$\int_a^b f(x)\,dx = F(b) - F(a)$

$\int_a^b f(x)\,dx$

LECTURE 17

Integration by Substitution

$\cancel{F' f}$

A substitution of variable can often dramatically simplify an integral.

When using the method of substitution to evaluate $\displaystyle\int_a^b f(x)dx$ don't forget to also substitute the limits and the increment dx.

$$\text{If } f \text{ is odd then } \int_{-a}^{a} f(x)\,dx = 0.$$

$$\text{If } f \text{ is even then } \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

SOME BASIC INTEGRALS

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^{ax} dx = \frac{e^{ax}}{a} + C$$

$$\int \sin ax\,dx = -\frac{\cos ax}{a} + C$$

$$\int \cos ax\,dx = \frac{\sin ax}{a} + C$$

$$\int \sec^2 ax\,dx = \frac{\tan ax}{a} + C$$

In this lecture we will develop some new integration techniques. But first a revision example on the fundamental theorems.

Example 1: Use the first fundamental theorem of calculus to find f if

$$\int_0^x t f(t)\, dt = \int_x^0 (t-1) f(t)\, dt + x^3.$$

$$\int_0^x t f(t)\, dt = \int_0^x (1-t) f(t)\, dt + x^3$$

$$\frac{d}{dx} \int_0^x t f(t)\, dt = \frac{d}{dx} \int_0^x (1-t) f(t)\, dt + 3x^2$$

$$x f(x) = (1-x) f(x) + 3x^2$$

$$x f(x) = f(x) - x f(x) + 3x^2$$

$$2x f(x) - f(x) = 3x^2$$

$$f(x)\{2x-1\} = 3x^2$$

$$\boxed{f(x) = \frac{3x^2}{2x-1}}$$

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x)$$

$$\bigstar \quad f(x) = \frac{3x^2}{2x-1} \quad \bigstar$$

Integration by Substitution

It is generally much harder to integrate functions than it is to differentiate them. A disturbing issue is that many functions have primitives which cannnot be expressed in terms of standard functions. This means that these functions can never be integrated via the second fundamental theorem of calculus. For example $\int e^{x^2}\,dx$ cannot be done regardless of how much mathematics you know. Looks simple but it's impossible! We do however have many tricks which help us to find the integrals of complicated functions.

The process of substitution through a change of variable can often clarify the nature of an integral. Substitutions do not solve the problem, but rather change the question into a format where the path to solution is a little clearer. When making a substitution always remember to take care of:

1) The function. ✓
2) The increment dx. ✓
3) The limits if they exist. ✓

Example 2: Find $\int (2x+7)^{11}\,dx$ by using the substitution $u = 2x+7$.

$\cancel{u = (2x+7)^{11}}$

$\cancel{u = 2x}$

$\cancel{\int u^{11}\,dx}$

Let $u = 2x+7$

$\frac{du}{dx} = 2 \Rightarrow du = 2dx$

$$\int = \int u^{11}\,\frac{du}{2}$$

$$= \frac{1}{2}\int u^{11}\,du = \frac{1}{2}\cdot\frac{u^{12}}{12} = \frac{u^{12}}{24} + c$$

$$= \quad \star \; \frac{(2x+7)^{12}}{24} + C \; \star$$

Note that it is crucial to return to the original variable. Note also ~~that the~~ increment dx plays a central role and should always be included when integrating by substitution.

We can always check our answer by differentiating.

$$\frac{d}{dx}\left[\frac{(2x+7)^{12}}{24} + \cancel{C}\right] = \frac{\cancel{12}\,(2x+7)^{11}(\cancel{2})}{\cancel{24}}$$

$$= (2x+7)^{11}$$

★

We will not always give you the substitution so you need to develop the skill of deciding what is appropriate. Generally we substitute away the piece of the integral which is causing the most concern.

Be aware that the method of substitution is not always appropriate and that there are plenty of integrals where no substitution will work!

Example 3: Find $\int x^2 \cos(x^3)\, dx$

Let $u = x^3$

$du = 3x^2 dx.$

$x^2 dx = \frac{du}{3}$

$= \int \cos(u) \frac{du}{3}$

$= \frac{1}{3} \int \cos(u) du$

$= \frac{1}{3} \sin u + c = \frac{1}{3} \sin(x^3) + C$

$$\bigstar \quad \frac{1}{3}\sin(x^3) + C \quad \bigstar$$

Example 4: Find $\int \frac{\sin(\ln x)}{x}\, dx$ and check your answer via differentiation.

du

Let $u = \ln x.$

$du = \frac{1}{x} dx.$

$\int = \int \sin(u) du$

$= -\cos(u) + c$

$= -\cos(\ln x) + c$

$$\bigstar \quad -\cos(\ln x) + C \quad \bigstar$$

Example 5: Find $\int \frac{e^t}{1+e^t}\,dt$ du

Let $u = 1+e^t$
$du = e^t dt$

$= \int \frac{du}{u} = \ln u$
$= \ln(1+e^t) + c$

★ $\ln(1+e^t)$ ★

When there are limits involved the process is much the same, but you must make sure to take care of the limits of integration.

Example 6: Evaluate $\int_3^4 x(x-3)^{10}\,dx$

You should never present an integral with more than one variable in it. It is either all x or all u.

Let $u = x-3$
$du = dx$.
$x = 3 \to u = 0$
$x = 4 \to u = 1$
$x = u+3$

!!!

$\int_0^1 (u+3)\,u^{10}\,du$

$\int_0^1 u^{11} + 3u^{10}\,du$

$\left[\frac{u^{12}}{12} + \frac{3u^{11}}{11}\right]_0^1 = \left(\frac{1}{12} + \frac{3}{11}\right) - (0)$

$= \frac{11+36}{132} = \frac{47}{132}$

★ $\frac{47}{132}$ ★

Note that if there are limits there is no need to return to the original variable as the final answer is a number.

Example 7: Evaluate $\int_0^1 \frac{2x}{(x+4)^2}dx$

Let $u = x+4$
$du = dx$.
$x = 0 \to u = 4$
$x = 1 \to u = 5$.

$\int_4^5 \frac{2(u-4)}{u^2}\,du = \int_4^5 \frac{2u}{u^2} - \frac{8}{u^2}\,du$

$= 2\int_4^5 \frac{1}{u}\,du - 8\int_4^5 u^{-2}\,du$

$= 2\left[\ln u\right]_4^5 - 8\left[\frac{u^{-1}}{-1}\right]_4^5 =$

$2\{\ln 5 - \ln 4\}$

$8\left[\frac{1}{u}\right]_4^5$

$8\{\frac{1}{5} - \frac{1}{4}\}$

$8\{\frac{4-5}{20}\}$

$= \frac{-8}{20}$

$= \boxed{\frac{-2}{5}}$

★ $\boxed{2\ln(\frac{5}{4}) - \frac{2}{5}}$ ★

Sometimes substitutions do their work in very mysterious ways.

Example 8: Use the substitution $u = \frac{\pi}{2} - x$ to evaluate

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin(x)}{\sin(x) + \cos(x)}\,dx$$

Let $u = \frac{\pi}{2} - x$
$du = -dx$
$x = 0 \to u = \frac{\pi}{2}$
$x = \frac{\pi}{2} \to u = 0$

$= \int_{\frac{\pi}{2}}^0 \frac{\sin(\frac{\pi}{2} - u)\,(-du)}{\sin(\frac{\pi}{2} - u) + \cos(\frac{\pi}{2} - u)}$

$= \int_0^{\frac{\pi}{2}} \frac{\cos(u)}{\cos(u) + \sin(u)}\,du = \int_0^{\frac{\pi}{2}} \frac{\cos(x)}{\cos(x) + \sin(x)}\,dx = I$

$I + I = 2I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx + \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x}\,dx$

$\boxed{2I = \frac{\pi}{2} \quad \therefore I = \frac{\pi}{4}}$

$= \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2}$ ★ $\frac{\pi}{4}$ ★

Remember that the variable of integration is always a dummy variable:

$\int_0^1 x^2\,dx = \int_0^1 t^2\,dt = \int_0^1 q^2\,dq = \ldots$ same

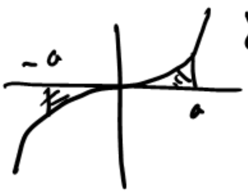
$\int_{-7}^{7} \text{odd}$!!!

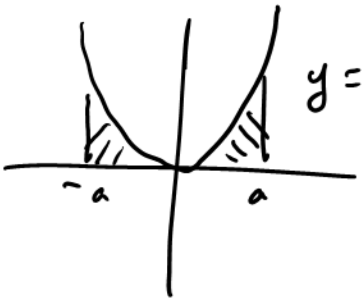
Fact: If f is odd then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0.$

Fact: If f is even then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$

You should always consider the issue of oddness and evenness when you are faced with a symmetric integral from $-a$ to a !

Why does this work?





odd: sin, , $x, x^3, x^5, \ldots$

even: cos, $|x|$, $1, x^2, x^4, \ldots$

★

Example 9: Evaluate $\displaystyle\int_{-1}^{1} \frac{\overbrace{x^8}^{\text{even}}\,\overbrace{\sin(x)}^{\text{odd}}}{\underbrace{\sqrt{1+x^2+x^4}}_{\text{even}}}\,dx.$ $=$!! odd !!

o × o = e
o × e = o
e × e = e
same ÷

$\int_{-1}^{1} \text{odd} = 0$

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