THE UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1131 Calculus

Section 4: - Differentiable Functions.

The subject we call *calculus* began with the ancient Greeks, who considered finding the area bounded by curved figures. Indeed, as well as being able to find the volume of a sphere, Archimedes was able to find the area bounded by a parabola and a line. This was done in a purely geometric way. In late antiquity the problem of finding tangents to curves arose (notably the curves were conics) and some progress was made on this problem. The methods used were ingenious and for the most part *ad hoc.*

The fact that these two problems were related, was not discovered and proven until the 17th century by Newton and about the same time by Leibniz. Our modern version of calculus and the notation we use were essentially due to Leibniz.

In this section we shall study the theoretical ideas of differentiation.

Definition: Suppose f is defined on some open interval containing x. We say that f is **differentiable** at x if

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

exists, and in this case, we denote the limit by $f'(x)$ of $\frac{df}{dx}$ or $\frac{d}{dx}(f(x))$.

It is called the derivative of f at x.

We say that f is **differentiable** on (a,b) is f is differentiable (write diffble) at each x in (a,b).

Ex: Find the derivatives of $f(x)=x^3$, and $f(x)=\sqrt{x}$, from first principles.

$$\frac{d}{dx}(x^3) = \lim_{h\to 0}\frac{(x+h)^3-x^3}{h} = \lim_{h\to 0}\frac{x^3+3hx^2+3h^2x+h^3-x^3}{h}$$

$$= \lim_{h\to 0}\frac{3hx^2+3h^2x+h^3}{h} = \lim_{h\to 0}(3x^2+3hx+h^2)$$

$$= 3x^2$$

$$\frac{d}{dx}(\sqrt{x}) = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0}\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Ex: Show that $f(x) = |x|$ is not differentiable at $x = 0$

$$\frac{d}{dx}(|x|) = \lim_{h \to 0} \frac{|x+h| - |x|}{h}$$

If $x = 0$, then we have

$$\lim_{h \to 0} \frac{|h|}{h}.$$

We know this limit does not exist (see page 2 of Chap. 2 of these notes).

Theorem: For $x \in \mathbb{R}$, $\frac{d}{dx}(\sin x) = \cos x$.

Proof:

$$\frac{d}{dx}(\sin x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h \to 0} \frac{\cos x \sin h}{h}.$$

The second limit is $\cos x \lim_{h \to 0} \frac{\sin h}{h} = \cos x$.

The first limit is

$$\sin x \lim_{h \to 0} \frac{\cos h - 1}{h} = \sin x \lim_{h \to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} = \sin x \lim_{h \to 0} \frac{-\sin^2 h}{h^2} \cdot \frac{h}{\cos h + 1}$$

Finally, $\lim_{h \to 0} \frac{-\sin^2 h}{h^2} = -\lim_{h \to 0} \left(\frac{\sin h}{h}\right)^2 = -1$. and

$\lim_{h \to 0} \frac{h}{\cos h + 1} = 0$. Hence $\frac{d}{dx}(\sin x) = \cos x$.

Ex: Check whether the following are differentiable at $x = 0$.

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and

$$g(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f'(0) = \lim_{h \to 0} \frac{h \sin \frac{1}{h}}{h} = \lim_{h \to 0} \sin \frac{1}{h}.$$

This limit does not exist, So f is not differentiable at $x = 0$.

$$g'(0) = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0.$$

Equivalent Definition: We could equally say that f is differentiable at $x = a$ if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists.

Proof: Replace $x - a$ by h.

Theorem: If f is differentiable at $x = a$ then f is cts at $x = a$.

Corollary: If f is NOT cts at $x = a$ then f is NOT diffble at $x = a$.

Ex:

$$f(x) = \begin{cases} e^x & x > 0 \\ x^2 & x \leq 0 \end{cases}$$

is not cts at $x = 0$ hence it cannot be diffble at $x = 0$.

Ex: Discuss whether or not $f(x) = x|x|$ is differentiable.

$$f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0. \end{cases}$$

The only potential problem is at $x=0$.

$$\lim_{h \to 0^\pm} \frac{\pm h^2}{h} = \lim_{h \to 0^\pm} \pm h = 0.$$

So $f(x)$ is differentiable everywhere.

Split Functions:

A function may be defined *piece-wise* using a split definiton.

Ex:

$$f(x) = \begin{cases} x & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

Again, the only potential problem is at $x=0$.

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h^2}{h} = 0$$

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$$

Theorem: Suppose

$$f(x) = \begin{cases} p(x) & x \geq a \\ q(x) & x < a \end{cases}$$

with $p(x)$ and $q(x)$ differentiable in some interval containing a. Then f is differentiable at $x = a$ iff f is cts at $x = a$ and $p'(a) = q'(a)$.

Ex:

$$f(x) = \begin{cases} \sin x & x < \pi \\ ax + b & x \geq \pi \end{cases}$$

Given that f is diffble at π find a and b.

$p(x) = \sin x \qquad q(x) = ax + b$

$p'(x) = \cos x \qquad q'(x) = a$

$p(\pi) = 0, \qquad q(\pi) = a\pi + b$

$p'(\pi) = 1, \qquad q'(\pi) = a.$

So $\begin{cases} a\pi + b = 0 \\ a = 1 \end{cases}$. Thus we have $a = 1,\ b = -\pi.$

Rules for Differentiation: Suppose f and g are both differentiable at $x = a$, then at $x = a$ we have

(i) $f \pm g$ is diffble and $(f \pm g)'(x) = f'(x) \pm g'(x)$.

(ii) $f(x) = C$ has derivative 0.

(iii) $cf(x)$ is differentiable and $(cf)'(x) = cf'(x)$ where c is a constant.

(iv) fg is differentiable and $(fg)'(x) = f(x)g'(x) + f'(x)g(x)$.

(v) $\frac{f}{g}$ is diffble, provided $g(a) \neq 0$ and $\left(\frac{f}{g}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$.

(vi) $f \circ g$ is differentiable and $(f \circ g)'(x) = f'(g(x)).g'(x)$.

Proof: We will only prove (iv) and (vi).

$$\frac{d}{dx}(fg)(x) = \lim_{h \to 0} \frac{fg(x+h) - fg(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x+h) + f(x)\,g(x+h) - f(x)\,g(x)}{h}$$

$$= \lim_{h \to 0} g(x+h) \frac{f(x+h) - f(x)}{h} + f(x) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} g(x+h) \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + f(x)\, g'(x)$$

$$= f'(x)\, g(x) + f(x)\, g'(x)$$

For (vi) we look at

$$\lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}.$$

Put $u = g(x)$, and $g(x+h) = u + \ell$. Now as $h \to 0, \ell \to 0$ and we can write the above limit as

$$\lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} = \lim_{\substack{h \to 0 \\ \ell \to 0}} \frac{f(u+\ell) - f(u)}{\ell} . \frac{\ell}{h}$$

$$= \lim_{\ell \to 0} \frac{f(u+\ell)) - f(u)}{\ell} . \lim_{h \to 0} \frac{g(x+h)) - g(x)}{h} = f'(g(x)).g'(x).$$

Rates of Change:

Suppose $y(t)$ is a quantity that can vary with time (t).

The **average** rate of change of y between times t and $t + h$ is clearly $\dfrac{y(t+h) - y(t)}{h}$.

If the limit of this rate exists as $h \to 0$, then that limit, $\frac{dy}{dt}$ will represent the *instantaneous rate of change* of y at time t.

You have already met this idea in relation to displacement, velocity, (which is the instantaneous rate of change of displacement) and acceleration (which is the instantaneous rate of change of velocity).

Ex: A point P is moving to the right along the x axis at a constant rate of 5 cm/sec and a point Q is moving up the y axis at a constant rate of 10 cm/sec. How fast is the distance between the two points changing when $OP = 30$ and $OQ = 40$?

Let $D(t)$ be the distance between the two points. So

$$D(t) = \sqrt{P(t)^2 + Q(t)^2}$$, where

$P(t)$ and $Q(t)$ are the distance to the origin of the points P and Q, respectively.

$$\frac{dD}{dt} = \frac{2(P(t)P'(t) + Q(t)Q'(t))}{2\sqrt{P^2(t) + Q^2(t)}}.$$ So if $P(t)=30$

and $Q(t) = 40$, $\frac{dD}{dt} = \frac{30 \cdot 5 + 40 \cdot 10}{\sqrt{30^2+40^2}} = 11$ cm/sec

Local maxima and minima and stationary points:

Local Maxima and Minima:

Definition: Suppose that f is a function and x_0 is a point in the domain of f. Consider the interval $(x_0 - \delta, x_0 + \delta)$, where we think of δ as being a small positive real number. If $f(x) \leq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta)$ then we say that x_0 is a **local maximum**.

Also, if $f(x) \geq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta)$ then we say that x_0 is a **local minimum**.

Theorem: Suppose that f is defined on the interval (a, b), and has a local maximum (or minimum) at $c \in (a, b)$.
If f is differentiable at c then $f'(c) = 0$.

Proof: Suppose that f has a minimum at c. Then $f(x) - f(c)$ must be positive (or 0) for all $x \in (c - \delta, c + \delta)$ for some $\delta > 0$. Hence $\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \geq 0$ since the denominator is also positive. Further $\lim_{x\to c^-} \dfrac{f(x)-f(c)}{x-c} \leq 0$ for all $x \in (c-\delta, c+\delta)$ since the denominator is always negative. Now f is differentiable, so the above limits exists as $x \to c$ hence by the pinching theorem we have $\lim_{x\to c} \dfrac{f(x)-f(c)}{x-c} = 0$, but this says $f'(c) = 0$.

Implicit Differentiation:

While most functions you have met were presented in an explicit form, that is, with y as the subject, a function may also be given implicitly.

For example $x^2 + y^2 = 9, y \geq 0$ gives an implicit form of the semi-circle radius 3 centre O. In order to find $\frac{dy}{dx}$, we could solve for y explicitly and then differentiate.

This is not always possible. For example, the function implicitly defined by $y^5 + xy = 3$ cannot be algebraically solved for y (although you could make x the subject and find $\frac{dx}{dy}$).

To differentiate such an expression, we use **implicit differentiation** which is really just an application of the chain rule.

Ex: Find $\frac{dy}{dx}$ for $x^4 + y^4 = 1$, and for $y^5 + xy = 3$.

$\frac{d}{dx}$: $x^4 + y^4 = 1$

$$4x^3 + 4y^3 \cdot \frac{dy}{dx} = 0 \text{ .. so } \frac{dy}{dx} = \frac{-x^3}{y^3}$$

$\frac{d}{dx}$: $y^5 + xy = 3$.

$$5y^4 \cdot \frac{dy}{dx} + y + x\frac{dy}{dx} = 0.$$

Hence $$\frac{dy}{dx} = \frac{-y}{5y^4 + x}.$$

Ex: Find the gradient at the point $(1, 1)$ for the curve (in fact it is an hyperbola).
$x^2 - 3xy + 2y^2 + y = 1$

Differentiating both sides, we get

$$2x - 3y - 3x\frac{dy}{dx} + 4y\frac{dy}{dx} + \frac{dy}{dx} = 0.$$

If $x = y = 1$, we get $2 - 3 - 3\frac{dy}{dx} + 4\frac{dy}{dx} + \frac{dy}{dx} = 0.$

So $\frac{dy}{dx} = \frac{1}{2}$.

Ex: Prove that $\frac{d}{dx}x^\alpha = \alpha x^{\alpha-1}$ then α is rational.

Write $\alpha = \frac{p}{q}$, then $y^q = x^p$. Hence $qy^{q-1}y' = px^{p-1}$. Solving for y' and substituting we have

$$y' = \frac{p}{q}x^{p-1}y^{1-q} = \frac{p}{q}x^{p-1}x^{\frac{p}{q}(1-q)} = \frac{p}{q}x^{\frac{p}{q}-1} = \alpha x^{\alpha-1}.$$