THE UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1131 Calculus

Section 10: - Hyperbolic Functions.

A number of centuries ago, one of the Bernoulli brothers asked the question: What curve do you get get when a piece of chain or rope is held loosely under gravity. It had been tacitly assumed that the curve must be a parabola, but Bernoulli showed that it wasn't, and indeed discovered a 'new' function which came to known as $\cosh x$. This function is not really 'new' since it is a combination of exponentials, but has enough important properties to be given its own special name.

Definition: We define the functions:

(i) $\sinh x = \dfrac{e^x - e^{-x}}{2}$

(ii) $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

(iii) $\tanh x = \dfrac{\sinh x}{\cosh x}$.

These are known as the **hyperbolic functions**. The name comes from the identity $\cosh^2 x - \sinh^2 x = 1$, so one can parametrize the hyperbola, $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ by $x = a\cosh t, y = b\sinh t$, in the same way that the trig. functions parametrize the circle (or ellipse).

The names are reminiscent of the trigonometric functions since they have properties that are very similar to the trigonometric functions. One can similarly define cosech x, sech x, coth x as the reciprocals of the above.

Simple differentiation yields:

$$\frac{d}{dx}\sinh x = \cosh x, \quad \frac{d}{dx}\cosh x = \sinh x, \quad \frac{d}{dx}\tanh x = \operatorname{sech}^2 x.$$

Graphs:

Identities: As with the trig functions, the hyperbolic functions satisfy a number of identities. We have already mentioned above

$$\cosh^2 x - \sinh^2 x = 1,$$

$$\cosh^2 x - \sinh^2 x$$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = 1.$$

Now $\cosh^2 x + \sinh^2 x$

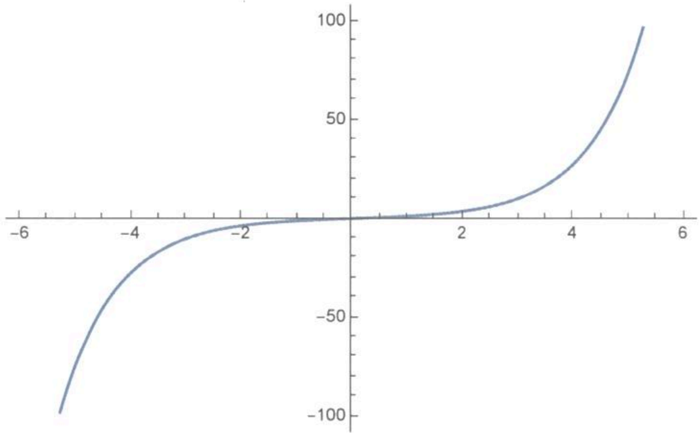
$$= \frac{e^{2x} + 2 + e^{-2x}}{4} + \frac{e^{2x} - 2 + e^{-2x}}{4}$$

$$= \frac{e^{2x} + e^{-2x}}{2} = \cosh 2\theta.$$

Ex: Derive the formula: $\cosh 2\theta = \cosh^2\theta + \sinh^2\theta$.

In[3] := `Plot[Sinh[x], {x, -6, 6}]`

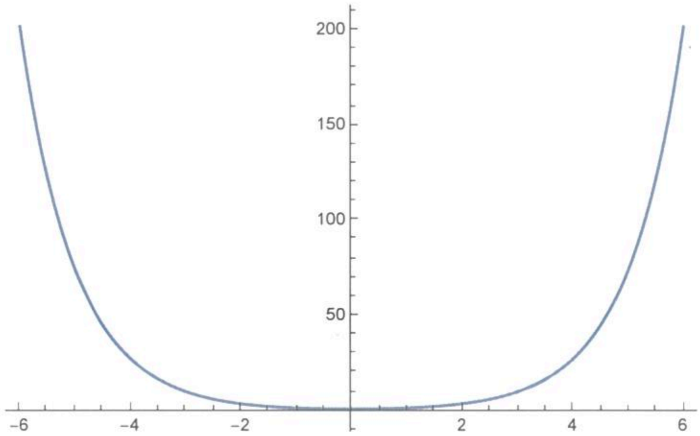
Out[3]=



$y = \sinh x$

In[4] := `Plot[Cosh[x], {x, -6, 6}]`

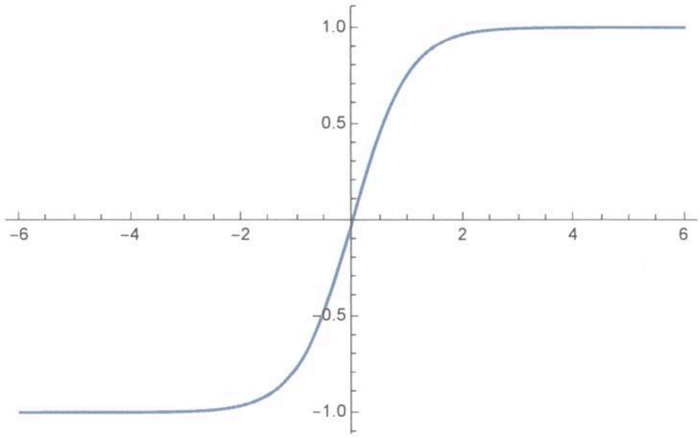
Out[4]=



$y = \cosh x$

In[5] := `Plot[Tanh[x], {x, -6, 6}]`

Out[5]=



$y = \tanh x$

Connection with Trig. functions

More over,

$$\cosh t \sinh t = \frac{e^{t}+e^{-t}}{2} \cdot \frac{e^{t}-e^{-t}}{2}$$

$$= \frac{e^{2t} - e^{-2t}}{4} = \frac{1}{2}\sinh(2t)$$

So $\sinh(2t) = 2\cosh t \sinh t.$

Inverse Hyperbolic Functions:

The functions $\sinh x$ and $\tanh x$ are increasing on $\mathbb{R}$ and so have inverses written as $\sinh^{-1} x$ and $\tanh^{-1} x$ respectively.
The function $\cosh x$ is not one-to-one, but if we restrict its domain to $x \geq 0$ then we can define an inverse, $\cosh^{-1} x$ for this section of the curve.

Ex: Find a. $\cosh(\sinh^{-1}(\frac{3}{4}))$, b. $\cosh(2\cosh^{-1} 2)$.

a) we know

$$\cosh^2(\sinh^{-1}\tfrac{3}{4}) - \sinh^2(\sinh^{-1}\tfrac{3}{4}) = 1$$

So $\cosh^2(\sinh^{-1}\frac{3}{4}) = 1 + \frac{3}{4}$ or

$$\cosh^2(\sinh^{-1}\tfrac{3}{4}) = \sqrt{\tfrac{7}{4}} = \sqrt{7}/2.$$

b) $\cosh(2\cosh^{-1}2) = \cosh^2(\cosh^{-1}2) + \sinh^2(\cosh^{-1}2)$

$$= 2\cosh^2(\cosh^{-1}2) + 1 = 5.$$

Inverse Hyperbolic and Logarithmic Functions:

Since the hyperbolic functions are defined in terms of exponentials, one would expect that there is some relationship between the inverse hyperbolic functions and the logarithmic functions.

Indeed, if we let $y = \cosh^{-1} x$ then

$$\frac{e^y + e^{-y}}{2} = x. \quad \text{Let } u = e^y. \text{ So}$$

$$u + \frac{1}{u} = 2x \text{ or } u^2 - 2ux + 1 = 0$$

$$u = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$$

Recall that $y \geq 0$ and $x \geq 1$. So we must have $u \geq 1$. So we must take

$$u = x + \sqrt{x^2-1}$$

$$\cosh^{-1} x = \log(x + \sqrt{x^2-1}).$$

There are similar formulae for the other inverse hyperbolic functions.

You do not need to memorize these formulae, but you might, for example, be asked to re-derive them.

$$\sinh^{-1} x = \log(x + \sqrt{x^2+1})$$

$$\tanh^{-1} x = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right).$$

Derivatives:

Let $y = \sinh^{-1} x$ then

$$y' = \left(\log(x + \sqrt{x^2+1})\right)'$$

$$= \frac{1}{x + \sqrt{x^2+1}} \cdot \left(1 + \frac{2x}{2\sqrt{x^2+1}}\right)$$

$$= \frac{1}{\sqrt{x^2+1}}$$

Similarly, $\dfrac{d}{dx}(\cosh^{-1} x) = \dfrac{1}{\sqrt{x^2-1}}$, and $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$.

These formulae immediately lead to the integrals:

$$\int \frac{1}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a} + C, \quad \int \frac{1}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C, \quad \int \frac{1}{a^2-x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C.$$

Of these, the first only is worth commiting to memory.

Ex: Find $\displaystyle\int_0^1 \frac{1}{\sqrt{x^2+4}}\,dx$. $\displaystyle = \int_0^1 \frac{1}{2}\frac{1}{\sqrt{(x/4)^2+1}}\,dx$, $u = \frac{x}{4}$, $2du = \frac{1}{2}dx$

$$= 2\int_0^{\frac{1}{4}} \frac{1}{\sqrt{u^2+1}}\,du = 2\sinh^{-1}\left(\tfrac{1}{4}\right) - 2\sinh^{-1}(0)$$

$$= 2\log\left(\tfrac{1}{4} + \sqrt{\left(\tfrac{1}{4}\right)^2+1}\right) = \frac{1+\sqrt{17}}{2}$$

Ex: Find $\displaystyle\int_0^1 \frac{1}{\sqrt{x^2+4x+13}}\,dx$.

$$\int_0^1 \frac{1}{\sqrt{(x+2)^2+9}}\,dx = \int_2^3 \frac{1}{\sqrt{u^2+9}}\,du = \sinh^{-1}\frac{u}{3}\Big|_2^3$$

$$= \log\frac{3(1+\sqrt{2})}{2+\sqrt{13}}$$

Ex: Find $\displaystyle\int \frac{x}{\sqrt{x^4+1}}\,dx$. (Put $u = x^2$ first).

$u = x^2$ So $u' = 2x\,dx$ or

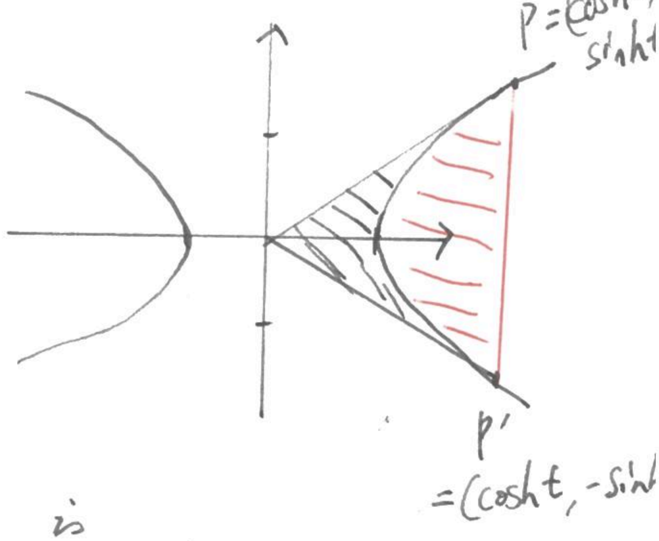
$\frac{1}{2}du = x\,dx$

The integral becomes

$$\frac{1}{2}\int \frac{1}{\sqrt{u^2+1}}\,du$$

$$= \frac{1}{2}\sinh^{-1} u + C = \frac{1}{2}\sinh^{-1} x^2 + C.$$

Ex: As mentioned above the hyperbolic functions can be used to parametrize the hyperbola. In fact, if we parametrize the point $P(x, y)$ on the hyperbola $x^2 - y^2 = 1$, by $x = \cosh t, y = \sinh t$, then the area bounded by the lines OP, OP' and the curve, where P' is the reflection of P in the x-axis, is t.



The area shaded in black is the area of the triangle minus the area of the red area. So it is

$$2\left(\frac{1}{2}\cosh t\cdot\sinh t - \int_1^{\cosh t}\sqrt{x^2-1}\,dx.\right)$$

$$= \cosh t\sinh t - 2\int_1^{\cosh t}\sqrt{x^2-1}\,dx.$$

Now set $x = \cosh u$, so $u = \cosh^{-1}x$

and $du = \dfrac{dx}{\sqrt{x^2-1}}$.

So $\displaystyle\int_1^{\cosh t}\sqrt{x^2-1}\,dx = \int_1^{\cosh t}\frac{x^2-1}{\sqrt{x^2-1}}\,dx = \int_0^t \sinh^2 u\,du$

Now $\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$. So

$\displaystyle\frac{1}{2}\int_0^t \cosh 2u - 1\,du = \frac{1}{4}\sinh(2t) - \frac{t}{2} = \frac{1}{2}\cosh t \sin t - \frac{t}{2}.$

The result follows from this.