School of Mathematics and Statistics

Math1131 Mathematics 1A

CALCULUS LECTURE 4

LIMITS TO INFINITY (PART 2)

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MATH1131 CALCULUS
Limit of Functions at Infinity Part 2

$\lim\limits_{x \longrightarrow \infty} f(x) = L$ if for every (small) positive real number ϵ there is a (large) real number M with the property that if $x > M$ then $|f(x) - L| < \epsilon$.

Limits of the form $\lim\limits_{x \longrightarrow a} f(x)$ are best attacked via factorisation.

$\lim\limits_{x \longrightarrow a^-} f(x)$ is the limit of $f(x)$ as x approaches a from **the left**

$\lim\limits_{x \longrightarrow a^+} f(x)$ is the limit of $f(x)$ as x approaches a from **the right**

$\lim\limits_{x \longrightarrow a} f(x)$ exists if and only if $\lim\limits_{x \longrightarrow a^-} f(x)$ and $\lim\limits_{x \longrightarrow a^+} f(x)$ both exist and are equal.

Just like differentiation from first principles we also have a formal definition of $\lim\limits_{x \longrightarrow \infty} f(x)$. It is crucial that we understand these fundamental definitions as they provide a solid foundation for the analysis. We can't just keep waving our hands around! If we want you to evaluate a limit formally we will always give you a clear warning.

Definition: $\lim\limits_{x \longrightarrow \infty} f(x) = L$ if for every (small) positive real number ϵ there is a (large) real number M with the property that if $x > M$ then $|f(x) - L| < \epsilon$.

Discussion

Note that $|f(x) - L| < \epsilon$ simply means that the difference between $f(x)$ and L is really really small. The limit definition is then simply saying that any degree of closeness between $f(x)$ and L can be generated by choosing x to be sufficiently large.

For $\lim\limits_{x \longrightarrow \infty} f(x)$ to be equal to L the function must eventually $(x > M)$ get into (and stay in!) an ϵ band of L.

Example 1: Consider $\lim_{x \longrightarrow \infty} \dfrac{10x-4}{5x+7}$.

a) Show that the value of the limit is $L = 2$.

b) Find M so that $f(x)$ is within $\dfrac{1}{1000}$ of its limit whenever $x > M$.

c) Does b) prove that $\lim_{x \longrightarrow \infty} \dfrac{10x-4}{5x+7} = 2$?

d) Prove from the limit definition that $\lim_{x \longrightarrow \infty} \dfrac{10x-4}{5x+7} = 2$.

That is given $\epsilon > 0$ find M with the property that if $x > M$ then $|f(x) - L| < \epsilon$.

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a)

b)

So $\left|\dfrac{10x-4}{5x+7} - 2\right| \leq \dfrac{1}{1000}$ provided we choose $x > 3598.6$. Note that we are not really trying to solve $\left|\dfrac{10x-4}{5x+7} - 2\right| \leq \dfrac{1}{1000}$. We are just trying to find an M with the property that the inequality is true for all $x > M$. This value of M is not unique and any $M > 3598.6$ would also work

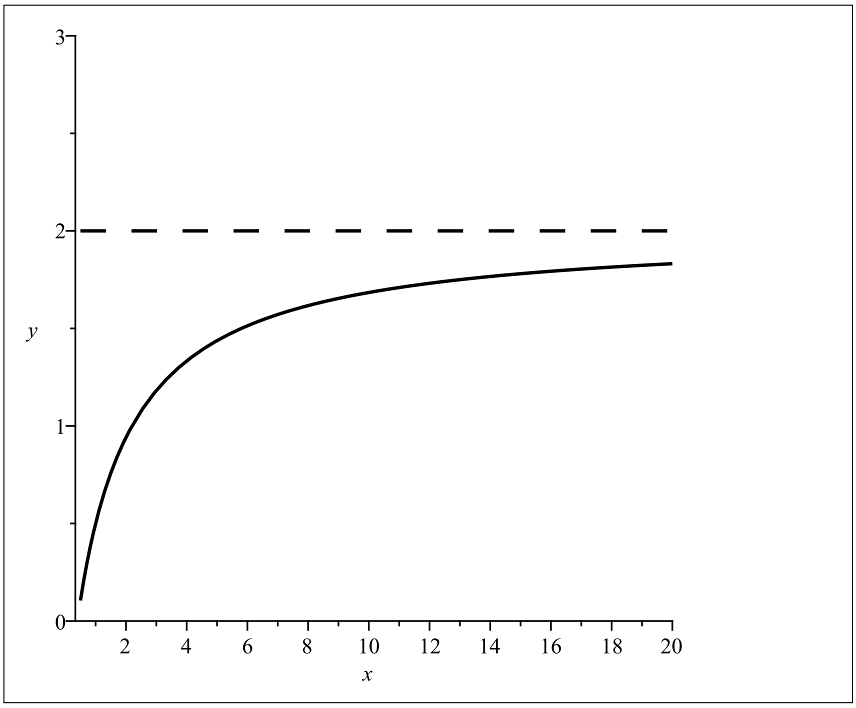
c) NO!!

d)

$$\bigstar \quad M = \frac{18}{5\epsilon} - \frac{7}{5}. \quad \bigstar$$

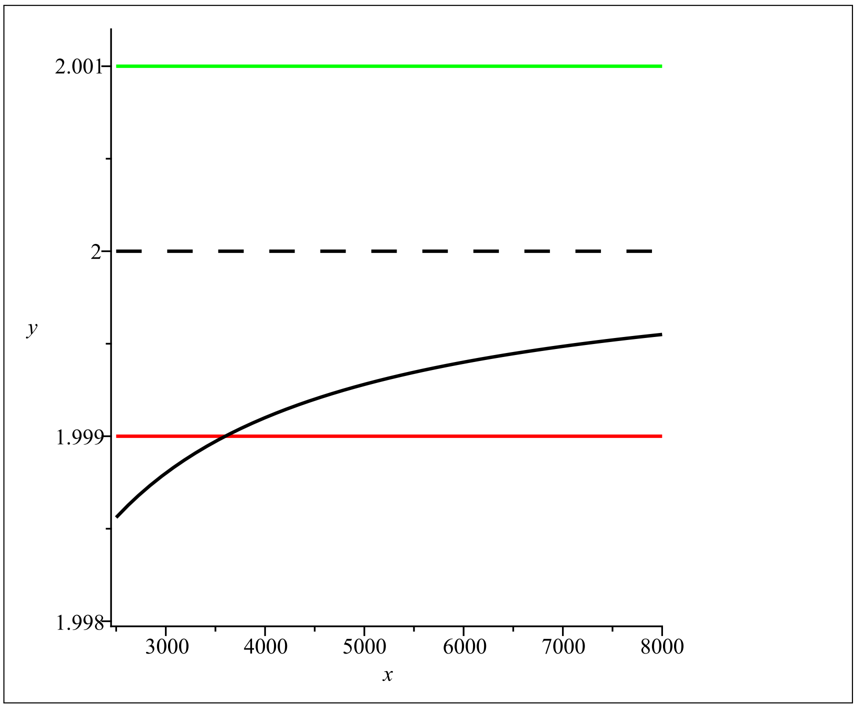
Note that since the above M depends on ϵ we sometimes write $M(\epsilon) = \dfrac{18}{5\epsilon} - \dfrac{7}{5}$.

Let's take a look at the graph for the situation of part b) where $\epsilon = \dfrac{1}{1000} = 0.001$. First the the function and its limit of $L = 2$ as a horizontal asymptote:



Next a band (red to green) of length $\pm\dfrac{1}{1000}$ either side of the limit $L = 2$.

Observe below that after $M = 3598.6$ the function is clearly locked within this limit band.



Example 2: Find $\lim_{x \longrightarrow \infty} \frac{\sin(x)}{x^2 + e^x}$ and prove from the limit definition that your answer is correct. Find a value of M for $\epsilon = \frac{1}{100}$ and sketch the behaviour of the function and its limit.

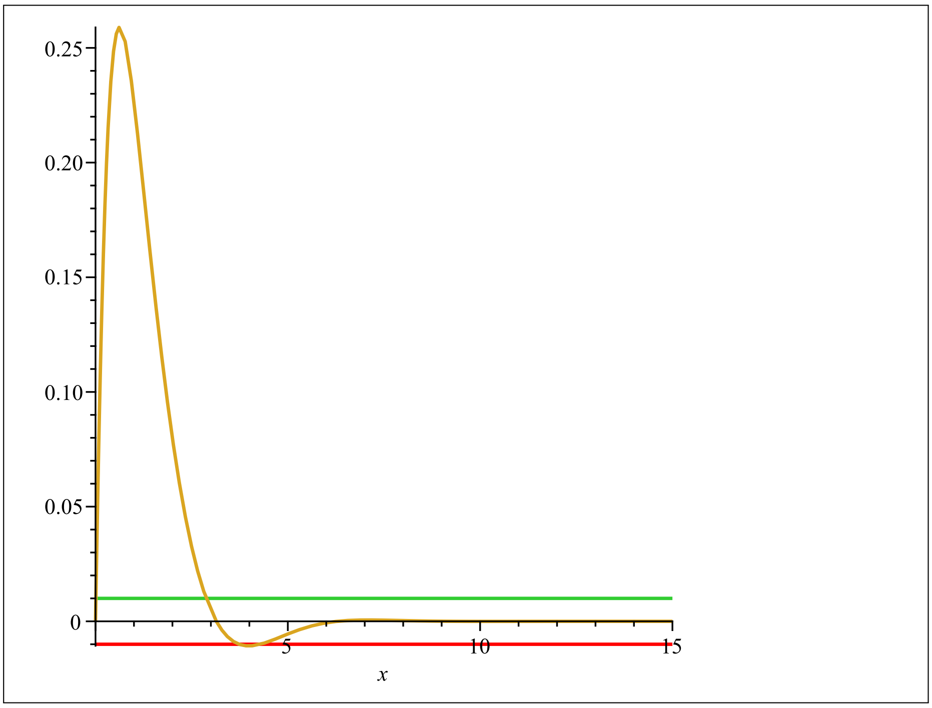
Clearly $\lim_{x \longrightarrow \infty} \frac{\sin(x)}{x^2 + e^x} =$

So $L = 0$. We now examine $|f(x) - L| = \left| \frac{\sin(x)}{x^2 + e^x} - 0 \right| =$

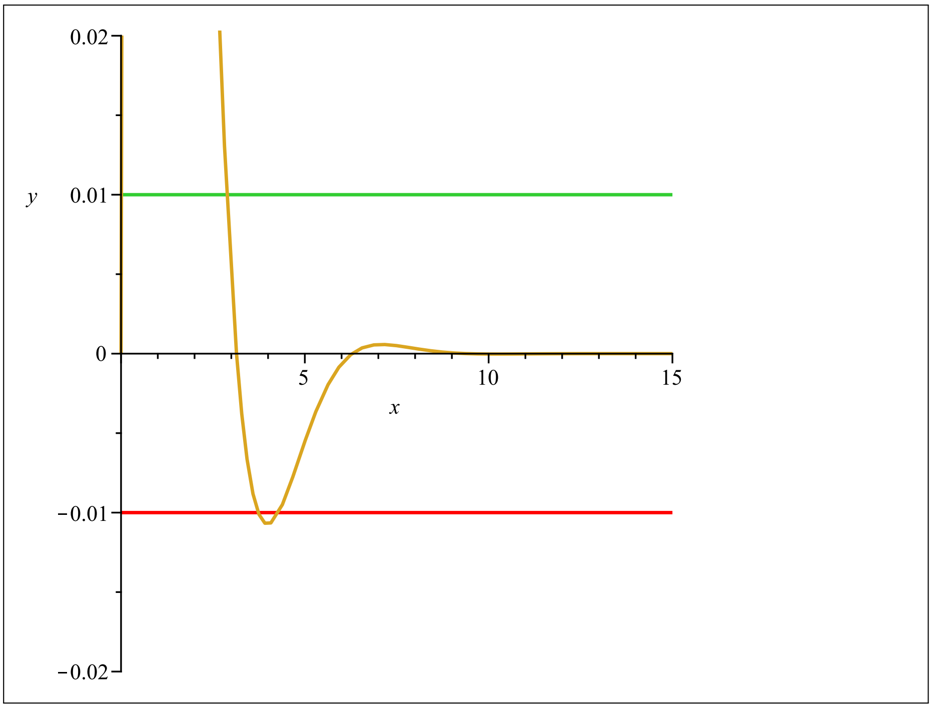
So we have $|f(x) - L| < \frac{1}{x^2}$. we therefore only need $\frac{1}{x^2} < \epsilon$. And hence:

★ $M = 10$ ★

Let's take a look at the graph:



Observe below that after $M = 10$ the function is clearly within its $\dfrac{1}{100}$ limit band although a smaller value of $M = 5$ would also do!!



Note that above we have most definitely **NOT** proven that $\lim\limits_{x \longrightarrow \infty} \dfrac{\sin(x)}{x^2 + e^x} = 0$! The full proof would demand that we use a general small ϵ rather than the specific $\dfrac{1}{100}$.

Limit of Functions at a Point

There are certain situations where a function fails to be defined **at** a point but it is perfectly happy **near** the point. We then use $\lim\limits_{x \longrightarrow a} f(x)$ to get a feeling for the behaviour of the function.

Consider the following table of values for $f(x) = \dfrac{x^2 - 25}{x - 5}$ near $x = 5$.

x	4.9	4.99	5	5.01	5.1
y	9.9	9.99	?	10.01	10.1

It is clear that the function is trying to get to 10 at $x = 5$ even though it is undefined there. We write

$$\lim_{x \longrightarrow 5} \frac{x^2 - 25}{x - 5} = 10$$

and say

"The limit as x approaches 5 of $\dfrac{x^2 - 25}{x - 5}$ is 10".

Example 3: Find $\lim\limits_{x \longrightarrow 5} \dfrac{x^2 - 25}{x - 5}$ and hence sketch the function $y = \dfrac{x^2 - 25}{x - 5}$.

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So for limits as x approaches a finite value our main technique is to factorise top and bottom. Always check first that you are facing the indeterminate form "$\frac{0}{0}$"

Example 4: Evaluate each of the following limits

a) $\displaystyle\lim_{x\longrightarrow 2} \frac{x^2-5x+6}{x^2-4}$

b) $\displaystyle\lim_{x\longrightarrow 1} \frac{x^2+1}{x^2+4}$

$$\bigstar \quad a)\ -\frac{1}{4} \quad b)\ \frac{2}{5} \quad \bigstar$$

In order to define these limits a little more formally we require the concept of a one-sided limit:

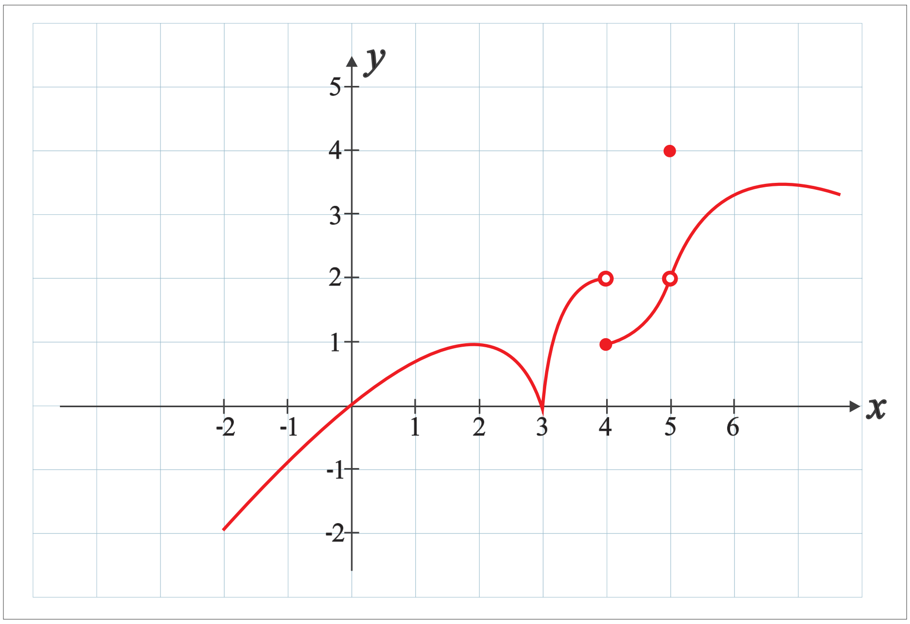
$\displaystyle\lim_{x\longrightarrow a^-} f(x)$ is the limit of $f(x)$ as x approaches a from **the left**

$\displaystyle\lim_{x\longrightarrow a^+} f(x)$ is the limit of $f(x)$ as x approaches a from **the right**

We can then say that the full limit $\displaystyle\lim_{x\longrightarrow a} f(x)$ formally exists if and only if $\displaystyle\lim_{x\longrightarrow a^-} f(x)$ and $\displaystyle\lim_{x\longrightarrow a^+} f(x)$ both exist and are equal.

Example 5: Consider the graph of $y = f(x)$ presented in red below:

For each of the following either evaluate the given quantity or explain why it does not exist:



a) $\lim\limits_{x \longrightarrow 4^-} f(x)$ b) $\lim\limits_{x \longrightarrow 4^+} f(x)$

c) $\lim\limits_{x \longrightarrow 4} f(x)$ d) $\lim\limits_{x \longrightarrow 5} f(x)$

e) $f(5)$ f) $\lim\limits_{x \longrightarrow 6} f(x)$

g) $\lim\limits_{x \longrightarrow 3} f(x)$