LECTURE 14
Polar Coordinates (r, θ)

$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

$$r = \sqrt{x^2 + y^2}$$

$$\tan(\theta) = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{r\cos(\theta) + \dfrac{dr}{d\theta}\sin(\theta)}{-r\sin(\theta) + \dfrac{dr}{d\theta}\cos(\theta)}$$

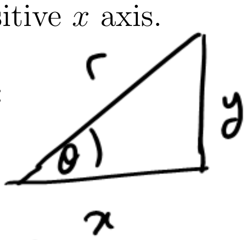
We have already seen in the complex number section of the algebra strand that it is sometimes advantageous to abandon the rectangular coordinate system (x, y) and replace it with polars (r, θ) .

This opens up a new world when sketching curves and also helps in the analysis of "round" objects such as circles, spirals and cardioids.

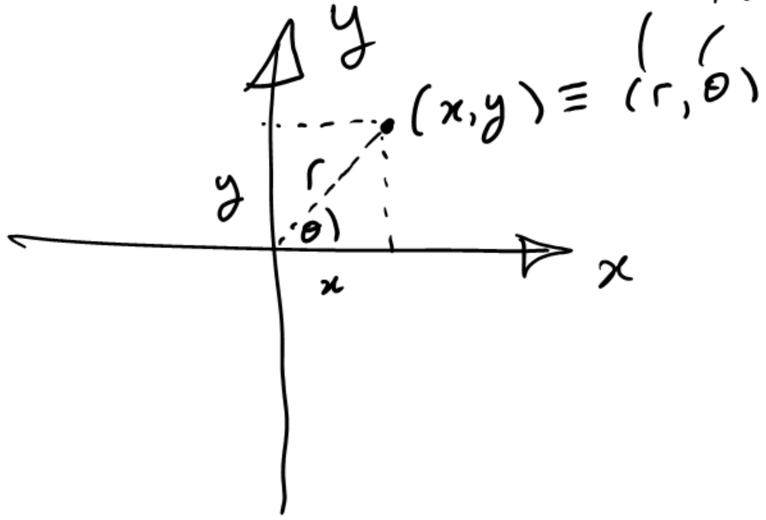
Let us begin by defining polar coordinates (r, θ) and proving the transformation equations above. Note that r is simply the distance from the point to the origin and θ is the angle to the positive x axis.

$|z|$ $\mathrm{Arg}(z)$

Discussion:



$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r &= \sqrt{x^2+y^2} \\ \tan\theta &= y/x. \end{aligned}\right\}$$



This should all sound very familiar! r is just $|\,z\,|$ and θ is nothing but $\mathrm{Arg}(z)$.

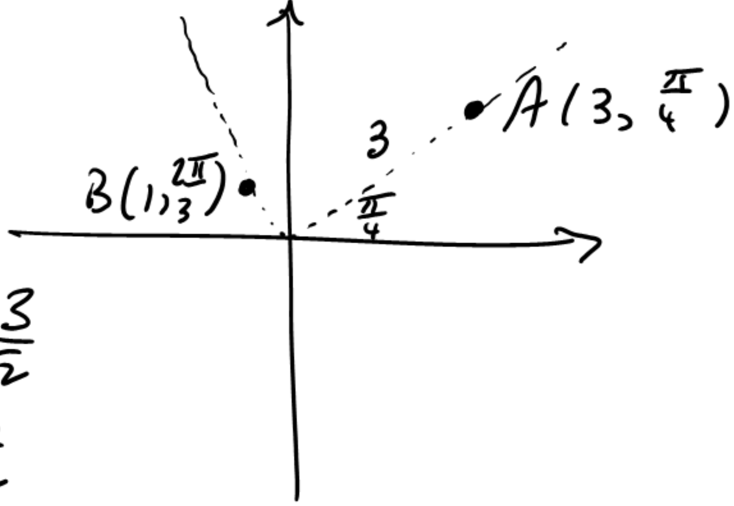
We do demand that $r \geq 0$ but are relaxed about θ. You can use $-\pi < \theta \leq \pi$ or $0 \leq \theta < 2\pi$. Sometimes we even let $\theta \geq 0$.

Example 1: Sketch the following points given in polar coordinates and convert each point over to Cartesian form:

(r, θ)

a) $A(3, \frac{\pi}{4})$.

b) $B(1, \frac{2\pi}{3})$.



a) $x = r\cos\theta = 3\cos\frac{\pi}{4} = \frac{3}{\sqrt{2}}$

$y = r\sin\theta = 3\sin\frac{\pi}{4} = \frac{3}{\sqrt{2}}$

$A(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}) = (x, y)$.

b) $x = 1\cos\frac{2\pi}{3} = -\frac{1}{2}$ ✓

$y = 1\sin 2\pi/3 = \sqrt{3}/2$

$B(-\frac{1}{2}, \frac{\sqrt{3}}{2}) = (x, y)$

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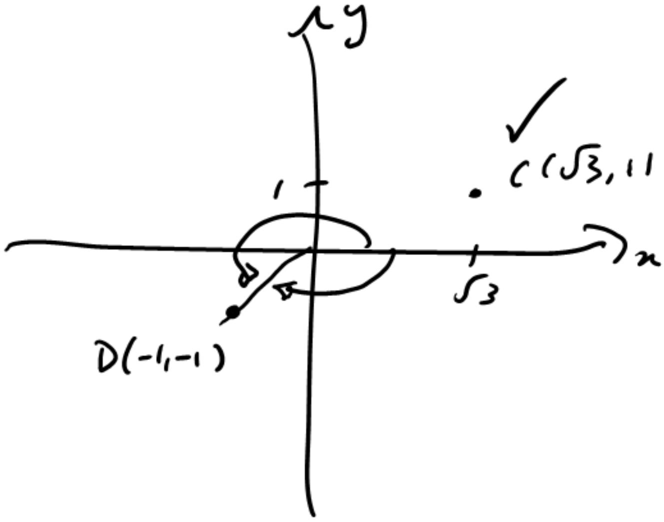
Example 2: Sketch the following points given in Cartesian coordinates and convert each point over to polar form:

c) $C(\sqrt{3}, 1)$. $= (x, y)$

d) $D(-1, -1)$. $= (x, y)$



c) $r = \sqrt{x^2+y^2} = \sqrt{3+1} = 2$

$\tan\theta = \frac{1}{\sqrt{3}}$

$\theta = 30°, 210° = \frac{\pi}{6}, \cancel{\frac{7\pi}{6}}$

$C(2, \frac{\pi}{6})$

d) $r = \sqrt{1+1} = \sqrt{2}$

$\tan\theta = \frac{-1}{-1} = 1$

$\theta = \cancel{\frac{\pi}{4}}, \frac{5\pi}{4}$

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$D(\sqrt{2}, \frac{5\pi}{4})$

Polar graphs are weird and wonderful objects. To sketch a polar graph expressed in terms of r and θ we almost never retreat to Cartesians and rarely use any calculus. Let's begin with some simple examples:

Example 3: Sketch each of the following polar equations in the $x - y$ plane.

a) $r = 5$.

b) $\theta = \dfrac{\pi}{4}$.

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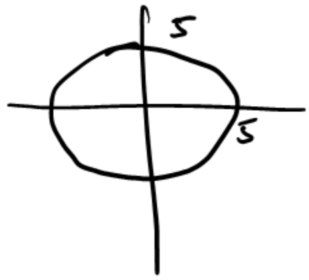
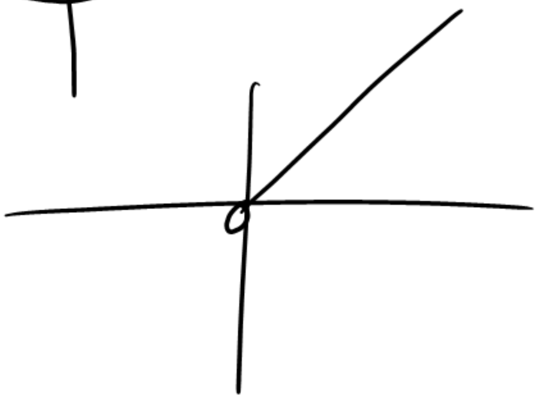
a) 

$\sqrt{x^2+y^2} = 5 \Rightarrow x^2+y^2 = 25$

b) 

Note that $r = 5 \to r^2 = 25 \to x^2 + y^2 = 25$ which makes sense! It's a circle.

b)

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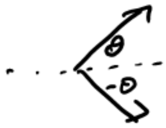


For more complicated graphs we have a procedure to follow:

1) Check for symmetry about the x axis. If replacing θ by $-\theta$ has no impact then the graph is symmetric about the x axis. That is the top is the same as the bottom.

2) Check for symmetry about the y axis. If replacing θ by $\pi - \theta$ has no impact then the graph is symmetric about the y axis. That is the left is the same as the right.

Two identities to keep in mind are

✓ $\cos(-\theta) = \cos(\theta)$. Symmetry about the x axis.

✓ $\sin(\pi - \theta) = \sin(\theta)$. Symmetry about the y axis.

3) Our final step is to plot a detailed table of θ vs r, usually using a calculator.

We can then use this table (and an appropriate scale if necessary) to step out all the angles and measure the appropriate distances along the rays. The final sketch is then simply a question of joining the dots and using the symmetry.

Example 4: Sketch in the $x-y$ plane the graph of $r = 2\cos(\theta)$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

The restriction on θ guarantees that $r \geq 0$.

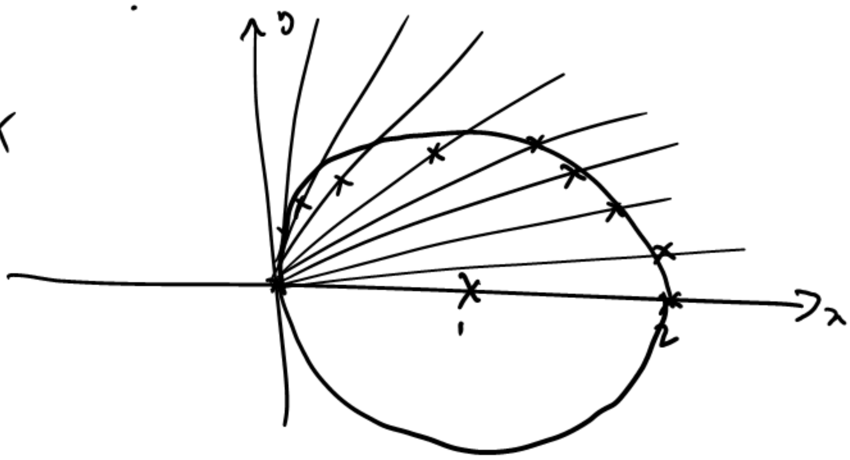
Sketching $y = 2\cos(x)$ will be awarded zero marks!! It's a polar plot!

$\theta \to -\theta$: $r = 2\cos(-\theta) = 2\cos(\theta) =$ same $\Rightarrow$ sym x-axis ✓✓

$\theta \to (\pi - \theta)$: $r = 2\cos(\pi - \theta) = -2\cos\theta$ ✗✗ $\Rightarrow$ no sym y-axis

We only need a table from 0 to 90°. Symmetry will take care of the rest.

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
r	2	1.97	1.88	1.73	1.53	1.29	1	.68	.35	0



Note that we can convert this to Cartesians! This is usually **not** a good idea.

$$r = 2\cos\theta$$
$$r^2 = 2r\cos\theta$$
$$x^2+y^2 = 2x \Rightarrow x^2-2x+y^2=0$$
$$\Rightarrow x^2-2x+1+y^2=1$$
$$\Rightarrow (x-1)^2+y^2=1$$

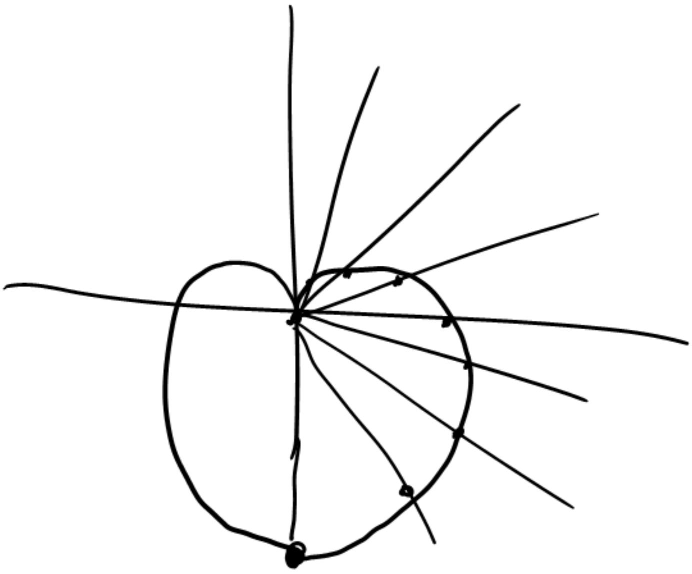
circle centre $(1,0)$ rad $= 1$ ★

Example 5: Sketch in the $x-y$ plane the graph of $r = 1 - \sin(\theta)$ for $-\pi < \theta \leq \pi$.

- $\theta \to -\theta$: $r = 1 - \sin(-\theta) = 1 + \sin\theta$ XX no x-axis sym.
- $\theta \to (\pi - \theta)$: $r = 1 - \sin(\pi - \theta) = 1 - \sin\theta$ = same
  $\Rightarrow$ symm about y axis

We only need a table from -90 to $90°$. Symmetry will take care of the rest.

θ	$-90°$	$-60°$	$-45°$	$-30°$	$0°$	$30°$	$45°$	$60°$	$90°$
r	2	1.87	1.71	1.50	1	0.5	.29	0.13	0



$r = 1 - \sin\theta$

$r^2 = r - r\sin\theta = r - y$

$x^2 + y^2 = \sqrt{x^2+y^2} - y$

For obvious reasons this curve is called a cardioid.

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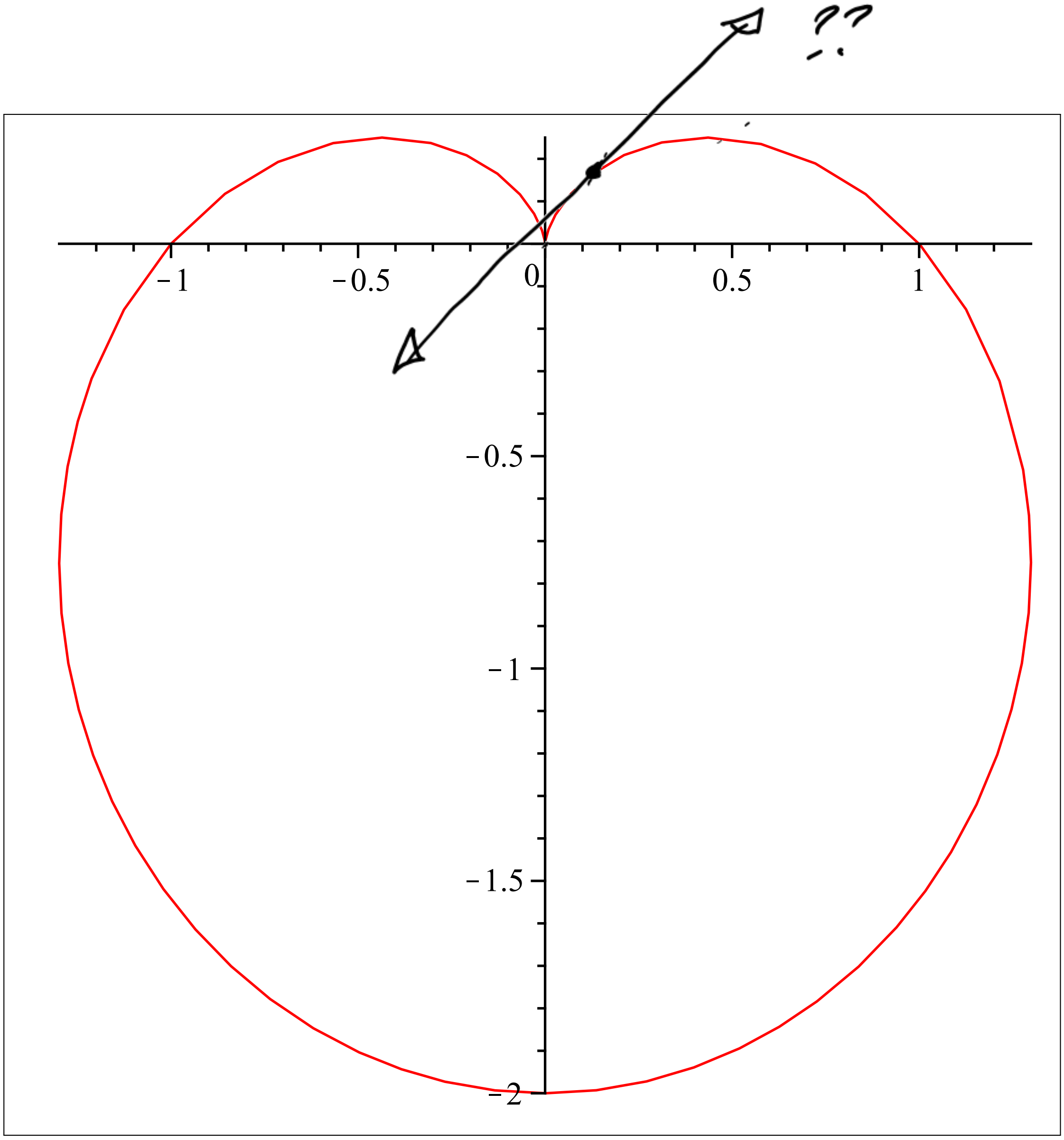
Let's see what Maple does with this sketch:

```
>plot([1-sin(theta),theta,theta=-Pi..Pi],coords=polar);
```



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Example 6: Sketch in the $x - y$ plane the graph of $\underset{\sim\sim\sim\sim\sim\sim\sim}{r = |\sin(2\theta)|}$ for $0 < \theta \leq 2\pi$. ✓

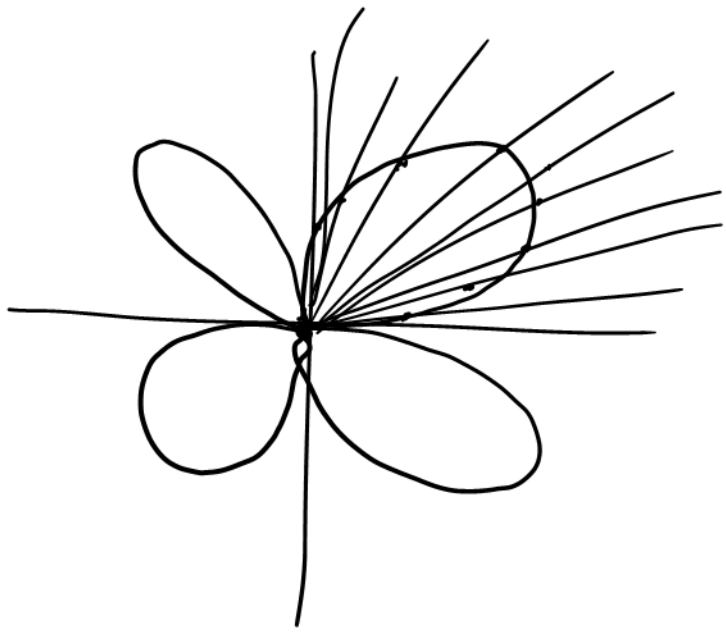
$\theta \to -\theta$: $r = |\sin(-2\theta)| = |-\sin(2\theta)| = |\sin 2\theta|$ ✓

$\theta \to (\pi - \theta)$: $r = |\sin(2(\pi - \theta))| = |\sin(2\pi - 2\theta)|$
$= |\sin(-2\theta)| = |\sin(2\theta)|$ ✓

Symm about x & y axes

Due to the double symmetry we only need a table from 0 to 90°.

θ	0°	10°	20°	30°	40°	45°	50°	60°	70°	80°	90°
r	0	.34	.64	.87	.98	1	.98	0.87	0.64	.34	0



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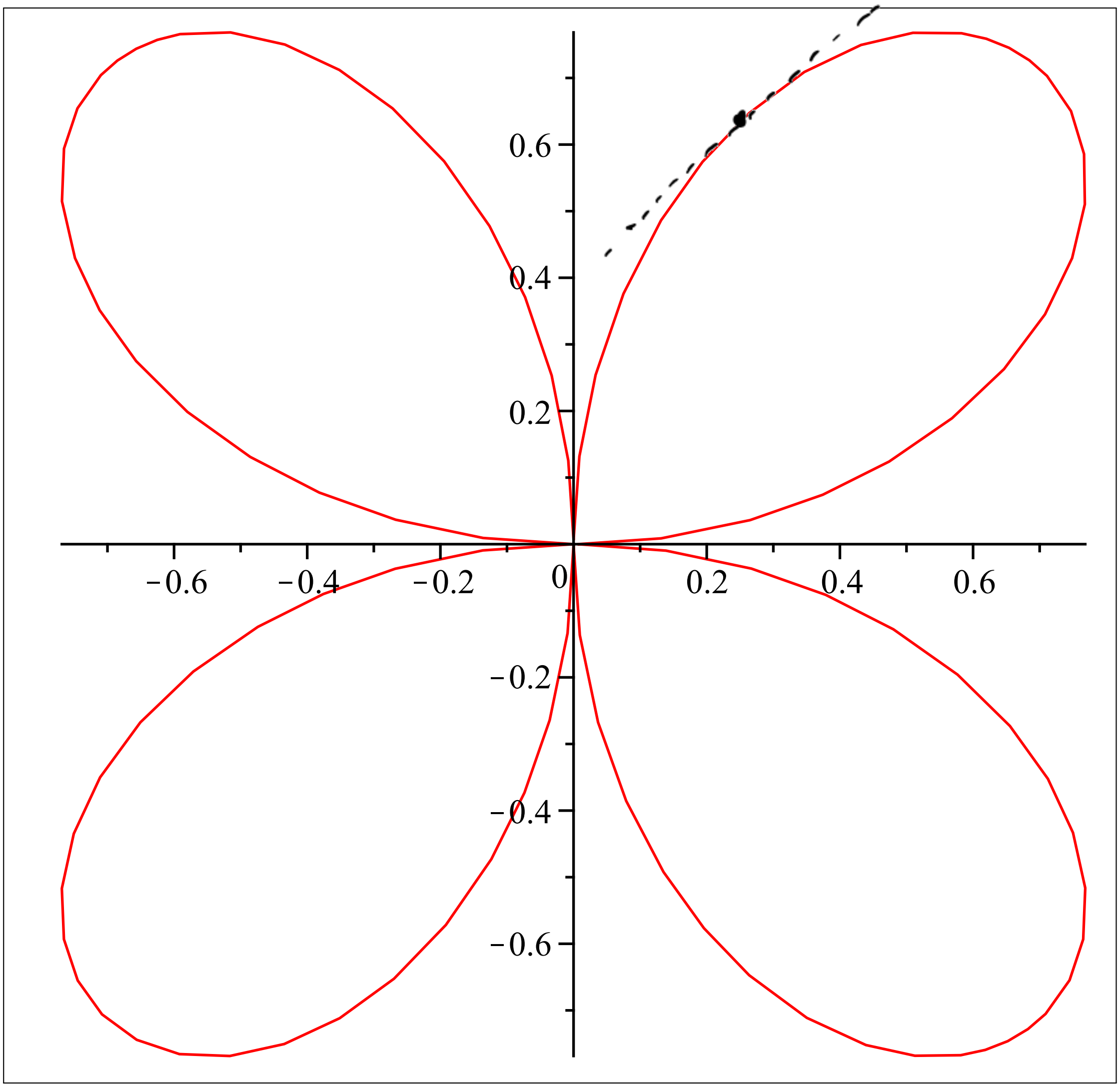
Let's see what Maple does with this sketch:

```
>plot([abs(sin(2*theta)),theta,theta=0..2*Pi],coords=polar);
```



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We close with a discussion of differentiation as it applies to polar curves. Note that despite their exotic shapes polar sketches are still curves, hence they have tangents and gradients. The first important thing to realise is that $\dfrac{dy}{dx} \neq \dfrac{dr}{d\theta}$. The situation is much more complicated than that!

Fact: If r is a function of θ then

$$\frac{dy}{dx} = \frac{r\cos(\theta) + \frac{dr}{d\theta}\sin(\theta)}{-r\sin(\theta) + \frac{dr}{d\theta}\cos(\theta)}$$

$y = r\sin\theta$

$\frac{dy}{d\theta} = r\cos\theta + \sin\theta\frac{dr}{d\theta}$

Proof: $\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$

$x = r\cos\theta$

$\frac{dx}{d\theta} = -r\sin\theta + \cos\theta\frac{dr}{d\theta}.$

$$= \frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta}.$$

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Example 7: Find the equation of the tangent to the curve $r = 1 - \sin(\theta)$ at $\theta = \frac{\pi}{3}$.

We already have a sketch of this from Example 5.

Now $r = 1 - \sin(\frac{\pi}{3}) = 1 - \frac{\sqrt{3}}{2}$ so the polar coordinates are $(r, \theta) = (1 - \frac{\sqrt{3}}{2}, \frac{\pi}{3})$.

Also $\frac{dr}{d\theta} = -\cos(\theta)$. So

$$\boxed{\frac{dy}{dx} = \frac{r\cos(\theta) + \frac{dr}{d\theta}\sin(\theta)}{-r\sin(\theta) + \frac{dr}{d\theta}\cos(\theta)}} = \frac{(1-\sin(\theta))\cos(\theta) + (-\cos(\theta))\sin(\theta)}{-(1-\sin(\theta))\sin(\theta) + (-\cos(\theta))\cos(\theta)}$$

Substituting $\theta = \frac{\pi}{3}$ yields

$\frac{(\frac{1}{2} - \frac{\sqrt{3}}{4}) - \frac{1}{2}\frac{\sqrt{3}}{2}}{(\frac{3}{4} - \frac{\sqrt{3}}{2}) - \frac{1}{4}} = 1$ amazingly. Hence the gradient of the tangent is 1 at $\theta = \frac{\pi}{3}$.

We now need to find a point (x, y).

$x = r\cos(\theta) = (1 - \frac{\sqrt{3}}{2})(\frac{1}{2}) = \frac{1}{2} - \frac{\sqrt{3}}{4}$.

$y = r\sin(\theta) = (1 - \frac{\sqrt{3}}{2})(\frac{\sqrt{3}}{2}) = \frac{\sqrt{3}}{2} - \frac{3}{4}$.

$y - y_1 = m(x - x_1)$

So the point is $(x, y) = (\frac{1}{2} - \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2} - \frac{3}{4})$.

Hence the equation of the tangent is $(y - (\frac{\sqrt{3}}{2} - \frac{3}{4})) = 1(x - (\frac{1}{2} - \frac{\sqrt{3}}{4}))$ ★