THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

2019 Term 3

MATH1131
Mathematics 1A

(1) TIME ALLOWED – Two (2) hours

(2) TOTAL NUMBER OF QUESTIONS – 3

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS IS ON THE LAST PAGE

(9) TO OBTAIN FULL MARKS, YOUR ANSWERS MUST NOT ONLY BE CORRECT, BUT ALSO ADEQUATELY EXPLAINED, CLEARLY WRITTEN AND LOGICALLY SET OUT.

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 1

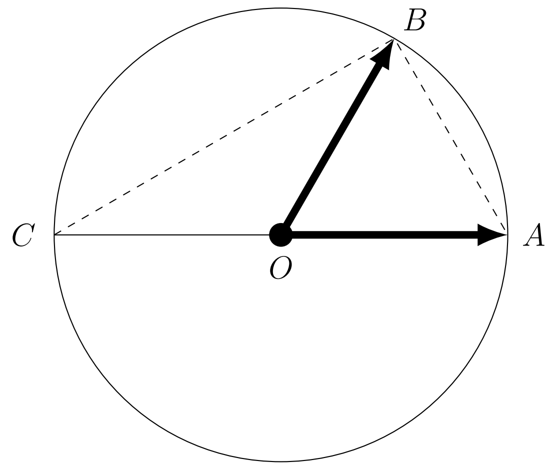
1. a) Evaluate the limit
$$\lim_{x\to 0} \frac{\sin x - x}{x^3}.$$

b) Use integration by parts twice to evaluate $\displaystyle\int_0^{\pi} e^{2x}\cos(3x)\,dx$.

c) i) State the Mean Value Theorem.

ii) Use the Mean Value Theorem to show that
$$9^{1/3} = 2 + \varepsilon$$
with $0 < \varepsilon < \dfrac{1}{12}$.

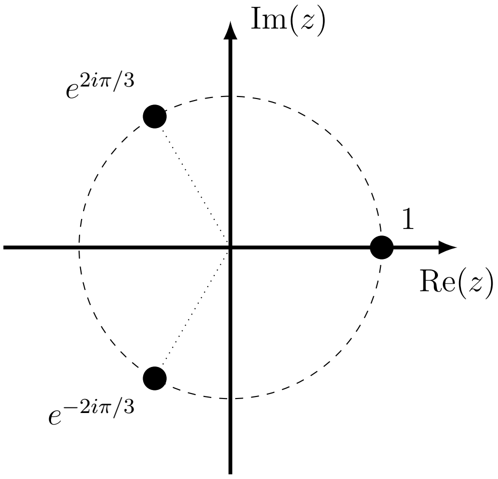
d) Consider a circle centred at O with diameter CA and another point B located on the circumference as shown.



i) Express $\overrightarrow{AB}$ and $\overrightarrow{BC}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

ii) By considering $\overrightarrow{AB} \cdot \overrightarrow{BC}$, prove that angle ABC is a right angle.

e) The cube roots of 1 are plotted on the diagram below.



i) Show that if ω is a cube root of 1 then $-2i\omega$ is a cube root of $8i$.

ii) Hence, by rotating and scaling the diagram above, plot and label the cube roots of $8i$. Indicate the angle of rotation and scaling factor.

iii) Find, in Cartesian form, z such that $(z+1-i)^3 = 8i$ and $\mathrm{Re}(z) > 0$.

f) Consider the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ defined as `v1` and `v2` in the following Maple session and then, using the Maple code or otherwise, answer the questions below.

```
> with(LinearAlgebra):
> v1 := <2,-1,3>;
```

$$v1 := \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

```
> v2 := <1,0,-2>;
```

$$v2 := \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

```
> CrossProduct(v1,v2);
```

$$\begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix}$$

i) Let P_0 be the plane through the origin spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. Find a condition on the components of $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ so that $\mathbf{b}$ lies in P_0.

ii) Find the Cartesian equation of the plane P_1 given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \qquad \text{for } \lambda_1, \lambda_2 \in \mathbb{R}.$$

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 2

2. a) Consider the line L with vector parametric form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \qquad \text{for } \lambda \in \mathbb{R},$$

and the plane P with Cartesian equation

$$x_1 + 2x_2 + 3x_3 = 4.$$

i) Show that the line L is parallel to the plane P.

ii) Find a point A on the line L and a point B on the plane P and write down the vector $\overrightarrow{AB}$.

iii) Find the shortest distance between the line L and the plane P.

b) A breakfast cereal manufacturer makes three types of muesli: Regular, Fruity and Nutty. The recipe for one bag of muesli involves mixing a whole number of portions of seeds, dried fruit and nuts. For each bag of muesli, the recipe is shown below.

- Regular:
  1 portion of seeds, 2 portions of dried fruit and 2 portions of nuts;

- Fruity:
  1 portion of seeds, 5 portions of dried fruit and 0 portions of nuts;

- Nutty:
  2 portions of seeds, 1 portion of dried fruit and 6 portions of nuts.

On a certain day, the manufacturer had 126 portions of seeds, 171 portions of dried fruit and 306 portions of nuts. The manager decided to use all these ingredients for the day to make exactly x bags of Regular, y bags of Fruity and z bags of Nutty.

i) Write a system of linear equations in x, y and z based on the above information.

ii) Use Gaussian elimination to solve the system of linear equations.

iii) The manager believes that Nutty is the most popular muesli. Find the solution which maximises the number of bags of Nutty.

c) Using the following Maple session, or otherwise, answer the questions below.

```
> p := z -> z^7-13*z^6+81*z^5-317*z^4+828*z^3-1380*z^2+1300*z-500:
> p(1+3*I);
                                    0
```

Let $p(z) = z^7 - 13z^6 + 81z^5 - 317z^4 + 828z^3 - 1380z^2 + 1300z - 500$.

i) Show that $z - 1$ is a factor of $p(z)$.

ii) Find a real quadratic factor of $p(z)$.

d) Consider the $n \times n$ matrix B and vector $\mathbf{y} \in \mathbb{R}^n$ such that $B\mathbf{y} \neq \mathbf{0}$ and $B^2\mathbf{y} = \mathbf{0}$.

i) Find a non-zero solution $\mathbf{x} \in \mathbb{R}^n$ to $B\mathbf{x} = \mathbf{0}$.

ii) What can be said about $\det(B)$? Give reasons for your answer.

iii) Show that the linear system $B^2\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 3

3. a) A string of length 1 is cut into two pieces of length x and $1-x$. An equilateral triangle is formed with the piece of length x and a circle of circumference $1-x$ is formed with the other piece. The total area of the two shapes is denoted A.

You may find the following Maple session useful in answering the questions below. You may assume that the formula given for A in the Maple code is correct.

```
> A := x^2*sqrt(3)/36 + (1-x)^2/4/Pi;
```

$$A := \frac{x^2\sqrt{3}}{36} + \frac{(1-x)^2}{4\pi}$$

```
> dAdx := diff(A,x);
```

$$dAdx := \frac{x\sqrt{3}}{18} - \frac{1-x}{2\pi}$$

```
> x0 := solve(dAdx=0,x);
```

$$x0 := \frac{9}{\sqrt{3}\pi + 9}$$

```
> evalf(subs(x=x0,A));
```

$$0.02998412613$$

i) By referring to an appropriate theorem, explain why there must be a maximum and a minimum total area A.

ii) Find the value of x that maximises the total area A and the value of x that minimises the total area A. Give reasons to justify your answers.

iii) For each of the extreme values you found in (ii), determine, giving reasons, whether the circle or the triangle has the larger area.

b) Apply the Limit Comparison Test to determine whether or not the improper integral

$$J = \int_1^\infty \frac{x}{\sqrt{1+x^4}}\, dx$$

converges or diverges.

c) In the following Maple session some hyperbolic trigonometric identities are verified. These identities may be useful in part (ii) below.

```
> simplify(cosh(u)^2 - cosh(2*u)/2);
```

$$\frac{1}{2}$$

```
> simplify(cosh(u)^2 - sinh(u)^2);
```

$$1$$

```
> simplify(sinh(2*u) - 2*sinh(u)*cosh(u));
```

$$0$$

i) Using the definition of $\cosh u$, prove the first of the hyperbolic trigonometric identities that are verified by Maple above.

ii) Use the substitution $x = \sinh u$ to evaluate the integral

$$\int \sqrt{1+x^2}\,dx.$$

d) i) Sketch the polar curve

$$r = 1 + \sin\theta.$$

ii) Find the coordinates of the points on the curve with a vertical tangent.

e) For a function $h : \mathbb{R} \to \mathbb{R}$, the definition of

$$\lim_{x\to\infty} h(x) = 0$$

states that for all positive ε, there is a real number M such that

$$x > M \implies |h(x) - 0| < \varepsilon.$$

Using the definition of the limit, prove that

$$\lim_{x\to\infty} \cos x \neq 0.$$

Please see over ...

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BASIC INTEGRALS

$$
\begin{aligned}
&\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k\\
&\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C\\
&\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1\\
&\int \sin ax\,dx = -\frac{1}{a}\cos ax + C\\
&\int \cos ax\,dx = \frac{1}{a}\sin ax + C\\
&\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C\\
&\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C\\
&\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C\\
&\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C\\
&\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C\\
&\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C\\
&\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C\\
&\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C\\
&\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C\\
&\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\\
&\int \frac{dx}{a^2-x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, \qquad |x| < a\\
&\qquad\qquad\quad = \tfrac{1}{a}\coth^{-1}\tfrac{x}{a} + C, \qquad |x| > a > 0\\
&\qquad\qquad\quad = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, \qquad x^2 \neq a^2\\
&\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C\\
&\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + C\\
&\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0
\end{aligned}
$$

END OF EXAMINATION