MATH1131 Mathematics 1A – Algebra

Lecture 17: Gaussian Elimination Examples

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Based on slides by Jonathan Kress

Example

Find the equation of the parabola that passes through the points $(1, 2)$, $(-1, 4)$, and $(2, 4)$.

Let the parabola be $f(x) = ax^2 + bx + c$ for some $a, b, c \in \mathbb{R}$. Then:

- $f(1) = a + b + c = 2$,
- $f(-1) = a - b + c = 4$, and
- $f(2) = 4a + 2b + c = 4$.

So the system of linear equations in variables a, b, and c is represented by the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 4 \\ 4 & 2 & 1 & 4 \end{array}\right)$$

Row-reducing the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 4 \\ 4 & 2 & 1 & 4 \end{array}\right) \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & 2 \\ 0 & -2 & -3 & -4 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -3 & -6 \end{array}\right)$$

There is a leading entry in every column left of the vertical line, and not in the last column. So there is a unique solution.

R_3 means $-3c = -6$, so $\boxed{c = 2}$.
R_2 means $-2b = 2$, so $\boxed{b = -1}$.
R_1 means $a + b + c = 2$, so $\boxed{a = 1}$.

Thus the parabola is given by $f(x) = x^2 - x + 2$.

Gaussian elimination examples

Example

Consider the following system of linear equations:

$$\begin{aligned} x + 2y + z &= 8 \\ 4x + 3y - z &= 7 \\ 3x + y - 2z &= -1 \end{aligned}$$

a) Find the general solution.

b) Find the solution which has an x-value of 10.

c) Given that x, y, and z must all be non-negative, find the maximum value of y.

The augmented matrix is: $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 4 & 3 & -1 & 7 \\ 3 & 1 & -2 & -1 \end{array}\right)$

a) Find the general solution.

Row-reducing the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 4 & 3 & -1 & 7 \\ 3 & 1 & -2 & -1 \end{array}\right) \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 4R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -5 & -5 & -25 \\ 0 & -5 & -5 & -25 \end{array}\right)$$

$$\xrightarrow[R_2 \to -\frac{1}{5}R_2]{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

There is not a leading entry in the last column, nor in every column left of the vertical line. So there are infinitely many solutions.

Since the third column has no leading entry, let $\boxed{z = \lambda}$.

R_2 means $y + z = 5$, so $\boxed{y = 5 - \lambda}$.

R_1 means $x + 2y + z = 8$, so $\boxed{x = \lambda - 2}$.

Thus the set of solutions is given by $x = \lambda - 2$, $y = 5 - \lambda$, and $z = \lambda$ for all $\lambda \in \mathbb{R}$.

b) Find the solution which has an x-value of 10.

The general set of solutions is given by $x = \lambda - 2$, $y = 5 - \lambda$, and $z = \lambda$ for all $\lambda \in \mathbb{R}$.

If $x = 10$, then $\lambda = 12$.

So $y = -7$ and $z = 12$ when $x = 10$.

c) Given that x, y, and z must all be non-negative, find the maximum value of y.

We want $x \geq 0$, $y \geq 0$, and $z \geq 0$, which respectively imply $\lambda \geq 2$, $5 \geq \lambda$, and $\lambda \geq 0$.

For all three conditions to be satisfied simultaneously, we require $5 \geq \lambda \geq 2$.

This is equivalent to $0 \leq y \leq 3$, so the maximum value of y is 3.

Gaussian elimination examples

Example

For the following system of linear equations, find conditions on the vector

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

such that the following system is consistent (i.e. has a solution):

$$\begin{aligned} x + 2y + 5z &= b_1 \\ 3x + 7y + 17z &= b_2 \\ x + 3y + 7z &= b_3 \end{aligned}$$

The augmented matrix is: $\left(\begin{array}{ccc|c} 1 & 2 & 5 & b_1 \\ 3 & 7 & 17 & b_2 \\ 1 & 3 & 7 & b_3 \end{array}\right)$

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Row-reducing the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 5 & b_1 \\ 3 & 7 & 17 & b_2 \\ 1 & 3 & 7 & b_3 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & 2 & 5 & b_1 \\ 0 & 1 & 2 & b_2 - 3b_1 \\ 0 & 1 & 2 & b_3 - b_1 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & 5 & b_1 \\ 0 & 1 & 2 & b_2 - 3b_1 \\ 0 & 0 & 0 & 2b_1 - b_2 + b_3 \end{array}\right)$$

The system will only be inconsistent if the last column contains a leading entry.

So the only requirement for the system to be consistent is that $2b_1 - b_2 + b_3 = 0$, since otherwise the bottom row would contain a leading entry right of the vertical bar (that is, the bottom row would imply $0x + 0y + 0z \neq 0$).

Example

For the following system of linear equations, determine which real values of λ (if any) will yield:

a) no solutions,

b) a unique solution,

c) infinitely many solutions.

$$\begin{aligned} x + y + z &= 4 \\ x + \lambda y + 2z &= 5 \\ 2x + (\lambda+1)y + (\lambda^2-1)z &= \lambda + 7 \end{aligned}$$

The augmented matrix is: $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & \lambda & 2 & 5 \\ 2 & \lambda+1 & \lambda^2-1 & \lambda+7 \end{array}\right)$

Row-reducing the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & \lambda & 2 & 5 \\ 2 & \lambda+1 & \lambda^2-1 & \lambda+7 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & \lambda-1 & 1 & 1 \\ 0 & \lambda-1 & \lambda^2-3 & \lambda-1 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & \lambda-1 & 1 & 1 \\ 0 & 0 & \lambda^2-4 & \lambda-2 \end{array}\right)$$

The system will have a unique solution only if there is a leading entry in each of the first three columns.

This can only occur if $\lambda - 1 \neq 0$ in R_2 and $\lambda^2 - 4 \neq 0$ in R_3.

So there is a unique solution only when $\lambda \neq 1$ and $\lambda \neq \pm 2$.

It remains to check what happens when λ does equal 1, 2, or -2...

When $\lambda = 2$, the REF matrix becomes:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & \lambda - 1 & 1 & 1 \\ 0 & 0 & \lambda^2 - 4 & \lambda - 2 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

There is no leading entry in the last column, nor in the third column. So there are infinitely many solutions.

When $\lambda = -2$, the REF matrix becomes:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & \lambda - 1 & 1 & 1 \\ 0 & 0 & \lambda^2 - 4 & \lambda - 2 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & -4 \end{array}\right)$$

There is a leading entry in the last column, so there are no solutions.

When $\lambda = 1$, the REF matrix becomes:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & \lambda - 1 & 1 & 1 \\ 0 & 0 & \lambda^2 - 4 & \lambda - 2 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -3 & -1 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 + 3R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

There is a leading entry in the last column, so there are no solutions.

So altogether, the system has no solutions if $\lambda = 1$ or $\lambda = -2$, infinitely many solutions if $\lambda = 2$, and a unique solution otherwise.

Example

Determine whether the line $\mathcal{L}$: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \ t \in \mathbb{R}$

meets the plane $\mathcal{P}$: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \ \lambda, \mu \in \mathbb{R}$.

$\mathcal{L}$ is parallel to $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and $\mathcal{P}$ is normal to $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}$.

Since $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix} = -12 \neq 0$, $\mathcal{L}$ is not perpendicular to $\mathcal{P}$'s normal.

That is, $\mathcal{L}$ is not parallel to $\mathcal{P}$, so it must intersect the plane at exactly one point.

Example

Determine where the line $\mathcal{L}$: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \ t \in \mathbb{R}$

meets the plane $\mathcal{P}$: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \ \lambda, \mu \in \mathbb{R}$.

To find any points of intersection, we can equate both expressions:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$

We want to solve:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$

Rearranging gives:

$$t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$$

This is a system of linear equations in variables t, λ, and μ, represented by the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & 2 & 1 \end{array}\right)$$

Row-reducing the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & 2 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 0 & -3 & 4 & -10 \\ 0 & -3 & 8 & -14 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 0 & -3 & 4 & -10 \\ 0 & 0 & 4 & -4 \end{array}\right)$$

There is no leading entry in the last column, and every other column contains a leading entry, so there is a unique solution.

R_3 means $4\mu = -4$, so $\boxed{\mu = -1}$.
R_2 means $-3\lambda + 4\mu = -10$, so $\boxed{\lambda = 2}$.
R_3 means $t + \lambda - 2\mu = 5$, so $\boxed{t = 1}$.

Substituting these parameter values into either equation for $\mathcal{L}$ or $\mathcal{P}$ reveals the point is $(2, 2, 2)$.

An alternative solution uses the Cartesian equation for $\mathcal{P}$:

Since $\begin{pmatrix}-1\\1\\0\end{pmatrix} \times \begin{pmatrix}2\\0\\-2\end{pmatrix} = -2\begin{pmatrix}1\\1\\1\end{pmatrix}$, a normal to the plane is $\begin{pmatrix}1\\1\\1\end{pmatrix}$.

So via the point-normal form of the plane, we have:

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}6\\0\\0\end{pmatrix}$$

which provides the Cartesian form $x + y + z = 6$.

Since the equation for the line is $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix} + t\begin{pmatrix}1\\2\\3\end{pmatrix}$,

it must intersect the plane when

$$x + y + z = (1 + t) + (2t) + (-1 + 3t) = 6.$$

Solving this yields $t = 1$, and substituting this into the equation for $\mathcal{L}$ reveals the point of intersection is $(2, 2, 2)$.