Chapter 9: The logarithmic and exponential functions

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(Alina Ostafe's notes, based on Fedor Sukochev's notes)

MATH1131

UNSW

Term 2 2019

Motivation

- In the preceding, we have manipulated functions such as

$$\ln x, \quad e^x, \quad x^\pi$$

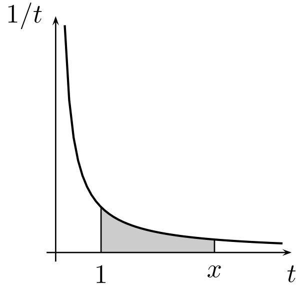
  even though we have not defined them formally.
- We will use the fundamental theorem of calculus to define the logarithmic function as an integral.
- The exponential function will be defined as the inverse of the log function.

Definition. The natural logarithm function

$$\ln : (0, \infty) \to \mathbb{R}$$

is defined by the formula

$$\ln x = \int_1^x \frac{1}{t}\, dt.$$



It is evident that $\ln x$ is simply the area of the shaded region shown above when $x \geq 1$.

Theorem

The function $\ln : (0, \infty) \to \mathbb{R}$ has the following properties:

(i) $\ln$ is differentiable on $(0, \infty)$ and

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

(ii) $\ln x > 0 \quad \text{for} \quad x > 1$,

$\ln 1 = 0$,

$\ln x < 0 \quad \text{for} \quad 0 < x < 1$.

(iii) $\ln x \to -\infty \quad \text{as} \quad x \to 0^+$,

$\ln x \to \infty \quad \text{as} \quad x \to \infty$.

(iv) For all $x, y > 0$: $\quad \ln(xy) = \ln x + \ln y$.

(v) For all $x, y > 0$: $\quad \ln\left(\dfrac{x}{y}\right) = \ln(x) - \ln(y)$.

(vi) For all $x > 0$ and $r \in \mathbb{Q}$: $\quad \ln(x^r) = r \ln x$.

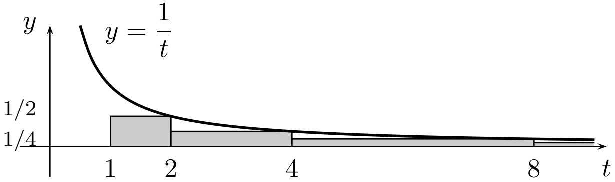
Proof.

(i) Apply the first fundamental theorem of calculus to the definition of $\ln$.

(ii) This follows from the definition of $\ln$ and the fact that $\frac{1}{t} > 0$ when $t > 0$.

(iii) The diagram below shows that

$$\int_1^2 \frac{dt}{t} \geq 1 \times \frac{1}{2}, \quad \int_2^4 \frac{dt}{t} \geq 2 \times \frac{1}{4}, \quad \int_4^8 \frac{dt}{t} \geq 4 \times \frac{1}{8}.$$



In general,

$$\int_1^{2^n} \frac{dt}{t} \geq \underbrace{\frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}}_{n \text{ terms}} = \frac{n}{2} \to \infty$$

as $n \to \infty$.

- Hence the improper integral $\displaystyle\int_1^\infty \frac{1}{t}\,dt$ 'diverges to infinity' and, therefore, $\ln x \to \infty$ as $x \to \infty$.

- As $x \to 0^+$: Let $u = 1/x$, then $u \to \infty$ as $x \to 0^+$.

By using property (iv) with $r = -1$ we obtain

$$\ln x = \ln(1/u) = -\ln u \to -\infty \quad \text{as } u \to \infty \ \text{ i.e. } x \to 0^+.$$

(iv), Prove...

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(iv), Prove...

(iv) Suppose that y is some fixed positive number and that $x > 0$. Then, the chain ru implies that

$$\frac{d}{dx}[\ln(xy)] = \frac{1}{xy}\frac{d}{dx}(xy) = \frac{y}{xy} = \frac{1}{x} = \frac{d}{dx}\ln x.$$

Accordingly,

$$\ln(xy) = \ln(x) + C$$

for some constant C.

Evaluation at $x = 1$ leads to

$$\ln(y) = C.$$

Hence

$$\ln(xy) = \ln(x) + \ln(y).$$

(v) Similar technique with the proof of (iv).

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(iv) Other way:

$$\begin{aligned}\ln(xy) &= \int_1^{xy} \frac{dt}{t} \\ &= \int_1^{x} \frac{dt}{t} + \int_x^{xy} \frac{dt}{t} \\ &= \ln x + \int_1^{y} \frac{du}{u}, \quad \text{substituting } t = xu \\ &= \ln x + \ln y\end{aligned}$$

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(v)

$$\begin{aligned}\ln(x/y) &= \int_1^{x/y} \frac{dt}{t} \\ &= \int_1^{x} \frac{dt}{t} + \int_x^{x/y} \frac{dt}{t} \\ &= \ln x - \int_1^{y} \frac{du}{u}, \quad \text{substituting } t = \frac{x}{u} \\ &= \ln x - \ln y.\end{aligned}$$

(vi) By letting $t = s^r$ we have

$$\ln(x^r) = \int_1^{x^r} \frac{dt}{t} = r \int_1^{x} \frac{ds}{s} = r \ln x.$$

Remark. The above properties imply that

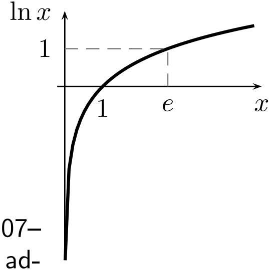
- $\mathrm{Range}(\ln) = \mathbb{R}$, and
- $\ln$ is increasing and hence invertible so that
- $\ln x = 1$ has a unique solution (Why?).

Definition. The real number e is defined to be the unique number x satisfying

$$\int_1^x \frac{1}{t}\,dt = 1.$$

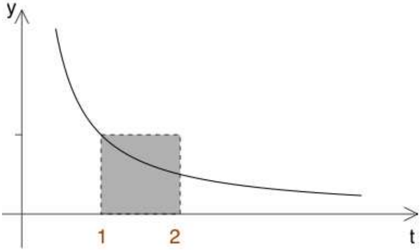
Hence

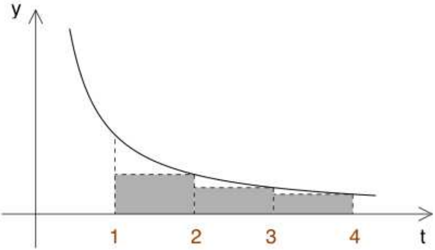
$$\ln(e) = 1$$



The letter e is used in honour of Leonhard Euler (1707–1783), who worked in many important branches of advanced calculus.

By evaluating the upper and lower Riemann sums in the following two diagrams





we see that

$$\ln 2 < 1 \quad \text{and} \quad \ln 4 > \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 \ .$$

By the Intermediate Value Theorem the equation $\ln x = 1$ has a solution between 2 and 4; therefore $2 < e < 4$.In fact the use of other methods (which you shall see in MATH1231) makes it very easy to calculate e and we have

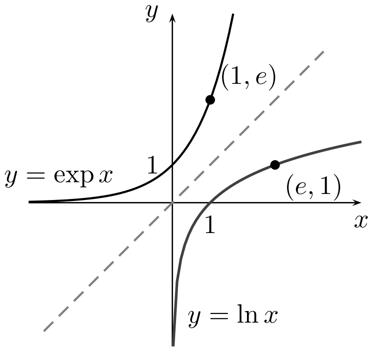
$$e = 2.718281828459045235360287471352662497757 2 \cdots$$

The exponential function

Definition. The function

$$\exp : \mathbb{R} \to (0, \infty)$$

is defined to be the inverse function of $\ln : (0, \infty) \to \mathbb{R}$.



Remark. For any rational number r, we can evaluate both

$$\exp r \qquad \text{and} \qquad e^r$$

but are these two numbers the same?

Theorem

The function $\exp : \mathbb{R} \to (0, \infty)$ has the following properties:

(i) $\exp(\ln x) = x$ for all $x \in (0, \infty)$,

$\ln(\exp x) = x$ for all $x \in \mathbb{R}$.

(ii) $\exp(1) = e$,

$\exp(0) = 1.$

(iii) $\exp x \to \infty$ as $x \to \infty$,

$\exp x \to 0$ as $x \to -\infty$.

(iv) $\exp$ is differentiable on $\mathbb{R}$ with

$$\frac{d}{dx}\exp x = \exp x.$$

(v) For all $x, y \in \mathbb{R}$:

$$\exp(x+y) = \exp x \, \exp y.$$

(vi) For all $x > 0$ and $r \in \mathbb{Q}$:

$$\exp(rx) = (\exp x)^r.$$

Proof.

(i) - (iii) Follows from the definition of $\exp$ as an inverse function.

(iv),(v),(vi)...

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(iv) The function $\exp$ is differentiable on $\mathbb{R}$ by virtue of the inverse function theorem and differentiation of

$$\ln(\exp x) = x$$

produces

$$\frac{1}{\exp x}\,\frac{d}{dx}\exp x = 1$$

as required.

(v) For any x and y, we have

$$\begin{aligned}\exp(x+y) &= \exp\Big(\ln(\exp x) + \ln(\exp y)\Big)\\ &= \exp\Big(\ln(\exp x\ \exp y)\Big)\\ &= \exp x\ \exp y\end{aligned}$$

as required.

Remark. In particular, the above theorem implies that

$$\exp(r) = (\exp 1)^r = e^r.$$

for every rational number r. It is therefore consistent to extend to non-rational numbers and make the following definition.

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(vi) Suppose that r is a rational number and x is a real number. Then,

$$\begin{aligned}\exp(rx) &= \exp\Big(r\ln(\exp x)\Big)\\ &= \exp\Big(\ln\left((\exp x)^r\right)\Big)\\ &= (\exp x)^r.\end{aligned}$$

Definition

For any $x \notin \mathbb{Q}$, we define the number e^x to be

$$e^x = \exp x.$$

Hence by construction, the function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \exp x = e^x$$

is differentiable (and continuous) and is called the exponential function.

Exponentials and logarithms with other bases

Question. How would one define b^x for $x \notin \mathbb{Q}$ and $b > 0$?

Since, for any rational number r,

$$b^r = \exp\big(\ln(b^r)\big) = \exp\big(r \ln b\big) = e^{r \ln b},$$

the following definition is natural for any real number.

Definition

Suppose that $b > 0$ and $x \in \mathbb{R}$.
Then, the number b^x is defined by the function

$$f_b : \mathbb{R} \to (0, \infty), \quad f_b(x) = b^x = \exp(x \ln b) = e^{x \ln b}$$

Example. It is seen that

$$f_3(2) = 3^2 = \exp(2\ln 3) = \exp(\ln 3)\ \exp(\ln 3) = 3 \times 3 = 9$$

as one would expect!

Remark. Since f_b is a combination of continuous and differentiable functions, it is also continuous and differentiable with

$$f_b'(x) = (\ln b)\exp(x \ln b) = (\ln b)b^x.$$

Accordingly,

- if $b > 1$ then $f_b'(x) > 0$ for all $x \in \mathbb{R}$,
- if $0 < b < 1$ then $f_b'(x) < 0$ for all $x \in \mathbb{R}$

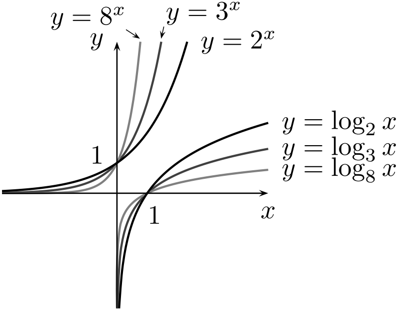
so that f_b is invertible for $b \neq 1$.

Definition. Suppose that b is a positive real number with $b \neq 1$. Then, the logarithm function to the base b

$$\log_b : (0, \infty) \to \mathbb{R}$$

is defined to be the inverse of the function

$$f_b : \mathbb{R} \to (0, \infty), \quad f_b(x) = b^x = \exp(x \ln b).$$



In particular, $\log_e x = \ln x$.

Remark. The above definition may be stated as

$$y = b^x \quad \Leftrightarrow \quad x = \log_b y.$$

The following theorem demonstrates that all logarithm functions are just scaled versions of the natural logarithm function.

Theorem

Suppose that b is a positive real number with $b \neq 1$. Then

$$\log_b x = \frac{\ln x}{\ln b}$$

for all $x > 0$.

Proof. Since $x = b^{\log_b x} = \exp(\ln(b^{\log_b x})) = \exp(\log_b x \ln b)$,

we conclude that

$$\ln x = \log_b x \ln b \qquad \text{and} \qquad \log_b x = \frac{\ln x}{\ln b}$$

It turns out that $\log_b$ shares all the properties of $\ln$ such as

$$\frac{d}{dx} \log_b x = \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) = \frac{1}{x \ln b} \qquad \text{or} \qquad \log_b(x^y) = y \log_b x.$$

Integration and the $\ln$ function

Since

$$\frac{d}{dx}\ln(x) = \frac{1}{x}, \quad x > 0, \quad \text{and} \quad \frac{d}{dx}\ln(-x) = \frac{1}{x}, \quad x < 0,$$

the function $\ln(x)$ is an antiderivative of $1/x$ if $x > 0$ and $\ln(-x)$ is an antiderivative of $1/x$ if $x < 0$ (or in other words, $\ln|x|$ is an antiderivative of $1/x$).

Thus,

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

provided that x is restricted to an interval which does NOT contain 0.

This may be generalised to

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

provided that f is differentiable and does not vanish on the interval of integration.

Example. On any interval not including zeros of $\cos x$, we have

$$\int \tan x \, dx = -\int \frac{-\sin x}{\cos x} \, dx = -\ln|\cos x| + C.$$

Example. Find the indefinite integral

$$\int \frac{1}{2\sec x + \tan x} \, dx.$$

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Solution. Rearranging

$$\int \frac{1}{2\sec x + \tan x} \, dx = \int \frac{1}{\frac{2}{\cos x} + \frac{\sin x}{\cos x}} \, dx = \int \frac{\cos x}{2 + \sin x} \, dx$$

and observing that

$$2 + \sin x \neq 0 \quad \text{for all } x \in \mathbb{R}$$

and

$$(2 + \sin x)' = \cos x,$$

we have

$$\int \frac{1}{2\sec x + \tan x} \, dx = \ln|2 + \sin x| + C.$$

Logarithmic differentiation

Logarithms are powerful in that they 'transform' powers into products, products into sums and quotients into differences.

Example. Find the derivative of

$$y = \left(\frac{(3x^2+4)(x+2)}{x^3+5x} \right)^{3/5}.$$

The idea is to take $\ln$ of both sides of the equation to obtain

$$\begin{aligned} \ln y &= \frac{3}{5} \ln \left(\frac{(3x^2+4)(x+2)}{x^3+5x} \right) \\ &= \frac{3}{5} \left(\ln(3x^2+4) + \ln(x+2) - \ln(x^3+5x) \right). \end{aligned}$$

Differentiating both sides with respect to x is relatively easy and leads to

$$\frac{1}{y}\,\frac{dy}{dx} = \frac{3}{5} \left(\frac{6x}{3x^2+4} + \frac{1}{x+2} - \frac{3x^2+5}{x^3+5x} \right).$$

Hence, we obtain

$$\frac{dy}{dx} = \frac{3}{5}\left(\frac{(3x^2+4)(x+2)}{x^3+5x}\right)^{3/5}\left(\frac{6x}{3x^2+4} + \frac{1}{x+2} - \frac{3x^2+5}{x^3+5x}\right).$$

Remark. The above procedure is only valid for intervals on which $y > 0$.

Example. Determine the derivative of the function

$$f : (0, \pi) \to \mathbb{R}, \quad f(x) = x^{\sin x}.$$

Hence, we obtain

$$\frac{dy}{dx} = \frac{3}{5}\left(\frac{(3x^2+4)(x+2)}{x^3+5x}\right)^{3/5}\left(\frac{6x}{3x^2+4} + \frac{1}{x+2} - \frac{3x^2+5}{x^3+5x}\right).$$

Remark. The above procedure is only valid for intervals on which $y > 0$.

Example. Determine the derivative of the function

$$f : (0, \pi) \to \mathbb{R}, \quad f(x) = x^{\sin x}.$$

Solution. Note first that $y = f(x) > 0$ on the domain. Taking logarithm of $y = x^{\sin x}$, we have

$$\ln y = \sin x \ \ln x.$$

Differentiating both sides of the equality above, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \cos x \ \ln x + \frac{1}{x}\sin x.$$

Multiplying by $y = x^{\sin x}$, we have

$$\frac{dy}{dx} = x^{\sin x}(\cos x \ \ln x + \frac{1}{x}\sin x).$$

Indeterminate forms with powers

Consider the limits

$$\lim_{x \to 0^+} x^x \qquad \text{and} \qquad \lim_{x \to \infty} x^{1/x}.$$

The first limit is of the form 0^0 while the second is of the form ∞^0.

Since each limit involves a power, it is natural to first take the logarithm of the limit and then bring l'Hôpital's rule into play.

Example. Evaluate the limit $\lim_{x \to 0^+} x^{2x}$.

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Example. Evaluate the limit $\lim_{x\to 0^+} x^{2x}$.

Solution. By taking the natural logarithm, we can transform the limit into an indeterminate form of the type $\frac{\infty}{\infty}$:

$$\begin{aligned}
\lim_{x\to 0^+} x^{2x} &= \lim_{x\to 0^+} \exp\left(\ln x^{2x}\right) && \text{(since } \ln \text{ and } \exp \text{ are inverses)}\\
&= \lim_{x\to 0^+} \exp\left(2x\ln x\right)\\
&= \exp\left(\lim_{x\to 0^+} 2x\ln x\right) && \text{(since } \exp \text{ is continuous)}\\
&= \exp\left(\lim_{x\to 0^+} \frac{\ln x}{1/(2x)}\right).
\end{aligned}$$

We can now apply l'Hôpital's rule to the problem. By differentiating the numerator and denominator and then simplifying we obtain

$$\lim_{x\to 0^+} x^{2x} = \exp\left(\lim_{x\to 0^+} \frac{1/x}{-1/(2x^2)}\right) = \exp\left(\lim_{x\to 0^+}(-2x)\right) = \exp(0) = 1.$$

The following example is of the indeterminate form "1^∞".

Example. Show that

$$\lim_{x\to\infty}\left(1+\frac{t}{x}\right)^x = e^t,$$

where t is a constant real parameter.

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where t is a constant real parameter.

Solution. Since $\ln$ is continuous function, we have

$$\exp\ln\left(\lim_{x\to\infty}\left(1+\frac{t}{x}\right)^x\right) = \exp\lim_{x\to\infty}\ln\left(1+\frac{t}{x}\right)^x = \exp\left(\lim_{x\to\infty} x\ln\left(1+\frac{t}{x}\right)\right) = ...$$

dividing by x, we obtain

$$... = \exp\left(\lim_{x\to\infty}\frac{\ln\left(1+\frac{t}{x}\right)}{\frac{1}{x}}\right) = \exp\left(,,\frac{0}{0},,\right) = ...$$

using L'Hopital's rule

$$... = \exp\left(\lim_{x\to\infty}\frac{\frac{1}{\left(1+\frac{t}{x}\right)}\left(-\frac{t}{x^2}\right)}{-\frac{1}{x^2}}\right) = \exp\left(\lim_{x\to\infty}\frac{t}{\left(1+\frac{t}{x}\right)}\right) = \exp t = e^t.$$

Remark.

Note:

'e^x grows faster than any positive power of x as $x \to \infty$' or

'e^{-x} decays to zero faster than any positive power of x can grow as $x \to \infty$'

Example.

Prove that

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} x^2 e^{-x} = 0.$$

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Indeterminate form $\dfrac{\infty}{\infty}$. Use l'Hôpital's rule repeatedly:

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0.$$