School of Mathematics and Statistics
Math1131-Algebra

Lec10: conjugate and division, polar form, modulus, argument

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2020 Term 1

Real and imaginary parts

$z = a + bi$

Special complex numbers: the real and purely imaginary ones

- $z = \mathrm{Re}(z) + \mathrm{Im}(z)i$ where both $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are real numbers.
- $z = \mathrm{Re}(z) \;\Leftrightarrow\; \mathrm{Im}(z) = 0 \;\Leftrightarrow\; z$ is *real*
- $z = \mathrm{Im}(z)i \;\Leftrightarrow\; \mathrm{Re}(z) = 0 \;\Leftrightarrow\; z$ is *purely imaginary*

Exercise 1.

(a) 3 is ....real....

(b) $3i$ is ....purely imaginary....

(c) Both 3 and $3i$ are ....complex....

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Equality of complex numbers

Equal complex numbers

Two complex numbers are *equal* *if and only if* their real parts are equal and their imaginary parts are equal.

Exercise 2. Find real numbers a and b such that $(3+4i)(a+bi) = 23+14i$.

LHS $(3+4i)(a+bi)$

$= (3a-4b) + i(3b+4a)$

Same real part: $3a - 4b = 23$ $\quad | \times 3 \quad | \times -4$

" Im " : $+\ 4a + 3b = 14$ $\quad | \times 4 \quad | \times 3$

$3① + 4②$

$(9+16)a = 23 \times 3 + 4 \times 14 = 125$

$(16+9)b = -4 \times 23 + 3 \times 14$

$25b = -50$ $\quad \boxed{b = -2}$

$25a = 125$ $\quad \boxed{a = 5}$

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Square roots of a complex number

$\boxed{-4}$

4 -1

Square roots of a complex number

A *square root* of a complex number w is a complex number z such that

$$z^2 = w.$$

Any non-zero complex number z has two square roots.

Exercise 3. a) Find the square roots of $-3 + 4i$

$(x+iy)^2 = -3+4i$

$x^2 - y^2 + 2ixy = -3+4i$

By equating the real and imaginary part, we get

$$\begin{cases} x^2 - y^2 = -3 & (1) \\ 2xy = 4 & (2) \end{cases}$$

(2) $y = \frac{2}{x}$

sub (2) into (1)

$x^2 - \frac{4}{x^2} = -3 \quad) \times x^2$

$x^4 + 3x^2 - 4 = 0$

$X = x^2$

$X^2 + 3X - 4 = 0$

$(x^2+4)(x^2-1) = 0$

$x^2+4=0$ impossible

$x^2 = 1$

$\boxed{x = \pm 1}$

If $x = 1$, $y = 2$

If $x = -1$, $y = -2$

$\boxed{z_1 = 1+2i}$

$\boxed{z_2 = -1-2i}$

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Quadratic equations

Quadratic formula with complex numbers

The solution(s) of the quadratic equation

$$az^2 + bz + c = 0$$

where $a, b, c \in \mathbb{C}$ with $a \neq 0$, is/are

$$\boxed{z = \frac{-b \pm \delta}{2a}}$$

where δ is a square root of $\Delta = b^2 - 4ac$, i.e. $\delta^2 = \Delta = b^2 - 4ac$.

Exercise 3, continued. b) Solve $\boxed{z^2 + 3z + (3 - i) = 0}$.

$\Delta = b^2 - 4ac = 9 - 4(3-i) = 9 - 12 + 4i = -3 + 4i = (1+2i)^2$

$\delta = 1+2i$ is a square root of Δ

$\boxed{z_1} = \dfrac{-3+1+2i}{2} = \dfrac{-2+2i}{2} = \boxed{-1+i}$

$\boxed{z_2} = \dfrac{-3-1-2i}{2} = \dfrac{-4-2i}{2} = \boxed{-2-i}$

Exercise 3, continued.

Find the square roots of $-3+4i$ and hence solve $z^2+3z+(3-i)=0$.

```
> # restart clears everything so z can be reused:

  restart
# We find the square roots of -3 + 4i

solve(z^2 = -3 + 4*I);
```

$$1+2\,\mathrm{I},\ -1-2\,\mathrm{I}$$

```
> solve(z^2 + 3*z +3-I = 0);
```

$$-1+\mathrm{I},\ -2-\mathrm{I}$$

Polar coordinates

- The *Cartesian form* of a complex number with real part x and imaginary part y is

$$z = x + yi.$$

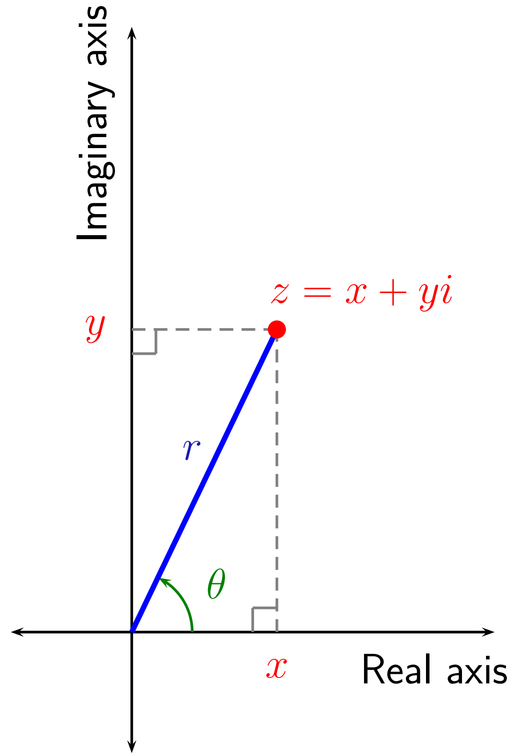
- We can also describe z by its distance r from the origin and its angle θ from the positive real axis as shown.

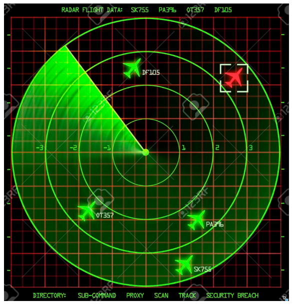
Trigonometry shows that

$$x = r\cos\theta, \quad y = r\sin\theta$$

and

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}.$$



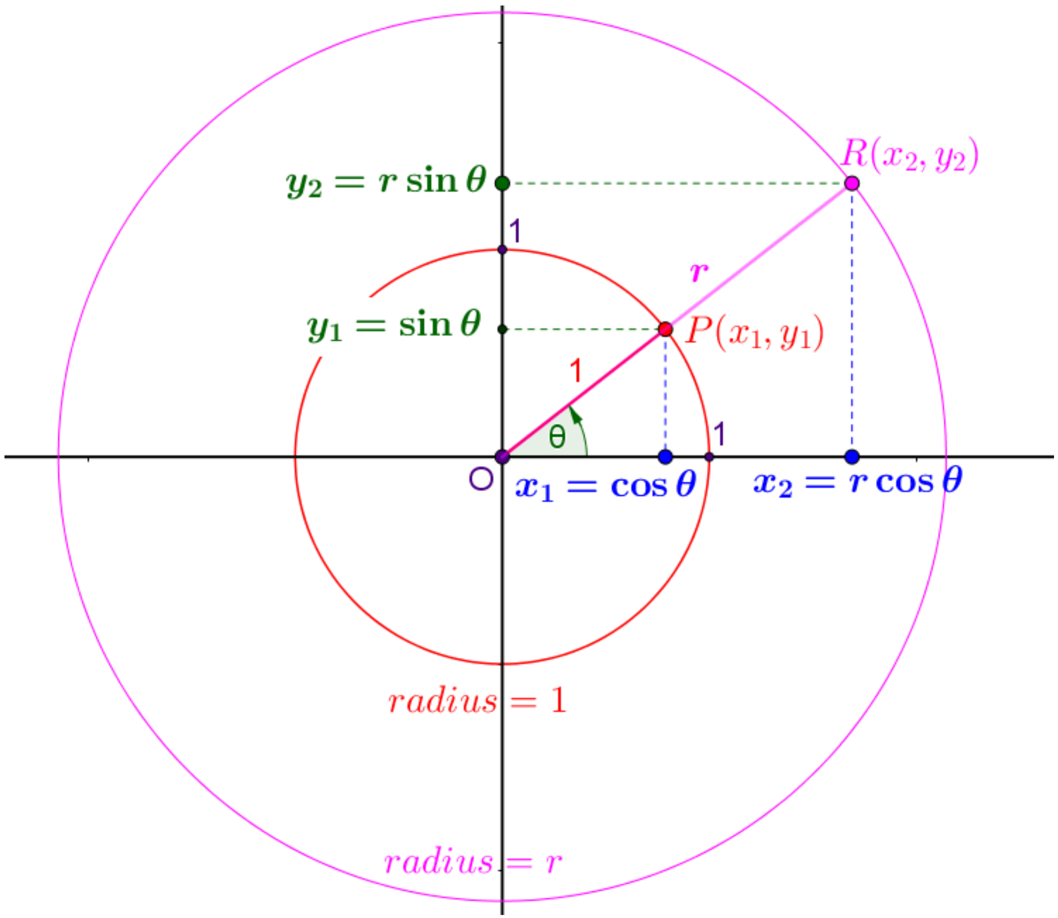


This radars display shows polar coordinates

Figure: Airport Air Traffic Control Radar Screen.

Polar coordinates

$x = r\cos\theta, \quad y = r\sin\theta, \quad \tan\theta = \frac{y}{x}$ illustrated



Modulus, arguments, principal argument

- We call r the *modulus* of $z = x + iy$ and denote it $|z|$.

$$|z| = \sqrt{x^2 + y^2}$$

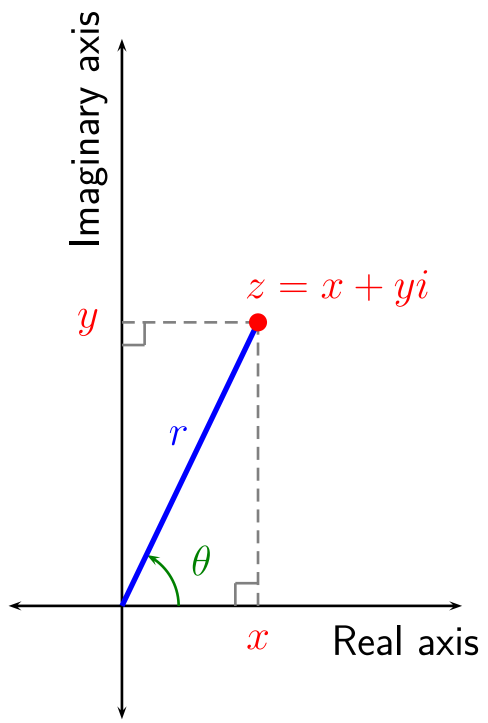
- We call θ ***an*** *argument* of $z = x + iy$ and denote it $\text{arg}(z)$.

$$\tan\left(\theta\right) = \frac{y}{x}$$

The argument is not unique: If θ is an argument of z, so is $\theta + 2k\pi$, $k \in \mathbb{Z}$.

The *principal argument* of z, denoted $\text{Arg}(z)$, is the only argument in the interval $(-\pi, \pi]$.

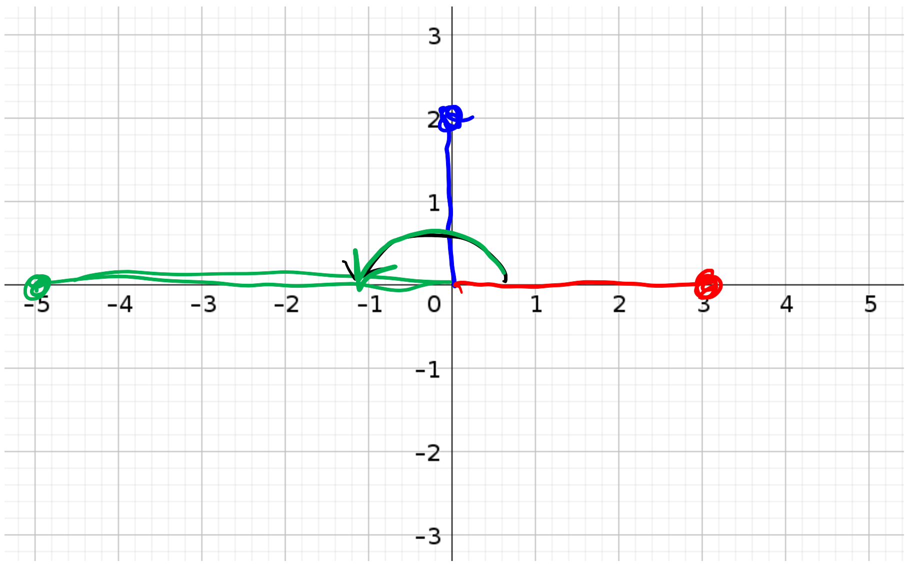
$$-\pi < \text{Arg}(z) \leqslant \pi.$$



Modulus and argument

Exercise 4. Plot each of the following complex numbers on an Argand diagram and find its modulus and principal argument.

a) $2i$ b) 3 c) -5



z	$\lvert z \rvert$	$\operatorname{Arg} z$
$2i$	2	$\frac{\pi}{2}$
3	3	0
-5	5	π

a) $z_a = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

b) $z_b = 3 = 3\left(\cos 0 + i\sin 0\right)$

c) $-5 = 5\left(\cos\pi + i\sin\pi\right)$

$|-5| = r > 0$

Modulus and argument

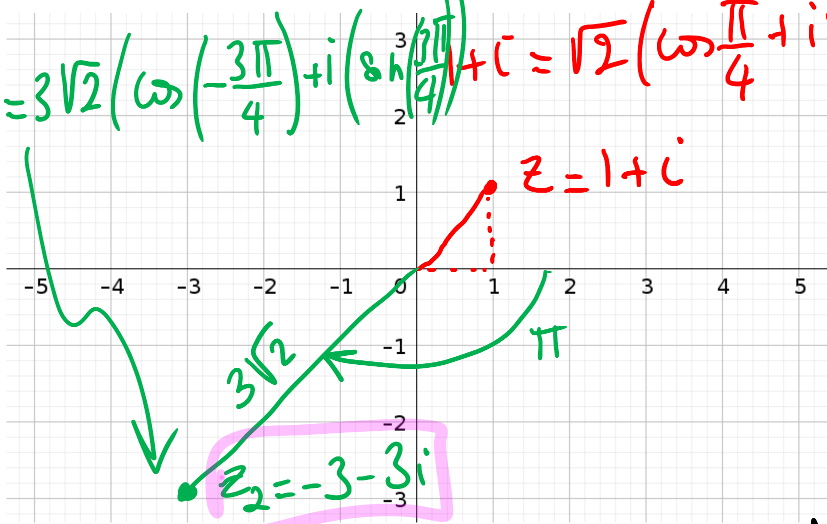
Exercise 4, continued. Plot each of the following complex numbers on an Argand diagram and find its modulus and principal argument.

d) $1+i$ e) $-3-3i$

You may use the revision notes about the unit circle on the last slide.

$-3-3i = 3\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\left(\sin\left(\frac{3\pi}{4}\right)\right)\right)$

$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

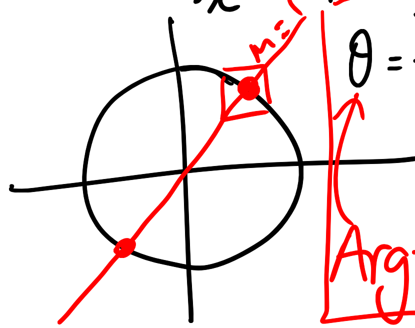


d) $|z| = \sqrt{x^2+y^2}$

$= \sqrt{1+1}$

$\boxed{= \sqrt{2}}$

$\tan\theta = \frac{y}{x} = \frac{1}{1} = 1$



$\theta = \frac{\pi}{4}$

Arg z

e) $|z| = \sqrt{x^2+y^2}$

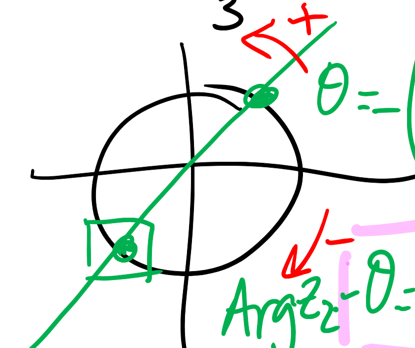
$= \sqrt{9+9}$

$= \sqrt{2\times 9}$

$\boxed{|z_2| = 3\sqrt{2}}$

. $\tan\theta = \frac{y}{x}$

$= \frac{-3}{-3} = 1$



$\theta = -\left(\frac{\pi}{2} + \frac{\pi}{4}\right)$

$\text{Arg}\, z_2 \; \boxed{\theta = -\frac{3\pi}{4}}$

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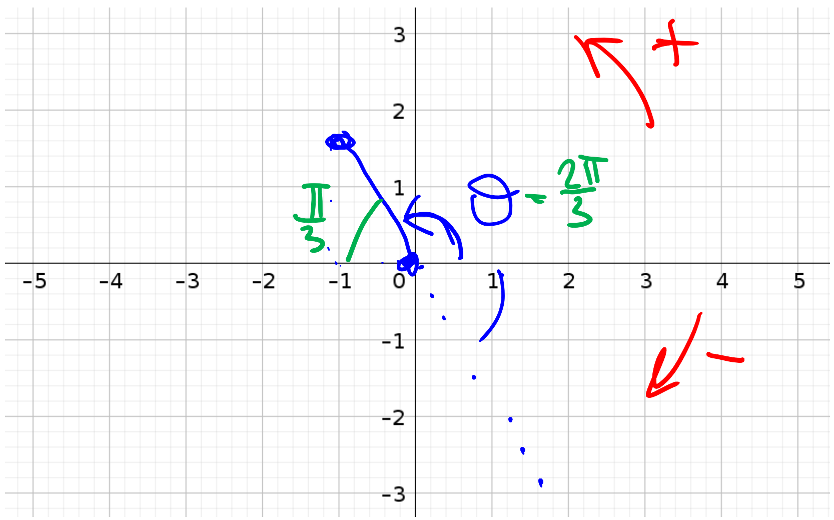
Modulus and argument

Exercise 4, continued. Plot each of the following complex numbers on an Argand diagram and find its modulus and principal argument.

f) $-1 + i\sqrt{3}$

g) 0

You may use the revision notes about the unit circle on the last slides.



$\tan(\theta) = \frac{\sqrt{3}}{-1} = -\sqrt{3}$

$\theta = \frac{2\pi}{3}$

$-\pi - \frac{\pi}{3} + 2\pi = \pi - \frac{\pi}{3}$

f) $z = -1 + i\sqrt{3}$

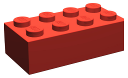
$|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\mathrm{Arg}(z) = \frac{2\pi}{3}$

g) $|0| = 0$

$\mathrm{Arg}(0) =$ undefined

Polar form of a non-zero complex number



Polar form of a non-zero complex number

Let z be a complex number, $z \neq 0$. If the modulus of z is $|z| = r$ and its principal argument is $\text{Arg}(z) = \theta$, then

$$x = r\cos\theta, \quad \text{and} \quad y = r\sin\theta$$

so for $z = x + iy$ we have,

$$z = r(\cos\theta + i\sin\theta).$$

We will call this the *polar form* of a complex number.

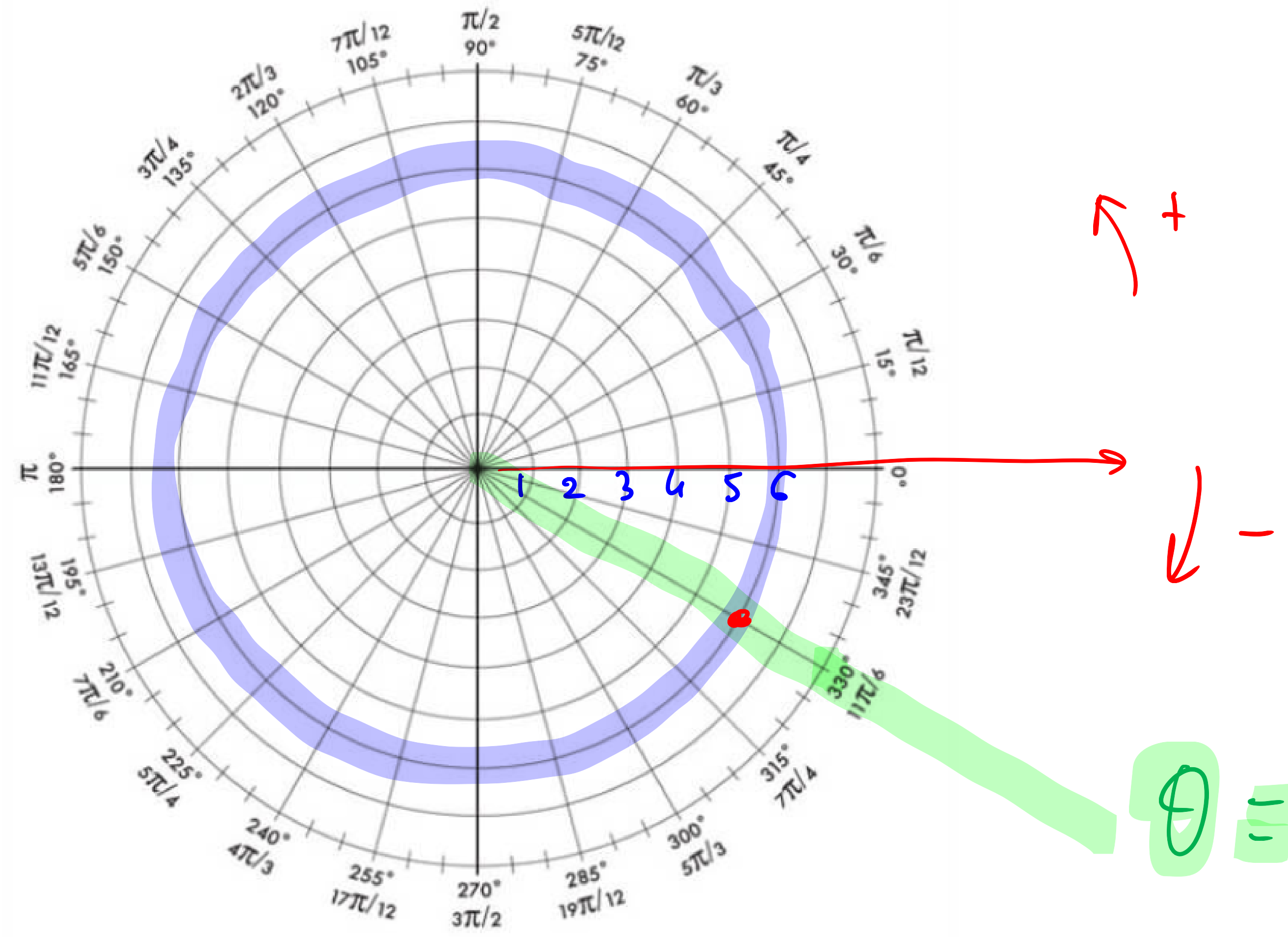
N.B. $z = r(\cos\theta + i\sin\theta)$ for **any** argument of the complex number z, not just the principal one.

Polar form

Exercise 5.

Sketch the following complex number on the complex plane and write it in Cartesian form.

$z = 6\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$



$\theta = -\frac{\pi}{6}$

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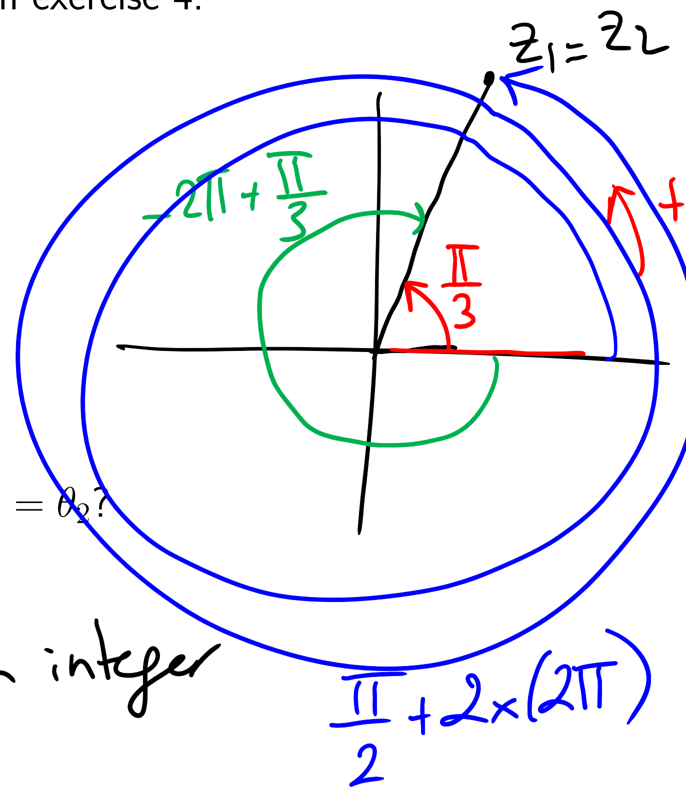
Polar form

Exercise 6. Find the polar form of each complex number in exercise 4.

Exercise 7.

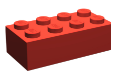
$$\begin{aligned} z_1 &= r_1(\cos\theta_1 + i\sin\theta_1) \\ z_2 &= r_2(\cos\theta_2 + i\sin\theta_2) \\ z_1 &= z_2 \end{aligned}$$

Is it true that since $z_1 = z_2$, we must have $r_1 = r_2$ and $\theta_1 = \theta_2$?



- $r_1 = r_2$ ✓
- $\theta_1 = \theta_2 + 2k\pi$ when k is an integer
  $k \in \mathbb{Z}$

Complex conjugates



Conjugate of a complex number

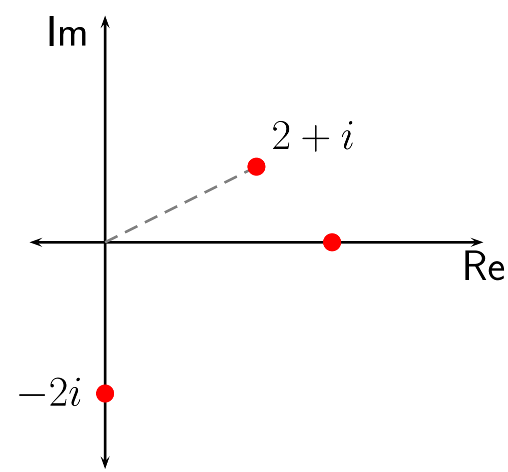
- Let $z = a + ib$ be a complex number with $a, b \in \mathbb{R}$.
  The *conjugate* of z, denoted $\overline{z}$, is

$$\boldsymbol{\overline{z} = a - ib}.$$

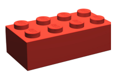
- On an Argand diagram, z and its conjugate $\overline{z}$ are symmetric with respect to the x-axis.

Example 2.

$$\overline{2+i} =$$
$$\overline{-2i} =$$
$$\overline{3} =$$



Complex conjugates



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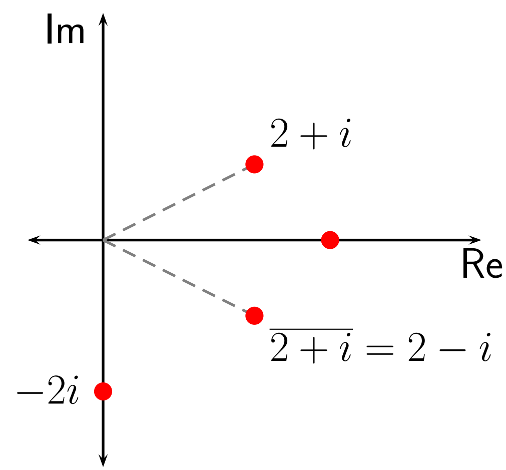
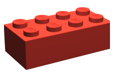
Example 2.

$$\begin{aligned} \overline{2+i} &= 2 - i \\ \overline{-2i} &= \\ \overline{3} &= \end{aligned}$$



Complex conjugates



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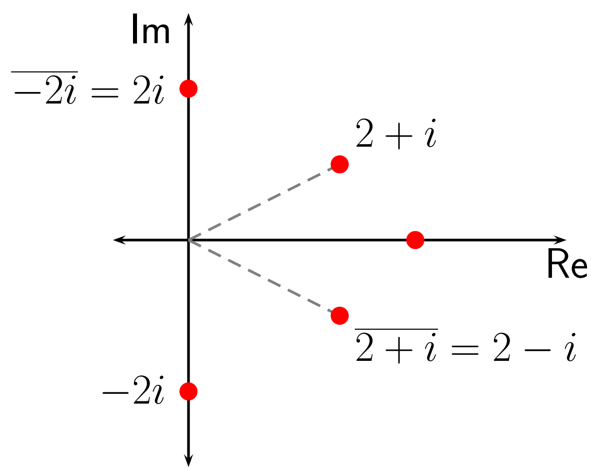


Complex conjugates

Conjugate of a complex number

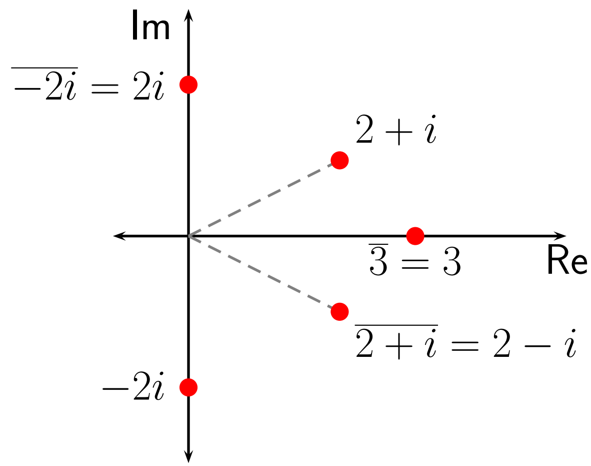
- Let $z = a + ib$ be a complex number with $a, b \in \mathbb{R}$.
  The *conjugate* of z, denoted $\overline{z}$, is

$$\overline{z} \qquad\qquad \boldsymbol{\overline{z} = a - ib}.$$

- On an Argand diagram, z and its conjugate $\overline{z}$ are symmetric with respect to the x-axis.

Example 2.

$$\begin{aligned} \overline{2+i} &= 2-i \\ \overline{-2i} &= 2i \\ \overline{3} &= 3 \end{aligned}$$



Complex conjugates

Conjugate of a complex number

- Let $z = a + ib$ be a complex number with $a, b \in \mathbb{R}$.
  The *conjugate* of z, denoted $\overline{z}$, is

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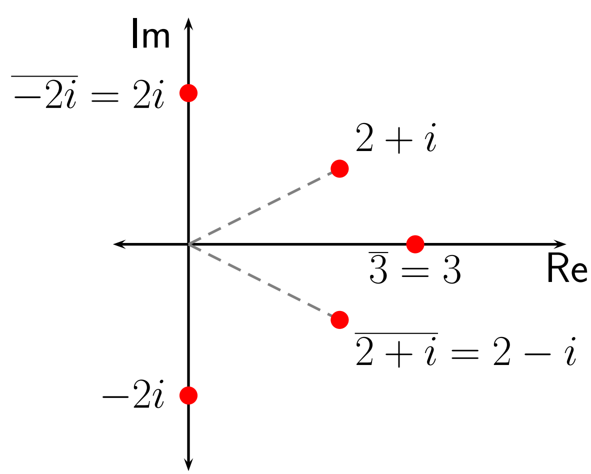


Exercise 3.

a) $\overline{1+7i} =$ 1 − 7i

b) $\overline{3-5i} =$ 3 + 5i

Theorems about the conjugate of a complex number

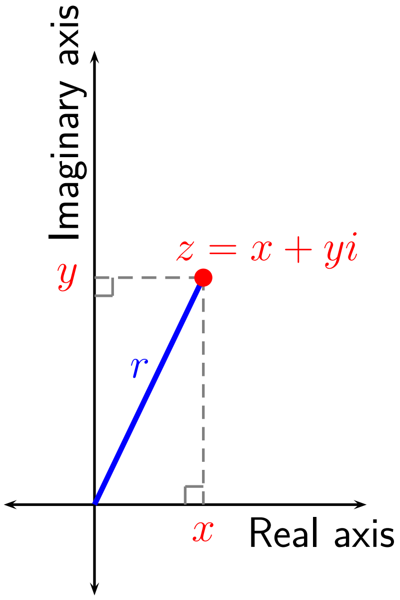
Prove the following useful results.

1. $\overline{\overline{z}} = z$ $\qquad \overline{\overline{z}} = z$
2. $\text{Re}(z) = \frac{1}{2}(z + \overline{z})$
3. $\text{Im}(z) = \frac{1}{2i}(z - \overline{z})$
4. $z\overline{z} = (\text{Re}(z))^2 + (\text{Im}(z))^2 = |z|^2$
5. $z\overline{z} \in \mathbb{R}$ and $z\overline{z} \geqslant 0$
6. $\overline{z + w} = \overline{z} + \overline{w}$
7. $\overline{zw} = \overline{z}\,\overline{w}$
8. $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$

$z = a + ib \quad \times 1$

$\overline{z} = a - ib \quad \times -1$

$z - \overline{z} = 2ib$



④ $(a+ib)(a-ib) = a^2 - (ib)^2 = a^2 + b^2 = \left(\sqrt{a^2+b^2}\right)^2$

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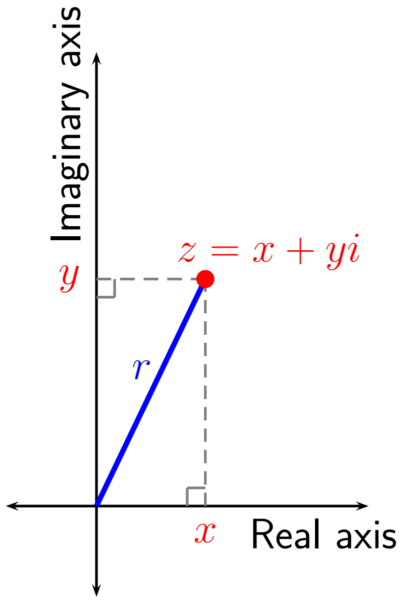
Theorems about the conjugate of a complex number

Some space for you to write the proofs (and a figure to help you visualise the modulus).



Example

Exercise 8.  [Challenge!] Let $z, w \in \mathbb{C}$ with $z\overline{z} = w\overline{w}$.

Prove that $\dfrac{z+w}{z-w}$ is purely imaginary.

Can we divide complex numbers? YES

How to divide complex numbers and get a result of the form $a + ib$

Multiply the numerator *and* the denominator by the conjugate of the denominator.

Exercise 4.

a) Find the real and imaginary parts of $\dfrac{3+4i}{2+5i}$.

b)  What about $\dfrac{c+di}{a+bi}$ for $a, b, c, d \in \mathbb{R}$ with a and b not both zero?

$$\frac{(3+4i)}{(2+5i)} \times \frac{(2-5i)}{(2-5i)}$$

$$= \frac{6 - 20i^2 + i(-15+8)}{2^2+5^2}$$

$$= \frac{26 - 7i}{29}$$

$$= \boxed{\frac{26}{29} - \frac{7}{29}i}$$

Checking our answer with Maple

Exercise 4, continued.

```
> # Division with complex numbers

  z := (3 + 4*I)/(2 + 5*I);
  # Note the uppercase I and the * for 'times'
```

$$z := \frac{26}{29} - \frac{7\,\mathrm{I}}{29}$$

```
> # General case

  z := (a + b*I)/(c + d*I);
```

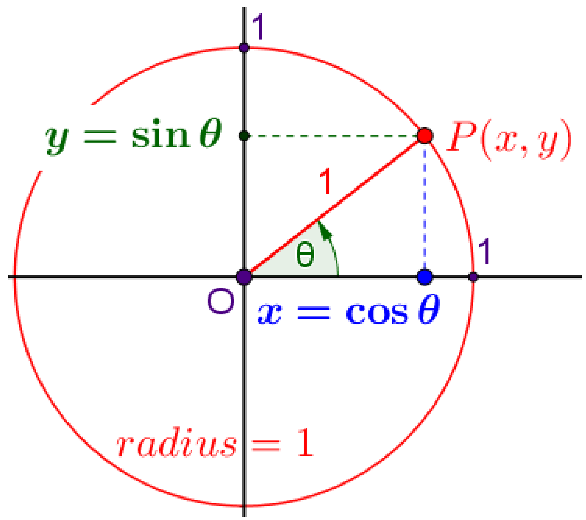
$$z := \frac{a + \mathrm{I}\,b}{c + \mathrm{I}\,d}$$

```
> # use evalc to get get the cartesian (= rectangular) form

  evalc(z);
```

$$\frac{a\,c}{c^2 + d^2} + \frac{b\,d}{c^2 + d^2} + \mathrm{I}\left(\frac{b\,c}{c^2 + d^2} - \frac{a\,d}{c^2 + d^2}\right)$$

Revision: The unit circle in a nutshell



Angles are counted counterclockwise from the positive x-axis.

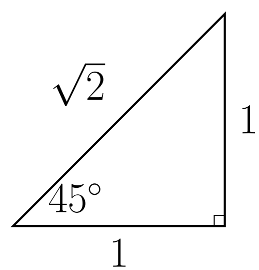
1. The x-coordinate of P is $\cos\theta$.
2. The y-coordinate of P is $\sin\theta$.
3. $\tan\theta$ is the gradient of the straight line OP.

$$m = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$$

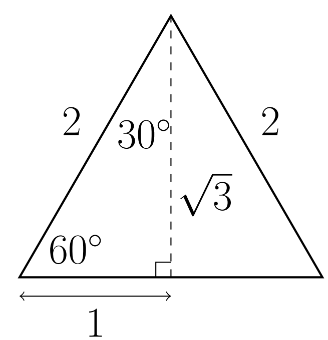
The Exact values you need to know:

θ	$\frac{\pi}{6}$ or $30°$	$\frac{\pi}{4}$ or $45°$	$\frac{\pi}{3}$ or $60°$
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

The Isosceles Right Triangle



The 30/60 Triangle



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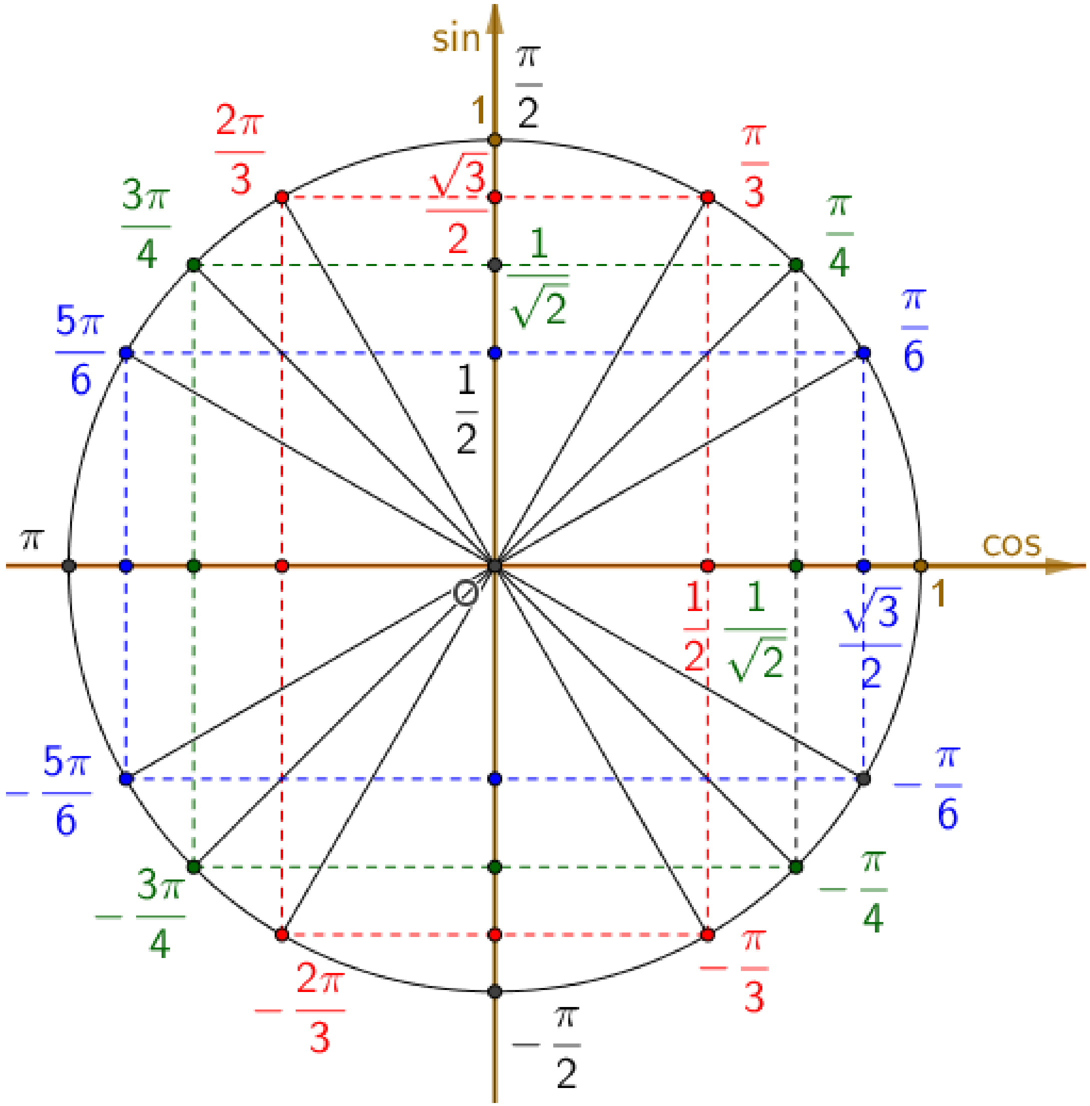

In the first quadrant are the exact values you need to know.