THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 1 2018

MATH1131
MATHEMATICS 1A

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS IS APPENDED TO THE PAPER

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 1

1. i) Calculate each of the following limits, or explain why it doesn't exist.

a) $\lim_{x\to\infty} \dfrac{3x^2 + \sin x^3}{x^2 + 2x + 1}$

b) $\lim_{x\to 0} \dfrac{e^x - 1}{x}$

c) $\lim_{x\to \ln 2} \dfrac{e^x - 2}{x}$

ii) Find the following integrals.

a) $I_1 = \displaystyle\int x \cos x \, dx$

b) $I_2 = \displaystyle\int \frac{1}{x(1 + \ln x)} \, dx$

iii) Consider the curve defined implicitly by

$$x^2y - xy^3 - x^2 = -2.$$

a) Calculate the value of $\dfrac{dy}{dx}$ at the point $(2, 1)$.

b) Hence write down a Cartesian equation of the tangent line at the point $(2, 1)$.

c) Express your tangent line from part (b) in parametric vector form.

iv) Let α be a positive real number. Write the complex number $-\alpha + \alpha i$ in polar form.

v) a) Solve $z^6 + 1 = 0$.

b) Factorise $z^6 + 1$ into linear factors with complex coefficients.

c) Factorise $z^6 + 1$ into linear or irreducible quadratic factors with real coefficients.

d) Factorise $z^6 + 1$ into irreducible factors with rational coefficients.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 2

2. i) Let $p : [0, \infty) \to \mathbb{R}$ be the function defined by

$$p(x) = \ln\left(x + \frac{1}{9}\right) + e^{-x}.$$

a) Show that p has at least one root in the interval $[0, 1]$.

b) Show that p has exactly one root in the interval $[0, 1]$.

c) Explain how your arguments in parts (a) and (b) can be extended to determine the number of roots of p in the interval $[0, \infty)$.

ii) a) Write down the formula for $\sinh(x)$ and $\cosh(x)$ in terms of exponentials.

b) Hence, prove that for $n \in \mathbb{Z}$ and $x \in \mathbb{R}$

$$\big(\cosh(x) + \sinh(x)\big)^n = \cosh(nx) + \sinh(nx)$$

iii) Sketch the graph of the polar curve

$$r = 1 + \cos(3\theta)$$

for $0 \leq \theta < 2\pi$.

iv) A function $f : [1, 5] \to \mathbb{R}$ has the following properties

- f has a global maximum at 3,
- f is continuous everywhere except 4,
- f has no global minimum.

Draw a sketch of the graph of a possible f.
(You do not need to give a formula for your function.)

v) Let

$$\mathbf{w} = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

You are given that the following vectors form an orthonormal set.
(You do not need to prove this.)

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix}, \quad \mathbf{v}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\1\end{pmatrix}, \quad \mathbf{v}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix}2\\1\\1\end{pmatrix}.$$

a) Calculate $\mathbf{v}_1 \cdot \mathbf{w}$.

b) Express $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$.

Please see over ...

vi) Consider the lines ℓ_1 and ℓ_2 in $\mathbb{R}^3$ defined below.

$$\ell_1 : \qquad \mathbf{x} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

$$\ell_2 : \qquad x_1 = 4, \quad \frac{x_2 - 4}{2} = \frac{x_3 + 1}{3}$$

a) Show that the lines ℓ_1 and ℓ_2 intersect.

b) Explain the Maple code below and how it is used to find the shortest distance from the point $P(1, 2, 3)$ to the plane containing the lines ℓ_1 and ℓ_2.

```
> with(LinearAlgebra):
> AP := <2,0,1> - <1,2,3>;
```

$$\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$$

```
> v1 := <1,2,-1>;
```

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

```
> v2 := <0,2,3>;
```

$$\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

```
> n := CrossProduct(v1,v2);
```

$$\begin{bmatrix} 8 \\ -3 \\ 2 \end{bmatrix}$$

```
> l := abs(n.AP)/sqrt(n.n);
```

$$l := \frac{10\sqrt{77}}{77}$$

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 3

3. i) Consider the points A, B, C, D and E with coordinate vectors

$$\mathbf{a} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

The plane containing A, B, C and D can be expressed in parametric vector form as

$$\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

a) Show that $ABCD$ is a parallelogram.

b) Describe the values of λ_1 and λ_2 in the parametric form of the plane given above that correspond to the edge CD of the parallelogram $ABCD$.

c) Compute the area of the parallelogram $ABCD$.

d) Compute the volume of the parallelepiped formed by the vectors $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{AE}$.

e) Does the point E lie in the plane containing the parallelogram $ABCD$? Give reasons for your answer.

ii) Suppose that z and w are complex numbers.

a) Show that $(z + \bar{w})(\bar{z} + w)$ is real.

b) What is the geometric relationship between $z + \bar{w}$ and $\bar{z} + w$?

iii) Consider the following system of equations.

$$\begin{aligned} x + 2y - z &= 3 \\ 2x - \lambda y - 2z &= 0 \\ x + 3y + \lambda z &= 5 \end{aligned}$$

a) For which values of λ does the system have no solution, a unique solution or infinitely many solutions.

b) For the value or values of λ for which the system has infinitely many solutions, write down the solutions in vector parametric form.

iv) Given that the invertible $n \times n$ matrix A satisfies

$$A^2 = 2A + I,$$

express the inverse of A in terms of A and I.

Please see over ...

v) Using the Maple code below, or otherwise, find the determinant of

$$B = \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 2 & 1 & -1 \\ 2 & 4 & 2 & -2 \\ 3 & 0 & 1 & 0 \end{pmatrix}.$$

```
> with(LinearAlgebra):
> B := <<1,0,2,3>|<2,2,4,0>|<1,1,2,1>|<4,-1,-2,0>>;
```

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 2 & 1 & -1 \\ 2 & 4 & 2 & -2 \\ 3 & 0 & 1 & 0 \end{bmatrix}$$

```
> RowOperation(B,[3,1],-2);
```

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & -10 \\ 3 & 0 & 1 & 0 \end{bmatrix}$$

```
> RowOperation(%,[4,1],-3);
```

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & -10 \\ 0 & -6 & -2 & -12 \end{bmatrix}$$

```
> RowOperation(%,[4,2],3);
```

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & -10 \\ 0 & 0 & 1 & -15 \end{bmatrix}$$

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 4

4. i) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \int_0^{x^4} \sin(t^3 + 1)dt.$$

(Do NOT try to evaluate this integral.)

a) Explain why f is an even function.

b) Find $f'(x)$.

ii) Find all real values of a and b such that the function defined by,

$$f(x) = \begin{cases} x^2 + bx & \text{if } x < a \\ 2x & \text{if } x \geq a \end{cases}$$

is differentiable for all $\mathbb{R}$.

iii) a) State the Mean Value Theorem.

b) Prove that, for all $x \geq 0$,

$$1 - x \leq e^{-x}.$$

iv) The function f is defined by

$$f : (0, 2) \to \mathbb{R} \quad \text{where} \quad f(x) = e^{-x}(1 - x).$$

a) Explain why f has an inverse $g = f^{-1}$.

b) Find the domain and range of g.

c) Evaluate $g'(0)$.

v) Do the following improper integrals converge or diverge? Give reasons for your answer.

a) $\displaystyle\int_0^{\infty} \frac{2}{x^2 + 4}\, dx$

b) $\displaystyle\int_1^{\infty} \frac{x^4}{\ln x}\, dx$

vi) Use the ϵ-definition of the limit to show that

$$\lim_{x \to \infty} (2 + e^{-x(\sin(x)+2)} \sin(x)\cos(x)) = 2.$$

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BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k$$
$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$
$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$
$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$
$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$
$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$
$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$
$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$
$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$
$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$
$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$
$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$
$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$
$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$
$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$
$$\begin{aligned}\int \frac{dx}{a^2-x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, \qquad |x| < a\\ &= \frac{1}{a}\coth^{-1}\frac{x}{a} + C, \qquad |x| > a > 0\\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, \qquad x^2 \neq a^2\end{aligned}$$
$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$
$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + C$$
$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$

END OF EXAMINATION

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