MATH1131 Mathematics 1A – Algebra

Lecture 14: Complex Polynomials

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Based on slides by Jonathan Kress

Complex polynomials

Definition

Suppose $n \in \mathbb{N}$. A complex polynomial of degree n is a complex-valued function p of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \qquad \text{for all } z \in \mathbb{C},$$

where $a_0, \ldots, a_n \in \mathbb{C}$ are the coefficients of p (with with $a_n \neq 0$).

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- The degree of the zero polynomial is undefined.

These are polynomials:

$$p(z) = z^3 + z + 1 \qquad q(z) = 2z^3 - iz^2 + 4$$

$$r(z) = 4z^5 - z^2 + 3i \qquad s(z) = i$$

$$f(z) = z + 1 \qquad g(z) = z^2 + 1$$

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These are not polynomials:

$$p(z) = \sin z \qquad q(z) = e^z$$
$$r(z) = \frac{z+1}{z-1} \qquad s(z) = z^2 + z - 1 + \sqrt{z}$$

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  For example, $p(z) = z^2 + 1 = (z + i)(z - i)$ has factors $z + i$ and $z - i$.

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Example

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So the remainder is $p(-4) = (-4)^3 + 5 \times (-4)^2 - 6 \times (-4) + 3 = 43$.

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where the n complex numbers $\alpha_1, \alpha_2, \ldots \alpha_n$ are the roots of p, and $a \in \mathbb{C}$.

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Note that this means every complex polynomial of degree n has exactly n complex roots, counting with multiplicity (i.e. counting repeated roots separately).

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$$\begin{aligned} \text{So } z^3 - 8i &= (z - 2e^{i\frac{\pi}{6}})(z - 2e^{i\frac{5\pi}{6}})(z - 2e^{-i\frac{\pi}{2}}) \\ &= (z - (\sqrt{3} + i))(z - (i - \sqrt{3}))(z + 2i). \end{aligned}$$

Real polynomials and conjugate roots

Theorem

If $\alpha \in \mathbb{C}$ is a root of a real polynomial p, then $\overline{\alpha}$ is also a root of p.

Proof

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Suppose p is a real polynomial and $\alpha \in \mathbb{C}$ is a root of p, that is,

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where $a_0, a_1, \ldots, a_n$ are real, and $p(\alpha) = 0$.

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where $a_0, a_1, \ldots, a_n$ are real, and $p(\alpha) = 0$. Then

$$\begin{aligned}
p(\overline{\alpha}) &= a_n \overline{\alpha}^n + \ldots + a_1 \overline{\alpha} + a_0 \\
&= \overline{a_n}\,\overline{\alpha}^n + \ldots + \overline{a_1}\,\overline{\alpha} + \overline{a_0} \\
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$$\begin{aligned}
p(\overline{\alpha}) &= a_n \overline{\alpha}^n + \ldots + a_1 \overline{\alpha} + a_0 \\
&= \overline{a_n}\, \overline{\alpha}^n + \ldots + \overline{a_1}\, \overline{\alpha} + \overline{a_0} \\
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Hence $\overline{\alpha}$ is also a root of p.

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Using this method, every real polynomial can be factored into real linear and quadratic factors.

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So the six (complex) roots are $z = 1\,,\ e^{i\frac{\pi}{3}}\,,\ e^{i\frac{2\pi}{3}}\,,\ -1\,,\ e^{-i\frac{2\pi}{3}}\,,\ e^{-i\frac{\pi}{3}}$.

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So as a product of linear factors,

$$z^6 - 1 = (z-1)(z-e^{i\frac{\pi}{3}})(z-e^{i\frac{2\pi}{3}})(z+1)(z-e^{-i\frac{2\pi}{3}})(z-e^{-i\frac{\pi}{3}}).$$

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To find the real quadratic factors, consider the non-real roots in pairs of conjugates: $e^{i\frac{\pi}{3}}$ with $e^{-i\frac{\pi}{3}}$, and $e^{i\frac{2\pi}{3}}$ with $e^{-i\frac{2\pi}{3}}$.

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To find the real quadratic factors, consider the non-real roots in pairs of conjugates: $e^{i\frac{\pi}{3}}$ with $e^{-i\frac{\pi}{3}}$, and $e^{i\frac{2\pi}{3}}$ with $e^{-i\frac{2\pi}{3}}$.

$$(z - e^{i\frac{\pi}{3}})(z - e^{-i\frac{\pi}{3}})$$

Real polynomials and conjugate roots – Example

Example

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$$(z - e^{i\frac{\pi}{3}})(z - e^{-i\frac{\pi}{3}}) = z^2 - (e^{i\frac{\pi}{3}} + e^{-i\frac{\pi}{3}})z + e^{i\frac{\pi}{3}}e^{-i\frac{\pi}{3}}$$

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$$\begin{aligned}(z - e^{i\frac{\pi}{3}})(z - e^{-i\frac{\pi}{3}}) &= z^2 - (e^{i\frac{\pi}{3}} + e^{-i\frac{\pi}{3}})z + e^{i\frac{\pi}{3}}e^{-i\frac{\pi}{3}} \\ &= z^2 - 2\mathrm{Re}(e^{i\frac{\pi}{3}})z + |e^{i\frac{\pi}{3}}|^2\end{aligned}$$

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$$
\begin{aligned}
(z - e^{i\frac{\pi}{3}})(z - e^{-i\frac{\pi}{3}}) &= z^2 - (e^{i\frac{\pi}{3}} + e^{-i\frac{\pi}{3}})z + e^{i\frac{\pi}{3}} e^{-i\frac{\pi}{3}} \\
&= z^2 - 2\mathrm{Re}(e^{i\frac{\pi}{3}})z + |e^{i\frac{\pi}{3}}|^2 \\
&= z^2 - 2\cos\left(\frac{\pi}{3}\right) z + 1
\end{aligned}
$$

Real polynomials and conjugate roots – Example

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&= z^2 - 2\cos\left(\frac{\pi}{3}\right) z + 1 \\
&= z^2 - z + 1
\end{aligned}$$

Real polynomials and conjugate roots – Example

Example

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&= z^2 - 2\cos\left(\tfrac{\pi}{3}\right)z + 1 \\
&= z^2 - z + 1
\end{aligned}$$

Similarly, $(z - e^{i\frac{2\pi}{3}})(z - e^{-i\frac{2\pi}{3}})$

Real polynomials and conjugate roots – Example

Example

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\end{aligned}$$

Similarly, $(z - e^{i\frac{2\pi}{3}})(z - e^{-i\frac{2\pi}{3}}) = z^2 - 2\cos\left(\frac{2\pi}{3}\right) z + 1$

Real polynomials and conjugate roots – Example

Example

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&= z^2 - 2\cos\left(\frac{\pi}{3}\right) z + 1 \\
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\end{aligned}
$$

Similarly, $(z - e^{i\frac{2\pi}{3}})(z - e^{-i\frac{2\pi}{3}}) = z^2 - 2\cos\left(\frac{2\pi}{3}\right) z + 1 = z^2 + z + 1$.

Real polynomials and conjugate roots – Example

Example

Express $z^6 - 1$ as a product of linear factors, and again as a product of **real** linear and quadratic factors.

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&= z^2 - 2\cos\left(\frac{\pi}{3}\right)z + 1 \\
&= z^2 - z + 1
\end{aligned}$$

Similarly, $(z - e^{i\frac{2\pi}{3}})(z - e^{-i\frac{2\pi}{3}}) = z^2 - 2\cos\left(\frac{2\pi}{3}\right)z + 1 = z^2 + z + 1.$

So $z^6 - 1 = (z-1)(z+1)(z^2 - z + 1)(z^2 + z + 1)$.