THE UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1131 Calculus

Section 7: - Curve Sketching.

Although one can now sketch curves on the computer, we still need to be able to give rough sketches of functions by hand.

In this section we will list a number of basic features one should look for when sketching the graph of a function.

Checklist:

A. Domain and Range

B. x and y intercepts

C. Symmetries (even, odd, periodic)

D. Horizontal and vertical asymptotes

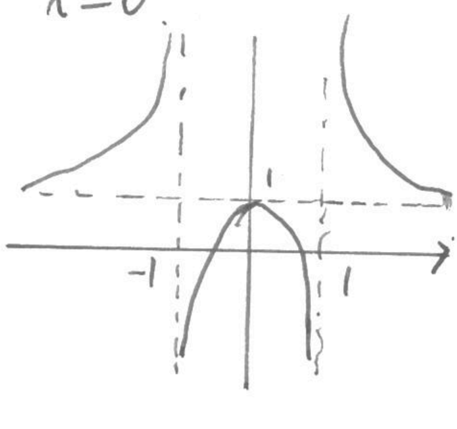
E. Oblique asymptotes and asymptotic behaviour

F. Stationary points and inflections of various types using Calculus.

Notice that I have placed the use of Calculus last!

Ex: Sketch $y = \frac{x^2}{x^2-1}$.

Domain : $\mathbb{R} - \{\pm 1\}$, Range : $\mathbb{R} \setminus [0, 1]$.
x - intercept. $y=0$. y-intercept, none.
even function.
asymptotes, $x=\pm 1$, $y=1$,
Stationary pt. at $x=0$



Ex: Sketch $y = x^2 e^x$.

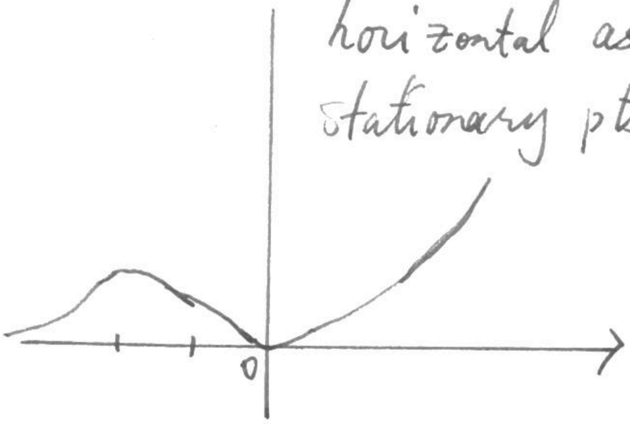
domain : $\mathbb{R}$, range : $[0, \infty)$.

x, y intercept at the origin.

horizontal asymptotes. $y = 0$.

stationary pts. $e^x(x^2+2x) = 0$ or

$x = 0$ and $x = -2$.



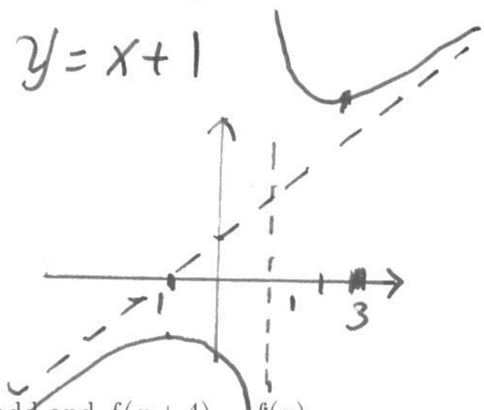
Ex: Sketch $y = \frac{x^2+3}{x-1}$ (Notice that this has oblique asymptote $y = x + 1$.)

domain $\mathbb{R} \setminus \{1\}$.

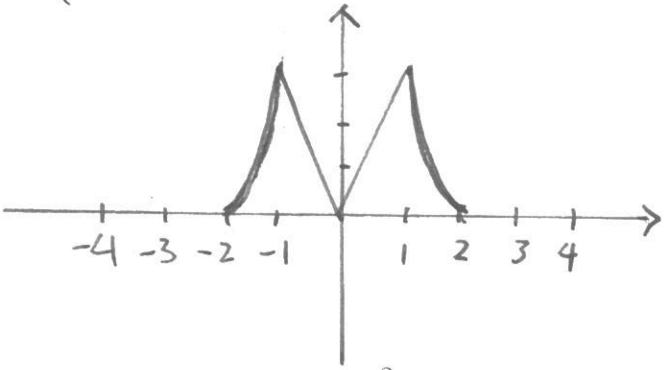
$y = x + 1 + \frac{4}{x-1}$

$y' = 1 - \frac{4}{(x-1)^2}$. So $y' = 0$ if $x = 3, -1$.

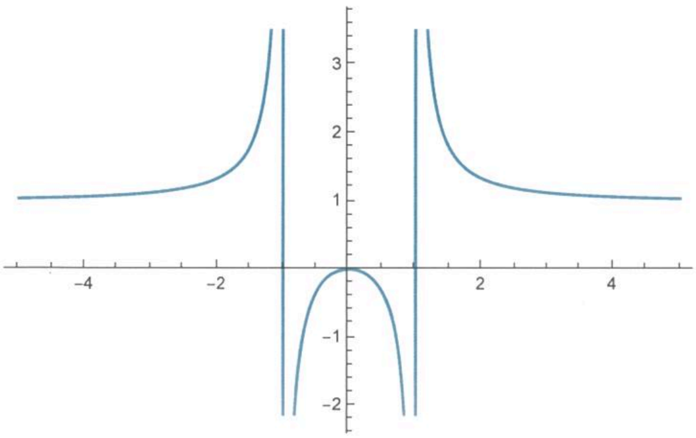
oblique asymptote $y = x + 1$



Ex: Sketch $f(x) = \begin{cases} 3x & 0 \le x < 1 \\ 4 - x^2 & 1 \le x < 2 \end{cases}$, with f odd and $f(x+4) = f(x)$.

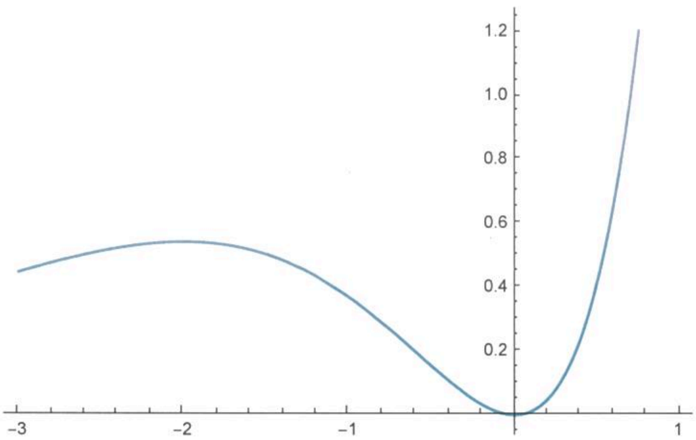


```
Plot[x^2 / (x^2 - 1), {x, -5, 5}]
```



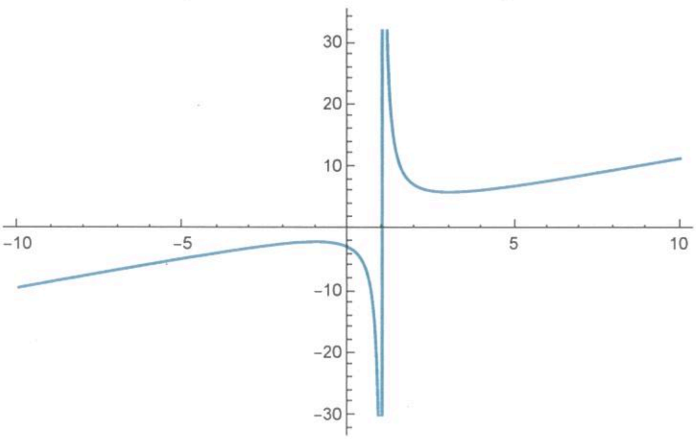
$$y = \frac{x^2}{x^2-1}$$

```
Plot[x^2 * Exp[x], {x, -3, 1}]
```



$$y = x^2 e^x$$

```
Plot[(x^2 + 3) / (x - 1), {x, -10, 10}]
```



$$y = \frac{x^2+3}{x-1}$$

Implicitly Defined Curves:

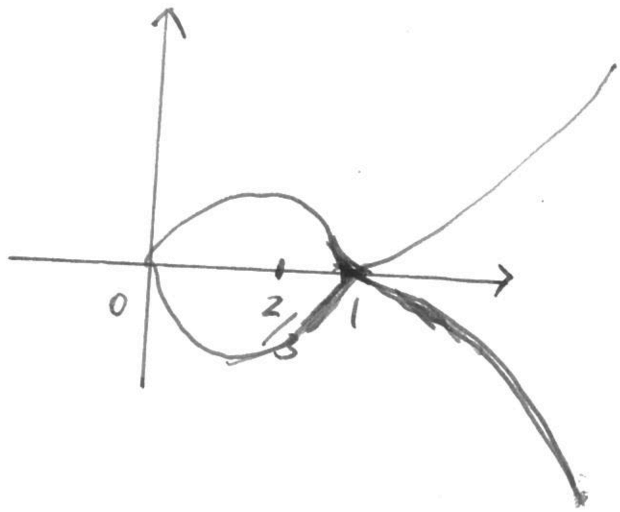
Ex: Sketch $y^2 = x(x-1)^2$.

The graph is symmetric about the x-axis and $x \geq 0$.

So it suffices to have the graph for $y \geqslant 0$.

$y = \sqrt{x}(x-1)$. $y' = \frac{3}{2} x^{\frac{1}{2}} - \frac{1}{\sqrt{x}}$.

Stationary pt at $x = \frac{2}{3}$



Parametric and Polar-Co-ordinates.

Given a function $y = f(x)$, then a **parametrization** of this function is a way of splitting the variables x and y into two separate equations which are linked by a new variable called a **parameter**.

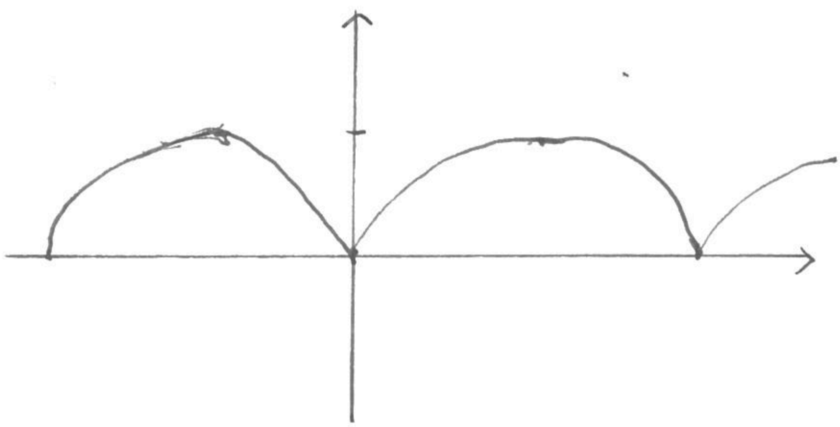
There are many reasons for wanting to do this. In the study of projectile motion, it is easier to look at the horizontal motion and vertical motion separately, despite the fact that they are clearly linked. The parameter in question is time t.

You have also seen the standard parametrization of the circle $x^2 + y^2 = r^2$, which is $x = r\cos\theta, y = r\sin\theta$, where the parameter θ measures the angle with the positive x axis made by a ray from the centre to the circumference of the circle.

In addition, you will have studied the standard parametrization of the parabola $x^2 = 4ay$, which is $x = 2at, y = at^2$, where the parameter t gives the derivative $\frac{dy}{dx}$.

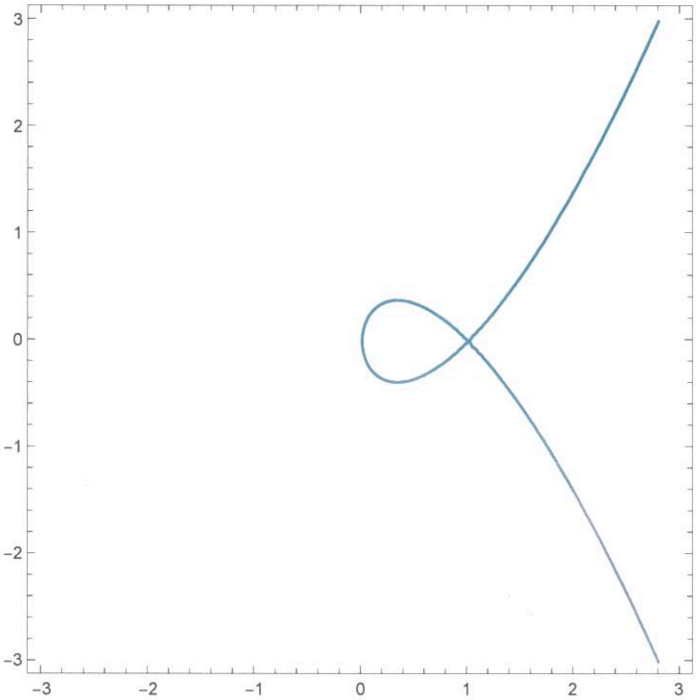
One can always trivially parametrize a curve of the form $y = f(x)$ by simply writing $x = t, y = f(t)$. Conversely, however, there are many parametrically defined curves which cannot easily be expressed explicitly in the form $y = f(x)$. For example, $x = at - a\sin t, y = a + at\cos t$.

Ex: Sketch $x = \theta - \sin\theta, y = 1 - \cos\theta$.



Ex: Sketch $\begin{cases} x = \cos 3t \\ y = \sin 2t \end{cases}$ using MAPLE.

```
ContourPlot[y^2 == x * (x - 1)^2, {x, -3, 3}, {y, -3, 3}]
```

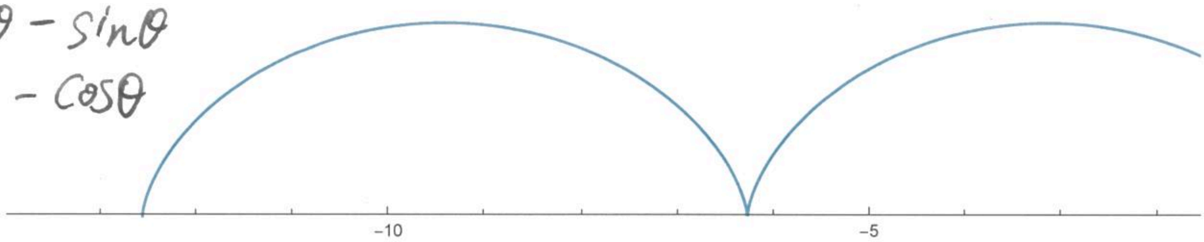


$y^2 = x(x-1)^2$

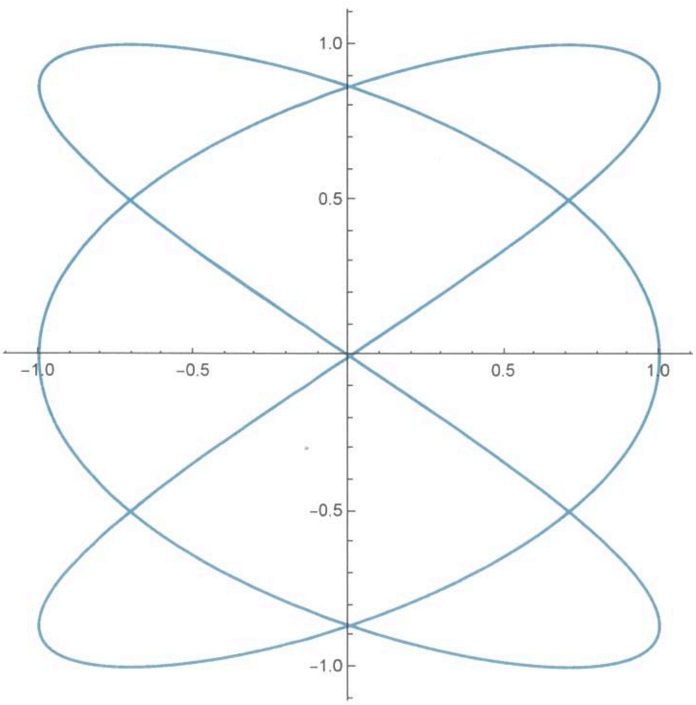
```
ParametricPlot[{u - Sin[u], 1 - Cos[ u]}, {u, -4 Pi, 4 Pi}]
```

$x = \theta - \sin\theta$

$y = 1 - \cos\theta$




```
ParametricPlot[{Cos[3 u], Sin[2 u]}, {u, -2 Pi, 2 Pi}]
```



$x = \cos 3t$

$y = \sin 3t$

To find the derivative $\frac{dy}{dx}$ for such curves we use the chain rule:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Ex: Find $\frac{dy}{dx}$ for $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$.

$$\frac{dy}{dt} = \cos t, \quad \frac{dx}{dt} = -\sin t$$

$$\frac{dy}{dx} = \frac{\cos t}{-\sin t} = -\cot t.$$

Ex: Find $\frac{dy}{dx}$ at the point $t = 1$ for the curve $x = \frac{3t}{1+t^3}, y = \frac{3t^2}{1+t^3}$. (This is the *Folium of Diocles/Descartes* and has cartesian eqn. $x^3 + y^3 = 3xy$.)

$$\frac{dx}{dt} = \frac{3(1+t^3) - 3t \cdot 3t^2}{(1+t^3)^2} = \frac{3 - 6t^3}{(1+t^3)^2}$$

$$\frac{dy}{dt} = \frac{6t(1+t^3) - 3t^2 \cdot 3t^2}{(1+t^3)^2} = \frac{6t - 3t^4}{(1+t^3)^2}$$

$$\frac{dy}{dx} = \frac{6t - 3t^4}{3 - 6t^3}.$$

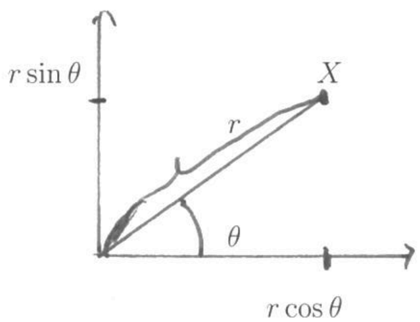
If $t = 1$, then

$$\frac{dy}{dx} = \frac{6-3}{3-6} = -1$$

Polar Co-ordinates:

As mentioned above, there is a standard way to parametrize the circle, using polar co-ordinates. We can generalise this slightly and arrive at a new co-ordinate system for describing every point in the plane.

Given a point $X(x, y) \neq (0, 0)$, then we measure its distance r from the origin and the angle θ it makes with the positive x axis. Knowing these two quantities, the point X is uniquely determined. The ordered pair (r, θ) are called the **polar co-ordinates** of the point X.



Using simple trigonometry and Pythagoras' theorem, we can write down the equations relating the two co-ordinate systems.

$$x = r\cos\theta, \quad y = r\sin\theta,$$

and

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}.$$

Note that the statement $\tan\theta = \frac{y}{x}$ needs to be used **carefully** and a diagram should be drawn!

NB. r is always positive, and in the case $r = 0$ then θ is not defined. The point $(0, 0)$ in Cartesian co-ordinates, is thus special. We take (by convention) $-\pi < \theta \leq \pi$.

Ex: a. Convert $(1, -1)$ to polar co-ordinates.
b. Convert $(3, \frac{5\pi}{6})$ to Cartesian co-ordinates.

a) $(1, -1)$ is $r = \sqrt{2}$, $\theta = \frac{3\pi}{2}$.

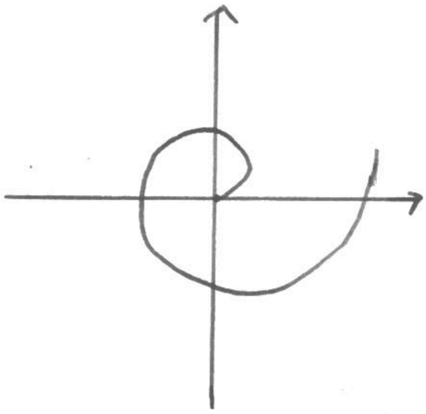
b) $(3, \frac{5\pi}{6}) = (3\cos\frac{5\pi}{6}, 3\sin\frac{5\pi}{6}) = (-\frac{3}{2}, \frac{3\sqrt{3}}{2})$.

Graphing Polar Curves:

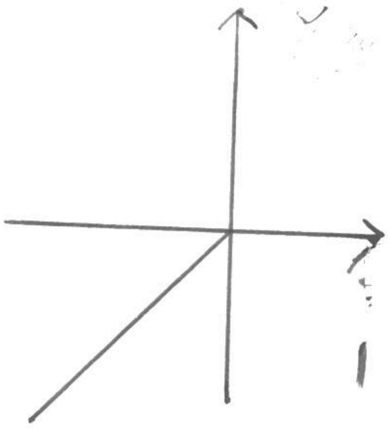
In this section we are going to learn how to graph curves which are given in polar form $r = f(\theta)$.

Here is a simple way to visualise what is happening.
Imagine an archer standing at the origin shooting arrows. The archer faces a certain direction (measured from the positive x axis), θ and shoots an arrow a distance r. As the archer moves through all possible values of θ and we place a marker at where the arrow lands. The locus of the marker is a curve.

Ex: $r = 2\theta$



Ex: $\theta = -\frac{3\pi}{4}$.



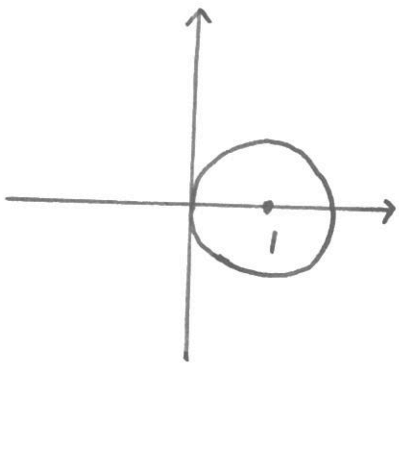
Ex: $r = 2\cos\theta$.

$\frac{r}{x} = \cos\theta$, so $r = 2 \cdot \frac{x}{r}$ or $r^2 = 2x$

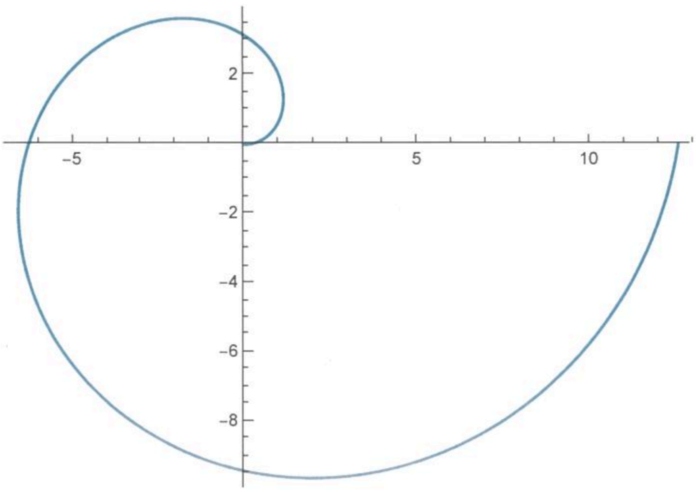
Hence $x^2 + y^2 - 2x = 0$

$x^2 - 2x + 1 + y^2 = 1$

$(x-1)^2 + y^2 = 1.$

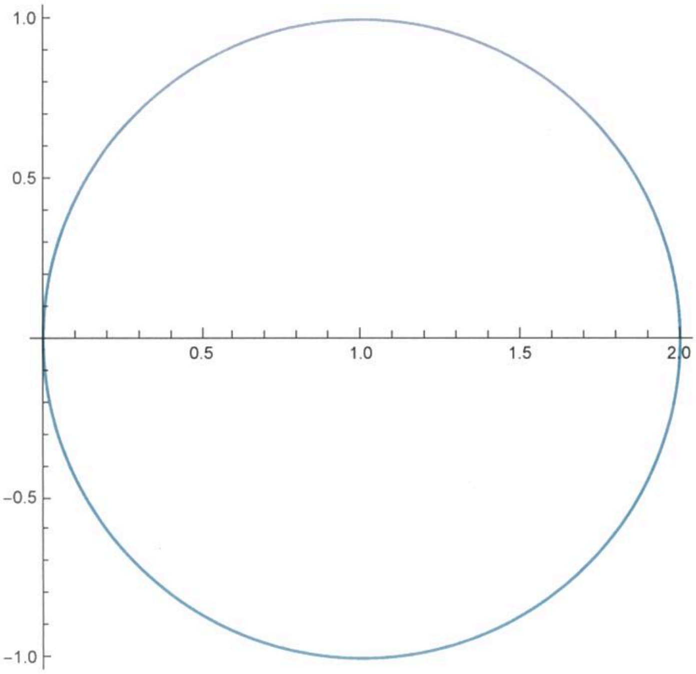


```
PolarPlot[2 * t, {t, 0, 2 * Pi}]
```



$r = 2\theta$

```
PolarPlot[2 * Cos[ t], {t, 0, Pi}]
```



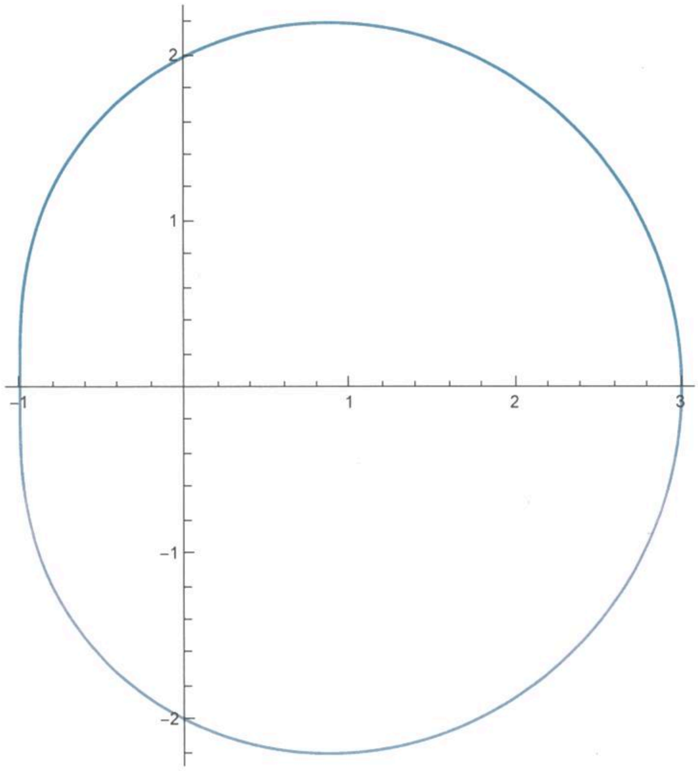
$r = 2\cos\theta$

Ex: $r = 2 + \cos\theta$. (This is a limacon, from the French for 'little snail').

Ex: $r = 2|\cos(4\theta)|$.

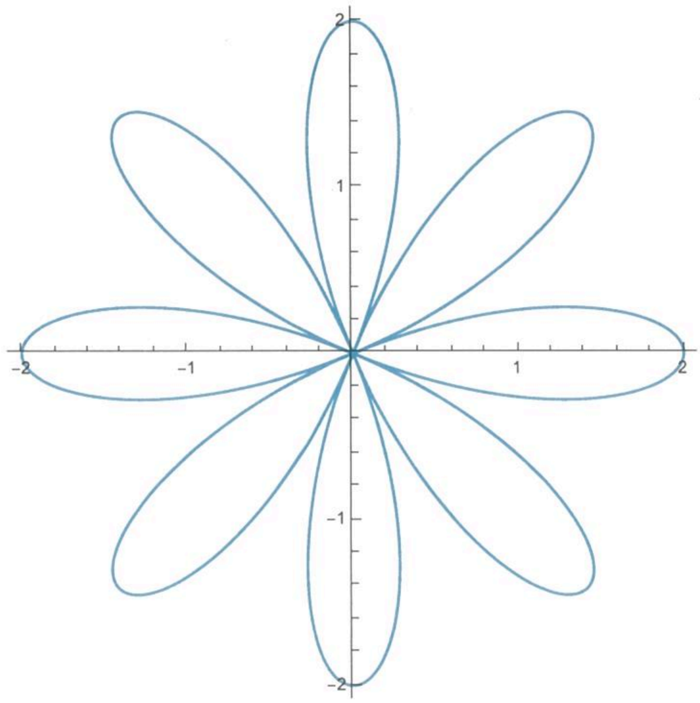
Ex: $r^2 = 4\sin(2\theta)$. (This is a lemniscate, from the Latin *lemniscatus* -ribbon shaped.)

```
PolarPlot[2 + Cos[ t], {t, 0, 2 * Pi}]
```



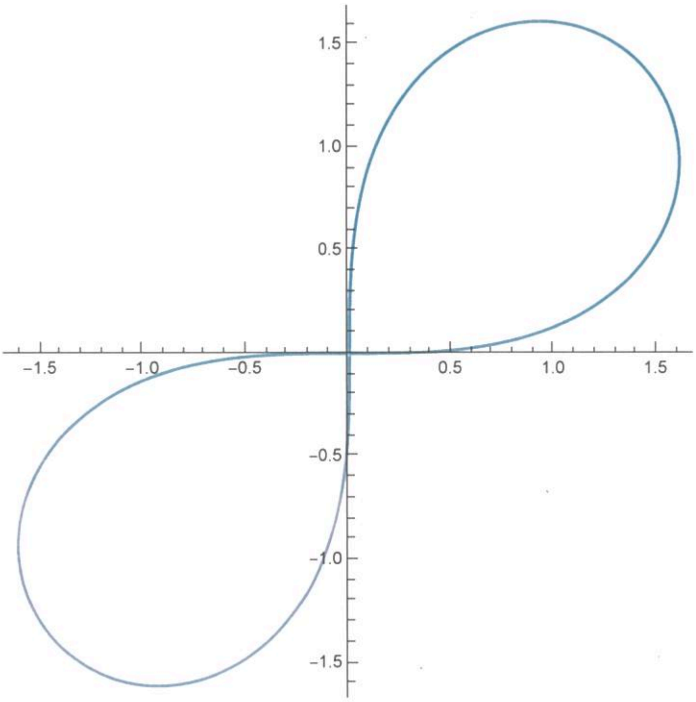
$r = 2 + \cos\theta$

```
PolarPlot[2 * Abs[Cos[4 t]], {t, 0, 8 * Pi}]
```



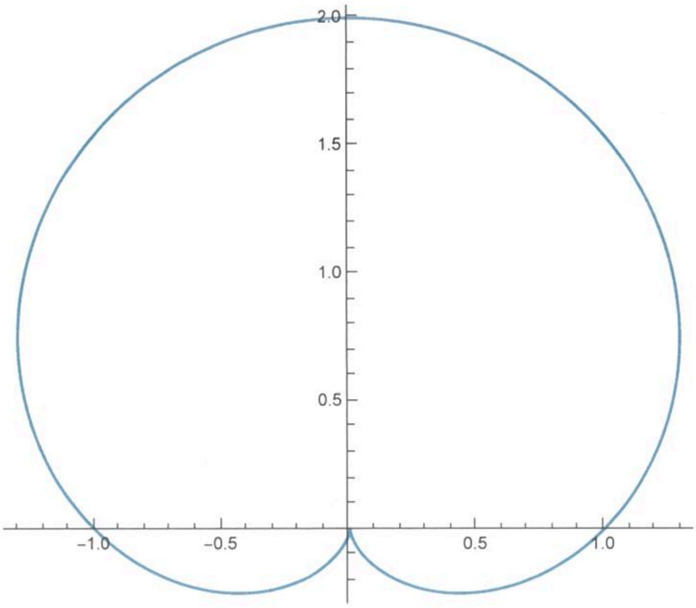
$r = 2|\cos(4\theta)|$

```
PolarPlot[2 * Sqrt[Sin[2 t]], {t, 0, 4 * Pi}]
```



$r^2 = 4\sin(2\theta)$

```
PolarPlot[1 + Sin[t], {t, 0, 2 * Pi}]
```



The Tangent to a Polar Curve:

Theorem: If $x = r(\theta)\cos\theta, y = r(\theta)\sin\theta$ are differentiable functions then

$$\frac{dy}{dx} = \frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta}.$$

proof: $\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$

$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$

The formula follows from this. $\boxed{}$

Ex: Find where the tangent to the polar curve $r = 1 + \sin\theta$ is horizontal and where it is vertical. What is happening at $\theta = 3\pi/2$?

$$\frac{dr}{d\theta} = \cos\theta. \quad \text{So}$$

$$\frac{dy}{dx} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta}$$

$$= \frac{\cos\theta\,(1+2\sin\theta)}{1-2\sin^2\theta - \sin\theta}$$

$$= \frac{\cos\theta\,(1+2\sin\theta)}{-2(\sin\theta+1)(\sin\theta-\frac{1}{2})}$$

The possible horizontal tangent lines are at

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}.$$

Possible vertical tangent lines are at

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}.$$

$\frac{3\pi}{2}$ appears in both lists and hence deserves more attention.

$$\lim_{x\to\frac{3\pi}{2}} \frac{dy}{dx} = \lim_{x\to\frac{3\pi}{2}} \frac{1+2\sin\theta}{-2(\sin\theta-\frac{1}{2})} \lim_{x\to\frac{3\pi}{2}} \frac{\cos\theta}{\sin\theta+1}$$

$$\overset{L'H}{=} -\frac{1}{3} \lim_{x\to\frac{3\pi}{2}} \frac{\sin\theta}{-\cos\theta}.$$

This last limit is $\pm\infty$, depending if we are looking at the left-hand or right-hand limit. So we have a vertical tangent line at $\frac{3\pi}{2}$.