School of Mathematics and Statistics
Math1131-Algebra

Lec01: Geometric Vectors

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2020 Term 1

Linear Algebra

Where are we going?

The algebra in MATH1131 and MATH1231 is Linear Algebra.

Linear Algebra is the study of vector spaces and the transformations between them. These transformation are called linear maps.

Linear Algebra underpins much of modern mathematics, including many engineering applications.
But that is for MATH1231 Mathematics 1B!

MATH1131 Algebra has many important applications and lays the ground work for the Linear Algebra in MATH1231.

Structure of the course

Chapter 1: Introduction to Vectors
Chapter 2: Vector Geometry
Chapter 3: Complex Numbers
Chapter 4: Linear Equations and Matrices
Chapter 5: Matrices

Introduction to Vectors

The abstract definition of vectors we are heading for is the following

What are vectors?

Vectors are things that can be added together and scaled (multiplied by a number).

Example 1: An example we won't see in MATH1131: solutions to the equation for an undamped vibrating string,

$$\frac{d^2y}{dx^2} + y = 0.$$

- Check that $y_1 = \sin x$ and $y_2 = \cos x$ satisfy this equation.

  (This icon means that you must write)

  $y_1 = \sin x$

  $\frac{dy_1}{dx} = \cos x$

  $\frac{d^2y}{dx} = -\sin x$

  LHS $= 0$

- What about

$$y = y_1 + y_2 \quad \text{or} \quad y = 2y_1 \quad \text{or} \quad y = -3y_2 \quad \text{or} \quad y = 2y_1 - 3y_2?$$

- More in MATH1231 and throughout your studies!

In MATH1131 we will study vectors geometrically in 2 and 3 dimensions . . . and algebraically in n dimensions.

Introduction to Vectors

Where does the idea of vectors come from? Why do we even need vectors?

- Certain things can be perfectly described with a single number.

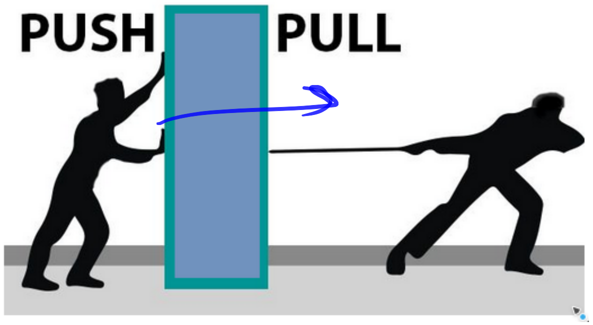
temperature

- For other quantities however, this is not good enough

Geometric Vectors

What is the model at the back of our head for geometric vectors?

Forces



Displacement

In order to describe how you go from A to B, you need to specify the length and the direction.

Geometric Vectors

A *geometric vector* can be thought of as being an arrow with a length and a direction (but we don't worry about where it is located.)

The length of the vector is sometimes called the magnitude or the norm of the vector.

Geometric Vectors

What does it mean for geometric vectors to be equal?

Two *geometric vectors are equal* if and only if they have the same length and the same direction.

We will now do a few exercises (on the next slide) using the Think-Pair-Share strategy:

Think-Pair-Share is a collaborative learning strategy where students work together to answer a question.
This strategy requires students to
(1) *think* individually about the question and write down their thoughts;
then (2) *pair* with a nearby student or a group of students
and finally (3) *share* their ideas with them.

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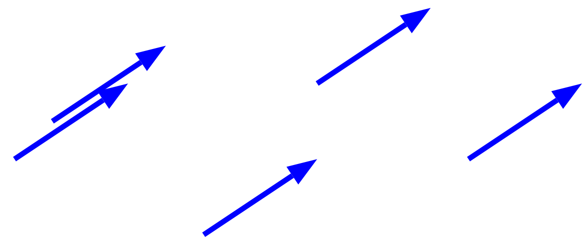
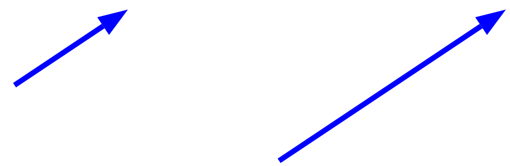
Geometric Vectors

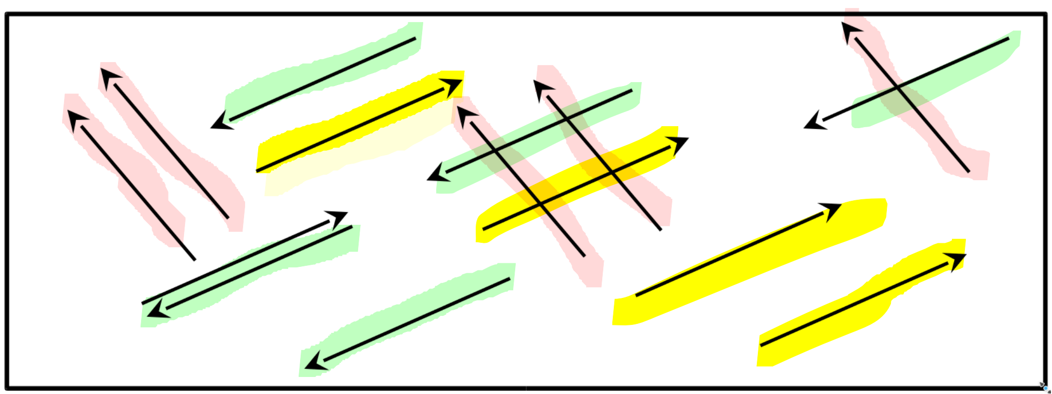
Example 2. Are all of the vectors shown below the same? yes



Example 3. Why are these vectors different? ≠ length



Example 4. How many different vectors are there inside this box? 3



Geometric Vectors

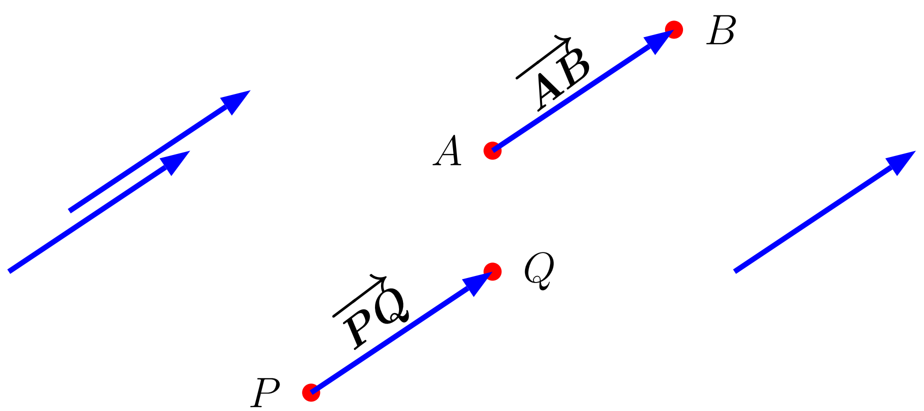
Vector from a point to another point

- The word "vector" comes from Latin in which is means "convey". The vector that conveys you from point A to point B is denoted $\overrightarrow{\boldsymbol{AB}}$.

- The direction of the arrow in the notation indicates that A is the tail, or initial point, and B is the head, or tip, or terminal point.

Example 5. In the picture below, the vector that takes you from A to B is the same as the vector that takes you from P to Q.

In that case we say $\overrightarrow{\boldsymbol{AB}} = \overrightarrow{\boldsymbol{PQ}}$.



Geometric Vectors

- **Scalars vs. vectors.** You are used to variables that take real values. These are called scalars.

  In linear algebra, we use the word scalar to mean "number as opposed to vector".

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  - The length of a vector is denoted with vertical bars.

$$\left|\overrightarrow{\boldsymbol{AB}}\right| = \text{the length of } \overrightarrow{\boldsymbol{AB}}.$$

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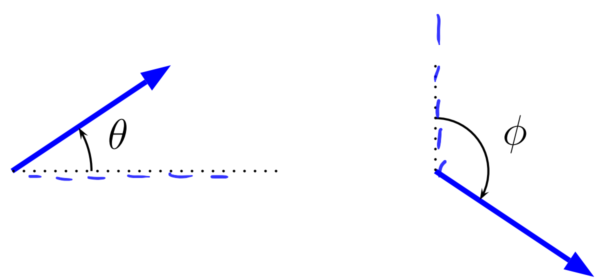
    $$\left|\overrightarrow{\boldsymbol{AB}}\right| = \text{the length of } \overrightarrow{\boldsymbol{AB}}.$$

  - In 2 dimensions, the direction of a vector can be easily described by the angle between a fixed direction such as North or the "x-axis" and the vector.



We won't try to describe a direction in 3 dimensions using angles. That can wait until second year.

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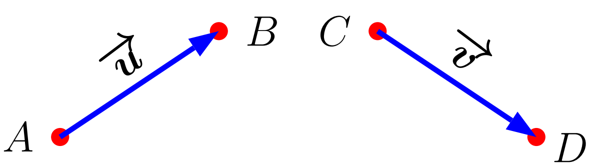
A Note on Notation

To distinguish vector quantities from scalar quantities we use different notation for vectors and scalars.

- Scalars take real values and are denoted with plain letters such as x, y, α, β.
- Vectors are written with an arrow over, a tilde under or in bold.

$$\overrightarrow{\boldsymbol{AB}} \qquad \vec{\boldsymbol{u}} \qquad \underset{\sim}{u} \qquad \mathbf{u}$$

$\underset{\sim}{u}$



$$\vec{\boldsymbol{u}} = \overrightarrow{\boldsymbol{AB}} \qquad\qquad \vec{\boldsymbol{v}} = \overrightarrow{\boldsymbol{CD}}$$

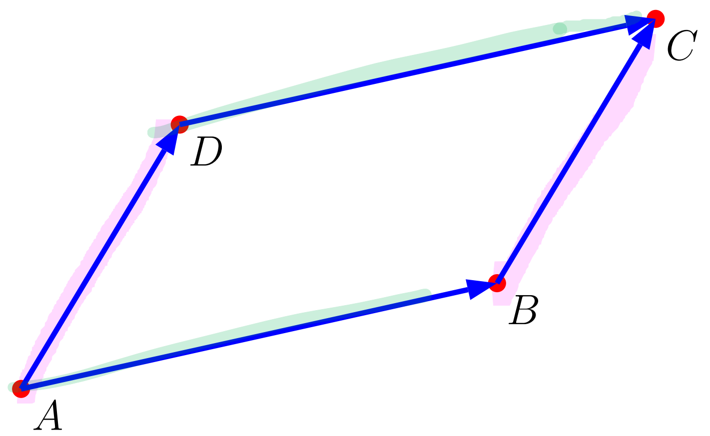
- Hand written vectors are written with a tilde under or an arrow over, eg $\vec{\boldsymbol{u}}$ or $\underset{\sim}{u}$.

A handwritten vector must have an arrow or a tilde.
The absence of those indicates that we are talking about a scalar.

Vectors and Parallelograms

Example 6. In the parallelogram below, which vectors are equal?

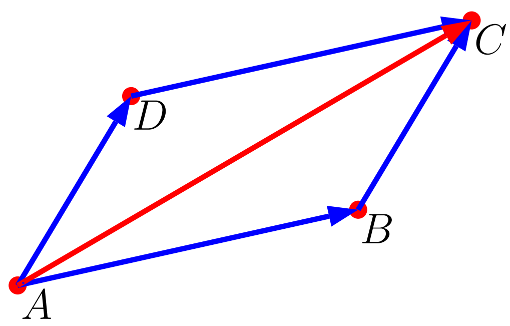
$\overrightarrow{AB} = \overrightarrow{DC}$ $\quad\quad$ $\overrightarrow{AD} = \overrightarrow{BC}$



Vector Addition: Investigation

What should the definition be? Let's look for an idea to guide our intuition...

How can we get from point A to point C?



$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}$

We could define vector addition by $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

Vector Addition: Investigation

Would that be consistent with our idea of vectors as forces?

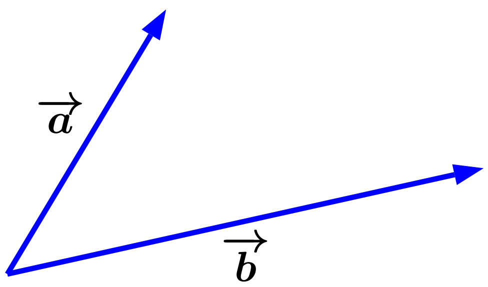
Example 7. Consider the situation of two individuals each pulling with a rope on a wagon. Person 1 pulls to the left with a force $\vec{F_1}$ of magnitude 20 Newtons (Newtons is the unit for forces). Person 2 pulls to the right with a force $\vec{F_2}$ of magnitude 30 Newtons.

What should the resulting force be?

Vector addition defined

Addition of vectors is “tip to tail” addition

We define vector addition for two geometric vectors by placing them tip to tail and form a new vector that “conveys” the point at the tail of the first to the point at the tip of the second.



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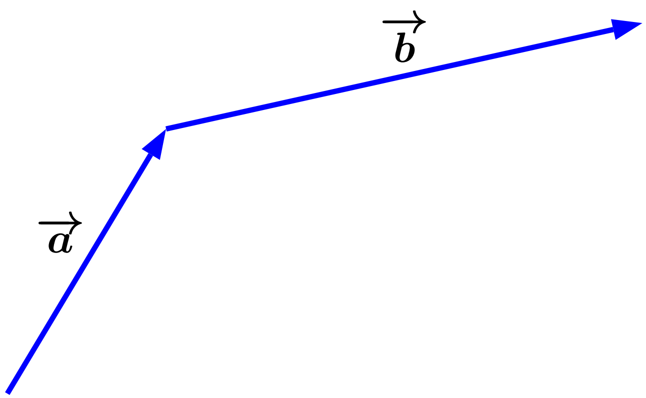

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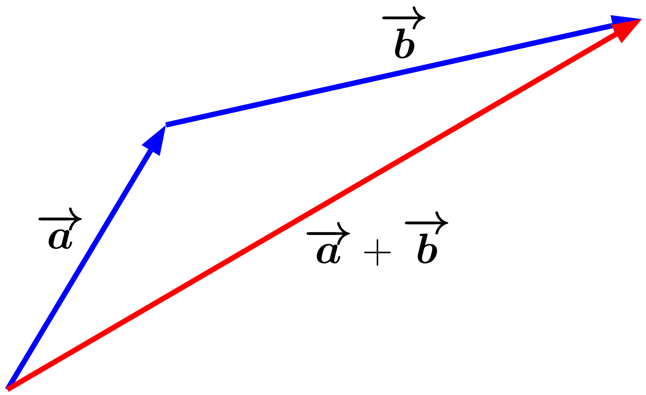

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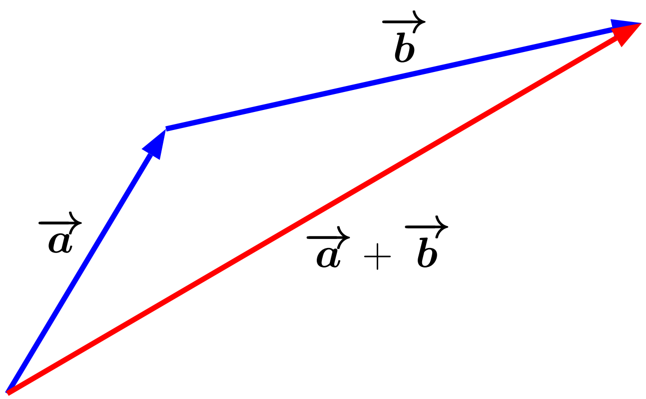

In other words, to add the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, we move one of them to put them tip to tail and then we complete the triangle.
This method of addition is known as t the triangle law of addition.

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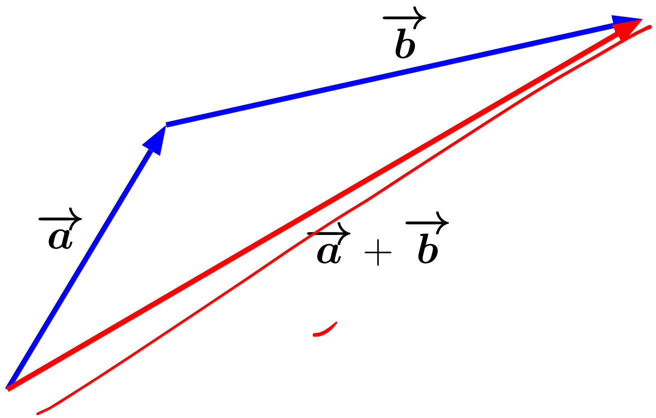

We are adding the vectors, NOT their lengths!! The length of $\overrightarrow{a} + \overrightarrow{b}$ is NOT the sum of the lengths of $\overrightarrow{a}$ and $\overrightarrow{b}$. It is shorter to travel in a straight line! $|\overrightarrow{a} + \overrightarrow{b}| \leqslant |\overrightarrow{a}| + |\overrightarrow{b}|$

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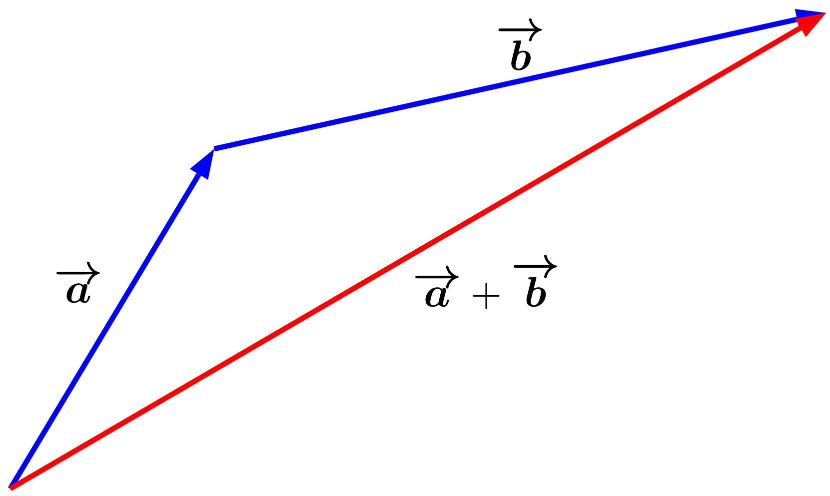
Vector addition is commutative



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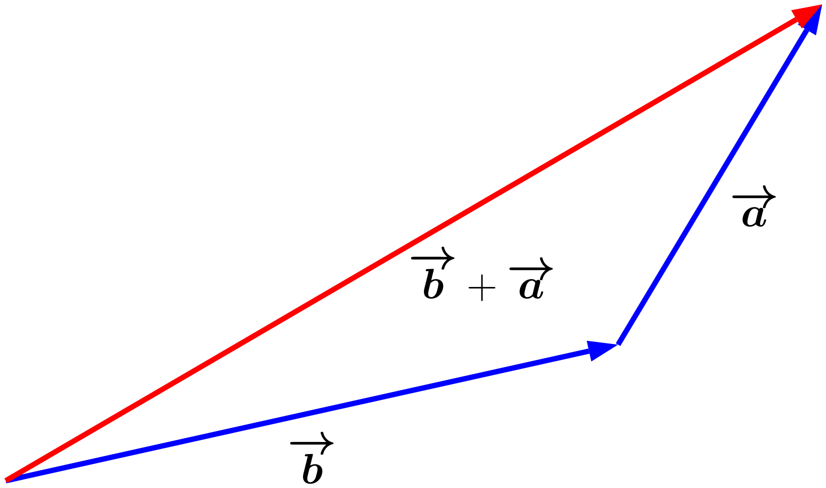

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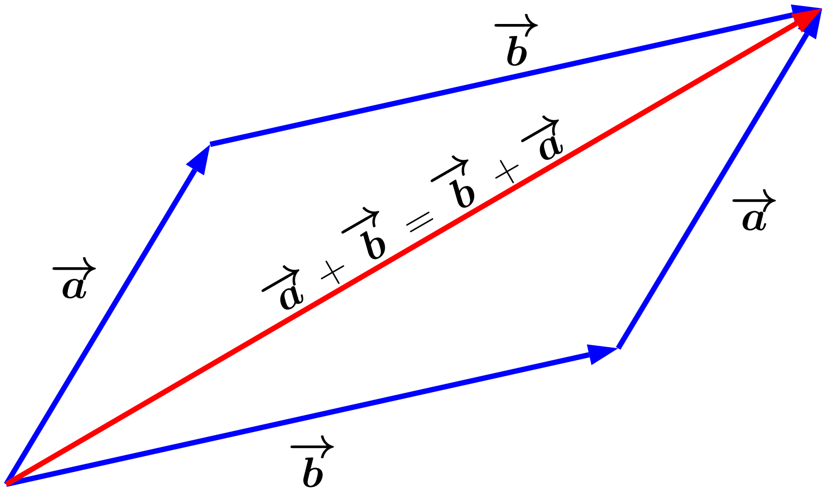

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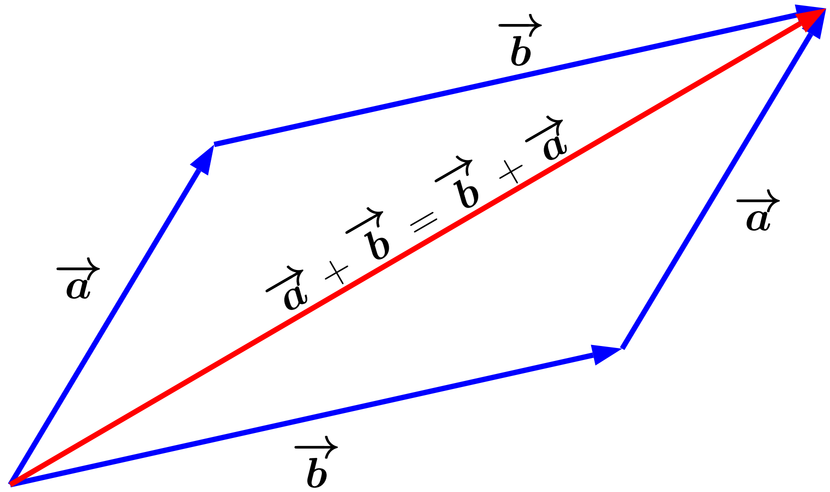


Note that this gives us a convenient method to add two vectors represented with the same initial point: Just complete the parallelogram.
This is sometimes referred to as the Parallelogram law of addition.

Vector Addition

Which properties of the addition of numbers still work for the addition of vectors?

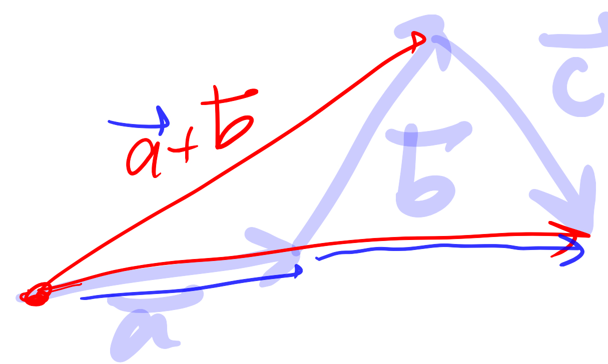
- **Vector Addition is commutative.**

  $$\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{b} + \overrightarrow{a}$$

- **Vector addition is also associative**, that is,

  $$(\overrightarrow{a} + \overrightarrow{b}) + \overrightarrow{c} = \overrightarrow{a} + (\overrightarrow{b} + \overrightarrow{a}).$$

  So it's safe to write $\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c}$.



The zero vector

- There is one vector that does not have a direction. Here it is . . .

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- It is written as $\mathbf{0}$ or $\underset{\sim}{0}$ or $\overrightarrow{\mathbf{0}}$.
- The zero vector added to any vector leaves the vector unchanged.

$$\overrightarrow{\boldsymbol{a}} + \overrightarrow{\mathbf{0}} = \overrightarrow{\boldsymbol{a}}.$$

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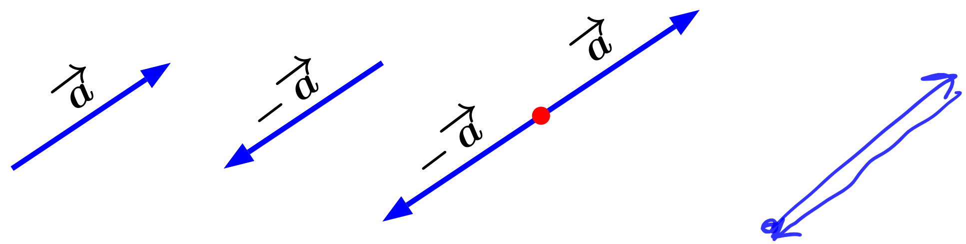
The negative of a vector

The negative of a vector

For each vector $\overrightarrow{\boldsymbol{a}}$ there is another called $-\overrightarrow{\boldsymbol{a}}$ ("minus $\overrightarrow{a}$ ") that has the property that

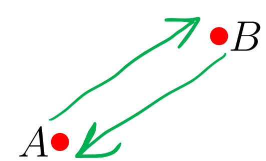
$$\overrightarrow{\boldsymbol{a}} + (-\overrightarrow{\boldsymbol{a}}) = \vec{\mathbf{0}}.$$

For example,



Note:

- $\overrightarrow{\boldsymbol{a}}$ and $-\overrightarrow{\boldsymbol{a}}$ have the same length, that is $|\overrightarrow{\boldsymbol{a}}| = |-\overrightarrow{\boldsymbol{a}}|$, but "opposite" directions.
- $\overrightarrow{\boldsymbol{BA}} = -\overrightarrow{\boldsymbol{AB}}$



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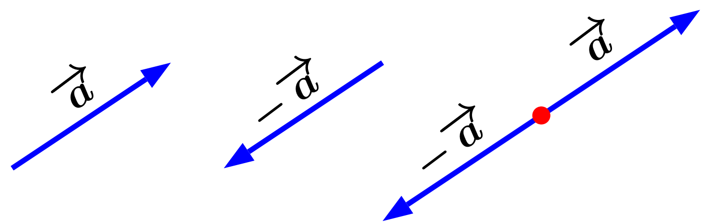

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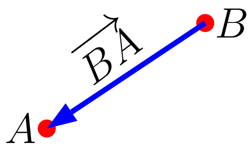


Exercise 8. Three ropes are attached to a block of wood, and a man is pulling on each rope. If the first man is pulling with a force of 30 Newtons due east and the second man is pulling with a force of 40 Newtons due north, find the force with which a third man must pull to stop the block from moving.

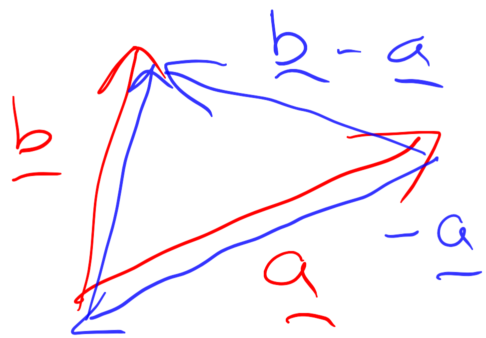
Think
Pair
Share

Subtracting vectors

Subtracting 2 is the same as adding -2
That gives us an idea !

Definition of vector subtraction

To subtract a vector we add its negative : $\overrightarrow{\boldsymbol{b}} - \overrightarrow{\boldsymbol{a}} = \overrightarrow{\boldsymbol{b}} + (-\overrightarrow{\boldsymbol{a}})$
which is also equal to $-\overrightarrow{\boldsymbol{a}} + \overrightarrow{\boldsymbol{b}}$.



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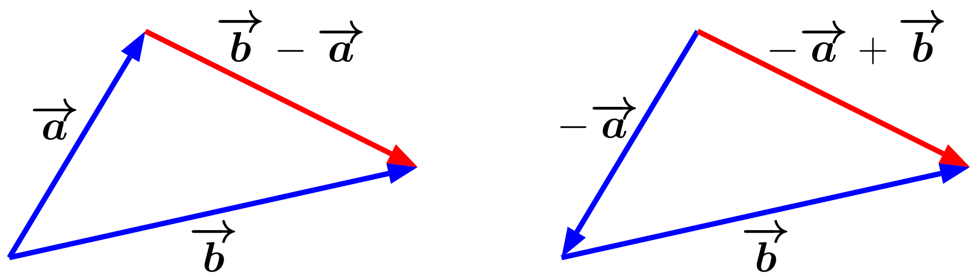
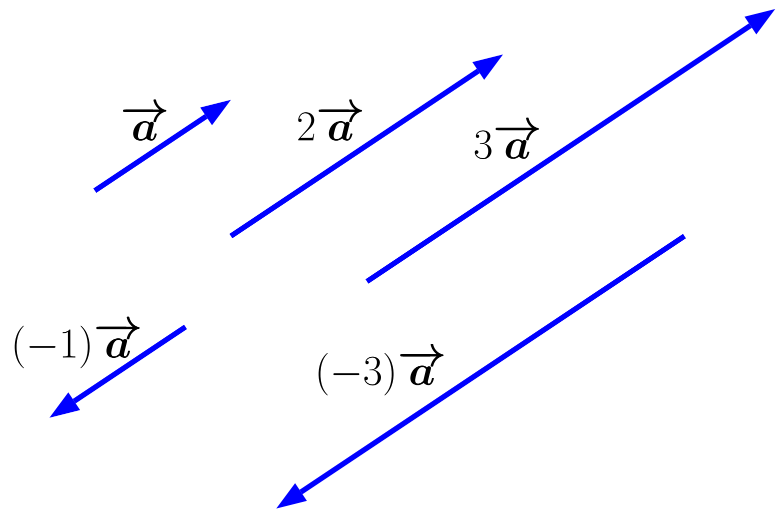


Scaling vectors = multiplying them by a scalar

Scalar multiplication changes a vector's length without changing its direction (or only changes it to the opposite direction).

Why? To guide your intuition, think forces or displacement. If the scalars are integers, this is a consequence of our definition of vector addition.



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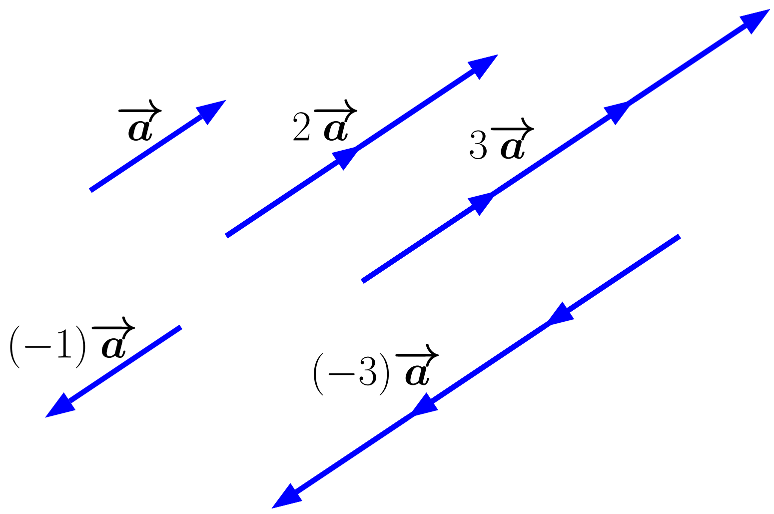


Scaling vectors and parallel vectors

Scalar Multiplication: what happens when we multiply a vector by a scalar?

For any scalar (real number) λ and vector $\overrightarrow{\boldsymbol{v}}$ the scalar multiple $\lambda\overrightarrow{\boldsymbol{v}}$ is the vector with length $|\lambda||\overrightarrow{\boldsymbol{v}}|$ and

- the same direction as $\overrightarrow{\boldsymbol{v}}$ when $\lambda > 0$
- the opposite direction as $\overrightarrow{\boldsymbol{v}}$ when $\lambda < 0$

Notes:

- When $\lambda = 0$, $\lambda\overrightarrow{\boldsymbol{v}} = \mathbf{0}$.
- For $\lambda = 1$, $1\overrightarrow{\boldsymbol{v}} = \overrightarrow{\boldsymbol{v}}$.

Here's a Geogebra demonstration of scalar multiplication — thanks Dr Mansfield!

https://www.geogebra.org/m/wFiCaOrE

Scaling vectors and parallel vectors

Definition of parallel vectors

Two vectors are said to be parallel if they have the same or opposite direction.

Example 9. Illustrate these two possible cases.

How to tell if vectors are parallel?

Two vectors are parallel *if and only if* one is a scalar multiple of the other.

Exercise 10. A, B, C and D are four points. Prove that no matter how the points A, B, C and D are chosen, $\overrightarrow{u}$ and $\overrightarrow{v}$ are parallel. Do they have the same direction or opposite direction?

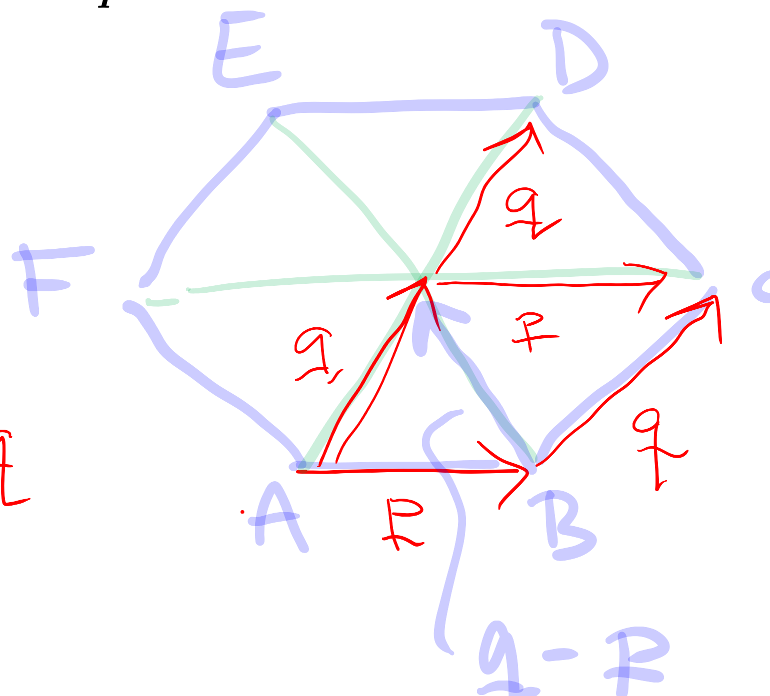
Exercises

Example 11. Suppose ABCDEF is a regular hexagon with $\overrightarrow{\boldsymbol{p}} = \overrightarrow{AB}$ and $\overrightarrow{\boldsymbol{q}} = \overrightarrow{BC}$.

Express the following vectors in terms of $\overrightarrow{\boldsymbol{p}}$ and $\overrightarrow{\boldsymbol{q}}$:

Think
Pair
Share

1. $\overrightarrow{CD}$
2. $\overrightarrow{DE} = -\vec{p}$
3. $\overrightarrow{EF} = -\vec{q}$
4. $\overrightarrow{FA}$
5. $\overrightarrow{AC} = p + q$
6. $\overrightarrow{AD} = 2q$
7. $\overrightarrow{AE}$



Example 12. [Challenge!] Suppose A and B are points with position vectors $\overrightarrow{\boldsymbol{a}}$ and $\overrightarrow{\boldsymbol{b}}$. Find a vector (in terms of $\overrightarrow{\boldsymbol{a}}$ and $\overrightarrow{\boldsymbol{b}}$) which bisects the angle AOB, where O is the origin.

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Is it true that $(-1)\overrightarrow{\boldsymbol{a}} = -\overrightarrow{\boldsymbol{a}}$?

Our intuition tells us it should be, so let's try to prove it:

We know

1. When $\lambda = 0$, $\lambda\overrightarrow{\boldsymbol{v}} = \overrightarrow{\mathbf{0}}$,
2. $1\overrightarrow{\boldsymbol{a}} = \overrightarrow{\boldsymbol{a}}$,
3. $(\lambda + \mu)\overrightarrow{\boldsymbol{a}} = \lambda\overrightarrow{\boldsymbol{a}} + \mu\overrightarrow{\boldsymbol{a}}$.

PROOF *This is a taste of what Mathematics really is...*

$$
\begin{aligned}
&\overrightarrow{a} + (-1)\overrightarrow{a} \\
&= 1\overrightarrow{a} + (-1)\overrightarrow{a} && \text{by (2)} \\
&= (1 + (-1))\overrightarrow{a} && \text{by (3)} \\
&= 0\,\overrightarrow{a} \\
&= \overrightarrow{0} && \text{by (1)}
\end{aligned}
$$

$-\overrightarrow{a}$ is the only vector which added to $\overrightarrow{a}$ gives $\overrightarrow{0}$

$\therefore\ (-1)\overrightarrow{a} = -\overrightarrow{a}$

Distributive and associative laws

Vectors Operations Laws

$$\begin{aligned}
(\overrightarrow{\boldsymbol{a}} + \overrightarrow{\boldsymbol{b}}) + \overrightarrow{\boldsymbol{c}} &= \overrightarrow{\boldsymbol{a}} + (\overrightarrow{\boldsymbol{b}} + \overrightarrow{\boldsymbol{c}}) && \text{associative law of vector addition} \\
\lambda(\mu \overrightarrow{\boldsymbol{a}}) &= (\lambda\mu)\overrightarrow{\boldsymbol{a}} && \text{associative law of scalar multiplication} \\
(\lambda + \mu)\overrightarrow{\boldsymbol{a}} &= \lambda \overrightarrow{\boldsymbol{a}} + \mu \overrightarrow{\boldsymbol{a}} && \text{scalar distributive law} \\
\lambda(\overrightarrow{\boldsymbol{a}} + \overrightarrow{\boldsymbol{b}}) &= \lambda \overrightarrow{\boldsymbol{a}} + \lambda \overrightarrow{\boldsymbol{b}} && \text{vector distributive law}
\end{aligned}$$

Example 13. Example from the notes...

- *What a rigorous solution with all the details would look like:*

$$\begin{aligned}
& 3(2\overrightarrow{\boldsymbol{a}} - \overrightarrow{\boldsymbol{b}}) + (\overrightarrow{\boldsymbol{a}} - 2\overrightarrow{\boldsymbol{b}}) \\
= {} & 3[2\overrightarrow{\boldsymbol{a}} + (-1)\overrightarrow{\boldsymbol{b}}] + [\overrightarrow{\boldsymbol{a}} + (-1)(2\overrightarrow{\boldsymbol{b}})] && \text{(Definition of subtraction)} \\
= {} & [3(2\overrightarrow{\boldsymbol{a}}) + 3((-1)\overrightarrow{\boldsymbol{b}})] + [\overrightarrow{\boldsymbol{a}} + (-1)(2\overrightarrow{\boldsymbol{b}})] && \text{(Vector distributive law)} \\
= {} & [6\overrightarrow{\boldsymbol{a}} + (-3)\overrightarrow{\boldsymbol{b}}] + [\overrightarrow{\boldsymbol{a}} + (-2)\overrightarrow{\boldsymbol{b}}] && \text{(Associative law of...)} \\
= {} & (6\overrightarrow{\boldsymbol{a}} + \overrightarrow{\boldsymbol{a}}) + [(-3)\overrightarrow{\boldsymbol{b}} + (-2)\overrightarrow{\boldsymbol{b}}] && \text{(Associative and comm...)} \\
= {} & (6+1)\overrightarrow{\boldsymbol{a}} + [(-3) + (-2)]\overrightarrow{\boldsymbol{b}} && \text{(Scalar distributive law)} \\
= {} & 7\overrightarrow{\boldsymbol{a}} - 5\overrightarrow{\boldsymbol{b}} && \text{(Definition of subtraction)}
\end{aligned}$$

- *In practice we would just do*

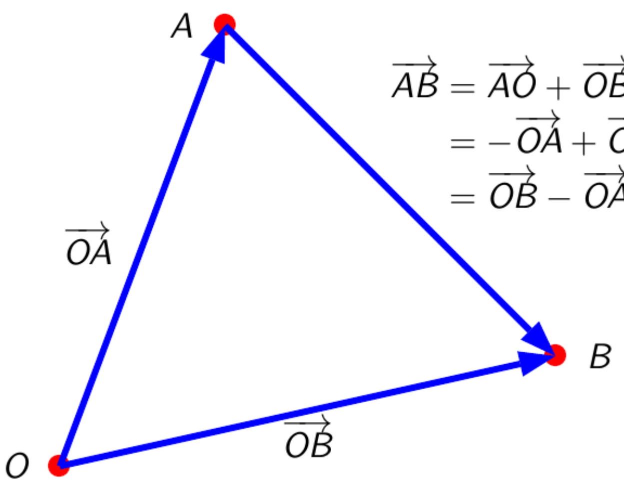
$$3(2\overrightarrow{\boldsymbol{a}} - \overrightarrow{\boldsymbol{b}}) + (\overrightarrow{\boldsymbol{a}} - 2\overrightarrow{\boldsymbol{b}}) = 6\overrightarrow{\boldsymbol{a}} - 3\overrightarrow{\boldsymbol{b}} + \overrightarrow{\boldsymbol{a}} - 2\overrightarrow{\boldsymbol{b}} = 7\overrightarrow{\boldsymbol{a}} - 5\overrightarrow{\boldsymbol{b}}.$$

Position vectors

These are used to describe the position of points once an origin has been chosen

Often we have a special point called the origin and denoted O.

- For points A and B the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are their position vectors.
- These position vectors (which are "attached" to the origin) can be used to express the vector $\overrightarrow{AB}$ (which is not "attached" to the origin).



$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -\overrightarrow{OA} + \overrightarrow{OB}\\ &= \overrightarrow{OB} - \overrightarrow{OA}\end{aligned}$$

$= \overrightarrow{OB} + \overrightarrow{AO}$

$= \overrightarrow{OB} + (-\overrightarrow{OA})$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$=$ position vector of the terminal point B minus position vector of the initial point A.

$= \overrightarrow{AO} + \overrightarrow{OB}$

$= \overrightarrow{AB}$

Extra exercises

Example 14. Prove (using vectors) that the midpoints of the sides of a quadrilateral form a parallelogram.

Extra exercises

Example 15. Suppose P, Q, R are the midpoints of AB, BC, CA respectivley in ΔABC. Let O be any point inside the triangle. Use vectors to prove that $\overrightarrow{\boldsymbol{OP}} + \overrightarrow{\boldsymbol{OQ}} + \overrightarrow{\boldsymbol{OR}} = \overrightarrow{\boldsymbol{OA}} + \overrightarrow{\boldsymbol{OB}} + \overrightarrow{\boldsymbol{OC}}$.