School of Mathematics and Statistics
Math1131-Algebra

Lec04: Linear Combinations and Planes

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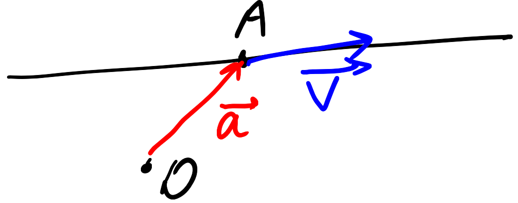
Red-Centre, Rooms 3090 and 3073

2019 Term 1

Warm up

$$\vec{x} = \underbrace{\vec{a}} + \lambda \underbrace{\vec{v}}$$



Exercise 1. Let $\vec{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R}$ be a line in $\mathbb{R}^3$. Write down a vector parametric form of the line ℓ' through $\underset{B}{(1, 2, 3)}$ that is parallel to the line ℓ above.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}} + \lambda \boxed{\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}} \qquad \lambda \in \mathbb{R}$$

Exercise 2. Find a vector parametric form of the line ℓ in $\mathbb{R}^4$ which passes through $A(2, -3, -1, 2)$ and $B(-1, 2, 2, 7)$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \boxed{\begin{pmatrix} 2 \\ -3 \\ -1 \\ 2 \end{pmatrix}} + t \boxed{\begin{pmatrix} -3 \\ 5 \\ 3 \\ 5 \end{pmatrix}}, \qquad t \in \mathbb{R}$$

$$\overrightarrow{AB} = \begin{pmatrix} -1-2 \\ 2+3 \\ 2+1 \\ 7-2 \end{pmatrix}$$

Learning outcomes for this lecture

At the the end of this lecture,

- ☐ you should know the difference between **Cartesian** equations and **parametric** equations;
- ☐ you should be able to go back and forth between **parametric** equations of a line and a **Cartesian** equation of this line in $\mathbb{R}^2$ and in $\mathbb{R}^3$;
- ☐ you should know what a **linear combination** is;
- ☐ you should know what the **span** of a set of vectors is, and what it means for a vector to be in the span of some vectors;
- ☐ you should be able to **write** a parametric vector equation of a **plane** given a point on the plane and two non-parallel vectors which are parallel to the plane;
- ☐ *conversely*, you should be able to **recognise** when a parametric vector equation describes a **plane** and from the equation, you should be able to find a point on the plane and two non-parallel vectors which are parallel to the plane;
- ☐ you should be able to go back and forth between a **parametric** equation of a plane in $\mathbb{R}^3$. and a **Cartesian** equation of this plane.

TIP!

You can use this list as a check list to get ready for our next class: After studying the lecture notes, come back to this list, and for each item, check that you have indeed mastered it. Then tick the corresponding box ... or go back to the notes.

Parametric to Cartesian equations of a line in $\mathbb{R}^3$

the cartesian equation that relates variables x, y, z (no parameter)

Example 3. Find a Cartesian equation for the line ℓ given parametrically as

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\2\\-5\end{pmatrix}+\lambda\begin{pmatrix}1\\3\\-1\end{pmatrix}\quad \lambda\in\mathbb{R}$$

$$\begin{cases}x=1+\lambda\\ y=2+3\lambda\\ z=-5-\lambda\end{cases}$$

$$\lambda=x-1$$
$$\lambda=\frac{y-2}{3}$$
$$\lambda=-z-5$$

$$(\lambda=\lambda)$$

π_2

$$\boxed{x-1=\frac{y-2}{3}=-z-5}$$

π_1

$\mathbb{R}^3$ $\quad ax+by+cz=d$ plane
provided $a\neq 0$ or $b\neq 0$ or $c\neq 0$

$a=b=c$

$a=b$
$b=c$
$a=c \leftarrow$ redunda

Parametric to Cartesian equation of a line in $\mathbb{R}^3$

Example 4. Find a Cartesian equation for the line ℓ given parametrically as

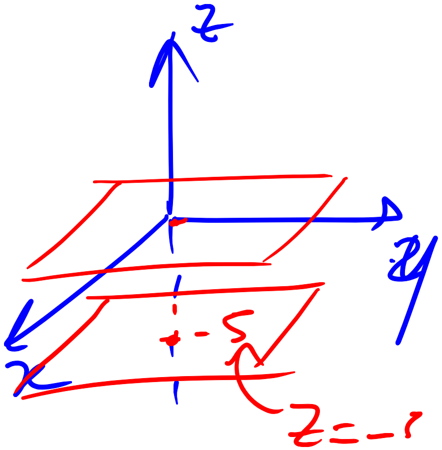
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \quad \lambda \in \mathbb{R}$$

$$\begin{cases} x = 1 + \lambda \\ y = 2 + 3\lambda \\ z = -5 \end{cases}$$

$$\lambda = x - 1$$

$$\lambda = \frac{y-2}{3}$$

?? $z = -5$



$$\boxed{\begin{cases} x - 1 = \dfrac{y-2}{3} & \pi_1 \\ z = 5 & \pi_2 \end{cases}}$$

A line is the intersection of 2 planes

Cartesian to parametric vector form of a line

Example 5. Find ~~the~~ a parametric vector equation for each of the following lines.

a) $\ell_1 : \dfrac{x-3}{5} = \dfrac{y+1}{2} = z-8$ in $\mathbb{R}^3$.

Let $\lambda = \dfrac{x-3}{5} \left(= \dfrac{y+1}{2} = z-8 \right)$

lambda

$\lambda = \dfrac{x-3}{5}$ $\quad x = 3 + 5\lambda$

$\lambda = \dfrac{y+1}{2}$ $\quad y = -1 + 2\lambda$

$\lambda = z-8$ $\quad z = 8 + \lambda$

$$\ell_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$$

b) $\ell_2 : \dfrac{x-3}{5} = \dfrac{y+1}{2}, \ z = 8$ in $\mathbb{R}^3$.

miu

Let $\mu = \dfrac{x-3}{5} = \dfrac{y+1}{2}$

$\mu = \dfrac{x-3}{5}$ $\quad x = 3 + 5\mu$

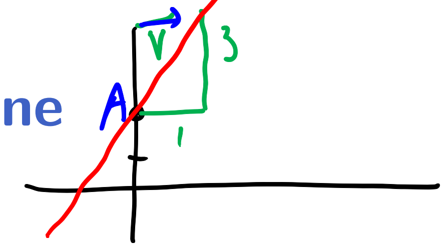
$\mu = \dfrac{y+1}{2}$ $\quad y = -1 + 2\mu$

μ

$$\ell_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}$$

$\mu \in \mathbb{R}$

Cartesian to parametric vector form of a line



Example 5, continued.

Find ~~the~~ a parametric vector equation for each of the following lines.

c) $\ell_3 : y = 3x + 2$ in $\mathbb{R}^2$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \underset{\uparrow \text{ param}}{\boxed{x}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \begin{matrix} \leftarrow x = x \\ \leftarrow y = 2 + 3x \end{matrix}$$

Let $\lambda = x$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underset{A}{\begin{pmatrix} 0 \\ 2 \end{pmatrix}} + \lambda \underset{v}{\begin{pmatrix} 1 \\ 3 \end{pmatrix}} \quad \lambda \in \mathbb{R}$$

d) $\ell_4 : x = 8$ in $\mathbb{R}^2$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix} + \underset{\uparrow \text{ param}}{\boxed{y}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftarrow x = 8$$

Let $\mu = y$

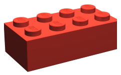
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

e) $\ell_5 : y = 2x + 1, \; z = 2$ in $\mathbb{R}^3$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \underset{\uparrow \text{ param}}{\boxed{x}} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

Let $\lambda = x$ . . .

Linear combinations

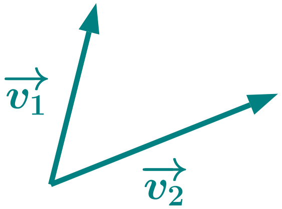


Linear combinations of two vectors

We say that a vector $\overrightarrow{\boldsymbol{v}}$ is a *linear combination* of two vectors $\overrightarrow{\boldsymbol{v_1}}$ and $\overrightarrow{\boldsymbol{v_2}}$ if $\overrightarrow{\boldsymbol{v}}$ is a sum of scalar multiples of $\overrightarrow{\boldsymbol{v_1}}$ and $\overrightarrow{\boldsymbol{v_2}}$, i.e. if $\overrightarrow{\boldsymbol{v}}$ can be written

$$\overrightarrow{\boldsymbol{v}} = \lambda_1 \overrightarrow{\boldsymbol{v_1}} + \lambda_2 \overrightarrow{\boldsymbol{v_2}},$$

where λ_1 and λ_2 are scalars.



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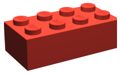

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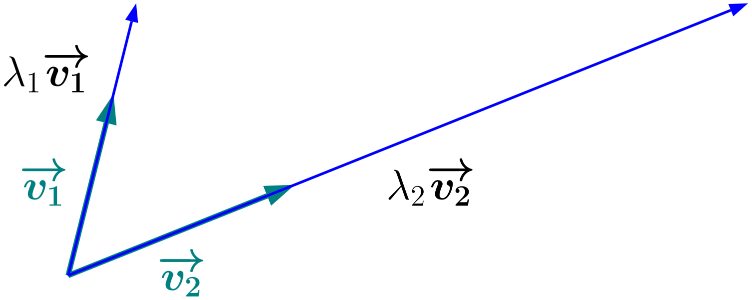

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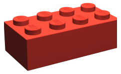

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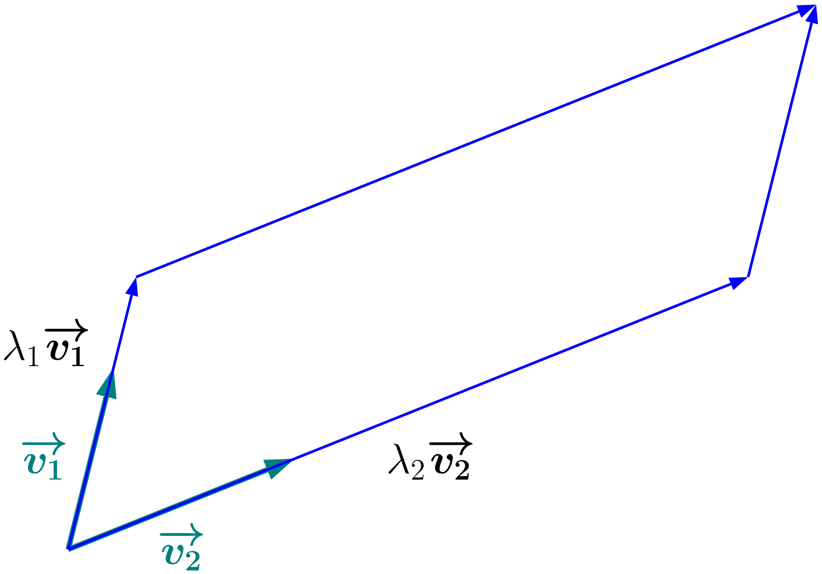

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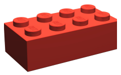

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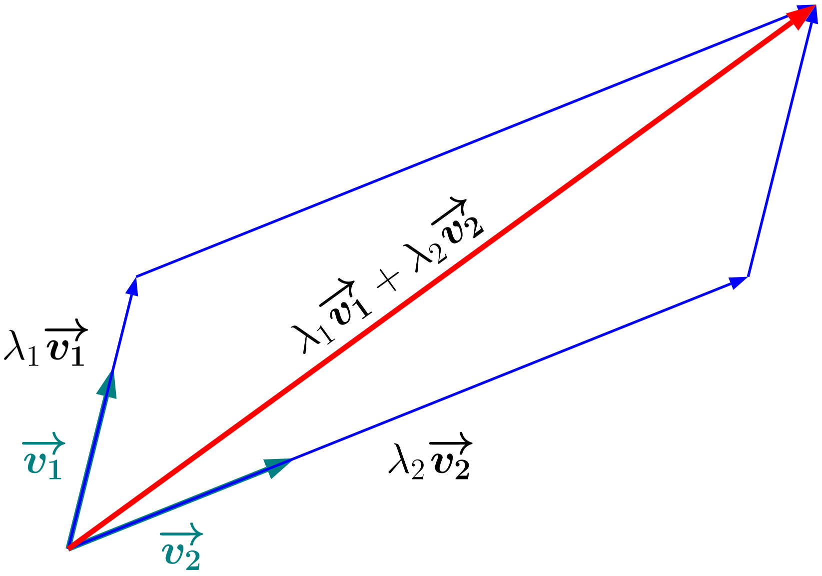
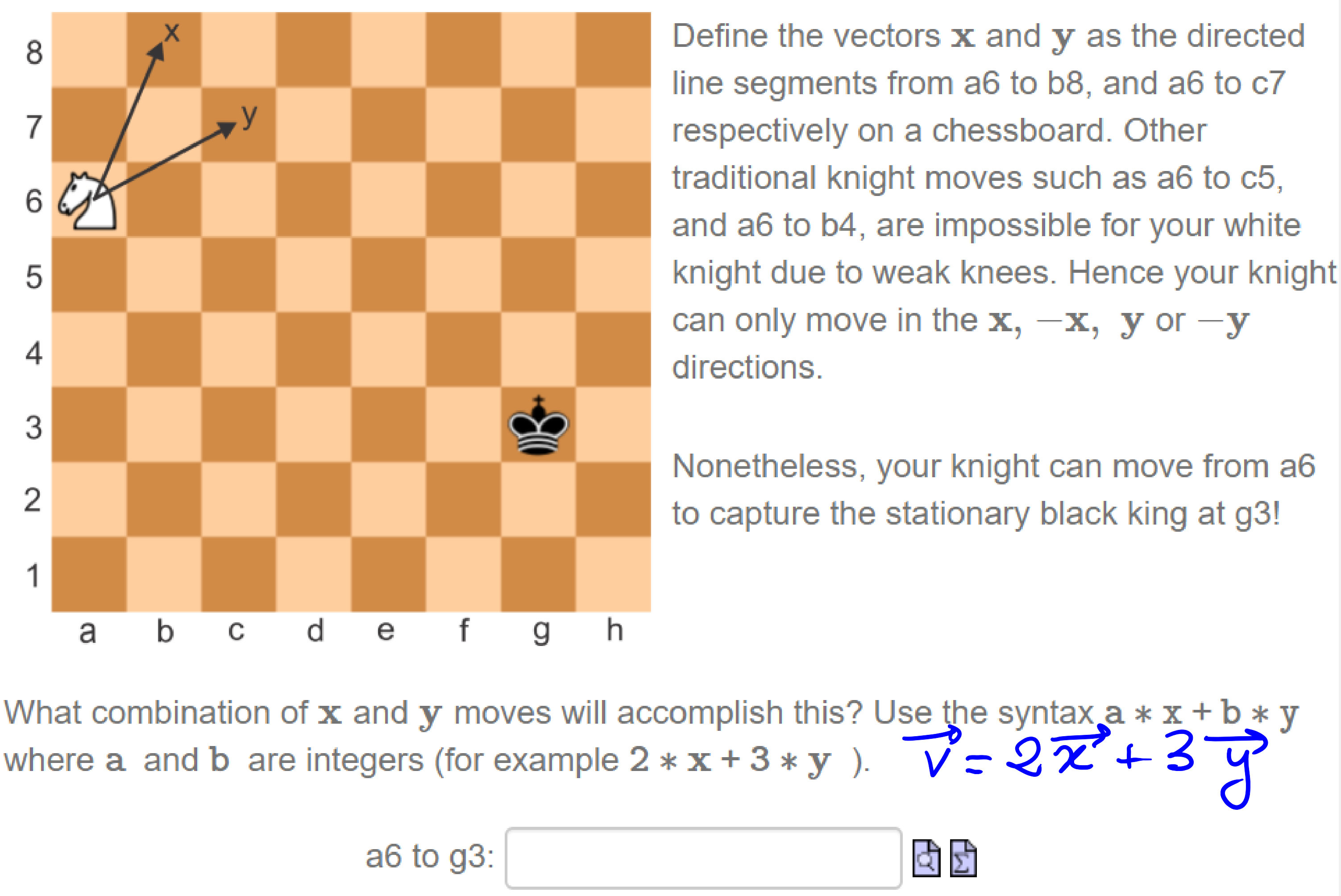


I have seen this before...



Define the vectors $\mathbf{x}$ and $\mathbf{y}$ as the directed line segments from a6 to b8, and a6 to c7 respectively on a chessboard. Other traditional knight moves such as a6 to c5, and a6 to b4, are impossible for your white knight due to weak knees. Hence your knight can only move in the $\mathbf{x}$, $-\mathbf{x}$, $\mathbf{y}$ or $-\mathbf{y}$ directions.

Nonetheless, your knight can move from a6 to capture the stationary black king at g3!

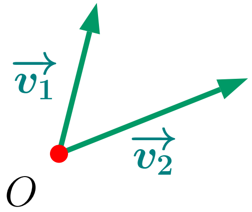
What combination of $\mathbf{x}$ and $\mathbf{y}$ moves will accomplish this? Use the syntax $\mathrm{a} * \mathrm{x} + \mathrm{b} * \mathrm{y}$ where a and b are integers (for example $2 * \mathrm{x} + 3 * \mathrm{y}$).

$\vec{v} = 2\vec{x} + 3\vec{y}$

a6 to g3:

Linear combinations

With $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ we can make many different linear combinations.



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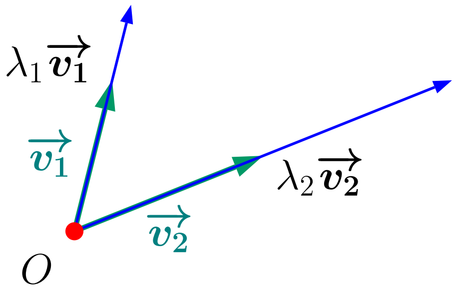

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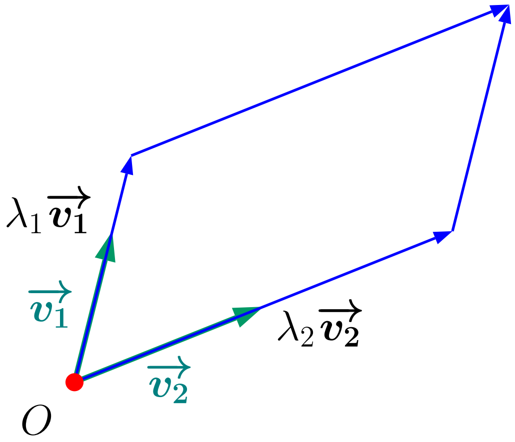

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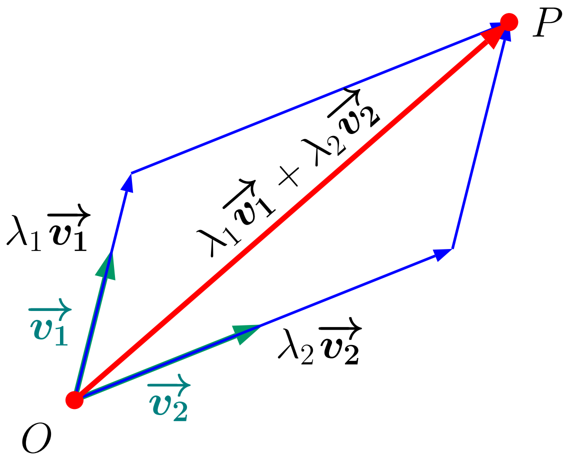

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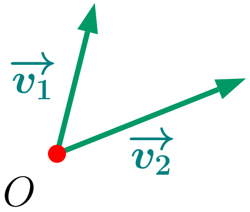

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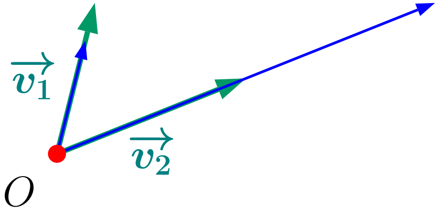

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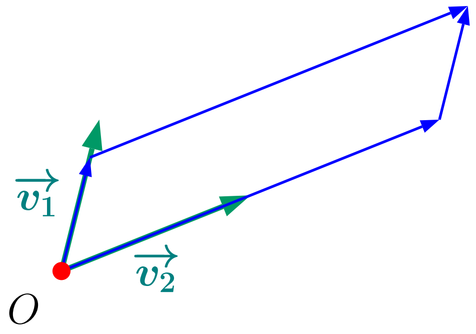

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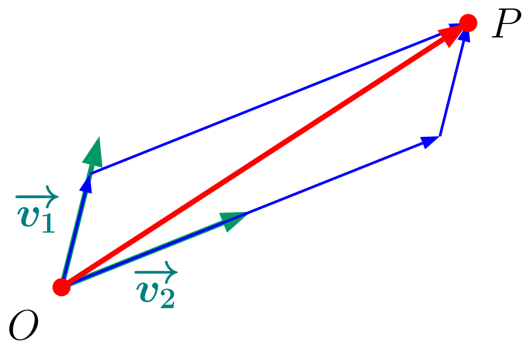

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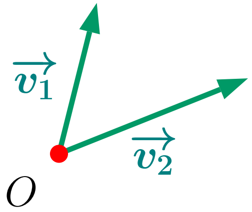

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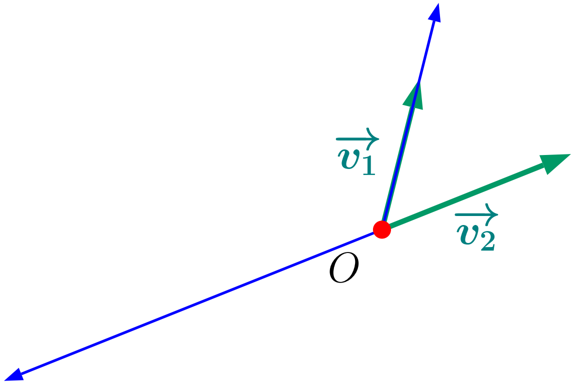

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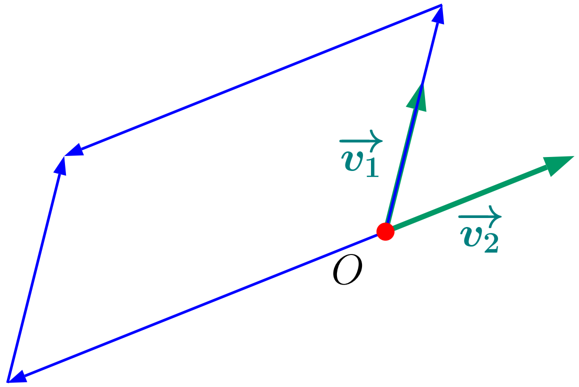

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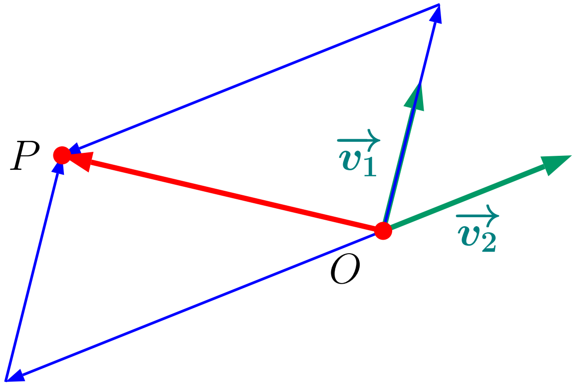

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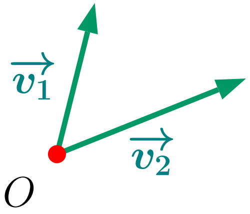

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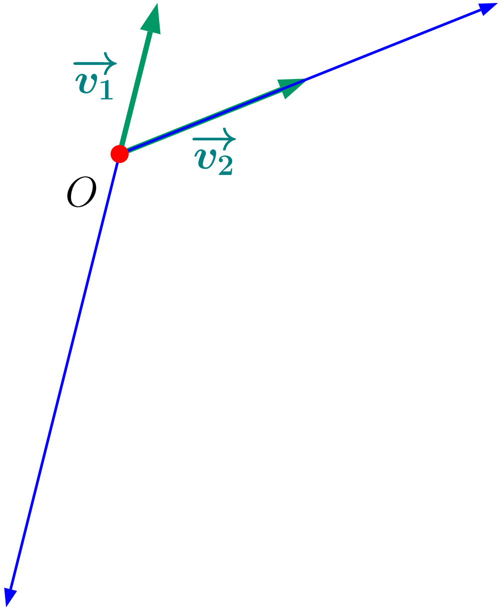

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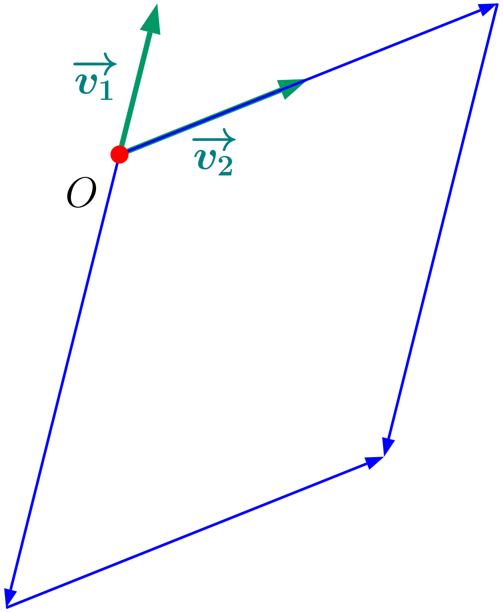

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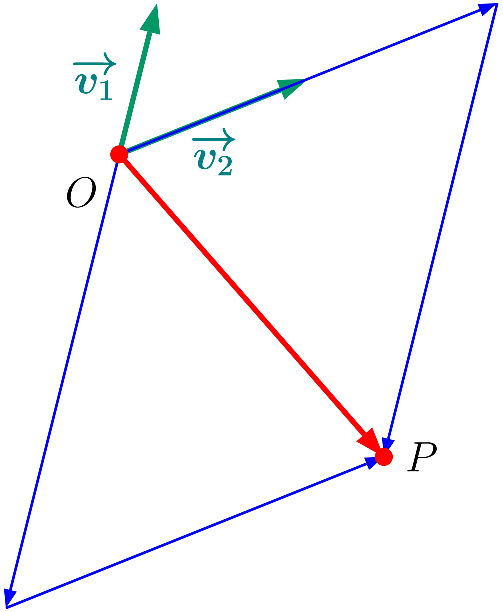
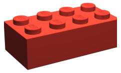


Span of two vectors



Span of two vectors

The set of all linear combinations of $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ is called the *span* of $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$.

$$\text{span}(\overrightarrow{v_1}, \overrightarrow{v_2}) = \{\overrightarrow{v} : \overrightarrow{v} = \lambda_1 \overrightarrow{v_1} + \lambda_2 \overrightarrow{v_2},\ \lambda_1, \lambda_2 \in \mathbb{R}\}.$$

Example 6.

a) Is $\overrightarrow{u} = \begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix}$ in the span of $\overrightarrow{u_1} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ and $\overrightarrow{u_2} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$?

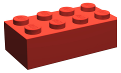
$\vec{u}$ is a linear combination of $\vec{u_1}$ and $\vec{u_2}$ if and only if $\vec{u}$ can be written $\vec{u} = \alpha_1 \vec{u_1} + \alpha_2 \vec{u_2}$ for some $\alpha_1, \alpha_2 \in \mathbb{R}$

$$\begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix} = \alpha_1 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

The first component would be $\alpha_1 \times 0 + \alpha_2 \times 0 = -3$ which is impossible.

$\therefore$ $\boxed{\vec{u} \text{ is not in the span of } \vec{v_1} \text{ and } v_2}$

Span of two vectors



Span of two vectors

The set of all linear combinations of $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ is called the *span* of $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$.

$$\text{span}(\overrightarrow{v_1}, \overrightarrow{v_2}) = \{\overrightarrow{v} : \overrightarrow{v} = \lambda_1 \overrightarrow{v_1} + \lambda_2 \overrightarrow{v_2},\ \lambda_1, \lambda_2 \in \mathbb{R}\}.$$

Example 6, continued.

b) Is $\overrightarrow{w} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ in the span of $\overrightarrow{w_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\overrightarrow{w_2} = \begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}$?

Can we find λ_1, λ_2 such that

$$\overrightarrow{w} = \lambda_1 \overrightarrow{w_1} + \lambda_2 \overrightarrow{w_2}$$

$$\overrightarrow{0} = 0\,\overrightarrow{w_1} + 0\,\overrightarrow{w_2}$$

$$\boxed{\overrightarrow{w} \in \text{span}(\overrightarrow{w_1}, \overrightarrow{w_2})}$$

Span

Example 6, continued.

c)  Is $\overrightarrow{v} = \begin{pmatrix}-3\\2\\6\end{pmatrix}$ in the span of $\overrightarrow{v_1} = \begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\overrightarrow{v_2} = \begin{pmatrix}5\\2\\-4\end{pmatrix}$?

$\vec{v}$ is in the span of $\vec{v_1}$ and $\vec{v_2}$ means there exists α_1 and α_2 in $\mathbb{R}$ such that $\vec{v} = \alpha_1\vec{v_1} + \alpha_2\vec{v_2}$

$\vec{v} = \alpha_1\vec{v_1} + \alpha_2\vec{v_2}$ means $\begin{pmatrix}-3\\2\\6\end{pmatrix} = \alpha_1\begin{pmatrix}1\\2\\1\end{pmatrix} + \alpha_2\begin{pmatrix}5\\2\\-4\end{pmatrix}$

$\alpha_1 + 5\alpha_2 = -3$ (i) $\quad \times 4 \quad \times 1$

$\alpha_1 + \alpha_2 = 1 \quad \times 1 \quad \times(-1)$

$+ + \quad \alpha_1 - 4\alpha_2 = 6$

$5\alpha_1 = 10$

$5\alpha_2 = -5$

$\boxed{\alpha_1 = 2}$

$\boxed{\alpha_2 = -1}$

sub in (i)

$2 + 5(-1) = -3 = RHS$

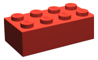
$\boxed{\text{The 3 equations are satisfied}}$

$\boxed{\text{yes}} \ \boxed{\vec{v} = 2\vec{v_1} - \vec{v_2} \text{ so } \vec{v} \text{ is in span}(\vec{v_1}, \vec{v_2})}$

Planes and lines through the origin

The span of two non-zero non-parallel vectors is a plane through the origin.

Planes and lines through the origin



The span of two non-zero non-parallel vectors is a plane through the origin.

We say that span$(\overrightarrow{v_1}, \overrightarrow{v_2})$ is the plane spanned by $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$.

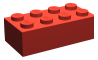
Planes and lines through the origin

The span of two non-zero non-parallel vectors is a plane through the origin.

We say that $\text{span}(\overrightarrow{v_1}, \overrightarrow{v_2})$ is the plane spanned by $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$.

In MATH1231, we will define the span of any number of vectors.

Planes and lines through the origin



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In MATH1231, we will define the span of any number of vectors.

Example 10 (Important!) Geometrically, what is the span of *one* non-zero vector

$$\text{span}(\overrightarrow{\boldsymbol{v_1}}) = \left\{\overrightarrow{\boldsymbol{x}} : \overrightarrow{\boldsymbol{x}} = \lambda_1 \overrightarrow{\boldsymbol{v_1}},\ \lambda_1 \in \mathbb{R}\right\}?$$

Planes and lines through the origin

The span of two non-zero non-parallel vectors is a plane through the origin.

We say that span$(\overrightarrow{\boldsymbol{v_1}}, \overrightarrow{\boldsymbol{v_2}})$ is the plane spanned by $\overrightarrow{\boldsymbol{v_1}}$ and $\overrightarrow{\boldsymbol{v_2}}$.

In MATH1231, we will define the span of any number of vectors.

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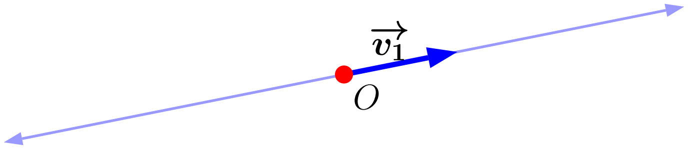
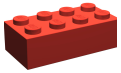
$$\text{span}(\overrightarrow{\boldsymbol{v_1}}) = \left\{\overrightarrow{\boldsymbol{x}} : \overrightarrow{\boldsymbol{x}} = \lambda_1 \overrightarrow{\boldsymbol{v_1}},\ \lambda_1 \in \mathbb{R}\right\}?$$

A line through the origin.



Planes

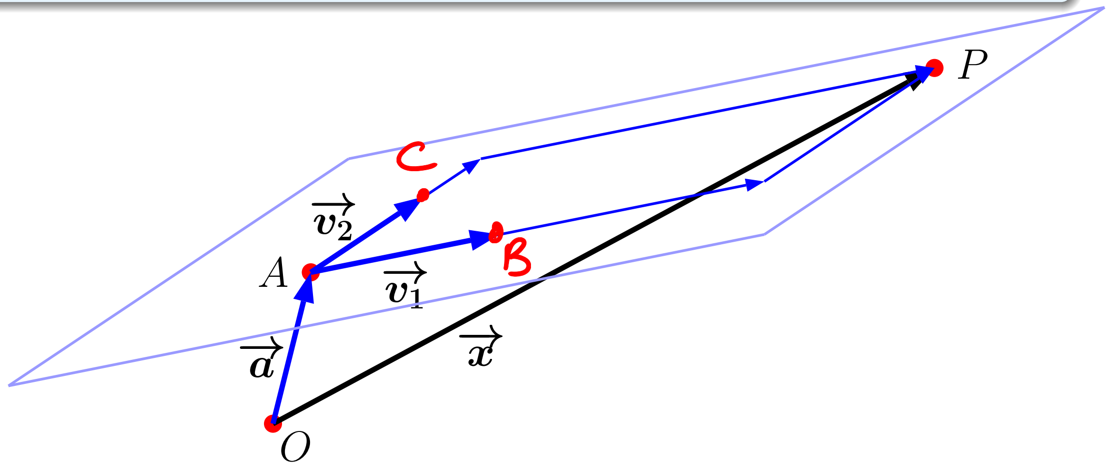


Planes in $\mathbb{R}^n$

A plane in $\mathbb{R}^n$ is the set of points or vectors $\overrightarrow{x}$ given by

$$\overrightarrow{x} = \overset{A,B \text{ or } C}{\overrightarrow{a}} + \lambda_1 \overset{\overrightarrow{AB}}{\overrightarrow{v_1}} + \lambda_2 \overset{\overrightarrow{AC}}{\overrightarrow{v_2}}, \quad \lambda_1, \lambda_2 \in \mathbb{R},$$

where $\overrightarrow{a} \in \mathbb{R}^n$ gives a point on the plane and $\overrightarrow{v_1}$, $\overrightarrow{v_2} \in \mathbb{R}^n$ are a pair of non-zero, non-parallel vectors that are directions in the plane.



Example 12. To be done on the next slide.

Find ~~the~~ a parametric vector form of the plane Π passing through the three points $A(1, -2, 1)$, $B(2, 1, 1)$ and $C(0, 3, 1)$.

Parametric equation of a plane from three points

Example 12.

Find ~~the~~ a parametric vector form of the plane Π passing through the three points $A(1,-2,1)$, $B(2,1,1)$ and $C(0,3,1)$.

$$\overrightarrow{AB} = \begin{pmatrix} 2-1 \\ 1+2 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \qquad \overrightarrow{AC} = \begin{pmatrix} 0-1 \\ 3+2 \\ 1-1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix}$$

$$\Pi : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underset{\substack{\uparrow \\ \text{a point} \\ \text{on the plane}}}{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}} + \lambda_1 \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix} \qquad \lambda_1, \lambda_2 \in \mathbb{R}$$

Parametric equation of a plane from 1 point and 2 lines

Example 13. Find ~~the~~ a parametric vector form of the plane Π passing through the point $(2,-1,2)$ and parallel to the lines

$$\ell_1 : \qquad \frac{x-2}{3} = \frac{y-1}{-3} = \frac{2z-3}{8}$$

and

$$\ell_2 : \qquad \vec{x} = \begin{pmatrix}-2\\3\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\1\end{pmatrix},\ \lambda \in \mathbb{R}.$$

$$\ell_1 \quad \text{let } \alpha = \frac{x-2}{3} = \frac{y-1}{-3} = \frac{2z-3}{8} \qquad 8\alpha = 2z-3$$

$$x = 2+3\alpha$$

$$y = 1-3\alpha$$

$$z = \frac{3}{2} + 4\alpha$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\1\\3/2\end{pmatrix} + \alpha\begin{pmatrix}3\\-3\\4\end{pmatrix}$$

$$\boxed{\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\-1\\2\end{pmatrix} + \alpha_1 \overset{\vec{v_2}}{\begin{pmatrix}2\\1\\1\end{pmatrix}} + \alpha_2 \overset{V_1}{\begin{pmatrix}3\\-3\\4\end{pmatrix}} \qquad \alpha_1, \alpha_2 \in \mathbb{R}}$$

Cartesian equation of a plane

Example 14. For the plane Π given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} + \lambda_2 \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R},$$

eliminate the parameters λ_1 and λ_2 to find an equation relating x_1, x_2 and x_3.

① $x_1 = 1 + 4\lambda_2 \iff \lambda_2 = \frac{x_1 - 1}{4}$

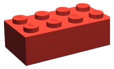
② $x_2 = 2 - \lambda_1 - 2\lambda_2$

③ $x_3 = 3 + 4\lambda_1 \iff \lambda_1 = \frac{x_3 - 3}{4}$

② $\iff x_2 = 2 - \frac{x_3 - 3}{4} - 2\,\frac{x_1 - 1}{4}$

$\iff \boxed{2x_1 + 4x_2 + x_3 = 13}$

Cartesian equation of a plane in $\mathbb{R}^3$



What does a Cartesian equation of a plane in $\mathbb{R}^3$ look like?

A Cartesian equation of a plane in $\mathbb{R}^3$ is an equation of the form

$$ax_1 + bx_2 + cx_3 = d$$

for some $a, b, c, d \in \mathbb{R}$ with at least one of a, b and c non-zero.

Example 15. Find ~~the~~ a vector equation for the plane Π in $\mathbb{R}^3$, $x_1 + 2x_2 - x_3 = 3.$

$$x_1 = 3 - 2x_2 + x_3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Let $\lambda_1 = x_2$ and $\lambda_2 = x_3$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \lambda_1, \lambda_2 \in \mathbb{R}}$$

Don't forget to read Chapter 1 of the Algebra Notes! (It is the big yellow book that was in the course pack, the one that contains the Algebra tutorial questions)