School of Mathematics and Statistics
Math1131-Algebra

Lec08: Triple scalar product/point normal form

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Learning outcomes for this lecture

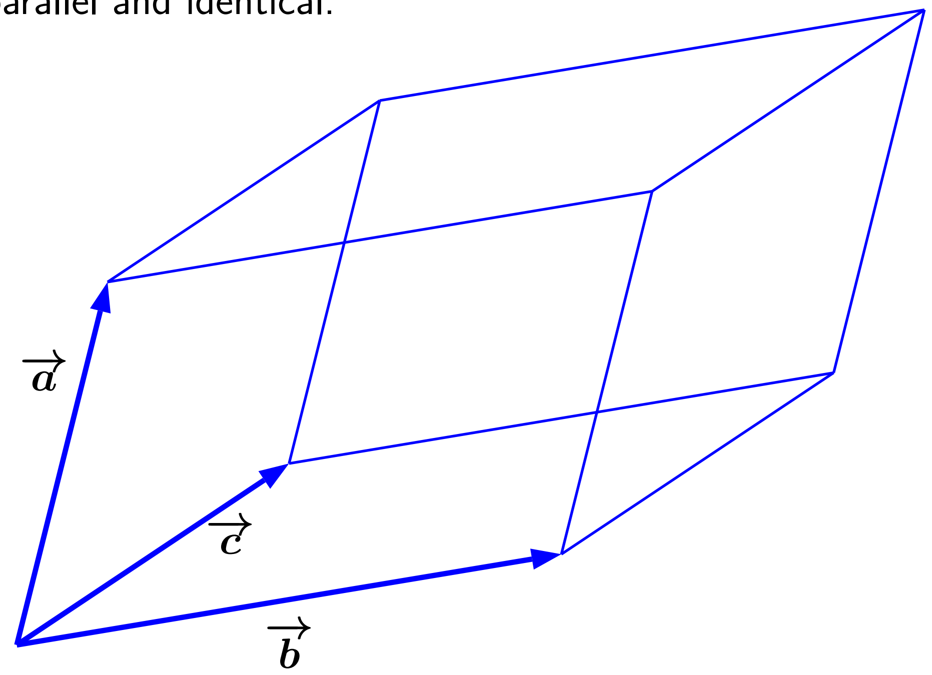
At he the end of this lecture

- ☐ you should be able to calculate the **triple scalar product** of three vectors in $\mathbb{R}^3$ and use it to calculate the **volume of a parallelepiped** in 3D;
- ☐ you should know that the **coefficients** of x, y, z in a **Cartesian** equation of a plane are the coordinates of a vector which is **normal** to that plane;
- ☐ you should be able to write and recognise an equation of a plane written in **point normal form**;
- ☐ you should be able to go any from any equation of a plane (cartesian, parametric, point-normal) to any other;
- ☐ you should be able to use the cross product to solve problems in **Geometry** (like finding the distance between a point and a plane);
- ☐ you should be more convinced than ever that **drawing is really helpful**!

You can use this list as a check list to get ready for our next class: After studying the lecture notes, come back to this list, and for each item, check that you have indeed mastered it. Then tick the corresponding box ... or go back to the notes.

Volume of a parallelepiped

The 3D version of a parallelogram is the parallelepiped. The six faces are parallelograms. They are pairwise parallel and identical.



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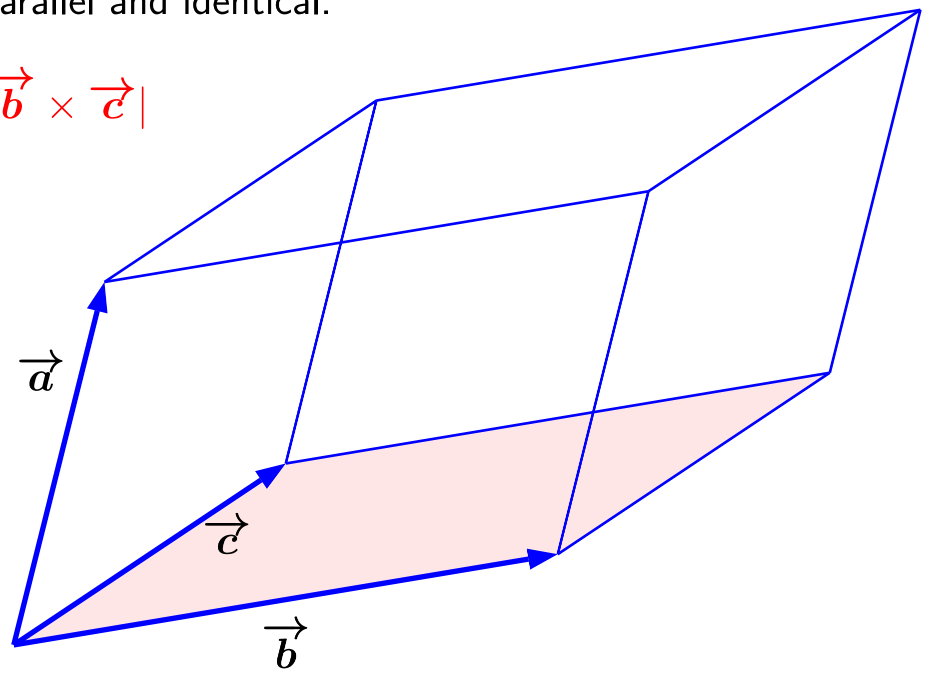
Area of base $= |\overrightarrow{b} \times \overrightarrow{c}|$



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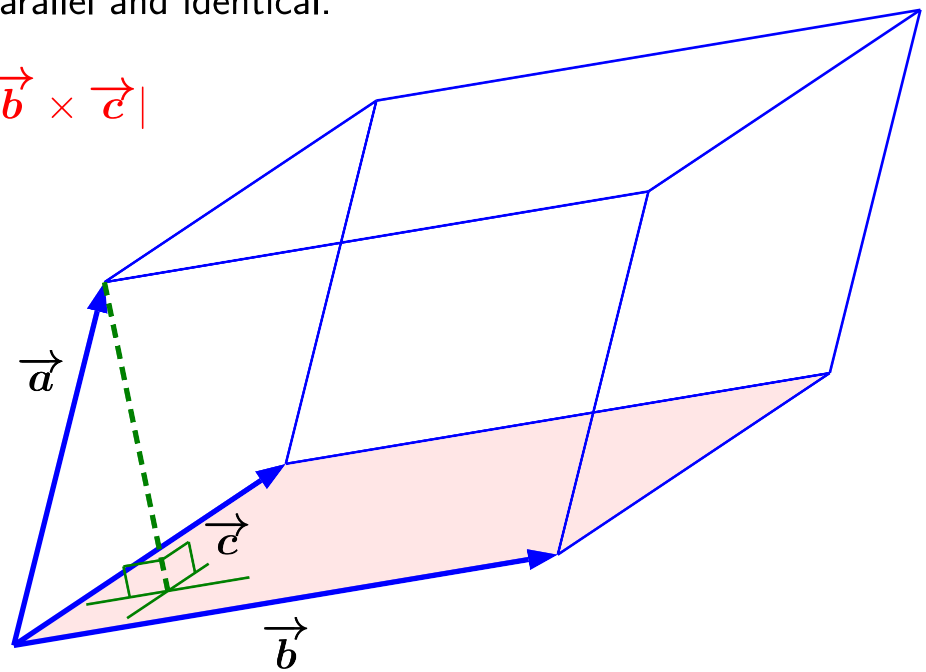

Volume $=$ Area of base $\times$ altitude

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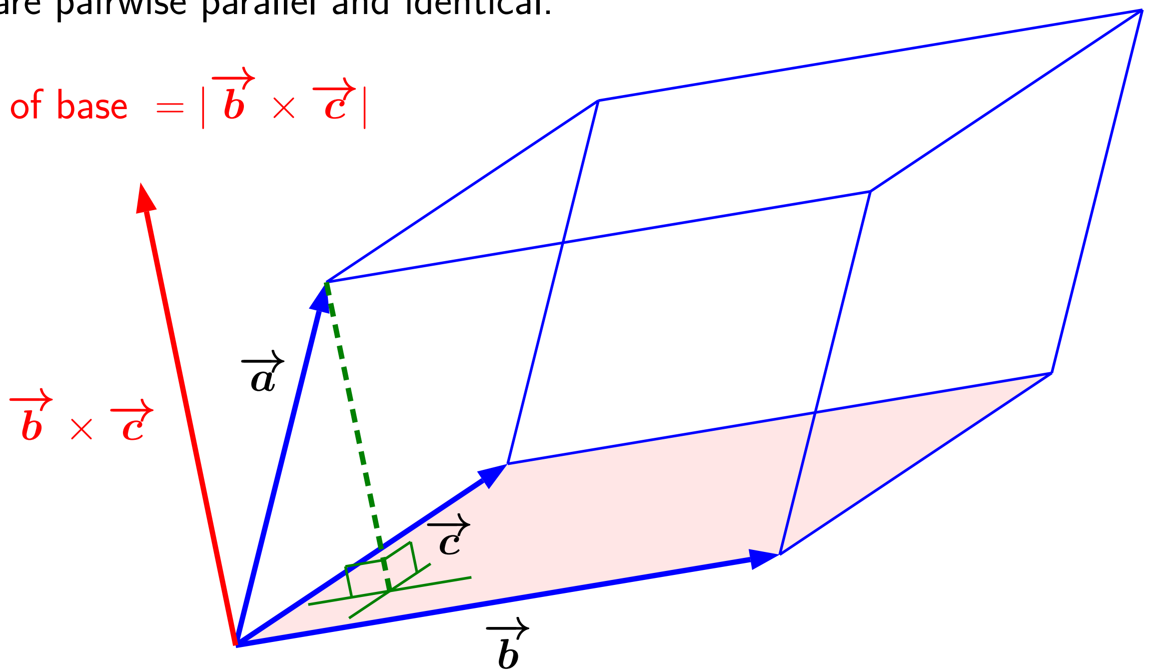

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Volume of a parallelepiped

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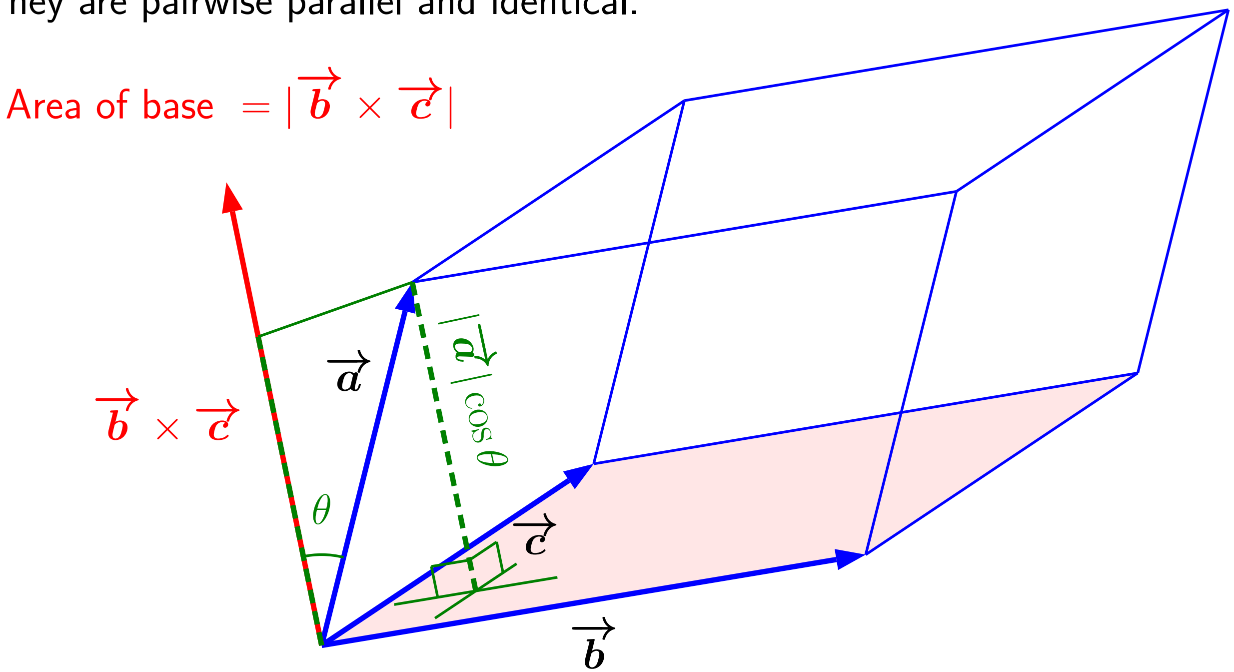

$$\begin{aligned} \text{Volume} &= \text{Area of base} \times \text{altitude} \\ &= |\overrightarrow{b} \times \overrightarrow{c}||\overrightarrow{a}||\cos\theta| \end{aligned}$$

Volume of a parallelepiped

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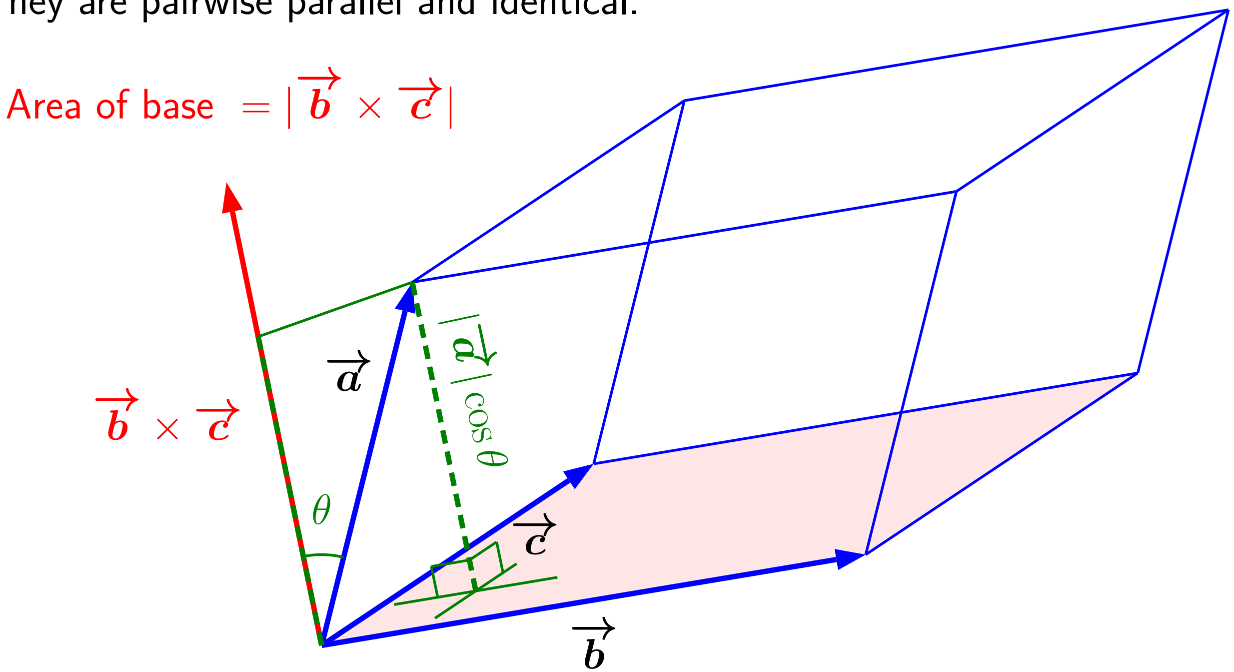


$$
\begin{aligned}
\text{Volume} &= \text{Area of base} \times \text{altitude} \\
&= |\overrightarrow{b} \times \overrightarrow{c}|\,|\overrightarrow{a}|\,|\cos\theta| \\
&= \left|\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c})\right|
\end{aligned}
$$

Scalar triple product

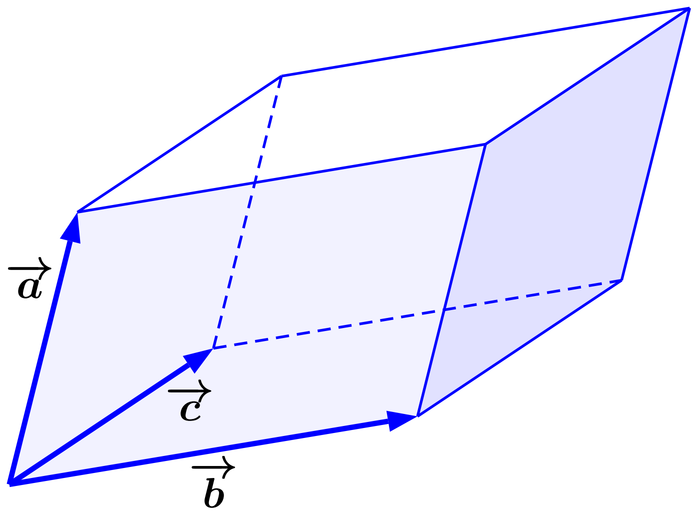
Scalar triple product of three vectors in $\mathbb{R}^3$

The triple scalar product of $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c} \in \mathbb{R}^3$ is the **number**

$$\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c}).$$

The volume of the parallelepiped with edges given by $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$ is

$$\left|\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c})\right|.$$



Application of the scalar triple product

Exercise 1. Consider the vectors

$$\overrightarrow{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \overrightarrow{v} = \begin{pmatrix} -3 \\ -2 \\ -7 \end{pmatrix}, \quad \overrightarrow{w} = \begin{pmatrix} -4 \\ 3 \\ 7 \end{pmatrix}.$$

a) Calculate
  (i) $\overrightarrow{u} \cdot (\overrightarrow{v} \times \overrightarrow{w})$
  (ii) $\overrightarrow{v} \cdot (\overrightarrow{u} \times \overrightarrow{w})$
  (iii) $\overrightarrow{w} \cdot (\overrightarrow{u} \times \overrightarrow{v})$

b) Find the volume of the parallelepiped with edges given by the vectors $\overrightarrow{u}$, $\overrightarrow{v}$ and $\overrightarrow{w}$.

Checking our answers with Maple

```
> with(LinearAlgebra):
> # use : instead of ; if you do not want the echo
  # SHIFT + ENTER gives a new line in the current execution
  block.

  u := <1,2,3>:
  v := <-3,-2,-7>:
  w := <-4,3,7>;
```

$$w := \begin{bmatrix} -4 \\ 3 \\ 7 \end{bmatrix}$$

```
> # First find the cross product of v and w and then the dot
  product with u.

  n := CrossProduct(v,w);
  u.n;
```

$$n := \begin{bmatrix} 7 \\ 49 \\ -17 \end{bmatrix}$$

$$54$$

```
> # There are two more equivalent ways to calculate this

  v.CrossProduct(w,u);
  w.CrossProduct(u,v);
```

$$54$$

$$54$$

Properties of the Scalar triple product

- Note that for $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

which we can calculate by developing along the first row or column.

- and that

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b}).$$

Note that we can obtain the second and then the third expression from the first one by permuting the vectors: $\vec{a} \to \vec{b} \to \vec{c} \to \vec{a}$.

As a result,

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}.$$

Application of the scalar triple product ... or otherwise

Exercise 2. Show that the points $A(3, 3, 5)$, $B(1, 0, 1)$, $C(2, 2, 4)$ and $D(2, 1, 2)$ are coplanar.

Checking our answers with Maple

```
> with(LinearAlgebra):
> # Define position vectors for the points A, B, C and D.

  a := <3,3,5>:
  b := <1,0,1>:
  c := <2,2,4>:
  d := <2,1,2>:
> # Find the cross product of vectors AB and AC.

  AB := b - a;
  AC := c - a;
  n := CrossProduct(AB,AC);
```

$$AB := \begin{bmatrix} -2 \\ -3 \\ -4 \end{bmatrix}$$

$$AC := \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

$$n := \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

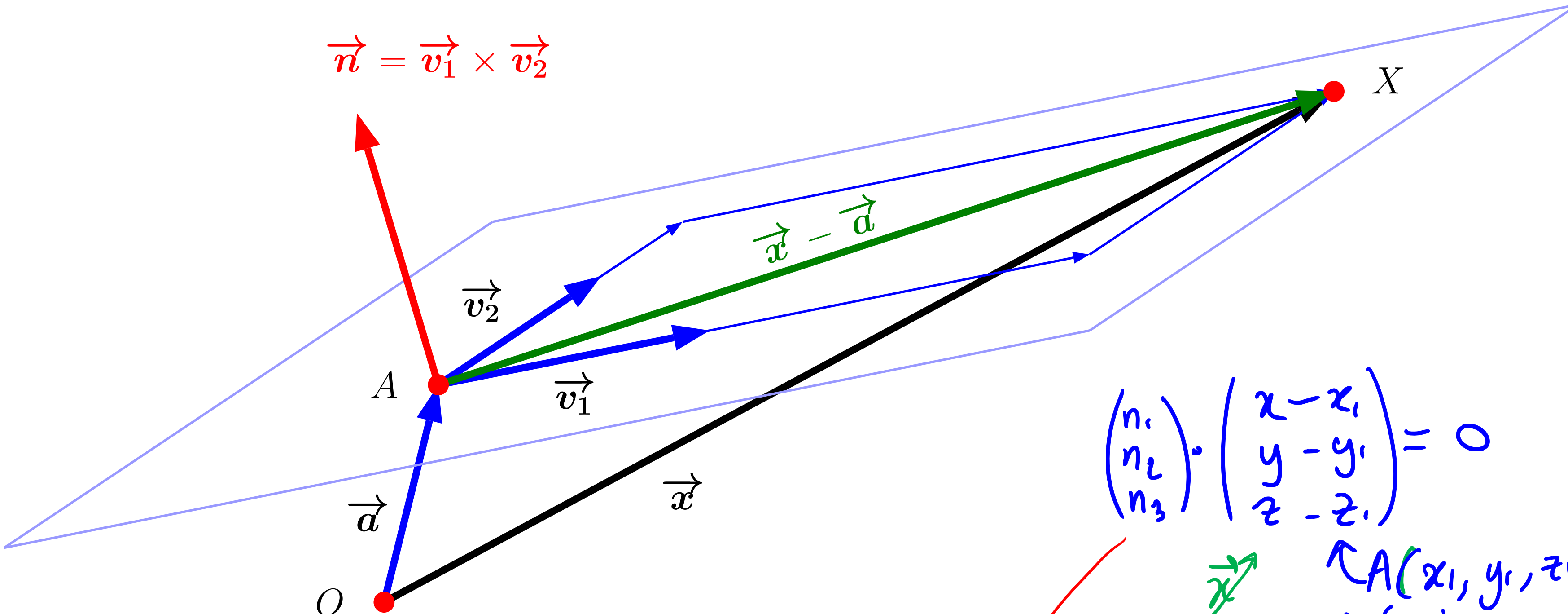
```
> # D is in the plane ABC iff AD is perpendicular to n.

  AD := d - a;
  n_dot_AD := n.AD;
```

$$AD := \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$$

$$0$$

Equation of a plane in point normal form



$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \cdot \begin{pmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{pmatrix} = 0$$

$\vec{x}$ $A(x_1, y_1, z_1)$

$\vec{a} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$

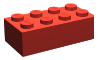
Every vector parallel to the plane is a linear combination of $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$.
The normal vector $\overrightarrow{n} = \overrightarrow{v_1} \times \overrightarrow{v_2}$ is perpendicular to all vectors parallel to the plane.

The point X is in the plane iff $\overrightarrow{x} - \overrightarrow{a}$ is perpendicular to $\overrightarrow{n}$.

$$X \text{ is in the plane} \iff \boxed{\overrightarrow{n} \cdot (\overrightarrow{x} - \overrightarrow{a}) = 0}$$

The boxed equation is a Cartesian equation for the plane written in point normal form.

Cartesian and point normal forms



Suppose $\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is a vector normal to a plane passing through the point $A(x_0, y_0, z_0)$.

- An equation in point normal form of this plane is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \right) = 0.$$

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$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \right) = 0.$$

- Expand this out to get

$$ax + by + cz = d,$$

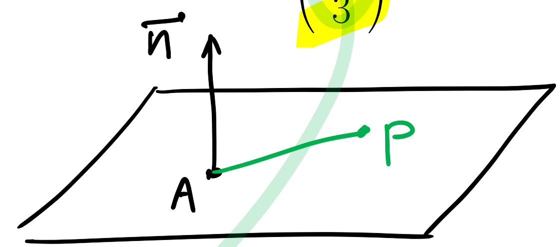
where $d = ax_0 + by_0 + cz_0$.

TIP!

Note that the **coefficients** of x, y and z in a **Cartesian** equation of the plane are the coordinates of $\overrightarrow{n}$, which is **normal** to the plane! This is always the case.

Expanded Cartesian and point normal forms

Exercise 3. Find an expanded Cartesian equation of the plane with normal $\overrightarrow{n} = \begin{pmatrix}4\\-2\\3\end{pmatrix}$ that passes through the point $A(1,2,1)$.



$P(x,y,z)$ is on the plane

$\iff \overrightarrow{AP}\cdot\vec{n}=0$

$\iff \boxed{\begin{pmatrix}4\\-2\\3\end{pmatrix}\cdot\begin{pmatrix}x-1\\y-2\\z-1\end{pmatrix}=0}$

$-4+4-3$

$\iff 4x-2y+3z-3=0$

Method 2: $4x-2y+3z+d=0$

A is on the plane so $4\times1-2\times2+3\times1+d=0$

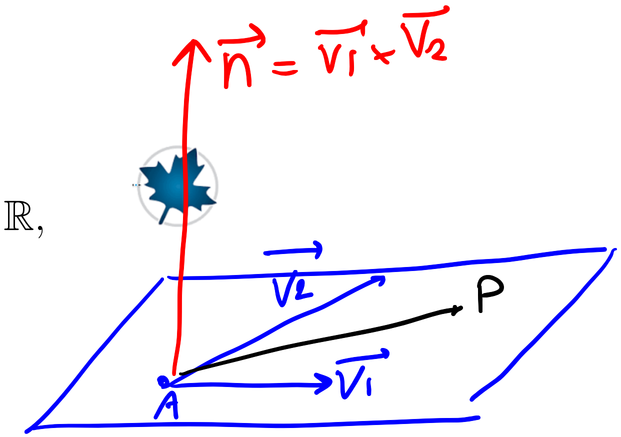
$3+d=0$

$d=-3$

Cartesian and point normal forms

Exercise 4. Write equations of the plane Π defined by

$$\vec{x} = \underbrace{\begin{pmatrix}1\\2\\3\end{pmatrix}}_{A} + \lambda_1 \underbrace{\begin{pmatrix}2\\-1\\3\end{pmatrix}}_{\vec{v_1}} + \lambda_2 \underbrace{\begin{pmatrix}5\\-4\\1\end{pmatrix}}_{\vec{v_2}}, \quad \lambda_1, \lambda_2 \in \mathbb{R},$$



in point normal form and in expanded Cartesian form.

Let $\vec{n} = \vec{v_1} \times \vec{v_2}$

$$= \begin{vmatrix} \vec{i} & 2 & 5 \\ -\vec{j} & -1 & -4 \\ \vec{k} & 3 & 1 \end{vmatrix}$$

$$= \begin{pmatrix} -1+12 \\ -(2-15) \\ -8+5 \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \\ -3 \end{pmatrix}$$

check $\vec{n} \cdot \vec{v_1} = 2 \times 11 - 13 - 9$
$= 22 - 13 - 9 = 0$

Equation of Π:

$\vec{n} \cdot (\vec{x} - \vec{a}) = 0$

$$\begin{pmatrix} 11 \\ 13 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y-2 \\ z-3 \end{pmatrix} = 0$$

$\cancel{-10} - 1 - 26$
$\cancel{+10} - 1$

$11x + 13y - 3z - 11 - 26 + 9 = 0$

$$\boxed{11x + 13y - 3z = 28}$$

Cartesian and point normal forms

Exercise 4, continued. Write equations of the plane Π defined by

$$\overrightarrow{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R},$$

in point normal form and in expanded Cartesian form.

Checking our answers with Maple

```
> with(LinearAlgebra):
> # Use : instead of ; if you do not want the echo.

  v1 := <2,-1,3>:
  v2 := <5,-4,1>:
> n := CrossProduct(v1,v2);
```

$$n := \begin{bmatrix} 11 \\ 13 \\ -3 \end{bmatrix}$$

```
> # Find a Cartesian equation in point normal form

  a := <1,2,3>:
  p := <x,y,z>:

  n.(p - a) = 0;
```

$$11\,x - 28 + 13\,y - 3\,z = 0$$

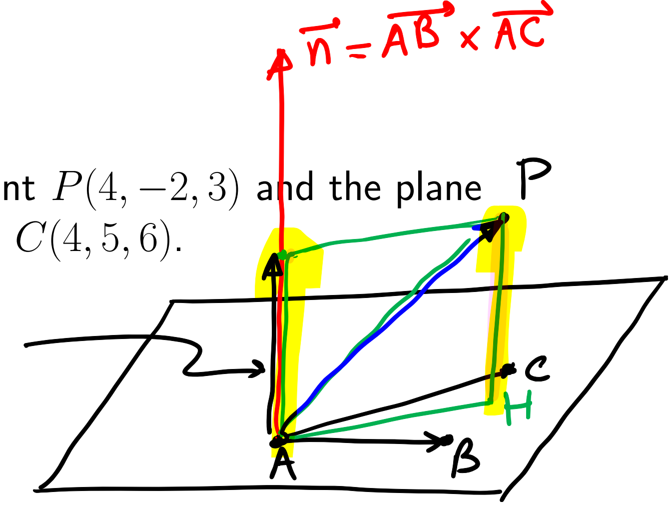
```
> # or ...

  n.p = n.a;
```

$$11\,x + 13\,y - 3\,z = 28$$

Distance from point to a plane

Exercise 5. Find the shortest distance between the point $P(4, -2, 3)$ and the plane passing through the points $A(1, 2, 3)$, $B(-3, 2, 1)$ and $C(4, 5, 6)$.



$$\overrightarrow{HP} = \text{proj}_{\vec{n}} \overrightarrow{AP}$$

$$\underset{\sim}{n} = \overrightarrow{AB} \times \overrightarrow{AC}$$

$$= \begin{vmatrix} \underset{\sim}{i} & -4 & 3 \\ \underset{\sim}{j} & 0 & 3 \\ \underset{\sim}{k} & -2 & 3 \end{vmatrix} = \begin{pmatrix} 0-(-6) \\ -(-12+6) \\ -12 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ -12 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\underset{\sim}{n}_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\overrightarrow{HP} = \text{proj}_{\underset{\sim}{n}_1} \overrightarrow{AP} = \frac{\overrightarrow{AP} \cdot \underset{\sim}{n}_1}{\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_1} \underset{\sim}{n}_1 = \frac{3-4}{1+1+4} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = -\frac{1}{6} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

shortest dist between P and plane is

$$|\overrightarrow{HP}| = \left| -\frac{1}{6} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right| = \frac{1}{6}\sqrt{1+1+4} = \frac{\sqrt{6}}{6}$$

Distance from point to a plane

Exercise 5, continued. Find the shortest distance between the point $P(4, -2, 3)$ and the plane passing through the points $A(1, 2, 3)$, $B(-3, 2, 1)$ and $C(4, 5, 6)$.

Checking our answers with Maple

```
> with(LinearAlgebra):
> # Define position vectors for the
  # points A, B, C and P.

  a := <1,2,3>:
  b := <-3,2,1>:
  c := <4,5,6>:
  p := <4,-2,3>:
> # Find the cross product of vectors AB and AC

  AB := b - a:
  AC := c - a:
  n := CrossProduct(AB,AC);
```

$$n := \begin{bmatrix} 6 \\ 6 \\ -12 \end{bmatrix}$$

```
> # Let's use a scalar multiple of n

  n1 := 1/6*n;
```

$$n1 := \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

```
> # Project vector AP onto n.

  AP := p - a:
  d := (AP.n1)/(n1.n1)*n1;
```

$$d := \begin{bmatrix} -\frac{1}{6} \\ -\frac{1}{6} \\ \frac{1}{3} \end{bmatrix}$$

```
> # The distance from P to the plane is
  # the length of vector d.

  dist := sqrt(d.d);
```

$$dist := \frac{\sqrt{6}}{6}$$