MATH1131 Mathematics 1A – Algebra

Lecture 18: Matrices

Lecturer: Sean Gardiner – sean.gardiner@unsw.edu.au

Based on slides by Jonathan Kress

Matrices

Matrices are rectangular arrays of numbers surrounded by a pair of brackets. Here are some examples:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 7 & 8 \\ -3 & 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{3} & \frac{3}{11} \\ -\frac{1}{4} & 4 \\ \frac{2}{9} & \frac{7}{11} \end{pmatrix} \qquad \begin{pmatrix} \pi & -1 \\ \sqrt{2} & e \end{pmatrix}$$

The sizes (or dimensions) of these matrices are:

$$3 \times 3 \qquad 3 \times 2 \qquad 2 \times 2.$$

We always refer to the row number before the column number when giving the size or describing the position of an entry.

It is often useful to think of matrices as column vectors placed side by side. An $n \times m$ matrix can be thought of as m vectors from $\mathbb{R}^n$ placed in an array.

The numbers in a matrix are called entries.

For a given matrix A, the entry in row i and column j is denoted by $[A]_{i,j}$. If unambiguous (i.e. when the number of rows and columns are single-digit numbers), the comma can be omitted.
For example, if

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 7 & 8 & -2 \\ -3 & 5 & 0 & 10 \end{pmatrix},$$

then

$$[A]_{23} = 8 \qquad \text{and} \qquad [A]_{32} = 5.$$

If the entries are unknown or of general nature, we may write the entry $[A]_{ij}$ as a_{ij}. (Note that we generally use capital letters for matrices, and lower-case letters for entries of matrices.)

$M_{mn}(\mathbb{R})$ denotes the set of all $m \times n$ matrices with real entries.
So for the A given above, $A \in M_{34}(\mathbb{R})$. (In fact, $A \in M_{34}(\mathbb{Z})$.)

Adding or scaling matrices

Just like we did for vectors, we can add matrices together and multiply matrices by scalars by applying the operation to corresponding entries of each matrix.

So if A and B are the same size, then every entry of the matrix $A+B$ is the sum of the corresponding entries of A and B:

$$[A+B]_{ij} = [A]_{ij} + [B]_{ij} \qquad \text{for all } i, j.$$

Similarly every entry of the matrix λA is the scalar multiple of the corresponding entry of A:

$$[\lambda A]_{ij} = \lambda [A]_{ij} \qquad \text{for all } i, j.$$

Also, we say two matrices are equal if and only if all their entries are equal. So

$$A = B \iff [A]_{ij} = [B]_{ij} \qquad \text{for all } i, j.$$

Note that if the matrices A and B have different sizes, $A \neq B$ and $A+B$ is not defined.

Adding or scaling matrices

Example

Given that

$$A = \begin{pmatrix} 2 & 3 \\ 1 & -3 \\ 0 & 4 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 3 \\ 3 & 0 \\ 2 & 5 \end{pmatrix}.$$

Find, if they exist, $A + C$, $A + B$, $4B$, and $2A - C$.

$$A + C = \begin{pmatrix} 2 & 3 \\ 1 & -3 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 3 & 0 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 4 & -3 \\ 2 & 9 \end{pmatrix}.$$

However $A + B$ does not exist, because A and B have different sizes (A is size 3×2 whereas B is size 2×3).

Adding or scaling matrices

Example

Given that

$$A = \begin{pmatrix} 2 & 3 \\ 1 & -3 \\ 0 & 4 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 3 \\ 3 & 0 \\ 2 & 5 \end{pmatrix}.$$

Find, if they exist, $A + C$, $A + B$, $4B$, and $2A - C$.

$$4B = 4 \begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 8 & 12 \\ -4 & 20 & 0 \end{pmatrix}.$$

$$2A - C = 2 \begin{pmatrix} 2 & 3 \\ 1 & -3 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 3 & 0 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 2 & -6 \\ 0 & 8 \end{pmatrix} + \begin{pmatrix} -1 & -3 \\ -3 & 0 \\ -2 & -5 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ -1 & -6 \\ -2 & 3 \end{pmatrix}$$

Zero matrices

The $m \times n$ zero matrix is the $m \times n$ matrix will all zero entries. It is written as simply 0, or sometimes $0_{m,n}$ to distinguish its size.

For example,

$$0_{2,3} = 0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is the 2×3 zero matrix, and

$$0_{3,3} = 0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is the 3×3 zero matrix.

For any matrix A,

$$A + 0 = 0 + A = A$$

where 0 is the zero matrix of the same size as A.

Properties of matrix addition and scalar multiplication

For all matrices $A, B, C \in M_{mn}$ and scalars λ, μ:

Associative Law of Addition $$(A + B) + C = A + (B + C)$$

Commutative Law of Addition $$A + B = B + A$$

Existence of Zero

Some $0_{mn} \in M_{mn}$ satisfies $A + 0_{mn} = A$ for all $A \in M_{mn}$

Existence of Negative

Some element $-A \in M_{mn}$ satisfies $A + (-A) = 0_{mn}$

Associative Law of Scalar Multiplication $$\lambda(\mu A) = (\lambda\mu)A$$

Multiplication by Scalar Identity $$1A = A$$

Scalar Distributive Law $$(\lambda + \mu)A = \lambda A + \mu A$$

Matrix Distributive Law $$\lambda(A + B) = \lambda A + \lambda B$$

0_{mn} is the $m \times n$ zero matrix

$-A$ is the $m \times n$ matrix with entries $[-A]_{ij} = -[A]_{ij}$ for all i, j

Properties of matrix addition and scalar multiplication

e.g. Proof of the first property

For all matrices $A, B, C \in M_{mn}$ and scalars λ, μ:

Associative Law of Addition $\qquad (A + B) + C = A + (B + C)$

Proof

Let $A, B, C \in M_{mn}$. Then for all $1 \leq i \leq$ and $1 \leq j \leq n$

$$\begin{aligned}
[(A+B)+C]_{ij} &= [A+B]_{ij} + [C]_{ij} && \text{(definition of matrix addition)}\\
&= ([A]_{ij} + [B]_{ij}) + [C]_{ij} && \text{(definition of matrix addition)}\\
&= [A]_{ij} + ([B]_{ij} + [C]_{ij}) && \text{(associative law of numbers)}\\
&= [A]_{ij} + ([B+C]_{ij}) && \text{(definition of matrix addition)}\\
&= [A+(B+C)]_{ij} && \text{(definition of matrix addition)}
\end{aligned}$$

This means the matrices $(A + B) + C$ and $A + (B + C)$ have the same entries. Hence they are equal. $\square$

Linear equations in matrix form

The system of linear equations

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 1 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & -1 \\ 7x_1 & - & 5x_2 & - & 9x_3 & = & 0 \end{array}$$

can be written in matrix form as

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -5 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Let's look at the left hand side and see how the "multiplication" works.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -5 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1x_1 + 2x_2 + 3x_3 \\ 4x_1 + 5x_2 + 6x_3 \\ 7x_1 - 5x_2 - 9x_3 \end{pmatrix}$$

This is the motivator for matrix multiplication.

Matrix multiplication

If

$$A = \begin{pmatrix} 7 & 2 \\ 1 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 5 \\ 6 & 8 \end{pmatrix}$$

then

$$AB = \begin{pmatrix} 7 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 7 \times 3 + 2 \times 6 & 7 \times 5 + 2 \times 8 \\ 1 \times 3 + 4 \times 6 & 1 \times 5 + 4 \times 8 \end{pmatrix} = \begin{pmatrix} 33 & 51 \\ 27 & 37 \end{pmatrix}.$$

The (i, j)th entry of AB comes from combining the ith row of A and the jth column of B. The "combination" is very similar to the dot product of two vectors.

We can also find

$$BA = \begin{pmatrix} 3 & 5 \\ 6 & 8 \end{pmatrix} \begin{pmatrix} 7 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 26 & 26 \\ 50 & 44 \end{pmatrix}.$$

Note: In general, $AB \neq BA$ for different matrices A and B. Matrix multiplication is not commutative!

Matrix multiplication

Example

Suppose

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ -1 & 0 & -2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}.$$

Find, if possible, CD and DC.

$$CD = \begin{pmatrix} 1 \times (-1) + 2 \times (-2) + 3 \times (-3) \\ 1 \times (-1) + 0 \times (-2) + 1 \times (-3) \\ (-1) \times (-1) + 0 \times (-2) + (-2) \times (-3) \end{pmatrix} = \begin{pmatrix} -14 \\ -4 \\ 7 \end{pmatrix}.$$

However DC does not exist, because the number of entries in a row of D does not match the number of entries in a column of C.

Note: The product AB of two matrices A and B is defined only if the number of columns of A is equal to the number of rows of B.
If A is size $m \times n$ and B is size $n \times q$, then AB is size $m \times q$.

Matrix multiplication

Exercise

Given

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 7 \\ 4 & 8 \end{pmatrix},$$

find both AB and BA.

$$AB = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 7 \\ 4 & 8 \end{pmatrix} = \begin{pmatrix} 17 & 38 \\ 32 & 67 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 0 \\ 2 & 7 \\ 4 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 44 & 39 & 44 \\ 52 & 48 & 44 \end{pmatrix}$$

Identity matrices

The diagonal entries of a matrix A are the entries $[A]_{11}, [A]_{22}, \dots, [A]_{ii}, \dots$.

The $n \times n$ identity matrix is the $n \times n$ matrix with ones in the diagonal entries and zeros everywhere else. It is denoted by the capital letter I, and sometimes I_n to distinguish its size. For example,

$$I_2 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is the 2×2 identity matrix, and

$$I_3 = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the 3×3 identity matrix.

For any matrix A,

$$AI = A \qquad \text{and} \qquad IA = A.$$

Identity matrices

Example

For

$$D = \begin{pmatrix} 3 & 5 & 8 \\ 2 & 4 & 8 \end{pmatrix}$$

show that $ID = DI = D$.

$$ID = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 5 & 8 \\ 2 & 4 & 8 \end{pmatrix} = \begin{pmatrix} 3 & 5 & 8 \\ 2 & 4 & 8 \end{pmatrix}$$

$$DI = \begin{pmatrix} 3 & 5 & 8 \\ 2 & 4 & 8 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 5 & 8 \\ 2 & 4 & 8 \end{pmatrix}$$

Properties of matrix multiplication

In general, if A is an $m \times n$ matrix and B is a $n \times q$ matrix, then AB is the $m \times q$ matrix given by

$$[AB]_{ij} = \sum_{k=1}^{n} [A]_{ik}[B]_{kj}$$

For example, when $i = 2$, $j = 1$, and $n = 3$:

$$[AB]_{21} = [A]_{21}[B]_{11} + [A]_{22}[B]_{21} + [A]_{23}[B]_{31}.$$

If we write the ith row of A as a column vector $\mathbf{a}_i$ and the jth column of B as a column vector $\mathbf{b}_j$, then $[AB]_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j$.

Properties of matrix multiplication

Properties

Suppose that A, B and C are matrices for which the relevant sums and products exist. Then,

- $A(BC) = (AB)C$ — Associative Law of Matrix Multiplication
- $(A + B)C = AC + BC$ — Right Distributive Law
- $A(B + C) = AB + AC$ — Left Distributive Law
- $A(\lambda B) = \lambda AB$ for any scalar λ — Scalar Distributivity
- $AI = A$ and $IA = A$, where I represents identity matrices of appropriate (possibly different) sizes — Matrix Identity

But remember: In general, $AB \neq BA$. That is, matrix multiplication is not commutative.

For example, $(A + B)^2 = A^2 + AB + BA + B^2 \neq A^2 + 2AB + B^2$.

Matrix multiplication – Example

Example

Let

$$B = \begin{pmatrix} 3 & 1 & -5 & 4 \\ 8 & 7 & 2 & 2 \\ 9 & -1 & -2 & -7 \end{pmatrix}.$$

a) Find a (column) vector $\mathbf{v}$ such that $B\mathbf{v}$ is the third column of B.

b) Find a row vector $\mathbf{w}$ such that $\mathbf{w}B$ is the second row of B.

c) Find a vector $\mathbf{u}$ such that $B\mathbf{u}$ is 2 times the first column of B plus 5 times the third column of B.

a) When $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $B\mathbf{v} = \begin{pmatrix} 3 & 1 & -5 & 4 \\ 8 & 7 & 2 & 2 \\ 9 & -1 & -2 & -7 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ -2 \end{pmatrix}$.

Matrix multiplication – Example

Example

Let

$$B = \begin{pmatrix} 3 & 1 & -5 & 4 \\ 8 & 7 & 2 & 2 \\ 9 & -1 & -2 & -7 \end{pmatrix}.$$

a) Find a (column) vector $\mathbf{v}$ such that $B\mathbf{v}$ is the third column of B.

b) Find a row vector $\mathbf{w}$ such that $\mathbf{w}B$ is the second row of B.

c) Find a vector $\mathbf{u}$ such that $B\mathbf{u}$ is 2 times the first column of B plus 5 times the third column of B.

b) When $\mathbf{w} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$,

$$\mathbf{w}B = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 & -5 & 4 \\ 8 & 7 & 2 & 2 \\ 9 & -1 & -2 & -7 \end{pmatrix} = \begin{pmatrix} 8 & 7 & 2 & 2 \end{pmatrix}.$$

Matrix multiplication – Example

Example

Let

$$B = \begin{pmatrix} 3 & 1 & -5 & 4 \\ 8 & 7 & 2 & 2 \\ 9 & -1 & -2 & -7 \end{pmatrix}.$$

a) Find a (column) vector $\mathbf{v}$ such that $B\mathbf{v}$ is the third column of B.

b) Find a row vector $\mathbf{w}$ such that $\mathbf{w}B$ is the second row of B.

c) Find a vector $\mathbf{u}$ such that $B\mathbf{u}$ is 2 times the first column of B plus 5 times the third column of B.

c) Following the same approach, we want $\mathbf{u} = 2\mathbf{e}_1 + 5\mathbf{e}_3 = \begin{pmatrix} 2 \\ 0 \\ 5 \\ 0 \end{pmatrix}$.