MATH1131 Mathematics 1A – Algebra

Lecture 15: Systems of Linear Equations

Lecturer: Sean Gardiner – sean.gardiner@unsw.edu.au

Based on slides by Jonathan Kress

Linear equations are equations like the following:

- $3x = 7$
- $2a + 3b = 0$
- $-3x + y = 7$
- $2x + 3y + 5z = -1$
- $3x_1 - x_2 + 7x_3 - x_4 = 10$
- $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$, for given scalars a_i and b.

i.e. The sum of scalar multiples of some variables equals a constant.

Notice the first three equations are lines in $\mathbb{R}^2$, and the first four equations are planes in $\mathbb{R}^3$.

Systems of linear equations can be solved systematically using an important algorithm known as Gaussian elimination.

Linear equations

A system of linear equations is a set of linear equations that all hold simultaneously. Solving such a system requires finding all possible solutions.

For example,

$$\begin{aligned} 3x + 2y &= 1 \\ 4x - 3y &= 7. \end{aligned}$$

Typically we would solve this in one of two ways:

- Use one equation to write x in terms of y and then substitute this into the other equation (substitution method).
- Subtract a multiple of one equation from a multiple of the other to eliminate one variable (elimination method).

We are going to concentrate on the method of elimination because it can be adapted into a powerful method called Gaussian elimination, which works for any number of linear equations and variables.

The augmented matrix

Notice that a system of linear equations like

$$\begin{aligned} 3x + 2y &= 1 \\ 4x - 3y &= 7 \end{aligned}$$

can also be written as a vector equation:

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} x + \begin{pmatrix} 2 \\ -3 \end{pmatrix} y = \begin{pmatrix} 1 \\ 7 \end{pmatrix}.$$

The augmented matrix for a system of linear equations is a simplified version of the above equation. We write a grid of numbers made up of each vector in order, omitting the variables x and y and drawing a vertical line to separate the left and right sides of the equation:

$$\left(\begin{array}{cc|c} 3 & 2 & 1 \\ 4 & -3 & 7 \end{array}\right) \begin{array}{l} \longleftarrow R_1 \\ \longleftarrow R_2 \end{array}$$

The ith row of the augmented matrix is denoted R_i.

Solving normally and with the augmented matrix

Solve the following system:

$$\begin{aligned} 3x + 2y &= 1 \quad ① \\ 4x - 3y &= 7 \quad ② \end{aligned}$$

Find $3 \times$ ②:

$$12x - 9y = 21 \quad ③$$

Find ③ $- 4 \times$ ①:

$$\begin{aligned} (12x - 9y) - 4(3x + 2y) &= 21 - 4 \times 1 \\ -17y &= 17 \quad ④ \end{aligned}$$

From ④, it follows that $y = -1$.

Substituting $y = -1$ into ① gives

$$\begin{aligned} 3x + 2 \times (-1) &= 1 \\ 3x &= 3. \end{aligned}$$

So $x = 1$ and $y = -1$.

Consider the system's augmented matrix:

$$\left(\begin{array}{cc|c} 3 & 2 & 1 \\ 4 & -3 & 7 \end{array}\right)$$

$R_2 \to 3 \times R_2$:

$$\left(\begin{array}{cc|c} 3 & 2 & 1 \\ 12 & -9 & 21 \end{array}\right)$$

$R_2 \to R_2 - 4 \times R_1$:

$$\left(\begin{array}{cc|c} 3 & 2 & 1 \\ 0 & -17 & 17 \end{array}\right)$$

R_2 means $-17y = 17$, so $y = -1$.

R_1 means $3x + 2y = 1$, so substituting:

$$\begin{aligned} 3x + 2 \times (-1) &= 1 \\ 3x &= 3. \end{aligned}$$

So $x = 1$ and $y = -1$.

In general, augmented matrices can be manipulated using elementary row operations (or EROs). There are three types of ERO:

- Swap two rows: $R_i \leftrightarrow R_j$

  e.g.
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 \end{array}\right)$$

- Multiply a row by a non-zero constant: $R_i \to \alpha R_i \quad (\alpha \neq 0)$

  e.g.
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{array}\right) \xrightarrow{R_1 \to 2R_1} \left(\begin{array}{ccc|c} 2 & 4 & 6 & 8 \\ 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 \end{array}\right)$$

- Add to one row the multiple of another row: $R_i \to R_i + \alpha R_j$

  e.g.
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 2 & 3 & 4 & 5 \end{array}\right)$$

Systems of linear equations in two variables

Example

Solve the following system of linear equations:

$$\begin{aligned} x + 3y &= 10 \\ 3x - 5y &= 2 \end{aligned}$$

The augmented matrix is:

$$\left(\begin{array}{cc|c} 1 & 3 & 10 \\ 3 & -5 & 2 \end{array}\right)$$

Row-reducing:

$$\left(\begin{array}{cc|c} 1 & 3 & 10 \\ 3 & -5 & 2 \end{array}\right) \xrightarrow{R_2 \to R_2 - 3R_1} \left(\begin{array}{cc|c} 1 & 3 & 10 \\ 0 & -14 & -28 \end{array}\right) \xrightarrow{R_2 \to -\frac{1}{14} R_2} \left(\begin{array}{cc|c} 1 & 3 & 10 \\ 0 & 1 & 14 \end{array}\right)$$

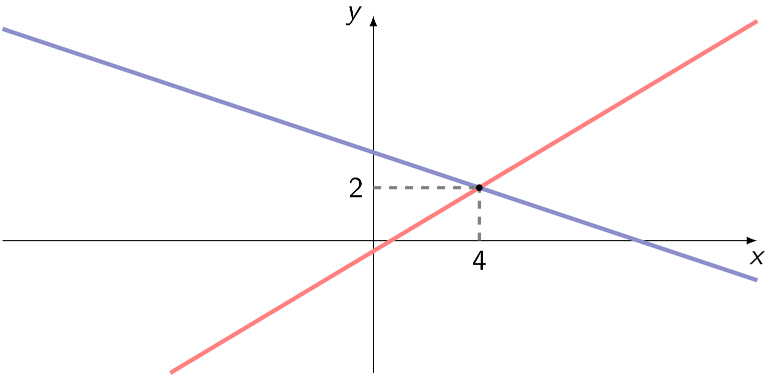
R_2 means $0x + 1y = 2$, so $\boxed{y = 2}$.

R_1 means $1x + 3y = 10$, so substituting $y = 2$ gives $\boxed{x = 4}$.

So the solution to the system of equations is $x = 4$ and $y = 2$.

We can check that it makes sense for there to be a unique solution by considering the system geometrically:

$$x + 3y = 10$$
$$3x - 5y = 2$$



The lines meet at a single point.

Example

Solve the following system of linear equations:

$$\begin{aligned} 3x - 5y &= 2 \\ 3x - 5y &= -15 \end{aligned}$$

The augmented matrix is:

$$\left(\begin{array}{cc|c} 3 & -5 & 2 \\ 3 & -5 & -15 \end{array}\right)$$

Row-reducing:

$$\left(\begin{array}{cc|c} 3 & -5 & 2 \\ 3 & -5 & -15 \end{array}\right) \xrightarrow{R_2 \to R_2 - R_1} \left(\begin{array}{cc|c} 3 & -5 & 2 \\ 0 & 0 & -17 \end{array}\right)$$

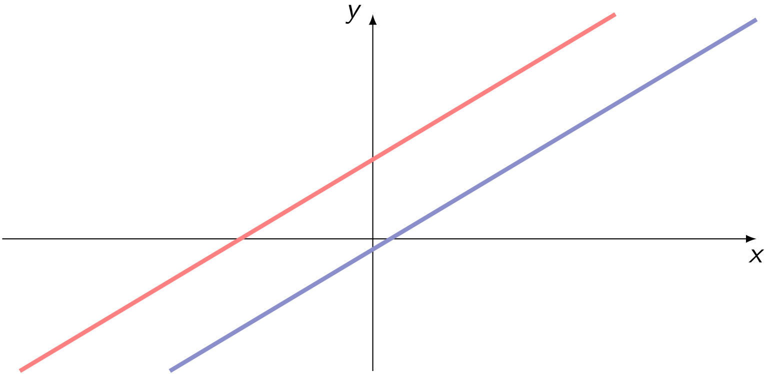
R_2 means $0x + 0y = -17$, which is impossible.

So there are no solutions to the system.

We say the system of equations is inconsistent.

We can again check that it makes sense for there to be no solutions by considering the system geometrically:

$$3x - 5y = 2$$
$$3x - 5y = -15$$



The lines never meet.

Systems of linear equations in two variables

Example

Solve the following system of linear equations:

$$\begin{aligned} 3x - 5y &= 2 \\ 6x - 10y &= 4 \end{aligned}$$

The augmented matrix is:

$$\left(\begin{array}{cc|c} 3 & -5 & 2 \\ 6 & -10 & 4 \end{array}\right)$$

Row-reducing:

$$\left(\begin{array}{cc|c} 3 & -5 & 2 \\ 6 & -10 & 4 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{cc|c} 3 & -5 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

R_2 tells us there is no second restriction on the set of solutions.

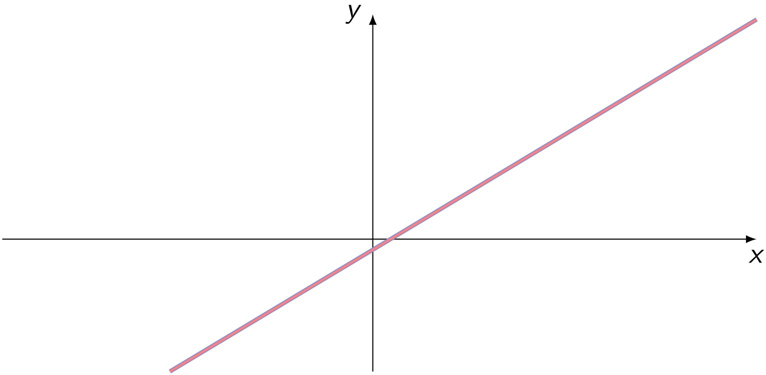
So all solutions are described by R_1, i.e. $3x - 5y = 2$.

The infinite set of solutions can be given parametrically, for example:

$$y = \lambda \text{ and } x = \frac{2+5\lambda}{3} \text{ for any } \lambda \in \mathbb{R}.$$

We can again check that it makes sense for there to be an infinite set of solutions by considering the system geometrically:

$$3x - 5y = 2$$
$$6x - 10y = 4$$

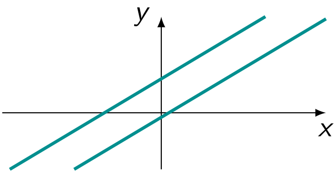


The lines are identical.

Systems of linear equations in two variables

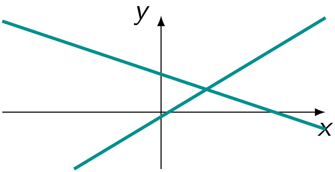
We found no solutions for

$$\begin{aligned} 3x - 5y &= -15 \\ 3x - 5y &= 2, \end{aligned}$$



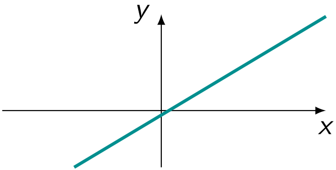
a unique solution for

$$\begin{aligned} x + 3y &= 10 \\ 3x - 5y &= 2, \end{aligned}$$



and infinitely many solutions for

$$\begin{aligned} 3x - 5y &= 2 \\ 6x - 10y &= 4. \end{aligned}$$



Example

Solve the following system of linear equations:

$$x + y + z = 5$$
$$3x + 4y + 7z = 20$$

Row-reducing the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 3 & 4 & 7 & 20 \end{array}\right) \xrightarrow{R_2 \to R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 4 & 5 \end{array}\right)$$

R_2 means $y + 4z = 5$, so letting $\boxed{z = \lambda}$, we find $\boxed{y = 5 - 4\lambda}$.

R_1 means $x + y + z = 5$, so substituting gives $\boxed{x = 3\lambda}$.

So the solution to the system of equations is:

$$x = 3\lambda,\ y = 5 - 4\lambda, \text{ and } z = \lambda \text{ for any } \lambda \in \mathbb{R}.$$

Here we found a parametric solution by setting z as the parameter λ. Then x and y were found via a process called back-substitution.

Systems of linear equations in three variables

The system

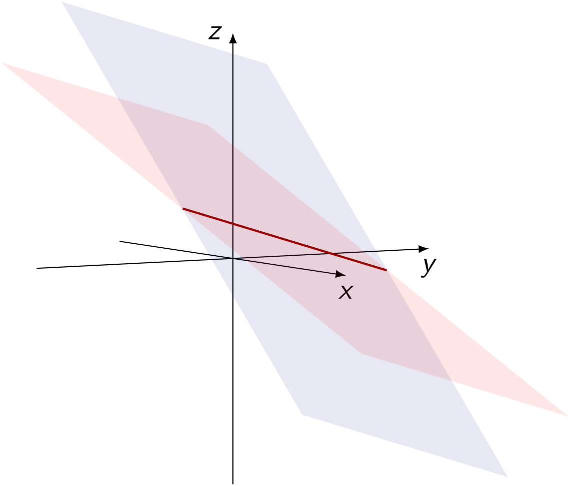
$$x + y + z = 5$$
$$3x + 4y + 7z = 20$$

has solution

$x = 3\lambda$, $y = 5 - 4\lambda$, and $z = \lambda$ for any $\lambda \in \mathbb{R}$,

or in vector form,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}.$$



The solution is a line because the planes are not parallel.

Systems of linear equations in three variables

Example

Solve the following system of linear equations:

$$\begin{aligned} 2x - y + 4z &= -3 \\ 4x - 2y + 8z &= 12 \end{aligned}$$

Row-reducing the augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & -1 & 4 & -3 \\ 4 & -2 & 8 & 12 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 2 & -1 & 4 & -3 \\ 0 & 0 & 0 & 18 \end{array}\right)$$

R_2 means $0x + 0y + 0z = 18$, which is impossible.

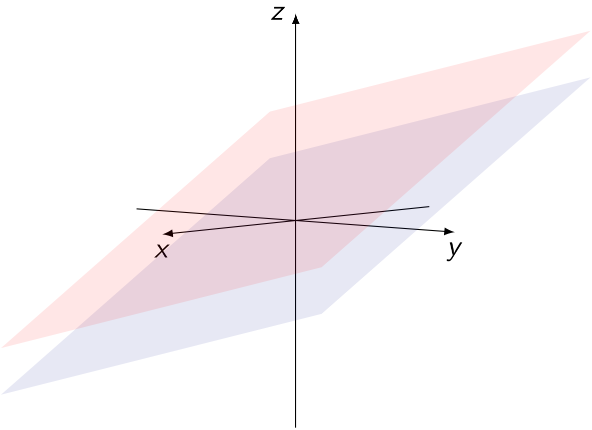
So there are no solutions to the system.

That is, the system is inconsistent.

The system

$$2x - y + 4z = -3$$
$$4x - 2y + 8z = 12$$

has no solution.



There is no solution because the planes are parallel and don't coincide.

Example

Solve the following system of linear equations:

$$2x - y + 4z = 5$$
$$4x - 2y + 8z = 10$$

Row-reducing the augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & -1 & 4 & 5 \\ 4 & -2 & 8 & 10 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 2 & -1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

R_2 tells us there is no second restriction on the set of equations.

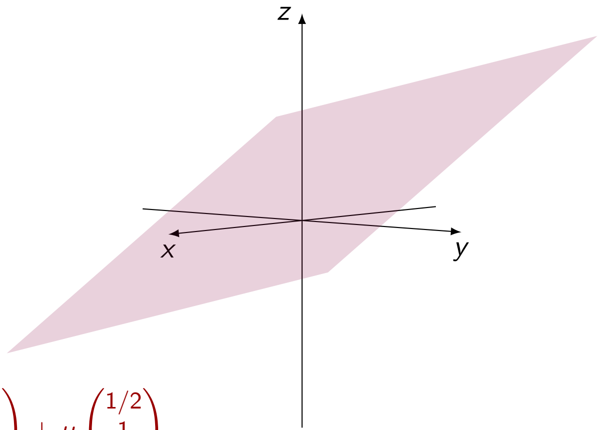
So all solutions are described by R_1, i.e. $2x - y + 4z = 5$.

The infinite set of solutions can be given parametrically, for example by setting $\boxed{z = \lambda}$ and $\boxed{y = \mu}$, giving the solution:

$$x = \frac{5 - 4\lambda + \mu}{2},\ y = \mu, \text{ and } z = \lambda \text{ for any } \lambda, \mu \in \mathbb{R}.$$

The system

$$2x - y + 4z = 5$$
$$4x - 2y + 8z = 10$$



has solution

$x = \frac{5-4\lambda+\mu}{2}$,
$y = \mu$, and $z = \lambda$
for any $\lambda, \mu \in \mathbb{R}$,

or in vector form,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5/2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}.$$

The solution is a plane because the planes are parallel and coincide.