LECTURE 19
Improper Integrals

$$\int_a^\infty f(x)\,dx = \lim_{R\longrightarrow\infty}\int_a^R f(x)\,dx$$

The improper integral $\displaystyle\int_a^\infty \frac{1}{x^p}\,dx$ is called a $p-$integral.

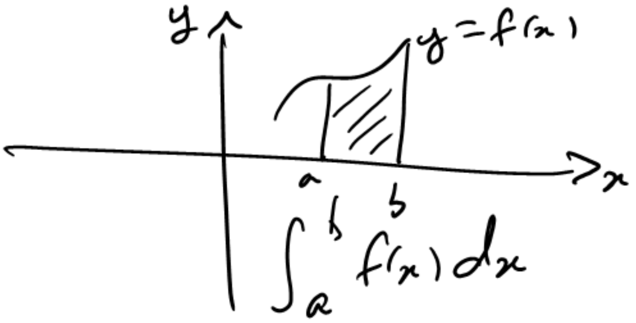
A $p-$integral converges if $p > 1$ and it diverges if $p \leq 1$.

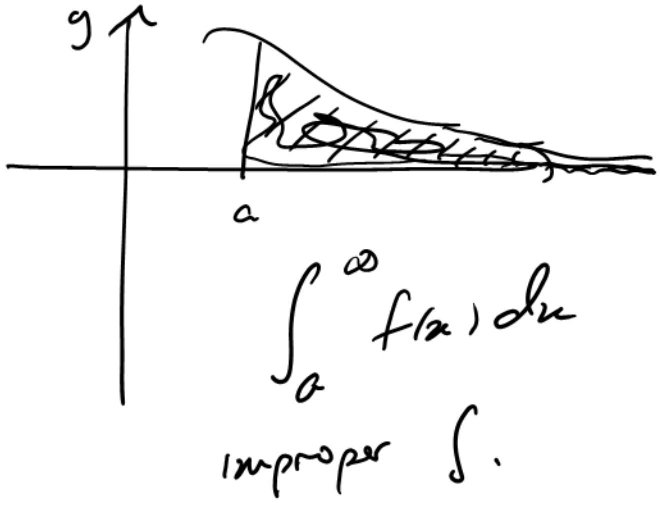
(The Comparison Test) Suppose that f and g are integrable functions and that $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

i) $\displaystyle\int_a^\infty g(x)\,dx$ converges $\Longrightarrow \displaystyle\int_a^\infty f(x)\,dx$ converges.

ii) $\displaystyle\int_a^\infty f(x)\,dx$ diverges $\Longrightarrow \displaystyle\int_a^\infty g(x)\,dx$ diverges.

Up until this point we have always integrated functions between two finite limits a and b. We now investigate the interesting possibility of letting b become ∞ to produce what is called an **improper** integral $\displaystyle\int_a^\infty f(x)\,dx$. As you may well imagine this causes lots of problems! The picture of the situation is





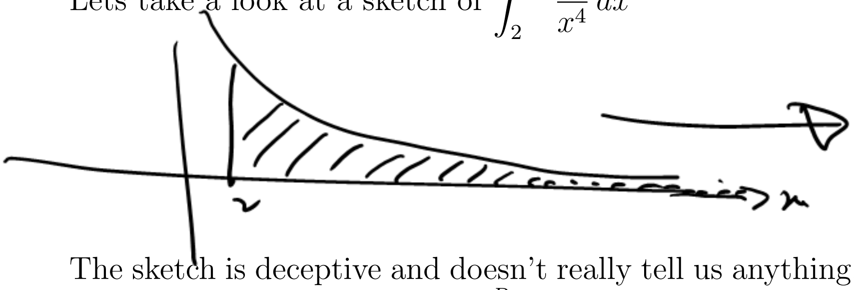
Your initial reaction may be that the region has an infinite tail, therefore its area is infinite. Remarkably this is not necessarily the case. It is possible for the area $\displaystyle\int_a^\infty f(x)\,dx$ to be finite! We then say that the improper integral converges. Else we say that the improper integral diverges.

How can something infinite generate something finite? You have already seen this happen with GP's where the infinite series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots\ldots$ converges to a finite value of 1. The key to evaluating improper integrals is to make effective use of limits. We make the following definition

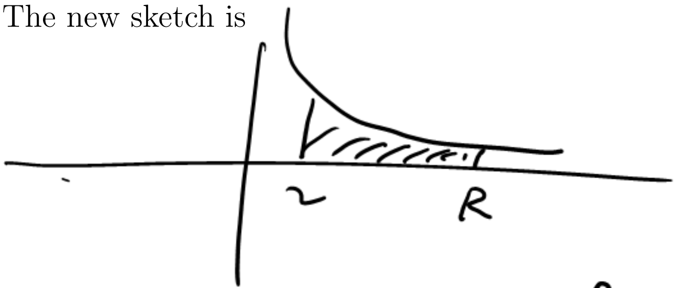
$$\int_a^\infty f(x)\,dx = \lim_{R\longrightarrow\infty}\int_a^R f(x)\,dx$$

Example 1: Show that the improper integral $\int_2^\infty \frac{1}{x^4}\,dx$ converges.

Lets take a look at a sketch of $\int_2^\infty \frac{1}{x^4}\,dx$



The sketch is deceptive and doesn't really tell us anything. Our method is to abandon $\int_2^\infty \frac{1}{x^4}\,dx$ and consider instead $\int_2^R \frac{1}{x^4}\,dx$. The new sketch is



This is easily evaluated.

$$\int_2^R x^{-4}\,dx = \left[\frac{x^{-3}}{-3}\right]_2^R = \left[-\frac{1}{3x^3}\right]_2^R$$

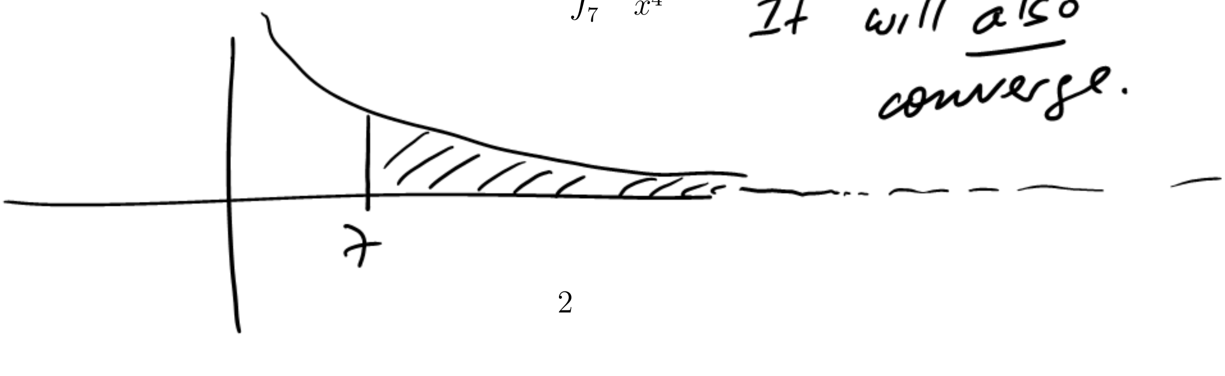
$$= -\frac{1}{3R^3} + \frac{1}{3}\cdot\frac{1}{8} = \frac{1}{24} - \frac{1}{3R^3}$$

Now we simply let $R \longrightarrow \infty$. Whatever happens happens. If we get a finite number the integral converges. Else the integral diverges.

$$\text{Now } \lim_{R\to\infty} \frac{1}{24} - \cancel{\frac{1}{3R^3}} = \boxed{\frac{1}{24}}\ u^2$$

★ The improper integral converges to $\frac{1}{24}$ ★

Question: What can we now say about $\int_7^\infty \frac{1}{x^4}\,dx$? It will also converge.

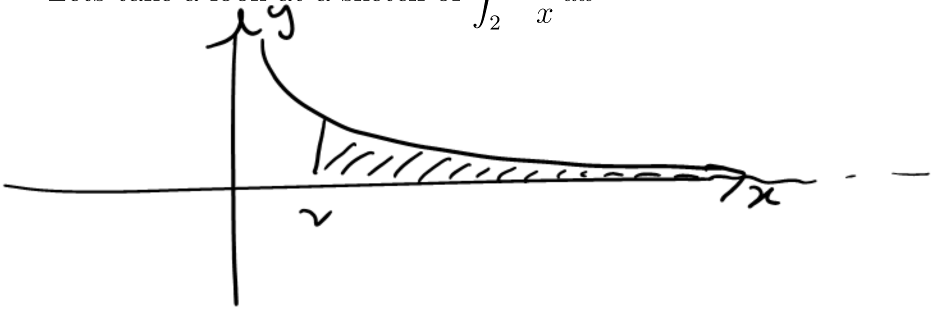


Remember to always have as the final step of your direct calculation of an improper integral the clear evaluation of a limit!

The next example shows you that the situation is fairly complex.

Example 2: Show that the improper integral $\int_2^\infty \frac{1}{x}\,dx$ diverges.

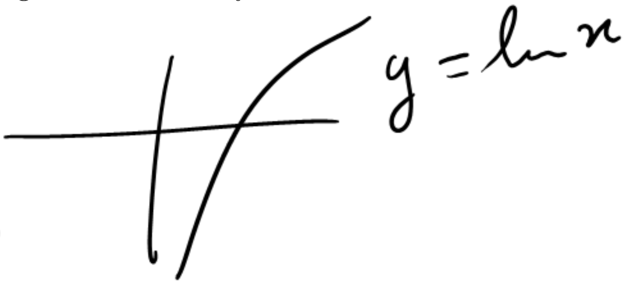
Lets take a look at a sketch of $\int_2^\infty \frac{1}{x}\,dx$



We abandon $\int_2^\infty \frac{1}{x}\,dx$ and consider instead $\int_2^R \frac{1}{x}\,dx$. The new sketch is

$$\int_2^R \frac{1}{x}\,dx = \left[\ln x\right]_2^R = \ln R - \ln(2)$$

This looks the same as before! But the improper integral is drastically different.



Letting $R \longrightarrow \infty$.

$$\lim_{R\to\infty} \underbrace{\ln R}_{\infty} - \ln(2) = \infty.$$

★ The improper integral diverges ★

Question: What can we now say about $\int_7^\infty \frac{1}{x}\,dx$?

It will also diverge. The convergence/divergence of an improper integral $\int_a^\infty f(x)\,dx$ is completely determined by its tail. The lower limit a is of little consequence

So why did the first integral converge and the second diverge? The fundamental difference is that $\frac{1}{x^4}$ goes to zero faster than $\frac{1}{x}$.

For example at $x = 5$, $\frac{1}{x^4} = \frac{1}{3125}$ while $\frac{1}{x} = \frac{1}{5}$ only.

This is the key. For $\int_a^\infty f(x)\,dx$ to converge, the function $f(x)$ certainly needs to go to 0 as $x \longrightarrow \infty$. But this is not enough. The function f needs to go to zero fast enough, to avoid the accumulation of an infinite amount of area.

We have a way of measuring all of this.

Definition: The improper integral $\int_a^\infty \frac{1}{x^p}\,dx$ is called a $p-$integral.

Fact: A $p-$integral converges if $p > 1$ and diverges if $p \leq 1$.

Thus for example $\int_3^\infty \frac{1}{x^7}\,dx$ and $\int_1^\infty \frac{1}{x^3}\,dx$ both converge (getting to zero fast).

But $\int_1^\infty \frac{1}{x}\,dx$ and $\int_4^\infty \frac{1}{\sqrt{x}}\,dx$ both diverge (too slow to 0).

We will use $p-$integrals extensively later on when using comparison tests. There are many improper integrals which are not however $p-$integrals:

Example 3: Determine whether the following integrals converge or diverge.

a) $\int_0^\infty e^{-2x}\,dx$

$\lim_{R\to\infty} \int_0^R e^{-2x}\,dx$

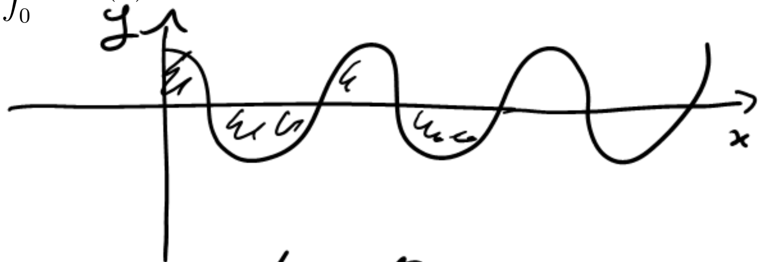
$= \lim_{R\to\infty} \left[-\frac{1}{2}e^{-2x}\right]_0^R$

$= \lim_{R\to\infty} -\frac{1}{2}e^{-2R} + \frac{1}{2}e^{-0}$

$= \lim_{R\to\infty} \frac{1}{2} - \cancel{\frac{1}{2e^{2R}}}$

$= \boxed{\frac{1}{2}}$

b) $\int_0^\infty \cos(x)\,dx$



$\cos(x) \not\to 0$

$\therefore \int_0^\infty \cos(x)\,dx$ diverges.

★ *a*) Converges to $\frac{1}{2}$ *b*) Diverges ★

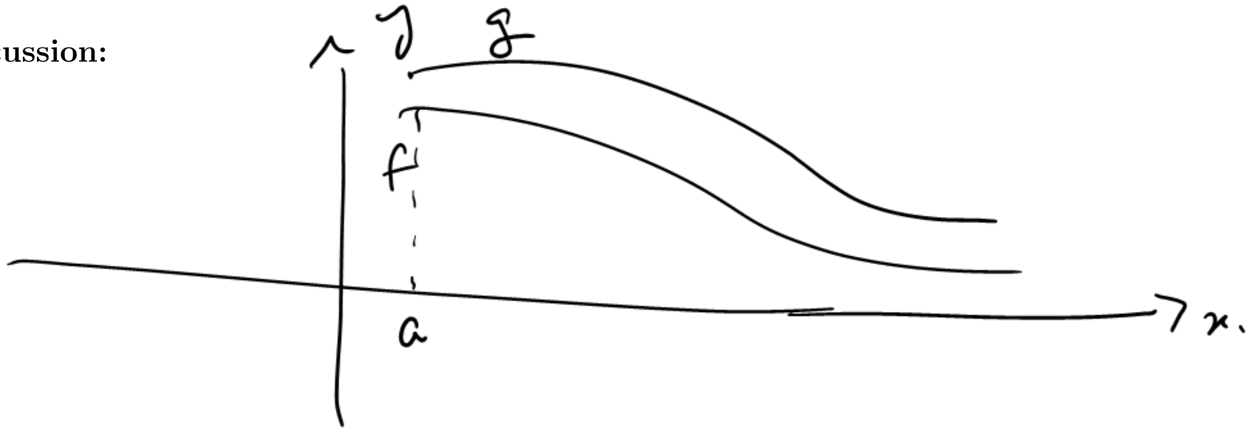
We now turn to the problem of convergence and divergence for improper integrals where we cannot actually find a primitive. Under these circumstance we try to establish whether or not they converge (without actually evaluating a final answer) by using strategic comparisons.

(The Comparison Test)

Suppose that f and g are integrable functions and that $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

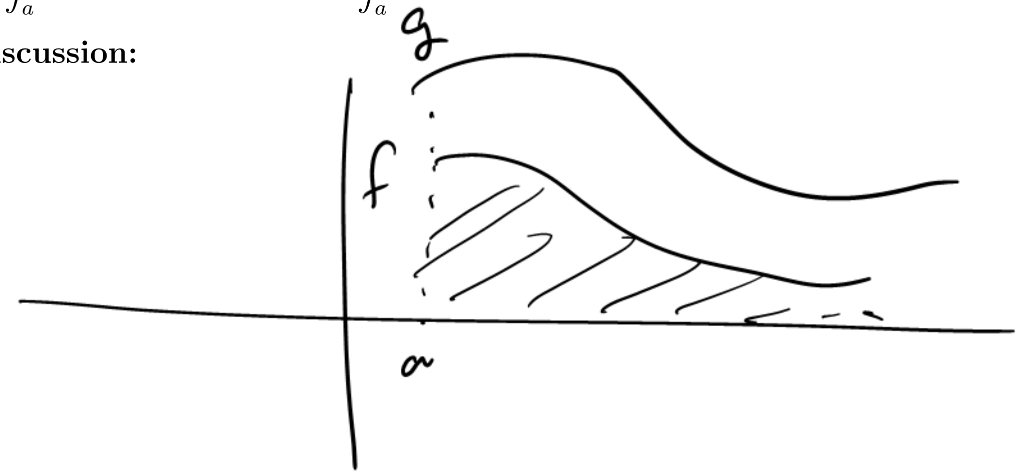
i) $\displaystyle\int_a^\infty g(x)\,dx$ converges $\Longrightarrow \displaystyle\int_a^\infty f(x)\,dx$ converges.

Discussion:



ii) $\displaystyle\int_a^\infty f(x)\,dx$ diverges $\Longrightarrow \displaystyle\int_a^\infty g(x)\,dx$ diverges.

Discussion:



Note carefully however that $\displaystyle\int_a^\infty g(x)\,dx$ diverging gives you **NO** information!!

Similarly $\displaystyle\int_a^\infty f(x)\,dx$ converging gives you **NO** information!!

To use the comparison test **your** integral must be smaller than a known converging integral or bigger than a known diverging integral, usually a $p-$integral.

When making comparisons remember that making the bottom of a fraction bigger will make the fraction smaller and vice versa. So for example $\frac{3}{7} > \frac{3}{10}$ because the denominator is larger on the right.

Example 4: Determine whether or not the following improper integrals converge or diverge by using the comparison test.

Note that there is no chance of actually evaluating the integral in either case. But the question is not one of evaluation! You are only being asked to investigate whether or not the integral converges.

a) $\displaystyle\int_3^\infty \frac{4}{x^2 + e^x + 7}\,dx$ $\displaystyle < \int_3^\infty \frac{4}{e^x} = 4\int^\infty e^{-x}\,dx.$

$$= 4 \lim_{R\to\infty} \int_3^R e^{-x}\,dx.$$

$$= 4 \lim_{R\to\infty} \left[-e^{-x}\right]_3^R = 4 \lim_{R\to\infty} \left\{\cancel{-e^{-R}} + e^{-3}\right\}$$

$$= 4/e^3$$

$\therefore$ By comparison test $\displaystyle\int_3^\infty \frac{4}{x^2+e^x+7}$ also converges.

★ The improper integral converges ★

b) $\displaystyle\int_{10}^\infty \frac{\cos(x) + 6}{\sqrt{x} - 2}\,dx$ $\displaystyle > \int_{10}^\infty \frac{1}{\sqrt{x} - 2}\,dx.$

since $\cos x + 6 \geq 5$.

$$> \int_{10}^\infty \frac{1}{\sqrt{x}}$$

$$= \int_{10}^\infty \frac{1}{x^{\frac{1}{2}}}\,dx$$

which is a diverging p-integral $p = \frac{1}{2} < 1$

$\therefore$ By comparison test $\displaystyle\int_{10}^\infty \frac{\cos(x)+6}{\sqrt{x}-2}$ also diverges

★ The improper integral diverges ★

Just two small notes to finish off.

• Firstly what do we do with $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$? The answer is quite simple. The improper integral $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ converges if and only if $\displaystyle\int_{0}^{\infty} f(x)\,dx$ and $\displaystyle\int_{-\infty}^{0} f(x)\,dx$ **both** converge.

So for example $\displaystyle\int_{-\infty}^{\infty} x\,dx$ is a divergent integral since both $\displaystyle\int_{0}^{\infty} x\,dx$ and $\displaystyle\int_{-\infty}^{0} x\,dx$ diverge. It is divergent even though $\displaystyle\int_{-R}^{R} x\,dx = 0$ for all R.

• Secondly we are often in a position where we suspect that an integral is divergent but can't quite get the inequalities pointing in the right direction.

For example consider $\displaystyle\int_{5}^{\infty} \frac{1}{x+1}\,dx$. We know that $\displaystyle\int_{5}^{\infty} \frac{1}{x}\,dx$ is a divergent $p-$integral and that $\displaystyle\frac{1}{x+1} < \frac{1}{x}$ but this doesn't help! The inequality is pointing in the wrong direction for the comparison test. In the next lecture we will facilitate these intuitive attacks by using the more subtle **limit form** of the comparison test.