School of Mathematics and Statistics
Math1131-Algebra

Lec19: Inverses and definition of determinants

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Learning outcomes for this lecture

At the the end of this lecture

- ☐ you should be able to calculate by hand the **inverse** of an invertible matrix of any size (not just 2×2);
- ☐ you should be able to calculate the **determinant** of a square matrix of any size (not just 2×2). In particular, you should be able to
  - ☐ know what a minor is;
  - ☐ know how elementary each row operation affects the determinant;
  - ☐ be able to expand a determinant along any row or column (and get the signs right!)
- ☐ you should know that the determinant tells you **if a matrix is invertible or not**;

You can use this list as a check list to get ready for our next class: After studying the lecture notes, come back to this list, and for each item, check that you have indeed mastered it. Then tick the corresponding box ... or go back to the notes.

Finding the inverse of a matrix using elementary row operations

Swapping rows can be achieved by matrix multiplication.

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

For example, if

- A is a 4×4 matrix and
- E is made by swapping rows 2 and 4 of a 4×4 identity matrix

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$$

then if EA is A with rows 2 and 4 swapped, that is,

$$EA = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 13 & 14 & 15 & 16 \\ 9 & 10 & 11 & 12 \\ 5 & 6 & 7 & 8 \end{pmatrix}.$$

Finding the inverse of a matrix using elementary row operations

$R_i \leftarrow R_i + \alpha R_j$ for $i \neq j$ can be achieved by matrix multiplication.

For example, if

- A is a 4×4 matrix and
- E is made by inserting 3 in row 2 column 4 of a 4×4 identity matrix

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$$

then if EA is A with R_2 replaced by $R_2 + 3R_4$, that is,

$$EA = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 44 & 48 & 52 & 56 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}.$$

Finding the inverse of a matrix using elementary row operations

A row can be scaled by matrix multiplication.

For example, if

- A is a 4×4 matrix and
- E is made by replacing the 3rd diagonal entry in a 4×4 identity matrix by a 2

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$$

then if EA is A with R_3 replaced by $2R_3$, that is,

$$EA = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 18 & 20 & 22 & 24 \\ 13 & 14 & 15 & 16 \end{pmatrix}.$$

Finding the inverse of a matrix using elementary row operations

Suppose A is a square matrix and the sequence of row operations that reduces it to **R**educed **R**ow **E**chelon **F**orm has corresponding matrices E_1, $E_2, \ldots E_r$. Then

$$E_r \ldots E_2 E_1 A = U$$

where U is a RREF matrix.

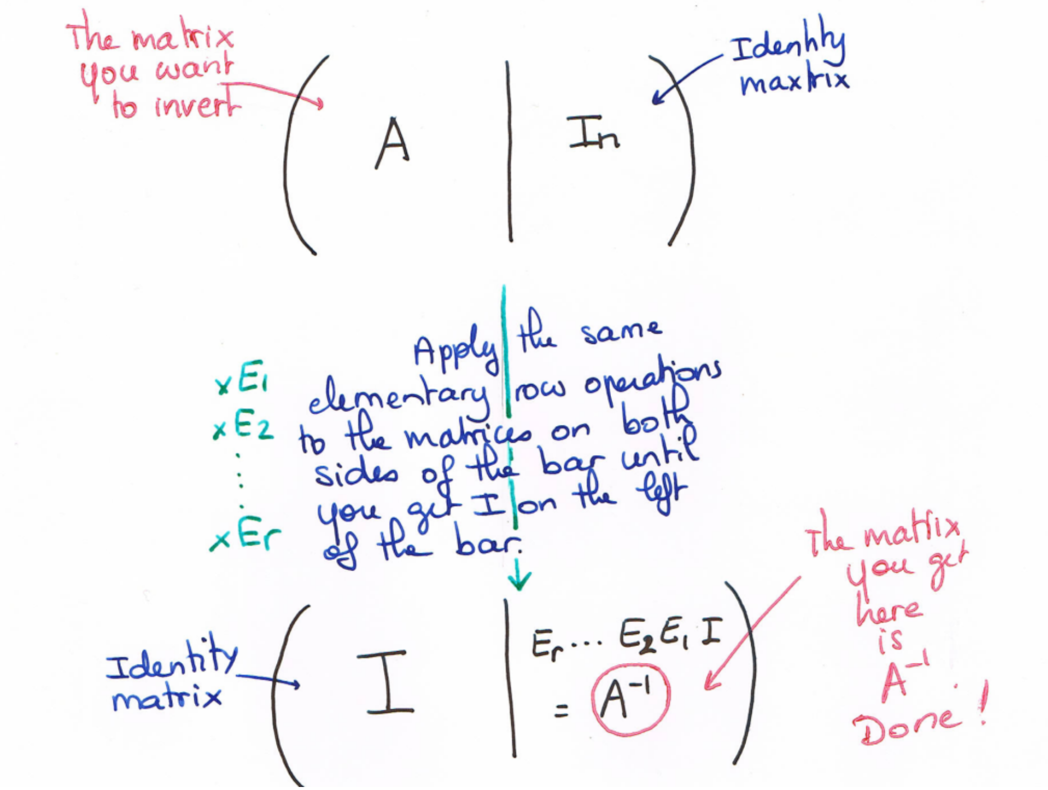
- If $U = I$ then A is invertible with
$$A^{-1} = E_r \ldots E_2 E_1 = E_r \ldots E_2 E_1 I.$$

- Conversely, if A is invertible, then the system of equations $A\overrightarrow{\boldsymbol{x}} = \overrightarrow{\boldsymbol{b}}$ has a unique solution
$$\overrightarrow{\boldsymbol{x}} = A^{-1}\overrightarrow{\boldsymbol{b}}$$

and so we know that the RREF of A is the identity matrix I.

SUMMARY.
◇ A matrix A is invertible *iff* its RREF is the identity matrix I.
◇ In that case, $A^{-1} = E_r \ldots E_2 E_1 = E_r \ldots E_2 E_1 I$
... This gives us an idea to find the inverse of a matrix! (explained on the next slide)

Finding the inverse of a matrix using elementary row operations on the augmented matrix $(A|I)$



If you **cannot** get the identity matrix on the left of the bar using Gaussian elimination, it means that your matrix A was **not** invertible in the first place.

Finding the inverse of a matrix using elementary row operations on the augmented matrix $(A|I)$

We can find the inverse of a square matrix A by row reducing A to RREF while at the same time applying those row operations to the identity matrix of the same size.

Example 1. To find the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$, we reduce $(A|I)$ to RREF $(A^{-1}|I)$.

If we can't get the left half to I then A^{-1} has no inverse.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\vdots$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{5}{6} & -\frac{1}{6} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right)$$

A^{-1}

Finding the inverse of a matrix

Exercise 2. Find the inverse of $B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}$.

Then compare this matrix B to the matrix A in the previous example and compare their inverses.

① $\begin{cases} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 3R_1 \end{cases}$

② $R_2 \leftarrow -\frac{1}{3} R_2$

③ $R_3 \leftarrow R_3 + 4R_1$

$$\left(\begin{array}{ccc|ccc} \boxed{1} & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l|l} \times -2 & \times -3 \\ \times 1 & \\ & \times 1 \end{array}$$

$$\xrightarrow{①} \left(\begin{array}{ccc|ccc} \boxed{1} & 2 & 1 & 1 & 0 & 0 \\ 0 & -3 & -3 & -2 & 1 & 0 \\ 0 & -4 & -2 & -3 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{②} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & -4 & -2 & -3 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \times 4 \\ \times 1 \end{array}$$

$$\xrightarrow{③} \left(\begin{array}{ccc|ccc} \boxed{1} & 2 & 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 2/3 & -1/3 & 0 \\ 0 & 0 & \boxed{2} & -\frac{1}{3} & -\frac{4}{3} & 1 \end{array}\right)$$

$\frac{8}{3} - 3$

$\frac{4}{3} - \frac{9}{3}$

Finding the inverse of a matrix

Exercise 2, continued.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 2 & -\frac{1}{3} & -\frac{4}{3} & 1 \end{array}\right)$$

④ $R_3 \leftarrow \frac{1}{2} R_3$

⑤ $\begin{cases} R_2 \leftarrow R_2 - R_3 \\ R_1 \leftarrow R_1 - R_3 \end{cases}$

⑥ $R_1 \leftarrow R_1 - 2R_2$

$$\xrightarrow{④} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & \boxed{1} & -\frac{1}{6} & -\frac{2}{3} & \frac{1}{2} \end{array}\right) \begin{array}{l} \times 1 \\ \times -1 \\ \\ \end{array} \begin{array}{l} \times 1 \\ \\ \times -1 \end{array}$$

$$\xrightarrow{⑤} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{7}{6} & \frac{2}{3} & -\frac{1}{2} \\ 0 & \boxed{1} & 0 & \frac{5}{6} & \frac{1}{3} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{6} & -\frac{2}{3} & \frac{1}{2} \end{array}\right) \begin{array}{l} \times 1 \\ \times -2 \\ \\ \end{array}$$

$$\xrightarrow{⑥} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 5/6 & \frac{1}{3} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{6} & -\frac{2}{3} & \frac{1}{2} \end{array}\right)$$

I_3 $\quad$ A^{-1}

$$\frac{2}{3} + \frac{1}{6} = \frac{4}{6} + \frac{1}{6}$$

Same process, step by step, using Maple

```
> with(LinearAlgebra):
> # Enter the matrix column by column
  # Create an augmented matrix with the Identity matrix to the
  right of the bar

  B := < <1,2,3>|<2,1,2>|<1,-1,1> >;
  BI := < B| IdentityMatrix(3)>;
```

$$B := \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{bmatrix}$$

$$BI := \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

```
> # We now apply row operation to get Identity matrix on the
  leftt of the bar
  # R2 <- R2 - 2*R1    and    R3 <- R3 - 3R1
  RowOperation(%, [2, 1], -2):
  RowOperation(%, [3, 1], -3);
```

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -3 & -3 & -2 & 1 & 0 \\ 0 & -4 & -2 & -3 & 0 & 1 \end{bmatrix}$$

```
> # Divide R2 by -3 (i.e. multiply by -1/3)
  RowOperation(%, 2, -1/3);
```

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & -4 & -2 & -3 & 0 & 1 \end{bmatrix}$$

```
> # R3 <- R3 +4R2
  RowOperation(%, [3, 2], 4);
```

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 2 & -\frac{1}{3} & -\frac{4}{3} & 1 \end{bmatrix}$$

```
> # Divide R3 by 2 (i.e. multiply by 1/2)
  RowOperation(%, 3, 1/2);
```

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{6} & -\frac{2}{3} & \frac{1}{2} \end{bmatrix}$$

```
> # The matrix is in REF. Now we want the RREF
  # Column 3: R2 <- R2 - R3  and   R1 <- R1 - R3
  RowOperation(%, [2, 3], -1);
  RowOperation(%, [1, 3], -1);
```

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{5}{6} & \frac{1}{3} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{6} & -\frac{2}{3} & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & \frac{7}{6} & \frac{2}{3} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{5}{6} & \frac{1}{3} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{6} & -\frac{2}{3} & \frac{1}{2} \end{bmatrix}$$

```
> # Column 2:  R1 <- R1 - 2R2
  RowOperation(%, [1, 2], -2);
```

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{5}{6} & \frac{1}{3} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{6} & -\frac{2}{3} & \frac{1}{2} \end{bmatrix}$$

Finding the inverse of a matrix

Exercise 2, continued.

Compare this matrix B to the matrix A in the previous example and compare their inverses.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{5}{6} & -\frac{1}{6} \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{6} & \frac{1}{3} & -\frac{1}{2} \\ -\frac{1}{6} & -\frac{2}{3} & \frac{1}{2} \end{pmatrix}$$

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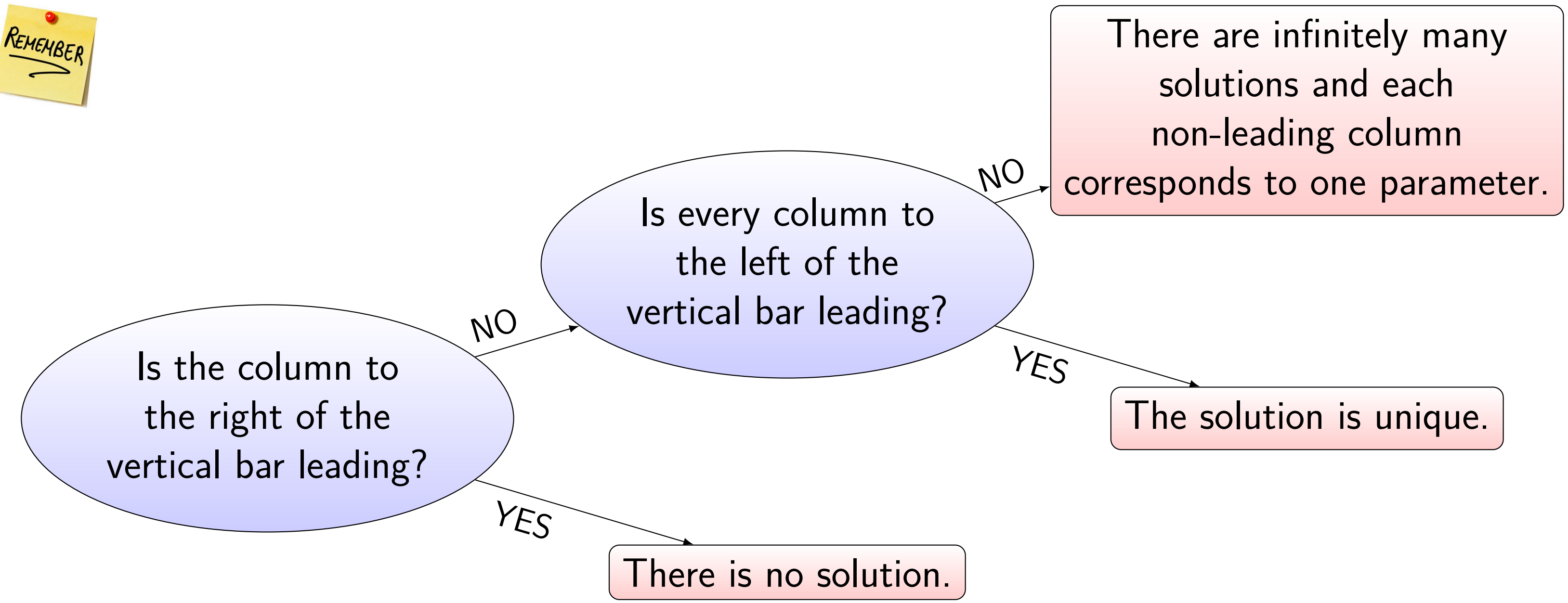
$B = A^T$ $\qquad\qquad$ $B^{-1} = (A^{-1})^T$

$\therefore (A^T)^{-1} = (A^{-1})^T$

Using matrices to solve $A\overrightarrow{\boldsymbol{x}} = \overrightarrow{\boldsymbol{b}}$

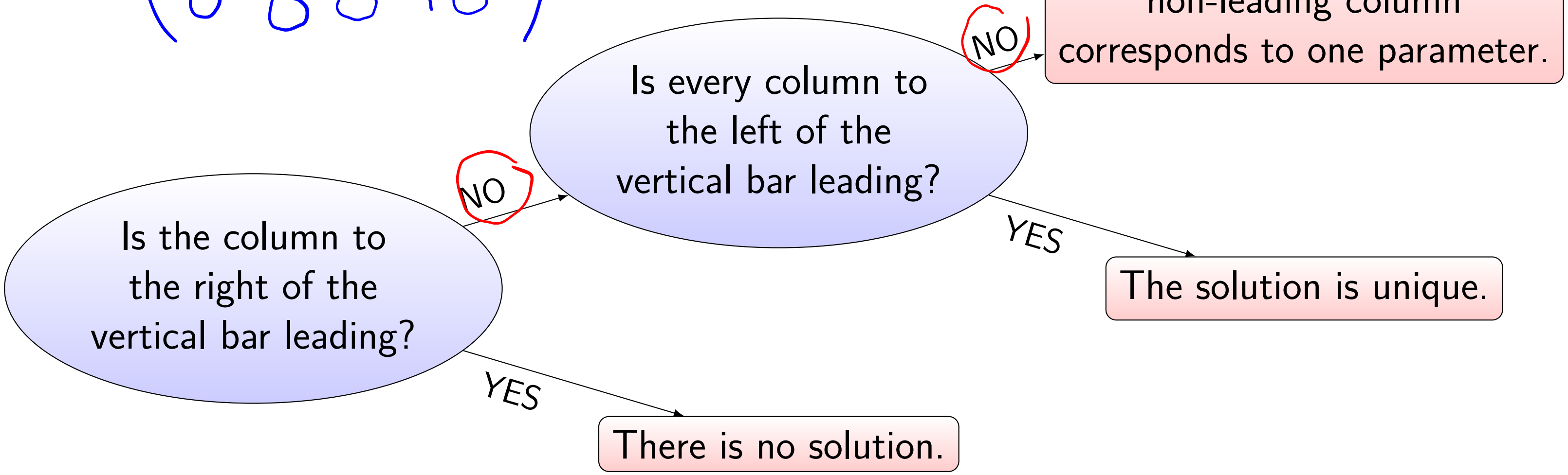


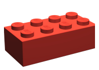
Using matrices to solve systems of equations.

- If A is invertible, then the system of equations $A\overrightarrow{\boldsymbol{x}} = \overrightarrow{\boldsymbol{b}}$ has a unique solution
$$\overrightarrow{\boldsymbol{x}} = A^{-1}\overrightarrow{\boldsymbol{b}}$$
- If A is not invertible then $A\overrightarrow{\boldsymbol{x}} = \overrightarrow{\boldsymbol{b}}$ has either no solutions or infinitely many solutions.

Using matrices to solve $A\overrightarrow{x} = \overrightarrow{b}$

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$





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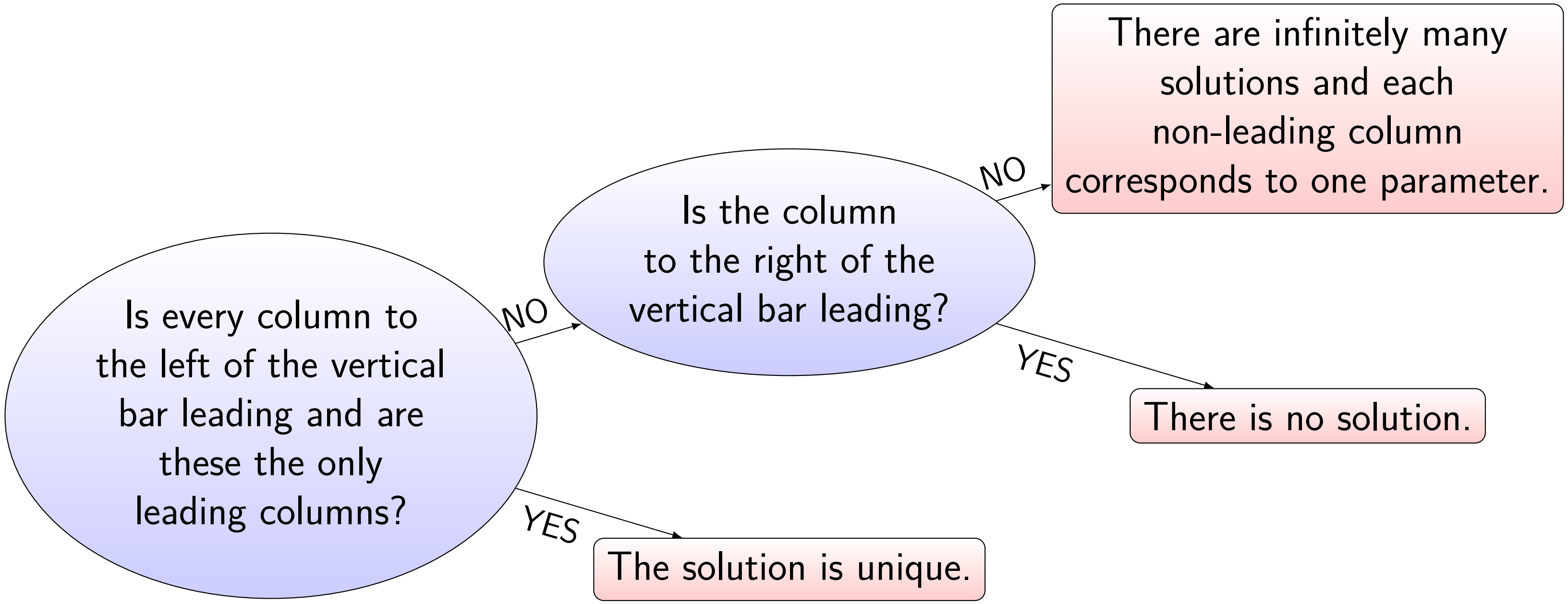

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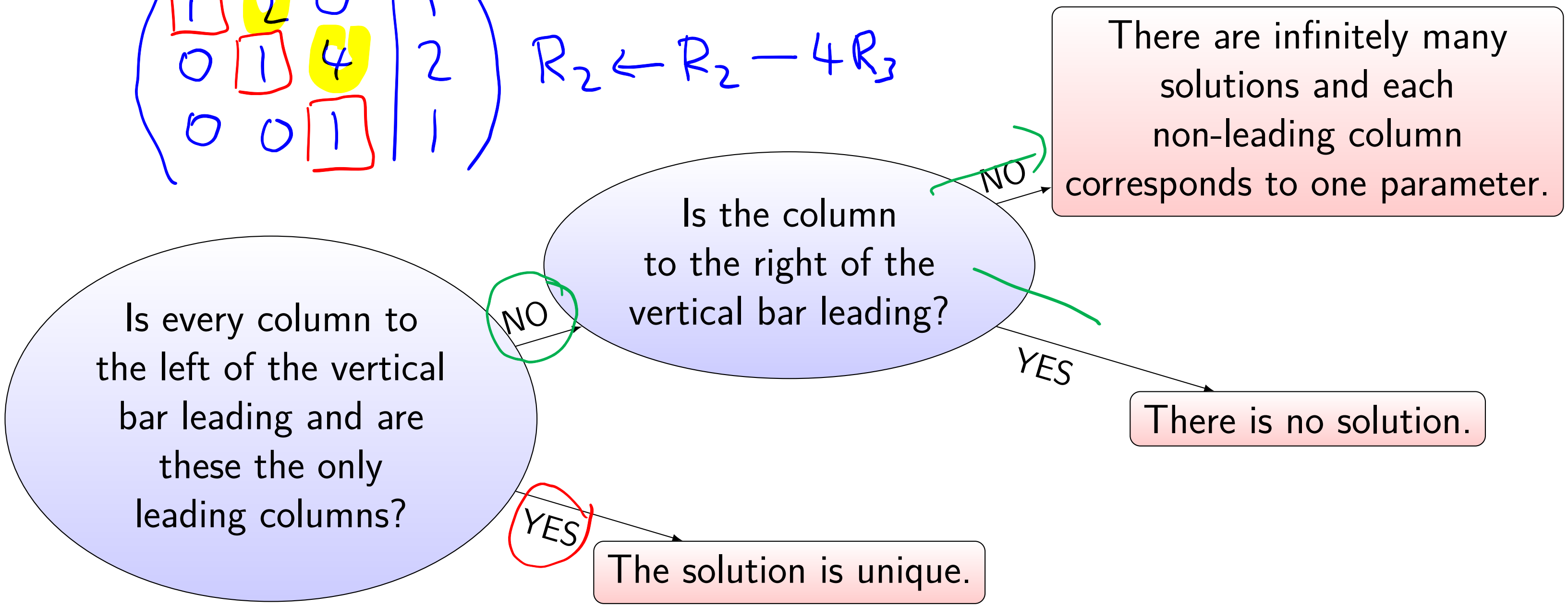
Using matrices to solve $A\vec{x} = \vec{b}$

A

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & 0 & 1 \\ 0 & \boxed{1} & 4 & 2 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right) \quad R_2 \leftarrow R_2 - 4R_3$$



Using matrices to solve systems of equations.

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$$\vec{x} = A^{-1}\vec{b}$$

- If A is not invertible then $A\vec{x} = \vec{b}$ has either no solutions or infinitely many solutions.

Matrix inverses

Exercise 3. Having found this inverse in the previous slide, we can use it to solve systems of linear equations. For example, solve

$$\begin{aligned} x + 2y + 3z &= 14 \\ 2x + y + 2z &= 10 \\ x - y + z &= 2. \end{aligned}$$

Answer : $x = 1, y = 2, z = 3$

Matrix inverses

Exercise 4. Try to find the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

Determinants

We have seen how to find the determinant of a 2×2 matrix using the "gamma rule ".
For a 1×1 matrix, the determinant is just the single entry itself.
We now look at larger matrices.

Definition (Minor)

For a square matrix A, the ij minor $|A_{ij}|$ is the determinant of the matrix obtained from A by deleting row i and column j.

Exercise 5. For

$$A = \begin{pmatrix} 1 & 4 & 6 \\ 1 & -8 & 7 \\ 5 & 2 & 0 \end{pmatrix}$$

we have

$$|A_{23}| = \begin{vmatrix} 1 & 4 \\ 5 & 2 \end{vmatrix}.$$

also

$|A_{31}| =$ and $|A_{13}| =$

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also

$$|A_{31}| = \begin{vmatrix} 4 & 6 \\ -8 & 7 \end{vmatrix} \quad \text{and} \quad |A_{13}| = \begin{vmatrix} 1 & -8 \\ 5 & 2 \end{vmatrix}$$

$$= 4 \times 7 - (-8) \times 6$$
$$= 28 + 48 = 76$$

$$= 1 \times 2 - 5 \times (-8)$$
$$= 2 + 40 = 42$$

Determinants

Recursive definition of the determinant of a square matrix.

The determinant of an $n \times n$ matrix A with entries a_{ij} is given by

$$|A| = \det(A) = a_{11}|A_{11}| - a_{21}|A_{21}| + a_{31}|A_{31}| - \cdots (-1)^{n+1}a_{n1}|A_{n1}|$$

Note that each term has a determinant of an $(n-1) \times (n-1)$ matrix so we need to apply this definition repeatedly.

This definition"expands" along the first column, and works like the way we calculated the cross-product.

Exercise 6. Find $\det \begin{pmatrix} 6 & 8 & 9 \\ 5 & 2 & 1 \\ 7 & 4 & 3 \end{pmatrix}$

Determinants

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Exercise 6. Find $\det \begin{pmatrix} 6 & 8 & 9 \\ 5 & 2 & 1 \\ 7 & 4 & 3 \end{pmatrix}$

$$= a_{11}|A_{11}| - a_{21}|A_{21}| + a_{31}|A_{31}|$$

$$= 6\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} - 5\begin{vmatrix} 8 & 9 \\ 4 & 3 \end{vmatrix} + 7\begin{vmatrix} 8 & 9 \\ 2 & 1 \end{vmatrix}$$

$$= 6(6-4) - 5(24-36) + 7(8-18)$$

$$= 6 \times 2 - 5 \times (-12) + 7 \times (-10)$$

$$= 12 + 60 - 70$$

$$= 2$$

Determinants

The definition on the previous slide "expands" along the the first column.

◇ A remarkable fact is that we can also expand along any other column or any row.

◇ The determinant is the sum of terms of the form $(-1)^{i+j}a_{ij}|A_{ij}|$ along any row or column.
It's easy to remember the signs as

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Exercise 7. Find the determinant of the matrix on the last slide by expanding along another row or a column.

$$\det \begin{pmatrix} 6 & 8 & 9 \\ 5 & 2 & 1 \\ 7 & 4 & 3 \end{pmatrix} =$$

Determinants

Exercise 8. Find the determinants of

$$A = \begin{pmatrix} 4 & 5 & 8 & -1 \\ 2 & 0 & -2 & -3 \\ 0 & 0 & 2 & 0 \\ 8 & 0 & -4 & -5 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 4 & 1 & 7 & -1 \\ 0 & 2 & -2 & -3 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Determinant of a triangular matrix

Determinant of a triangular matrix

The determinant of an upper triangular matrix (Row Echelon Form matrix) is the product of the diagonal elements. That is,

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & 0 & 0 & a_{44} & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

$$= a_{11} \times \det \begin{pmatrix} a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ 0 & a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & 0 & a_{44} & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} + 0 + \cdots + 0$$

$$\cdots = a_{11}a_{22}a_{33}a_{44}\cdots a_{nn}.$$

TIP!

The determinant of an *lower triangular matrix* (i.e. with zeros *above* the diagonal rather than below) is also the product of the diagonal elements.

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$$= a_{11} \times \det \begin{pmatrix} a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ 0 & a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & 0 & a_{44} & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} \underline{+ 0 + \cdots + 0}$$

$$\cdots = a_{11}a_{22}a_{33}a_{44}\cdots a_{nn}.$$

TIP!

The determinant of an *lower triangular matrix* (i.e. with zeros *above* the diagonal rather than below) is also the product of the diagonal elements.

Properties of determinants

Properties of determinants

(a) $\det(A^T) = \det(A)$

(b) $\det(AB) = \det(A)\det(B)$

(c) $R_i \leftarrow R_i + \alpha R_j$ for $i \neq j$ does not change the determinant.

(d) $R_i \leftrightarrow R_j$ for $i \neq j$ changes the sign of the determinant.

(e) $R_i \leftarrow \alpha R_i$ scales the determinant by α.

(f) If A has a zero row or column then $\det(A) = 0$.

Some important consequences:

(g) $\det(\alpha A) = \alpha^n \det(A)$ for an $n \times n$ matrix. ⚠ Note that it is α^n not α.

(h) $\det(A^{-1}) = 1/\det(A)$.

(i) Swapping two columns changes the sign of the determinant.

(j) Row operations can simplify the calculation of determinants.

(k) A is invertible *if and only* $\det(A) \neq 0$.

(l) If one row of A is a multiple of another row then $\det(A) = 0$.

(m) If one column of A is a multiple of another column then $\det(A) = 0$.

Determinants

Exercise 9. For which values of the numbers a, b and c is the matrix A invertible?

$$A = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}$$

Determinant with Maple

```
> with(LinearAlgebra):
> # Enter the matrices columnwise

  A := <<1, 1, 1> | <a, b, c> | <a^2, b^2, c^2>>;
```

$$A := \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$$

```
> # Calculate the determinant of A

  Det_of_A := Determinant(A);
```

$$Det\_of\_A := -a^2 b + a^2 c + a b^2 - a c^2 - b^2 c + b c^2$$

```
> # Factorise the determinant of A

  factor(Det_of_A);
```

$$-(b - c)(a - c)(a - b)$$