School of Mathematics and Statistics

Math1131 Mathematics 1A

CALCULUS LECTURE 7

DIFFERENTIABLE FUNCTIONS

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MATH1131 CALCULUS
Differentiable Functions

$$\frac{dy}{dx} = f'(x) = \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h}$$

If $f'(x) > 0$ for all x in an interval (a, b) then f is increasing on (a, b).

If $f'(x) < 0$ for all x in an interval (a, b) then f is decreasing on (a, b).

If $f'(x) = 0$ for all x in an interval (a, b) then f is constant on (a, b).

If $y = f(x)g(x)$ then $\dfrac{dy}{dx} = f'(x)g(x) + g'(x)f(x)$.

If $y = \dfrac{f(x)}{g(x)}$ then $\dfrac{dy}{dx} = \dfrac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$.

If $y = f(g(x))$ then $\dfrac{dy}{dx} = f'(g(x))g'(x)$.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

$$\frac{d}{dx}(\tan(x)) = \sec^2(x)$$

$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$$

* Note that we must use radians when dealing with the calculus of trig functions.

We turn now to differentiation and the calculus. This deals with the central problem of calculating the gradient of a function at any point. First developed by Gottfried Leibniz and Sir Isaac Newton in the late 1600's the derivative provides us with the perfect tool for calculating instantaneous rates of change.

Almost every application of mathematics to the physical sciences involves calculus in one way or another, so we will make a very detailed and formal attack on the theory. I will assume that you are already familiar with the basics of calculus from your high school studies. If not you will need to devote some extra time to make sure you understand this lecture, which is essentially just revision of your high school calculus.

When faced with the problem of calculating the gradient of $y = f(x)$, Newton's great realization was that it was possible to quickly and accurately calculate the gradient function $\frac{dy}{dx}$. The gradient function is called the derivative of $y = f(x)$ and is also often denoted by $f'(x)$. The formal limit definition of the derivative

$$\frac{dy}{dx} = f'(x) = \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h}$$

is rarely used! Instead we calculate $\frac{dy}{dx}$ through an increasingly sophisticated system of algorithms!

The term "gradient" is also referred to as slope or rate of change of the function. If the gradient (that is derivative) is positive the curve is increasing. A negative derivative signals a decreasing function and a zero derivative is usually an indication of a local max or min.

Example 1: Find the gradient of $y = 7x^2 - 5x + 3$ at the point where $x = 2$.

Using the facts that $\frac{d}{dx}(x^n) = nx^{n-1}$ and that the process of differentiation respects linearity we can easily find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = 7(2x^1) - 5 + 0$$

$$= 14x - 5$$

$$\frac{dy}{dx} = 14x - 5$$

at $x = 2$: $\frac{dy}{dx} = 14(2) - 5 = \underline{\underline{23}}$

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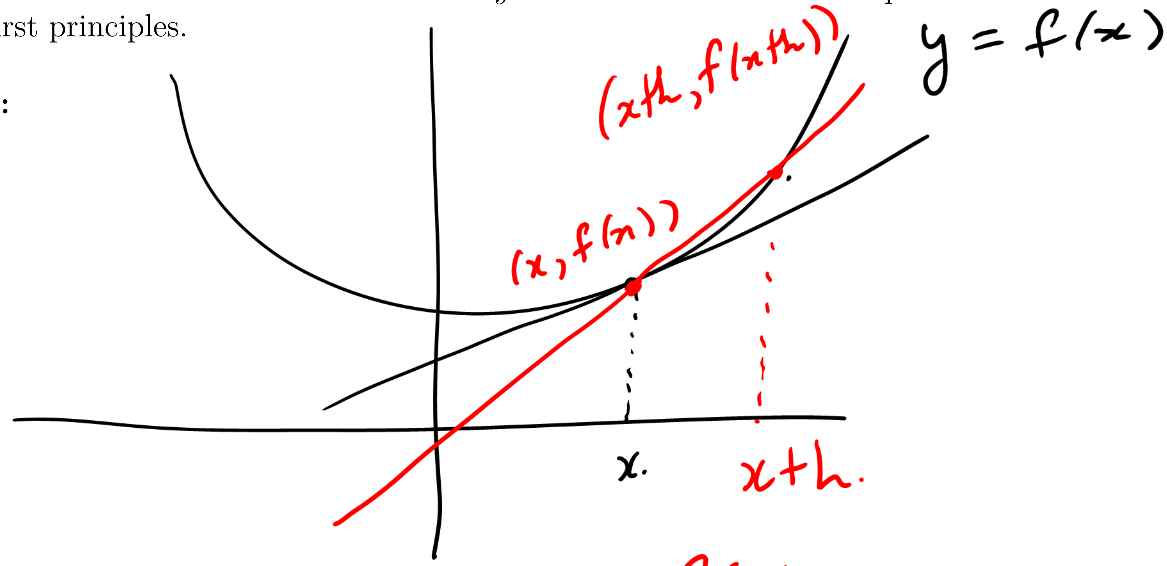
Example 2: Find the derivative of the function $y = 7x^2 - 5x + 3$ from the previous example from first principles.

Discussion:



$$m_{\text{secant}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x+h) - f(x)}{(x+h) - x}$$

$$= \frac{f(x+h) - f(x)}{h}$$

$$m_{\text{tangent}} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = 7x^2 - 5x + 3$$

$$f(x+h) = 7(x+h)^2 - 5(x+h) + 3 = 7(x^2 + 2xh + h^2) - 5x - 5h + 3$$

$$= 7x^2 + 14xh + 7h^2 - 5x - 5h + 3.$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{7x^2} + 14xh + 7h^2 - \cancel{5x} - 5h + \cancel{3}) - (\cancel{7x^2} - \cancel{5x} + \cancel{3})}{h}$$

$$= \lim_{h \to 0} \frac{14xh + 7h^2 - 5h}{h} = \text{"}\frac{0}{0}\text{"}$$

$$= \lim_{h \to 0} \frac{14x + 7h - 5}{1} = \underline{\underline{14x - 5}}$$

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Example 3: Find the derivative for each of the following:

a) $y = 8x^2 + 2\sqrt{x} + \dfrac{1}{x^3} + 4$

b) $y = e^x \sin(x)$

c) $y = \dfrac{2t+1}{3t-2}$

d) $y = (\ln(x) + 1)^{14}$

a) $y = 8x^2 + 2x^{\frac{1}{2}} + x^{-3} + 4$

$y' = 8(2x) + 2\left(\frac{1}{2}x^{-\frac{1}{2}}\right) + (-3x^{-4}) + 0$

$= 16x + x^{-\frac{1}{2}} - 3x^{-4} = 16x + \frac{1}{\sqrt{x}} - \frac{3}{x^4}$ #

$(uv)' = u'v + v'u.$ product rule

b) $y = e^x \sin(x)$

$\frac{dy}{dx} = e^x \sin(x) + \cos(x)\, e^x.$

c) $y = \dfrac{2t+1}{3t-2}$

$\left(\frac{u}{v}\right)' = \dfrac{vu' - uv'}{v^2}$

$\dfrac{(\text{bott})(\text{top})' - (\text{top})(\text{bott})'}{(\text{bott})^2}$

quotient rule

$\frac{dy}{dx} = \dfrac{(3t-2)(2) - (2t+1)(3)}{(3t-2)^2}$

$= \dfrac{6t - 4 - 6t - 3}{(3t-2)^2} = \dfrac{-7}{(3t-2)^2}$

d) $y = (\ln(x) + 1)^{14}$

$\frac{dy}{dx} = 14(\ln(x) + 1)^{13}\left(\frac{1}{x} + 0\right)$

$= \frac{14}{x}(\ln(x) + 1)^{13}$

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Example 4: For the following graph of $y = f(x)$ determine the value(s) of x for which:

a) $\dfrac{dy}{dx}$ is positive b) $\dfrac{dy}{dx}$ is negative

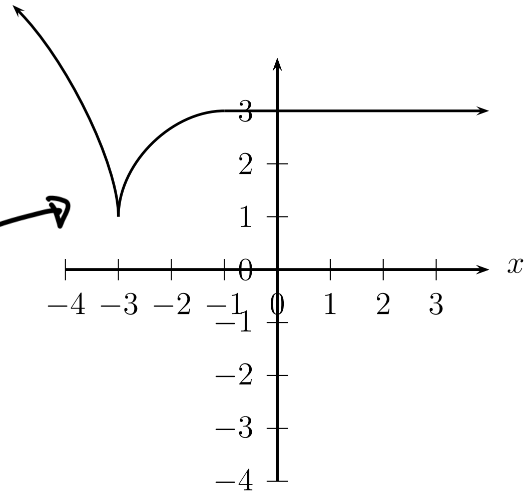
c) $\dfrac{dy}{dx}$ is zero d) $\dfrac{dy}{dx}$ is undefined

a) $-3 < x < -1$

b) $x < -3$

c) $x \geq -1$

d) $x = -3$



cusp

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Note that the function above is continuous but not differentiable. Differentiability is a much stronger condition that continuity! A differentiable function is always continuous but a continuous function is not always differentiable (consider the absolute value function). Note also that any point where a function is discontinuous will automatically be a point of non-differentiability.

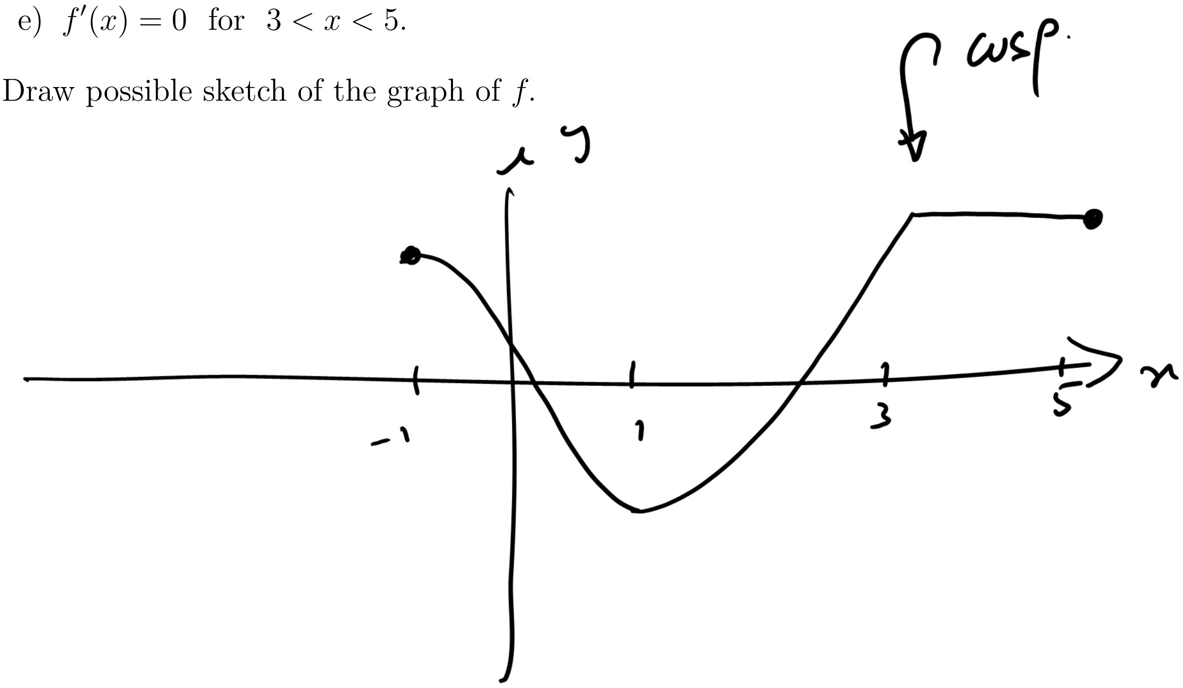
Example 5: A function $y = f(x)$ defined over the interval $-1 \le x \le 5$ has the the following 5 properties:

a) $f'(x) < 0$ for $-1 < x < 1$. b) $f'(1) = 0$.

c) $f'(x) > 0$ for $1 < x < 3$. d) $f'(3)$ is undefined; and

e) $f'(x) = 0$ for $3 < x < 5$.

Draw possible sketch of the graph of f.



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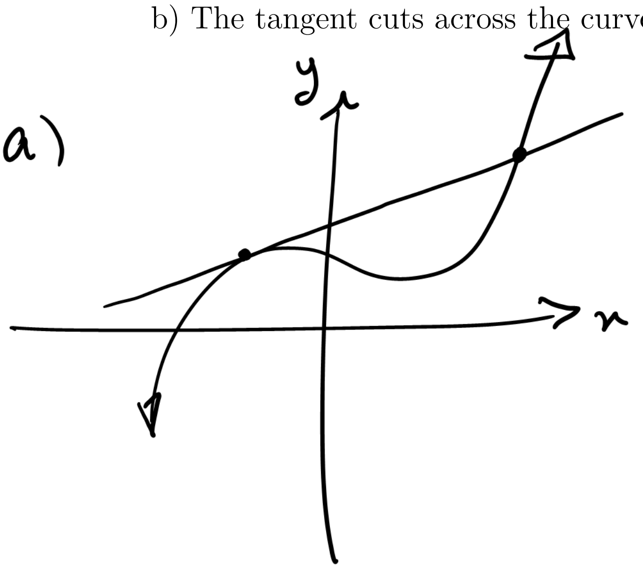
Example 6: A function $y = f(x)$ has a tangent at the point P on its graph. Sketch an example where:

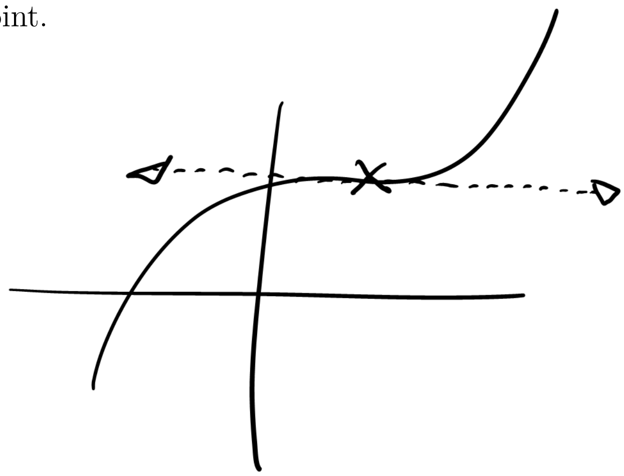
a) The tangent meets the curve at more than one point.

b) The tangent cuts across the curve at P.

a)



b)



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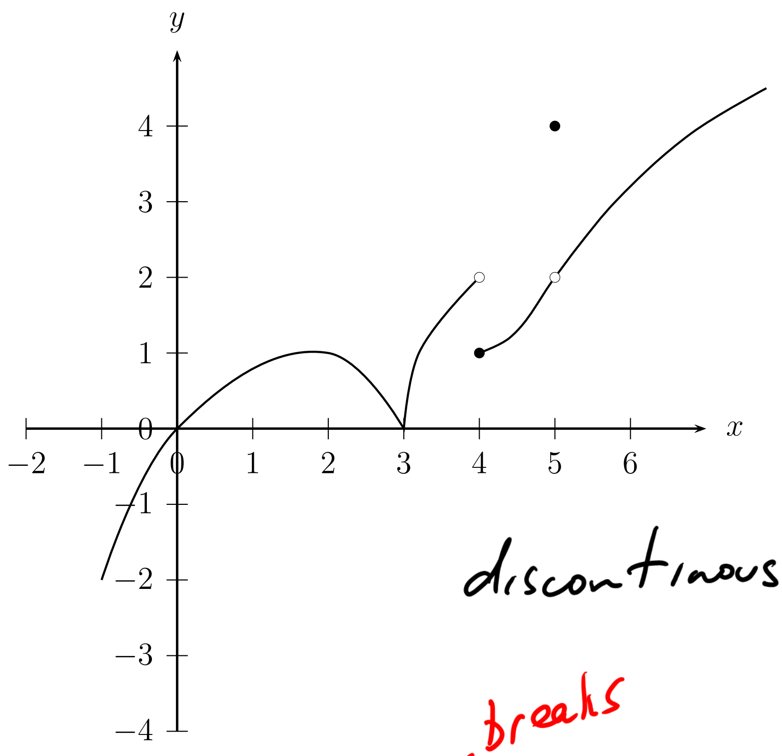
Example 7: Consider the graph of $y = f(x)$ presented below:



discontinous => non-diff

breaks

a) For which value(s) of x is the function discontinuous. $x = 4, 5$

b) For which value(s) of x is the function non-differentiable. $x = 4, 5, 3$

breaks and cusps

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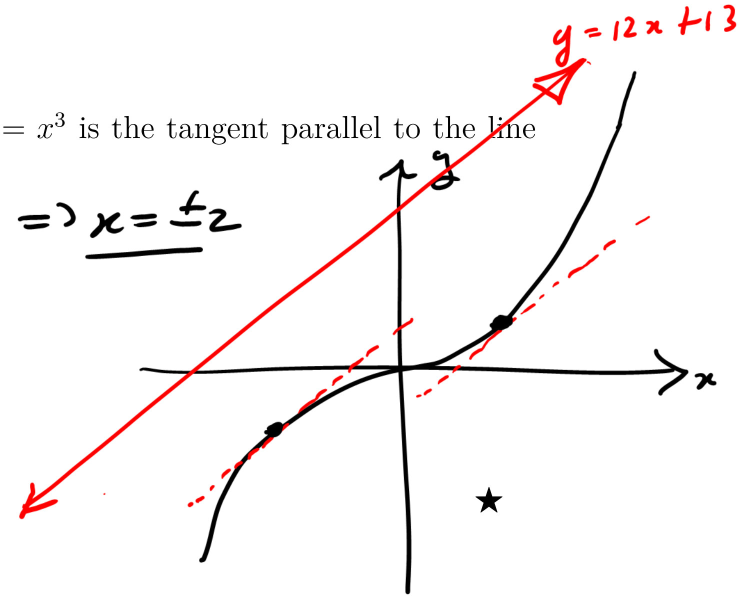
Example 8: At which point(s) on the curve $y = x^3$ is the tangent parallel to the line $y = 12x + 13$?

$\frac{dy}{dx} = 3x^2 = 12 \Rightarrow x^2 = 4 \Rightarrow \underline{x = \pm 2}$

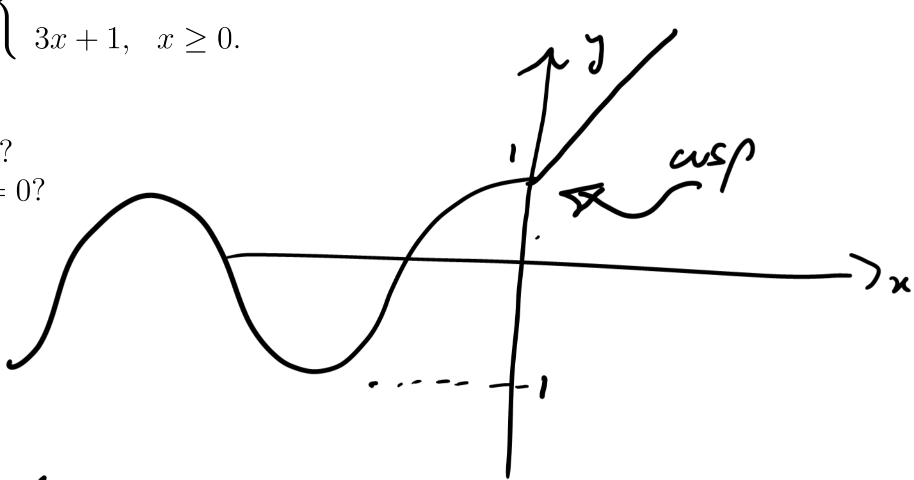
$\therefore$ Points $(2, 8)$, $(-2, -8)$



Example 9: Let $f(x) = \begin{cases} \cos(x), & x < 0; \\ 3x + 1, & x \geq 0. \end{cases}$

a) Is f continuous at $x = 0$?
b) Is f differentiable at $x = 0$?

A Sketch:



a) $\underline{\text{Yes.}}$ $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 1 = f(0)$

$\therefore$ limit exists $\underline{\text{and}}$ is equal to function value.

b) Let $p(x) = \cos(x)$ and $q(x) = 3x + 1$.

$p'(x) = -\sin(x) \to p'(0) = 0$

$q'(x) = 3 \quad \to q'(0) = 3$

Derivatives do $\underline{\text{not}}$ match