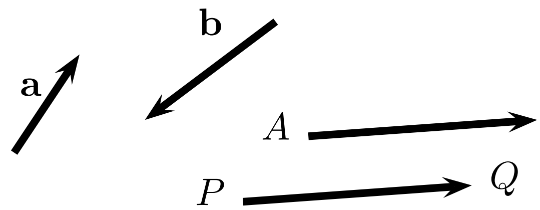
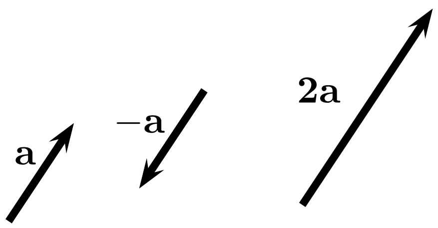
§1 Introduction to Vectors (2020T1: W1-Tu-We-Th, W2-Tu)

- **Geometric vectors.**
  - A *scalar* quantity is specified by a single number.
  - A (*geometric*) *vector* quantity $\mathbf{a}$ (*typed with boldface, or handwritten with undertilde* $\underset{\sim}{a}$) is specified by two attributes:
    - a *magnitude*, denoted by $|\mathbf{a}|$; and
    - a *direction*.



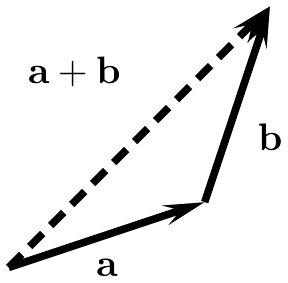
  - We represent a vector graphically with an arrow.
  - We can specify a vector by stating an *initial point* A and a *terminal point* B, and we denote it by $\overrightarrow{AB}$.
  - Two vectors are *equal* if and only if they have the same magnitude and the same direction.
    This is regardless of their initial and terminal points. The position does not matter!

- **Multiplication by a scalar.**
  - The *zero vector* $\mathbf{0}$ is the vector with magnitude zero and undefined direction.
  - The *negative* of a vector $\mathbf{a}$, denoted by $-\mathbf{a}$, is the vector with the same magnitude as $\mathbf{a}$ but in the opposite direction to $\mathbf{a}$.

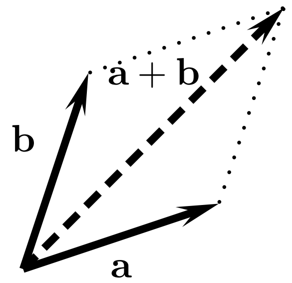


  - The *multiplication* $\lambda\mathbf{a}$ of a vector $\mathbf{a}$ *by a scalar* $\lambda \in \mathbb{R}$ is defined as follows:
    - If $\lambda > 0$ then $\lambda\mathbf{a}$ is the vector with magnitude $\lambda|\mathbf{a}|$ and in the same direction as $\mathbf{a}$.
    - If $\lambda < 0$ then $\lambda\mathbf{a}$ is the vector with magnitude $|\lambda||\mathbf{a}|$ and in the opposite direction to $\mathbf{a}$.
    - If $\lambda = 0$ then $\lambda\mathbf{a} = \mathbf{0}$.

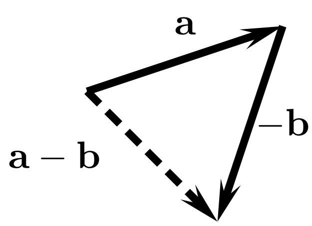
- **Vector addition.** There are two equivalent definitions:
  - *Triangle law*. The vector $\mathbf{a}+\mathbf{b}$ is obtained by joining the initial point of $\mathbf{b}$ to the terminal point of $\mathbf{a}$ and then taking the arrow which goes from the initial point of $\mathbf{a}$ to the terminal point of $\mathbf{b}$.



  - *Parallelogram law*. Join the initial points of the vectors $\mathbf{a}$ and $\mathbf{b}$ and form a parallelogram using these vectors as adjacent sides. The vector $\mathbf{a}+\mathbf{b}$ is obtained by joining the common initial point of $\mathbf{a}$ and $\mathbf{b}$ to the opposite corner of the parallelogram.



- **Vector laws.** Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be vectors, and λ and μ be real numbers. Then
  - *Commutative law of vector addition*: $\mathbf{a}+\mathbf{b} \;=\; \mathbf{b}+\mathbf{a}$.
  - *Associative law of vector addition*: $(\mathbf{a}+\mathbf{b})+\mathbf{c} \;=\; \mathbf{a}+(\mathbf{b}+\mathbf{c})$.
  - *Associative law of multiplication by scalar*: $\lambda(\mu\mathbf{a}) \;=\; (\lambda\mu)\mathbf{a}$.
  - *Scalar distributive law*: $(\lambda+\mu)\mathbf{a} = \lambda\mathbf{a}+\mu\mathbf{a}$.
  - *Vector distributive law*: $\lambda(\mathbf{a}+\mathbf{b}) = \lambda\mathbf{a}+\lambda\mathbf{b}$.
  - The zero vector satisfies $\mathbf{a}+\mathbf{0} \;=\; \mathbf{a} \;=\; \mathbf{0}+\mathbf{a}$.
  - The negative of vector satisfies $\mathbf{a}+(-\mathbf{a}) \;=\; -\mathbf{a}+\mathbf{a} \;=\; \mathbf{0}$.
- **Vector subtraction** is defined by $\mathbf{a}-\mathbf{b} \;=\; \mathbf{a}+(-\mathbf{b})$.

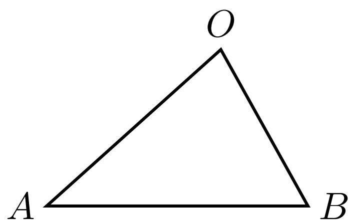


Exercise. Verify the associative law of vector addition using diagrams.

Exercise. Simplify $2(3\mathbf{a} - \mathbf{b}) - (2\mathbf{b} - \mathbf{a})$.

Example. Let OAB be a triangle. We can write

$$\begin{aligned} \overrightarrow{OA} + \overrightarrow{AB} &= \overrightarrow{OB}, \\ \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA}. \end{aligned}$$

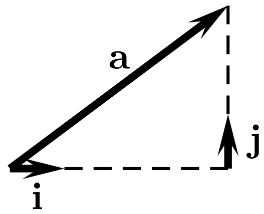


Exercise. Let OAB be a triangle. Let M be the midpoint between O and A. Let P and Q divide the line segment OB into three equal sections, with P closer to O and Q closer to B. Show that the line segment AQ is parallel to the line segment MP and has twice its length.

Exercise. Three ropes are attached to a block of wood, and a man is pulling on each rope. If the first man is pulling with a force of 30 Newtons due east and the second man is pulling with a force of 40 Newtons due north, find the force with which a third man must pull to stop the block from moving.

- **Geometric vectors v.s. coordinate vectors.**
  - *Standard basis vectors* in the plane: choose two unit-length (geometric) vectors, call them $\mathbf{i}$ and $\mathbf{j}$, with $\mathbf{j}$ at an angle of $\frac{\pi}{2}$ anticlockwise from $\mathbf{i}$.
  - Every (geometric) vector $\mathbf{a}$ in the plane can be uniquely expressed as

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}.$$



    - a_1, a_2 are called the *components* or *coordinates* of $\mathbf{a}$.
  - Every (geometric) vector $\mathbf{a}$ in the plane has a corresponding *coordinate vector* $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ with respect to the chosen basis vectors $\mathbf{i}$ and $\mathbf{j}$.
    - $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ is a *(two-dimensional) column vector* or a 2-*vector*.
    - a_1, a_2 are called the *components* or *coordinates* of $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.
  - We identify a (geometric) vector $\mathbf{a}$ with its coordinate vector $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, and write simply

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

- **The set $\mathbb{R}^2$.**
  - We define the set of 2-*vectors* by

$$\mathbb{R}^2 = \left\{ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} : a_1, a_2 \in \mathbb{R} \right\}.$$

  - For any $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}^2$, and $\lambda \in \mathbb{R}$, we define
    - *Multiplication by a scalar*: $\lambda\mathbf{a} = \lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \end{pmatrix}$.
    - *Addition*: $\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$.
  - These algebraic definitions are consistent with the geometric interpretations.

Exercise. Solve the “three ropes tied to a block of wood” problem using coordinate vectors.

- **Generalisation to n-dimensions – the set $\mathbb{R}^n$.**
  - We define the set of *n-vectors* by
$$\mathbb{R}^n = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \ : \ a_1, a_2, \ldots, a_n \in \mathbb{R} \right\}.$$
  - For any $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{R}^n$, and $\lambda \in \mathbb{R}$, we define
    - *Zero vector*: $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, i.e., all n components are 0.
    - *Negative of vector*: $-\mathbf{a} = \begin{pmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_n \end{pmatrix}$.
    - *Multiplication by a scalar*: $\lambda \mathbf{a} = \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \\ \vdots \\ \lambda a_n \end{pmatrix}$.
    - *Addition* or *sum*: $\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}$.
    - *Subtraction* or *difference*: $\mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ \vdots \\ a_n - b_n \end{pmatrix} = \mathbf{a} + (-\mathbf{b})$.
  - Two vectors $\mathbf{a}$ and $\mathbf{b}$ are *equal*, $\mathbf{a} = \mathbf{b}$, if the corresponding components are equal.
  - Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are *parallel* if there is a non-zero real number λ such that $\mathbf{a} = \lambda \mathbf{b}$. They are said to be *in the same direction* if $\lambda > 0$.
  - Vectors in $\mathbb{R}^n$ satisfy the **vector laws** on page 2.

Example.

(a) $\begin{pmatrix}1\\2\end{pmatrix}$ is a 2-vector; $\begin{pmatrix}-1\\\pi\\e\\\sqrt{2}\end{pmatrix}$ is a 4-vector; $\begin{pmatrix}1\\2\\3\end{pmatrix} \in \mathbb{R}^3$.

(b) $\begin{pmatrix}0\\0\end{pmatrix}$ is the zero vector in $\mathbb{R}^2$; $\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ is the zero vector in $\mathbb{R}^4$.

(c) $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}2\\1\\3\end{pmatrix}$ are *not* equal.

(d) $\begin{pmatrix}1\\-2\end{pmatrix}$ is parallel to $\begin{pmatrix}2\\-4\end{pmatrix}$, and $\begin{pmatrix}1\\-2\end{pmatrix}$ is in the same direction as $\begin{pmatrix}2\\-4\end{pmatrix}$.

(e) $\begin{pmatrix}1\\-2\\3\end{pmatrix}$ and $\begin{pmatrix}-2\\4\\-6\end{pmatrix}$ are parallel, but they are *not* in the same direction.

Exercise.

(a) $2\begin{pmatrix}2\\-1\\5\end{pmatrix}$

(b) $\begin{pmatrix}1\\2\\3\end{pmatrix} + \begin{pmatrix}2\\-1\\5\end{pmatrix}$

(c) $\begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}2\\-1\\5\end{pmatrix}$

(d) $\begin{pmatrix}1\\2\\3\end{pmatrix} + \begin{pmatrix}2\\-1\\5\\4\end{pmatrix}$

(e) Are $\begin{pmatrix}1\\-2\\0\\3\end{pmatrix}$ and $\begin{pmatrix}-4\\8\\0\\-12\end{pmatrix}$ parallel?

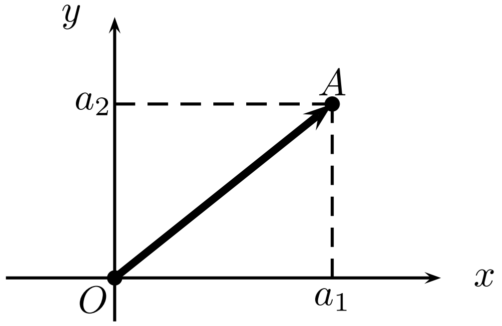
(f) Are $\begin{pmatrix}1\\-2\\0\\3\end{pmatrix}$ and $\begin{pmatrix}3\\-6\\3\\9\end{pmatrix}$ parallel?

Example. Prove the commutative law of vector addition $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$ in $\mathbb{R}^n$.

Proof. Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ for some positive integer n, where the components a_j and b_j are all real numbers for $j = 1, 2, \ldots, n$. Then

$$\begin{aligned}
\mathbf{a}+\mathbf{b} &= \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \\
&= \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{pmatrix} && \text{definition of vector addition} \\
&= \begin{pmatrix} b_1+a_1 \\ b_2+a_2 \\ \vdots \\ b_n+a_n \end{pmatrix} && \text{commutative law of real numbers} \\
&= \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} && \text{definition of vector addition} \\
&= \mathbf{b}+\mathbf{a}\,.
\end{aligned}$$

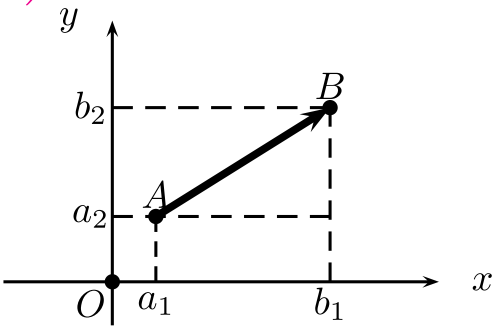
- **Coordinate system for points in $\mathbb{R}^2$.**
  - A *(Cartesian) coordinate system* in the *plane* consists of an origin O, a unit of length, and two directions (the x direction and the y direction) at right angle to each other.



  - The standard basis vectors $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the x and y directions.
  - Every point A on the plane has a unique pair of *coordinates* (a_1, a_2) with respect to our chosen coordinate system.
    - A is a_1 units from the origin in the x direction and a_2 units from the origin in the y direction.
    - The vector $\mathbf{a} = \overrightarrow{OA} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ is called the *position vector* or *coordinate vector* of A.
    - (Pythagoras' Theorem) The *length* of the vector $\mathbf{a} = \overrightarrow{OA}$ is

$$|\mathbf{a}| = |\overrightarrow{OA}| = \sqrt{a_1^2 + a_2^2}.$$

  - For any two points A and B with coordinates (a_1, a_2) and (b_1, b_2):
    - A has coordinate vector $\mathbf{a} = \overrightarrow{OA} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.
    - B has coordinate vector $\mathbf{b} = \overrightarrow{OB} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.
    - $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$.

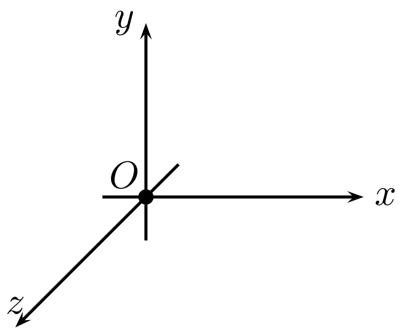


    - The *distance* between A and B is

$$\mathrm{dist}(A, B) = |\overrightarrow{AB}| = |\mathbf{b} - \mathbf{a}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}.$$

- **Coordinate system for points in $\mathbb{R}^3$.**
  - A *coordinate system* in the *space* $\mathbb{R}^3$ consists of an origin O, a unit of length, and three directions (the x, y and z directions) each at right angles to the others.
  - We follow the "right-handed" system:

    index finger – x direction,
    middle finger – y direction,
    thumb – z direction.



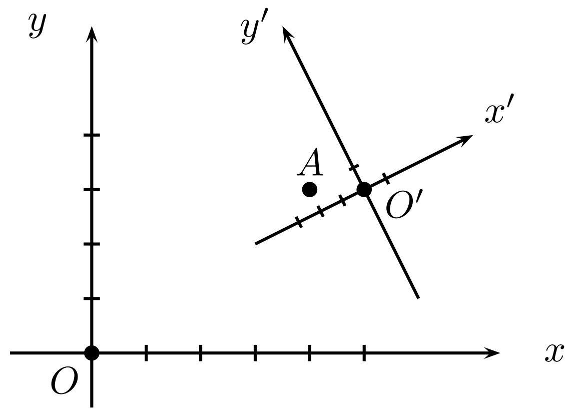
  - The standard basis vectors in the space $\mathbb{R}^3$ are denoted by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.
  - Analogously to $\mathbb{R}^2$, every point in $\mathbb{R}^3$ has a position vector, and we can define the length of a vector and the distance between two points...
- **Generalisation to points in the n-dimensional space.**
  - The *standard basis vectors* $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ in $\mathbb{R}^n$ are defined as follows:

    $\mathbf{e}_j$ has a 1 in the jth component and 0 everywhere else.

  - Every point A in the space $\mathbb{R}^n$ has a *position vector* or *coordinate vector* $\mathbf{a} = \overrightarrow{OA} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ so that A is a_j units from the origin in the direction of $\mathbf{e}_j$.
  - The *length* of a vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$.
  - If two points A and B have coordinate vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$, then $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ \vdots \\ b_n - a_n \end{pmatrix}$, and the *distance* between A and B is

$$\operatorname{dist}(A, B) = |\overrightarrow{AB}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + \cdots + (b_n - a_n)^2}.$$

Exercise. Find the coordinate vector of A with respect to each coordinate system in the figure.



Example. The three standard basis vectors in $\mathbb{R}^3$ are

$$\mathbf{i} = \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{j} = \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{k} = \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Exercise. Write down the standard basis vectors in $\mathbb{R}^5$.

Exercise. Let $\mathbf{a} = \begin{pmatrix} 5 \\ 1 \\ 2 \\ 0 \\ -3 \end{pmatrix}$.

(a) Write the vector $\mathbf{a}$ as a linear combination of standard basis vectors.

(b) Find the length of the vector $\mathbf{a}$.

(c) Find a *unit vector* (a vector of length one) parallel to the the vector $\mathbf{a}$.

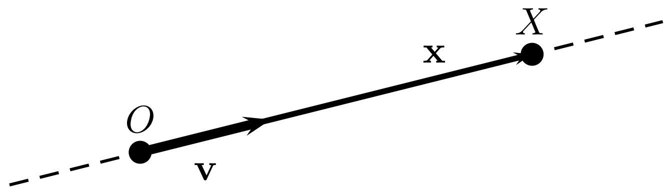
Exercise. If the points A and B have coordinates $(1, -2, 3)$ and $(-3, 1, 5)$, find the vector $\overrightarrow{AB}$ and the distance between A and B.

Exercise. Three (or more) points are said to be *collinear* if they lie on the same line. Show that the points $A(1, -2, 0, 4)$, $B(3, 1, 2, -5)$, $C(5, 4, 4, -14)$ are collinear.

Equation of lines – parametric vector form. Let $\mathbf{v}$ be a non-zero vector in $\mathbb{R}^n$.

- The line *through the origin* O and *parallel to the vector* $\mathbf{v}$ has *parametric vector form*

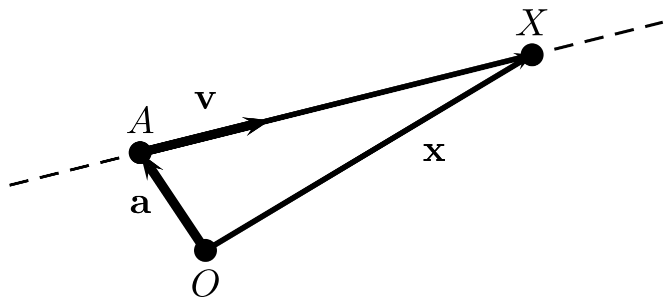
$$\mathbf{x} = \lambda\mathbf{v}, \qquad \lambda \in \mathbb{R}.$$



(In words, any point X on the line has coordinate vector $\mathbf{x} = \overrightarrow{OX}$ given by some scalar multiple of $\mathbf{v}$.)

- The line *through the point* A with coordinate vector $\mathbf{a} = \overrightarrow{OA}$ and *parallel to the vector* $\mathbf{v}$ has *parametric vector form*

$$\mathbf{x} = \mathbf{a} + \lambda\mathbf{v}, \qquad \lambda \in \mathbb{R}.$$



Notes:

(i) Given a vector $\mathbf{v} \in \mathbb{R}^n$, the *span* of $\mathbf{v}$ is the set

$$\text{span}(\mathbf{v}) = \{\mathbf{x} \in \mathbb{R}^n \;:\; \mathbf{x} = \lambda\mathbf{v},\, \lambda \in \mathbb{R}\}.$$

When $\mathbf{v} \neq \mathbf{0}$, $\text{span}(\mathbf{v})$ is a line through the origin in $\mathbb{R}^n$, and we say that the line is *spanned by* $\mathbf{v}$.

(ii) A general parametric vector form for a line is

$$\mathbf{x} = (\text{one point on the line}) + \lambda\,(\text{direction of the line}), \qquad \lambda \in \mathbb{R}.$$

(iii) A parametric vector form for a line is *not unique*.

Example. A parametric vector form of the line through the point $(1, 2, 3)$ and parallel to the vector $\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}, \qquad \lambda \in \mathbb{R}.$$

Exercise. Find a parametric vector form of the line spanned by the vector $\begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}$.

Exercise. Find a parametric vector form of the line through the two points $(1, 2, 3)$ and $(-1, 4, 1)$.

Exercise. Find a parametric vector form of the line through the point $(1, 2, 3)$ and parallel to the line $\mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

Exercise. Find a parametric vector form of the line through the point $(1, 2, 3)$ and parallel to the line through the two points $(1, 1, 2)$ and $(3, -4, 1)$.

Exercise. Let A and B have coordinates $(1, 2, 3)$ and $(4, -4, 0)$. Let P and Q divide the line segment AB into three equal sections, with P closer to A and Q closer to B. Find a parametric vector form of the line through A and B, hence obtain the coordinates of P and Q.

Equation of lines – Cartesian form.

- *Parametric vector form* $\Rightarrow$ *Cartesian form*: extract each variable in $\mathbb{R}^n$ and rearrange the equations so that λ becomes the subject (thus eliminating λ).
- *Cartesian form* $\Rightarrow$ *parametric vector form*: set some expression (e.g. one variable) to λ and write each variable in terms of λ.

Note. The Cartesian form is unique, but a parametric vector form is not unique.

Example. To convert a parametric vector form

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \qquad \lambda \in \mathbb{R},$$

into Cartesian form, let $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and we extract

$$\begin{cases} x = 1 + 4\lambda & \Rightarrow \lambda = \dfrac{x-1}{4}, \\ y = 2 - \lambda & \Rightarrow \lambda = \dfrac{y-2}{-1}. \end{cases}$$

Thus the Cartesian form is

$$\frac{x-1}{4} = \frac{y-2}{-1} \qquad \text{or} \qquad x + 4y = 9.$$

Example. To convert the Cartesian form $x+4y = 9$ into a parametric vector form, let $y = \lambda$. Then $x + 4\lambda = 9$, so we have

$$\begin{cases} x = 9 - 4\lambda, \\ y = 0 + 1\lambda. \end{cases}$$

Thus a parametric vector form is

$$\mathbf{x} = \begin{pmatrix} 9 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 1 \end{pmatrix}, \qquad \lambda \in \mathbb{R}.$$

Alternatively, we can let $x = \mu$ so

$$\mu + 4y = 9 \quad \Rightarrow \quad y = \frac{9-\mu}{4},$$

and this yields

$$\begin{cases} x = 0 + 1\mu, \\ y = \dfrac{9}{4} - \dfrac{1}{4}\mu. \end{cases}$$

Thus another parametric vector form is

$$\mathbf{x} = \begin{pmatrix} 0 \\ \frac{9}{4} \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -\frac{1}{4} \end{pmatrix}, \qquad \mu \in \mathbb{R}.$$

Exercise. Obtain the Cartesian form for the line

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \ \lambda \in \mathbb{R}.$$

Exercise. Obtain a parametric vector form for the line

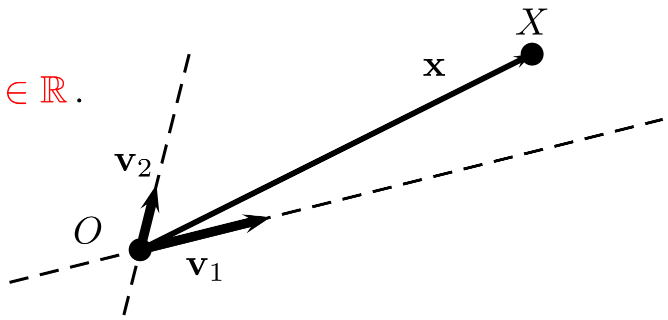
$$\frac{x+2}{3} = \frac{y-1}{2} = 3 - z.$$

Exercise. Find the equation of the line in $\mathbb{R}^4$ through the point (a_1, a_2, a_3, a_4) and parallel to the vector $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$ in parametric vector form and Cartesian form, assuming that all components v_1, v_2, v_3, v_4 are non-zero.

Equation of planes – parametric vector form. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be non-zero and non-parallel vectors in $\mathbb{R}^n$.

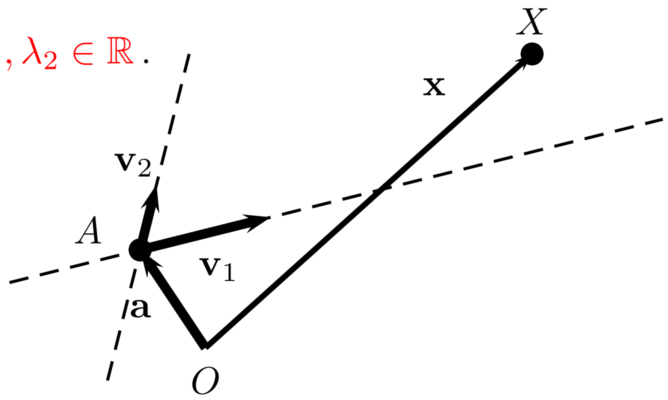
- The plane *through the origin* O and *parallel to the vectors* $\mathbf{v}_1$ and $\mathbf{v}_2$ has *parametric vector form*

$$\mathbf{x} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2, \qquad \lambda_1, \lambda_2 \in \mathbb{R}.$$



- The plane *through the point* A with coordinate vector $\mathbf{a} = \overrightarrow{OA}$ and *parallel to the vectors* $\mathbf{v}_1$ and $\mathbf{v}_2$ has *parametric vector form*

$$\mathbf{x} = \mathbf{a} + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2, \qquad \lambda_1, \lambda_2 \in \mathbb{R}.$$



Notes:

(i) A *linear combination* of two vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$ is a sum of scalar multiples of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 \quad \text{for some } \lambda_1, \lambda_2 \in \mathbb{R}.$$

(ii) The *span* of two vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$ is the set

$$\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2) = \{\mathbf{x} \in \mathbb{R}^n \;:\; \mathbf{x} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2,\ \lambda_1, \lambda_2 \in \mathbb{R}\}.$$

(iii) When two vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$ are non-zero and non-parallel, $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$ is a plane through the origin in $\mathbb{R}^n$, and we say that the plane is *spanned by* $\mathbf{v}_1$ and $\mathbf{v}_2$.

(iv) If one vector is $\mathbf{0}$ (say, $\mathbf{v}_2 = \mathbf{0}$) or if the two vectors are parallel (that is, $\mathbf{v}_2 = \lambda\mathbf{v}_1$ for some $\lambda \in \mathbb{R}$), then $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$ is a line through the origin in $\mathbb{R}^n$.

(v) A general parametric vector form for a plane is

$$\begin{aligned}\mathbf{x} = {} & (\text{one point on the plane}) + \lambda_1\,(\text{one vector on the plane}) \\ & + \lambda_2\,(\text{another vector on the plane}), \qquad \lambda_1, \lambda_2 \in \mathbb{R}.\end{aligned}$$

(vi) A parametric vector form for a plane is *not unique*.

Example. The equation of the plane through the point $(1,2,3,4)$ and parallel to the vectors $\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}2\\1\\-1\\1\end{pmatrix}$ has a parametric vector form

$$\mathbf{x} = \begin{pmatrix}1\\2\\3\\4\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\0\\1\\0\end{pmatrix} + \lambda_2 \begin{pmatrix}2\\1\\-1\\1\end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

Exercise. Find a parametric vector form for the plane passing through the points $(1,2,3)$, $(2,1,-2)$, and $(4,4,1)$.

Exercise. Find a parametric vector form for the plane passing through the point $(1,1,3)$ and parallel to the lines

$$\mathbf{x} = \begin{pmatrix}2\\1\\-2\end{pmatrix} + \lambda \begin{pmatrix}4\\0\\1\end{pmatrix}, \lambda \in \mathbb{R},$$

and

$$\frac{x-2}{-2} = \frac{y+1}{3} = z.$$

Exercise. Describe geometrically the following sets:

(a) $\operatorname{span}\left\{\begin{pmatrix}1\\3\\2\\4\end{pmatrix}, \begin{pmatrix}2\\6\\4\\8\end{pmatrix}\right\}$.

(b) $\mathbf{x} = \lambda_1 \begin{pmatrix}1\\0\\1\\0\end{pmatrix} + \lambda_2 \begin{pmatrix}2\\3\\4\\-1\end{pmatrix}, \ \lambda_1, \lambda_2 \in \mathbb{R}$.

(c) $\mathbf{x} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\0\\1\end{pmatrix} + \lambda_2 \begin{pmatrix}-2\\0\\-2\end{pmatrix}, \ \lambda_1, \lambda_2 \in \mathbb{R}$.

- **Equation of planes – Cartesian form in $\mathbb{R}^3$.**
  - In the special case of $\mathbb{R}^3$, the *Cartesian form* of a plane is a linear equation

$$ax + by + cz \;=\; d\,,$$

    where $a, b, c, d \in \mathbb{R}$.
  - The conversion between parametric vector form and Cartesian form can be done in a similar way as for lines.

Exercise. Obtain a parametric vector form for the plane $3x - 2y + z = 4$.

Exercise. Obtain the Cartesian form for the plane

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

- **Intersection of lines and planes.**
  - *Intersection of a line and a plane* can be
    - empty set – the line is parallel to the plane but is not in the plane; or
    - a point – the line intersects the plane at one point; or
    - a line – the line lies in the plane.
  - *Intersection of two lines* can be
    - empty set – two lines are parallel or skewed; or
    - a point – two lines intersect at one point; or
    - a line – two lines coincide.
  - *Intersection of two planes* can be
    - empty set – two planes are parallel and do not meet; or
    - a line – two planes intersect in a line; or
    - a plane – two planes coincide.

Exercise. Find the point of intersection of the line

$$\mathbf{x} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\1\end{pmatrix}, \quad \lambda \in \mathbb{R},$$

and the plane $3x - 2y + z = 6$.

Exercise. Find the point of intersection of the two lines

$$\mathbf{x} = \begin{pmatrix}1\\0\\-2\end{pmatrix} + \lambda_1\begin{pmatrix}2\\1\\0\end{pmatrix}, \ \lambda_1 \in \mathbb{R}, \qquad \text{and} \qquad \mathbf{x} = \begin{pmatrix}2\\3\\3\end{pmatrix} + \lambda_2\begin{pmatrix}-1\\0\\1\end{pmatrix}, \ \lambda_2 \in \mathbb{R}.$$