LECTURE 13
Curve Sketching

$y = x^2$

A function f is said to be **odd** if $f(-x) = -f(x)$ over its domain.

A function f is said to be **even** if $f(-x) = f(x)$ over its domain.

To sketch an unknown graph of a function $y = f(x)$ we use a checklist to investigate its structure. We find:

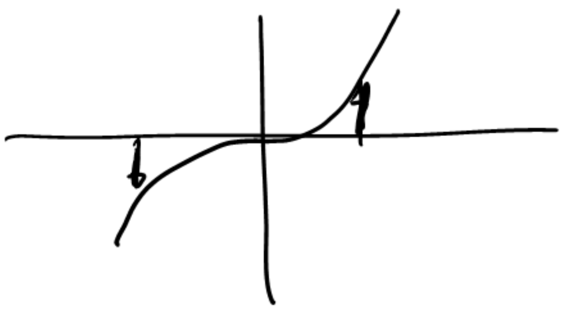
✓ the y intercept by setting $x = 0$.
✓ the x intercept(s) by solving $f(x) = 0$ for x.
✓ whether or not the function is odd or even.
✓ (V.A.) the vertical asymptotes (usually a consequence of division by zero).
✓ (H.A.) the behaviour of the function for large $|x|$ by considering $\lim_{x \longrightarrow \pm\infty} f(x)$.
(H.A. will determine horizontal and oblique asymptotes).
✓ the position and nature of any stationary points (this is crucial).
✓ !! If stuck plot points !!

Odd and Even Functions

A function f is said to be **odd** if $f(-x) = -f(x)$ over its domain.
Examples of odd functions are $y = x$, $y = x^3$,$y = x^5$, $y = \sin(x)$ and $y = \sin^{-1}(x)$.

Odd functions exhibit skew symmetry in their graphs.

Example 1:



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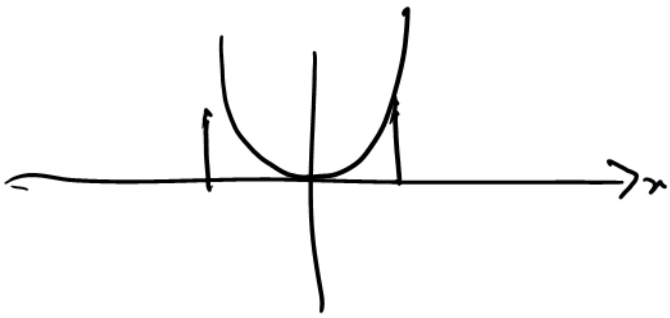
A function f is said to be **even** if $f(-x) = f(x)$ over its domain.

Examples of even functions are $y = 1, \quad y = x^2, \quad y = x^4$ and $y = \cos(x)$. $y = |x|$

Even functions exhibit symmetry about the y axis in their graphs.

Example 2:



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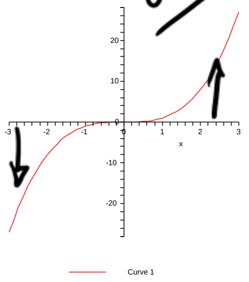
odd×odd=even, odd×even=odd, even×even=even.

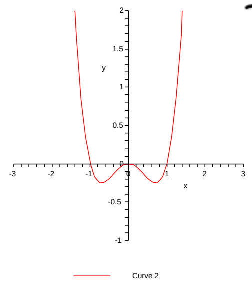
odd±odd=odd, even±even=even, even±odd=neither.

$$\int_{-a}^{a} odd = 0, \quad \int_{-a}^{a} even = 2\int_{0}^{a} even.$$

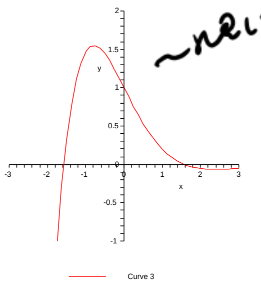
Example 3: Identify each of the following functions as odd, even or neither:

a) 

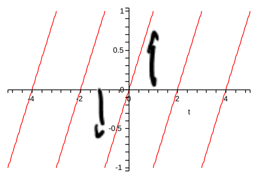
odd

b) 

even

c) 

neither

d) 

odd

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Example 4: Identify each of the following functions as odd, even or neither:

i) $f(x) = \cos(3x)$ even

ii) $f(x) = x\cos(x)$ = odd

iii) $f(x) = \sin^2(x)$ $= \sin x \times \sin x$ = even

o × o

iv) Prove your answer in iii) from the definition.

$f(x) = \sin^2(x)$

$$\begin{aligned} f(-x) &= \sin^2(-x) \\ &= \sin(-x) \times \sin(-x) \\ &= -\sin x \times -\sin x \\ &= \sin^2(x) = f(x) \end{aligned}$$

$\therefore$ f is even

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Example 5: Prove that the derivative of an even function is an odd function.

Proof: Let f be an even fn.

$\therefore \quad f(-x) = f(x)$

$f'(-x)(-1) = f'(x)$

chain rule.

$\therefore f'(-x) = -f'(x)$

$\therefore f'$ is odd

$\bigstar$

We now turn to the problem of sketching an unknown function. This can be quite tricky and our approach is to piece together a graph using a host of different clues as to the shape of the curve. Our checklist is to find: $y = f(x)$

✓ the y intercept by setting $x = 0$.

✓ the x intercept(s) by solving $f(x) = 0$ for x.

✓ whether or not the function is odd or even. – saves time!

✓ (V.A.) the vertical asymptotes (usually a consequence of division by zero).

✓ (H.A.) the behaviour of the function for large $|x|$ by considering $\lim_{x \longrightarrow \pm\infty} f(x)$.
(H.A. will determine horizontal and oblique asymptotes).

✓ the position and nature of any stationary points (this is crucial). $\cdot\,\cdot\,\cdot$

✓ !! If stuck plot points !!

Let's have a look at a batch of examples.

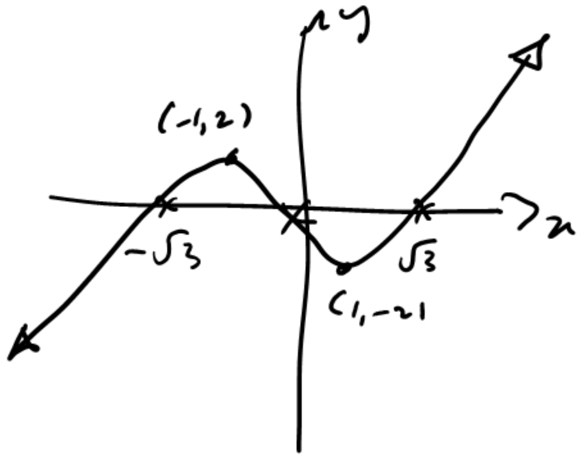
odd → ⌐ 2n

Example 6: Sketch the graph of $y = f(x) = x^3 - 3x$.

$x = 0 \rightarrow y = 0$, $y = 0 \Rightarrow x^3 - 3x = 0$
$\Rightarrow x(x^2 - 3) = 0 \Rightarrow x = 0, \pm\sqrt{3}$

VA: X

HA: $\lim_{x \to \infty} x^3 - 3x = \infty$



$y' = 3x^2 - 3 = 0 \Rightarrow x^2 = 1$
$\Rightarrow x = \pm 1$

$x = 1 \rightarrow y = 1 - 3 = -2 \Rightarrow (1, -2)$
$x = -1 \Rightarrow y = 2 \Rightarrow (-1, 2)$

$y'' = 6x$
$y''(1) = 6(1) = 6 > 0$ ∪ ⇒ local min

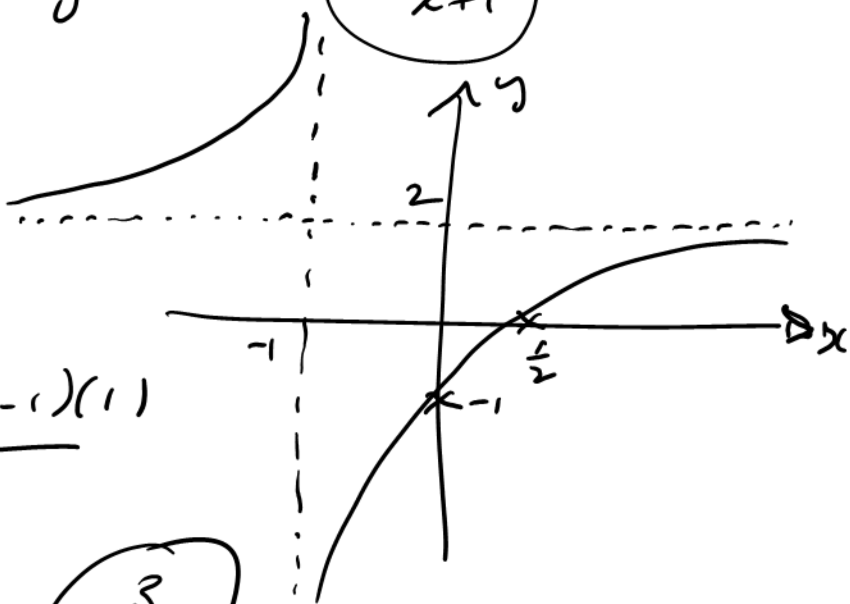
Example 7: Sketch the graph of $\boxed{y = f(x) = \dfrac{2x-1}{x+1}}$.

$x = 0 \Rightarrow y = -1$, $\quad y = 0 \Rightarrow \dfrac{2x-1}{x+1} = 0 \Rightarrow x = \frac{1}{2}$

orse XX

V.A. $x = -1$

H.A. $\lim_{x \to \infty} \dfrac{2x-1}{x+1} = 2$



$$y' = \frac{(x+1)(2) - (2x-1)(1)}{(x+1)^2}$$

$$= \frac{\cancel{2x} + 2 - \cancel{2x} + 1}{(x+1)^2} = \frac{3}{(x+1)^2} = 0$$

(st. pnts)

No stat. pnts.

$y' > 0$

$\Rightarrow$ increasing.

Example 8: Consider the function $f(x) = \dfrac{x^2 - 5x + 29}{x - 4}$.

a) Sketch the graph of $y = f(x)$.

b) Hence find the domain and range of f.

c) By considering your sketch find all values of k for which the equation

$$x^2 - (5 + k)x + (29 + 4k) = 0$$

2a

has exactly one solution.

d) **Homework:** Check your answer to c) by using the discriminant.

a) $x = 0 \Rightarrow y = -\frac{29}{4} = -7\frac{1}{4}$

$y = 0 \Rightarrow \dfrac{x^2 - 5x + 29}{x - 4} = 0 \Rightarrow x^2 - 5x + 29 = 0$

$\Delta = 25 - 4(29) < 0$

$\Rightarrow$ no sol^{n}.

VA : $x = 4$

HA: $\lim_{x \to \infty} \dfrac{x^2 - 5x + 29}{x - 4} = \infty.$

$$\begin{array}{r|l} & x - 1 \\ \hline (x-4) & x^2 - 5x + 29 \\ & x^2 - 4x \\ \hline & -x + 29 \\ & -x + 4 \\ \hline & 25. \end{array}$$

$$\frac{x^2 - 5x + 29}{x - 4} = \underbrace{(x - 1)}_{\text{inclined asymptote}} + \underbrace{\frac{25}{x - 4}}_{\text{large } x \to 0}$$

$\frac{13}{4} = 3\frac{1}{3}$

$$y = (x-1) + \frac{25}{x-4}$$

$$y' = 1 - 0 + \frac{0 - 25(1)}{(x-4)^2} = 0 \quad \text{(st. pnts)}$$

$$1 = \frac{25}{(x-4)^2} \Rightarrow (x-4)^2 = 25$$

$$\Rightarrow x - 4 = \pm 5$$

$$\Rightarrow x = 9, -1$$

$y = 13$, $y = -7$

$$\Rightarrow (9, 13), (-1, -7)$$

min, max

c)

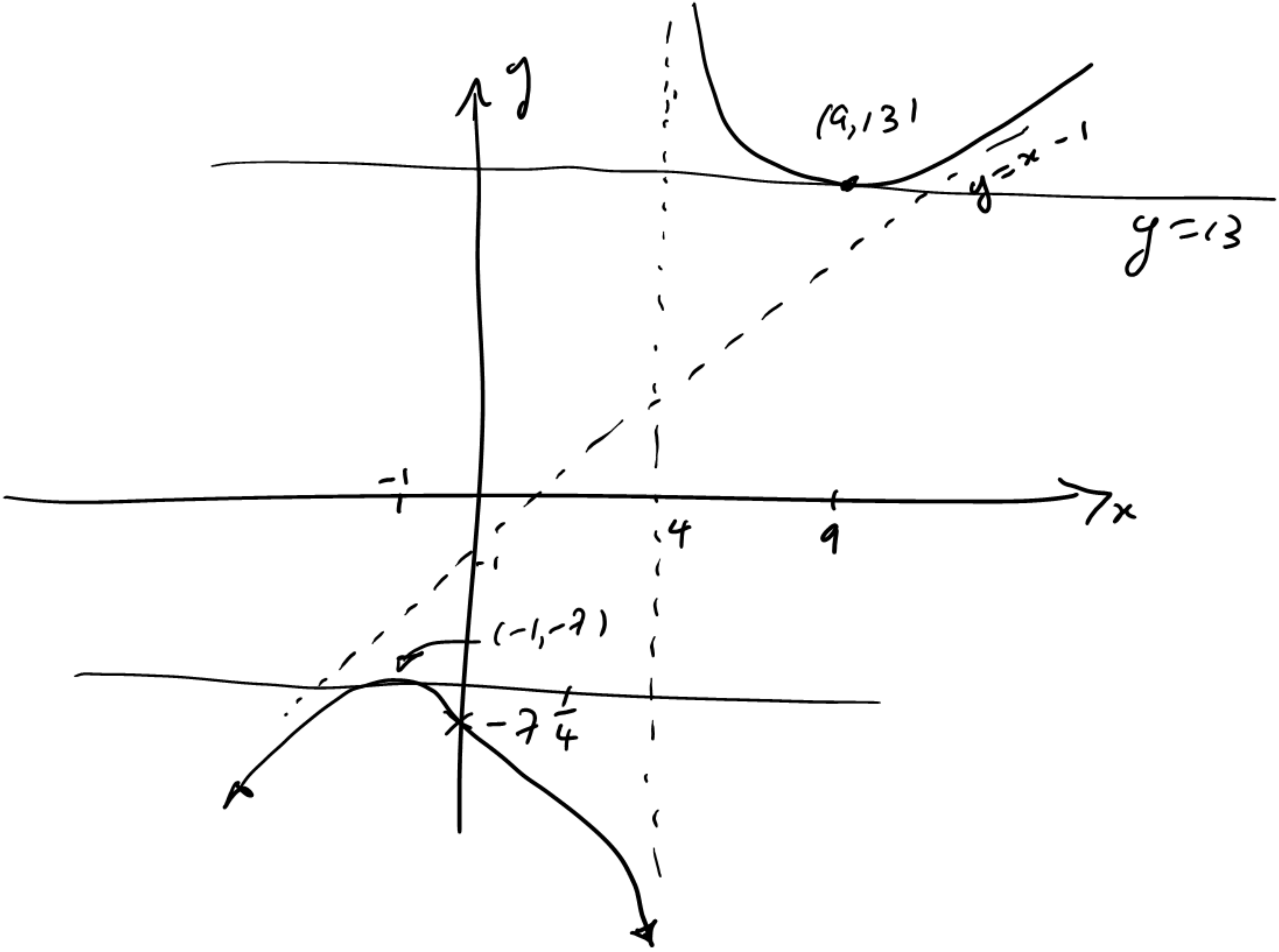
$$x^2 - (5+k)x + (29+4k) = 0$$

$$x^2 - 5x - kx + 29 + 4k = 0$$

$$x^2 - 5x + 29 = k(x-4)$$

$$\Rightarrow \frac{x^2 - 5x + 29}{x-4} = k. \quad !!$$

✓ hor. line.



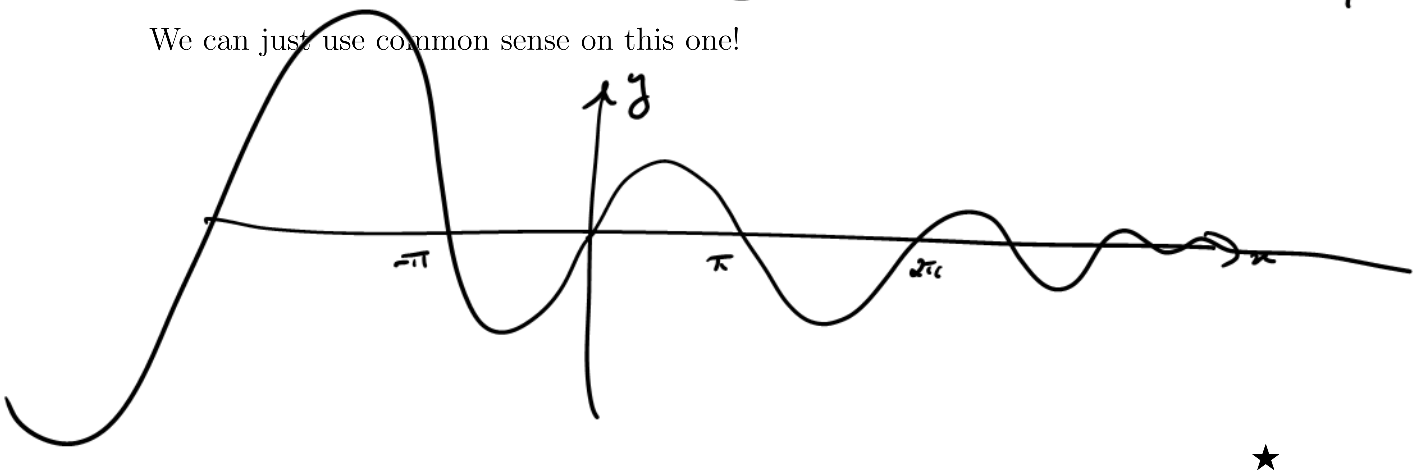
★ $(-1,-7)$ is a local max and $(9,13)$ is a local min. ★

★ Dom(f) $= \{x \in \mathbb{R} : x \neq 4\}$ Range(f) $= [13,\infty) \cup (-\infty,-7]$ ★

★ $k = -7, 13$ ★

Example 9: Sketch the graph of $y = f(x) = e^{-x}\sin(x)$.

We can just use common sense on this one!



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