MATH1131 Mathematics 1A – Algebra

Lecture 6: Orthogonality and Projections

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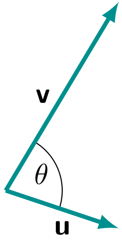
Based on slides by Jonathan Kress

Perpendicular vectors

Recall that for two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.



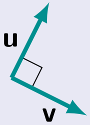
For non-zero vectors $\mathbf{u}$ and $\mathbf{v}$,

$$\theta = \frac{\pi}{2} \quad \text{if and only if} \quad \mathbf{u} \cdot \mathbf{v} = 0$$

and we then say that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.

Example

The vectors $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ are perpendicular since $\mathbf{u} \neq \mathbf{0}$, $\mathbf{v} \neq \mathbf{0}$, and $\mathbf{u} \cdot \mathbf{v} = 1 \times 2 + 2 \times (-1) = 0$.



Orthogonal vectors

Definition

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

Example

The vectors $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ are orthogonal since $\mathbf{u} \cdot \mathbf{v} = 1 \times 2 + 2 \times (-1) = 0$.

Note: The zero vector is orthogonal to every vector including itself. This is how the definition differs from perpendicularity.

Orthogonal and orthonormal sets

Definition

A set of vectors in $\mathbb{R}^n$ is said to be an orthogonal set of vectors if all vectors are mutually orthogonal.

So a set $S \subseteq \mathbb{R}^n$ is orthogonal if

$$\mathbf{u} \cdot \mathbf{v} = 0 \qquad \text{for all } \mathbf{u}, \mathbf{v} \in S.$$

Definition

A set of vectors in $\mathbb{R}^n$ is said to be an orthonormal set if it is an orthogonal set, and all vectors in the set are unit vectors.

So a set $S \subseteq \mathbb{R}^n$ is orthonormal if for all $\mathbf{u}, \mathbf{v} \in S$,

$$\mathbf{u} \cdot \mathbf{v} = \begin{cases} 0 & \text{if } \mathbf{u} \neq \mathbf{v} \\ 1 & \text{if } \mathbf{u} = \mathbf{v} \end{cases}$$

(since if $\mathbf{u} = \mathbf{v}$, then $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 = 1$).

Are each of the following orthogonal or orthonormal sets?

- $\left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$
- $\left\{ \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$
- $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$
- $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$
- $\left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{42}} \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}, \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \right\}$

Orthogonal and orthonormal sets

Examples

$$\left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

Here $\begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2 \times 1 + (-1) \times 2 = 0$, so the set of vectors is orthogonal.

But $\left| \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right| = \sqrt{2^2 + (-1)^2} = \sqrt{5} \neq 1$, so not all vectors in the set are unit vectors and therefore the set cannot be orthonormal.

$$\left\{ \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

Here $\frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{5} \times 0 = 0$, so the set of vectors is again orthogonal.

(Note we could have just ignored the scalar multipliers.)

Furthermore $\left| \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right| = \frac{1}{\sqrt{5}} \times \sqrt{5} = 1$, and similarly $\left| \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right| = \frac{1}{\sqrt{5}} \times \sqrt{5} = 1$, so all vectors in the set are unit vectors and therefore the set is orthonormal.

$\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ or $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$

We already know that the standard basis vectors form an orthonormal set by definition.

For example, in $\mathbb{R}^3$, $\mathbf{i} \cdot \mathbf{j} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$, so $\mathbf{i}$ and $\mathbf{j}$ are orthogonal.

Similarly all pairs of vectors will be orthogonal.

Furthermore $|\mathbf{i}| = \left| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right| = \sqrt{1^2 + 0^2 + 0^2} = 1$, so we can see that in general the set of vectors is orthonormal.

$$\left\{ \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\-1\end{pmatrix}, \frac{1}{\sqrt{42}}\begin{pmatrix}4\\1\\5\end{pmatrix}, \frac{1}{\sqrt{14}}\begin{pmatrix}2\\-3\\-1\end{pmatrix} \right\}$$

Recall that we can ignore scalar multipliers when checking orthogonality, so we can just check that:

$$\begin{pmatrix}1\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}4\\1\\5\end{pmatrix} = 1\times 4 + 1\times 1 + (-1)\times 5 = 0,$$

$$\begin{pmatrix}1\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\-1\end{pmatrix} = 1\times 2 + 1\times(-3) + (-1)\times(-1) = 0, \text{ and}$$

$$\begin{pmatrix}4\\1\\5\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\-1\end{pmatrix} = 4\times 2 + 1\times(-3) + 5\times(-1) = 0.$$

So the set of vectors is orthogonal.

Orthogonal and orthonormal sets

Examples

$$\left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{42}} \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}, \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \right\}$$

Now we consider the scalar multipliers when checking lengths:

$$\left| \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right| = \frac{1}{\sqrt{3}} \times \sqrt{1^2 + 1^2 + (-1)^2} = 1,$$

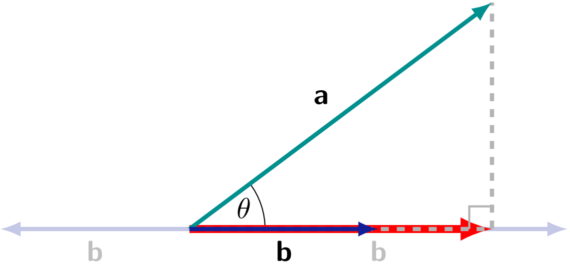
$$\left| \frac{1}{\sqrt{42}} \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} \right| = \frac{1}{\sqrt{42}} \times \sqrt{4^2 + 1^2 + 5^2} = 1, \quad \text{and}$$

$$\left| \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \right| = \frac{1}{\sqrt{14}} \times \sqrt{2^2 + (-3)^2 + (-1)^2} = 1.$$

So the set of vectors is orthonormal.

Projections

For vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ with $\mathbf{b} \neq \mathbf{0}$, the projection of $\mathbf{a}$ on $\mathbf{b}$ is denoted $\text{proj}_{\mathbf{b}}\mathbf{a}$.



$$\begin{aligned}
\text{proj}_{\mathbf{b}}\mathbf{a} &= \text{length of red arrow} \times \text{unit vector in direction of } \mathbf{b} \\
&= |\mathbf{a}| \cos\theta \times \frac{1}{|\mathbf{b}|}\, \mathbf{b} \\
&= |\mathbf{a}||\mathbf{b}| \cos\theta \; \frac{1}{|\mathbf{b}|^2}\, \mathbf{b} \\
&= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\, \mathbf{b}
\end{aligned}$$

Projections

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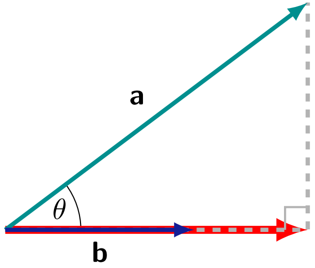


So $\quad \text{proj}_{\mathbf{b}}\mathbf{a} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\,\mathbf{b}.$

Note that the length of the projection is given by:

$$|\text{proj}_{\mathbf{b}}\mathbf{a}| = \frac{|\mathbf{a} \cdot \mathbf{b}|}{|\mathbf{b}|^2}\,|\mathbf{b}| = \frac{|\mathbf{a} \cdot \mathbf{b}|}{|\mathbf{b}|}$$

Projections – Examples

Example

Find the projection of $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ on $\mathbf{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

$$
\begin{aligned}
\text{proj}_{\mathbf{b}}\mathbf{a} &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\ \mathbf{b} \\
&= \frac{1 \times 3 + 2 \times 1}{(\sqrt{3^2 + 1^2})^2}\ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\
&= \tfrac{5}{10} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\
&= \tfrac{1}{2} \begin{pmatrix} 3 \\ 1 \end{pmatrix}
\end{aligned}
$$

Example

Find the projection of $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ on $\begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix}$.

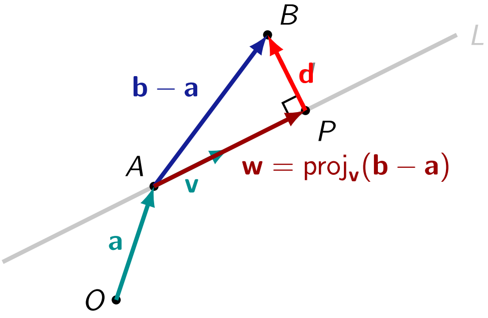
$$
\begin{aligned}
\mathrm{proj}_{\mathbf{b}}\mathbf{a} &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\ \mathbf{b} \\
&= \frac{1\times(-1)+2\times(-1)+4\times(-2)}{(-1)^2+(-1)^2+(-2)^2}\ \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix} \\
&= \frac{-11}{6}\begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix} \\
&= \frac{11}{6}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}
\end{aligned}
$$

Shortest distance to a line

How might we find the shortest distance between a point B and the line L given by

$$\mathbf{x} = \mathbf{a} + \lambda \mathbf{v}, \qquad \lambda \in \mathbb{R},$$

and/or find the point P on the line that is closest to B?



$$\mathbf{b} = \overrightarrow{OB}$$

$$\mathbf{p} = \overrightarrow{OP} = \mathbf{a} + \mathbf{w}$$

$$\begin{aligned} \mathbf{d} = \overrightarrow{PB} &= \mathbf{b} - \mathbf{a} - \mathbf{w} \\ &= \mathbf{b} - \mathbf{p} \end{aligned}$$

That is, the closest point has position vector $\mathbf{a} + \text{proj}_{\mathbf{v}}(\mathbf{b} - \mathbf{a})$, and the shortest distance is the length $|\mathbf{b} - \mathbf{a} - \text{proj}_{\mathbf{v}}(\mathbf{b} - \mathbf{a})|$.

Example

Find the shortest distance between the point $B(15, -7, 4)$ and the line

$$\mathbf{x} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}, \qquad \lambda \in \mathbb{R}.$$

Also find the point P on the line that is closest to B.

$$\begin{aligned} \mathrm{proj}_{\mathbf{v}}(\mathbf{b} - \mathbf{a}) &= \frac{(\mathbf{b} - \mathbf{a}) \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} \\ &= \frac{(15 - 1) \times 4 + (-7 - 4) \times 2 + (4 - 5) \times 6}{4^2 + 2^2 + 6^2} \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} \\ &= \frac{28}{56} \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \end{aligned}$$

Example

Find the shortest distance between the point $B(15, -7, 4)$ and the line

$$\mathbf{x} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}, \qquad \lambda \in \mathbb{R}.$$

Also find the point P on the line that is closest to B.

So the position vector of P is given by:

$$\begin{aligned} \overrightarrow{OP} &= \mathbf{a} + \operatorname{proj}_{\mathbf{v}}(\mathbf{b} - \mathbf{a}) \\ &= \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix} \end{aligned}$$

So P is the point $(3, 5, 8)$.

Example

Find the shortest distance between the point $B(15, -7, 4)$ and the line

$$\mathbf{x} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}, \qquad \lambda \in \mathbb{R}.$$

Also find the point P on the line that is closest to B.

The shortest distance is given by:

$$\begin{aligned} |\mathbf{b} - \mathbf{a} - \text{proj}_{\mathbf{v}}(\mathbf{b} - \mathbf{a})| &= |\mathbf{b} - \overrightarrow{OP}| \\ &= \left| \begin{pmatrix} 15 \\ -7 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix} \right| \\ &= \sqrt{12^2 + (-12)^2 + (-4)^2} \;=\; 4\sqrt{19} \end{aligned}$$