MATH1131 Mathematics 1A
and
MATH1141 Higher Mathematics 1A

PAST EXAM PAPERS
AND
SOLUTIONS

Contents

The 2017 to 2019 exam papers will be provided on Moodle.

PAST EXAM PAPERS

Please note that the emphasis of the exam may change from year to year and that it is not possible to test all aspects of the course in a 2-hour examination. Hence, these papers are only a guide as to the style and level of difficulty of our first year examinations.

The exam format may be different in 2020 — check Moodle for details.

The solutions to the examination papers contained here have been written by many members of staff of the School of Mathematics and Statistics. While every care is taken to excluded errors, we cannot guarantee that the solutions are error-free. Please report any serious errors to the Director of First Year Mathematics.

Exam papers from 2019 will be provided on Moodle for practice before the exam. Students are encouraged to produce their own solutions to the 2019 exam papers and discuss these on Moodle.

MATH1131 November 2010

1. i) Let $z = 1 + i$.
   a) Find $|z|$.
   b) Find $\mathrm{Arg}(z)$.
   c) Use the polar form of z to evaluate z^{28} and then express your answer in **Cartesian form**.

   ii) Suppose that $(x + iy)(3 + 2i) = (4 + 7i)$ where $x, y \in \mathbb{R}$.
   Find the value of x and the value of y.

   iii) a) Find all solutions to the equation $z^6 = 1$, where $z \in \mathbb{C}$.
   b) Hence, or otherwise, express $z^6 - 1$ as a product of three quadratic polynomials with real coefficients.

   iv) Let the region S be defined as

$$S = \{z \in \mathbb{C} : |z + i| = 1\}.$$

   Sketch the region S on a carefully labelled Argand diagram.

   v) Consider the following MAPLE session.

```
> with(LinearAlgebra):
> A:=<<1,0,-sqrt(3)>|<0,2,0>|<sqrt(3),0,1>>;
```

$$A := \begin{bmatrix} 1 & 0 & \sqrt{3} \\ 0 & 2 & 0 \\ -\sqrt{3} & 0 & 1 \end{bmatrix}$$

```
> A^2;
```

$$\begin{bmatrix} -2 & 0 & 2\sqrt{3} \\ 0 & 4 & 0 \\ -2\sqrt{3} & 0 & -2 \end{bmatrix}$$

```
> A^4;
```

$$\begin{bmatrix} -8 & 0 & -8\sqrt{3} \\ 0 & 16 & 0 \\ 8\sqrt{3} & 0 & -8 \end{bmatrix}$$

```
> A^6;
```

$$\begin{bmatrix} 64 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 64 \end{bmatrix}$$

   Use the above Maple session to find the inverse of the matrix A^2.

vi) A car dealer sells three brands of cars: Audis, BMWs and Chevrolets. Her costs consist of registration, GST and the manufacturer's price of the car. She has to pay registration costs of \$200 per Audi, \$600 per BMW and \$400 per Chevrolet. The GST charges are $\$1,800$ per Audi, $\$2,400$ per BMW and $\$2,000$ per Chevrolet. To the manufacturer she pays $\$20,000$ per Audi, $\$30,000$ per BMW and $\$26,000$ per Chevrolet. Last year her total registration bill was $\$12,000$, the total GST tax bill was $\$65,600$ and she paid a total of \$784,000 to manufacturers.

Let a, b and c denote the number of Audis, BMWs and Chevrolets she sold last year.

a) Write down a system of linear equations in a, b and c determined by the above information.

b) Convert the system of equations to an augmented matrix form and hence find a, b and c by performing Gaussian elimination on the augmented matrix.

2. i) Let $P = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \end{pmatrix}$.

a) Evaluate the matrix product PP^T.

b) State the size of the matrix $P^T P$.

ii) Let A, B and C be points in $\mathbb{R}^3$ with position vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} \quad \text{respectively.}$$

a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b) Find a parametric vector equation of the plane that passes through the points A, B and C.

iii) Determine the coordinates of the point of intersection of the line

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{for} \quad t \in \mathbb{R}$$

and the plane $x - 3y + z = 15$.

iv) Let $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 5 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -2 \\ -6 \\ \beta \\ -10 \end{pmatrix}$ be two vectors in $\mathbb{R}^4$.

Find the value of β so that the vectors $\mathbf{v}$ and $\mathbf{w}$ are perpendicular.

v) Let $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}$ be a vector in $\mathbb{R}^3$.

a) Find the magnitude of $\mathbf{u}$.

b) Write down a vector with a magnitude of 10 which is parallel to $\mathbf{u}$.

vi) Find the determinant of the matrix

$$C = \begin{pmatrix} 3 & 1 & 0 \\ 4 & 1 & 7 \\ 1 & 2 & 0 \end{pmatrix}.$$

vii) Suppose that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are distinct non-zero vectors in $\mathbb{R}^n$ with the property that

$$\text{proj}_{\mathbf{w}}(\mathbf{u}) = \text{proj}_{\mathbf{w}}(\mathbf{v}).$$

Prove that $(\mathbf{u} - \mathbf{v})$ is perpendicular to $\mathbf{w}$.

3. i) Evaluate the limit

$$\lim_{x \to 0} \frac{x^2 e^x}{1 - \cos(\pi x)}.$$

ii) Determine all real values of a and b such that the function f given by

$$f(x) = \begin{cases} x^2 & \text{for } x \leq 1, \\ -x^2 + ax + b & \text{for } x > 1, \end{cases}$$

is differentiable at $x = 1$.

iii) Let

$$L = \lim_{x \to \infty} \frac{x^2 - 2}{x^2 + 1}.$$

a) Evaluate L.

b) Given any $\varepsilon > 0$, find a constant M (which may depend on ε) such that we have

$$\left| \frac{x^2 - 2}{x^2 + 1} - L \right| < \varepsilon$$

whenever $x > M$.

iv) Let $f(x) = x^3 + \sqrt{3}\, x - 5$ for all real x.

a) Use the Intermediate Value Theorem to prove that f has at least one positive real root.

b) By considering f', or otherwise, show that f has only one real root.

v) Use logarithmic differentiation to calculate $\dfrac{dy}{dx}$ for $y = (\sin x)^x$.

vi) A curve in $\mathbb{R}^2$ is given in polar coordinates as

$$r = 6 \sin \theta, \quad \text{where } 0 \leq \theta \leq \pi/2.$$

a) Express the equation of the curve using Cartesian coordinates x and y and state the range of x and the range of y.

b) Hence, or otherwise, sketch the curve in the xy-plane.

4. i) Evaluate the following integrals:

a) $I_1 = \displaystyle\int \frac{1-x}{(1+x)^3}\,dx;$

b) $I_2 = \displaystyle\int_0^{\pi} x\cos 2x\;dx.$

ii) Determine whether the improper integral

$$K = \int_1^{\infty} \frac{1+\sin x}{3x^2}\,dx$$

converges or diverges.

iii) a) State the definitions of $\sinh x$ and $\cosh x$ in terms of the exponential function.

b) Hence find an expression for $\dfrac{d}{dx}\cosh(ax)$ in terms of $\sinh(ax)$, where a is a constant.

c) Simplify the expression $\cosh^{-1}\big(\cosh(-4726)\big)$.

iv) A continuous function f satisfies the equation

$$\int_0^x f(t)\,dt = \int_x^1 t^2 f(t)\,dt + \frac{x^{16}}{8} + \frac{x^{18}}{9} - \frac{1}{9}.$$

By differentiating this equation with respect to x, and using the First Fundamental Theorem of Calculus where needed, find $f(x)$.

v) Let g be a function defined by

$$g(x) = \tan^{-1}(x) + \tan^{-1}(1/x).$$

a) What is the (maximal) domain of g? Give reasons for your answer.

b) By examining $g'(x)$, show that g is piecewise constant on its domain.

c) Hence, or otherwise, determine the exact value of

$$\tan^{-1}(-e^{\alpha}) + \tan^{-1}(-e^{-\alpha}), \quad \text{for all } \alpha \in \mathbb{R}.$$

MATH1131 JUNE 2011

1. i) Let $z = -1 - i$.
   a) Find $|z|$.
   b) Find $\text{Arg}(z)$.
   c) Use the polar form of z to evaluate z^{102} and then express your answer in **Cartesian form**.

   ii) a) Simplify $(2 + 4i)^2$.
   b) Hence, or otherwise, solve the quadratic equation $z^2 - 4z + (7 - 4i) = 0$.

   iii) Sketch the following region on the Argand diagram

$$S = \{z \in \mathbb{C} : 0 \leq \text{Arg}(z - i) \leq \frac{\pi}{4}\}.$$

   iv) Evaluate the limit

$$\lim_{x\to\infty} \frac{1}{x - \sqrt{x^2 - 6x - 4}} \ .$$

   v) Evaluate the improper integral

$$\int_1^\infty x^{-5/4}\, dx \ .$$

   vi) A curve in the plane is defined implicitly by the equation

$$x^2 - 3xy^2 + 11 = 0 \ .$$

   a) Show that the curve has slope at the point (x, y) given by

$$\frac{dy}{dx} = \frac{2x - 3y^2}{6xy} \ .$$

   b) Find the equation of the tangent to the curve at the point $(1, 2)$.
   c) Write a Maple command to plot the curve in the region
   $1 \leq x \leq 4$ and $-5 \leq y \leq 5$.

2. i) Use De Moivre's Theorem to prove that

$$\cos(4\theta) = 8\cos^4\theta - 8\cos^2\theta + 1.$$

   ii) Consider the line ℓ with parametric vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} t, \quad \text{for} \quad t \in \mathbb{R}.$$

   a) Give two points on the line.
   b) Give a vector parallel to the line.
   c) Explain why the line ℓ is perpendicular to the plane P with Cartesian equation

$$9x + 6y + 15z = 24.$$

d) Find a point on the line whose y-coordinate is 0.

iii) Consider the following Maple session:

```
> with(LinearAlgebra):
> A:=<<1,1,0>|<-1,1,0>|<0,0,sqrt(2)>>;
```

$$A := \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

```
> A^2;
```

$$\begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

```
> A^4;
```

$$\begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

```
> A^8;
```

$$\begin{bmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

Use the above MAPLE session to find the inverse of A^7.

iv) Evaluate the limit

$$\lim_{x\to 1} \frac{(x-1)^2}{1+\cos(\pi x)} .$$

v) Evaluate the indefinite integral

$$\int x \sin(2x)\, dx.$$

vi) The function f has domain $[0, 1]$ and is defined by $f(x) = e^x + ax$, where a is a positive constant.

a) Prove that 2 is in the range of f.

b) Prove that f has an inverse function f^{-1}.

c) Find the domain of f^{-1}.

3. i) Let $A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ 0 & 5 \\ 1 & 0 \end{pmatrix}$.

a) Find AB.

b) What is the size of the matrix BA?

ii) A pet shop has x hamsters, y rabbits and z guinea pigs.
Each hamster eats $50g$ of dry food and $40g$ of fresh vegetables, and needs $1m^2$ of space.
Each rabbit eats $300g$ dry food and $320g$ of fresh vegetables, and needs $5m^2$ of space.
Each guinea pig eats $100g$ of dry food and $200g$ of fresh vegetables, and needs $3m^2$ of space.
Altogether they eat $2900g$ of dry food and $3920g$ of fresh vegetables, and need $63m^2$ of space.

a) Explain why $5x + 30y + 10z = 290$.

b) Write down a system of linear equations that determine x, y and z.

c) Reduce your system to echelon form and solve to find the number of hamsters, rabbits and guinea pigs.

iii) The points A, B and C in $\mathbb{R}^3$ have position vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$$

respectively.
Write down a parametric vector equation of the plane passing through A, B and C.

iv) Consider the following system of linear equations.

$$\begin{aligned} x + y - z &= 2 \\ 2x + 3y + z &= 6 \end{aligned}$$

a) Using Gaussian Elimination find the general solution to the system of equations.

b) Hence, or otherwise, find a solution to the system with the property that the sum of the x, y and z coordinates of the solution is 0.

v) Suppose A, B are two points in $\mathbb{R}^3$ with position vectors $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$ respectively.

We let O denote the origin.

a) Find $|\overrightarrow{OB}|$.

b) Find the area of triangle AOB.

c) Hence, or otherwise, find the perpendicular distance from A to the line through O and B.

vi) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are non-zero, non-parallel vectors in $\mathbb{R}^3$ of the same magnitude. Prove that $\mathbf{u} - \mathbf{v}$ is perpendicular to $\mathbf{u} + \mathbf{v}$.

4. i) a) Give the definition of $\cosh x$.

b) Use the definition to prove that

$$4\cosh^3 x = \cosh 3x + 3\cosh x \ .$$

ii) Find

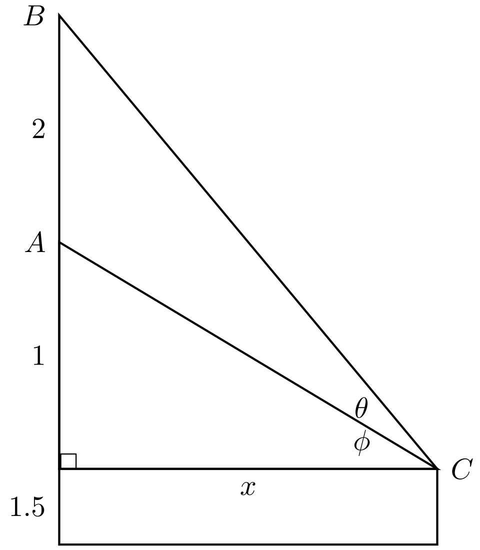
a) $\dfrac{d}{dx}\displaystyle\int_0^x \frac{\cos t}{\sqrt{1+t^2}}\, dt$;

b) $\dfrac{d}{dx}\displaystyle\int_0^{\sinh x} \frac{\cos t}{\sqrt{1+t^2}}\, dt$. Give your answer in simplest form.

iii) a) Sketch the curve whose equation in polar coordinates is $r = 6\sin 2\theta$.

b) Find the gradient, $\frac{dy}{dx}$, of this curve at the point where $\theta = \frac{1}{6}\pi$.

iv) A statue 2 metres high stands on a pillar 2.5 metres high. A person, whose eye is 1.5m above the ground, stands at a distance x metres from the base of the pillar.



The diagram shows the above information, with the person's eye being at C.

a) Prove that

$$\frac{d}{dt}\left(\cot^{-1} t\right) = \frac{-1}{1+t^2}.$$

b) Show that

$$\theta = \cot^{-1}\left(\frac{x}{3}\right) - \cot^{-1} x$$

c) Hence find the distance x that maximises the angle θ.

MATH1131 June 2012

1. i) Let $u = 3 + 2i$ and $w = 1 - 5i$.
   a) Find $u - 2w$ in **Cartesian form**.
   b) Find u/w in **Cartesian form**.

   ii) Let $z = \sqrt{3} - i$.
   a) Calculate $|z|$ and $\text{Arg}(z)$.
   b) Express z in polar form.
   c) Hence, or otherwise, express $z^{10} + (\overline{z})^{10}$ in **Cartesian form**.

   iii) Evaluate the determinant
$$\begin{vmatrix} 1 & -1 & 4 \\ 0 & 2 & 7 \\ 0 & 3 & 1 \end{vmatrix}.$$

   iv) a) Evaluate $\displaystyle\lim_{x\to\infty} \frac{3x^2 + \sin(2x^2)}{x^2}$.
   b) Evaluate $\displaystyle\lim_{x\to 0} \frac{3x^2 + \sin(2x^2)}{x^2}$.

   v) Consider the curve in the plane defined by
$$x^2 - 5x \sin y + y^2 = 4.$$
   Find the equation of the tangent line to this curve at the point $(2, 0)$.

   vi) Let $p(x) = x^5 + 5x + 7$.
   a) Explain why p has at least one real root.
   b) Prove that p has exactly one real root.

2. i) Use De Moivre's Theorem to show that
$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta.$$

   ii) Find the intersection of the line $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$, with the plane $5x - 2y + z = 17$.

   iii) Consider the following MAPLE session:

```
> with(LinearAlgebra):
> A:=<<0,0,1>|<0,1,0>|<-1,0,0>>;
```
$$A := \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
```
> A^2;
```
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

```
> A^3;
```

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

```
> A^4;
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using the Maple session above, find the **inverse** of A^{2001}.

iv) The points C and D have position vectors

$$\mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{d} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}.$$

a) Find the cross product $\mathbf{c} \times \mathbf{d}$.

b) Hence, or otherwise, find the area of the parallelogram with adjacent sides OC and OD, where O is the origin.

v) Evaluate the indefinite integral

$$\int x^4 \ln x \, dx.$$

vi) Consider the three functions:

$$\begin{aligned} &f : \mathbb{R} \to \mathbb{R}, && f(x) = \frac{x^2}{1+x^2}, \\ &g : (0,3) \to \mathbb{R}, && g(x) = (x-1)^2, \\ &h : [1,5] \to \mathbb{R}, && h(x) = \sqrt{1 + \ln x + \sin x \cos x}. \end{aligned}$$

Only **one** of these functions has a maximum value (on its given domain). Which one is it? Give reasons for your answer.

vii) Sketch the polar curve $r = 2 - 2\cos\theta$. You should show any lines of symmetry, and clearly identify where the curve intersects the x and y axes.

viii) Suppose that $y = x^{\sin x}$. Find $\dfrac{dy}{dx}$.

3. i) The points A and B in $\mathbb{R}^3$ have position vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

a) Find a parametric vector equation of the line l passing through A and B.

b) By evaluating an appropriate dot product, show that the line l from part (a) is perpendicular to the line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} t;\ t \in \mathbb{R}$.

ii) Let $P = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 4 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 5 & 0 \end{pmatrix}$.

a) Evaluate PQ^T.

b) What is the size of P^TQ?

c) Does the matrix product PQ exist? Explain your answer.

iii) A system of three equations in three unknowns x, y and z has been reduced to the following echelon form

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & \alpha^2 - 9 & \alpha - 3 \end{array}\right).$$

a) For which value of α will the system have no solution?

b) For which value of α will the system have infinitely many solutions?

c) For the value of α determined in part (b), find the general solution.

iv) The number of \$10, \$20, and \$50 notes in the cash register at Bill's Burger Barn is x, y and z respectively. The total value of all the notes in the register is \$1020. There are 44 notes in total. Also, the number of \$10 notes is equal to the sum of the number of \$20 notes and the number of \$50 notes.

a) Explain why $x + 2y + 5z = 102$.

b) By setting up two further equations and solving the system of three equations in three unknowns x, y and z, determine how many of each type of note is in the cash register.

v) The (non-zero) point Q has position vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. The vector $\overrightarrow{OQ}$ makes angles α, β and γ respectively with the X, Y and Z axes.

a) By considering the vector $\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ show that

$$a = \sqrt{a^2 + b^2 + c^2} \cos \alpha.$$

b) Deduce that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

c) If the angles α and β are complementary, that is, their sum is 90°, what can be said about the vector $\overrightarrow{OQ}$?

4. i) Find a quadratic function $q(x) = x^2 + bx + c$ such that the function

$$h(x) = \begin{cases} e^{3x}, & \text{if } x \leq 0 \\ q(x), & \text{if } x > 0 \end{cases}$$

is differentiable at $x = 0$.

ii) Each of the following calculations is expressed in MAPLE. Write each in normal mathematical notation and **evaluate**.

a) `arcsin(sin(7*Pi/3));`

b) `diff(int(exp(t^2),t=0..x^2),x);`

iii) Prove that $\displaystyle\lim_{x\to\infty} \frac{x^2-2}{x^2+3} = 1$ as follows:
Given any real number $\epsilon > 0$, find a real number M (expressed in terms of ϵ), such that if $x > M$ then $\left|\dfrac{x^2-2}{x^2+3} - 1\right| < \epsilon$.

iv) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^3 + \sinh x + 1.$$

a) Explain why f has a differentiable inverse g.

b) What is the domain of g?

c) Evaluate $g'(1)$.

v) A chemical process produces Factor X, which flows into a 50 litre tank which is initially empty.

At time $t \geq 0$, Factor X flows into the tank at the rate of $\dfrac{100}{10+t^2}$ litres per hour.

Will the tank eventually overflow? Explain your answer.

MATH1131 June 2013

1. i) Evaluate the following limits:

   a)
$$\lim_{x\to\infty} \frac{10x^2 + 3x + \sin x}{5x^2 + 3x - 2},$$

   b)
$$\lim_{x\to 0} \frac{e^{3x} - 1}{\sin(7x)}.$$

   ii) A function $f : [0,5] \to \mathbb{R}$ has the following properties:

   - $\lim_{x\to 2^+} f(x) = 3,$
   - $\lim_{x\to 2^-} f(x) = 1,$
   - $f(2) = 4.$

   Draw a possible sketch of the graph of f. (You do not need to give a formula for your function.)

   iii) a) State the definitions of $\cosh x$ and $\sinh x$ in terms of the exponential function.

   b) Prove that $\cosh^2 x - \sinh^2 x = 1$.

   iv) Let $z = 5 + 5i$ and $w = 2 + i$.

   a) Find $2z + 3\overline{w}$.

   b) Find $z(w - 1)$.

   c) Find z/w.

   v) Suppose that $(x + iy)(3 + 4i) = 13 + 9i$, where $x, y \in \mathbb{R}$.
   Find the value of x and the value of y.

   vi) Let the set S in the complex plane be defined by

$$S = \left\{ z \in \mathbb{C} \ : \ |z| \leq 3 \text{ and } 0 \leq \operatorname{Im}(z) \leq 3 \right\}.$$

   a) Sketch the set S on a labelled Argand diagram.

   b) By considering your sketch, or otherwise, find the area of the region defined by S.

vii) Consider the following MAPLE session.

```
> with(LinearAlgebra):
> A:=<<0,1,-1>|<1,0,1>|<-1,1,0>>;
```

$$A := \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

```
> B:=A^2;
```

$$B := \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

```
> C:=A^3;
```

$$C := \begin{bmatrix} -2 & 3 & -3 \\ 3 & -2 & 3 \\ -3 & 3 & -2 \end{bmatrix}$$

```
> F:=3*A-C;
```

$$F := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Without carrying out any row reduction, use the above Maple session to find the inverse of the matrix A.

2. i) A function h is defined by

$$h(x) = \begin{cases} ax^2 + 3x, & \text{if } x \geq 1 \\ 2x + d, & \text{if } x < 1. \end{cases}$$

Given that h is differentiable at $x = 1$, find the values of a and d.

ii) Evaluate

$$\int_0^{\ln 2} 9xe^{3x}\,dx.$$

iii) Find the equation of the tangent at the origin to the curve implicitly defined by

$$e^x + \sin y = xy + 1.$$

iv) Sketch the polar curve whose equation in polar coordinates is given by

$$r = 1 + \cos 2\theta.$$

v) Let $z = \sqrt{2} - \sqrt{2}i$.

a) Find $|z|$.

b) Find $\mathrm{Arg}(z)$.

c) Use the polar form of z to evaluate z^6. Express your answer in **Cartesian form**.

vi) Let $A = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$.

a) Evaluate AB or explain why this product does not exist.

b) Evaluate AB^T or explain why this product does not exist.

vii) a) Find, in polar form, all solutions to the equation $z^5 = -1$, where $z \in \mathbb{C}$.

b) Hence, or otherwise, express $z^5 + 1$ as a product of real linear and real quadratic factors.

3. i) Determine the point of intersection of the line

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{for} \quad t \in \mathbb{R}\,.$$

and the plane $4x - 5y + 3z = 0$.

ii) Let $M = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 2 & \alpha \end{pmatrix}$.

a) Evaluate the determinant of M.

b) Determine the value(s) of α for which M does **not** have an inverse.

c) Find the inverse of M when $\alpha = 1$.

iii) Let $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$.

a) Find the cross product $\mathbf{u} \times \mathbf{v}$.

b) Hence find the **Cartesian** equation of the plane parallel to $\mathbf{u}$ and $\mathbf{v}$ and passing through the point $\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$.

iv) Let $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 0 \\ 3 \\ -3 \\ \beta \end{pmatrix}$ be two vectors in $\mathbb{R}^4$.

a) Find the value of β so that the vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

b) For the value $\beta = 1$, find the projection, $\text{proj}_{\mathbf{u}}(\mathbf{v})$, of $\mathbf{v}$ onto $\mathbf{u}$.

v) Let A, B and D be three points on some circle with centre C in $\mathbb{R}^2$ with position vectors

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{d} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

a) Let M be the midpoint of the line joining A and B. Find the position vector $\mathbf{m}$ of the point M.

b) Find a non-zero vector $\mathbf{u}$ that is perpendicular to $\overrightarrow{AB}$.

c) Hence or otherwise, find the parametric vector equation for the line whose points are equidistant from A and B.

d) Given that the parametric vector equation for the line whose points are equidistant from B and D is $\mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mu \in \mathbb{R}$, find the centre C of the circle.

4. i) Find $\displaystyle\int \frac{\cos(\ln(x))}{x}\,dx$.

ii) Use the Pinching theorem to evaluate

$$\lim_{x\to\infty} e^{-x}\sin(x).$$

iii) Show that the improper integral $\displaystyle\int_0^\infty \frac{dx}{x^2+e^x}$ converges.

iv) The following calculation is expressed in MAPLE

```
> F:=diff(int(sin(sqrt(t)),t=0..x^2),x);
```

a) Write the calculation using standard mathematical notation.

b) Evaluate F.

v) Let $p(x) = x^3 + 4x - 7$.

a) Use the Intermediate Value theorem to show that p has at least one real root in the interval $[1, 2]$.

b) Show that p has **exactly** one real root in the interval $[1, 2]$.

c) Let g be the inverse of p and α be the unique root of p, whose existence is guaranteed in part b). Express $g'(0)$ in terms of α.

vi) Use the Mean Value Theorem to prove that, for $x > 0$,

$$\ln(1+x) > \frac{x}{1+x}.$$

MATH1131 June 2014

1. i) Let $z = 7 + i$ and $w = 4 + 3i$.
   a) Find $2z - \overline{w}$ in $a + ib$ form.
   b) Find $5(w - i)/z$ in $a + ib$ form.
   c) Find $|zw|$.
   d) Find $\text{Arg}(zw)$.
   e) Hence, or otherwise, show that $\text{Arg}(z) + \text{Arg}(w) = \dfrac{\pi}{4}$.
   f) Use the polar form of zw to evaluate $(zw)^{40}$.

   ii) Consider the following system of equations:

$$\begin{array}{rcrcrcl} x & - & y & - & z & = & 1 \\ x & - & 3y & + & z & = & 1 \\ 2x & - & 3y & - & z & = & 2. \end{array}$$

   a) Write the system in augmented matrix form and reduce it to row echelon form.
   b) Solve the system.

   iii) Evaluate the limits
   a)
$$\lim_{x\to\infty} \frac{6x^2 + \sin x}{4x^2 + \cos x};$$
   b)
$$\lim_{x\to 0} \frac{e^{2x} - 2x - 1}{4x^2}.$$

   iv) A function g is defined by

$$g(x) = \begin{cases} \dfrac{|x^2 - 16|}{x - 4} & \text{if } x \neq 4 \\ \alpha & \text{if } x = 4. \end{cases}$$

   By considering the left and right hand limits at $x = 4$, show that no value of α can make g continuous at the point $x = 4$.

   v) Let $f(x) = x^5 + x^3 + x - 2$.
   a) Prove that f has at least one real root in the interval $[0, 2]$, naming any theorems you use.
   b) State, with reasons, the number of real roots of f.

2. i) Use a substitution to find the integral

$$\int \frac{dx}{x(1 + (\log x)^2)}.$$

   ii) a) Give the definitions of $\sinh x$ and $\cosh x$ in terms of the exponential function.
   b) Use your definitions to prove that $\sinh(2x) = 2\sinh x \cosh x$.

   iii) Evaluate the integral

$$\int_0^{\pi/3} x \sin(2x)\, dx.$$

iv) Simplify the matrix expression $(A^T A)^{-1}(A^T A)^T$, where A is an invertible matrix.

v) Consider the three points A, B, C in $\mathbb{R}^3$ with position vectors $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$, and $\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ respectively.

a) Find a parametric vector form for the plane Π that passes through points A, B, and C.

b) Calculate the cross product $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$, showing your working.

c) Hence, or otherwise, find a **Cartesian** equation for the plane Π.

d) Find the area of the triangle ABC.

e) Find the minimal distance from the point $P \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}$ to the plane Π.

vi) Consider the following MAPLE session.

```
> with(LinearAlgebra):
> A := <<0,1,1>|<1,-1,-1>|<-2,1,0>>;
```

$$A := \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

```
> B := A^2;
```

$$B := \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -3 \\ -1 & 2 & -3 \end{bmatrix}$$

```
> C:=A^3;
```

$$C := \begin{bmatrix} 2 & -3 & 3 \\ -2 & 2 & 1 \\ -1 & 0 & 4 \end{bmatrix}$$

```
> F := 2A + B + C;
```

$$F := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using the above MAPLE session, or otherwise, find the 3×3 matrix which is the inverse of the matrix A.

3. i) A block of wood is subject to 3 vector forces:
$\mathbf{F}_1 = 1$, in the direction West, $\mathbf{F}_2 = 1$ in the direction South, $\mathbf{F}_3 = 2$ in the direction North-East, each measured in Newtons.
Let $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$ be the resultant force on the block.

a) On a scale diagram draw the 3 forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and the resultant force $\mathbf{F}$.

b) Find the exact value of $|\mathbf{F}|$ and the direction of $\mathbf{F}$.

ii) Let $M = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ -1 & 2 & 1 \end{pmatrix}$ and $N = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$.

a) Evaluate the determinant of M.

b) Write down the inverse of N.

iii) Let $p(z) = z^4 + 2z^2 - 3$.

a) Show that $p(1) = p(-1) = 0$.

b) Factor $p(z)$ into two real quadratic polynomials $q(z)$ and $r(z)$.

c) Find the roots of $p(z)$.

d) Factor $p(z)$ into four complex linear polynomials.

iv) Consider the line ℓ and the plane Π given by the following equations:

$$\ell : \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \lambda \in \mathbb{R},$$

$$\Pi : 6x + 8y - 9z = 0.$$

Determine the point of intersection of the line ℓ and the plane Π.

v) a) Use De Moivre's Theorem to prove that $4\cos^3\theta = \cos 3\theta + 3\cos\theta$.

b) Deduce that $2\cos\dfrac{\pi}{9}$ is a root of the polynomial $q(z) = z^3 - 3z - 1$.

vi) Let $\mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ \beta \end{pmatrix}$ be two vectors in $\mathbb{R}^3$.

a) Find the value of β so that the vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

b) For the value $\beta = 0$, find the projection, $\mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$, of $\mathbf{u}$ onto $\mathbf{v}$.

c) Find the value of β so that the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\dfrac{\pi}{4}$.

4. i) Use the Fundamental Theorem of Calculus to find

$$\frac{d}{dx}\int_{x^2}^{x^3} \cos\left(\frac{1}{t}\right)\, dt.$$

ii) Let $f(x) = \dfrac{175x^2 - 350x + 10}{x^2 - 2x + 2}$.

Consider the following MAPLE session.

```
> f:=(175*x^2-350*x+10)/(x^2-2*x+2);
```

$$\frac{175\,x^2 - 350\,x + 10}{x^2 - 2\,x + 2}$$

```
> subs(x=0.0,f);
```

$$5.0$$

```
> subs(x=4.0,f);
```

$$141.0$$

```
> solve(f=0,x);
```

$$1+\sqrt{1155}/35,\ 1-\sqrt{1155}/35$$

```
> evalf(%);
```

$$1.971008312,\ 0.0289916875$$

```
> fdash:=diff(f,x);
```

$$\frac{350\,x-350}{x^2-2\,x+2}-\frac{\left(175\,x^2-350\,x+10\right)(2\,x-2)}{(x^2-2\,x+2)^2}$$

```
> solve(fdash=0,x);
```

$$1$$

```
> subs(x=1.0,f);
```

$$-165.0$$

a) Use the information in the MAPLE output to give a rough sketch of $f(x) = \dfrac{175x^2-350x+10}{x^2-2x+2}$, for $0 \le x \le 4$.

b) Hence, or otherwise, find the maximum and minimum values of $f(x) = \left|\dfrac{175x^2-350x+10}{x^2-2x+2}\right|$ over the closed interval $[0, 4]$.

iii) Determine, with reasons, whether the improper integral

$$K = \int_0^\infty \frac{dx}{e^{2x}+\cos^2 x}$$

converges or diverges.

iv) Sketch in the xy-plane, the graph of the polar curve given by $r = 1-\cos\theta$. (You are NOT required to find the slope of the curve.)

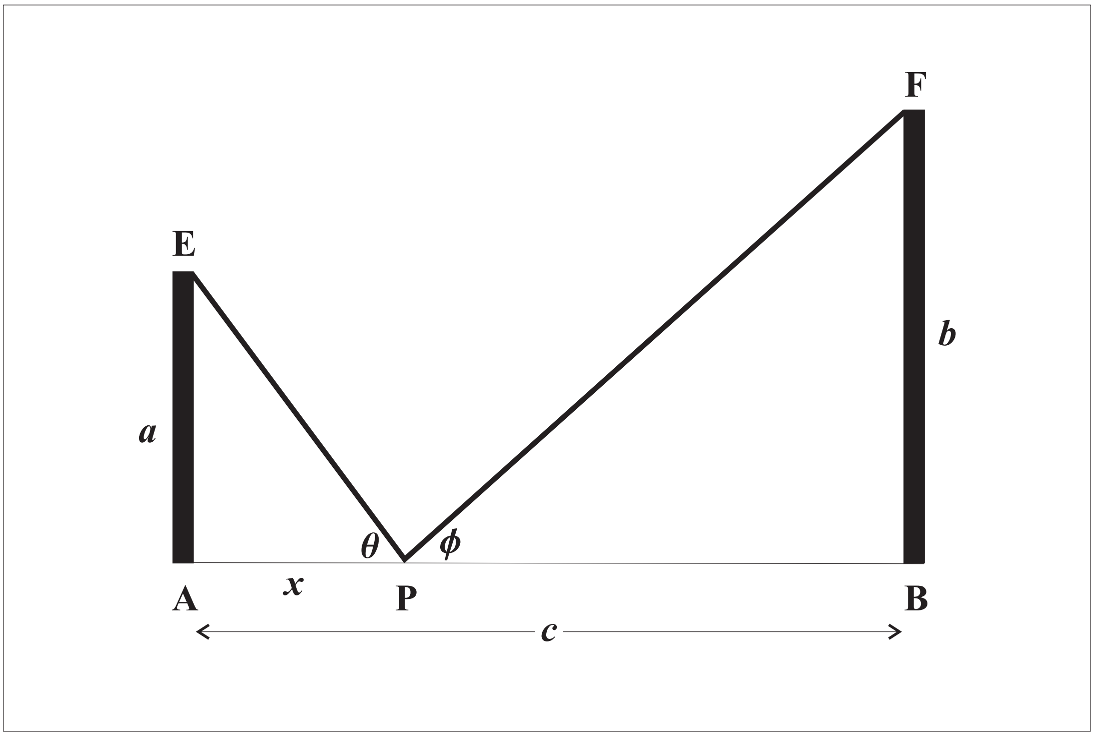
v) a) State carefully the Mean Value Theorem.

b) Suppose $-1 < x < y < 1$. By applying the Mean Value Theorem to the function $f(t) = \sin^{-1} t$ on the interval $[x, y]$, prove that

$$\sin^{-1} y - \sin^{-1} x \ge y - x.$$

vi) Two poles A and B, of heights a metres and b metres respectively, are c metres apart on the horizontal ground. A single tight rope runs from the top of pole A to the point P on the ground between A and B and then to the top of pole B.

Assume that the distance from A to P is x, and that the angles θ and ϕ are as shown in the diagram.



a) Explain why the length L of the rope is given by

$$L = \sqrt{a^2 + x^2} + \sqrt{b^2 + (c - x)^2}.$$

b) Prove that $\cos\theta = \cos\phi$ when $\dfrac{dL}{dx} = 0$.

c) Assuming that $\cos\theta = \cos\phi$ minimizes L, using similar triangles, or otherwise, find the value of x that minimizes L.

MATH1131 June 2015

1. i) Find $\lim_{x\to\infty} \dfrac{x+\sin x}{2x}$. (Give brief reasons for your answer.)

   ii) The function f is defined by

$$f(x) = \begin{cases} 3-x, & 0 \le x < 1, \\ (x-2)^2+1, & 1 \le x \le 3, \end{cases}$$
$$\text{and} \quad f(-x) = f(x) \quad \text{for all } x.$$

   a) Find $f(\frac{3}{2})$.

   b) Sketch the graph of $f(x)$ over the interval $-3 \le x \le 3$.

   c) Which, if any, of the following exists? If it exists, state its value. (Give brief reasons for your answer.)

$$\lim_{x\to 1} f(x), \quad f'(0).$$

   iii) Find, for $x > 0$,

$$I_1 = \int \frac{dx}{1+\sqrt{x}}.$$

   iv) Evaluate

$$\lim_{x\to 0} \frac{1-\cos\left(\frac{x}{2}\right)}{x^2}.$$

   v) Let $z = 5+i$ and $w = 3+2i$.

   a) Find $z - \overline{w}$ in $a+ib$ form.

   b) Find $10w/(z-2)$ in $a+ib$ form.

   c) Find $|(z/w)^8|$.

   d) Find $\text{Arg}(zw)$.

   e) Use the polar form of zw to evaluate $(zw)^8$.

   vi) Consider the three points A, B, C in $\mathbb{R}^3$ with position vectors

$$\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix},$$

   respectively.

   Find a parametric vector form for the plane Π that passes through points A, B, and C.

vii) Consider the following MAPLE session.

```
> with(LinearAlgebra):
> A := <<1,1,3>|<1,0,1>|<4,1,5>>;
```

$$A := \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 4 & 1 & 5 \end{bmatrix}$$

```
> B := A^2;
```

$$B := \begin{bmatrix} 14 & 4 & 19 \\ 5 & 2 & 8 \\ 25 & 9 & 38 \end{bmatrix}$$

```
> C := A^3;
```

$$C := \begin{bmatrix} 94 & 33 & 141 \\ 39 & 13 & 57 \\ 186 & 63 & 274 \end{bmatrix}$$

```
> F := C - 6B - 9A;
```

$$F := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) Using the above MAPLE session, or otherwise, find the 3×3 matrix which is the inverse of the matrix A.

b) State $(A^T)^2$.

2. i) By writing $z = a + ib$, or otherwise, solve $z^2 = 40 + 42i$, giving your answers in Cartesian form.

ii) Find condition(s) on b_1, b_2, b_3 to ensure that the following system has a solution.

$$\begin{array}{rcrcrcl} x & + & 2y & & & = & b_1 \\ x & + & y & - & z & = & b_2 \\ 2x & + & y & - & 3z & = & b_3 \end{array}$$

iii) Sketch the following region on the Argand diagram:

$$S = \Big\{ z \in \mathbb{C} \ : \ |z - i - 1| \leq 1 \quad \text{or} \quad |\mathrm{Im}(z)| \geq 1 \Big\}.$$

iv) Let

$$\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}.$$

a) Calculate the acute angle θ between the vectors $\mathbf{u}$ and $\mathbf{v}$.

b) Find $\mathbf{u} \times \mathbf{v}$.

v) Consider the function

$$f(x) = \begin{cases} 1 + ax^2, & x \leq 1, \\ bx + 2x^3, & x > 1, \end{cases}$$

where $a, b \in \mathbb{R}$.

a) Find all values of a and b such that f is continuous at $x = 1$.

b) Find all values of a and b such that f is differentiable at $x = 1$.

vi) Evaluate

$$I_2 = \int x^3 \ln x \, dx.$$

vii) Consider the polar curve $r = 1 + \cos\theta$, for $0 \le \theta \le 2\pi$.

a) Find the slope of the tangent to the curve at the point with Cartesian coordinates $(0, 1)$.

b) Find the polar coordinates of the points at which the tangent to the curve is horizontal.

c) Sketch the polar curve.

3. i) Show that the set of points z in the complex plane that satisfy the equation

$$2|z-3i| = |z+3i|$$

lie on a circle. State the radius and centre of the circle.

ii) Consider the complex polynomial $p(z) = z^4 - 3z^3 + 6z^2 - 12z + 8$.

a) Given that $p(2i) = 0$, factorise p into linear and quadratic factors with real coefficients.

b) Find all roots of p.

iii) Consider the plane Π in $\mathbb{R}^3$ with Cartesian equation

$$x - 3y + 2z = 1\,.$$

a) Find a point-normal form for the plane Π.

b) Show that the line ℓ with parametric vector form

$$\mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \qquad \lambda \in \mathbb{R}$$

is parallel to the plane Π.

c) Find the shortest distance between the point $P\,(4,2,2)$ and the plane Π.

iv) a) Find the determinant of

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 3 & a & 2 \\ 2 & 1 & 1 \end{pmatrix}.$$

b) For what value of a does the matrix A **not** have an inverse.

c) Determine the value of a and the value of b for which $A^{-1} = B$, where

$$B = \begin{pmatrix} -1 & 4 & -5 \\ 1 & b & 7 \\ 1 & -3 & 4 \end{pmatrix}.$$

v) Let X be an $n \times n$ matrix with determinant $|X| = 1$.
Prove that the inverse of X also has determinant equal to 1.

4. i) a) Define the functions $\sinh x$ and $\cosh x$ in terms of the exponential function.

b) From the definitions in (a), show that

$$\frac{d}{dx}(\cosh 6x) = 6\sinh 6x.$$

c) Show that, for $x > 0$,

$$\ln(\sinh x) < x - \ln 2.$$

ii) Prove carefully that the improper integral $\displaystyle\int_1^\infty \frac{\ln x}{x^3}\,dx$ converges.

iii) Find $\displaystyle\frac{d}{dx}\left(\int_0^{x^3} \cos\left(t^2\right)\,dt\right)$.

iv) Consider the polynomial $p(x) = x^3 + 3x + 1$ defined on $\mathbb{R}$.

a) Use the Intermediate Value Theorem to show that the equation $p(x) = 0$ has at least one real root.

b) Show that the function p has an inverse function g.

c) Find the value of $g'(1)$.

v) a) State the Mean Value Theorem.

b) Apply the Mean Value Theorem to $f(t) = \cos t$, on a suitably chosen interval, to show that $|\cos y - \cos x| \le |y - x|$, for all $x, y \in \mathbb{R}$.

PAST HIGHER EXAM PAPERS

MATH1141 JUNE 2011

2. i) You may use the following Maple session to assist you in answering the question below.

```
> with(LinearAlgebra):
> A:=<<1,2,1>|<1,3,a>|<-1,a,3>>;
```

$$A := \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & a \\ 1 & a & 3 \end{bmatrix}$$

```
> t:=<1,3,2>;
```

$$t := \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

```
> B:=<A|t>;
```

$$B := \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & a & 3 \\ 1 & a & 3 & 2 \end{bmatrix}$$

```
> G:=GaussianElimination(B);
```

$$G := \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & a+2 & 1 \\ 0 & 0 & 6-a^2-a & 2-a \end{bmatrix}$$

For which values of a will the system $A\mathbf{x} = \mathbf{t}$ have:

a) no solutions,

b) unique solution,

c) infinitely many solutions?

ii) Evaluate the determinant

$$\begin{vmatrix} 2 & 0 & -1 \\ 1 & 3 & 0 \\ 5 & 7 & 3 \end{vmatrix}.$$

iii) Find the point of intersection, if any, of the line $\boldsymbol{x} = \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\lambda \in \mathbb{R}$, with the plane

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \nu \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mu, \nu \in \mathbb{R}.$$

iv) Consider the function f defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

a) Given that $\lim_{x\to\infty} xe^{-x} = 0$, evaluate the limit

$$\lim_{h\to 0} \frac{e^{-1/h^2}}{h}.$$

b) Using the definition of a derivative, determine whether f is differentiable at $x = 0$.

v) Consider the function f defined by

$$f(x) = \begin{cases} \dfrac{2}{x} & \text{for } x < -1, \\ x^2 - 1 & \text{for } -1 \leq x \leq \dfrac{6}{\pi}, \\ \left(\dfrac{72}{\pi^2} - 2\right) \sin \dfrac{1}{x} & \text{for } x > \dfrac{6}{\pi}. \end{cases}$$

a) Find all critical points of f and determine their nature. In particular, distinguish between local and global extrema.

b) Find all horizontal asymptotes for f.

c) Given that f does not have a point of inflexion, sketch the graph of f, clearly indicating all of the above information.

3. i) Let

$$S = e^{i\theta} + \frac{e^{3i\theta}}{3} + \frac{e^{5i\theta}}{3^2} + \frac{e^{7i\theta}}{3^3} + \cdots .$$

a) Prove that

$$S = \frac{3(3e^{i\theta} - e^{-i\theta})}{10 - 6\cos(2\theta)}.$$

b) Hence, or otherwise, find the sum

$$T = \sin(\theta) + \frac{\sin(3\theta)}{3} + \frac{\sin(5\theta)}{3^2} + \frac{\sin(7\theta)}{3^3} + \cdots .$$

ii) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are non-zero, non-parallel vectors in $\mathbb{R}^3$ of the same magnitude. Prove that $\mathbf{u} - \mathbf{v}$ is perpendicular to $\mathbf{u} + \mathbf{v}$.

iii) Let $A = (a_{ij})$ be a real $n \times n$ matrix and let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be the standard basis vectors for $\mathbb{R}^n$.

a) Prove that $\boldsymbol{e}_i^T A \boldsymbol{e}_j = a_{ij}$ for all $1 \le i, j \le n$.

b) Prove that if A is symmetric then $\boldsymbol{x}^T A \boldsymbol{y} = (A\boldsymbol{x})^T \boldsymbol{y}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

c) Conversely, prove that if $\boldsymbol{x}^T A \boldsymbol{y} = (A\boldsymbol{x})^T \boldsymbol{y}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, then A is symmetric.

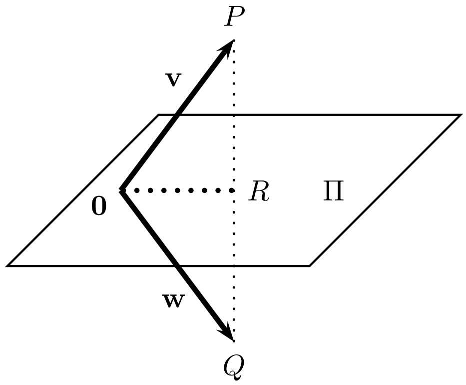
iv) The matrix

$$M = \frac{1}{9}\begin{pmatrix} 1 & 4 & -8 \\ 4 & 7 & 4 \\ -8 & 4 & 1 \end{pmatrix}$$

has the following property.

If $\boldsymbol{v}$ is the position vector of any point P in $\mathbb{R}^3$, then $\boldsymbol{w} = M\boldsymbol{v}$ is the vector obtained by reflecting $\boldsymbol{v}$ in a fixed plane Π, which passes through the origin.

Hence, in the diagram, $|\boldsymbol{v}| = |\boldsymbol{w}|$ and $|RP| = |RQ|$, where R is the foot of the perpendicular from P (or Q) to the plane.



a) Show that M reflects the vector $\boldsymbol{a} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ to the vector $-\boldsymbol{a}$.

b) Hence write down the Cartesian equation of Π.

c) Find a **non-zero** vector $\boldsymbol{u}$ such that $M\boldsymbol{u} = \boldsymbol{u}$.

d) Find the shortest distance from the point B with position vector $\boldsymbol{b} = \begin{pmatrix} 6 \\ -6 \\ 0 \end{pmatrix}$ to the plane.

4. i) a) Using L'Hôpital's rule or otherwise, indicate why

$$\lim_{x \to \infty} e^{-x} x^n = 0$$

for any $n \in \mathbb{N}$.

b) Show that for all $x \geq 1$,

$$e^{-x} x^n < C x^{-2},$$

where C is a positive constant.

c) Hence, or otherwise, show that the improper integral

$$\int_1^\infty e^{-x} x^n \, dx$$

converges for any $n \in \mathbb{N}$.

ii) Suppose that $f : [0, 2] \to [0, 12]$ is continuous on its domain and twice differentiable on $(0, 2)$. Further suppose that $f(0) = 0, f(2) = 12$.

a) Explain why $f'(c) = 6$ for some real number $c \in (0, 2)$.

b) Suppose further that $f'(0) = 0$, prove that $f''(d) > 3$ for some real number $d \in (0, c)$.

iii) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = x - a \tanh x$$

for some constant $a \in \mathbb{R}$.

a) Explain why $\operatorname{sech} x < 1$ for $x > 0$.

b) By considering the derivative, or otherwise, find the values of a for which $f(x) > 0$ for all $x > 0$? Give reasons for your answer

c) For which values of a does the equation

$$x = a \tanh x$$

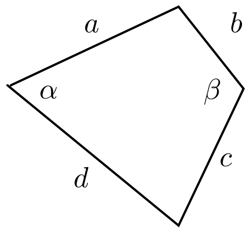
have a positive solution?

iv) The area $A(t)$ of an arbitrary convex quadrilateral with given side lengths a, b, c, d depends on the sum $t = \alpha + \beta$ of either pair of opposite angles, and is given by

$$A(t) = \sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{2}abcd(1 + \cos t)},$$

where

$$s = \frac{1}{2}(a + b + c + d).$$



a) Explain why the area A of a convex quadrilateral, with fixed side lengths a, b, c, d, is maximal if the sum of either pair of opposite angles is π.

b) Show that the area function $A : [0, \pi] \to \mathbb{R}$ as defined above is invertible, and that the inverse function B is differentiable on $(A(0), A(\pi))$.

c) Show that

$$B'(A_0) = \frac{4A_0}{abcd},$$

where

$$A_0 = \sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{2}abcd}.$$

MATH1141 JUNE 2012

2. i) Consider the function $f : (0, 2\sqrt{\pi}\,] \to \mathbb{R}$ defined by

$$f(x) = x^2 + \cos(x^2).$$

a) Find all critical points of f and determine their nature.

b) Explain why f is invertible, state the domain of f^{-1} and find $f^{-1}(5\pi/2)$.

c) Where is f^{-1} differentiable?

ii) Let

$$f(x) = \int_0^{x^2-9x} e^{-t^2}\,dt.$$

a) Use the Mean Value Theorem to show that f has a stationary point x_0 in the interval $[0, 9]$.

b) Find the value of x_0 and determine the nature of the stationary point.

iii) Suppose that z lies on the unit circle in the complex plane.

a) Show that $z + \dfrac{1}{z}$ is real.

b) Find the maximum value of $z + \dfrac{1}{z}$.

iv) Use De Moivre's theorem to prove that

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.$$

v) Consider the plane P with parametric vector form

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}, \qquad \lambda_1, \lambda_2 \in \mathbb{R}.$$

a) Does the point $\mathbf{a} = \begin{pmatrix} 4 \\ 4 \\ -7 \end{pmatrix}$ lie on P?

b) Is the vector $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ parallel to P?

c) Is the vector $\mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ orthogonal to P?

3. i) Suppose that A is a 4×4 matrix with $\det(A) = 5$. The matrix B is the result of performing the following three elementary row operations on A:

1. first multiply the 3rd row by 7;
2. then replace the second row with twice the first row plus the second row;
3. then swap the first and last rows.

What is the value of $\det(B)$?

ii) Consider the line in $\mathbb{R}^3$,

$$x - 4 = -y = z - 5.$$

a) Write this line in parametric vector form.

b) Find the point on the line closest to the origin $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

iii) Consider the following Maple session, which defines a matrix A and a vector $\mathbf{b} \in \mathbb{R}^3$:

```
> with(LinearAlgebra):
> A:=<<1,3,2>|<1,2,a>|<-2,2*a,4>>;
```

$$A := \begin{bmatrix} 1 & 1 & -2 \\ 3 & 2 & 2a \\ 2 & a & 4 \end{bmatrix}$$

```
> b:=<1,2,-2>;
```

$$\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

```
> GaussianElimination(<A|b>);
```

$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & -1 & 2a+6 & -1 \\ 0 & 0 & -4+2a^2+2a & -2-a \end{bmatrix}$$

For which values of a will the system $A\mathbf{x} = \mathbf{b}$ have

a) a unique solution,

b) no solutions,

c) infinitely many solutions?

iv) A matrix $Q \in M_{nn}(\mathbb{R})$ is said to be *nilpotent* (of degree 2) if $Q^2 = \mathbf{0}$, the zero matrix.

a) Give an example of a non-zero 2×2 nilpotent matrix.

b) Explain why a nilpotent matrix cannot be invertible.

Suppose now that $S, Q \in M_{nn}(\mathbb{R})$ commute, that S is invertible and that Q is nilpotent (of degree 2).

c) Prove that $S^{-1}Q = QS^{-1}$.

d) Show that $S + Q$ is invertible by finding an integer k such that

$$(S + Q)(S^{-1} - S^{-k}Q) = I.$$

v) Consider the non-zero vector $\mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ in $\mathbb{R}^3$ which makes angles α, β, γ with the three coordinate axes respectively.

By considering dot products with the standard basis vectors $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$, (or otherwise), prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

vi) Suppose that A is an $n \times n$ matrix with the property that every vector $\mathbf{b} \in \mathbb{R}^n$ can be written uniquely as a linear combination of the **columns** of A. Prove that every vector $\mathbf{b} \in \mathbb{R}^n$ can also be written uniquely as a linear combination of the **rows** of A.

4. i) Consider the polar curve $r = 1 + \cos 2\theta$.

a) Prove that the curve is symmetric about the x axis and also about the y axis.

b) Sketch the curve.
(You are NOT required to find the derivative.)

ii) Let $f(x) = \tanh x$ (the hyperbolic tangent).

a) Express $\tanh x$ in terms of exponentials.

b) Sketch the graph $y = f(x)$.

c) Show that

$$\lim_{x\to\infty} \frac{1 - \tanh x}{e^{-2x}} = 2\,.$$

d) Explain why the improper integral

$$\int_0^\infty (1 - \tanh x)\,dx$$

converges.

e) Compute

$$\int_0^\infty (1 - \tanh x)\,dx,$$

justifying your calculations.

iii) Suppose that f is a function whose derivative is continuous and hence bounded on $[a, b]$, with $|f'(x)| \leq L$ for all $x \in [a, b]$.

a) Show that for any $n > 0$,

$$\int_a^b f(x) \sin nx\,dx = \frac{K(n)}{n} + \frac{1}{n}\int_a^b f'(x) \cos nx\,dx,$$

where $K(n) = f(a)\cos(na) - f(b)\cos(nb)$.

b) Explain why

$$\left|\int_a^b f'(x) \cos nx\,dx\right| \leq (b - a)L.$$

c) Find, with reasons,

$$\lim_{n\to\infty} \int_a^b f(x) \sin nx\,dx.$$

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2. i) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^3 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0. \end{cases}$$

a) Explain why f is differentiable everywhere and determine $f'(x)$.

b) Explain why the function g defined by $g(x) = f'(x)$ is continuous at $x = 0$.

c) Use the definition of the derivative to determine whether g is differentiable at $x = 0$.

ii) Consider the function $f(x) = \dfrac{1}{1+x}$ defined on $[0, 1]$ and let P be the partition $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\}$.

a) Show that the lower Riemann sum $L_P(f)$ is given by

$$L_P(f) = \sum_{k=1}^{n} \frac{1}{n+k}.$$

b) Assuming that the limits of the upper and lower Riemann sums are equal, evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k}.$$

iii) Let A, B and C be three points in the plane with corresponding position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

a) Let M be the midpoint of the line joining A and B. What is the position vector $\mathbf{m}$ of M?

b) Write a parametric vector equation for the line through C and M.

c) Suppose that

$$(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a}) = \frac{1}{2}|\mathbf{b} - \mathbf{a}|\,|\mathbf{c} - \mathbf{a}| \quad \text{and} \quad (\mathbf{c} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \frac{1}{2}|\mathbf{c} - \mathbf{b}|\,|\mathbf{a} - \mathbf{b}|.$$

Explain why the triangle ABC is equilateral.

iv) Consider the system of equations

$$\begin{aligned} x + y - z &= 2 && (1) \\ x - y + 3z &= 6 && (2) \\ x^2 + y^2 + z^2 &= 10 && (3) \end{aligned}$$

[Note that equation (3) is NOT linear.]

a) Give, in parametric vector form, the set of points which satisfy the first two equations (that is, (1) and (2)).

b) Describe this solution set geometrically.

c) Using the answer to (a), or otherwise, find all the points which satisfy all three equations.

3. i) Find the shortest distance from the plane

$$\mathbf{x} = \begin{pmatrix}1\\2\\4\end{pmatrix} + \lambda_1 \begin{pmatrix}0\\2\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}-1\\1\\3\end{pmatrix}, \qquad \lambda_1, \lambda_2 \in \mathbb{R}$$

to the point $\mathbf{p} = \begin{pmatrix}3\\2\\3\end{pmatrix}$.

ii) Let $p(z) = z^4 - z^3 - z^2 - z + 2$. Denote the roots of p by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, where α_1 is an **integer**.

a) Find the value of α_1.

b) Given that at least one of the roots of p is not real, deduce how many of the roots are real.

c) By considering the sum of the roots, or otherwise, prove that at least one of the roots has negative real part.

d) Prove that $|\alpha_j| > \frac{1}{2}$ for $j = 1, 2, 3, 4$.

iii) Which of the following statements are true **for all** non-zero 2×2 matrices $A, B, C \in M_{2,2}(\mathbb{R})$? For those statements which are not always true, give a counterexample.

a) $AB = BA$.

b) $\det(AB) = \det(BA)$.

c) If $\det(AB) = \det(AC)$ then $\det(B) = \det(C)$.

d) If $AB = AC$ then $B = C$.

iv) a) Define what it means for a set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ to be an **orthonormal set** in $\mathbb{R}^n$.

b) Let M be the matrix whose columns consist of the n orthonormal vectors, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbb{R}^n$. By considering $M^T M$ or otherwise, find, with reasons, all possible values for $\det(M)$.

4. i) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \int_0^{x^3} e^{-t^2}\, dt.$$

a) Determine, with reasons,

$$\lim_{t\to\infty} t^2 e^{-t^2}.$$

b) Does the improper integral

$$I = \int_0^{\infty} e^{-t^2}\, dt$$

converge? Give reasons for your answer.

c) Find all critical points and asymptotes of f.

d) Carefully sketch the graph of f, clearly indicating the above information and any other relevant features.

ii) Let f be a differentiable function on (a,b), and take $c \in (a,b)$. Define

$$q(x) = \frac{f(x) - f(c) - f'(c)(x-c)}{(x-c)^2},$$

where $a < x < b$ and $x \neq c$.

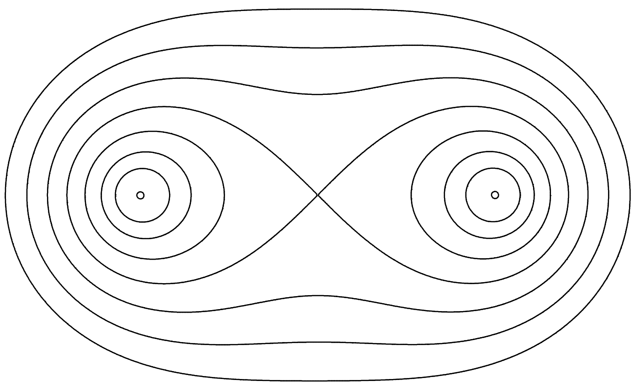
Show that if $f''(c)$ exists, then

$$\lim_{x \to c} q(x) = \frac{f''(c)}{2}.$$

iii) An oval of Cassini is a curve on the (x,y)-plane defined implicitly by

$$\left(x^2+y^2\right)^2 - 2\left(x^2-y^2\right) + 1 = b.$$

The shape of the curve depends on the value of the positive constant b. The plot below shows ovals of Cassini for several different values of b.



a) Show that the points on an oval of Cassini where the tangent is horizontal either lie on the unit circle $x^2+y^2=1$ or lie on the y-axis.

b) Determine all such points (x,y) for which the corresponding tangent is horizontal and state carefully for which values of $b > 0$ these exist.

iv) Suppose $f : [0,2] \to [0,8]$ is continuous and differentiable on its domain.

a) By considering the function $g(x) = f(x) - x^3$, prove that there is a real number $\xi \in [0,2]$ such that $f(\xi) = \xi^3$, stating any theorems you use.

b) Now suppose that $f(0) = 0$ and $f(2) = 8$. Explain why $f'(\eta) = 4$ for some real $\eta \in (0,2)$, stating any theorems you use.

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3. i) Let $g(x) = 3x - \cos 2x - 1$, $x \in \mathbb{R}$. Explain why g has a differentiable inverse function $h = g^{-1}$ and calculate $h'(-2)$.

ii) a) State carefully the Mean Value Theorem.

b) Use the Mean Value Theorem to prove that if $a < b$ then

$$0 < \tan^{-1} b - \tan^{-1} a \leq b - a.$$

c) Using (b) or otherwise, prove that the improper integral

$$I = \int_1^{\infty} \tan^{-1}\left(t + \frac{1}{t^2}\right) - \tan^{-1} t \, dt$$

converges.

iii) Use the ϵ-M definition of the limit to prove that

$$\lim_{x \to \infty} \frac{e^x}{\cosh x} = 2.$$

iv) Consider the polar curve $r = 1 + \cos 4\theta$.

a) Determine the values of $\theta \in [0, 2\pi]$ for which r has the smallest and largest values.

b) Hence, or otherwise, sketch this polar curve. (You are not required to find the slope.)

v) For $x > 0$, let $f(x) = x^{x \ln x}$.

a) Evaluate $f'(x)$.

b) Determine the values of x for which $f'(x) > 0$ and the values of x for which $f'(x) < 0$.

c) Given that $\lim_{x \to 0^+} f(x) = 1$, sketch the graph $y = f(x)$ for $0 \leq x \leq 2$.

4. i) Find the conditions on b_1, b_2, b_3 which ensure that the following system has a solution.

$$\begin{array}{rcrcrcl} 2x & & & - & 4z & = & b_1 \\ 3x & + & y & - & 2z & = & b_2 \\ -2x & - & y & & & = & b_3 \end{array}$$

ii) Let I, J and K be the points in $\mathbb{R}^3$ whose position vectors are the three standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ respectively.

By considering vectors of the form $\begin{pmatrix} x \\ x \\ x \end{pmatrix}$, find the position vector of a point A, not the origin, such that the distances from A to I, J and K are all 1.

iii) Consider the complex matrix $A = \begin{pmatrix} 2 & i \\ 1+i & \alpha \end{pmatrix}$.

a) Find A^{-1} in the case when $\alpha \in \mathbb{R}$.

b) Find all values of $\alpha \in \mathbb{C}$ for which $\det(A^2) = -1$.

iv) You may assume that $(z^9 - 1) = (z^3 - 1)(z^6 + z^3 + 1)$.

a) Explain why the roots of $z^6 + z^3 + 1 = 0$ are $e^{\pm \frac{2\pi i}{9}}, e^{\pm \frac{4\pi i}{9}}, e^{\pm \frac{8\pi i}{9}}$.

b) Divide $z^6 + z^3 + 1$ by z^3 and let $x = z + \dfrac{1}{z}$.
Find a cubic equation satisfied by x.

c) Deduce that $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} = 0$.

v) The norm $\|M\|$ of an $n \times n$ matrix M is the maximum value that $|M\mathbf{u}|$ takes for all unit vectors $\mathbf{u} \in \mathbb{R}^n$.

a) Show that for any vector $\mathbf{x} \in \mathbb{R}^n$,

$$|M\mathbf{x}| \leq \|M\||\mathbf{x}|.$$

b) Suppose that M and N are any two $n \times n$ matrices. By considering $MN\mathbf{u}$, or otherwise, show that

$$\|MN\| \leq \|M\|\|N\|.$$

c) What is the norm of the matrix $\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$

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3. i) a) Carefully state the Mean Value Theorem.

b) Assume that a differentiable function f on $\mathbb{R}$ is such that $f'(x) \le 1$ for all $x \in \mathbb{R}$. Given that $f(2) = 2$, show that $f(x) \ge x$ for all $x \le 2$.

ii) Let f be a continuous function on $\mathbb{R}$ and

$$g(x) = \frac{\int_0^x f(t)\,dt - xf(0)}{x^2}.$$

Use L'Hôpital's rule to show that if $f'(0)$ exists then

$$\lim_{x \to 0} g(x) = \frac{f'(0)}{2}.$$

iii) Let

$$f(x) = \int_0^{x^3} (t^2 - 1)e^{t^2}\,dt.$$

a) Show that f is an odd function, that is, $f(-x) = -f(x)$.

b) Find the stationary points of f.

c) By examining the sign of $f'(x)$ in a neighbourhood of each stationary point, or otherwise, determine the nature of each stationary point.

iv) Consider the curve in the (x, y)-plane defined by the relation

$$xy(x^2 + y^2) = 1. \tag{1}$$

a) Wherever y is an implicit function of x, determine $\frac{dy}{dx}$ in terms of x and y.

b) For any point (x, y) on the curve with $x, y \ge 0$, consider the area $A = xy$ of a rectangle of edge lengths x and y. Find the point (x, y) on the curve for which A is stationary.

c) Find the polar form of the curve (1) and express A in terms of the radial polar coordinate r.

d) Explain why the stationary point of A corresponds to the rectangle which has the largest area.

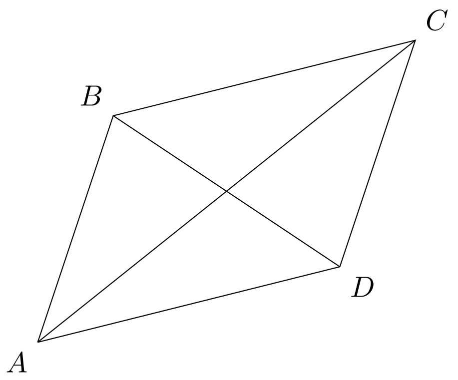
4. i) Find the shortest distance from the point $P(1, 2, 0)$ to the line with parametric vector equation

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

ii) a) By using the dot product, or otherwise, show that for any vectors $\mathbf{a}$ and $\mathbf{b}$, the following identity holds:

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2.$$

b) Consider a parallelogram $ABCD$, as pictured below.



If $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$, describe clearly in words the geometric interpretation of the result in (a).

iii) a) Use De Moivre's Theorem to express $\sin\,5\theta$ as a polynomial in $x = \sin\theta$.

b) Consider the polynomial $p(x) = 16x^5 - 20x^3 + 5x - 1$.
Show that $\sin\frac{\pi}{10}$ is a root of $p(x)$.

c) Using the fact that

$$16x^5 - 20x^3 + 5x - 1 = (x-1)(4x^2 + 2x - 1)^2$$

find the distinct roots of $p(x)$.

d) Evaluate $\sin\frac{\pi}{10}$ in surd form.

iv) Consider the matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. Suppose that $\det(A) = 7$.

Find the value of the determinant of each of the following matrices

$$B = \begin{pmatrix} d & e & f \\ a & b & c \\ d-3a & e-3b & f-3c \end{pmatrix} \text{ and } C = \begin{pmatrix} g-3a & 2a & a-d \\ h-3b & 2b & b-e \\ i-3c & 2c & c-f \end{pmatrix}.$$

v) Recall that the dot product of two vectors $\mathbf{a}$ and $\mathbf{b}$ can be written as $\mathbf{a}^T\mathbf{b}$ and that a square matrix Q is said to be *orthogonal* if $Q^TQ = I$.

Let Q be a real orthogonal $n \times n$ matrix and suppose $\mathbf{v}$ is a non-zero vector in $\mathbb{R}^n$.

a) Prove that the vectors $Q\mathbf{v}$ and $\mathbf{v}$ have the same length.

b) If $\mathbf{v}$ is a non-zero vector in $\mathbb{R}^n$, such that $Q\mathbf{v} = \lambda\mathbf{v}$, for some $\lambda \in \mathbb{R}$, show that $\lambda = 1$ or $\lambda = -1$.

PAST EXAM SOLUTIONS

MATH1131 NOVEMBER 2010 Solutions

1. i) a) $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

b) $\text{Arg } z = \frac{\pi}{4}$

c)

$$\begin{aligned} z &= \sqrt{2}e^{i\frac{\pi}{4}} \\ z^{28} &= \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^{28} \\ &= (2)^{\frac{28}{2}} e^{i\frac{28\pi}{4}} \\ &= 2^{14} (\cos 7\pi \ + i\sin 7\pi \ ) \\ &= -2^{14} (= -16384) \end{aligned}$$

ii)

$$\begin{aligned} (x+iy)(3+2i) &= 4+7i \\ x+iy &= \frac{4+7i}{3+2i} \\ &= \frac{(4+7i)}{(3+2i)}\frac{(3-2i)}{(3-2i)} \\ &= \frac{26+13i}{13} \\ &= 2+i \end{aligned}$$

Equating real and imaginary parts, $x = 2$ and $y = 1$.

iii) a)

$$z^6 = 1 = e^{2k\pi i}, \qquad k \in \mathbb{Z}$$

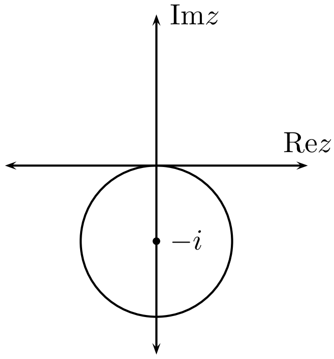
Hence $z = e^{2ki\pi/6}$. Substituting appropriate values for k, the solutions are

$$1, e^{i\pi/3}, e^{-i\pi/3}, e^{2i\pi/3}, e^{-2i\pi/3}, -1.$$

b)

$$\begin{aligned} \left(z^6 - 1\right) &= (z-1)\left(z - e^{\frac{i\pi}{3}}\right)\left(z - e^{-i\frac{\pi}{3}}\right)\left(z - e^{i\frac{2\pi}{3}}\right)\left(z - e^{-i\frac{2\pi}{3}}\right)(z+1) \\ &= (z-1)(z+1)\left(z^2 - 2z\cos(\pi/3) + 1\right)\left(z^2 - 2z\cos(2\pi/3) + 1\right) \\ &= \left(z^2 - z + 1\right)\left(z^2 + z + 1\right). \end{aligned}$$

iv) $|z + i| = 1$ represents a circle, centred at $z = -i$, radius 1.



v) From MAPLE, $A^6 = 64I$. Multiplying by $(A^2)^{-1}$, we have

$$A^4 = 64(A^2)^{-1}$$

$$so \quad \left(A^2\right)^{-1} = \frac{1}{64}A^4 = \frac{1}{64}\begin{bmatrix} -8 & 0 & -8\sqrt{3} \\ 0 & -16 & 0 \\ 8\sqrt{3} & 0 & -8 \end{bmatrix}$$

vi) a)

$$\begin{aligned} 200a + 600b + 400c &= 12000 \\ 1800a + 2400b + 2000c &= 65600 \\ 20000a + 30000b + 26000c &= 784000 \end{aligned}$$

b)

$$\left[\begin{array}{ccc|c} 200 & 600 & 400 & 12000 \\ 1800 & 2400 & 2000 & 65600 \\ 20000 & 30000 & 26000 & 784000 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 2 & 6 & 4 & 120 \\ 18 & 24 & 20 & 656 \\ 20 & 30 & 26 & 784 \end{array}\right]$$

$$\xrightarrow[R_3 - 10R_1]{R_2 - 9R_1} \left[\begin{array}{ccc|c} 2 & 6 & 4 & 120 \\ 0 & -30 & -16 & -424 \\ 0 & -30 & -14 & -416 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 2 & 6 & 4 & 120 \\ 0 & 30 & 16 & 424 \\ 0 & 0 & 2 & 8 \end{array}\right]$$

Hence,

$$\begin{aligned} c &= 4 \\ 30b + 16c &= 424 \\ b &= 12 \\ 2a + 6b + 4c &= 120 \\ a &= 16 \end{aligned}$$

i.e. $(a, b, c) = (16, 12, 4)$.

2. i) a)

$$PP^T = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 13 \end{pmatrix}.$$

b) P^TP is a 3×3 matrix.

ii) a) $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix}, \overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}.$

b) The plane has equation $\mathbf{x} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}0\\-2\\-2\end{pmatrix} + \mu\begin{pmatrix}2\\2\\4\end{pmatrix}, \qquad \lambda, \mu \in \mathbb{R}.$

iii) Substituting the $x = 1 + t, y = 2 - y, z = 5 + t$ into the equation of the plane, we have

$$(1 + t) - 3(2 - t) + (5 + t) = 15 \Rightarrow t = 3.$$

Hence, the point of intersection is $\mathbf{x} = \begin{pmatrix}1\\2\\5\end{pmatrix} + 3\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}4\\-1\\8\end{pmatrix}.$

iv) Since the vectors are perpendicular, their dot product is 0, hence

$$1 \times -2 + 3 \times -6 - \beta + 5 \times -10 = 0 \Rightarrow \beta = -70.$$

v) a) The magnitude of $\mathbf{u}$ is $\sqrt{2^2 + 1^2 + 7^2} = \sqrt{54} = 3\sqrt{6}$.
b) The vector $\frac{10}{3\sqrt{6}}\mathbf{u}$ is parallel to $\mathbf{u}$ and has length 10.

vi) One possible method is to row-reduce first, but in this case it is probably easier to expand across the first row.

$$\det(C) = 3\det\begin{pmatrix}1 & 7\\2 & 0\end{pmatrix} - 1 \times \det\begin{pmatrix}4 & 7\\1 & 0\end{pmatrix} + 0 = -35.$$

vii) If $\text{proj}_{\mathbf{w}}(\mathbf{u}) = \text{proj}_{\mathbf{w}}(\mathbf{v})$ then

$$\begin{aligned}\frac{\mathbf{u}.\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w} &= \frac{\mathbf{v}.\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}\\ \Rightarrow \mathbf{u}.\mathbf{w} &= \mathbf{v}.\mathbf{w}\end{aligned}$$

since $\mathbf{w}$ is a non-zero vector. Hence, using the properties of dot product, $(\mathbf{u} - \mathbf{v}).\mathbf{w} = 0$ and so $\mathbf{u} - \mathbf{v}$ is perpendicular to $\mathbf{w}$.

3. i) Since both the numerator and denominator are 0 at $x = 0$, we apply L'Hôpital's rule.

$$L = \lim_{x\to 0} \frac{x^2 e^x}{1 - \cos(\pi x)} = \lim_{x\to 0} \frac{(2x + x^2)e^x}{\pi \sin(\pi x)}.$$

Again, both the numerator and denominator are 0 at $x = 0$, so we apply L'Hôpital's rule a second time.

$$L = \lim_{x\to 0} \frac{(2 + 4x + x^2)e^x}{\pi^2 \cos(\pi x)} = \frac{2}{\pi^2}.$$

ii) As a necessary condition, f must be continuous at $x = 1$ so

$$\lim_{x\to 1^+} f(x) = f(1).$$

Hence

$$\lim_{x\to 1^+} (-x^2 + ax + b) = 1 \Rightarrow a + b = 2.$$

For differentiability, we need to show that

$$\lim_{h\to 0^-} \frac{f(1 + h) - f(1)}{h} = \lim_{h\to 0^+} \frac{f(1 + h) - f(1)}{h}.$$

Now

$$\text{LHS} = \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{(x+h)^2-x^2}{h} = 2$$

and

$$\text{RHS} = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{-(x+h)^2+a(x+h)+b-(-x^2+ax+b)}{h} = -2+a.$$

Hence we have $a = 4$ and from $a+b = 2$ it follows that $b = -2$.

iii) a)

$$L = \lim_{x\to\infty} \frac{x^2-2}{x^2+1} = \lim_{x\to\infty} \frac{1-\frac{2}{x^2}}{1+\frac{1}{x^2}} = 1.$$

b)

$$\left|\frac{x^2-2}{x^2+1} - 1\right| = \frac{3}{x^2+1} < \frac{3}{x^2}.$$

Hence, if $\frac{3}{x^2} < \varepsilon$ then $\left|\frac{x^2-2}{x^2+1} - 1\right| < \varepsilon$. Now $\frac{3}{x^2} < \varepsilon \Rightarrow x > \sqrt{\frac{3}{\varepsilon}}$ so let $M = \sqrt{\frac{3}{\varepsilon}}$ and then if $x > M$, we have $\left|\frac{x^2-2}{x^2+1} - 1\right| < \varepsilon$.

iv) a) The function f is continuouson the closed interval $[0, 2]$ and $f(0) = -5 < 0$, while $f(2) = 3 + 2\sqrt{3} > 0$. Hence by the intermediate value theorem, f has at least one positive real root in the interval $[0, 2]$.

b) Since $f'(x) = 3x^2 + \sqrt{3} > 0$, the function f is increasing. Hence f has exactly one real positive root.

v) Using logarithms,

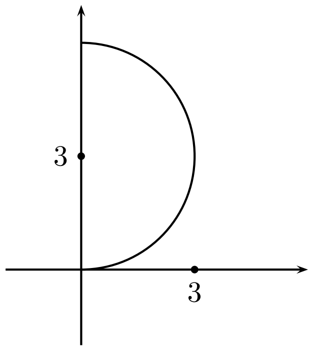
$$\log y = \log(\sin x)^x = x \log(\sin x).$$

Hence

$$\frac{1}{y}\frac{dy}{dx} = \log(\sin x) + x\frac{\cos x}{\sin x} \Rightarrow \frac{dy}{dx} = (\sin x)^x(\log(\sin x) + x \cot x).$$

vi) a) $x = r\cos\theta = 6\sin\theta\cos\theta, y = r\sin\theta = 6\sin^2\theta$,
so $x^2 + y^2 = 36\sin^2\theta(\sin^2\theta + \cos^2\theta) = 36\sin^2\theta = 6y$. Completing the square, $x^2 + (y-3)^2 = 9$.
From the above, since $0 \le \theta \le \frac{\pi}{2}$, we see that $x \in [0, 3], y \in [0, 6]$ and so we have a semicircle centre 3 radius 3 in the right half plane.

b)



4. i) a) Using integration by parts with $u = 1 - x, v' = \frac{1}{(1+x)^3}$, we have

$$\begin{aligned} I_1 &= (1-x).\left(\frac{-1}{2(1+x)^2}\right) - \int (-1).\frac{(-1)}{2(1+x)^2}\,dx \\ &= \frac{x-1}{2(1+x)^2} + \frac{1}{2(1+x)} + C. \end{aligned}$$

Alternatively you can make the substitution $x = u - 1$.

b) Using integration by parts with $u = x, v' = \cos 2x$ we have

$$I_2 = \left[x(\frac{1}{2}\sin(2x))\right]_0^\pi - \int_0^\pi 1.\frac{1}{2}\sin(2x)\,dx = \left[\frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x)\right]_0^\pi = \frac{1}{4} - \frac{1}{4} = 0.$$

ii)

$$\begin{aligned} K &= \int_1^\infty \frac{1+\sin x}{3x^2}\,dx \le \int_1^\infty \frac{2}{3x^2}\,dx \\ &= \lim_{N\to\infty} \int_1^N \frac{2}{3x^2}\,dx = \lim_{N\to\infty} \frac{2}{3} - \frac{2}{3N} = \frac{2}{3}. \end{aligned}$$

Hence the improper integral converges by comparison.

iii) a)

$$\sinh x = \frac{e^x - e^{-x}}{2}, \cosh x = \frac{e^x + e^{-x}}{2}.$$

b) Replacing x with ax, we have

$$\frac{d}{dx}\cosh(ax) = \frac{d}{dx}\frac{e^{ax} + e^{-ax}}{2} = \frac{ae^{ax} - ae^{-ax}}{2} = a\sinh(ax).$$

c) Since cosh is an even function,

$$\cosh^{-1}\big(\cosh(-4726)\big) = \cosh^{-1}\big(\cosh(4726)\big) = 4726.$$

iv) Using the First Fundamental Theorem of Calculus,

$$\begin{aligned} \frac{d}{dx}\int_0^x f(t)\,dt &= \frac{d}{dx}\int_x^1 t^2 f(t)\,dt + \frac{d}{dx}\left(\frac{x^{16}}{8} + \frac{x^{18}}{9} - \frac{1}{9}\right) \\ \Rightarrow f(x) &= -\frac{d}{dx}\left(\int_1^x t^2 f(t)\,dt\right) + 2x^{15} + 2x^{17} = -x^2 f(x) + 2x^{15} + 2x^{17}. \end{aligned}$$

Solving for $f(x)$, we have

$$f(x) = \frac{2x^{15} + 2x^{17}}{1+x^2} = 2x^{15}.$$

v) a) Since $\tan^{-1}$ is defined for all real x, $g(x) = \tan^{-1}(x) + \tan^{-1}(1/x)$ is defined for all real x except $x = 0$. Hence the maximal domain for g is $(-\infty, 0) \cup (0, \infty)$.

b) For $x > 0$,

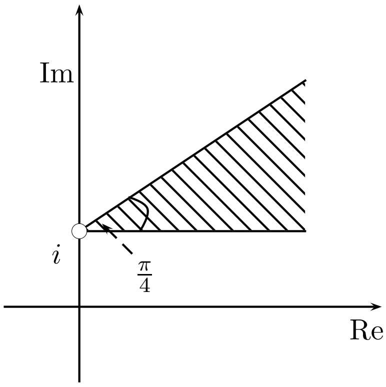
$$g'(x) = \frac{1}{1+x^2} + \frac{\frac{-1}{x^2}}{1+(\frac{1}{x})^2} = \frac{1}{x^2+1} - \frac{1}{1+x^2} = 0.$$

Hence on $(0, \infty)$ g is constant. The same calculations hold for $x \in (-\infty, 0)$. Hence g is piecewise constant on its domain. (Note that the constant is not the same in the two intervals. On $(0, \infty)$, $g(x) = \frac{\pi}{2}$, while on $(-\infty, 0)$, $g(x) = -\frac{\pi}{2}$.

c) Since the expression is constant, put $\alpha = 0$ then $\tan^{-1}(-1) + \tan^{-1}(-1) = -\frac{\pi}{2}$.

MATH1131 June 2011 Solutions

1. i) a) $|z| = \sqrt{2}$.
   b) $\text{Arg}(z) = -\frac{3\pi}{4}$.
   c) $z = \sqrt{2}e^{-3\pi i/4}$ and hence $z^{102} = (\sqrt{2})^{102}e^{-153\pi i/2} = 2^{51}e^{-\pi i/2} = -2^{51}i$.
   ii) a) $(2+4i)^2 = -12+16i$.
   b) Applying the quadratic formula, (or by completing the square), we have $z = \frac{4\pm\sqrt{-12+16i}}{2}$. By (a),this simplifies to $z = \frac{4\pm(2+4i)}{2}$, and so the two solutions are $z = 3+2i$ or $z = 1-2i$.
   iii) Diagram as below



   iv) Rationalising the denominator,

$$\lim_{x\to\infty} \frac{1}{x-\sqrt{x^2-6x-4}} = \lim_{x\to\infty} \frac{x+\sqrt{x^2-6x-4}}{(x-\sqrt{x^2-6x-4})(x+\sqrt{x^2-6x-4})}$$

$$= \lim_{x\to\infty} \frac{x+\sqrt{x^2-6x-4}}{(x^2-(x^2-6x-4))}$$

$$= \lim_{x\to\infty} \frac{x+\sqrt{x^2-6x-4}}{6x+4}.$$

We now divide by the highest power of x in the denominator to obtain:

$$\lim_{x\to\infty} \frac{1+\sqrt{1-\frac{6}{x}-\frac{4}{x^2}}}{6+\frac{4}{x}} = \frac{1}{3}.$$

   v) Applying the definition,

$$\int_1^\infty x^{-5/4}\,dx = \lim_{M\to\infty}\int_1^M x^{-5/4}\,dx = \lim_{M\to\infty}\left[-4x^{-\frac{1}{4}}\right]_1^M = \lim_{M\to\infty} 4-\frac{4}{M^{\frac{1}{4}}} = 4.$$

   vi) a) Differentiating implicitly,

$$2x-3y^2-6xy\frac{dy}{dx} = 0$$

and the result follows.
   b) At the point $(1,2)$, $\dfrac{dy}{dx} = -\dfrac{5}{6}$, so the equation of the tangent is $y-2 = -\frac{5}{6}(x-1)$ or $5x+6y = 17$.

c) implicitplot($x^2 - 3xy^2 + 11 = 0, x = 1..4, y = -5..5$).

2. i) By De Moivre's Theorem,

$$\cos(4\theta) + i\sin(4\theta) = (\cos\theta + i\sin\theta)^4.$$

Expanding and equating the real parts on both sides, we have

$$\cos(4\theta) = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$

Replacing the sine terms,

$$\cos(4\theta) = \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$

and the result follows.

ii) a) Using the values $t = 0, t = 1$ we obtain the two points $\begin{pmatrix}1\\2\\4\end{pmatrix}$ and $\begin{pmatrix}4\\4\\9\end{pmatrix}$.

b) $\begin{pmatrix}3\\2\\5\end{pmatrix}$

c) The vector $\begin{pmatrix}9\\6\\15\end{pmatrix} = 3\begin{pmatrix}3\\2\\5\end{pmatrix}$ is normal to the plane and parallel to the direction of the line. Hence the line is perpendicular to the plane.

d) $t = -1$ gives the required point $\begin{pmatrix}-2\\0\\-1\end{pmatrix}$.

iii) From the MAPLE, we see that $A^8 = 16I$. Multiplying both sides by $(A^7)^{-1}$, we have $(A^7)^{-1} = \frac{1}{16}A = \frac{1}{16}\begin{pmatrix}-1 & -1 & 0\\1 & 1 & 0\\0 & 0 & \sqrt{2}\end{pmatrix}$.

iv) The expression is indeterminate at $x = 1$ and so we apply L'Hôpital's Rule (twice),

$$\lim_{x\to1}\frac{(x-1)^2}{1+\cos(\pi x)} = \lim_{x\to1}\frac{2(x-1)}{-\pi\sin(\pi x)} = \lim_{x\to1}\frac{2}{-\pi^2\cos(\pi x)} = \frac{2}{\pi^2}.$$

v) Using integration by parts,
$\int x\sin(2x)\,dx = -\frac{x}{2}\cos 2x + \frac{1}{2}\int\cos 2x\,dx = -\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x + C.$

vi) a) $f(0) = 1 < 2$ and $f(1) = e + a > 2$ for $a > 0$. Since f is continuous, by the intermediate value theorem, there is a $c \in (0,1)$ such that $f(c) = 2$ and so 2 is in the range of f.

b) $f'(x) = e^x + a > 0$ for $a > 0$ and for all x. Thus the function f is continuous and increasing and so f has an inverse.

c) The domain of f^{-1} is a the range of f and since f is increasing, the range is $[f(0), f(1)] = [1, e + a]$.

3. i) a) $AB = \begin{pmatrix}8 & 23\\5 & 11\end{pmatrix}$.

b) BA is a 3×3 matrix.

ii) a) Looking at the dry fruit eaten, we have

$$50x + 300y + 100z = 2900.$$

Dividing this equation by 2 gives the desired result.

b)

$$\begin{aligned} 5x + 30y + 10z &= 290 \\ 4x + 32y + 20z &= 392 \\ x + 5y + 3z &= 63 \end{aligned}$$

c)

$$\left(\begin{array}{ccc|c} 5 & 30 & 10 & 290 \\ 4 & 32 & 20 & 392 \\ 1 & 5 & 3 & 63 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 5 & 3 & 63 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 20 \end{array}\right)$$

Back substitution yields, $x = 8, y = 5, z = 10$ so there are 8 hamsters, 5 rabbits, 10 guinea pigs.

iii) $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \ \lambda, \mu \in \mathbb{R}.$

iv) a)

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & 3 & 1 & 6 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 3 & 2 \end{array}\right)$$

Let $z = \lambda$, then by back substitution, $x = 4\lambda, y = 2 - 3\lambda$.

So $\mathbf{x} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}.$

b) $x + y + z = 2 + 2\lambda = 0$ when $\lambda = -1$ and then $\mathbf{x} = \begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix}$

v) a) $|\overrightarrow{OB}| = \sqrt{35}$.

b) Area of triangle $AOB = \frac{1}{2}|\mathbf{a} \times \mathbf{b}$

$$= \frac{1}{2} \left\| \begin{vmatrix} i & j & k \\ 1 & 2 & 4 \\ 3 & 1 & 5 \end{vmatrix} \right\| = \frac{1}{2} \begin{pmatrix} 6 \\ 7 \\ -5 \end{pmatrix} = \frac{1}{2}\sqrt{110}.$$

c) Since the area of a triangle is half the base time perpendicular height, the perpendicular distance from A to the line through O and B is $\frac{1}{2}\sqrt{110} \div \frac{1}{2}\sqrt{35} = \sqrt{154}/7$.

vi) $(\mathbf{u} - \mathbf{v})(\mathbf{u} + \mathbf{v}) = \mathbf{u}.\mathbf{u} + \mathbf{u}.\mathbf{v} - \mathbf{v}.\mathbf{u} - \mathbf{v}.\mathbf{v} = |\mathbf{u}|^2 - |\mathbf{v}|^2 = 0$ since the vectors have the same magnitude. Hence the vectors $\mathbf{u} - \mathbf{v}$ and $\mathbf{u} + \mathbf{v}$ are perpendicular.

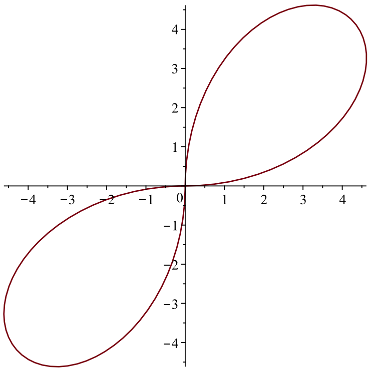
4. i) a) $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

b) $4\cosh^3 x = 4(\frac{1}{2}(e^x + e^{-x}))^3 = \frac{1}{2}(e^{3x} + e^{-3x} + 3e^x + 3e^{-x}) = \cosh 3x + 3\cosh x$.

ii) a) By the Fundamental Theorem of Calculus, $\dfrac{d}{dx}\displaystyle\int_0^x \frac{\cos t}{\sqrt{1+t^2}}\,dt = \frac{\cos x}{\sqrt{1+x^2}}$.

b) By the Fundamental Theorem of Calculus, and the chain rule, $\dfrac{d}{dx}\displaystyle\int_0^{\sinh x} \frac{\cos t}{\sqrt{1+t^2}}\,dt = \cosh x \times \frac{\cos(\sinh x)}{\sqrt{1+\sinh^2 x}} = \cos(\sinh x)$.

iii) a) Diagram as below,



b) $x = r\cos\theta = 6\cos\theta\sin(2\theta)$ and $y = r\sin\theta = 6\sin\theta\sin(2\theta)$.
Hence $\frac{dy}{d\theta} = 12\cos(2\theta)\sin\theta + 6\sin(2\theta)\cos\theta = \frac{15}{2}$ at $\theta = \frac{\pi}{6}$. Similarly, $\frac{dx}{d\theta} = 12\cos(2\theta)\cos\theta - 6\sin(2\theta)\sin\theta = \frac{3}{2}\sqrt{3}$ at $\theta = \frac{\pi}{6}$.
Hence, $\frac{dy}{dx} = \frac{dy}{d\theta}/\frac{dx}{d\theta} = 5\sqrt{3}/3$.

iv) a) Write $y = \cot^{-1} t$, then $t = \cot y$ so $\dfrac{dt}{dy} = -\operatorname{cosec}^2 y = -(1+\cot^2 y) = -(1+t^2)$. The result follows.

b) Write $\theta = (\theta+\phi) - \phi$ and use the two right triangles.

c) $\dfrac{d\theta}{dx} = -\dfrac{3}{9+x^2} + \dfrac{1}{1+x^2} = 0$ for a maximum. Solving this yields $x = \sqrt{3}$ (since $x > 0$). To show it is a maximum, we note that $\frac{d^2\theta}{dx^2}$ is negative at $x = \sqrt{3}$.

MATH1131 June 2012 Solutions

1. i) a) $u - 2w = 1 + 12i$.

   b) $u/w = -\frac{7}{26} + i\frac{17}{26}$.

   ii) a) $|z| = 2$ and $\text{Arg}(z) = -\frac{\pi}{6}$..

   b) $z = 2e^{-i\frac{\pi}{6}}$.

   c) $z^{10} = 2^{10}e^{-\frac{5\pi}{3}}$ and so $\overline{z}^{10} = 2^{10}e^{\frac{5\pi}{3}}$. Hence $z^{10} + (\overline{z})^{10} = 2 \times 2^{10} \cos \dfrac{5\pi}{3} = 2^{10}$.

   iii) Expanding down the first column,

$$\begin{vmatrix} 1 & -1 & 4 \\ 0 & 2 & 7 \\ 0 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 7 \\ 3 & 1 \end{vmatrix} = -19.$$

   iv) a) $\displaystyle\lim_{x\to\infty} \frac{3x^2 + \sin(2x^2)}{x^2} = \lim_{x\to\infty} \frac{3 + \sin(2x^2)/x^2}{1} = 3.$

   b) Applying L'Hopital's Rule,
   $\displaystyle\lim_{x\to 0} \frac{3x^2 + \sin(2x^2)}{x^2} = \lim_{x\to 0} \frac{6x + 4x\cos(2x^2)}{2x} = \lim_{x\to 0} \frac{6 + 4\cos(2x^2)}{2} = 5.$

   v) Differentiating implicitly,

$$2x - 5\sin y - 5x\cos y\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

   and so $\dfrac{dy}{dx} = \dfrac{2}{5}$ at $(2, 0)$. Hence the equation of the tangent at this point is $y = \frac{2}{5}(x - 2)$ or $2x - 5y = 4$.

   vi) a) p is a polynomial and so is continuous. Since p has degree 5, for large positive x, $p(x) > 0$ and for large negative x, $p(x) < 0$, so by the Intermediate Value Theorem, $p(x)$ has at least one real root.

   b) $p'(x) = 5x^4 + 5 > 0$ for all x and so $p(x)$ is strictly increasing. Hence p has at most one real root.

2. i) We use De Moivre's theorem and then the binomial formula to find

$$\begin{aligned} \cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta. \end{aligned}$$

   Equating real parts then gives

$$\begin{aligned} \cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta) \\ &= 4\cos^3\theta - 3\cos\theta. \end{aligned}$$

   The result is shown.

   ii) We solve equations simultaneously using $x = 1 + \lambda, y = \lambda, z = 2 + 2\lambda$ to find

$$17 = 5(1 + \lambda) - 2\lambda + (2 + 2\lambda) = 5\lambda + 7.$$

Solving for λ we find $\lambda = 2$. The point of intersection can now be obtained from the parametric equation for the line as

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}.$$

iii) Below we use the MAPLE output giving $A^4 = I$ and the output for A^3. The inverse to A^{2001} is

$$A^{-2001} = A^{4\times(-501)+3} = (A^4)^{-501} A^3 = I^{-501} A^3 = A^3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

iv) a) We compute

$$\mathbf{c} \times \mathbf{d} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 2 & 3 \\ 4 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ -7 \end{pmatrix}.$$

b) The area of the parallelogram with sides $\mathbf{c}, \mathbf{d}$ is $|\mathbf{c} \times \mathbf{d}| = \sqrt{7^3 + 7^3 + (-7)^3} = 7\sqrt{3}$.

v) We use the integration by parts formula $\int u dv = uv - \int v du$ with $u = \ln x, v = \frac{1}{5}x^5$ so $du = \frac{1}{x}dx, dv = x^4 dx$.

$$\begin{aligned} \int x^4 \ln x dx &= \frac{1}{5}x^5 \ln x - \int \frac{1}{5}x^5 \frac{1}{x} dx \\ &= \frac{1}{5}x^5 \ln x - \int \frac{1}{5}x^4 dx \\ &= \frac{1}{5}x^5 \ln x - \frac{1}{25}x^5 dx + C \end{aligned}$$

where C is an arbitrary constant

vi) We first note that the functions $\ln x, \sin x, \cos x$ are all continuous on their domains. Hence the same is true of $h(x)$. Furthermore, the domain of h is the closed interval $[1, 5]$ which has finite length. It follows by the max-min theorem that $h(x)$ must attain a maximum value.

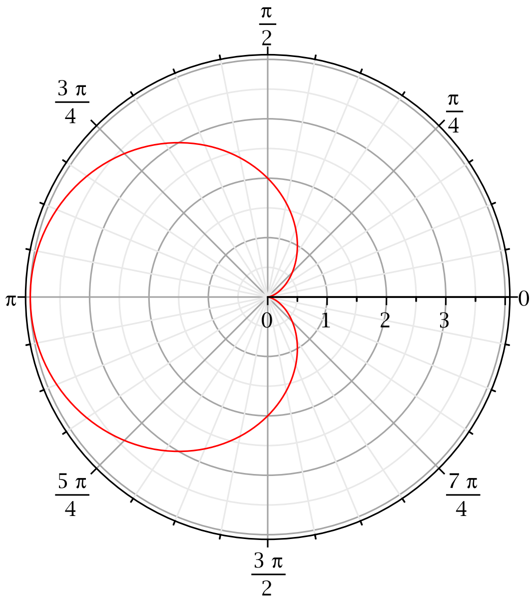
vii) First note that r is an even function of θ so the graph symmetric about the x-axis.

We now compute some values

$$\begin{aligned} r(0) &= 2 - 2\cos 0 = 0 \\ r(\pi/2) &= 2 - 2\cos \pi/2 = 2 - 0 = 2 \\ r(\pi) &= 2 - 2\cos \pi = 2 - 2(-1) = 4 \end{aligned}$$

Note that in general, as θ increases from 0 to π, $r(\theta)$ decreases from 0 to 4. Hence the polar curve spirals out as you rotate anti-clockwise or clockwise from the positive x-axis to the negative x-axis.

Note that the x-intercepts are ± 2 while the y-intercepts are $0, -4$. The only axis of symmetry is the x-axis.



viii) We differentiate implicitly with respect to x, the equation $\ln y = \ln(x^{\sin x}) = \sin x \ln x$.

$$\frac{1}{y}\frac{dy}{dx} = \cos x \ln x + \frac{\sin x}{x}$$

Hence

$$\frac{dy}{dx} = y(\cos x \ln x + \frac{\sin x}{x}) = x^{\sin x}(\cos x \ln x + \frac{\sin x}{x})$$

3. i) a) We have $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$. Therefore l is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, \lambda \in \mathbb{R}.$$

b) $\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = 0 - 6 + 6 = 0$. Therefore the two lines are perpendicular.

ii) a) Note that $Q^T = \begin{pmatrix} 1 & 2 \\ -1 & 5 \\ 1 & 0 \end{pmatrix}$. Thus

$$PQ^T = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 5 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 12 \\ 8 & 1 \end{pmatrix}.$$

b) P^T is a 3×2 and Q is a 2×3 matrix. Therefore, P^TQ is a 3×3 matrix.

c) No as the number of columns of P doesn't equal to the number of rows of Q.

iii) a) Note that $\alpha^2 - 9 = (\alpha - 3)(\alpha + 3)$ and so the matrix becomes:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & (\alpha-3)(\alpha+3) & \alpha - 3 \end{array}\right).$$

If $\alpha = -3$ the last column will be leading, and so this will lead to no solutions.

b) If $\alpha = 3$ both the thirst and last columns will be non-leading, implying infinitely many solutions.

c) If $\alpha = 3$ we get

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Let $z = \lambda$. Back substituting, we get $2y + 4\lambda = 8$ and so $y = 4 - 2\lambda$. Also, $x + 2(4 - 2\lambda) + 3\lambda = 5$ and thus we get $x = -3 + \lambda$. Thus

$$\mathbf{x} = \begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$$

iv) a) Since the total value if \$1020 we have

$$10x + 20y + 50z = 1020$$

and after dividing by 10 we get

$$x + 2y + 5x = 102.$$

b) Since there are 44 notes in total, we get

$$x + y + z = 44.$$

The last line in the question implies $x = y + z$ or equivalently

$$x - y - z = 0.$$

Putting this in augmented matrix form and row reducing we get:

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 44 \\ 1 & 2 & 5 & 102 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 2 & 2 & 44 \\ 0 & 3 & 6 & 102 \end{array}\right)$$

$$\xrightarrow[R_3 \to \frac{1}{3}R_3]{R_3 \to \frac{1}{2}R_3} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 22 \\ 0 & 1 & 2 & 34 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 22 \\ 0 & 0 & 1 & 12 \end{array}\right).$$

Back substituting we get $z = 12, y = 10, x = 22$.

v) a) We have

$$\mathbf{a} \cdot \mathbf{e_1} = |\mathbf{a}||\mathbf{e_1}| \cos \alpha$$

and so $a = \sqrt{a^2 + b^2 + c^2} \cos \alpha$.

b) Similarly to (a) we get $b = \sqrt{a^2+b^2+c^2}\cos\beta$ and $c = \sqrt{a^2+b^2+c^2}\cos\gamma$. Squaring and adding the above three equations gives $a^2+b^2+c^2 = (a^2+b^2+c^2)(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$ and so we get

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

c) If $\alpha+\beta = 90^\circ$ then $\cos\beta = \sin\alpha$. From this we get $\cos^2\alpha+\cos^2\beta = \cos^2\alpha+\sin^2\alpha = 1$ and so from (b) we see that $\cos^2\gamma = 0$. This implies $\gamma = \pi/2$ i.e. $\overrightarrow{OQ}$ is parallel to the XY-plane.

4. i) Let $p(x) = e^{3x}$. For h to be differentiable at $x = 0$, we need

- $p(0) = q(0)$, so that h is continuous at 0;
- $p'(0) = q'(0)$.

As $p'(x) = 3e^{3x}$ and $q'(x) = 2x + b$, these conditions say

- $1 = c$;
- $3 = b$,

and so $q(x) = x^2 + 3x + 1$.

ii) a) In mathematical notation this is $\sin^{-1}(\sin(7\pi/3))$. Evaluating (or just writing this down from thinking about the quadrant)

$$\sin^{-1}(\sin(7\pi/3)) = \sin^{-1}(\sqrt{3}/2) = \frac{\pi}{3}.$$

b) In mathematical notation this is $\displaystyle\frac{d}{dx}\int_0^{x^2} e^{t^2}\,dt$.

To evaluate this, let $F(u) = \int_0^u e^{t^2}\,dt$ and let $u(x) = x^2$. Then

$$\begin{aligned}\frac{d}{dx}\int_0^{x^2} e^{t^2}\,dt &= \frac{d}{dx}F(u(x)) \\ &= F'(u(x))\,u'(x) \qquad \text{(by the chain rule)} \\ &= e^{u(x)^2}\,2x \qquad \text{(by the FTC)} \\ &= 2xe^{x^4}.\end{aligned}$$

iii) Suppose that $\epsilon > 0$ is given. Put $M = \sqrt{\frac{5}{\epsilon}}$ then for $x \ge M$,

$$\left|\frac{x^2-2}{x^2+3} - 1\right| = \left|\frac{x^2-2-x^2-3}{x^2+3}\right| = \frac{5}{x^2+3} \le \frac{5}{x^2} \le \frac{5}{M^2} = \epsilon.$$

iv) a) f is differentiable and

$$f'(x) = 3x^2 + \cosh x \ge \cosh x \ge 1$$

and so $f'(x)$ is never zero. It follows by the inverse function theorem that f has a differentiable inverse g.

b) As $f(x) \to -\infty$ as $f \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$

$$\mathrm{Dom}(g) = \mathrm{Ran}(f) = \mathbb{R}.$$

c) The inverse function theorem says that

$$g'(x) = \frac{1}{f'(g(x))}.$$

We need to find $y = g(1)$. That is, solve $f(y) = y^3 + \sinh y + 1 = 1$. By inspection, the solution is $y = 0$. Plugging this into $f'(x)$ from (a):

$$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(0)} = \frac{1}{1} = 1.$$

v) Let $V(t)$ denote the volume of Factor X in the tank at time $t \geq 0$.
We are given that $V'(t) = \dfrac{100}{10+t^2}$.
Thus after T hours the amount in the tank is

$$V(T) = \int_0^T \frac{100}{10+t^2}\,dt = \left[10\sqrt{10}\tan^{-1}\frac{t}{\sqrt{10}}\right]_0^T = 10\sqrt{10}\tan^{-1}\frac{T}{\sqrt{10}}.$$

As $T \to \infty$,

$$\tan^{-1}\frac{T}{\sqrt{10}} \to \frac{\pi}{2}$$

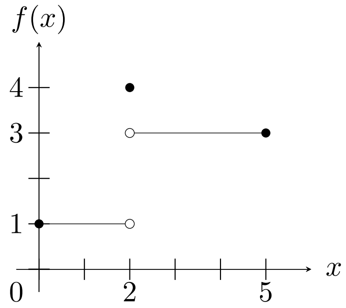
and so $V(T)$ increases towards its limit $5\sqrt{10}\pi \approx 49.6729 < 50$.
In particular, the tank is just big enough to never overflow.

MATH1131 June 2013 Solutions

1. i) a) 2

   b) $\dfrac{3}{7}$

   ii)



   iii) a) $\cosh x = \frac{1}{2}(e^x + e^{-x}) \qquad \sinh x = \frac{1}{2}(e^x - e^{-x})$

   b) **Proof**.

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{1}{2}(e^x + e^{-x})\right)^2 - \left(\frac{1}{2}(e^x - e^{-x})\right)^2 \\ &= \frac{1}{4}(e^{2x} + e^{-2x} + 2) - \frac{1}{4}(xe^{2x} + e^{-2x} - 2) = \frac{1}{4} \times 4 = 1\,.\end{aligned}$$ ■

   iv) a) $16 + 7i$

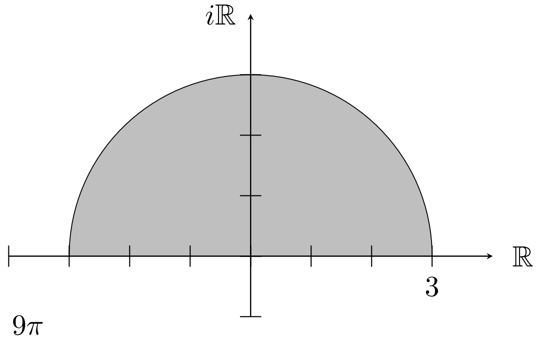
   b) $10i$

   c) $3 + i$

   v) $x = 3$

   $y = -1$

   vi) a)



   b) $\dfrac{9\pi}{2}$

   vii) $A = \dfrac{1}{2}(3I - A^2) = \dfrac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$

2. i)

$$h(x) = \begin{cases} ax^2 + 3x, & \text{if } x \geq 1 \\ 2x + d, & \text{if } x < 1. \end{cases}$$

Both branches are polynomials and are therefore elementary functions. This means they are both continuous and differentiable $x \neq 1$.

Differentiating for $x \neq 1$:

$$h'(x) = \begin{cases} 2ax+3, & \text{if } x > 1 \\ 2, & \text{if } x < 1. \end{cases}$$

If $h(x)$ is differentiable at $x = 1$ then $h(x)$ is continuous at $x = 1$. i.e.

$$\begin{aligned} h(1) = \lim_{x\to 1^+} h(x) &= \lim_{x\to 1^-} h(x) \\ \lim_{x\to 1^+} ax^2 + 3x &= \lim_{x\to 1^-} 2x + d \\ a + 3 &= 2 + d \\ d &= a + 1. \end{aligned}$$

Also, $h'(x)$ has to be continuous at $x = 1$, i.e. we require

$$\begin{aligned} \lim_{x\to 1^+} h'(x) &= \lim_{x\to 1^-} h'(x) \\ \lim_{x\to 1^+} 2ax + 3 &= \lim_{x\to 1^-} 2 \\ 2a + 3 &= 2 \\ a &= -\frac{1}{2}. \end{aligned}$$

But $d = a + 1$, so

$$a = -\frac{1}{2} \quad \text{and} \quad d = \frac{1}{2}.$$

ii)

$$\begin{aligned} \int_0^{\ln 2} 9xe^{3x}\,dx &= \left[9x\frac{e^{3x}}{3}\right]_0^{\ln 2} - \int_0^{\ln 2} 9\frac{e^3x}{3}dx \\ &= 3(\ln 2)e^{3\ln 2} - 0 - 3\left[\frac{e^{3x}}{3}\right]_0^{\ln 2} \\ &= 3(\ln 2)2^3 - e^{3\ln 2} + e^0 \\ &= 24\ln 2 - 7. \end{aligned}$$

iii)

$$e^x + \sin y = xy + 1.$$

Differentiating implicitly with respect to x throughout, using the chain rule and the product rule:

$$e^x + \cos y\frac{dy}{dx} = x\frac{dy}{dx} + 1y + 0$$

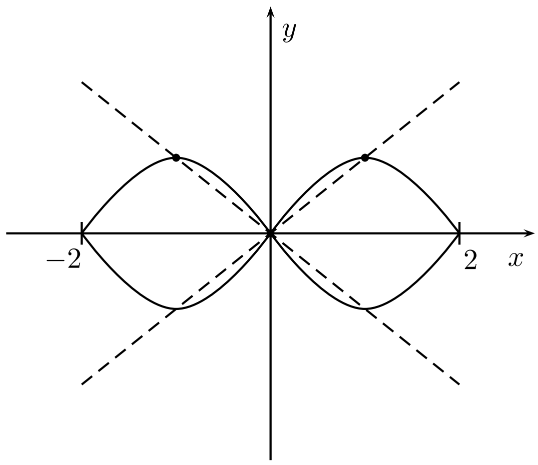
Evaluate this at $(x, y) = (0, 0)$:

$$\begin{aligned} e^0 + \cos 0\frac{dy}{dx} &= 0\frac{dy}{dx} + 0 + 0 \\ 1 + \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -1. \end{aligned}$$

The tangent is a line passing through $(x, y) = (0, 0)$ with gradient 1. The equation is therefore

$$\begin{aligned} y - 0 &= -1(x - 0) \\ y &= -x. \end{aligned}$$

iv) Diagram as below,



v) a)

$$|z| = \sqrt{2+2} = 2.$$

b)

$$\text{Arg}(z) = \tan^{-1}\left(\frac{-\sqrt{2}}{\sqrt{2}}\right) = -\frac{\pi}{4}.$$

c)

$$\begin{aligned} z &= 2e^{-\frac{\pi}{4}i} \\ z^6 &= 2^6 e^{-\frac{6\pi}{4}i} \\ &= 2^6 e^{-\frac{3\pi}{2}i} \\ &= 2^6 i \\ &= 64i. \end{aligned}$$

vi) a)

$$\begin{aligned} AB &= \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 0 & 2 \\ 2 & -2 & 1 \end{pmatrix}. \end{aligned}$$

b) AB^T does not exist as A is order (2×2) and B^T is order (3×2), and so the number of columns in A is not equal to the number of rows in B^T.

vii) a)

$$z^5 = -1 = e^{i(\pi + 2k\pi)}, \quad k \in \mathbb{Z}.$$

So,

$$\begin{aligned} z &= e^{i\frac{\pi}{5}(2k+1)} \\ &= e^{i\frac{\pi}{5}}, e^{-i\frac{\pi}{5}}, e^{i\frac{3\pi}{5}}, e^{-i\frac{3\pi}{5}}, -1, \quad \text{listing the principal set.} \end{aligned}$$

b) Using these solutions,

$$\begin{aligned} z^5+1 &= \left(z-e^{i\frac{\pi}{5}}\right)\left(z-e^{-i\frac{\pi}{5}}\right)\left(z-e^{i\frac{3\pi}{5}}\right)\left(z-e^{-i\frac{3\pi}{5}}\right)(z+1) \\ &= \left(z^2-2\cos\left(\frac{\pi}{5}\right)z+1\right)\left(z^2-2\cos\left(\frac{3\pi}{5}\right)z+1\right)(z+1). \end{aligned}$$

3. i) The line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ can be written as $\begin{cases} x = 1+t \\ y = -1+2t \\ z = -1+t \end{cases}$.

For the point of intersection, we substitute the parametric equations for the line into the Cartesian equation of the plane.

$$\begin{aligned} 4(1+t)-5(-1+2t)+3(-1+t) &= 0 \\ -3t+6 &= 0 \\ t &= 2 \end{aligned}$$

Hence the point of intersection is $\begin{pmatrix} 1+2 \\ -1+2(2) \\ -1+2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$.

ii) a) Expand $\det(M)$ along the first row.

$$\begin{vmatrix} 1 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 2 & \alpha \end{vmatrix} = \begin{vmatrix} 5 & 1 \\ 2 & \alpha \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 0 & \alpha \end{vmatrix} = (5\alpha-2)-2(2\alpha) = \alpha-2.$$

b) M does not have an inverse if and only if $\det(M)=0$. That is, $\alpha = 2$.

c) When $\alpha = 1$, the matrix M is invertible. We can find the inverse of M by reducing the augmented matrix $(M|I)$ to reduced row-echelon form.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 5 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 4 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 = -R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -4 & 2 & -1 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 - R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & -4 & 2 & -1 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & -2 \\ 0 & 1 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & -4 & 2 & -1 \end{array}\right)$$

Hence $M^{-1} = \begin{pmatrix} -3 & 2 & -2 \\ 2 & -1 & 1 \\ -4 & 2 & -1 \end{pmatrix}$.

iii) a) The cross product

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{vmatrix} = \mathbf{e}_1 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - \mathbf{e}_2 \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} + \mathbf{e}_3 \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$$

b) The vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to the plane. Hence, a point-normal form of the plane is

$$\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, the Cartesian equation of the plane is $x - 2y - z = -9$.

iv) a) The vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. That is,

$$0 + 3 - 12 + 3\beta = 0, \qquad \text{i.e.} \quad \beta = 3.$$

b) For the value $\beta = 1$,

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}^2|} \right) \mathbf{u} = \frac{0 + 3 - 12 + 3}{2^2 + 1^2 + 4^2 + 3^2} \begin{pmatrix} 2 \\ 1 \\ 4 \\ 3 \end{pmatrix} = -\frac{1}{5} \begin{pmatrix} 2 \\ 1 \\ 4 \\ 3 \end{pmatrix}.$$

v) a) $\mathbf{m} = \dfrac{1}{2}(\mathbf{a} + \mathbf{b}) = \begin{pmatrix} 2 \\ \frac{3}{2} \end{pmatrix}$.

b) Since $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we may take $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which is perpendicular to $\overrightarrow{AB}$.

c) The perpendicular bisector of AB is the line whose points are equidistant from A and B. A parametric vector equation for this line is

$$\mathbf{x} = \begin{pmatrix} 2 \\ \frac{3}{2} \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

d) Since the centre is equidistant from A, B and D, the centre is the intersection of the two perpendicular bisectors. At the point of intersection, we have

$$\begin{aligned} \begin{pmatrix} 2 \\ \frac{3}{2} \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \end{aligned}$$

At the intersection, the value of λ is 1. Hence the position vector of the point of intersection is $\begin{pmatrix} 2 \\ \frac{3}{2} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ \frac{3}{2} \end{pmatrix}$.

4. i) Let $u = \ln(x)$, and so $du = \dfrac{dx}{x}$. Substituting:

$$\int \frac{\cos(\ln(x))}{x}\, dx = \int \cos u\, du = \sin u + C = \sin(\ln(x)) + C.$$

ii) Let $f(x) = e^{-x}$ and $g(x) = -e^{-x}$. Then for all x

$$g(x) \le e^{-x}\sin(x) \le f(x).$$

Since

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} f(x) = 0$$

the Pinching theorem implies that $\lim\limits_{x\to\infty} e^{-x}\sin(x)$ exists and also equals 0.

iii) For all $x \ge 0$, we have $0 \le \dfrac{1}{x^2 + e^x} \le e^{-x}$.

Now $\displaystyle\int_0^\infty e^{-x}\, dx = \lim_{R\to\infty} [-e^{-x}]_0^R = \lim_{r\to\infty} \left(-e^{-R} + 1\right) = 1$ converges.

Thus, by the comparison test $\displaystyle\int_0^\infty \frac{dx}{x^2 + e^x}$ converges too.

iv) a) $F = \dfrac{d}{dx} \displaystyle\int_0^{x^2} \sin(\sqrt{t})\, dt$.

b) Let $u(x) = x^2$ and let $g(u) = \int_0^u \sin(\sqrt{t})\, dt$. Then, using the chain rule, $F = \dfrac{d}{dx} g(u(x)) = g'(u(x))\, u'(x)$.

By the Fundamental Theorem of Calculus $g'(u) = \sin(\sqrt{u})$, and so

$$F = \sin(\sqrt{x^2}) \cdot 2x = 2x\sin(|x|).$$

v) a) Since p is a polynomial it is continuous on $[1, 2]$. Now $p(1) = -2 < 0$ and $p(2) = 9 > 0$, so by the Intermediate Value theorem there is a value $c \in (1, 2)$ so that $p(c) = 0$.

b) $p'(x) = 3x^2 + 4 > 0$ for all $x \in [1, 2]$. Therefore p is increasing and hence can only take on any value at most once. Combined with (a) this implies that p has exactly one root in $[1, 2]$.

c) The Inverse Function theorem says that g is differentiable with

$$g'(0) = \frac{1}{p'(\alpha)} = \frac{1}{3\alpha^2 + 4}.$$

vi) Let

$$f(x) = \ln(1 + x) - \frac{x}{1 + x}.$$

We need to show that $f(x) > 0$ for all $x > 0$.

Suppose then that $x > 0$. Since f is continuous on $[0, x]$ and differentiable on $(0, x)$, the Mean Value Theorem says that there exists $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}.$$

Now, differentiating f at c,

$$f'(c) = \frac{1}{1 + c} - \frac{(1 + c) - c}{(1 + c)^2} = \frac{(1 + c) - 1}{(1 + c)^2} = \frac{c}{(1 + c)^2}$$

and so for any $c \in (0, x)$, $f'(c) > 0$. It follows then that

$$f(x) = xf'(c) > 0.$$

MATH1131 June 2014 Solutions

1. i) a) $2z - \overline{w} = 10 + 5i$

   b) $5(w - i)/z = 3 + i$.

   c) $|zw| = |z||w| = \sqrt{50} \times 5 = 25\sqrt{2}$.

   d) $zw = 25 + 25i$ and so $\text{Arg}(zw) = \frac{\pi}{4}$.

   e) Since both z and w lie in the first quadrant, the sum of their arguments lies between 0 and π. Hence $\text{Arg}(z)+$ $\text{Arg}(w)$ $=\text{Arg}(zw) = \pi/4$.

   f) $zw = 25\sqrt{2}e^{\frac{i\pi}{4}}$ so $(zw)^{40} = (25\sqrt{2})e^{10\pi i} = 5^{80}2^{20}$.

   ii) a)

$$[A|\mathbf{b}] = \left(\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 1 & -3 & 1 & 1 \\ 2 & -3 & -1 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

   b) Let $z = \lambda$ then $y = \lambda$ and $x = 1 + 2\lambda$. Hence the general solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}.$$

   iii) a)

$$\lim_{x\to\infty} \frac{6x^2 + \sin x}{4x^2 + \cos x} = \lim_{x\to\infty} \frac{6 + \frac{\sin x}{x^2}}{4 + \frac{\cos x}{x^2}} = \frac{3}{2}$$

   since $\cos x$ and $\sin x$ are bounded.

   b) Using L'Hôpital's Rule twice, and checking the necessary conditions,

$$\lim_{x\to 0} \frac{e^{2x} - 2x - 1}{4x^2} = \lim_{x\to 0} \frac{2e^{2x} - 2}{8x} = \lim_{x\to 0} \frac{4e^{2x}}{8} = \frac{1}{2}.$$

   iv) For $x > 4$, $g(x) = \frac{x^2-16}{x-4} = x + 4$ and so $\lim_{x\to 4^+} g(x) = 8$.

   For $3 < x < 4$, $g(x) = \frac{-(x^2-16)}{x-4} = -(x + 4)$ and so $\lim_{x\to 4^-} g(x) = -8$.

   Since these limits have different values, so value of $g(4)$ may be given to make g continuous at $x = 4$.

   v) a) f is a polynomial and so is continuous on $[0, 2]$.
   Now $f(0) = -2 < 0$ and $f(2) = 40 > 0$ and so by the intermediate value theorem, f has at least one zero in the interval $(0, 2)$.

   b) $f'(x) = 5x^4 + 3x^2 + 1 > 1$ for all real x and so f is an increasing function. Hence f has only one real root.

2. i) Let $u = \log x$, $\frac{du}{dx} = \frac{1}{x}$

$$\begin{aligned}\int \frac{dx}{x\left(1 + (\log x)^2\right)} &= \int \frac{du}{1 + u^2} \\ &= \tan^{-1} u + C \\ &= \tan^{-1}(\log x) + C\end{aligned}$$

ii) a)

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

b)

$$\begin{aligned} RHS &= 2\sinh x \cosh x \\ &= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) \\ &= \frac{1}{2}\left(e^{2x} - e^{-2x}\right) \\ &= \sinh(2x) \\ &= RHS \end{aligned}$$

i.e. $\sinh(2x) = 2\sinh x \cosh x$.

iii)

$$\begin{aligned} \int_0^{\frac{\pi}{3}} x\sin(2x)dx &= \left[x\left(-\frac{\cos(2x)}{2}\right)\right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} 1\left(-\frac{\cos(2x)}{2}\right)dx \\ &= \frac{\pi}{3}\left(-\frac{\cos\left(\frac{2\pi}{3}\right)}{2}\right) - 0 + \frac{1}{2}\left[\frac{\sin(2x)}{2}\right]_0^{\frac{\pi}{3}} \\ &= \frac{\pi}{12} + \frac{1}{4}\frac{\sqrt{3}}{2} \\ &= \frac{\pi}{12} + \frac{\sqrt{3}}{8}. \end{aligned}$$

iv)

$$\left(A^TA\right)^{-1}\left(A^TA\right)^T = \left(A^TA\right)^{-1}\left(A^TA\right) = I$$

v) a)

$$\begin{aligned} \Pi &= \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + \lambda_1\left[\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}\right] + \lambda_2\left[\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}\right], \quad \lambda_1, \lambda_2 \in \mathbb{R} \\ &= \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + \lambda_1\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda_2\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}. \end{aligned}$$

b)

$$\begin{aligned}
\mathbf{n} &= \vec{AB} \times \vec{AC} \\
&= \left[\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}\right] \times \left[\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}\right] \\
&= \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ 3 & 0 & 1 \\ 3 & 1 & 1 \end{vmatrix} \\
&= \mathbf{e_1}(0 \times 1 - 1 \times 1) - \mathbf{e_2}(3 \times 1 - 1 \times 3) + \mathbf{e_3}(3 \times 1 - 0 \times 3) \\
&= \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}.
\end{aligned}$$

c) Π in cartesian coordinates is

$$\begin{aligned}
-1(x-0) + 0(y-1) + 3(z-3) &= 0 \\
-x + 3(z-3) &= 0.
\end{aligned}$$

d)

$$\begin{aligned}
\text{Area } ABC &= \frac{1}{2}\left|\vec{AB} \times \vec{AC}\right| \\
&= \frac{1}{2}|\mathbf{n}| \\
&= \frac{1}{2}\sqrt{(-1)^2 + (0)^2 + (3)^2} \\
&= \frac{\sqrt{10}}{2}.
\end{aligned}$$

e)

$$\vec{PA} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \\ 3 \end{pmatrix}.$$

$$
\begin{aligned}
\text{min distance} &= \frac{\left|\vec{PA}\cdot\mathbf{n}\right|}{|\mathbf{n}|}\\
&= \frac{\left|\begin{pmatrix}-5\\-2\\3\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\3\end{pmatrix}\right|}{\left|\begin{pmatrix}-1\\0\\3\end{pmatrix}\right|}\\
&= \frac{5+0+9}{\sqrt{(-1)^2+(0)^2+(3)^2}}\\
&= \frac{14}{\sqrt{10}}.
\end{aligned}
$$

vi)

$$
\begin{aligned}
F &= 2A+B+C\\
I &= 2A+A^2+A^3\\
A^{-1} &= 2I+A+A^2\\
&= 2I+A+B\\
&= 2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\begin{pmatrix}0&1&-2\\1&-1&1\\1&-1&0\end{pmatrix}+\begin{pmatrix}-1&1&1\\0&1&-3\\-1&2&-3\end{pmatrix}\\
&= \begin{pmatrix}1&2&-1\\1&2&-2\\0&1&-1\end{pmatrix}.
\end{aligned}
$$

3. i) a) DIAGRAM

b) $|\mathbf{F}| = \sqrt{(\sqrt{2}-1)^2+(\sqrt{2}-1)^2} = \sqrt{6-4\sqrt{2}}$;
Equivalently, $|\mathbf{F}| = 2-\sqrt{2}$ (these two expressions are equal) and $\mathbf{F}$ has direction North-East.

ii) a)

$$
\begin{vmatrix}1&-1&1\\2&1&3\\-1&2&1\end{vmatrix} = \begin{vmatrix}1&-1&1\\0&3&1\\0&1&2\end{vmatrix} = -\begin{vmatrix}1&-1&1\\0&1&2\\0&3&1\end{vmatrix} = -\begin{vmatrix}1&-1&1\\0&1&2\\0&0&-5\end{vmatrix} = 5
$$

b) $N^{-1} = \dfrac{1}{3\times 2-1\times 4}\begin{pmatrix}2&-1\\-4&3\end{pmatrix} = \dfrac{1}{2}\begin{pmatrix}2&-1\\-4&3\end{pmatrix}$

iii) a) $p(1) = 1^4+2\cdot 1^2-3 = 0$ and $p(-1) = (-1)^4+2(-1)^2-3 = 0$

b) $p(z) = (z^2-1)(z^2+3)$.

c) $\pm 1, \pm\sqrt{3}\,i$.

d) $p(z) = (z-1)(z+1)(z-\sqrt{3}\,i)(z+\sqrt{3}\,i)$

iv) Substitute $x = 3+\lambda$, $y = 2+2\lambda$, and $z = 1+3\lambda$ into the Cartesian equation:

$$
0 = 6(3+\lambda)+8(2+2\lambda)-9(1+3\lambda) = 25-5\lambda\,.
$$

We see that $\lambda = 5$,

so the point of intersection is represented by $\vec{x} = \begin{pmatrix}3\\2\\1\end{pmatrix} + 5\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}8\\12\\16\end{pmatrix}$.

v) a) **Proof**. By De Moivre's Theorem,

$$\begin{aligned}\cos 3\theta + i\,\sin 3\theta = e^{i3\theta} = (e^{i\theta})^3 &= (\cos\theta + i\,\sin\theta)^3\\ &= \cos^3\theta + i3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta\,.\end{aligned}$$

By comparing the real parts of the equation, we see that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta) = 4\cos^3\theta - 3\cos\theta\,.$$

Hence, $4\cos^3\theta = 3\cos\theta + \cos 3\theta$. $\square$

b) By part a) with $\theta = \dfrac{\pi}{9}$,

$$q\Big(2\cos\frac{\pi}{9}\Big) = 8\cos^3\frac{\pi}{9} - 6\cos\frac{\pi}{9} - 1 = 2\Big(3\cos\frac{\pi}{9} + \cos\frac{3\pi}{9}\Big) - 6\cos\frac{\pi}{9} - 1 = 2\frac{1}{2} - 1 = 0\,.$$

Hence, $2\cos\dfrac{\pi}{9}$ is a root of the polynomial $q(z) = z^3 - 3z - 1$.

vi) a) $\vec{u}\cdot\vec{v} = 1-\beta$, so the vectors are orthogonal when $\beta = 1$.

b)

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} = \frac{\begin{pmatrix}0\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\0\end{pmatrix}}{\left|\begin{pmatrix}2\\1\\0\end{pmatrix}\right|^2}\begin{pmatrix}2\\1\\0\end{pmatrix} = \frac{1}{5}\begin{pmatrix}2\\1\\0\end{pmatrix}$$

c)

$$\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} = \frac{\begin{pmatrix}0\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\\beta\end{pmatrix}}{\left|\begin{pmatrix}0\\1\\-1\end{pmatrix}\right|\left|\begin{pmatrix}2\\1\\\beta\end{pmatrix}\right|} = \frac{1-\beta}{\sqrt{2}\sqrt{5+\beta^2}}\,,$$

so $5+\beta^2 = (1-\beta)^2 = \beta^2 - 2\beta + 1$. Hence, $\beta = -2$.

4. i) Let $F(t) = \int \cos\left(\frac{1}{t}\right)\,dt$ so $F'(t) = \cos\left(\frac{1}{t}\right)$. By the fundamental theorem of calculus

$$\int_{x^2}^{x^3}\cos\left(\frac{1}{t}\right)\,dt = F(x^3) - F(x^2).$$

Then

$$\begin{aligned}\frac{d}{dx}\int_{x^2}^{x^3}\cos\left(\frac{1}{t}\right)\,dt &= \frac{d}{dx}\left(F(x^3) - F(x^2)\right)\\ &= F'(x^3)\,\frac{d\,x^3}{dx} - F'(x^2)\,\frac{d\,x^2}{dx}\\ &= \cos\left(\frac{1}{x^3}\right)\,3x^2 - \cos\left(\frac{1}{x^2}\right)\,2x.\end{aligned}$$

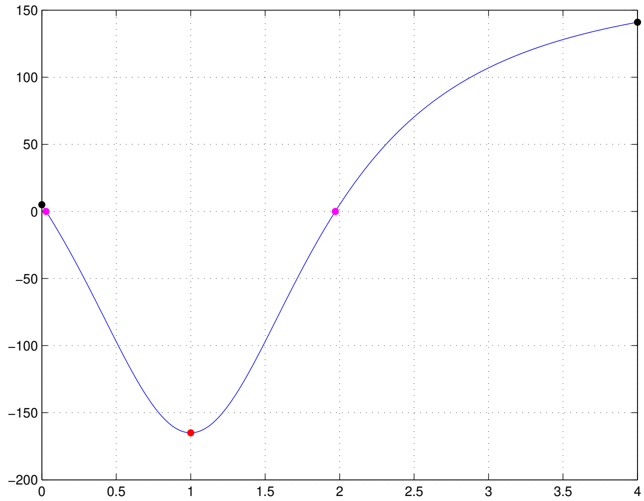
ii) a) The MAPLE session defines the function

$$f(x) = \frac{175x^2 - 350x + 10}{x^2 - 2x + 2}.$$

and you are given that f is differentiable for all real x.
The MAPLE session gives the following information

- $f(0) = 5$ and $f(4) = 141$.
- f has two zeros where $f(x) = 0$, close to $x = 0.029$ and $x = 1.971$.
- f has one stationary point $f'(1) = 0$ where $f(1) = -165$.

Thus a sketch of the function f over the interval $[0, 4]$ is



b)

$$\left| \frac{175x^2 - 350x + 10}{x^2 - 2x + 2} \right| = |f(x)|$$

is the absolute value of the function $f(x)$ from part a). As $|f(x)|$ is continuous on the closed interval $[0, 4]$ the (global) minimum and maximum both exist, and $|f(x)|$ has

- a minimum value of 0 which occurs at the two points $x = 1 \pm \sqrt{1155}/35$
- a maximum value of 165 at $x = 1$.

iii) As $\cos^2 x \geq 0$ for all x,

$$0 \leq \frac{1}{e^{2x} + \cos^2 x} \leq \frac{1}{e^{2x}}.$$

Hence

$$0 \leq K = \int_0^\infty \frac{dx}{e^{2x} + \cos^2 x} \leq \int_0^\infty \frac{dx}{e^{2x}}.$$

Now

$$\int_0^\infty \frac{dx}{e^{2x}} = \lim_{b \to \infty} \int_0^b e^{-2x}\, dx = \lim_{b \to \infty} \left[-\frac{1}{2} e^{-2x} \right]_0^b = \lim_{b \to \infty} -\frac{1}{2} e^{-2b} + \frac{1}{2} = \frac{1}{2}.$$

As $K \leq \int_0^\infty \frac{dx}{e^{2x}}$ which converges, the comparison test implies that

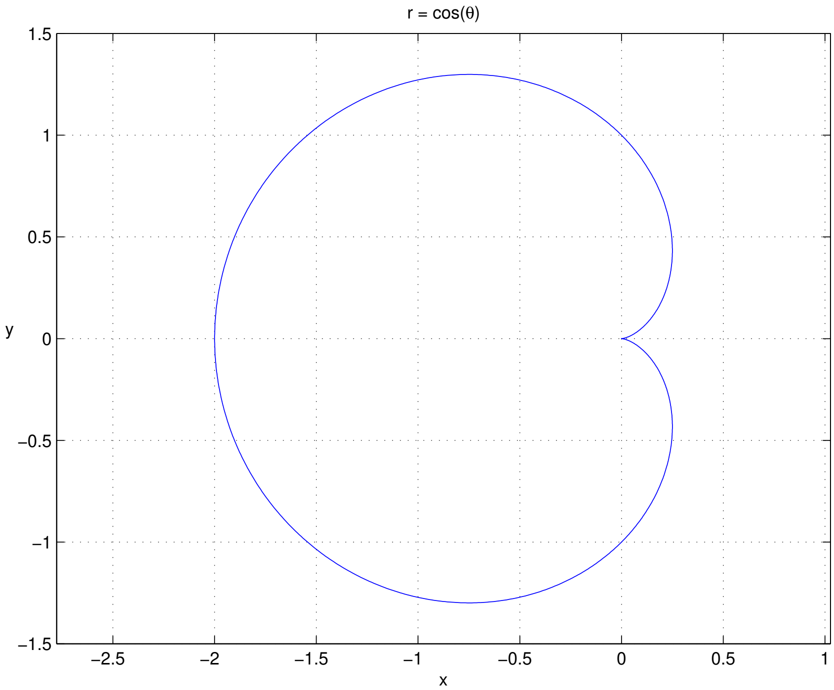
$$K = \int_0^\infty \frac{dx}{e^{2x} + \cos^2 x}$$

also converges.

iv) The polar curve $r = 1 - \cos\theta$ and has the following values

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = 1 - \cos\theta$	0	1	2	1
$x = r\cos\theta$	0	0	-2	0
$y = r\sin\theta$	0	1	0	-1

Noting that the curve is symmetric about $\theta = 0$ ($y = 0$), a plot in the xy-plane is



v) a) Mean Value Theorem: If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), there there exists a $c \in (a, b)$, such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

b) Let $f(t) = \sin^{-1} t$, so

$$f'(t) = \frac{1}{\sqrt{1 - t^2}}.$$

Moreover for any t with $-1 < t < 1$, we have $0 < 1 - t^2 \leq 1$, so $f'(t) \geq 1$. Using the mean value theorem on $[x, y]$ where $-1 < x < y < 1$, there exists a c with $-1 < x < c < y < 1$ such that

$$\frac{\sin^{-1} y - \sin^{-1} x}{y - x} = f'(c) = \frac{1}{\sqrt{1 - c^2}} \geq 1.$$

As multiplying both sides by $y - x > 0$ preserves the inequality, this gives

$$\sin^{-1} y - \sin^{-1} x \geq y - x.$$

vi) a) Let the length of $EP = L_1$ and the length of $FP = L_2$. As the triangles AEP and BFP are right-angled, by Pythagoras $L_1^2 = a^2 + x^2$ and $L_2^2 = b^2 + (c - x)^2$, so

$$L = L_1 + L_2 = \sqrt{a^2 + x^2} + \sqrt{b^2 + (c - x)^2}.$$

b) Differentiating $L = \sqrt{a^2 + x^2} + \sqrt{b^2 + (c-x)^2}$ gives

$$\begin{aligned} \frac{dL}{dx} &= \frac{1}{2}\frac{2x}{\sqrt{a^2+x^2}} + \frac{1}{2}\frac{-2(c-x)}{\sqrt{b^2+(c-x)^2}} \\ &= \frac{x}{L_1} - \frac{(c-x)}{L_2}. \end{aligned}$$

Hence

$$\frac{dL}{dx} = 0 \Longrightarrow \cos\theta = \frac{x}{L_1} = \frac{(c-x)}{L_2} = \cos\phi.$$

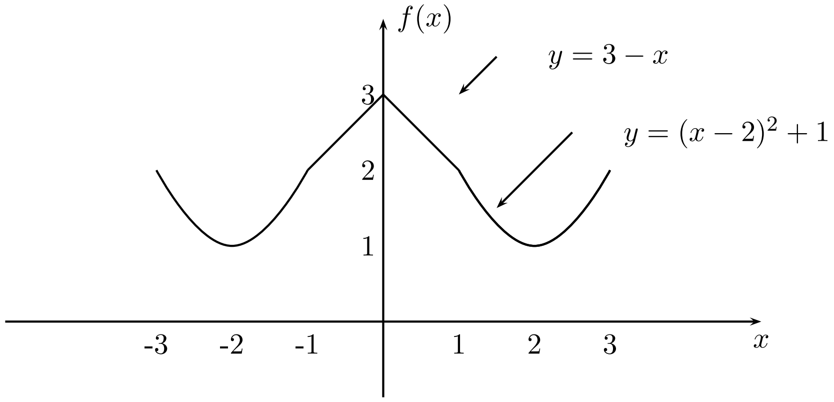
c) As the triangles AEP and BFP are similar when $\frac{dL}{dx} = 0$ (right angles, $\theta = \phi$)

$$\frac{x}{a} = \frac{c-x}{b} \Longrightarrow x = \frac{ac}{a+b}.$$

MATH1131 June 2015 Solutions

1. i) $\displaystyle\lim_{x\to\infty}\frac{x+\sin x}{2x}=\lim_{x\to\infty}\frac{1+\frac{\sin x}{x}}{2}=\frac{1}{2}.$

ii) a) $f(\frac{3}{2})=\frac{5}{4}$

b) 

c)

$$\lim_{x\to 1}f(x)=f(1)=2.$$

$f'(x)=-1$ for $0<x<1$, and $f'(1)=1$ for $-1<x<0$, hence $f'(0)$ does not exist.

iii) Let $x=t^2, t>0$, then $\frac{dx}{dt}=2t$ and

$$I_1=\int\frac{2t}{1+t}\,dt=\int 2-\frac{2}{1+t}\,dt=2t-2\log(1+t)+C$$

$$=2\sqrt{x}-2\log(1+\sqrt{x})+C.$$

iv) Since the limit is of indeterminate form, we apply L'Hopital's rule.

$$\lim_{x\to 0}\frac{1-\cos\left(\frac{x}{2}\right)}{x^2}=\lim_{x\to 0}\frac{\frac{1}{2}\sin\left(\frac{x}{2}\right)}{2x}=\lim_{x\to 0}\frac{\frac{1}{4}\cos\left(\frac{x}{2}\right)}{2}=\frac{1}{8}.$$

v) a) $z-\overline{w}=2+3i.$

b) $10w/(z-2)=11+3i.$

c) Since $|z|=\sqrt{26}$ and $|w|=\sqrt{13}$. Hence $|(z/w)^8|=2^4=16.$

d) $zw=13(1+i)$ and so $\operatorname{Arg}(zw)=\frac{\pi}{4}.$

e) $|zw|^8=13^8\times 16$ and $\operatorname{Arg}((zw)^8)=8\times\frac{\pi}{4}=2\pi$. Hence $(zw)^8=13^8\times 16.$

vi)

$$\mathbf{x}=\begin{pmatrix}1\\1\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\-3\end{pmatrix}+\mu\begin{pmatrix}-2\\3\\-1\end{pmatrix},\lambda,\mu\in\mathbb{R}.$$

vii) a) From the MAPLE, $A^3-6A^2-9A=I$. Hence A^{-1} exists and $A^{-1}=A^2-6A-9I$

$$=\begin{pmatrix}-1&-2&1\\-1&-7&2\\1&3&-1\end{pmatrix}$$

b) $(A^T)^2 = (A^2)^T = B^T$

$$= \begin{pmatrix} 14 & 5 & 25 \\ 4 & 2 & 9 \\ 19 & 8 & 38 \end{pmatrix}$$

2. i)

$$\begin{aligned} 40 + 42i &= z^2 \\ &= (a+ib)^2 \\ &= a^2 - b^2 + i2ab \\ \text{i.e. } a^2 - b^2 &= 40 \\ 2ab &= 42 \\ b &= \frac{21}{a} \\ \text{by inspection, or} \\ a^2 - \frac{21^2}{a^2} &= 40 \\ a^4 - 441 &= 40a^2 \\ a^2 &= 20 \pm \frac{1}{2}\sqrt{3364} \\ &= -9, 49 \\ a &= \pm 3i, \pm 7 \\ \text{but } a, b \in \mathbb{R} \text{ so } a &= \pm 7, \text{ and} \\ b &= \frac{21}{\pm 7} = \pm 3. \end{aligned}$$

So $z = 7 + 3i, -7 - 3i$.

ii) Converting to augmented form,

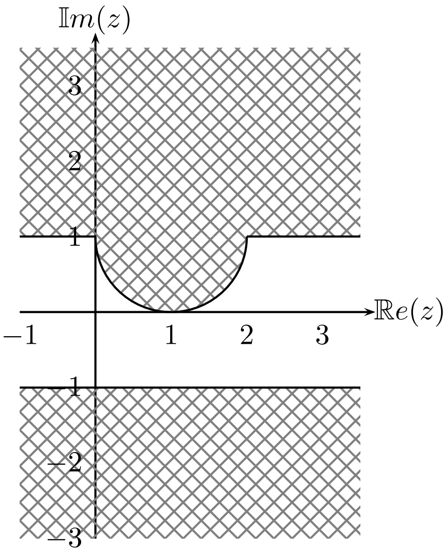
$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 1 & 1 & -1 & b_2 \\ 2 & 1 & -3 & b_3 \end{array}\right)$$

Row reducing,

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -3 & -3 & b_3 - 2b_1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - 2b_1 - 3(b_2 - b_1) \end{array}\right)$$

In order for a solution to exist, $b_3 - 2b_1 - 3(b_2 - b_1) = 0$, i.e. $b_3 + b_1 - 3b_2 = 0$.

iii) 

iv) a)

$$\begin{aligned}
\mathbf{u}.\mathbf{v} &= |\mathbf{u}||\mathbf{v}|\cos\theta \\
2-1+2 &= \sqrt{4+1+1}\sqrt{1+1+4}\cos\theta \\
3 &= 6\cos\theta \\
\cos\theta &= \frac{1}{2} \\
\theta &= \frac{\pi}{3}.
\end{aligned}$$

b)

$$\begin{aligned}
\mathbf{u}\times\mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & -1 & -2 \end{vmatrix} \\
&= \mathbf{i}(-2-1) - \mathbf{j}(-4+1) + \mathbf{k}(-2-1) \\
&= \begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix} = 3\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.
\end{aligned}$$

v) a)

$$f(x) = \begin{cases} 1+ax^2, & x \le 1, \\ bx + 2x^3, & x > 1, \end{cases}$$

a)

$$\left.\begin{array}{rclcl}
\lim_{x\to1^-} f(x) & = & \lim_{x\to1^-} bx+2x^3 & = & b+2 \\
\lim_{x\to1^+} f(x) & = & \lim_{x\to1^+} 1+ax^2 & = & 1+a
\end{array}\right]$$

$$\text{i.e. } b+2 = 1+a \text{ for } \lim_{x\to1} f(x) \text{ to exist.}$$

$$a = b+1.$$

b)

$$f'(x) = \begin{cases} 2ax, & x < 1, \\ b + 6x^2, & x > 1, \end{cases}$$

$$\left.\begin{array}{rcccl} \lim_{x\to 1^-} f'(x) & = & \lim_{x\to 1^-} b + 6x^2 & = & b+6 \\ \lim_{x\to 1^+} f'(x) & = & \lim_{x\to 1^+} 2ax & = & 2a \end{array}\right]$$

i.e. $b + 6 = 2a$ for $\lim\limits_{x\to 1} f'(x)$ to exist.

Also require f to be continuous at $x = 1$, so additionally $a = b + 1$ from part (a).

$$\begin{aligned} b + 6 &= 2a \\ b + 6 &= 2(b+1) \\ b &= 4 \\ a &= b + 1 = 5. \end{aligned}$$

So $a = 5$ and $b = 4$ for f to be differentiable at $x = 1$.

vi)

$$\begin{aligned} I_2 &= \int x^3 \ln x dx \\ &= \frac{x^4}{4}\ln x - \int \frac{x^4}{4}\frac{1}{x} dx \\ &= \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3 dx \\ &= \frac{x^4}{4}\ln x - \frac{1}{4}\frac{x^4}{4} + C \\ &= \frac{x^4}{4}\left(\ln x - \frac{1}{4}\right) + C. \end{aligned}$$

vii) a) $(x, y) = (0, 1) \implies (r, \theta) = \left(1, \frac{\pi}{2}\right)$.

$$\begin{aligned} x &= r\cos\theta = (1 + \cos\theta)\cos\theta \text{ and} \\ y &= r\sin\theta = (1 + \cos\theta)\sin\theta. \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\ &= \frac{\frac{d}{d\theta}\left((1+\cos\theta)\sin\theta\right)}{\frac{d}{d\theta}\left((1+\cos\theta)\cos\theta\right)} \\ &= \frac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta - 2\cos\theta\sin\theta} \\ &= \frac{2\cos^2\theta + \cos\theta - 1}{-\sin\theta - 2\cos\theta\sin\theta} \end{aligned}$$

When $\theta = \frac{\pi}{2}$,

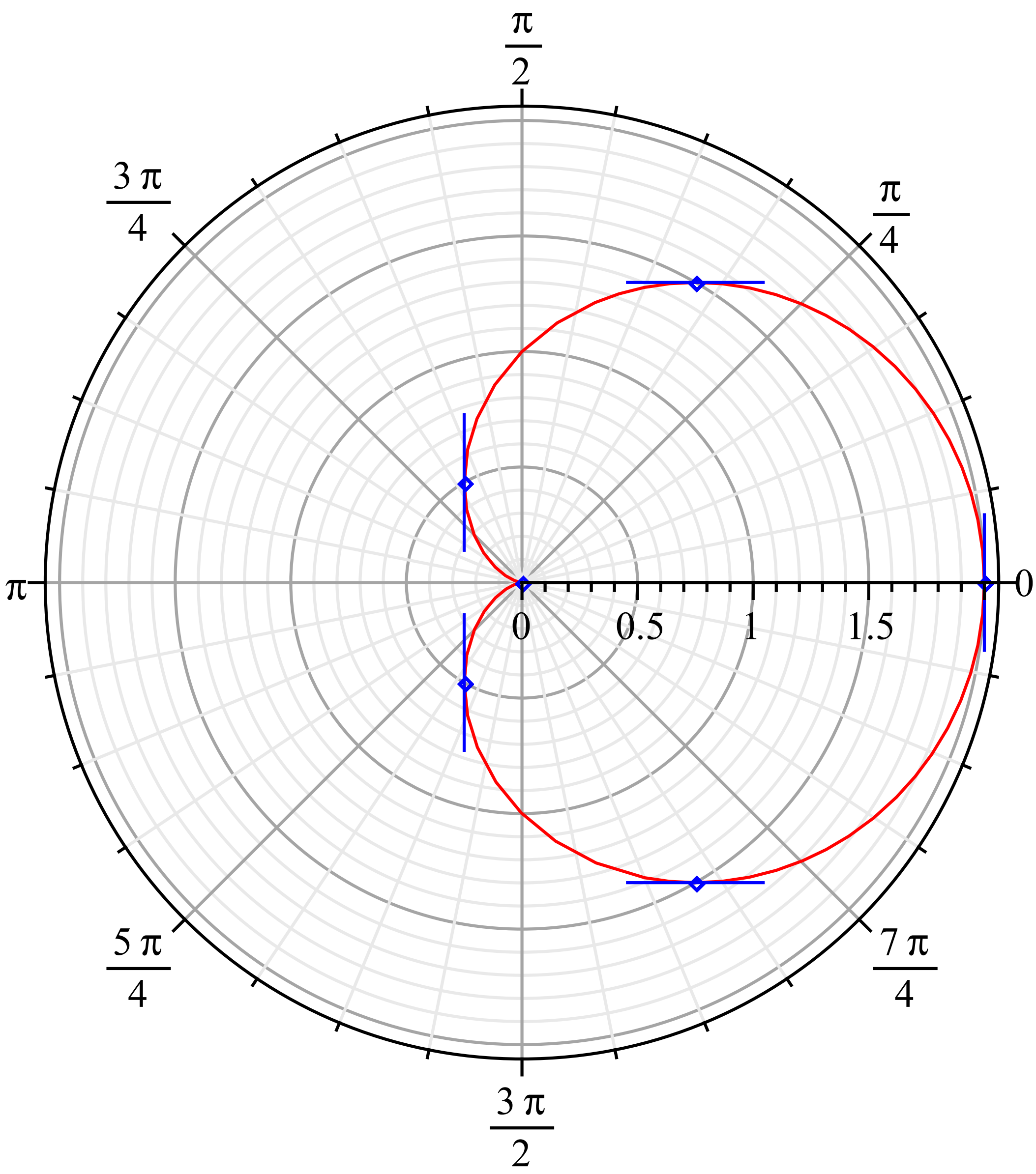
$$\frac{dy}{dx} = \frac{-1}{-1} = 1.$$

i.e. the slope of the tangent at $(x, y) = (0, 1)$ is 1.

b)

$$\begin{aligned} \frac{dy}{dx} = 0 \implies \frac{dy}{d\theta} &= 2\cos^2\theta + \cos\theta - 1 = 0 \\ (2\cos\theta - 1)(\cos\theta + 1) &= 0 \\ \cos\theta &= \frac{1}{2}, -1 \\ \theta &= \frac{\pi}{3}, \frac{5\pi}{3}, \pi. \end{aligned}$$

However, at $\theta = \pi$, both $\frac{dy}{d\theta} = 0$, and $\frac{dx}{d\theta} = 0$.
If we calculate the limiting value of $\frac{dy}{dx}$ at $\theta = \pi$ we get 0, indicating a cusp point.
So the points where the tangent is horizontal are $(r, \theta) = \left(\frac{3}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$.



c)

3. i) Consider the equation $2|z-3i| = |z+3i|$. Squaring both sides and using the identity $|z|^2 = z\bar{z}$ yields

$$\begin{aligned} 4\left(z\bar{z} + 3i(z-\bar{z}) + 9\right) &= z\bar{z} - 3i(z-\bar{z}) + 9 \\ \Rightarrow \quad 3z\bar{z} + 15i(z-\bar{z}) + 18 &= 0 \\ \Rightarrow \quad z\bar{z} + 5i(z-\bar{z}) + 9 &= 0 \\ \Rightarrow \quad z\bar{z} + 5i(z-\bar{z}) + 25 &= 16 \\ \Rightarrow \quad |z-5i|^2 &= 16 \\ \Rightarrow \quad |z-5i| &= 4\,. \end{aligned}$$

This is the equation of a circle, centre $(0,5)$ (on an Argand diagram) with radius 4.

ii) a) Since $z = 2i$ is a root of a polynomial with real coefficients then another root of the polynomial is the complex conjugate, i.e., $z = -2i$. Hence a quadratic factor of the polynomial is z^2+4. Using long division yields

$$\begin{aligned} z^4 - 3z^3 + 6z^2 - 12z + 8 &= (z^2+4)(z^2-3z+2) \\ &= (z^2+4)(z-1)(z-2)\,. \end{aligned}$$

b) Thus the four roots of $p(z)$ are $\pm 2i, 1, 2$.

iii) a) To find a point-normal form for the equation of the plane Π we need a point Q on the plane (to then create the position vector $\overrightarrow{OQ}$ where O is the origin) and a normal vector $\mathbf{n}$ to the plane. A point on the plane Π is $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and hence a position vector to Q on the plane Π is $\overrightarrow{OQ} = \begin{pmatrix}1\\0\\0\end{pmatrix}$. The coefficients of the cartesian form of the plane Π yield the normal vector $\mathbf{n} = \begin{pmatrix}1\\-3\\2\end{pmatrix}$. Hence a point-normal form for the plane Π is given by

$$\left(\mathbf{x} - \overrightarrow{OQ}\right)\cdot\mathbf{n} = 0 \quad \Rightarrow \quad \begin{pmatrix}x-1\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\-3\\2\end{pmatrix} = 0\,.$$

b) To show that the line ℓ is parallel to the plane Π we can show that the normal $\mathbf{n}$ to the plane Π and the direction vector $\mathbf{v}$ for the line ℓ are perpendicular, i.e., $\mathbf{n}\cdot\mathbf{v} = 0$. The direction vector $\mathbf{v}$ for the line ℓ is $\mathbf{v} = \begin{pmatrix}2\\0\\-1\end{pmatrix}$. Hence

$$\begin{pmatrix}1\\-3\\2\end{pmatrix}\cdot\begin{pmatrix}2\\0\\-1\end{pmatrix} = 2 + 0 - 2 = 0\,.$$

Thus the plane Π and line ℓ are parallel.

c) The shortest distance between a point P and a plane is given by the length of the projection of a vector from a point on the plane, say Q, and P, i.e., $\overrightarrow{QP}$ and a normal vector $\mathbf{n}$ to the plane. From part a) we have $\mathbf{n} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ and $\overrightarrow{QP} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$.

Thus

$$|\operatorname{proj}_{\mathbf{n}} \overrightarrow{QP}| = \frac{\left| \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \right|}{\sqrt{\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}}} = \frac{1}{\sqrt{14}}.$$

Hence the shortest distance between the point $P(4,2,2)$ and the plane Π is $\dfrac{1}{\sqrt{14}}$.

iv) a) Using elementary row operations to calculate the determinant yields

$$\begin{aligned} \begin{vmatrix} 1 & -1 & 3 \\ 3 & a & 2 \\ 2 & 1 & 1 \end{vmatrix} &= \begin{vmatrix} 1 & -1 & 3 \\ 0 & a+3 & -7 \\ 0 & 3 & -5 \end{vmatrix} \qquad R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1 \\ &= \begin{vmatrix} a+3 & -7 \\ 3 & -5 \end{vmatrix} \\ &= 6 - 5a\,. \end{aligned}$$

b) The matrix A will not have an inverse if $\det(A) = 0$, i.e., $a = \dfrac{6}{5}$.

c) If matrix B is an inverse for matrix A then $AB = I$ where I is the 3×3 identity matrix, i.e.,

$$\begin{pmatrix} 1 & -1 & 3 \\ 3 & a & 2 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 4 & -5 \\ 1 & b & 7 \\ 1 & -3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If we multiply row 2 of the first matrix by the 1st column of the second matrix yields $-3 + a + 2 = 0$ with $a = 1$. Also if we multiply the 1st row of the first matrix by the 2nd column of the second matrix yields $4 - b - 9 = 0$ with $b = -5$. (A quick check is to multiply the 2nd row of the first matrix by the 2nd column of the second matrix to verify it is equal to 1). Thus

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 3 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix} \quad \text{and} \quad A^{-1} = B = \begin{pmatrix} -1 & 4 & -5 \\ 1 & -5 & 7 \\ 1 & -3 & 4 \end{pmatrix}.$$

v) By definition, $XX^{-1} = I$, where I is the $n \times n$ identity matrix. Hence

$$\begin{aligned} \det(XX^{-1}) = \det(I) &= 1 \\ \Rightarrow \quad \det(X)\det(X^{-1}) &= 1 \qquad \text{since} \det(AB) = \det(A)\det(B) \\ \Rightarrow \quad 1 \times \det(X^{-1}) &= 1 \qquad \text{since} \det(X) = 1 \\ \Rightarrow \quad \det(X^{-1}) &= 1 \qquad \text{as required.} \end{aligned}$$

4. i) a)

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

b)

$$\frac{d(\cosh(6x))}{dx} = \frac{d}{dx}(\frac{e^{6x} + e^{-6x}}{2}) = 6(\frac{e^{6x} - e^{-6x}}{2}) = 6\sinh(6x).$$

c) Consider the definition in a) and note that

$$\sinh x = \frac{e^x - e^{-x}}{2} < e^x/2.$$

Taking logs of both sides yields

$$\ln(\sinh(x)) < \ln\left(\frac{e^x}{2}\right) = x - \ln 2.$$

ii) Using the fact that for $x > 0$,

$$\ln x < x$$

it implies that

$$\frac{\ln x}{x^3} < \frac{1}{x^2}.$$

Integrating both sides of the inequality from 1 ti ∞ yields

$$\int_1^\infty \frac{\ln x}{x^3}\,dx < \int_1^\infty \frac{dx}{x^2}.$$

To determine the integral, we write it terms of a proper integral, that is

$$\lim_{R\to\infty} \int_1^R \frac{dx}{x^2} = \lim_{R\to\infty}\left[1 - \frac{1}{R}\right] = 1.$$

Thus, the improper integral is bounded by 1,

$$\int_1^\infty \frac{\ln x}{x^3}\,dx < 1.$$

Alternatively,
we can tackle this question directly by parts:

$$\int_1^\infty \frac{\ln x}{x^3}\,dx = \lim_{R\to\infty}\left(\left[-\frac{\ln x}{2x^2}\right]_1^R + \int_1^R \frac{-1}{2x^3}\,dx\right) = 1/4.$$

iii) (a) Using the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\left(\int_0^{x^3} \cos(t^2)\,dt\right) = 3x^2\cos(x^6).$$

iv) a) Let $p(x) := x^3 + 3x + 1$.
Note that on the interval $[-1, 0]$ that $p(-1) = -3$ and $p(0) = 1$. The function p is a continuous function and thus by the Intermediate Value Theorem there is a point $c \in (-1, 0)$ such that $p(c) = 0$.

b) Notice that the function is always increasing since

$$p'(x) = 3x^2 + 3 > 0.$$

This means that the function is $1 - 1$ and has an inverse function $g(x)$.

c) Using the chain rule,

$$p'(x)g'(p(x)) = 1.$$

That is,

$$g'(p(x)) = \frac{1}{p'(x)}.$$

Now, if we solve the equation

$$p(x) = 1$$

then $x = 0$. Thus,

$$g'(1) = g'(p(0)) = \frac{1}{3}.$$

v) a) If f is differentiable on the open interval (a, b) and continuous on the $[a, b]$, then there exists a point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

b) Let $f(t) = \cos t$ and consider the interval $[x, y]$ then by the MVT, we have

$$\frac{\cos(y) - \cos(x)}{y - x} = \sin(c).$$

Now, the $|\sin c| \leq 1$ for any $c \in (x, y)$. If we take the absolute value of both sides, we obtain

$$\left|\frac{\cos(y) - \cos(x)}{y - x}\right| = |\sin(c)| \leq 1.$$

Rearranging, proves the inequality

$$|\cos(y) - \cos(x)| \leq |y - x|.$$

PAST HIGHER EXAM SOLUTIONS

MATH1141 June 2011 Solutions

2. i) a) No solutions requires both $6 - a^2 - a = 0$ and $2 - a \neq 0$. i.e.

$$\begin{aligned} -a^2 - a + 6 &= 0 \\ -(a-2)(a+3) &= 0 \\ a &= 2, -3 \end{aligned}$$

AND

$$\begin{aligned} 2 - a &\neq 0 \\ a &\neq 0. \end{aligned}$$

Putting these together we have, for no solutions, $a = -3$.

b) For a unique solution, we require $6 - a^2 - a \neq 0$, which (using part a) gives

$$a \neq 2 \quad \text{and} \quad a \neq -3.$$

c) For infinite solutions we require both $6 - a^2 - a = 0$ and $2 - a = 0$.
i.e. to satisfy both $a = 2$.

ii)

$$\begin{aligned} \begin{vmatrix} 2 & 0 & -1 \\ -1 & 3 & 0 \\ 5 & 7 & 3 \end{vmatrix} &= 2\begin{vmatrix} 3 & 0 \\ 7 & 3 \end{vmatrix} - 0\begin{vmatrix} -1 & 0 \\ 5 & 3 \end{vmatrix} - 1\begin{vmatrix} -1 & 3 \\ 5 & 7 \end{vmatrix} \\ &= 2(9-0) - 0(3-0) + (-1)(7-15) \\ &= 18 + 8 \\ &= 26. \end{aligned}$$

Alternatively an upper triangular form could be used, multiplying the pivots and adjusting for the number of swaps of rows performed to get to the reduced form.

iii)

$$\lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \nu \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

i.e.

$$\begin{aligned} \lambda &= 1 + \mu + \nu \\ 2\lambda &= \mu \\ 3\lambda &= -\nu \end{aligned}$$

Substituting the last 2 equations into the first we have

$$\begin{aligned} \lambda &= 1 + 2\lambda - 3\lambda \\ 2\lambda &= 1 \\ \lambda &= \frac{1}{2}. \end{aligned}$$

i.e.

$$\boldsymbol{x} = \frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{3}{2} \end{pmatrix},$$

Alternatively you could rearrange

$$\lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \mu \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \nu \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & -1 \\ 2 & -1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and put the problem in an augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right)$$

This can be row-reduced and back-substituted to find (λ, μ, ν) which can be substituted into the line (or plane) to find $\boldsymbol{x}$.

Another alternative is to convert the plane to cartesian form and then substitute the line and solve for λ.

iv) a)

$$\begin{aligned} \lim_{h\to 0} \frac{e^{-1/h^2}}{h} &= \lim_{h\to 0} \frac{e^{-1/h^2}}{h^2} h \\ &= \lim_{h\to 0} \frac{e^{-1/h^2}}{h^2} \lim_{h\to 0} h. \end{aligned}$$

Letting $x = \frac{1}{h^2}$, when $h \to 0$ then $x \to \infty$. i.e.

$$\begin{aligned} \lim_{h\to 0} \frac{e^{-1/h^2}}{h} &= \lim_{h\to 0} \frac{e^{-1/h^2}}{h^2} \lim_{h\to 0} h \\ &= \lim_{x\to\infty} xe^{-x} \lim_{h\to 0} h \\ &= 0 \cdot 0 \\ &= 0. \end{aligned}$$

Alternatively, you could use

$$\begin{aligned} \lim_{h\to 0} \frac{e^{-1/h^2}}{h} &= \lim_{h\to 0} \frac{h^{-1}}{e^{1/h^2}} \\ &= \text{"}\frac{\infty}{\infty}\text{"} \qquad \Longrightarrow \text{ use L'Hopital's rule} \\ &= \lim_{h\to 0} \frac{-h^{-2}}{-2h^{-3}e^{1/h^2}} \\ &= \lim_{h\to 0} \frac{h}{2e^{1/h^2}} \\ &= \frac{0}{\infty} \\ &= 0. \end{aligned}$$

b) Firstly, you should check $f(x)$ is continuous at $x = 0$.

$$\begin{aligned}
\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} e^{-1/x^2} = 0\\
\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^+} e^{-1/x^2} = 0\\
f(0) &= 0.
\end{aligned}$$

i.e. $f(x)$ is continuous at $x = 0$.
The definition of the derivative at $x = 0$ is

$$\begin{aligned}
f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}\\
&= \lim_{h\to 0} \frac{e^{-1/h^2} - 0}{h}\\
&= \lim_{h\to 0} \frac{e^{-1/h^2}}{h} \quad \text{which from part a)}\\
&= 0.
\end{aligned}$$

Thus $f(x)$ is differentiable at $x = 0$ (with $f'(0) = 0$).

v) a) The derivative at points other than the split points is

$$f(x) = \begin{cases} -\frac{2}{x^2} & \text{for } x < -1,\\ 2x & \text{for } -1 < x < \frac{6}{\pi},\\ -\frac{1}{x^2}\left(\frac{72}{\pi^2} - 2\right)\cos\frac{1}{x} & \text{for } x > \frac{6}{\pi}. \end{cases}$$

This is a split funciton, so the positions of the splits are possible critical points.

$x = -1$:

$$\begin{aligned}
\lim_{x\to -1^-} f(x) &= \frac{2}{-1} = -2\\
\lim_{x\to -1^+} f(x) &= (-1)^2 - 1 = 0
\end{aligned}$$

i.e. $\lim_{x\to -1^-} f(x) \neq \lim_{x\to -1^+} f(x)$ so $x = -1$ **is a critical point** due to the function being discontinuous (derivative undefined) at the is point.

$x = \frac{6}{\pi}$:

$$\begin{aligned}
\lim_{x\to \frac{6}{\pi}^-} f(x) &= \left(\frac{6}{\pi}\right)^2 - 1 = \frac{36}{\pi^2} - 1\\
\lim_{x\to \frac{6}{\pi}^+} f(x) &= \left(\frac{72}{\pi^2} - 2\right)\sin\frac{\pi}{6} = \left(\frac{72}{\pi^2} - 2\right)\frac{1}{2} = \frac{36}{\pi^2} - 1
\end{aligned}$$

i.e. $\lim_{x\to \frac{6}{\pi}^-} f(x) = \lim_{x\to \frac{6}{\pi}^+} f(x)$ so the function is continuous at $x = \frac{6}{\pi}$. This point is, however the global maximum of the function.

Checking the derivative:

$$\lim_{x\to\frac{6}{\pi}^-} f'(x) = 2\frac{6}{\pi} = \frac{12}{\pi}$$
$$\lim_{x\to\frac{6}{\pi}^+} f'(x) = -\frac{\pi^2}{36}\left(\frac{72}{\pi^2}-2\right)\cos\frac{\pi}{6} = -\frac{\pi^2}{36}\left(\frac{72}{\pi^2}-2\right)\frac{\sqrt{3}}{2}$$

i.e. $\lim_{x\to\frac{6}{\pi}^-} f'(x) \neq \lim_{x\to\frac{6}{\pi}^+} f'(x)$, i.e. the derivative is not continuous at $x = \frac{6}{\pi}$, so $x = \frac{6}{\pi}$ **is critical point**.

Now look for any stationary points in the intervals:

$x < -1$:
$f'(x) = -\frac{2}{x^2} \neq 0$ for any $x \in \mathbb{R}$. i.e. no stationary points.
$-1 < x < \frac{6}{\pi}$:
$f'(x) = 2x = 0$ for $x = 0$, so there is a stationary point at $x = 0$. As $f''(0) = 2 > 0$, this is a local minimum. Thus $x = 0$ **is also a critical point**.

$x > \frac{6}{\pi}$:
$f'(x) = -\frac{1}{x^2}\left(\frac{72}{\pi^2}-2\right)\cos\frac{1}{x} = 0$ for $x \to \pm\infty$ or $\cos\frac{1}{x} = 0$. For the latter case this means $x = \dfrac{2}{(2k+1)\pi}$, for $k \in \mathbb{Z}$. However, we are restricted to $x > \frac{6}{\pi}$, i.e.

$$\begin{aligned}
\frac{2}{(2k+1)\pi} &> \frac{6}{\pi}\\
\frac{1}{2k+1} &> 3\\
2k+1 &> 3(2k+1)^2\\
0 &> 12k^2+10k+2\\
0 &> (2k+1)(3k+1)
\end{aligned}$$

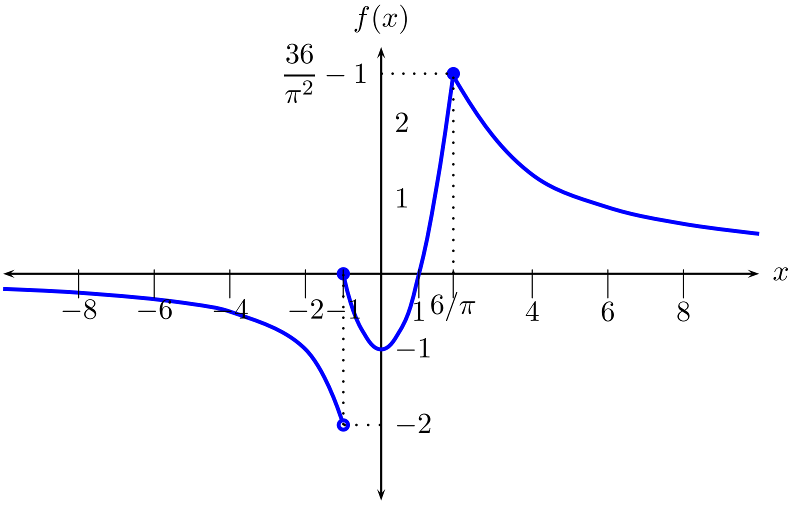
i.e. $k \in (-1/2, -1/3)$, but $k \in \mathbb{Z}$, so there is no solution. i.e. there are no stationary points for $x > \frac{6}{\pi}$.

Putting it altogether, we have 3 critical points:
$x = -1$, where the function is discontinuous, and the derivative is undefined,
$x = 0$, where we have a local minimum, and
$x = \dfrac{6}{\pi}$, where the derivative is undefined.

b)

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2}{x} = 0$$
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(\frac{72}{\pi^2}-2\right)\sin\frac{1}{x} = 0$$

Thus there is a horizontal asymptote at $f(x) = 0$.



c)

3. i) a) We recognise that S is the sum of a geometric progression with common ratio $e^{2i\theta}/3$. Therefore

$$\begin{aligned} S &= e^{i\theta} \sum_{k=0}^{\infty} \left(\frac{e^{2i\theta}}{3}\right)^k \\ &= \frac{e^{i\theta}}{1 - e^{2i\theta}/3} \\ &= \frac{3e^{i\theta}}{3 - e^{2i\theta}} \cdot \frac{3 - e^{-2i\theta}}{3 - e^{-2i\theta}} \\ &= \frac{3\left(3e^{i\theta} - e^{-i\theta}\right)}{(3 - \cos(2\theta))^2 + \sin^2(2\theta)} \\ &= \frac{3\left(3e^{i\theta} - e^{-i\theta}\right)}{10 - 6\cos(2\theta)} \end{aligned}$$

as required.

b) We observe that $T = \mathrm{Im}(S)$, and hence using part (a) we calculate

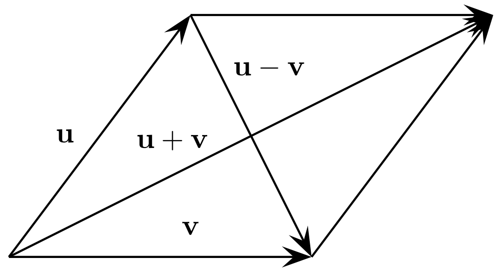
$$\begin{aligned} T &= \frac{3(3\sin(\theta) - \sin(-\theta))}{10 - 6\cos(2\theta)} \\ &= \frac{3 \times 4\sin(\theta)}{10 - 6\cos(2\theta)} \\ &= \frac{6\sin(\theta)}{5 - 3\cos(2\theta)}. \end{aligned}$$

ii) To show that $\mathbf{u} - \mathbf{v}$ is perpendicular to $\mathbf{u} + \mathbf{v}$, we show that the dot product of the two vectors is zero. Now, using properties of the dot product and the fact that $\mathbf{u}$ and $\mathbf{v}$ have the same length, we have

$$\begin{aligned}(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) &= \mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v}\\ &= \mathbf{u}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v}\\ &= \|\mathbf{u}\|^2-\|\mathbf{v}\|^2\\ &= 0,\end{aligned}$$

as required.

Alternatively, note that $\mathbf{u}-\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ are the diagonals of the rhombus spanned by the vectors $\mathbf{u}$ and $\mathbf{v}$, as shown in the figure.



We know that the diagonals of a rhombus are perpendicular, completing the proof.

iii) a) Recall that $\mathbf{e}_j$ is the $n\times 1$ vector with a 1 in the jth position and zeros everywhere else. Therefore

$$A\mathbf{e}_j = \begin{pmatrix} a_{1j}\\ a_{2j}\\ \vdots\\ a_{nj}\end{pmatrix},$$

which is the jth column of A. Similarly, $\mathbf{e}_i^T$ is a $1\times n$ matrix with a 1 in the ith column and zeros everywhere else. Hence

$$\mathbf{e}_i^T A\mathbf{e_j} = \mathbf{e}_i^T\begin{pmatrix} a_{1j}\\ a_{2j}\\ \vdots\\ a_{nj}\end{pmatrix} = a_{ij},$$

since a_{ij} is the ith entry of the jth column of A. This completes the proof.

b) If A is symmetric then $A^T = A$, and hence

$$(A\mathbf{x})^T\mathbf{y} = \mathbf{x}^T A^T\mathbf{y} = \mathbf{x}^T A\mathbf{y},$$

as required.

c) Suppose that $\mathbf{x}^T A\mathbf{y} = (A\mathbf{x})^T\mathbf{y}$ for all vectors $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$. Rewriting this gives

$$\mathbf{x}^T A\mathbf{y} = \mathbf{x}^T A^T\mathbf{y} \qquad (*)$$

for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$. Now let $\mathbf{x}=\mathbf{e}_i$ and $\mathbf{y}=\mathbf{e}_j$, where i,j are arbitrary integers in $\{1,\dots,n\}$. Applying part (a) to condition $(*)$ gives

$$a_{ij} = (A^T)_{ij} = a_{ji}.$$

This proves that A is symmetric, since i and j were arbitrary.

iv) a) To show that M reflects the vector $\mathbf{a}$ to the vector $-\mathbf{a}$, we must show that $M\mathbf{a} = -\mathbf{a}$. We calculate

$$M\mathbf{a} = \frac{1}{9}\begin{pmatrix} 1 & 4 & -8 \\ 4 & 7 & 4 \\ -8 & 4 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 2-4-16 \\ 8-7+8 \\ -16-4+2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = -\mathbf{a}$$

as required.

b) Every vector in the plane is perpendicular to the vector $\mathbf{a}$. Therefore the Cartesian equation of the plane Π is

$$2x - y + 2z = 0.$$

c) Any nonzero vector $\mathbf{u}$ which is the position vector of a point in the plane Π satisfies $M\mathbf{u} = \mathbf{u}$. One such vector is $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

d) The shortest distance from B to the plane is equal to the length of the projection of $\mathbf{b}$ onto $\mathbf{a}$. We calculate

$$|\operatorname{proj}_{\mathbf{a}}(\mathbf{b})| = \frac{|\mathbf{a}\cdot\mathbf{b}|}{\|\mathbf{a}\|} = \frac{|12+6+0|}{\sqrt{4+1+4}} = \frac{18}{3} = 6.$$

The shortest distance from B to the plane Π is 6 units.

4. i) a) We apply L'Hôpital's rule repeatedly (checking each time that he required conditions are satisfied)

$$\lim_{x\to\infty}\frac{x^n}{e^x} = \lim_{x\to\infty}\frac{nx^{n-1}}{e^x} = \ldots = \lim_{x\to\infty}\frac{n!}{e^x} = 0.$$

b) By (a), $e^{-x}x^{n+2} \to 0$ as $n \to \infty$ and so $e^{-x}x^{n+2} < 1$ for sufficiently large n, say $n > M$. Also by continuity, $e^{-x}x^{n+2} < K$ on the interval $[1, M]$. Thus if $C = \max\{K, 1\}, e^{-x}x^{n+2} < C$ for all $x \geq 1$ and the result follows.

c)

$$\int_1^\infty e^{-x}x^n\,dx < \int_1^\infty \frac{C}{x^2}\,dx.$$

The latter integral converges by the p-test and so the given integral converges for any $n \in \mathbb{N}$.

ii) a) Applying the Mean Value Theorem to f on $[0, 2]$, we have, for some $c \in (0, 2)$,

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = 6.$$

b) Applying the Mean Value Theorem to f' on $[0, c]$, we have, for some $d \in (0, c)$,

$$f''(d) = \frac{f(c) - f(0)}{c - 0} = \frac{6}{c} > 3.$$

iii) a) Since $\cosh x > 1$ for all $x > 0$,the result follows.

b)

$$f'(x) = 1 - a\ \mathrm{sech}^2 x > 0$$

if $a \leq 1$. Thus f is strictly increasing on $(0, \infty)$. Also $f(0) = 0$, hence $f(x) > 0$ for all $x > 0$ when $a \leq 1$.

c) For f to have a positive zero, $f'(x) = 1 - a\ \text{sech}^2 x = 0$ for some $x > 0$ so $a = \frac{1}{\text{sech}^2 x} > 1$.

iv) a)

$$A(t) = \sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{2}abcd(1+\cos t)}$$

has a maximum when $1 + \cos t = 0$, that is, when $\alpha + \beta = \pi$.

b) Now $1 + \cos t$ is decreasing on $[0, \pi]$ and so $-(1 + \cos t)$ is increasing. Thus $A(t)$ is increasing and continuous on $[0, \pi]$ and so has an inverse.

Furthermore, $A'(t) = \frac{\frac{1}{2}abcd \sin t}{2A(t)}$ so that A is differentiable. By the inverse function theorem, $B = A^{-1}$ is differentiable on $(A(0), A(\pi))$.

c) If $t = \frac{\pi}{2}$, then $A(\frac{\pi}{2}) = \sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{2}abcd} = A_0$. By the inverse function theorem,

$$B'(A_0) = \frac{1}{A'(\frac{\pi}{2})} = \frac{1}{\frac{abcd}{4A(\pi)}} = \frac{4A_0}{abcd}.$$

MATH1141 June 2012 Solutions

2. i) a)

$$\begin{aligned}
f(x) &= x^2+\cos\left(x^2\right), \quad x\in\left(0,2\sqrt{\pi}\right)\\
f'(x) &= 2x\left(1-\sin\left(x^2\right)\right)\\
\text{Note } f'(x) &= 0, \quad x\in\left(0,2\sqrt{\pi}\right]\\
f''(x) &= 2\left(1-\sin\left(x^2\right)\right)+4x^2\cos\left(x^2\right)\\
f'(x) &= 0 \ \Rightarrow\\
x &= 0, \quad \text{which is not in the domain, or}\\
x^2 &= \frac{\pi}{2},\frac{5\pi}{2}\\
x &= \sqrt{\frac{\pi}{2}},\sqrt{\frac{5\pi}{2}}. \qquad \text{Note } 3\sqrt{\frac{\pi}{2}} \text{ is not in the domain.}\\
f''\left(\sqrt{\frac{\pi}{2}}\right) &= 0, f''\left(\sqrt{\frac{5\pi}{2}}\right)=0.
\end{aligned}$$

So the critical points are $x=\sqrt{\frac{\pi}{2}},\sqrt{\frac{5\pi}{2}}$, the stationary points (both points of inflection) and $x=2\sqrt{\pi}$, the endpoint, and global maximum of the function.

b)

$$\begin{aligned}
f\left(2\sqrt{\pi}\right) &= 4\pi+1\\
f(0) &= 1\\
f'(x) &= 2x\left(1-\sin\left(x^2\right)\right)\geq 0 \quad \text{for } x\in\left(0,2\sqrt{\pi}\right]
\end{aligned}$$

So f is an increasing function on its domain.
It is also continuous (combination of elementary functions) and thus is invertible.
$Range\,(f)=(1,4\pi+1]$, so $Dom\left(f^{-1}\right)=(1,4\pi+1]$.
From above we can see $f\left(\sqrt{\frac{5\pi}{2}}\right)=\frac{5\pi}{2}$. Thus$f^{-1}\left(\frac{5\pi}{2}\right)=\sqrt{\frac{5\pi}{2}}$.

c) From the inverse function theorem, $\frac{d}{dx}f^{-1}(x)=-\frac{1}{f'(f^{-1}(x))}$. So $f^{-1}(x)$ is not differentiable where $f'\left(f^{-1}(x)\right)=0$. i.e. $f^{-1}(x)=\sqrt{\frac{\pi}{2}},\sqrt{\frac{5\pi}{2}}$, or $x=\frac{\pi}{2},\frac{5\pi}{2}$. i.e. $f^{-1}(x)$ is differentiable in $(1,4\pi+1)$, $x\neq\frac{\pi}{2},\frac{5\pi}{2}$.

ii) a)

$$\begin{aligned}
f(x) &= \int_0^{x^2-9x} e^{-t^2}\,dt\\
f'(x) &= e^{-\left(x^2-9x\right)^2}(2x-9)
\end{aligned}$$

Stationary point: $f'(x)=0$.
Mean value theorem: $f(x)$ is continuous on $[0,9]$ and differentiable on $(0,9)$. There-

fore

$$\frac{f(9)-f(0)}{9-0} = f'(x_0), \quad x_0 \in (0,9).$$
$$f(9) = \int_0^0 e^{-t^2}\,dt = 0$$
$$f(0) = \int_0^0 e^{-t^2}\,dt = 0$$

Therefore,

$$f'(x_0) = 0, \quad x_0 \in (0,9).$$

b)

$$f'(x) = e^{-\left(x^2-9x\right)^2}(2x-9) = 0$$

i.e. the stationary point is $x = \frac{9}{2}$.

$$f''(x) = \left(2 - 2\left(x^2-9x\right)(2x-9)\right)e^{-\left(x^2-9x\right)^2}$$
$$f''\left(\frac{9}{2}\right) = \left(2 - 2\left(\left(\frac{9}{2}\right)^2 - 9\frac{9}{2}\right)\left(2\frac{9}{2}-9\right)\right)e^{-\left(\left(\frac{9}{2}\right)^2-9\frac{9}{2}\right)^2} = 2e^{-\left(\frac{81}{4}\right)^2} > 0$$

iii) a) Since z lies on the unit circle, $z = e^{i\theta}$ for some $\theta \in [0,2\pi]$. Hence $z + \dfrac{1}{z} = e^{i\theta} + e^{-i\theta} = 2\cos\theta \in \mathbb{R}$.

b) By (a), $z + \dfrac{1}{z}$ has a maximum value of 2 (when $\theta = 0$).

iv)

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$
$$e^{i4\theta} = (\cos\theta + i\sin\theta)^4 \quad \text{by de Moivre's theorem}$$
$$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta \quad \text{(binomial expansion)}$$

$\cos 4\theta$

$$= \mathbb{R}e\left(e^{i4\theta}\right)$$
$$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$
$$= \cos^4\theta - 6\cos^2\theta\left(1-\cos^2\theta\right) + \left(1-\cos^2\theta\right)^2$$
$$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$$
$$= 8\cos^4\theta - 8\cos^2\theta + 1$$

v) a)

$$x = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}$$

$$\begin{pmatrix} 4 \\ 4 \\ -7 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 2 \\ -7 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 3 & 3 \\ 1 & -1 & 2 \\ -3 & -1 & -7 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & -4 & -1 \\ 0 & 8 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & -4 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

This system has unique solution, so the point lies on the plane.

b) If $\mathbf{b}$ is parallel to P then $\mathbf{b}= \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 3 & 1 \\ 1 & -1 & 3 \\ -3 & -1 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & -4 & 2 \\ 0 & 8 & 5 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 9 \end{array}\right)$$

i.e. no solution, and $\mathbf{b}$ is not parallel to P.

c)

$$\mathbf{c} \cdot \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} = 0, \quad \text{and} \mathbf{c} \cdot \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = 0.$$

Therefore $\mathbf{c}$ is orthogonal to P.

3. i) $\det(B) = -35$.

ii) a) $\mathbf{x} = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ where $\lambda \in \mathbb{R}$.

b) The quadratic distance from $\mathbf{x}$ to $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is

$$(4+\lambda)^2 + (-\lambda)^2 + (5+\lambda)^2 = 3\lambda^2 + 18\lambda + 41$$

which is minimal when $\lambda = -\frac{18}{2\cdot 3} = -3$.

The closest point is thus given by $\mathbf{x} = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$.

iii) a) All $a \neq 1, -2$.

b) $a = 1$.

c) $a = -2$.

iv) a) For example, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

b) If Q is nilpotent (of degree 2), then $|\det Q| = \sqrt{|\det Q^2|} = \sqrt{|\mathbf{0}|} = 0$, so Q is not invertible.

c) PROOF. $S^{-1}Q = S^{-1}QI = S^{-1}QSS^{-1} = S^{-1}SQS^{-1} = IQS^{-1} = QS^{-1}$. $\square$

d) By c),

$$(S+Q)(S^{-1}-S^{-2}Q) = I+QS^{-1}-S^{-1}Q-QS^{-2}Q = I-S^{-1}QQS^{-1} = I-S^{-1}\mathbf{0}S^{-1} = I\,.$$

Hence, $S+Q$ is invertible, and $k = 2$.

v) PROOF. We see that

$$a = \mathbf{x} \cdot \mathbf{e}_1 = |\mathbf{x}||\mathbf{e}_1| \cos\alpha = |\mathbf{x}| \cos\alpha\,;$$
$$b = \mathbf{x} \cdot \mathbf{e}_2 = |\mathbf{x}||\mathbf{e}_2| \cos\beta = |\mathbf{x}| \cos\beta\,;$$
$$c = \mathbf{x} \cdot \mathbf{e}_3 = |\mathbf{x}||\mathbf{e}_3| \cos\gamma = |\mathbf{x}| \cos\gamma\,.$$

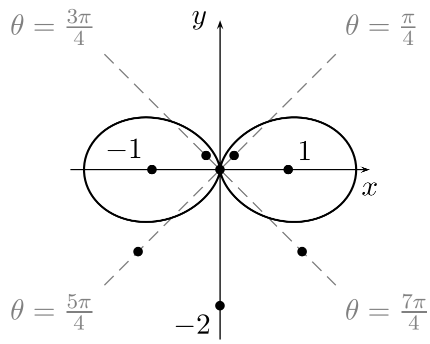
Hence,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a^2}{|\mathbf{x}|^2} + \frac{b^2}{|\mathbf{x}|^2} + \frac{c^2}{|\mathbf{x}|^2} = \frac{|\mathbf{x}|^2}{|\mathbf{x}|^2} = 1\,. \quad \square$$

vi) PROOF. Since $\det(A^T) = \det(A) \neq 0$, every vector $\mathbf{b} \in \mathbb{R}$ can be written uniquely as a linear combination of columns of A^T, or in other words, rows of A. $\square$

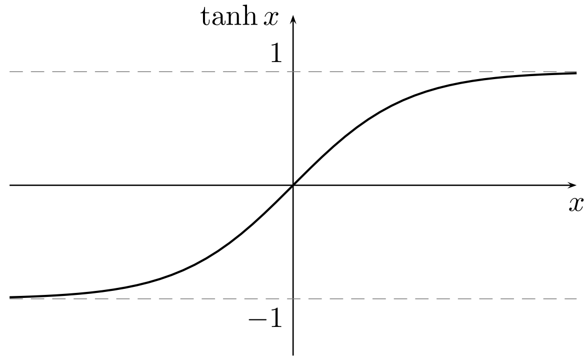
4. i) a) Since $\cos(-2\theta) = \cos\theta$ the curve is symmetric about the x axis and since $\cos 2(\pi - \theta) = \cos 2\theta$ the curve is symmetric about the y axis.

b) Diagram as below



ii) a) $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

b) Diagram as below



c) Since $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = 1 - \dfrac{2e^{-x}}{e^x + e^{-x}}$, $(1 - \tanh x)/e^{-2x} = \dfrac{2e^x}{e^x + e^{-x}}$ which tends to 2 as $x \to \infty$.

d) Since $\displaystyle\int_0^\infty e^{-2x}\,dx$ converges, it follows from (c) by the limit comparison test that

$$\int_0^\infty (1 - \tanh x)\,dx \text{ converges also.}$$

e)

$$\int_0^\infty (1 - \tanh x)\,dx = \lim_{R\to\infty} \int_0^R (1 - \tanh x)\,dx = \lim_{R\to\infty} \left[x - \log(\cosh x)\right]_0^R$$

$$= \lim_{R\to\infty} R - \log(\cosh R) = \lim_{R\to\infty} \log \frac{2e^R}{e^R + e^{-R}} = \log 2$$

iii) a) Integrating by parts with $u = f(x)$, we have

$$\int_a^b f(x)\sin nx\, dx = \left[-\frac{\cos(nx)}{n}\right]_a^b + \frac{1}{n}\int_a^b f'(x)\cos(nx)\, dx$$

$$= \frac{f(a)\cos(nx) - f(b)\cos(nb)}{n} + \frac{1}{n}\int_a^b f'(x)\cos(nx)\, dx.$$

b)

$$\left|\int_a^b f'(x)\cos nx\, dx\right| \le \int_a^b \left|f'(x)\cos nx\right|\, dx \le \int_a^b |f'(x)|\, dx \le \int_a^b L\, dx = L(b-a).$$

c)

$$\lim_{n\to\infty}\int_a^b f(x)\sin nx\, dx = \lim_{n\to\infty}\frac{f(a)\cos(nx) - f(b)\cos(nb)}{n} + \frac{1}{n}\int_a^b f'(x)\cos(nx)\, dx$$

$$\le \lim_{n\to\infty} K(n)/n + L(b-a)/n = 0$$

since $K(n)$ is bounded.

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2. i) a) Away from 0 f is differentiable. Since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 0$, f is continuous at $x = 0$ and since the derivatives of the two constituent functions have the same derivative at $x = 0$, by the 'split function theorem', f is differentiable at $x = 0$ and hence everywhere.

$$f'(x) = \begin{cases} 3x^2 & \text{if } x < 0 \\ 2x & \text{if } x \geq 0. \end{cases}$$

b) Since $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^-} f'(x) = 0$, so f' is continuous at $x = 0$

c) For $x < 0$, $\lim_{h\to 0^-} \dfrac{f'(0+h) - f'(0)}{h} = 0$, while for $x > 0$, $\lim_{h\to 0^+} \dfrac{f'(0+h) - f'(0)}{h} = 2$. Hence f' is not differentiable at $x = 0$.

ii) a)

$$L_P(f) = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} = \sum_{k=1}^{n} \frac{1}{n+k}.$$

b) Under the given assumption,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \int_0^1 \frac{1}{1+x} = \log 2.$$

iii) a) $\frac{1}{2}(\mathbf{a}+\mathbf{b})$.

b) $\mathbf{x} = \mathbf{c} + \lambda(\frac{1}{2}(\mathbf{a}+\mathbf{b}) - \mathbf{c}), \lambda \in \mathbb{R}$.

c) In triangle ABC, usin the dot product formula, $\cos A = \frac{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})}{|\mathbf{b}-\mathbf{a}|\,|\mathbf{c}-\mathbf{a}|} = \frac{1}{2}$ and so $A = \pi/3$. Similarly $\cos B = \frac{1}{2}$ so $B = \pi/3$. Hence ABC is equilateral.

iv) a)

$$\mathbf{x} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}.$$

b) This is the line of intersection of two planes.

c) Substituting the line into the thrid equation (which is a sphere), gives $3\lambda^2 - 8\lambda + 5 = 0$ and so $\lambda = 1, 5/3$. The value $\lambda = 1$ gives the point $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ and $\lambda = 5/3$ gives the point $\frac{1}{3}\begin{pmatrix} 7 \\ 4 \\ 5 \end{pmatrix}$.

3. i) A vector normal to the plane

$$\boldsymbol{n} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix}.$$

A vector from the point $\begin{pmatrix}1\\2\\4\end{pmatrix}$ on the plane to $\mathbf{p}$:

$$\begin{pmatrix}1\\2\\4\end{pmatrix} - \begin{pmatrix}3\\2\\3\end{pmatrix} = \begin{pmatrix}-2\\0\\1\end{pmatrix}.$$

The length of the projection of this vector onto $\boldsymbol{n}$ is the shortest distance from $\mathbf{p}$ to the plane

$$\frac{\left|\begin{pmatrix}-2\\0\\1\end{pmatrix}\cdot\begin{pmatrix}7\\1\\2\end{pmatrix}\right|}{\left|\begin{pmatrix}7\\1\\2\end{pmatrix}\right|} = \frac{4}{\sqrt{6}}.$$

ii) a) $\alpha_1 = 1$.

b) Complex roots come in conjugate pair because p has real coefficients. Hence there are two complex roots and two real roots.

c) Assume that α_2 is the second real root, and that α_3 and α_4 are the conjugate complex roots ($\alpha_4 = \bar{\alpha}_3$). Since $\alpha_1 + \cdots + \alpha_4 = 1$ and $\alpha_1 = 1$, there holds

$$\alpha_2 + \alpha_3 + \alpha_4 = \alpha_2 + 2a = 0, \tag{2}$$

where we denote by a the real part of α_3. Noting that 0 is not a root, we deduce that either $\alpha_2 < 0$ or $a < 0$.

d) If α is a root satisfying $|\alpha| \le 1/2$ then from $p(\alpha) = 0$ we deduce $\alpha^4 - \alpha^3 - \alpha^2 - \alpha = -2$, so that

$$2 = |\alpha^4 - \alpha^3 - \alpha^2 - \alpha| \le |\alpha|^4 + |\alpha|^3 + |\alpha|^2 + |\alpha| \le \frac{1}{2^4} + \frac{1}{2^3} + \frac{1}{2^2} + \frac{1}{2} = \frac{15}{16},$$

contradiction! Hence $|\alpha_j| > 1/2$ for $j = 1, 2, 3, 4$.

iii) a) False. Take for example

$$A = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}$$

b) True. (Note: $\det(AB) = \det(A)\det(B)$.)

c) False. Since $\det(AB) = \det(A)\det(B)$ and $\det(AC) = \det(A)\det(C)$ we need to choose an example such that $\det(A) = 0$. For example

$$A = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}, \quad B = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

d) False. Choose an example such that A is not invertible, i.e., $\det(A) = 0$ as in c). (The statement is true if A is invertible.)

iv) a) $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthonormal set if

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \ne j. \end{cases}$$

b) Since the jth row of M^T is $\mathbf{v}_j$, we have

$$M^TM = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1\cdot\mathbf{v}_1 & \mathbf{v}_1\cdot\mathbf{v}_2 & \cdots & \mathbf{v}_1\cdot\mathbf{v}_n \\ \mathbf{v}_2\cdot\mathbf{v}_1 & \mathbf{v}_2\cdot\mathbf{v}_2 & \cdots & \mathbf{v}_2\cdot\mathbf{v}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_n\cdot\mathbf{v}_1 & \mathbf{v}_n\cdot\mathbf{v}_2 & \cdots & \mathbf{v}_n\cdot\mathbf{v}_n \end{pmatrix} = I$$

i.e., M is an orthogonal matrix. Hence

$$\det(M)^2 = \det(M^T)\det(M) = \det(M^TM) = \det(I) = 1,$$

implying $\det(M) = \pm 1$.

4. i) a)

$$\lim_{t\to\infty} t^2e^{-t^2} = \lim_{t\to\infty}\frac{t^2}{e^{t^2}} = \lim_{t\to\infty}\frac{2t}{2te^{t^2}} = \lim_{t\to\infty}\frac{1}{e^{t^2}} = 0.$$

b) Since $e^{t^2} \geq t^2$ for $t \geq 1$, we have that $\displaystyle\int_1^\infty e^{-t^2}dt \leq \int_1^\infty \frac{1}{t^2}dt$, which converges by the p-test. Also $\displaystyle\int_0^1 e^{-t^2}dt < \int_0^1 1\,dt = 1$ and so the original integral converges.

c) $f'(x) = 3x^2e^{-x^6}$ and so the only critical point is $(0,0)$ which is an inflection point. As $x \to \infty$ $f(x) \to I$.

d) DIAGRAM

ii)

$$\lim_{x\to c}\frac{f(x)-f(c)-f'(c)(x-c)}{(x-c)^2} = \lim_{x\to c}\frac{f'(x)-f'(c)}{2(x-c)} = \lim_{x\to c}\frac{f''(x)}{2} = \frac{f''(c)}{2}.$$

iii) a) Differentiating implicitly, $2(x^2+y^2)(2x+2yy')-2(2x-2yy') = 0$, so $y' = \dfrac{x(1-(x^2+y^2))}{y(1+x^2+y^2)}$. The tangents correspond to when the numerator is 0, and this occurs when $x = 0$, i.e. on the y axis or when $x^2+y^2 = 1$.

b) The condition $x = 0$ yields the point, $(x,y) = (0, \pm\sqrt{-1+\sqrt{b}})$, provided $b \geq 1$. The condition $x^2+y^2 = 1$ yields $4y^2 = b$; $0 < b \leq 4$ and hence the points $(x,y) = (\frac{\sqrt{4-b}}{2}, \pm\frac{\sqrt{b}}{2})$.

iv) a) $g(0) = f(0) \geq 0$ and $g(2) = f(2) - 8 \leq 0$ and so by the IVT, g has at least one root ξ in the interval $[0,2]$. Thus $f(\xi) = \xi^3$.

b) Applying the MVT to f on $[0,2]$ we have $\frac{f(2)-f(0)}{2-0} = f'(\eta)$ for some $\eta \in (0,2)$ and the result follows.

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3. i)

$$g'(x) = 3 + 2\sin 2x = 2(1 + \sin 2x) + 1 > 0.$$

Therefore, $g(x)$ is an increasing function. It is also continuous (because it is a composition of elementary functions) so it is invertible. By the inverse function theorem, the inverse is also continuous and differentiable.

By inspection, $g(0) = -2$ so $h(-2) = 0$. Thus

$$h'(-2) = \frac{1}{g'(h(-2))} = \frac{1}{g'(0)} = \frac{1}{3}.$$

ii) a) Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists at least one real number c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

b) Let $f(x) = \tan^{-1} x$ *(1 mark)*. Then $f'(x) = 1/(1+x^2)$ and the Mean Value Theorem states that there exists a c in (a, b) such that

$$f'(c) = \frac{\tan^{-1} b - \tan^{-1} a}{b - a}. \qquad \textit{(1 mark)}$$

But $0 < 1/(1 + c^2) \leq 1$ so

$$0 < \frac{\tan^{-1} b - \tan^{-1} a}{b - a} \leq 1$$

or

$$0 < \tan^{-1} b - \tan^{-1} a \leq b - a.$$

c) Let $I = \int_1^\infty g(t)\, dt$ where $g(t) = \tan -1(t+t^{-2}) - \tan^{-1} t$. From (b), $0 \leq g(t) \leq h(t)$ where $h(t) = t + t^{-2} - t = t^{-2}$.

Since $\int_1^\infty h(t)dt = \int_1^\infty t^{-2}dt$ converges (by the p-test) then so does $\int_1^\infty g(t)dt$ (by the comparison test).

iii) Let $L = 2$. Then

$$\begin{aligned} |f(x) - L| &= \left|\frac{e^x}{\cosh x} - 2\right| \\ &= \frac{e^x}{\cosh x}\left|1 - 2\frac{\cosh x}{e^x}\right| \\ &= \frac{e^x}{\cosh x}\left|1 - 1 - e^{-2x}\right| \\ &= \frac{e^x}{\cosh x}\left|-e^{-2x}\right| \\ &= \frac{e^{-x}}{\cosh x} \leq e^{-x}, \end{aligned}$$

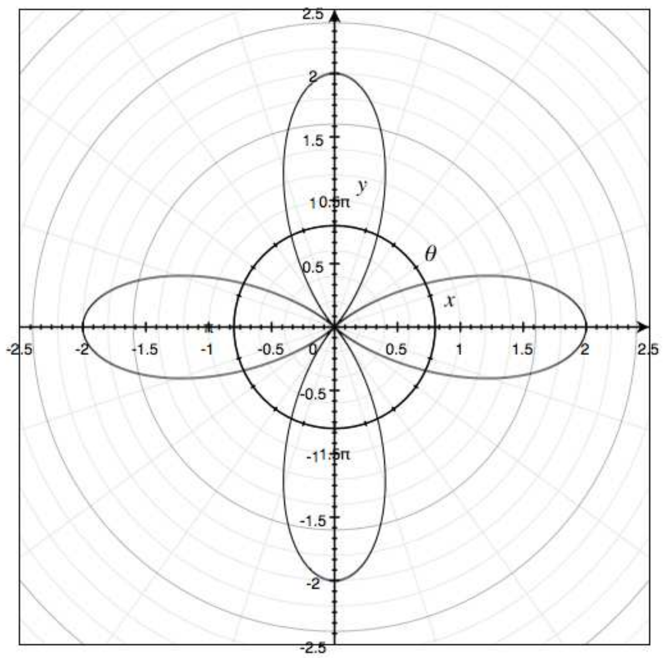
since $\cosh x \geq 1$.

Let $\epsilon > 0$. Then $e^{-x} < \epsilon$ if and only if $-x < \ln \epsilon$ or $x > -\ln \epsilon = \ln \epsilon^{-1}$.

Let $M = \ln \epsilon^{-1}$. Then we have shown that if $x > M$ then there exists an $\epsilon > 0$ such that $|f(x) - L| < \epsilon$, as required.

iv) a) $r' = -4\sin 4\theta = 0$ when $\theta = n\pi/4$ for $n = 0, 1, 2, \cdots$. Thus:
r takes a maximum value of 2 at $\theta = 0, \pi/2, \pi, 3\pi/2, 2\pi$.
r takes a minimum value of 0 at $\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.

b) Diagram as below

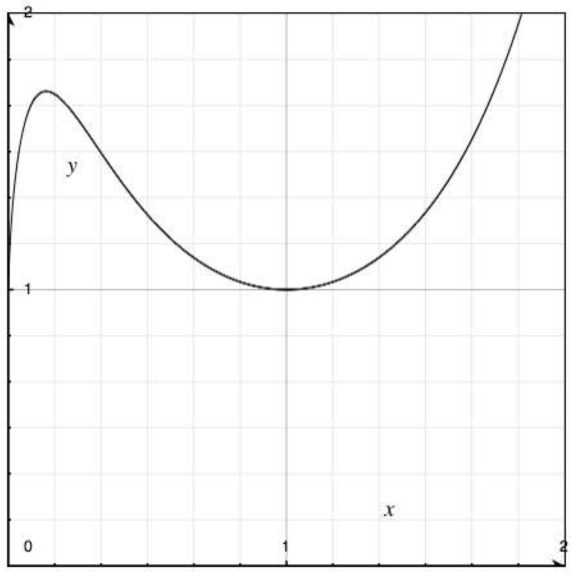


v) a) $f(x) = x^{x\ln x} = (e^{\ln x})^{x\ln x} = e^{x(\ln x)^2}$. So

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left(x(\ln x)^2\right) e^{x(\ln x)^2} \\
&= \left((\ln x)^2 + x\frac{2\ln x}{x}\right) e^{x(\ln x^2} \\
&= \left((\ln x)^2 + 2\ln x\right) x^{x\ln x}.
\end{aligned}
$$

b) Since $x^{x\ln x} = e^{x(\ln x)^2} > 0$ for $x > 0$, the sign of $f'(x)$ will be determined by the sign of $g(x) = (\ln x)^2 + 2\ln x$.
Let $z = \ln x$. Then $g = z^2 + 2z = z(z+2)$. This negative for $-2 < z < 0$. Thus, $f'(x)$ is negative for $e^{-2} < x < 1$ and positive for $x < e^{-2}$ or $x > 1$.

c) Diagram as below



4. i) We first write the augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 0 & -4 & b_1 \\ 3 & 1 & -2 & b_2 \\ -2 & -1 & 0 & b_3 \end{array}\right).$$

and row reduce to get:

$$\left(\begin{array}{ccc|c} 2 & 0 & -4 & b_1 \\ 0 & 2 & 8 & 2b_2 - 3b_1 \\ 0 & 0 & 0 & 2b_3 + 2b_2 - b_1 \end{array}\right).$$

The system has a solution if the final column in a row reduced form of the augmented matrix is not a leading column; this condition corresponds to the equation

$$-b_1 + 2b_2 + 2b_3 = 0.$$

ii) The distance from the point represented by the vector $\begin{pmatrix} x \\ x \\ x \end{pmatrix}$ to I (which is represented by the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$) is given by the distance formula:

$$d = \sqrt{(x-1)^2 + (x-0)^2 + (x-0)^2} = \sqrt{3x^2 - 2x + 1}.$$

The distance will be 1 if and only if the square of the distance is 1, so we get the equation

$$3x^2 - 2x + 1 = 1,$$

or

$$3x^2 - 2x = 0,$$

which has the solution

$$x = 0 \text{ or } x = \frac{2}{3}.$$

To get a point distinct from the origin, we take $x = \frac{2}{3}$ and the corresponding vector is $\begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$.

iii) a) The inverse is given by

$$\frac{1}{\det(A)} \begin{pmatrix} \alpha & -i \\ -1-i & 2 \end{pmatrix} = \frac{1}{2\alpha + 1 - i} \begin{pmatrix} \alpha & -i \\ -1-i & 2 \end{pmatrix}.$$

b) Suppose that $\det(A^2) = -1$. Then we have

$$-1 = \det(A^2) = (\det(A))^2,$$

so $\det(A) = \pm i$. On the other hand, we have

$$\det(A) = 2\alpha + 1 - i = \pm i,$$

so we get the solutions

$$\alpha = -\frac{1}{2} \text{ or } \alpha = \frac{2i-1}{2}.$$

iv) a) The roots of $z^9 - 1$ are the ninth roots of unity: 1 and $e^{\pm \frac{2\pi k}{9}}, k = 1, 2, 3, 4$. From the factorization

$$(z^9 - 1) = (z^3 - 1)(z^6 + z^3 + 1)$$

we see that the roots $z^6 + z^3 + 1$ are those ninth roots of unity which are not also cube roots of unity, which leaves the six roots $e^{\pm \frac{2\pi k}{9}}, k = 1, 2, 4$.

b) Dividing $z^6 + z^3 + 1 = 0$ by z^3 gives the equation

$$z^3 + 1 + \frac{1}{z^3} = 0,$$

or

$$z^3 + \frac{1}{z^3} = -1.$$

We have

$$x^3 = (z + \frac{1}{z})^3 = z^3 + \frac{1}{z^3} + 3z + \frac{3}{z} = -1 + 3x,$$

so

$$x^3 - 3x + 1 = 0.$$

c) Let $z_1 = e^{\frac{2\pi i}{9}}$, $z_2 = e^{\frac{4\pi i}{9}}$, $z_3 = e^{\frac{8\pi i}{9}}$. Then by parts (a) and (b), the three numbers $z_1 + \frac{1}{z_1}, z_2 + \frac{1}{z_2}, z_3 + \frac{1}{z_3}$ are the roots of the cubic polynomial $x^3 - 3x + 1$. Since this polynomial has no quadratic term, the sum of these roots must be 0, and we have

$$0 = z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + z_3 + \frac{1}{z_3} = 2(\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9}).$$

v) a) We can write any vector as $\mathbf{x} = |\mathbf{x}|\mathbf{u}$, with $\mathbf{u}$ a unit vector. Then

$$|M\mathbf{x}| = |M(|\mathbf{x}|\mathbf{u})| = |\mathbf{x}| \cdot |M\mathbf{u}| \leq |\mathbf{x}|||M||$$

by the definition of $||M||$.

b) For any unit vector $\mathbf{u}$, we have

$$|(MN)\mathbf{u}| = |M(N\mathbf{u})| \leq ||M|| \cdot |N\mathbf{u}| \leq ||M|| \cdot ||N||,$$

where we have used part (a) for the first inequality and the definition of norm for the second.
Therefore $||M|| \cdot ||N||$ is at least as big as the maximum of $|(MN)\mathbf{u}|$ for all unit vectors $\mathbf{u}$, which is the norm of MN.

c) The matrix given takes an arbitrary unit vector $\begin{pmatrix} u \\ v \end{pmatrix}$ to the vector $\begin{pmatrix} v \\ -2u \end{pmatrix}$. The second vector is clearly not more than twice as long as the first vector. On the other hand, if $v = 0$ then is it exactly twice as long. Therefore the norm is 2.

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2. i) a) If f is differentiable on the open interval (a, b) and continuous on the $[a, b]$, then there exists a point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

b) Let x be a real number, $x \leq 2$. The function f satisfies the requirements of the Mean Value Theorem on $[x, 2]$ so

$$\frac{f(2) - f(x)}{2 - x} = f'(c)$$

for some $c \in (2, x)$. Hence

$$\frac{f(2) - f(x)}{2 - x} \leq 1 \Rightarrow f(x) \geq x.$$

ii) The limit satisfies the conditions for L'Hopital's rule, so $\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{f(x)-f(0)}{2x} = \lim_{x\to 0} \frac{f'(x)}{2} = \frac{f'(0)}{2}$.

iii) a) Now

$$f(-x) = \int_0^{(-x)^3} (t^2 - 1)e^{t^2}\, dt.$$

Replace $-t$ with t in the integral, so replace dt with $-dt$, then

$$f(-x) = -\int_0^{x^3} (t^2 - 1)e^{t^2}\, dt = -f(x).$$

Hence f is odd.

b) Set $f'(x) = 3x^2[(x^6 - 1)e^{x^6}] = 0$ for a stationary point, giving $x = 0, 1, -1$.

c) Around $x = 0$, $f'(x)$ is positive so we have an inflection point at $x = 0$.
Around $x = 1$, $f'(x)$ moves from negative to positive so we have a minimum at $x = 1$.
Around $x = -1$, $f'(x)$ moves from positive to negative so we have a maximum at $x = -1$.

iv) a) Differentiating implicitly and solving for y' we have

$$y' = -\frac{y}{x}\frac{3x^2 + y^2}{x^2 + 3y^2}.$$

b)

$$\frac{dA}{dx} = y + xy' = 0$$

for a stationary point. Substituting and noting that $x, y > 0$ we have $y = x$. Substituting this back into the equation fo curve we obtain $x = y = \alpha$, where $\alpha = \frac{1}{\sqrt{42}}$.

c) Passing to polar coordinates, we have $r^4 \sin\theta\cos\theta = 1$ or $r^4 \sin(2\theta) = 2$.
Also $A = xy = r^2 \sin\theta\cos\theta = \frac{1}{r^2}$.

d) We can write $A = \frac{\sqrt{\sin(2\theta)}}{\sqrt{2}}$ which has a maximum when $\theta = \frac{\pi}{4}$. Thus, $y = x$ and this is the line on which A has the maximum point $x = y = \alpha$, where $\alpha = \frac{1}{\sqrt[4]{2}}$.

3. i) Let A be the point $(1, 0, 1)$ and let $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$; then $\overrightarrow{AP} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$,

$\overrightarrow{AP} \cdot \mathbf{v} = \mathbf{1}$ and $\mathbf{v} \cdot \mathbf{v} = \mathbf{2}$.
The distance is then

$$|\overrightarrow{AP} - \text{proj}_{\mathbf{v}}\overrightarrow{AP}| = \left|\overrightarrow{AP} - \frac{\overrightarrow{AP} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}\right| = \left|\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right| = \frac{3}{2}\left|\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right| = \frac{3}{\sqrt{2}}.$$

ii) a) $|\mathbf{a}+\mathbf{b}|^{\mathbf{2}} + |\mathbf{a}-\mathbf{b}|^{\mathbf{2}} = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) + (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \mathbf{2a}\cdot\mathbf{a} + \mathbf{2a}\cdot\mathbf{b} = \mathbf{2}|\mathbf{a}|^{\mathbf{2}} + \mathbf{2}|\mathbf{b}|^{\mathbf{2}}$

b) Now, $|\mathbf{a}|$ and $|\mathbf{b}|$ are the side lengths of the parallellogram
whereas $|\mathbf{a} + \mathbf{b}| = |\overrightarrow{AB} + \overrightarrow{AD}| = |\overrightarrow{AB} + \overrightarrow{BC}| = |\overrightarrow{AC}|$
and $|\mathbf{a} - \mathbf{b}| = |\overrightarrow{AB} - \overrightarrow{AD}| = |\overrightarrow{AB} + \overrightarrow{DA}| = |\overrightarrow{DB}|$ are the diagonal lengths.
Therefore, a) states that the sum of the squared diagonal lengths is twice the sum of the squared side lengths.

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iii) a)

$$\begin{aligned}
&\cos 5\theta + i\,\sin 5\theta = e^{i5\theta} = (e^{i\theta})^5 = (\cos\theta + i\,\sin\theta)^5 \\
&= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta \\
&= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)
\end{aligned}$$

By comparing the imaginary parts, we see that

$$\begin{aligned}
\sin 5\theta &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \\
&= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta \\
&= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta \\
&= 16x^5 - 20x^3 + 5x\,.
\end{aligned}$$

b) Set $\theta = \frac{\pi}{10}$. By a), $p(x) = 16x^5 - 20x^3 + 5x - 1 = \sin 5\theta - 1 = \sin\frac{5\pi}{10} - 1 = 1 - 1 = 0$.

c) 1, $\frac{-1\pm\sqrt{5}}{4}$.

d) By parts b) and c), $\sin\frac{\pi}{10}$ must be one of the roots 1, $\frac{-1\pm\sqrt{5}}{4}$.
Since it is not 1 and is not negative, we see that

$$\sin\frac{\pi}{10} = \frac{-1+\sqrt{5}}{4}\,.$$

iv) $\det(B) = 0$ and $\det(C) = -2(-1)^2\det(A) = -14$
since the third row of B is a linear combination of the first two rows
and since swapping rows twice on the transpose C^T gives A with rows scaled by $-1, 2$.

v) a) **Proof**. $|Q\mathbf{v}|^2 = (Q\mathbf{v}) \cdot (Q\mathbf{v}) = \mathbf{v}^T Q^T Q\mathbf{v} = \mathbf{v}^T I\mathbf{v} = \mathbf{v}^T\mathbf{v} = \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$,
so $|Q\mathbf{v}| = |\mathbf{v}|$.

b) **Proof**. By a), $|\lambda||\mathbf{v}| = |\lambda\mathbf{v}| = |\mathbf{Q}\mathbf{v}| = |\mathbf{v}|$.
Since $|\mathbf{v}| \neq \mathbf{0}$, we see that $|\lambda| = 1$, so $\lambda = \pm 1$.

THE UNIVERSITY OF NEW SOUTH WALES
BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k$$
$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$
$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$
$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$
$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$
$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$
$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$
$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$
$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$
$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$
$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$
$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$
$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$
$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$
$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$
$$\begin{aligned}\int \frac{dx}{a^2 - x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, \qquad |x| < a \\ &= \frac{1}{a}\coth^{-1}\frac{x}{a} + C, \qquad |x| > a > 0 \\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, \qquad x^2 \neq a^2\end{aligned}$$
$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$$
$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} + C$$
$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$