School of Mathematics and Statistics
Math1131-Algebra

Lec11: Euler and De Moivre's formulae

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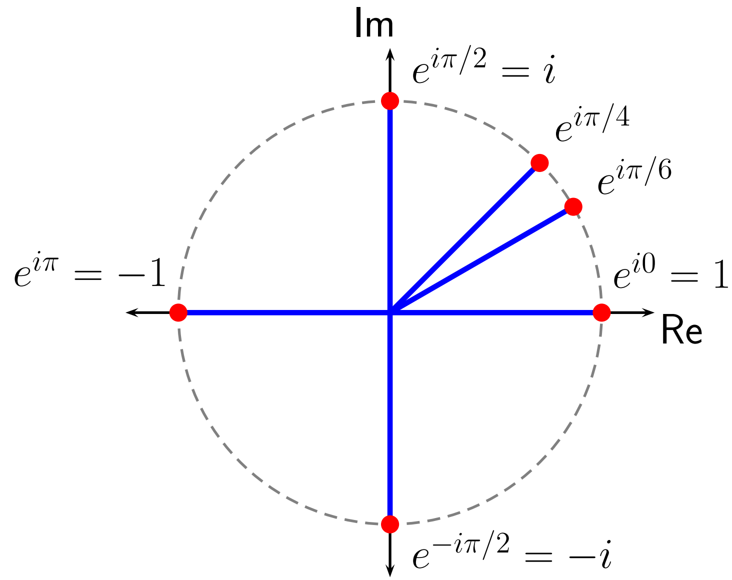
Euler's formula: A new notation for $\cos\theta + i\sin\theta$

A new notation for $\cos\theta + i\sin\theta$: $e^{i\theta}$

We define the *complex exponential* by :

$$e^{i\theta} \stackrel{def}{=} \cos\theta + i\sin\theta. \qquad \cancel{\text{cis}\,\theta}$$

This is *Euler's formula*.



Polar form : $z = re^{i\theta}$

The *polar form* of a non-zero complex number $z = a + ib$
with modulus $r = |z| = \sqrt{a^2 + b^2}$
and principal argument $\text{Arg}(z) = \theta$ is

$$z = re^{i\theta}$$

$$\begin{aligned} e^{i\pi/6} &= \cos\left(\tfrac{\pi}{6}\right) + i\sin\left(\tfrac{\pi}{6}\right) \\ &= \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i \end{aligned}$$

$$e^{i\pi/4} = \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i$$

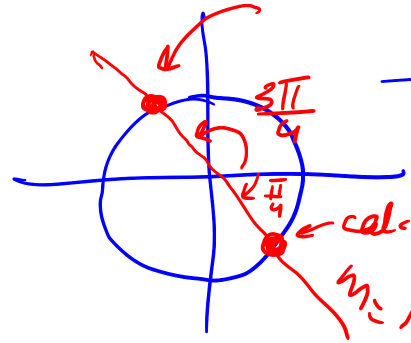
Examples: Cartesian form to polar form and vice-versa

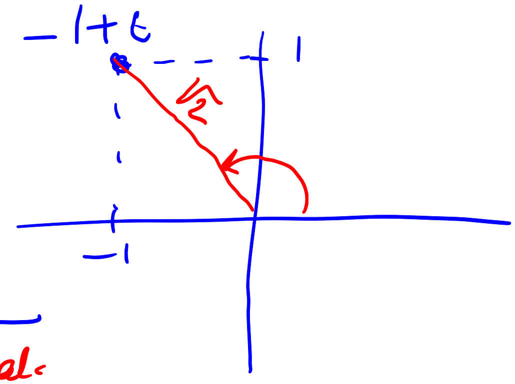
Exercise 1. Find the polar form of $z = -1 + i$.

$r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\tan\theta = \frac{y}{x} = \frac{1}{-1} = -1$

$$\boxed{z = \sqrt{2}\, e^{\frac{3\pi}{4} i}}$$



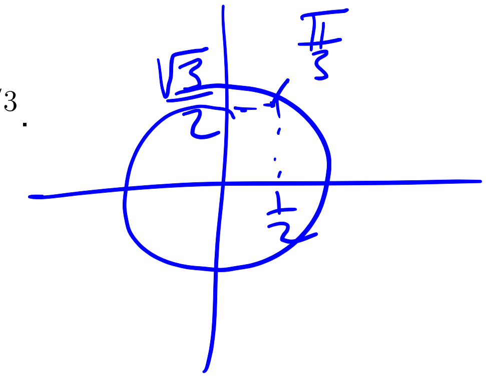


Exercise 2. Find the Cartesian form of $w = 6e^{i\pi/3}$.

$w = 6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= 6\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= \boxed{3 + i\,3\sqrt{3}}$



Checking our answer with Maple

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> z := -1 + I;
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$$z := -1 + \mathrm{I}$$

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> # Convert to polar coordinates

  convert(%, polar);
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$$\mathrm{polar}\left(\sqrt{2}, \frac{3\pi}{4}\right)$$

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> # Polar form to Cartesian form

  w := 6*exp(I*Pi/3);
```

$$w := 3 + 3\,\mathrm{I}\sqrt{3}$$

Examples: When the argment is a multiple of 2π

Exercise 3.

a) Evaluate $e^{2i\pi} = 1$

b) Evaluate $e^{-6i\pi} = 1$

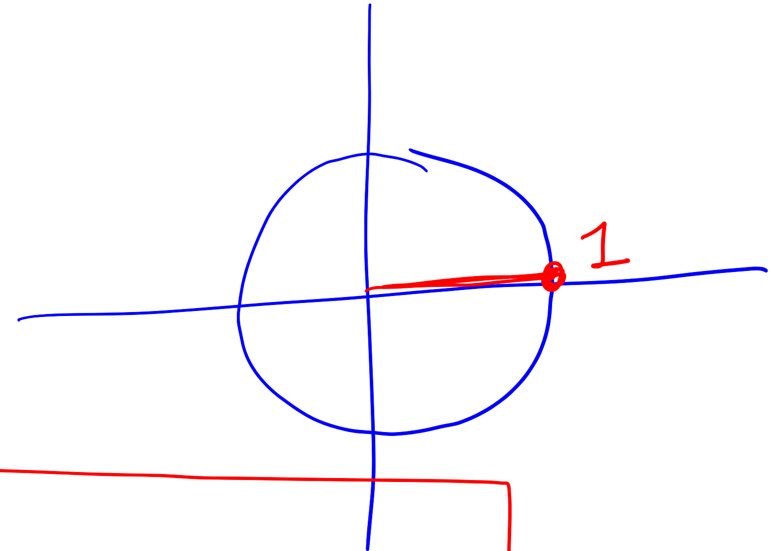
c) Find a generalisation of these two results.

$$\boxed{e^{i0} = e^{0} = 1}$$

$$e^{2i\pi} = 1 \times e^{2i\pi}$$

$$= \cos 2\pi + i \sin 2\pi$$

a) $e^{2i\pi} = 1 + i0 = 1$



$$\boxed{e^{i(2k\pi)} = 1 \quad \text{for all } k \in \mathbb{Z}}$$

Euler's formula: Why is it a good notation?

The function $f(\theta) = \cos\theta + i\sin\theta$ has properties that are very similar to the properties of the exponential function.

This is what has lead to the choice of adopting the notation $e^{i\theta} = \cos\theta + i\sin\theta$.

$$e^{i0} = \cos 0 + i\sin 0 = 1 + i \times 0 = 1$$

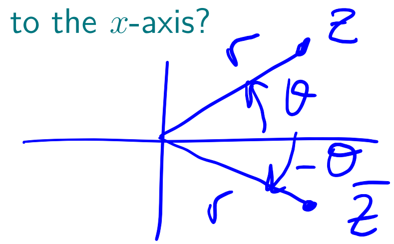
$$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}$$

$$\begin{aligned}\frac{d}{d\theta}e^{i\theta} &= \frac{d}{d\theta}\Big(\cos\theta + i\sin\theta\Big)\\ &= -\sin\theta + i\cos\theta\\ &= i(\cos\theta + i\sin\theta)\\ &= ie^{i\theta}\end{aligned}$$

Conjugates and Products in polar form

Note the following : $\overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}$

Recall how we said a complex and its conjugate are symmetric with respect to the x-axis? Same message here!



and

$$
\begin{aligned}
e^{i\theta}e^{i\phi} &= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\
&= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi) \\
&= \cos(\theta+\phi) + i\sin(\theta+\phi) \\
&= e^{i(\theta+\phi)}
\end{aligned}
$$

TIP! In other words, the usual index Laws apply to the complex exponential.

This gives us an easy way to multiply complex numbers in polar form:

Product of complex numbers in polar form

For $r_1, r_2, \theta_1, \theta_2 \in \mathbb{R}$,

$$z_1 = r_1 e^{i\theta_1} \quad \text{and} \quad z_2 = r_2 e^{i\theta_2} \quad \Longrightarrow \quad z_1 z_2 = r_1 e^{i\theta_1} \times r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

The moduli multiply and the arguments add.

Division in polar form

Firstly note that

$$e^{-i\theta}e^{i\theta} = (\cos\theta - i\sin\theta)(\cos\theta + i\sin\theta) = \cos^2\theta + \sin^2\theta = 1$$

and hence

$$\frac{1}{e^{i\theta}} = e^{-i\theta}.$$

This gives us an easy way to divide complex numbers in polar form:

Quotient of complex numbers in polar form

For $r_1, r_2, \theta_1, \theta_2 \in \mathbb{R}$,

$$z_1 = r_1 e^{i\theta_1} \quad \text{and} \quad z_2 = r_2 e^{i\theta_2} \quad \Longrightarrow \quad \frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

The modulus of the quotient is the quotient of the moduli
and the argument of the quotient is the difference of the arguments.

TIP! Again, this says that the usual index Laws apply to the complex exponential.

Multiplication and division in polar form

SUMMARY: Product and Quotient of complex numbers in polar form

For $z, w \in \mathbb{C}$,

$$\textit{Product}: \quad |zw| = |z||w| \quad \text{and} \quad \text{Arg}(zw) = \text{Arg}(z) + \text{Arg}(w) + 2k\pi$$

$$\textit{Quotient}: \quad \left|\frac{z}{w}\right| = \frac{|z|}{|w|} \quad \text{and} \quad \text{Arg}\left(\frac{z}{w}\right) = \text{Arg}(z) - \text{Arg}(w) + 2k\pi$$

for suitable $k \in \mathbb{Z}$.

Exercise 4. Let $z = 2e^{2i\pi/3}$ and $w = 5e^{3i\pi/4}$. Find each of the following in polar form, state their modulus and principal argument and sketch them on the Argand Diagram.

(a) zw (b) $\frac{z}{w}$ (c) $\overline{z}$

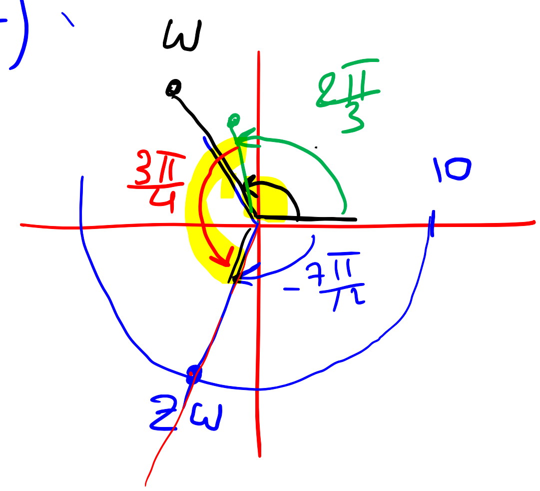
(a) $zw = 2e^{\frac{2i\pi}{3}} \times 5e^{\frac{3i\pi}{4}}$

$= 2 \times 5\, e^{i\left(\frac{2\pi}{3} + \frac{3\pi}{4}\right)}$

$= 10\, e^{i\left(\frac{17\pi}{12}\right)}$

$= 10\, e^{i\left(\frac{17\pi}{12} - \frac{24\pi}{12}\right)}$

$= \boxed{10\, e^{-\frac{7i\pi}{12}}}$



Multiplication and division in polar form

Exercise 4, continued. Let $z = 2e^{2i\pi/3}$ and $w = 5e^{3i\pi/4}$. Find each of the following in polar form, state their modulus and principal argument and sketch them on the Argand Diagram.

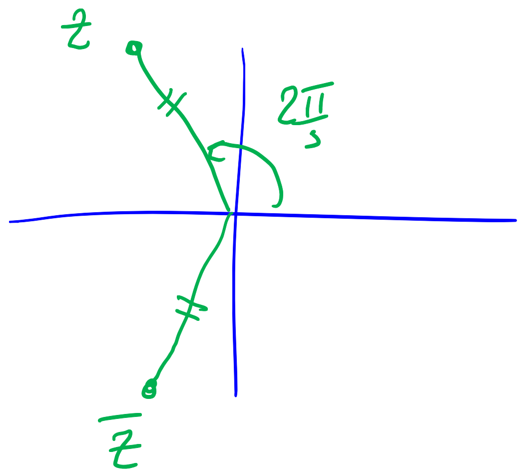
(a) zw (b) $\frac{z}{w}$ (c) $\overline{z}$

$$\frac{z}{w} = \frac{2e^{\frac{2i\pi}{3}}}{5e^{\frac{3i\pi}{4}}}$$

$$= \frac{2}{5}e^{i\left(\frac{2\pi}{3} - \frac{3\pi}{4}\right)}$$

$$= \frac{2}{5}e^{i\left(-\frac{\pi}{12}\right)}$$

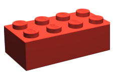
(c) $\overline{z} = \overline{2e^{\frac{2i\pi}{3}}} = 2e^{-\frac{2i\pi}{3}}$



De Moivre's Theorem

So far we have seen that index laws for the product and quotients of real exponentials hold for complex exponentials.

The fact that this extends to **integer powers** is De Moivre's Theorem.



De Moivre's Theorem

For any $\theta \in \mathbb{R}$ and any $n \in \mathbb{Z}$

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

which can be re-written

$$(e^{i\theta})^n = e^{in\theta}.$$

Proof of De Moivre's Theorem

RECALL De Moivre's Theorem: For any $\theta \in \mathbb{R}$ and any $n \in \mathbb{Z}$, $\quad (e^{i\theta})^n = e^{in\theta}$.

PROOF

- For $n \in \mathbb{N}$, we prove it by induction on n.

Base case: $n = 0$. Note that for $\theta \in \mathbb{R}$, $(e^{i\theta})^0 = 1 = e^{i \times 0 \times \theta}$.

Induction step : Suppose that for some integer $n \in \mathbb{N}$ we have $(e^{i\theta})^n = e^{in\theta}$. Then,

$$(e^{i\theta})^{n+1} = (e^{i\theta})^n e^{i\theta} = e^{in\theta} e^{i\theta} = e^{i(n\theta + \theta)} = e^{i(n+1)\theta}.$$

So by induction we have shown that

$$(e^{i\theta})^n = e^{in\theta} \text{ for all } n \in \mathbb{N}.$$

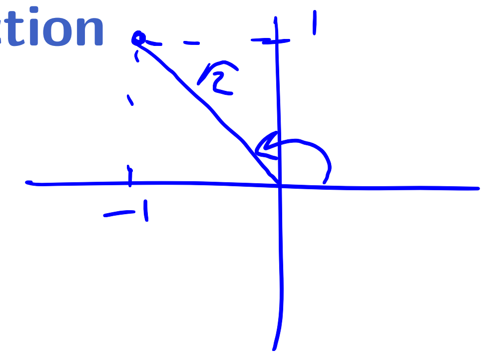
- Now suppose that n is a negative integer. Let $m = -n > 0$

$$(e^{i\theta})^n = (e^{i\theta})^{-(-n)} = (e^{i\theta})^{-m} = \frac{1}{(e^{i\theta})^m} = \frac{1}{e^{im\theta}} = e^{-im\theta} = e^{in\theta}.$$

So

$$(e^{i\theta})^n = e^{in\theta} \text{ for all } n \in \mathbb{Z}.$$

De Moivre's Theorem in action



Exercise 5. Find $(-1+i)^{202}$.

$$z := -1 + i = \sqrt{2}\, e^{i\frac{3\pi}{4}}$$

$$z^{202} = \left(\sqrt{2}\, e^{i\frac{3\pi}{4}}\right)^{202}$$

$$= \left(\sqrt{2}\right)^{202} \times e^{i\frac{3\pi}{4} \times 202}$$

$$= 2^{101} \times e^{i\left(150\pi + \frac{3\pi}{2}\right)}$$

$$= 2^{101} \times \underbrace{e^{i150\pi}}_{1\ (*)} \times \boxed{e^{i\frac{3\pi}{2}}}$$

$$= 2^{101}(-i)$$

$$= \boxed{-2^{101} \times i}$$

$$\frac{606}{4} = \frac{400 + 200 + 6}{4}$$

$$=$$

$$\boxed{e^{2ik\pi} = 1} \quad *$$