School of Mathematics and Statistics

Math1131 Mathematics 1A

CALCULUS LECTURE 4

LIMITS TO INFINITY (PART 2)

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MATH1131 CALCULUS
Limit of Functions at Infinity Part 2

$\lim\limits_{x \longrightarrow \infty} f(x) = L$ if for every (small) positive real number ϵ there is a (large) real number M with the property that if $x > M$ then $|f(x) - L| < \epsilon$.

Limits of the form $\lim\limits_{x \longrightarrow a} f(x)$ are best attacked via factorisation.

$\lim\limits_{x \longrightarrow a^-} f(x)$ is the limit of $f(x)$ as x approaches a from **the left**

$\lim\limits_{x \longrightarrow a^+} f(x)$ is the limit of $f(x)$ as x approaches a from **the right**

$\lim\limits_{x \longrightarrow a} f(x)$ exists if and only if $\lim\limits_{x \longrightarrow a^-} f(x)$ and $\lim\limits_{x \longrightarrow a^+} f(x)$ both exist and are equal.

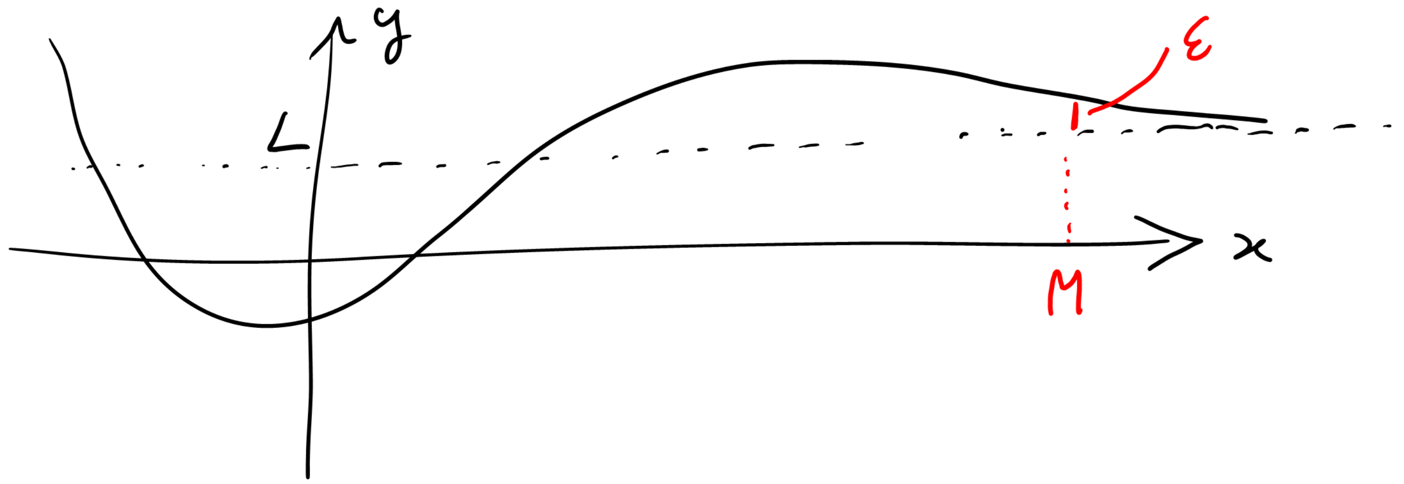
Just like differentiation from first principles we also have a formal definition of $\lim\limits_{x \longrightarrow \infty} f(x)$. It is crucial that we understand these fundamental definitions as they provide a solid foundation for the analysis. We can't just keep waving our hands around! If we want you to evaluate a limit formally we will always give you a clear warning.

Definition: $\lim\limits_{x \longrightarrow \infty} f(x) = L$ if for every (small) positive real number ϵ there is a (large) real number M with the property that if $x > M$ then $|f(x) - L| < \epsilon$.

Difference between $f(x)$ and L

Discussion

Note that $|f(x) - L| < \epsilon$ simply means that the difference between $f(x)$ and L is really really small. The limit definition is then simply saying that any degree of closeness between $f(x)$ and L can be generated by choosing x to be sufficiently large.



For $\lim\limits_{x \longrightarrow \infty} f(x)$ to be equal to L the function must eventually ($x > M$) get into (and stay in!) an ϵ band of L.

Example 1: Consider $\lim_{x\longrightarrow\infty} \frac{10x-4}{5x+7}$.

a) Show that the value of the limit is $L = 2$.

b) Find M so that $f(x)$ is within $\frac{1}{1000}$ of its limit whenever $x > M$.

c) Does b) prove that $\lim_{x\longrightarrow\infty} \frac{10x-4}{5x+7} = 2$?

d) Prove from the limit definition that $\lim_{x\longrightarrow\infty} \frac{10x-4}{5x+7} = 2$.

That is given $\epsilon > 0$ find M with the property that if $x > M$ then $|f(x) - L| < \epsilon$.

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a) $$\lim_{x\to\infty} \frac{10x-4}{5x+7} = \lim_{x\to\infty} \frac{10-4/x}{5+7/x} = \frac{10}{5} = 2$$

b) $$\left|\frac{10x-4}{5x+7} - 2\right| < \frac{1}{1000}$$

$$\left|\frac{10x-4-2(5x+7)}{5x+7}\right| < \frac{1}{1000}$$

$$\left|\frac{\cancel{10x}-4-\cancel{10x}-14}{5x+7}\right| < \frac{1}{1000} \Rightarrow \left|\frac{-18}{5x+7}\right| < \frac{1}{1000}$$

$$\frac{18}{|5x+7|} < \frac{1}{1000} \Rightarrow \frac{18}{5x+7} < \frac{1}{1000}$$

$$\frac{5x+7}{18} > 1000 \Rightarrow 5x+7 > 18000$$

$$5x > 17993 \Rightarrow x > \underline{\underline{3598.6}} \leftarrow M$$

So $\left|\frac{10x-4}{5x+7} - 2\right| \leq \frac{1}{1000}$ provided we choose $x > 3598.6$. Note that we are not really trying to solve $\left|\frac{10x-4}{5x+7} - 2\right| \leq \frac{1}{1000}$. We are just trying to find an M with the property that the inequality is true for all $x > M$. This value of M is not unique and any $M > 3598.6$ would also work

c) NO!!

d) $$\left| \frac{10x-4}{5x+7} - 2 \right| \leq \varepsilon$$

$$\left| \frac{10x-4 - 2(5x+7)}{5x+7} \right| \leq \varepsilon \Rightarrow \frac{|-18|}{|5x+7|} \leq \varepsilon$$

$$\frac{18}{5x+7} \leq \underbrace{\varepsilon}_{\text{little}} \Rightarrow \frac{5x+7}{18} \geq \underbrace{\frac{1}{\varepsilon}}_{\text{big}}$$

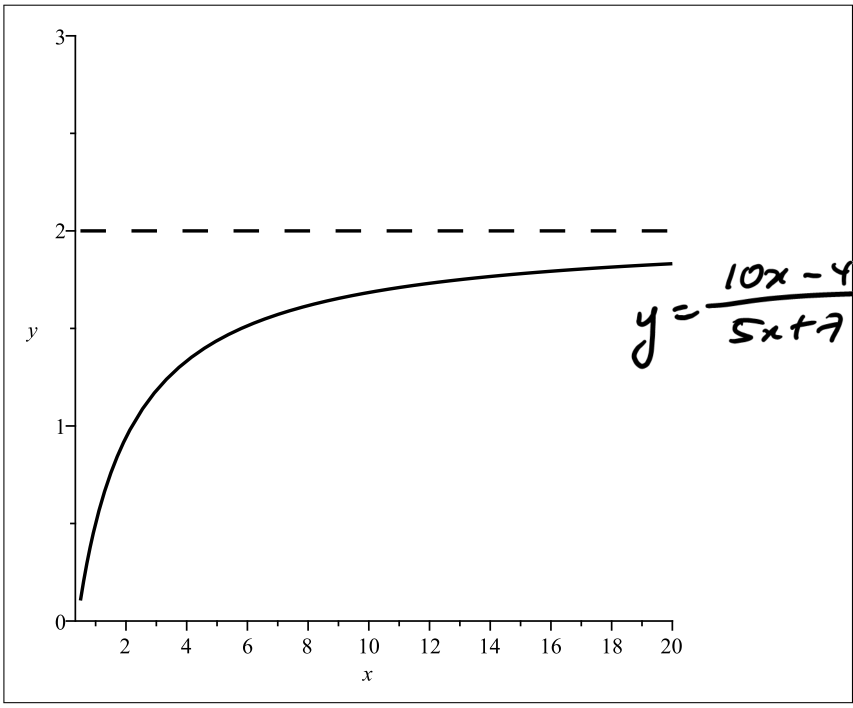
$$5x+7 \geq \frac{18}{\varepsilon} \Rightarrow 5x \geq \frac{18}{\varepsilon} - 7$$

$$x \geq \frac{18}{5\varepsilon} - 7/5 = \underline{M}$$

$$\star \quad M = \frac{18}{5\epsilon} - \frac{7}{5}. \quad \star$$

Note that since the above M depends on ϵ we sometimes write $M(\epsilon) = \dfrac{18}{5\epsilon} - \dfrac{7}{5}$.

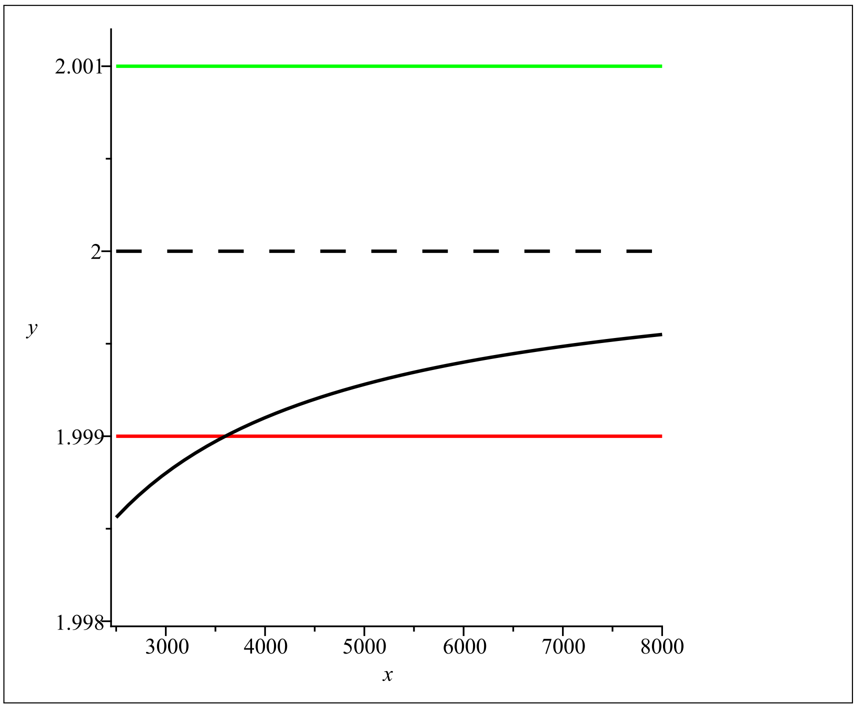
Let's take a look at the graph for the situation of part b) where $\epsilon = \dfrac{1}{1000} = 0.001$. First the the function and its limit of $L = 2$ as a horizontal asymptote:



$y = \dfrac{10x-4}{5x+7}$

Next a band (red to green) of length $\pm\dfrac{1}{1000}$ either side of the limit $L = 2$.

Observe below that after $M = 3598.6$ the function is clearly locked within this limit band.



Example 2: Find $\lim_{x \longrightarrow \infty} \frac{\sin(x)}{x^2 + e^x}$ and prove from the limit definition that your answer is correct. Find a value of M for $\epsilon = \frac{1}{100}$ and sketch the behaviour of the function and its limit.

Clearly $\lim_{x \longrightarrow \infty} \frac{\sin(x)}{x^2 + e^x} =$

$|\sin(x)| < 1$

$$\left| \lim_{x \to \infty} \frac{\sin x}{x^2 + e^x} \right| = \lim_{x \to \infty} \left| \frac{\sin(x)}{x^2 + e^x} \right|$$

$$\leq \lim_{x \to \infty} \frac{1}{x^2 + e^x} = 0.$$

$$\therefore \lim_{x \to \infty} \frac{\sin(x)}{x^2 + e^x} = 0$$

So $L = 0$. We now examine $|f(x) - L| = \left| \frac{\sin(x)}{x^2 + e^x} - 0 \right| =$ $\left| \frac{\sin(x)}{x^2 + e^x} \right|$

$0 < |\sin(x)| < 1$

$$\leq \frac{1}{|x^2 + e^x|} \leq \frac{1}{x^2}$$

So we have $|f(x) - L| < \frac{1}{x^2}$. we therefore only need $\frac{1}{x^2} < \epsilon$. And hence: We need

$|f(x) - L| < \varepsilon$

$\left| \frac{\sin(x)}{x^2 + e^x} \right| < \varepsilon$

Suffices to consider $\frac{1}{x^2} < \varepsilon$

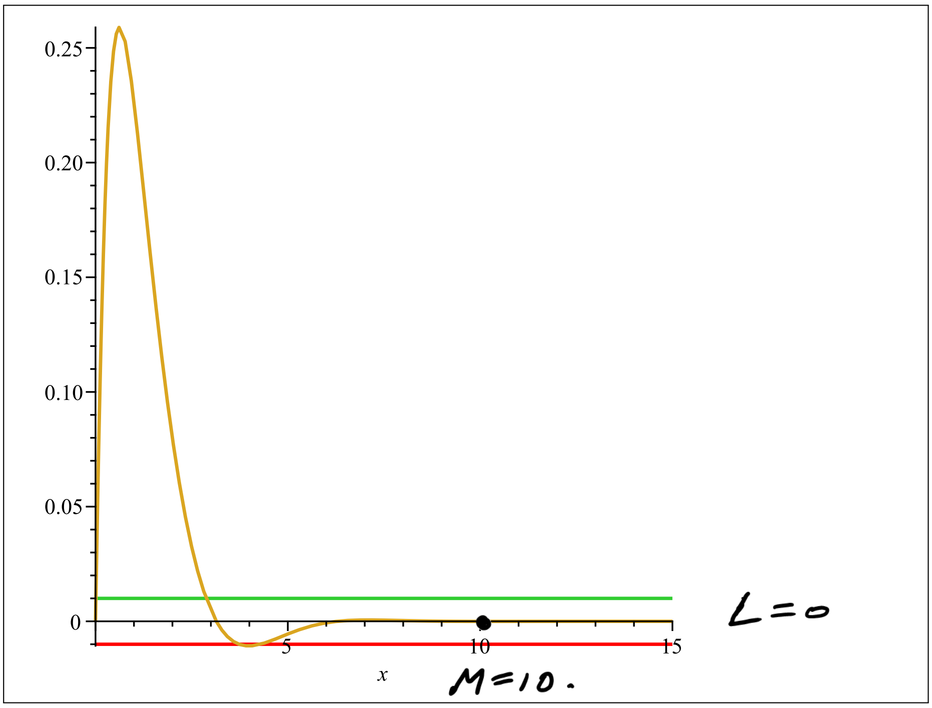
$$x^2 > \frac{1}{\varepsilon} \Rightarrow x > \frac{1}{\sqrt{\varepsilon}}$$

$M.$

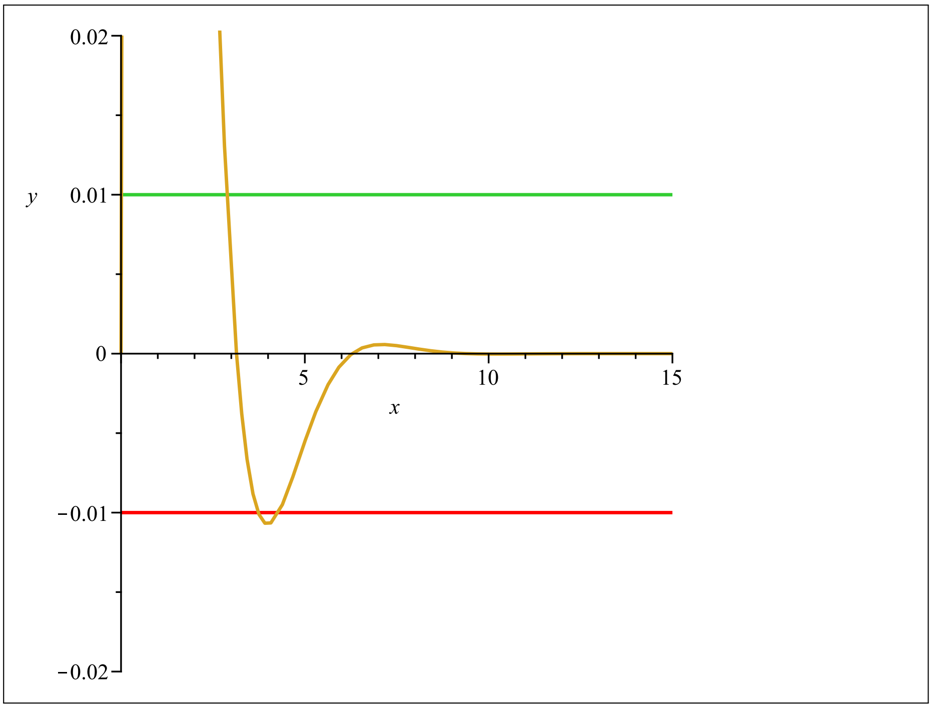
$\underline{\underline{\varepsilon = \frac{1}{100}}} \Rightarrow x > \frac{1}{\sqrt{\frac{1}{100}}} = 10 = M.$

★ $M = 10$ ★

Let's take a look at the graph:



Observe below that after $M = 10$ the function is clearly within its $\frac{1}{100}$ limit band although a smaller value of $M = 5$ would also do!!



Note that above with $\varepsilon = \frac{1}{100}$ we have most definitely **NOT** proven that $\lim_{x \longrightarrow \infty} \frac{\sin(x)}{x^2 + e^x} = 0$! The full proof would demand that we use a general small ϵ rather than the specific $\frac{1}{100}$.

Limit of Functions at a Point

There are certain situations where a function fails to be defined **at** a point but it is perfectly happy **near** the point. We then use $\lim\limits_{x\longrightarrow a} f(x)$ to get a feeling for the behaviour of the function.

Consider the following table of values for $f(x) = \dfrac{x^2-25}{x-5}$ near $x = 5$.

x	4.9	4.99	5	5.01	5.1
y	9.9	9.99	?	10.01	10.1

It is clear that the function is trying to get to 10 at $x = 5$ even though it is undefined there. We write

$$\lim_{x\longrightarrow 5} \frac{x^2-25}{x-5} = 10$$

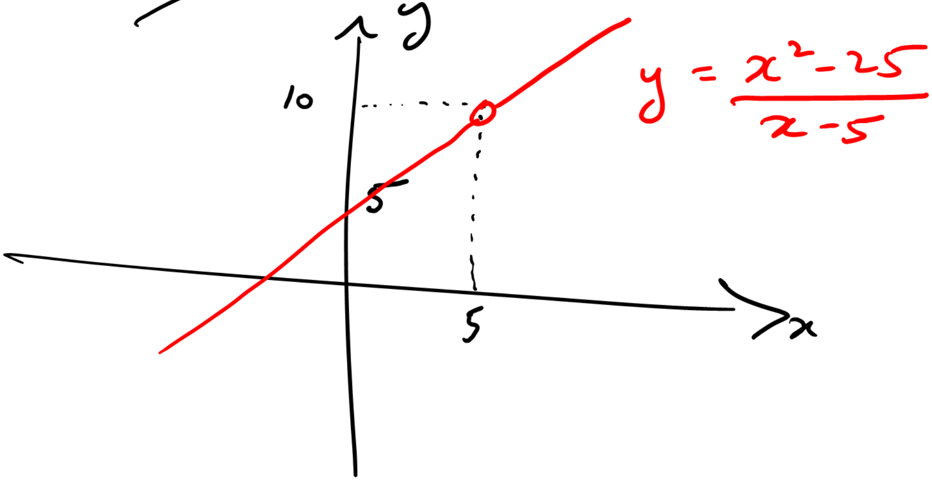
and say

"The limit as x approaches 5 of $\dfrac{x^2-25}{x-5}$ is 10".

Example 3: Find $\lim\limits_{x\longrightarrow 5} \dfrac{x^2-25}{x-5}$ and hence sketch the function $y = \dfrac{x^2-25}{x-5}$.

$$\lim_{x\to 5} \frac{x^2-25}{x-5} = \text{"}\frac{0}{0}\text{"}$$

$$= \lim_{x\to 5} \frac{\cancel{(x-5)}(x+5)}{\cancel{(x-5)}} = \lim_{x\to 5} (x+5) = \underline{\underline{10}}$$



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So for limits as x approaches a finite value our main technique is to factorise top and bottom. Always check first that you are facing the indeterminate form "$\frac{0}{0}$"

Example 4: Evaluate each of the following limits

a) $\lim\limits_{x \longrightarrow 2} \dfrac{x^2 - 5x + 6}{x^2 - 4}$

b) $\lim\limits_{x \longrightarrow 1} \dfrac{x^2 + 1}{x^2 + 4}$

a) $\lim\limits_{x \to 2} \dfrac{x^2-5x+6}{x^2-4} = \text{"}\dfrac{0}{0}\text{"}$

$= \lim\limits_{x \to 2} \dfrac{\cancel{(x-2)}(x-3)}{\cancel{(x-2)}(x+2)} = \lim\limits_{x \to 2} \dfrac{(x-3)}{(x+2)}$

$= \dfrac{-1}{4} = -\dfrac{1}{4}$

b) $\lim\limits_{x \to 1} \dfrac{x^2+1}{x^2+4} = \dfrac{2}{5}$

$\bigstar \quad a)\ -\dfrac{1}{4} \quad b)\ \dfrac{2}{5} \quad \bigstar$

In order to define these limits a little more formally we require the concept of a one-sided limit:

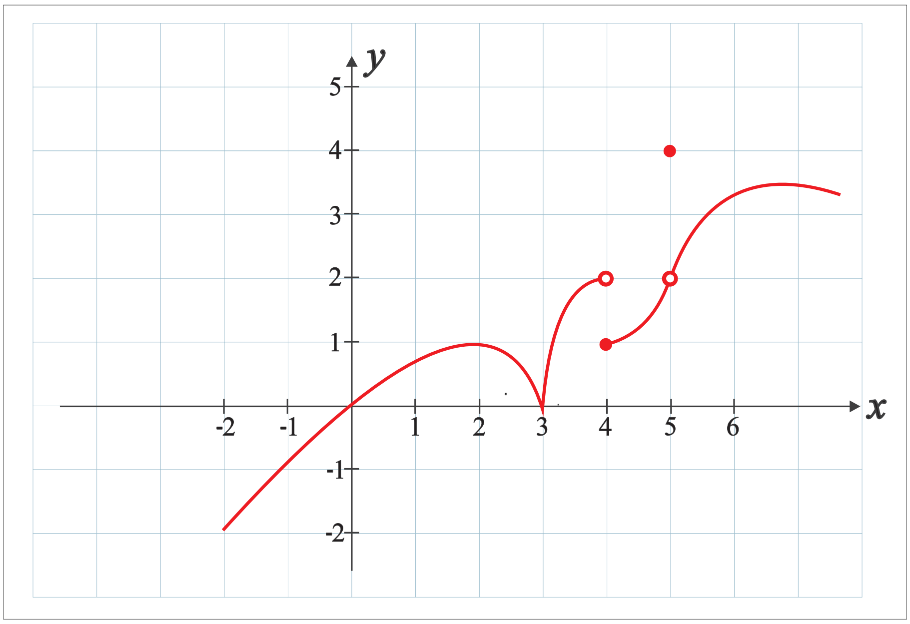
$\lim\limits_{x \longrightarrow a^-} f(x)$ is the limit of $f(x)$ as x approaches a from **the left**

$\lim\limits_{x \longrightarrow a^+} f(x)$ is the limit of $f(x)$ as x approaches a from **the right**

We can then say that the full limit $\lim\limits_{x \longrightarrow a} f(x)$ formally exists if and only if $\lim\limits_{x \longrightarrow a^-} f(x)$ and $\lim\limits_{x \longrightarrow a^+} f(x)$ both exist and are equal.

Example 5: Consider the graph of $y = f(x)$ presented in red below:

For each of the following either evaluate the given quantity or explain why it does not exist:



a) $\lim_{x \longrightarrow 4^-} f(x) = 2$ b) $\lim_{x \longrightarrow 4^+} f(x) = 1$

c) $\lim_{x \longrightarrow 4} f(x)$ d) $\lim_{x \longrightarrow 5} f(x)$

e) $f(5)$ f) $\lim_{x \longrightarrow 6} f(x)$

g) $\lim_{x \longrightarrow 3} f(x)$

LH limit $\neq$ RH limit

c) $\lim_{x \to 4} f(x)$ does not exist

d) $\lim_{x \to 5^-} f(x) = 2$, $\lim_{x \to 5^+} f(x) = 2$

$\therefore \lim_{x \to 5} f(x) = 2$

e) $f(5) = 4$

f) $\lim_{x \to 6} f(x) \doteqdot 3\frac{1}{3}$

g) $\lim_{x \to 3} f(x) = 0$ since $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 0$