School of Mathematics and Statistics
Math1131-Algebra

Lec10: conjugate and division, polar form, modulus, argument

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2020 Term 1

Real and imaginary parts

Special complex numbers: the real and purely imaginary ones

- $z = \text{Re}(z) + \text{Im}(z)i$ where both $\text{Re}(z)$ and $\text{Im}(z)$ are real numbers.
- $z = \text{Re}(z) \Leftrightarrow \text{Im}(z) = 0 \Leftrightarrow z$ is *real*
- $z = \text{Im}(z)i \Leftrightarrow \text{Re}(z) = 0 \Leftrightarrow z$ is *purely imaginary*

Exercise 1.

(a) 3 is ....

(b) $3i$ is ....

(c) Both 3 and $3i$ are ....

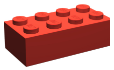
Equality of complex numbers

Equal complex numbers

Two complex numbers are *equal* *if and only if* their real parts are equal and their imaginary parts are equal.

Exercise 2. Find real numbers a and b such that $(3+4i)(a+bi) = 23+14i.$

Square roots of a complex number



Square roots of a complex number

A *square root* of a complex number w is a complex number z such that

$$z^2 = w.$$

Any non-zero complex number z has two square roots.

Exercise 3. a) Find the square roots of $-3 + 4i$

Quadratic equations

Quadratic formula with complex numbers

The solution(s) of the quadratic equation

$$az^2 + bz + c = 0$$

where $a, b, c \in \mathbb{C}$ with $a \neq 0$, is/are

$$z = \frac{-b \pm \delta}{2a}$$

where δ is a square root of $\Delta = b^2 - 4ac$, i.e. $\delta^2 = \Delta = b^2 - 4ac$.

Exercise 3, continued. b) Solve $z^2 + 3z + (3 - i) = 0$.

Exercise 3, continued.

Find the square roots of $-3 + 4i$ and hence solve $z^2 + 3z + (3 - i) = 0$.

```
> # restart clears everything so z can be reused:

  restart
# We find the square roots of -3 + 4i

solve(z^2 = -3 + 4*I);
```

$$1 + 2\,\mathrm{I},\ -1 - 2\,\mathrm{I}$$

```
> solve(z^2 + 3*z +3-I = 0);
```

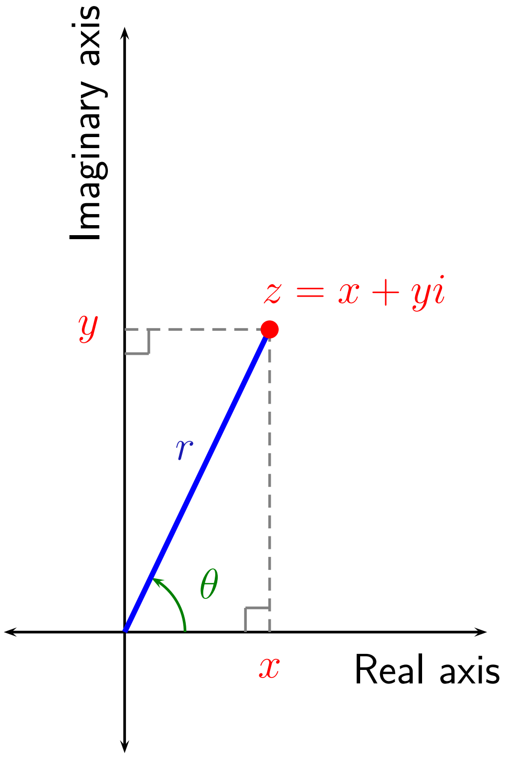
$$-1 + \mathrm{I},\ -2 - \mathrm{I}$$

Polar coordinates

• The *Cartesian form* of a complex number with real part x and imaginary part y is

$$z = x + yi.$$

• We can also describe z by its distance r from the origin and its angle θ from the positive real axis as shown.

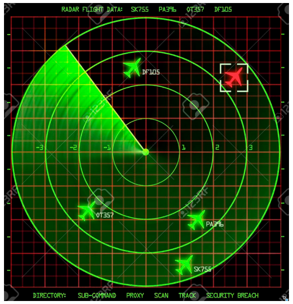


Trigonometry shows that

$$x = r\cos\theta, \quad y = r\sin\theta$$

and

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}.$$

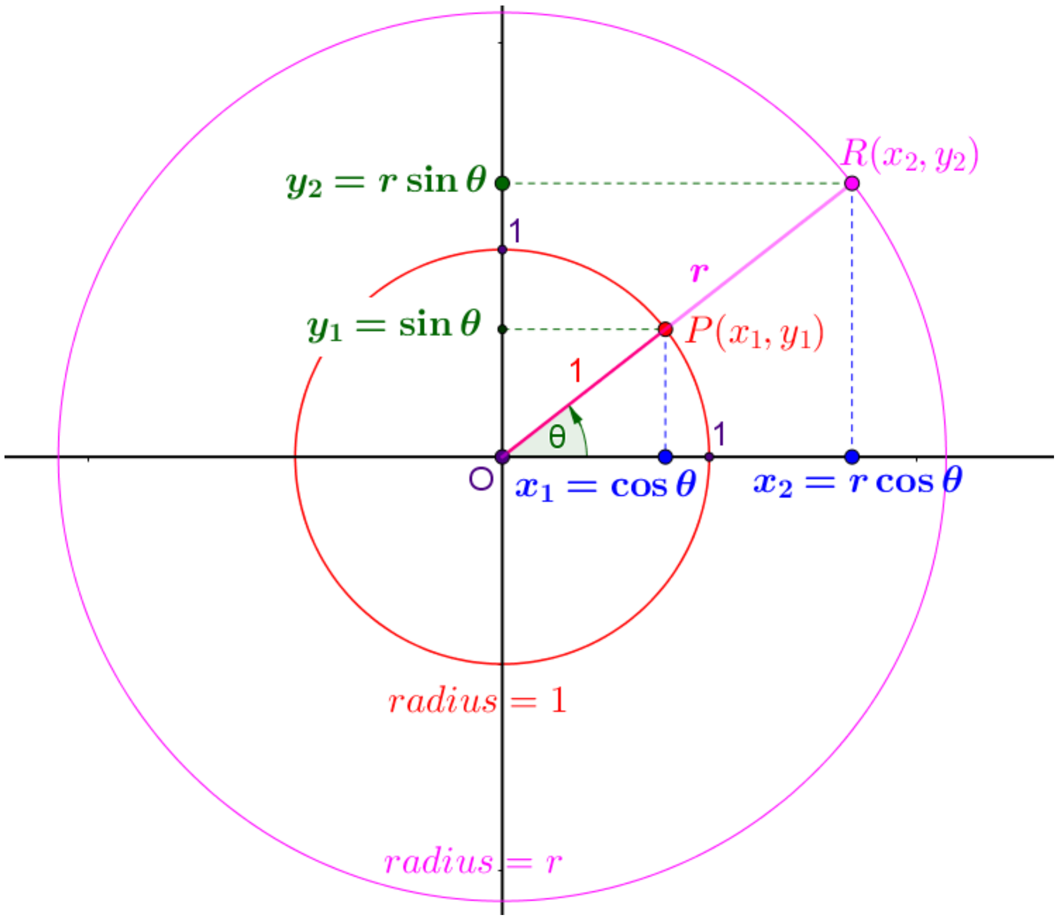


This radars display shows polar coordinates

Figure: Airport Air Traffic Control Radar Screen.

Polar coordinates

$x = r\cos\theta, \quad y = r\sin\theta, \quad \tan\theta = \frac{y}{x}$ illustrated



Modulus, arguments, principal argument

- We call r the *modulus* of $z = x + iy$ and denote it $|z|$.

$$|z| = \sqrt{x^2 + y^2}$$

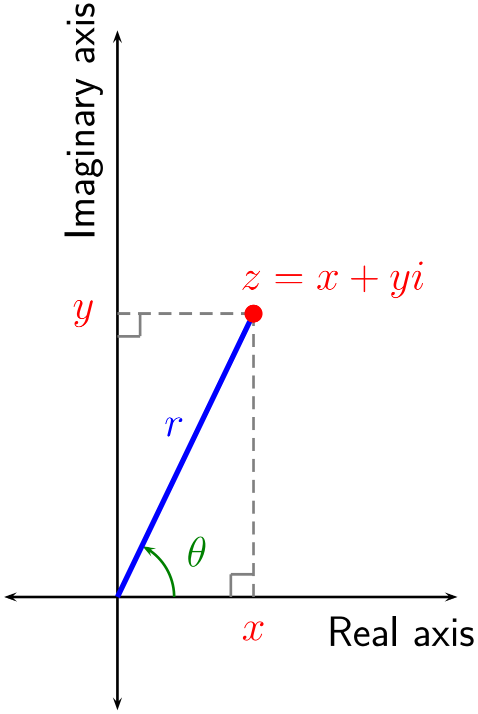
- We call θ ***an*** *argument* of $z = x + iy$ and denote it $\arg(z)$.

$$\tan\left(\theta\right) = \frac{y}{x}$$

The argument is not unique: If θ is an argument of z, so is $\theta + 2k\pi$, $k \in \mathbb{Z}$.

The *principal argument* of z, denoted $\mathrm{Arg}(z)$, is the only argument in the interval $(-\pi, \pi]$.

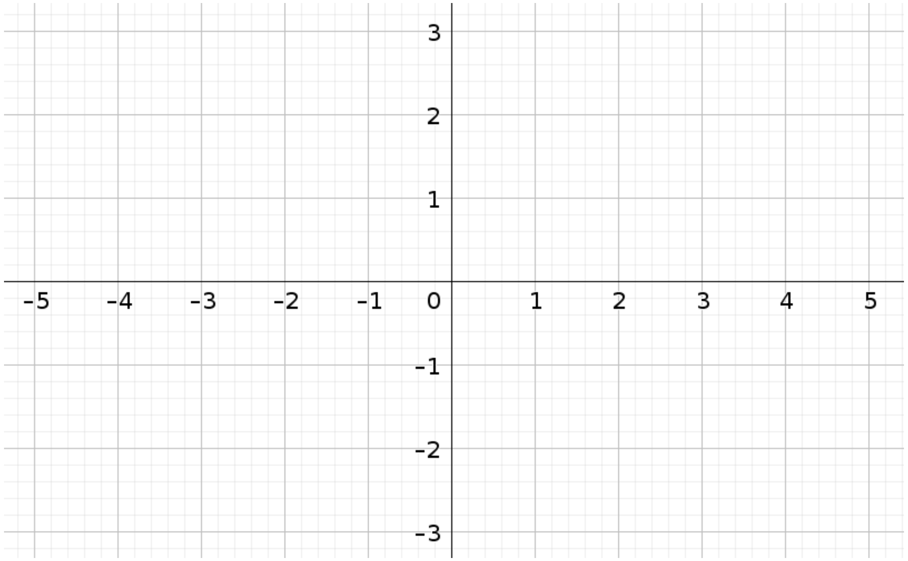
$$-\pi < \mathrm{Arg}(z) \leqslant \pi.$$



Modulus and argument

Exercise 4. Plot each of the following complex numbers on an Argand diagram and find its modulus and principal argument.

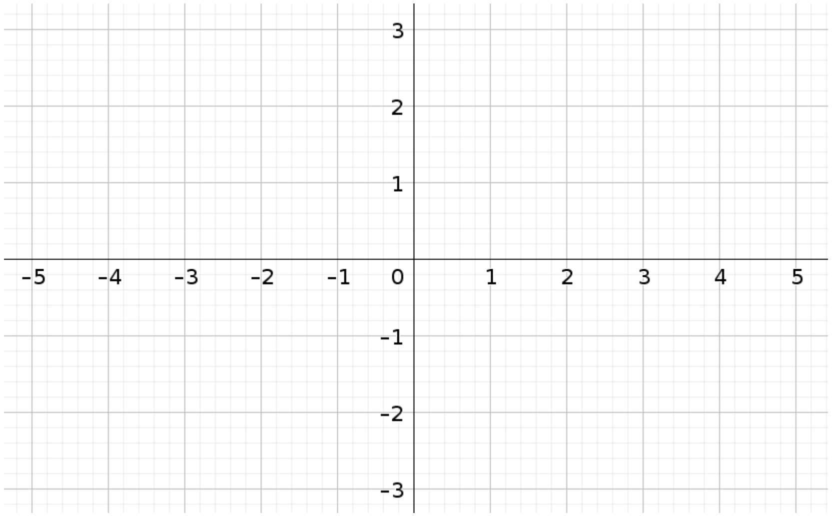
a) $2i$ b) 3 c) -5



Modulus and argument

Exercise 4, continued. Plot each of the following complex numbers on an Argand diagram and find its modulus and principal argument.

d) $1 + i$ e) $-3 - 3i$



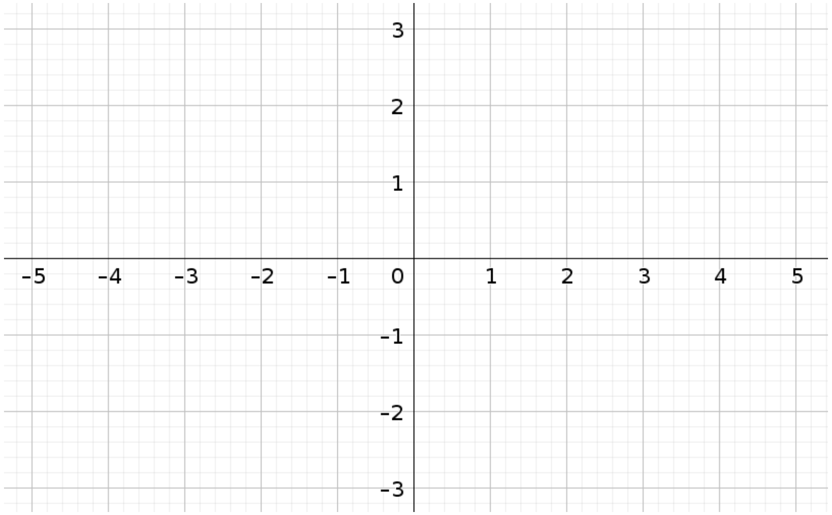
You may use the revision notes about the unit circle on the last slide.

Modulus and argument

Exercise 4, continued. Plot each of the following complex numbers on an Argand diagram and find its modulus and principal argument.

f) $-1 + i\sqrt{3}$

g) 0



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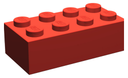
Polar form of a non-zero complex number



Polar form of a non-zero complex number

Let z be a complex number, $z \neq 0$. If the modulus of z is $|z| = r$ and its principal argument is $\text{Arg}(z) = \theta$, then

$$x = r\cos\theta, \quad \text{and} \quad y = r\sin\theta$$

so for $z = x + iy$ we have,

$$z = r(\cos\theta + i\sin\theta).$$

We will call this the *polar form* of a complex number.

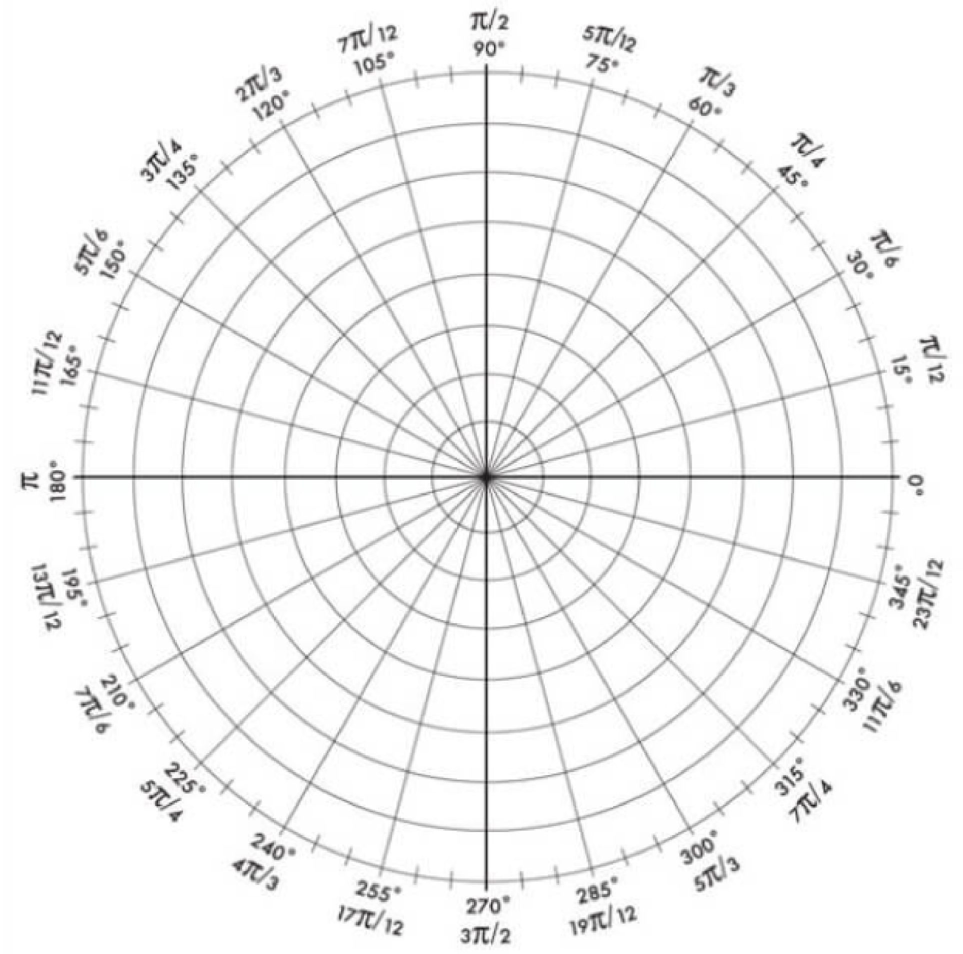
N.B. $z = r(\cos\theta + i\sin\theta)$ for **any** argument of the complex number z, not just the principal one.

Polar form

Exercise 5.

Sketch the following complex number on the complex plane and write it in Cartesian form.

$z = 6\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$



Polar form

Exercise 6. Find the polar form of each complex number in exercise 4.

Exercise 7.

$$\begin{aligned} z_1 &= r_1(\cos\theta_1 + i\sin\theta_1) \\ z_2 &= r_2(\cos\theta_2 + i\sin\theta_2) \\ z_1 &= z_2 \end{aligned}$$

Is it true that since $z_1 = z_2$, we must have $r_1 = r_2$ and $\theta_1 = \theta_2$?

Complex conjugates

Conjugate of a complex number

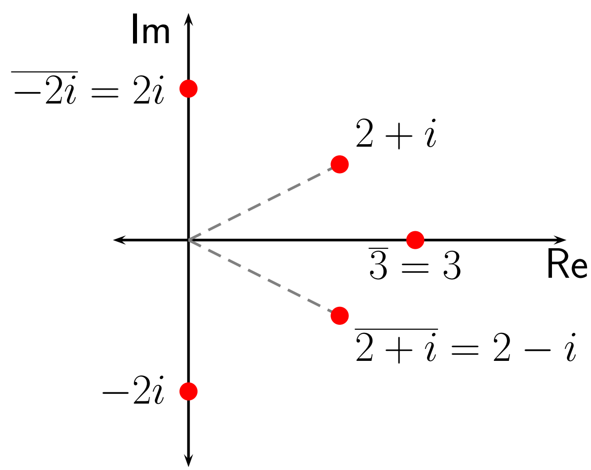
- Let $z = a + ib$ be a complex number with $a, b \in \mathbb{R}$.
  The *conjugate* of z, denoted $\overline{z}$, is

$$\boldsymbol{\overline{z} = a - ib}.$$

- On an Argand diagram, z and its conjugate $\overline{z}$ are symmetric with respect to the x-axis.

Example 2.

$$\begin{aligned} \overline{2+i} &= 2 - i \\ \overline{-2i} &= 2i \\ \overline{3} &= 3 \end{aligned}$$



Exercise 3.

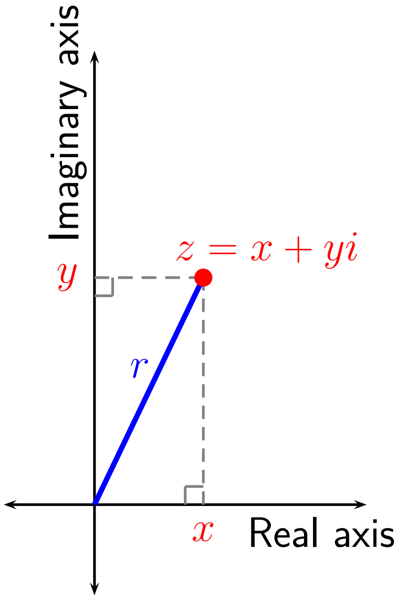
a) $\overline{1+7i} =$ ....

b) $\overline{3-5i} =$ ....

Theorems about the conjugate of a complex number

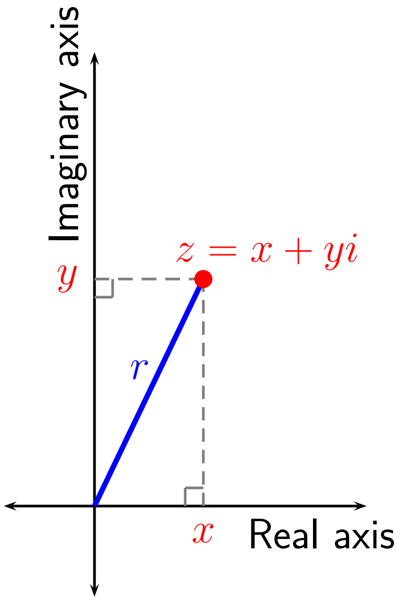
Prove the following useful results.

1. $\overline{\overline{z}} = z$
2. $\text{Re}(z) = \frac{1}{2}(z + \overline{z})$
3. $\text{Im}(z) = \frac{1}{2i}(z - \overline{z})$
4. $z\overline{z} = (\text{Re}(z))^2 + (\text{Im}(z))^2 = |z|^2$.
5. $z\overline{z} \in \mathbb{R}$ and $z\overline{z} \geqslant 0$.
6. $\overline{z + w} = \overline{z} + \overline{w}$
7. $\overline{zw} = \overline{z}\ \overline{w}$
8. $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$



Theorems about the conjugate of a complex number

Some space for you to write the proofs (and a figure to help you visualise the modulus).



Example

Exercise 8.  [Challenge!] Let $z, w \in \mathbb{C}$ with $z\overline{z} = w\overline{w}$.

Prove that $\dfrac{z+w}{z-w}$ is purely imaginary.

Can we divide complex numbers? YES

How to divide complex numbers and get a result of the form $a + ib$

Multiply the numerator *and* the denominator by the conjugate of the denominator.

Exercise 4.

a) Find the real and imaginary parts of $\dfrac{3+4i}{2+5i}$.

b)  What about $\dfrac{c+di}{a+bi}$ for $a, b, c, d \in \mathbb{R}$ with a and b not both zero?

Checking our answer with Maple

Exercise 4, continued.

```
> # Division with complex numbers

  z := (3 + 4*I)/(2 + 5*I);
  # Note the uppercase I and the * for 'times'
```

$$z := \frac{26}{29} - \frac{7\,\mathrm{I}}{29}$$

```
> # General case

  z := (a + b*I)/(c + d*I);
```

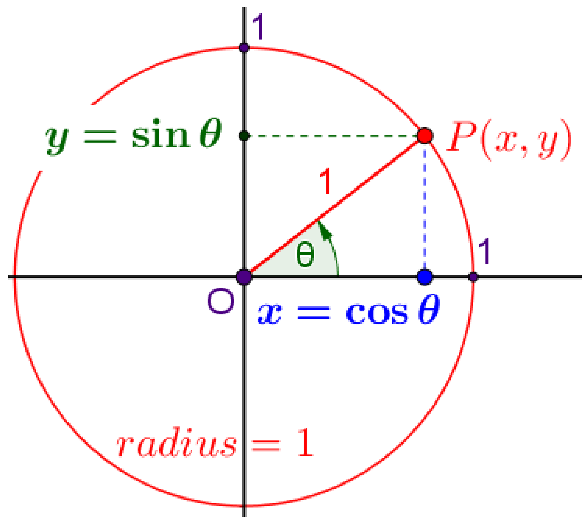
$$z := \frac{a + \mathrm{I}\, b}{c + \mathrm{I}\, d}$$

```
> # use evalc to get get the cartesian (= rectangular) form

  evalc(z);
```

$$\frac{a\,c}{c^2 + d^2} + \frac{b\,d}{c^2 + d^2} + \mathrm{I}\left(\frac{b\,c}{c^2 + d^2} - \frac{a\,d}{c^2 + d^2}\right)$$

Revision: The unit circle in a nutshell



Angles are counted counterclockwise from the positive x-axis.

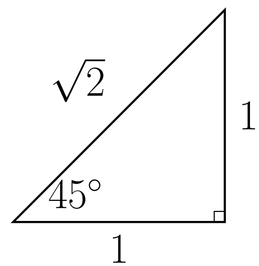
1. The x-coordinate of P is $\cos\theta$.
2. The y-coordinate of P is $\sin\theta$.
3. $\tan\theta$ is the gradient of the straight line OP.

$$m = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$$

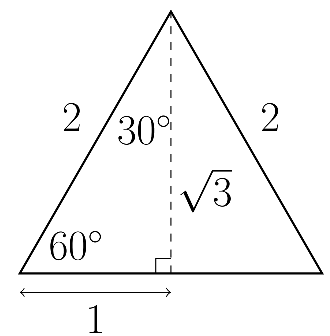
The Exact values you need to know:

θ	$\frac{\pi}{6}$ or $30°$	$\frac{\pi}{4}$ or $45°$	$\frac{\pi}{3}$ or $60°$
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

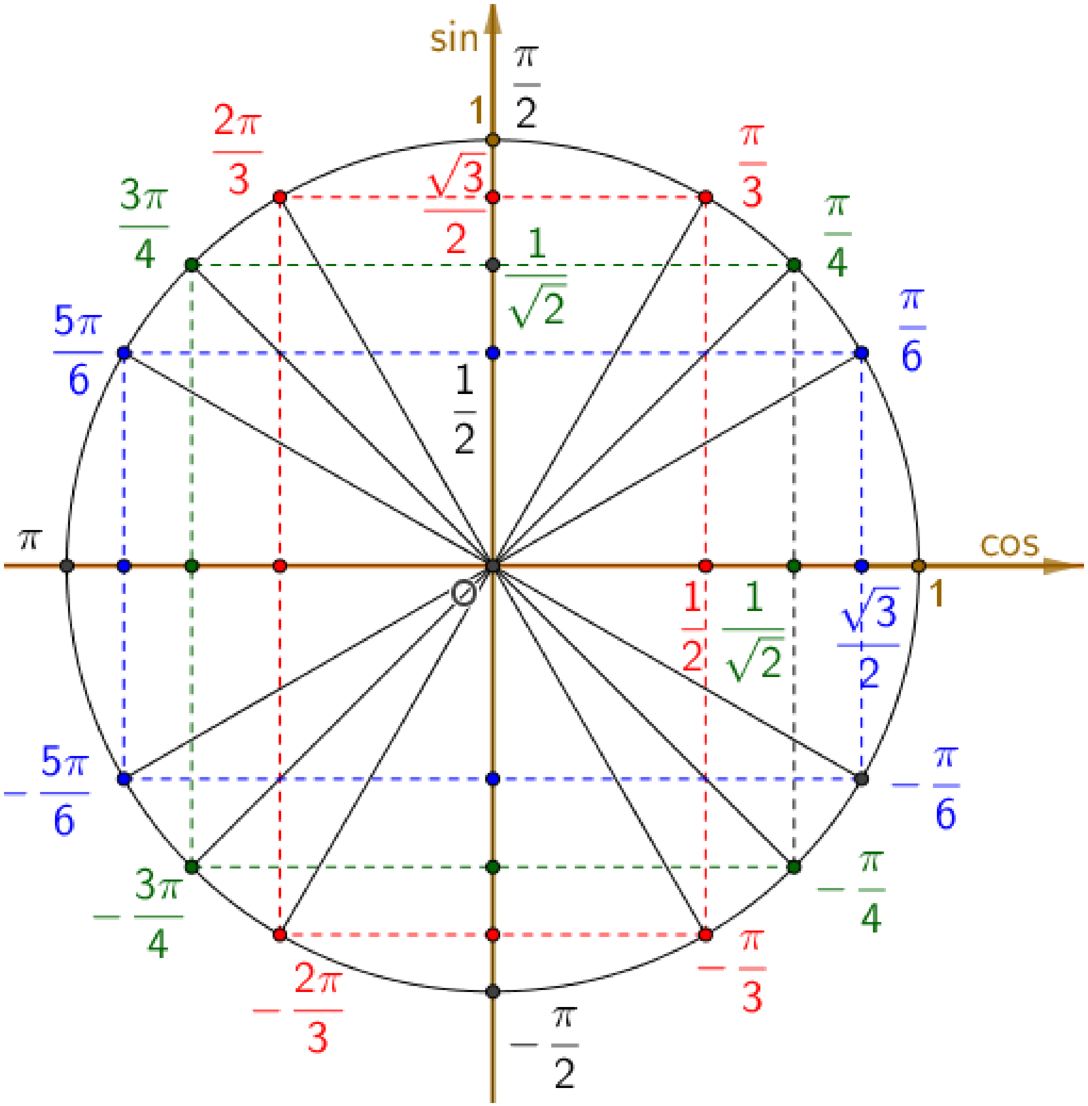
The Isosceles Right Triangle



The 30/60 Triangle



Revision: The unit circle in a nutshell



In the first quadrant are the exact values you need to know.