School of Mathematics and Statistics

Math1131 Mathematics 1A

CALCULUS LECTURE 3

LIMIT OF FUNCTIONS AT INFINITY (PART 1)

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MATH1131 CALCULUS
Limit of Functions at Infinity Part 1

> $\pm\dfrac{x^2}{a^2} \pm \dfrac{y^2}{a^2} = 1$ is an ellipse with two ++ and a hyperbola with one +.
>
> $\displaystyle\lim_{x\longrightarrow\infty} \frac{f(x)}{g(x)}$ can be attacked by dividing top and bottom by the dominant term.
>
> $\dfrac{"0"}{0}$, $\dfrac{"\infty"}{\infty}$ and $"\infty - \infty"$ are called indeterminate forms.
>
> If $\displaystyle\lim_{x\longrightarrow\infty} f(x) = L$ then $y = L$ is a horizontal asymptote to the graph of $y = f(x)$.

In this lecture we will look carefully at the concept of a limit as $x \to \infty$. But first a few sketches of relations. Ideally we like to start with nice clean functions $y = f(x)$. But sometimes we begin with a horrible tangled mess in the x and y variables.

For example

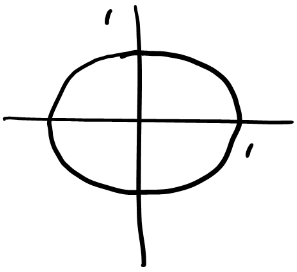
$$x^2y^3 + e^x \sin(y) = \ln(x^3 + y^4) + \frac{1}{xy}$$

These relations are usually not functions and it is often algebraically impossible to express y in terms of x. It is best not to even try! Later on we will see that calculus may still be used on such exotic creatures through the process of implicit differentiation. For the moment let's just have a look at the graph of a few standard relations.

Example 1: Sketch the graph of each of the following relations. Find the domain of the relation in each case:

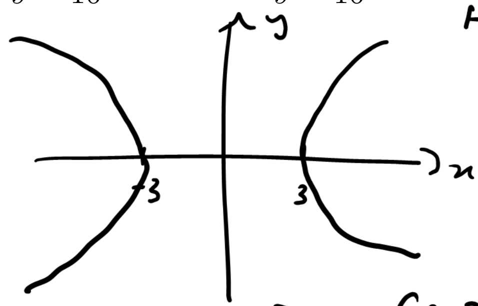
a) $x^2+y^2 = 1$ b) $\dfrac{x^2}{9}+\dfrac{y^2}{16} = 1$ c) $\dfrac{x^2}{9}-\dfrac{y^2}{16} = 1$ d) $-\dfrac{x^2}{9}+\dfrac{y^2}{16} = 1$ e) $-\dfrac{x^2}{9}-\dfrac{y^2}{16} = 1$.

a)



Dom: $[-1,1]$

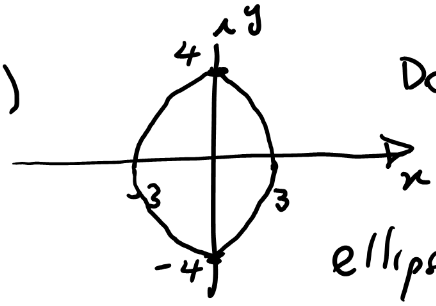
c)



Hyperbola

Dom: $[3,\infty) \cup (-\infty,-3]$

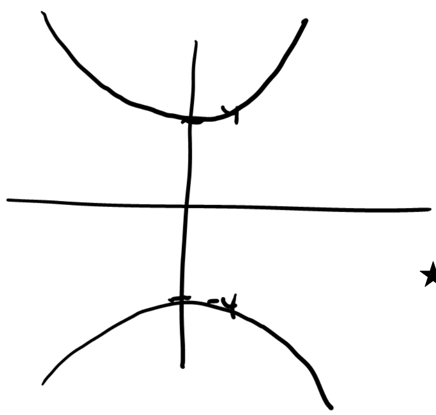
b)



Dom: $[-3,3]$

ellipse: ++

d)



Dom: $\mathbb{R}$.

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Hyperbola
+ −

e) Graph is empty.

The limit of a function at infinity $\lim_{x \longrightarrow \infty} f(x)$

We now examine the behaviour of functions for large values of x through the limit $\lim_{x \longrightarrow \infty} f(x)$. Infinity is not really a number so we cannot just evaluate the function as normal. These limits can be done by inspection by considering dominance in numerator and denominator or a little more formally by simply dividing everything by the dominant term. Much like differentiation there is also an extremely formal approach (from first principles) which also must be mastered.

Note that if $\lim_{x \longrightarrow \infty} f(x) = L$ it means that $y = L$ is a horizontal asymptote for the function. That is for large x the function will settle down to $y = L$.

Consider the problem $\lim_{x \longrightarrow \infty} \dfrac{12x^2 - 2x + 1}{4x^2 + 7x + 6}$.

This is asking "What happens to $\dfrac{12x^2 - 2x + 1}{4x^2 + 7x + 6}$ when x gets really big".

There are a number of possible answers. Maybe the function also gets big, maybe it gets really small or perhaps it approaches a non-zero number. There is no ∞ button on your calculator so lets try evaluating $\dfrac{12x^2 - 2x + 1}{4x^2 + 7x + 6}$ at something really big.

At $x = 100$ $\quad \dfrac{12x^2 - 2x + 1}{4x^2 + 7x + 6} \approx 2.943$

At $x = 10000$ $\quad \dfrac{12x^2 - 2x + 1}{4x^2 + 7x + 6} \approx 2.999$

It looks like $\lim_{x \longrightarrow \infty} \dfrac{12x^2 - 2x + 1}{4x^2 + 7x + 6} = 3$. But how do we prove this?

We have a number of methods. First a really sloppy approach:

Method 1 (only look at powerful terms in the numerator and denominator)

$$\lim_{x \to \infty} \frac{12x^2 - 2x + 1}{4x^2 + 7x + 6} \quad \underline{\underline{= 3}}$$

Method 2 (divide top and bottom by the dominant term)

$$\lim_{x \to \infty} \frac{12 - \cancel{\frac{2}{x}} + \cancel{\frac{1}{x^2}}}{4 + \cancel{7/x} + \cancel{6/x^2}} = \frac{12}{4} = 3$$

Method 3 (Next Lecture: A formal attack)

Some terminology

If you "imagine" putting infinity into $\dfrac{12x^2-2x+1}{4x^2+7x+6}$ you get $\dfrac{\text{“}\infty\text{”}}{\infty}$.

We call $\dfrac{\text{“}\infty\text{”}}{\infty}$ an indeterminate form, as it has no real meaning.

Other indeterminate forms are $\dfrac{\text{“}0\text{”}}{0}$, 0^∞, ∞^0, , $\infty-\infty$, and $0\times\infty$.

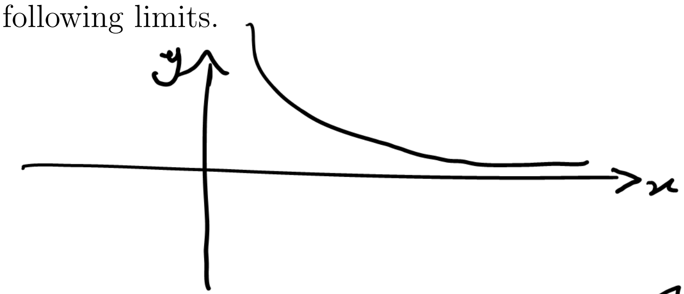
Note that $\dfrac{\text{“}0\text{”}}{0}$ and $\dfrac{\text{“}\infty\text{”}}{\infty}$ must not naively be assigned a value of 1 !!
It is generally true however that $\dfrac{0}{\infty}$ can be evaluated as 0 and $\dfrac{\infty}{0}$ is unbounded.

We have a host of tricks to knock these off.

Remember to keep the $\lim\limits_{x\longrightarrow\infty}$ in your argument to the very end, that is until you actually take the limit! Marks are often lost for letting $\lim\limits_{x\longrightarrow\infty}$ disappear too early. Let's take a look:

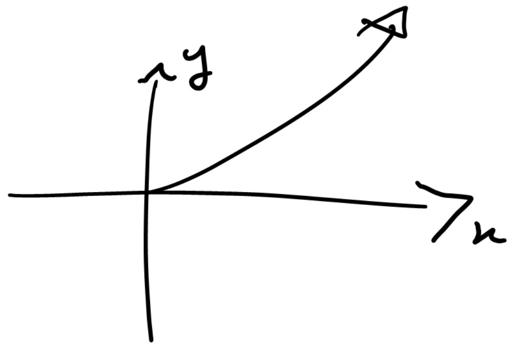
Example 2: Evaluate each of the following limits.

a) $\lim\limits_{x\longrightarrow\infty}\dfrac{1}{x^3} = 0$



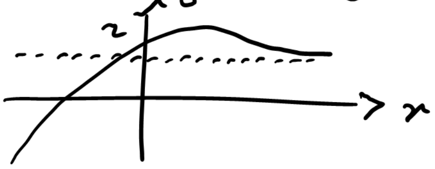
b) $\lim\limits_{x\longrightarrow\infty}\dfrac{x^3}{1} = \infty$

= unbounded.

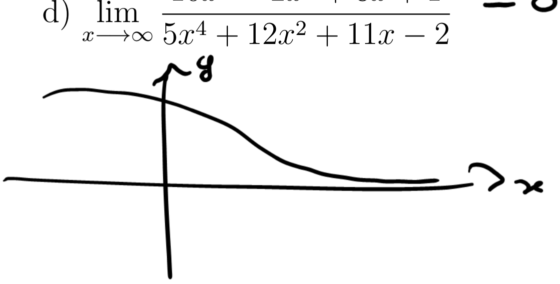


c) $\lim\limits_{x\longrightarrow\infty}\dfrac{10x^3-2x^2+8x+1}{5x^3+12x^2+11x-2}$

$$= \lim_{x\to\infty} \frac{10 - 2/x + 8/x^2 + 1/x^3}{5 + \frac{12}{x} + \frac{11}{x^2} - 2/x^3} = \frac{10}{5} = 2$$

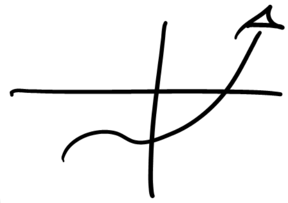


d) $\lim\limits_{x\longrightarrow\infty}\dfrac{10x^3-2x^2+8x+1}{5x^4+12x^2+11x-2} = 0$ (denominator dominates)



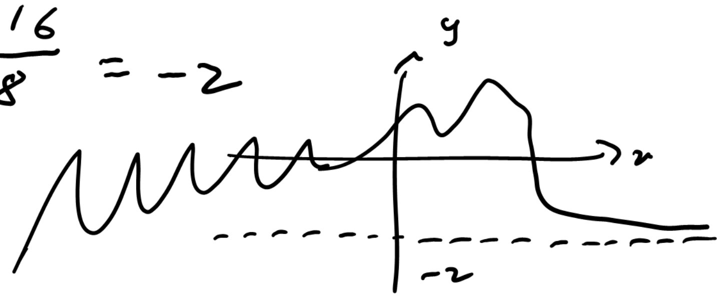
e) $\lim\limits_{x \longrightarrow \infty} \dfrac{10x^4 - 2x^2 + 8x + 1}{5x^3 + 12x^2 + 11x - 2}$ = unbounded

(numerator dominates)



f) $\lim\limits_{x \longrightarrow \infty} \dfrac{12 + x - 16x^2}{8x^2 - 2x + 5}$ $= 2$

$$= \lim_{x \to \infty} \frac{\cancel{12/x^2} + \cancel{1/x} - 16}{8 - \cancel{2/x} + \cancel{5/x^2}} = \frac{-16}{8} = -2$$



g) $\lim\limits_{x \longrightarrow \infty} \dfrac{4x - 3}{\sqrt{25x^2 + 24x + 23}}$

$$= \lim_{x \to \infty} \sqrt{\frac{(4x-3)^2}{25x^2+24x+23}} = \sqrt{\lim_{x \to \infty} \frac{16x^2 - 24x + 9}{25x^2 + 24x + 23}}$$

$$= \sqrt{\frac{16}{25}} = \frac{4}{5} \ \#$$

h) $\lim\limits_{x \longrightarrow \infty} \dfrac{x^2 - 3x + \sin(x)}{25x^2 + \cos(33x) + 1}$

$$= \lim_{x \to \infty} \frac{1 - 3/x + \frac{\sin x}{x^2}}{25 + \frac{\cos(33x)}{x^2} + \frac{1}{x^2}} = \frac{1}{25}$$

since
$-1 \le \sin(x) \le 1$
$-1 \le \cos(33x) \le 1$

i) $\lim\limits_{x \longrightarrow \infty} \dfrac{x^2 - 3x + 10000000000000000000000000000000000000000000000000000 \sin(x)}{25x^2 + 10000000000000000000000000000000000000000000000000000000000 \cos(33x) + 1}$

$$= \frac{1}{25}$$ $-1 \le \sin(x) \le 1$

j) $\lim_{x \longrightarrow \infty} \sqrt{x^2 - 8x} - x$ $=$ "$\infty - \infty$"

This is "$\infty - \infty$". There is a special trick for this one!

Let's first cheat: $x = 10000 \Longrightarrow \sqrt{x^2 - 8x} - x = -4.0008$. Looks like the limit is -4.

$$\lim_{x \to \infty} \left(\sqrt{x^2-8x} - x\right)\left(\frac{\sqrt{x^2-8x}+x}{\sqrt{x^2-8x}+x}\right)$$

$$= \lim_{x \to \infty} \frac{\cancel{x^2} - 8x - \cancel{x^2}}{\sqrt{x^2-8x} + x}$$

$$= \lim_{x \to \infty} \frac{-8x}{\sqrt{x^2-8x} + x}$$

$$= \lim_{x \to \infty} \frac{-8}{\frac{\sqrt{x^2-8x}}{\sqrt{x^2}} + 1}$$

$$= \lim_{x \to \infty} \frac{-8}{\sqrt{\frac{x^2-8x}{x^2}} + 1}$$

$$= \lim_{x \to \infty} \frac{-8}{\sqrt{1 - 8/x} + 1} = \frac{-8}{2} = \underline{-4}$$

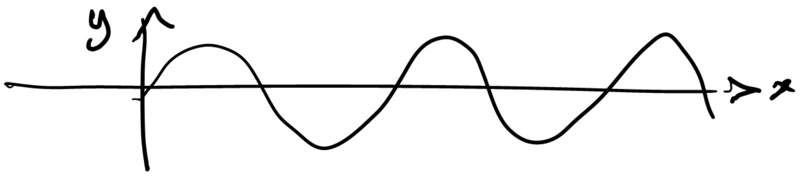
k) $\lim_{x \longrightarrow \infty} \dfrac{30e^x + 7x^2 + \sin(x)}{5e^x + 25x^2 + 1}$

$$= \lim_{x \to \infty} \frac{30 + \frac{7x^2}{e^x} + \frac{\sin(x)}{e^x}}{5 + \frac{25x^2}{e^x} + \frac{1}{e^x}} = \frac{30}{5} = 6$$

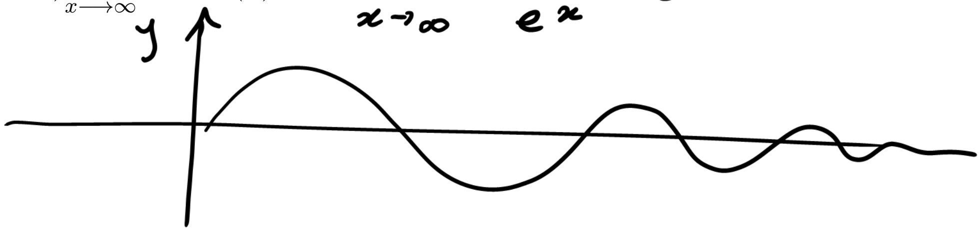
(exponential function dominates $\underline{\text{all}}$ polys for large x)

Always remember that for large x, e^x is more powerful than ANY polynomial.

l) $\lim_{x \longrightarrow \infty} \sin(x)$ Does not exist D.N.E.



m) $\lim_{x \longrightarrow \infty} e^{-x} \sin(x) = \lim_{x \to \infty} \dfrac{\sin(x)}{e^x} = 0$



n) $\lim_{x \longrightarrow \infty} \cos\left(\dfrac{1}{x}\right) =$

$$= \cos\left(\lim_{x \to \infty} \frac{1}{x}\right) = \cos(0) = 1$$

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