MATH1131 Mathematics 1A – Algebra

Lecture 4: Linear Combinations and Planes

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Based on slides by Jonathan Kress

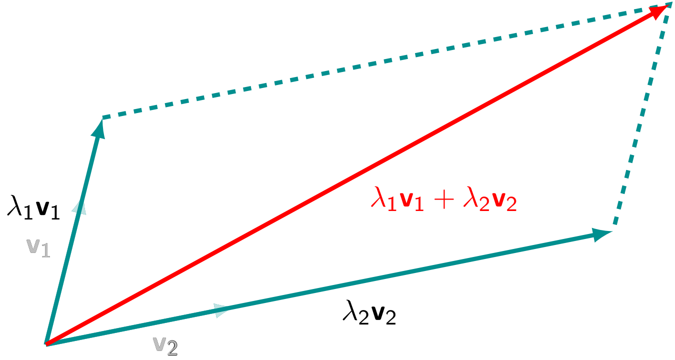
Linear combinations

Definition

A linear combination of two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is a sum of scalar multiples of $\mathbf{v}_1$ and $\mathbf{v}_2$,

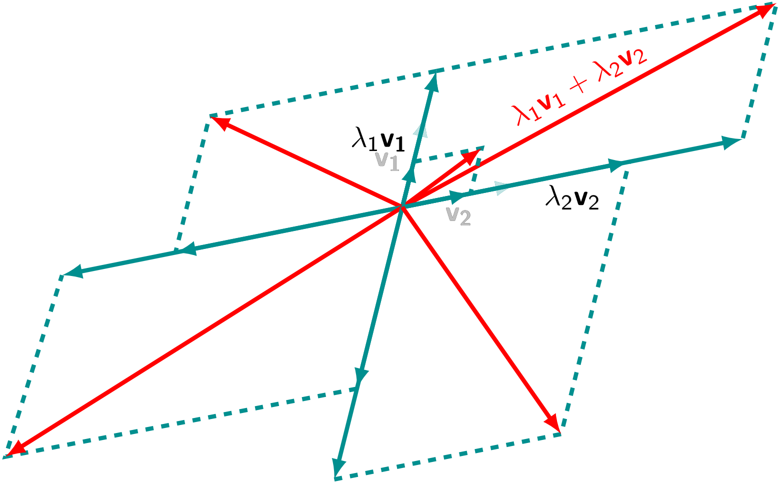
$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2,$$

where λ_1 and λ_2 are scalars.



Linear combinations

With $\mathbf{v}_1$ and $\mathbf{v}_2$ we can make many different linear combinations.



Span

Definition

The set of all linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$ is called the span of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2) = \{\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 \mid \lambda_1, \lambda_2 \in \mathbb{R}\}$$

Example

Show that $\text{span}(\mathbf{i}, \mathbf{j}) = \mathbb{R}^2$.

If $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2$, then $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ which is certainly in $\text{span}(\mathbf{i}, \mathbf{j})$.

Conversely, if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} \in \text{span}(\mathbf{i}, \mathbf{j})$, then $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2$.

Span - Examples

Example

Is $\begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix}$ in the span of $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$?

If $\begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix} \in \operatorname{span}\left(\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}\right)$, there must exist scalars $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix} = \lambda_1 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$.

But comparing the first components on each side, we have $-3 = \lambda_1 \times 0 + \lambda_2 \times 0 = 0$, which is impossible.

So $\begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix} \notin \operatorname{span}\left(\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}\right)$.

Span - Examples

Example

Is $\begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix}$ in the span of $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}$?

If so, $\begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$.

Comparing the components on each side gives three equations:

$$\begin{aligned} -3 &= \lambda_1 + 5\lambda_2 \\ 2 &= 2\lambda_1 + 2\lambda_2 \\ 6 &= \lambda_1 - 4\lambda_2 \end{aligned}$$

This has a single solution:

$$\lambda_1 = 2,\ \lambda_2 = -1.$$

So $\begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix} \in \operatorname{span}\left(\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}\right)$.

Example

Is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ in the span of $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}$?

If so, $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$.

This obviously has the solution $\lambda_1 = \lambda_2 = 0$.

So $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \text{span}\left(\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix} \right)$.

Indeed, $\mathbf{0}$ is in the span of any two vectors in $\mathbb{R}^n$.

The span of two non-zero non-parallel vectors is a plane through the origin.

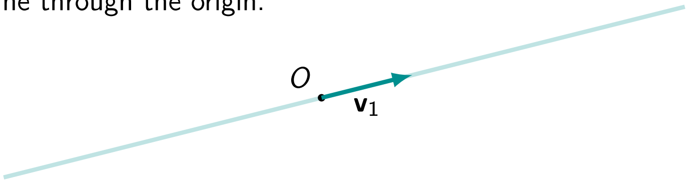
We say that $\text{span}(\mathbf{v}_1, \mathbf{v}_2)$ is the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

We can define the span of any number of vectors.

Geometrically, what is the span of one non-zero vector

$$\text{span}(\mathbf{v}_1) = \{\lambda_1 \mathbf{v}_1 \mid \lambda_1 \in \mathbb{R}\}?$$

It's a line through the origin.



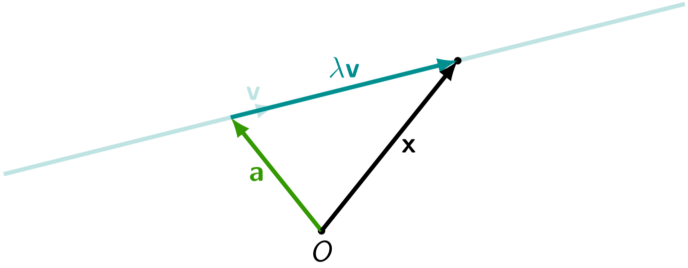
Lines

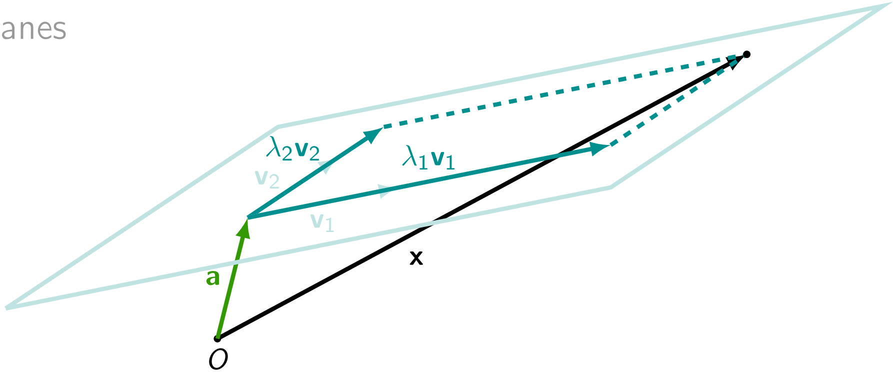
We found that the position vectors $\mathbf{x}$ of points on a line in $\mathbb{R}^n$ are given by

$$\mathbf{x} = \mathbf{a} + \lambda\mathbf{v}, \quad \lambda \in \mathbb{R}.$$

position vector of a point on the line

vector in span(v)





Similarly, a plane in $\mathbb{R}^n$ is the set of points with position vectors $\mathbf{x}$ given by

$$\mathbf{x} = \mathbf{a} + \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2, \quad \lambda_1, \lambda_2 \in \mathbb{R},$$

where $\mathbf{a} \in \mathbb{R}^n$ is the position vector for any point on the plane, and $\mathbf{v}_1$, $\mathbf{v}_2 \in \mathbb{R}^n$ are a pair of non-zero, non-parallel vectors that are directions within the plane.

This is the parametric vector form of a plane in $\mathbb{R}^n$.

Planes

Example

Find a parametric vector form of the plane passing through the point $(2, -1, 2)$ and parallel to the lines

$$\frac{x_1 - 2}{3} = \frac{x_2 - 1}{-3} = \frac{x_3 - 3}{8}$$

and

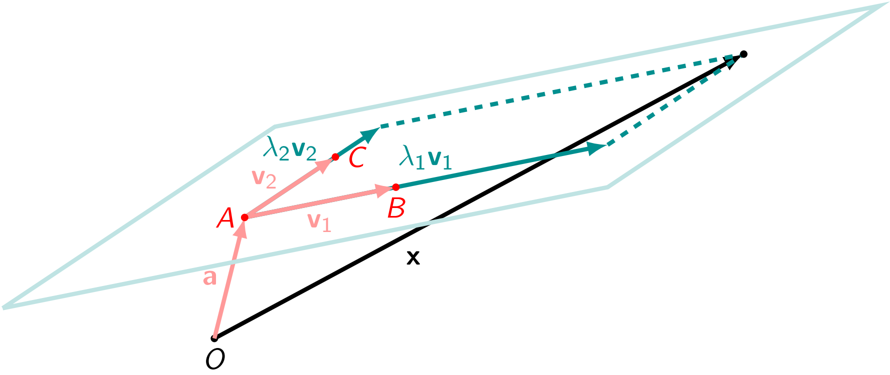
$$\mathbf{x} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ -3 \\ 8 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

Planes

Example

Find a parametric vector form of the plane passing through the three points $A(1, -2, 1)$, $B(2, 1, 1)$ and $C(0, 3, 1)$.



Example

Find a parametric vector form of the plane passing through the three points $A(1, -2, 1)$, $B(2, 1, 1)$ and $C(0, 3, 1)$.

So one solution would be:

$$\begin{aligned}
\mathbf{x} &= \overrightarrow{OA} + \lambda_1 \overrightarrow{AB} + \lambda_2 \overrightarrow{AC} \\
&= \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2-1 \\ 1-(-2) \\ 1-1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0-1 \\ 3-(-2) \\ 1-1 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}.
\end{aligned}$$

Cartesian equation of a plane

Example

For the plane given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} + \lambda_2 \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R},$$

eliminate the parameters λ_1 and λ_2 to find an equation relating x_1, x_2 and x_3.

We have the simultaneous equations:

$$\begin{aligned} x_1 &= 1 + 4\lambda_2 & &\implies \lambda_2 = \frac{x_1 - 1}{4} \\ x_2 &= 2 - \lambda_1 - 2\lambda_2 & &\implies x_2 = 2 - \frac{x_3 - 3}{4} - 2\frac{x_1 - 1}{4} \\ x_3 &= 3 + 4\lambda_1 & &\implies \lambda_1 = \frac{x_3 - 3}{4} \end{aligned}$$

Cartesian equation of a plane

Example

For the plane given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} + \lambda_2 \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R},$$

eliminate the parameters λ_1 and λ_2 to find an equation relating x_1, x_2 and x_3.

From $x_2 = 2 - \dfrac{x_3 - 3}{4} - 2\dfrac{x_1 - 1}{4}$, simplifying and rearranging yields:

$$2x_1 + 4x_2 + x_3 = 13.$$

We call this a Cartesian equation of the plane.

Cartesian equation of a plane

Definition

A Cartesian equation of a plane in $\mathbb{R}^3$ is an equation of the form

$$ax_1 + bx_2 + cx_3 = d$$

for some $a, b, c, d \in \mathbb{R}$ with at least one of a, b and c non-zero.

The analogous construction in $\mathbb{R}^n$ is called the Cartesian equation of a hyperplane in $\mathbb{R}^n$.

We will see a simpler way to find the Cartesian form from the vector form of a plane in a few more lectures.

Cartesian equation of a plane

Example

Find a vector equation for the plane in $\mathbb{R}^3$ given by

$$x_1 + 2x_2 - x_3 = 3.$$

Let $x_2 = \lambda_1$ and $x_3 = \lambda_2$ behave as the paramaters. Then we have:

$$\begin{array}{lllll} x_1 & = 3 & -2\lambda_1 & +\lambda_2 \\ x_2 & = & \lambda_1 & \\ x_3 & = & & \lambda_2 \end{array}$$

So we can write a parametric vector equation as:

$$\mathbf{x} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}$$

Cartesian equation of a plane

Example

Find a vector equation for the plane in $\mathbb{R}^3$ given by

$$3x_1 + x_2 = 3.$$

Let $x_2 = \lambda_1$ and $x_3 = \lambda_2$ behave as the paramaters. Then we have:

$$\begin{aligned} x_1 &= 1 \quad -\tfrac{1}{3}\lambda_1 \\ x_2 &= \qquad\quad \lambda_1 \\ x_3 &= \qquad\qquad\quad \lambda_2 \end{aligned}$$

So we can write a parametric vector equation as:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1/3 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}$$

Notice x_3 had to be one of the parameters in this case, because there are no restrictions on x_3 in the Cartesian equation.