilon}{\alpha} > \frac{x - \alpha^n}{\alpha^n n!n} \quad \text{and} \quad -1 < -\frac{\varepsilon}{\alpha} \leq 0.$$

Now, we have

$$(\alpha - \varepsilon)^n = \alpha^n \left(1 - \frac{\varepsilon}{\alpha}\right)^n = \alpha^n \left[1 + \sum_{i=1}^n \binom{n}{i} \left(-\frac{\varepsilon}{\alpha}\right)^i\right] \geq \alpha^n \left[1 + \sum_{i=1}^n \binom{n}{i} \left(-\frac{\varepsilon}{\alpha}\right)\right]$$

$$\geq \alpha^n \left[1 + n!n \left(-\frac{\varepsilon}{\alpha} \right) \right] > \alpha^n \left(1 + n!n \frac{x - \alpha^n}{\alpha^n n!n} \right) = \alpha^n \left(1 + \frac{x - \alpha^n}{\alpha^n} \right) = x.$$

Therefore, $(\alpha - \varepsilon)$ is also an element of B . However, $\alpha - \varepsilon < \alpha$, α can not be the lower bound of B . This is a contradiction, so α^n can not be more than x .

5. From the third part and the fourth part, we can conclude that $\alpha^n = x$, which contradicts with our assumption that there is no positive real y such that $y^n = x$.

To sum up, for every real $x > 0$ and every integer $n > 0$ there is at least one positive real y such that $y^n = x$. With the conclusion in the first part, for two positive real numbers y and z , if $y \neq z$, then $y^n \neq z^n$, so the real number that satisfies this is unique.

The identity $b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$ yields the inequality

$$b^n - a^n < (b - a)nb^{n-1}$$

when $0 < a < b$. Assume $\alpha^n < x$. Choose ε so that $0 < \varepsilon < 1$ and

$$\varepsilon < \frac{x - \alpha^n}{n(\alpha + 1)^{n-1}}.$$

Put $a = \alpha$, $b = \alpha + \varepsilon$. Then

$$(\alpha + \varepsilon)^n - \alpha^n < \varepsilon n(\alpha + \varepsilon)^{n-1} < \varepsilon n(\alpha + 1)^{n-1} < x - \alpha^n.$$

Thus $(\alpha + \varepsilon)^n < x$, and $\alpha + \varepsilon \in A$. Since $\alpha + \varepsilon > \alpha$, this contradicts the fact that α is an upper bound of A . Assume $\alpha^n > x$. Put

$$\varepsilon = \frac{\alpha^n - x}{n\alpha^{n-1}}.$$

Then $0 < \varepsilon < \alpha$. If $t \geq \alpha - \varepsilon$, we conclude that

$$\alpha^n - t^n \leq \alpha^n - (\alpha - \varepsilon)^n < \varepsilon n\alpha^{n-1} = \alpha^n - x.$$

Thus $t^n > x$, and $t \in A$. It follows that $\alpha - \varepsilon$ is an upper bound of A . But $\alpha - \varepsilon < \alpha$, which contradicts the fact that α is the least upper bound of A . Hence $\alpha^n = x$, and the proof is complete. ■

Corollary 1.1 If a and b are positive real numbers and n is a positive integer, then

$$(ab)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}.$$

Proof. Suppose that α and β are positive real numbers. Then

$$(\alpha\beta)^n = \alpha\beta\alpha\beta\cdots\alpha\beta \stackrel{(M2)}{=} \alpha\alpha\cdots\alpha\beta\beta\cdots\beta = \alpha^n\beta^n.$$

Let $\alpha = a^{1/n}, \beta = b^{1/n}$, then

$$\left(a^{\frac{1}{n}}b^{\frac{1}{n}}\right)^n = \left(a^{\frac{1}{n}}\right)^n \left(b^{\frac{1}{n}}\right)^n = ab \quad \Rightarrow \quad (ab)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}.$$

■

Theorem 1.5 These definitions of addition and multiplication turn the set of all complex numbers into a field, with $(0, 0)$ and $(1, 0)$ in the role of 0 and 1.

Proof. Suppose that x, y and z are any complex numbers. Let $x = (a, b), y = (c, d)$ and $z = (e, f)$.

(A1) If $x = (a, b) \in F$ and $y = (c, d) \in F$, then their sum $x + y = (a + c, b + d) \in F$.

(A2) For all $x = (a, b)$ and $y = (c, d)$

$$x + y = (a + c, b + d) = (c + a, d + b) = y + x.$$

(A3) For all $x = (a, b), y = (c, d)$ and $z = (e, f)$

$$x + y = (a + c, b + d) \quad \Rightarrow \quad (x + y) + z = (a + c + e, b + d + f),$$

$$y + z = (c + e, d + f) \quad \Rightarrow \quad x + (y + z) = (a + c + e, b + d + f).$$

Therefore, $(x + y) + z = x + (y + z)$.

(A4) For every $x = (a, b) \in F$,

$$(0, 0) + x = (0, 0) + (a, b) = (0 + a, 0 + b) = (a, b) = x.$$

(A5) For every $x = (a, b) \in F$,

$$-x = (-1, 0)x = (-1, 0) \cdot (a, b) = (-1 \cdot a - 0 \cdot b, -1 \cdot b + 0 \cdot b) = (-a, -b).$$

Then

$$x + (-x) = (a, b) + (-a, -b) = (a - a, b - b) = (0, 0).$$

(M1) If $x = (a, b) \in F$ and $y = (c, d) \in F$, then their product $xy = (ac - bd, ad + bc) \in F$.

(M2) For all $x = (a, b)$ and $y = (c, d)$

$$xy = (ac - bd, ad + bc) = (ac - bd, bc + ad) = (ca - db, cb + da) = yx.$$

(M3) For all $x = (a, b), y = (c, d)$ and $z = (e, f)$

$$\begin{aligned}(xy)z &= (ac - bd, ad + bc)(e, f) \\ &= (ace - bde - adf - bcf, acf - bdf + ade + bce) \\ &= (a, b)(ce - df, cf + de) \\ &= x(yz).\end{aligned}$$

Therefore, $(xy)z = x(yz)$.

(M4) For every $x = (a, b) \in F$,

$$(1, 0)x = (1 \cdot a - 0 \cdot b, 1 \cdot b + 0 \cdot a) = (a, b) = x.$$

(M5) For every $x = (a, b) \in F$ and $a^2 + b^2 \neq 0$, let

$$\frac{1}{x} = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right).$$

Then

$$x \cdot \frac{1}{x} = (a, b) \cdot \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right) = \left(\frac{a^2 + b^2}{a^2 + b^2}, \frac{-ab + ab}{a^2 + b^2} \right) = (1, 0).$$

(D) For all $x = (a, b), y = (c, d)$ and $z = (e, f)$

$$\begin{aligned}x(y + z) &= (a, b) \cdot (c + e, d + f) \\ &= (ac + ae - bd - bf, ad + af + bc + be) \\ &= (ac - bd, ad + bc) + (ae - bf, af + be) \\ &= xy + xz.\end{aligned}$$

■

Theorem 1.6 For any real numbers a and b we have

$$(a, 0) + (b, 0) = (a + b, 0), \quad (a, 0)(b, 0) = (ab, 0).$$

Proof.

$$\begin{aligned}(a, 0) + (b, 0) &= (a + b, 0 + 0) = (a + b, 0), \\ (a, 0)(b, 0) &= (ab - 0 \cdot 0, a \cdot 0 + 0 \cdot b) = (ab - 0, 0 + 0) = (ab, 0).\end{aligned}$$

■

Theorem 1.7 $i^2 = -1$.

Proof. From Definition 1.12, $i = (0, 1)$, we have

$$i^2 = (0, 1)(0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (0 - 1, 0 + 0) = (-1, 0) = -1.$$

■

Theorem 1.8 If a and b are real, then $(a, b) = a + bi$.

Proof. We have

$$\begin{aligned}a + bi &= (a, 0) + (b, 0)(0, 1) \\ &= (a, 0) + (b \cdot 0 - 0 \cdot 1, b \cdot 1 + 0 \cdot 0) \\ &= (a, 0) + (0 - 0, b + 0) \\ &= (a, 0) + (0, b) \\ &= (a + 0, 0 + b) \\ &= (a, b).\end{aligned}$$

■

Theorem 1.9 If z and w are complex, then

- (a) $\overline{z + w} = \bar{z} + \bar{w}$,
- (b) $\overline{zw} = \bar{z} \cdot \bar{w}$,
- (c) $z + \bar{z} = 2 \operatorname{Re}(z)$, $z - \bar{z} = 2i \operatorname{Im}(z)$,
- (d) $z\bar{z}$ is real and positive (except when $z = 0$).

Proof. Let $z = a + bi$ and $w = c + di$, we have $\bar{z} = a - bi$ and $\bar{w} = c - di$. With Definition 1.13, we have

(a)

$$\left. \begin{aligned} z + w = (a + c) + (b + d)i &\Rightarrow \overline{z + w} = (a + c) - (b + d)i \\ \overline{z} + \overline{w} = (a + c) - (b + d)i &\end{aligned} \right\} \Rightarrow \overline{z + w} = \overline{z} + \overline{w}.$$

(b)

$$\left. \begin{aligned} zw = (ac - bd) + (ad + bc)i &\Rightarrow \overline{zw} = (ac - bd) - (ad + bc)i \\ \overline{z} \cdot \overline{w} = (ac - bd) - (ad + bc)i &\end{aligned} \right\} \Rightarrow \overline{zw} = \overline{z} \cdot \overline{w}.$$

(c)

$$\begin{aligned} z + \overline{z} &= (a + bi) + (a - bi) = 2a = 2 \operatorname{Re}(z), \\ z - \overline{z} &= (a + bi) - (a - bi) = 2bi = 2i \operatorname{Im}(z). \end{aligned}$$

(d)

$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2.$$

If $z = 0 = 0 + 0i$, then $z\overline{z} = a^2 + b^2 = 0$; if $z \neq 0$, then $z\overline{z} = a^2 + b^2 > 0$, so $z\overline{z}$ is real and positive.

■

Theorem 1.10 Let z and w be complex numbers. Then

(a) $|z| > 0$ unless $z = 0$, $|0| = 0$,

(b) $|\overline{z}| = |z|$,

(c) $|zw| = |z| |w|$,

(d) $|\operatorname{Re}(z)| \leq |z|$,

(e) $|z + w| \leq |z| + |w|$.

Proof. Let $z = a + bi$ and $w = c + di$, we have $\overline{z} = a - bi$. With Definition 1.13 and Definition 1.14, we have

(a) Because $|z| = \sqrt{z\overline{z}} = \sqrt{(a + bi)(a - bi)} = \sqrt{a^2 + b^2}$, if $z = 0 = 0 + 0i$, then $a = b = 0$, $|z| = \sqrt{|0|} = \sqrt{a^2 + b^2} = 0$; if $z \neq 0$, then a and b can not be both 0, so $|z| = \sqrt{a^2 + b^2} > 0$.

(b) Because $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$, then

$$|\bar{z}| = \sqrt{\bar{z}\bar{\bar{z}}} = \sqrt{\bar{z}z} = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2} = |z|.$$

(c) Because $zw = (ac - bd) + (ad + bc)i$, then

$$\begin{aligned} |zw| &= \sqrt{(ac - bd)^2 + (ad + bc)^2} \\ &= \sqrt{a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2} \\ &= \sqrt{(a^2 + b^2)(c^2 + d^2)} \\ &= \sqrt{a^2 + b^2}\sqrt{c^2 + d^2} \\ &= |z||w|. \end{aligned}$$

(d) Because $|\operatorname{Re}(z)| = |a|$ and $|z| = \sqrt{a^2 + b^2}$, we have

$$a^2 \leq a^2 + b^2 \quad \Rightarrow \quad \sqrt{a^2} = |a| = |\operatorname{Re}(z)| \leq \sqrt{a^2 + b^2} = |z|.$$

(e) Because $z\bar{w} + \bar{z}w = 2 \operatorname{Re}(z\bar{w})$, we have

$$\begin{aligned} |z + w|^2 &= (z + w)(\overline{z + w}) \\ &= (z + w)(\bar{z} + \bar{w}) \\ &= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\ &= |z|^2 + 2 \operatorname{Re}(z\bar{w}) + |w|^2 \\ &\leq |z|^2 + 2|z\bar{w}| + |w|^2 \\ &= |z|^2 + 2|z||w| + |w|^2 \\ &= (|z| + |w|)^2. \end{aligned}$$

Therefore, $|z + w| \leq |z| + |w|$.

■

Theorem 1.11 (Cauchy–Schwarz Inequality) If a_1, \dots, a_n and b_1, \dots, b_n are complex numbers, then

$$\left| \sum_{j=1}^n a_j \bar{b}_j \right|^2 \leq \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |b_j|^2.$$

Proof. With Theorem 1.12, we have

$$\left| \sum_{j=1}^n a_j \overline{b_j} \right|^2 \leq \left(\sum_{j=1}^n |a_j \overline{b_j}| \right)^2 \leq \left(\sum_{j=1}^n |a_j| |\overline{b_j}| \right)^2 = \left(\sum_{j=1}^n |a_j| |b_j| \right)^2.$$

We will use mathematical induction to prove that

$$\left(\sum_{j=1}^n |a_j| |b_j| \right)^2 \leq \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |b_j|^2.$$

- **Basis**

For $n = 1$,

$$\left(\sum_{j=1}^1 |a_j| |b_j| \right)^2 = (|a_1| |b_1|)^2 = |a_1|^2 |b_1|^2 = \sum_{j=1}^1 |a_j|^2 \sum_{j=1}^1 |b_j|^2.$$

Because $(|a_1| |b_2| - |a_2| |b_1|)^2 \geq 0$ and

$$(|a_1| |b_2| - |a_2| |b_1|)^2 = |a_1|^2 |b_2|^2 + |a_2|^2 |b_1|^2 - 2 |a_1| |a_2| |b_1| |b_2|,$$

we have

$$2 |a_1| |a_2| |b_1| |b_2| \leq |a_1|^2 |b_2|^2 + |a_2|^2 |b_1|^2.$$

Then we add $|a_1|^2 |b_1|^2 + |a_2|^2 |b_2|^2$ on both two sides of the inequation to get

$$(|a_1| |b_1| + |a_2| |b_2|)^2 \leq (|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2).$$

Therefore, for $n = 2$, we have

$$\left(\sum_{j=1}^2 |a_j| |b_j| \right)^2 \leq \sum_{j=1}^2 |a_j|^2 \sum_{j=1}^2 |b_j|^2.$$

- **Inductive Assumption**

Assume that for $i = 1, 2, \dots, n$ where n is a positive integer, there is

$$\left(\sum_{j=1}^i |a_j| |b_j| \right)^2 \leq \sum_{j=1}^i |a_j|^2 \sum_{j=1}^i |b_j|^2.$$

- **Inductive Step**

Let

$$\sum_{j=1}^n |a_j| |b_j| = \Sigma_{|a||b|}, \quad \sum_{j=1}^n |a_j|^2 = \Sigma_{|a|}, \quad \sum_{j=1}^n |b_j|^2 = \Sigma_{|b|},$$

so we have $(\Sigma_{|a||b|})^2 \leq \Sigma_{|a|}\Sigma_{|b|}$. Moreover, $(|a_{n+1}|\sqrt{\Sigma_{|b|}} - |b_{n+1}|\sqrt{\Sigma_{|a|}})^2 \geq 0$ and

$$(|a_{n+1}|\sqrt{\Sigma_{|b|}} - |b_{n+1}|\sqrt{\Sigma_{|a|}})^2 = |a_{n+1}|^2\Sigma_{|b|} + |b_{n+1}|^2\Sigma_{|a|} - 2|a_{n+1}||b_{n+1}|\sqrt{\Sigma_{|a|}}\sqrt{\Sigma_{|b|}},$$

we have

$$2|a_{n+1}||b_{n+1}|\Sigma_{|a||b|} \leq 2|a_{n+1}||b_{n+1}|\sqrt{\Sigma_{|a|}}\sqrt{\Sigma_{|b|}} \leq |a_{n+1}|^2\Sigma_{|b|} + |b_{n+1}|^2\Sigma_{|a|}.$$

For $i = n + 1$, then

$$\begin{aligned} \left(\sum_{j=1}^{n+1} |a_j||b_j| \right)^2 &= (\Sigma_{|a||b|} + |a_{n+1}||b_{n+1}|)^2 \\ &= (\Sigma_{|a||b|})^2 + |a_{n+1}|^2|b_{n+1}|^2 + 2|a_{n+1}||b_{n+1}|\Sigma_{|a||b|} \\ &\leq \Sigma_{|a|}\Sigma_{|b|} + |a_{n+1}|^2|b_{n+1}|^2 + |a_{n+1}|^2\Sigma_{|b|} + |b_{n+1}|^2\Sigma_{|a|} \\ &= (\Sigma_{|a|} + |a_{n+1}|)(\Sigma_{|b|} + |b_{n+1}|) \\ &= \sum_{j=1}^{n+1} |a_j|^2 \sum_{j=1}^{n+1} |b_j|^2. \end{aligned}$$

To sum up,

$$\left| \sum_{j=1}^n a_j \overline{b_j} \right|^2 \leq \left(\sum_{j=1}^n |a_j||b_j| \right)^2 \leq \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |b_j|^2.$$

■

Theorem 1.12 Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in R^k$, and α is real. Then

- (a) $|\mathbf{x}| \geq 0$;
- (b) $|\mathbf{x}| = 0$ if and only if $\mathbf{x} = \mathbf{0}$;
- (c) $|\alpha\mathbf{x}| = |\alpha||\mathbf{x}|$;
- (d) $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$;
- (e) $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$;
- (f) $|\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|$.

Proof. Suppose that $\mathbf{x} = (x_1, x_2, \dots, x_k)$, $\mathbf{y} = (y_1, y_2, \dots, y_k)$ and $\mathbf{z} = (z_1, z_2, \dots, z_k)$.

Then

(a)

$$|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{\sum_{i=1}^k x_i^2} \geq 0.$$

(b) • $|\mathbf{x}| = 0 \Rightarrow \mathbf{x} = \mathbf{0}$. We have

$$|\mathbf{x}| = \sqrt{\sum_{i=1}^k x_i^2} \geq 0 \Rightarrow x_1 = x_2 = \cdots = x_k = 0 \Rightarrow \mathbf{x} = \mathbf{0}.$$

• $|\mathbf{x}| = 0 \Leftarrow \mathbf{x} = \mathbf{0}$. We have

$$\mathbf{x} = \mathbf{0} \Rightarrow x_1 = x_2 = \cdots = x_k = 0 \Rightarrow |\mathbf{x}| = \sqrt{\sum_{i=1}^k x_i^2} = 0.$$

(c) Because $\alpha\mathbf{x} = (\alpha x_1, \alpha x_2, \cdots, \alpha x_k)$, we have

$$|\alpha\mathbf{x}| = \sqrt{\sum_{i=1}^k (\alpha x_i)^2} = \sqrt{\alpha^2 \sum_{i=1}^k x_i^2} = |\alpha| \sqrt{\sum_{i=1}^k x_i^2} = |\alpha| |\mathbf{x}|.$$

(d) With Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=1}^k |x_i| |y_i| \right)^2 \leq \sum_{i=1}^n |x_i|^2 \sum_{i=1}^n |y_i|^2.$$

so we can get

$$|\mathbf{x} \cdot \mathbf{y}| = \left| \sum_{i=1}^k x_i y_i \right| \leq \left| \sum_{i=1}^k |x_i| |y_i| \right| \leq \sqrt{\sum_{i=1}^n |x_i|^2} \sqrt{\sum_{i=1}^n |y_i|^2} = |\mathbf{x}| |\mathbf{y}|.$$

(e) Because $\mathbf{x} \cdot \mathbf{y} \leq |\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}| |\mathbf{y}|$, we have

$$|\mathbf{x} + \mathbf{y}|^2 = (\mathbf{x} + \mathbf{y})^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2 \mathbf{x} \cdot \mathbf{y} \leq |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2 |\mathbf{x}| |\mathbf{y}| = (|\mathbf{x}| + |\mathbf{y}|)^2.$$

Therefore, we get $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$.

(f) Because $\mathbf{x} - \mathbf{y}, \mathbf{y} - \mathbf{z} \in R^k$, we have

$$|\mathbf{x} - \mathbf{z}| = |(\mathbf{x} - \mathbf{y}) + (\mathbf{y} - \mathbf{z})| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|.$$

■