LECTURE 22
The Hyperbolic Trig Functions

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\frac{d}{dx}\sinh(x) = \cosh(x)$$

$$\frac{d}{dx}\cosh(x) = \sinh(x)$$

$$\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$$

$$\cosh^2(x) - \sinh^2(x) = 1$$

$$1 - \tanh^2(x) = \operatorname{sech}^2(x)$$

$$\sinh(2x) = 2\sinh(x)\cosh(x)$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x)$$

$$\tanh(2x) = \frac{2\tanh(x)}{1 + \tanh^2(x)}$$

$$\int \cosh(ax+b)\,dx = \frac{1}{a}\sinh(ax+b) + C$$

$$\int \sinh(ax+b)\,dx = \frac{1}{a}\cosh(ax+b) + C$$

$$\int \operatorname{sech}^2(ax+b)\,dx = \frac{1}{a}\tanh(ax+b) + C$$

We close Math1131 Calculus with a pair of lectures examining a fascinating class of functions known as the hyperbolic trigonometric functions. These functions are defined in terms of the exponential function e^x but remarkably look, feel and taste almost exactly like trig functions. They are fake trig functions!

Their names are

$$f(x) = \sinh(x) \quad \text{(pronounced } \textit{shine} \text{ of } x\text{)}$$

$$f(x) = \cosh(x) \quad \text{(pronounced } \textit{cosh} \text{ of } x\text{)}$$

$$f(x) = \tanh(x) \quad \text{(pronounced } \textit{than} \text{ of } x \text{ or } \textit{tanch} \text{ of } x\text{).}$$

The definitions are:

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$

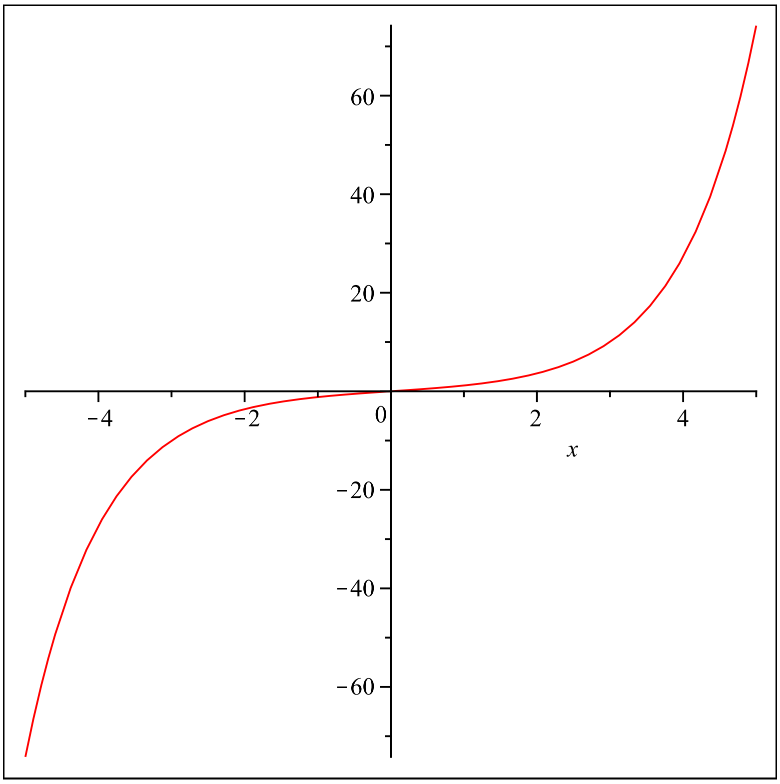
and

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

The extra h stands for hyperbolic and we will soon see the amazing connections with the trig functions. But first some graphs and properties.

$f(x) = \sinh(x)$

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$



$\sinh(x)$ is an odd function.

$\sinh(x)$ is an increasing function.

$\sinh(0) = 0.$

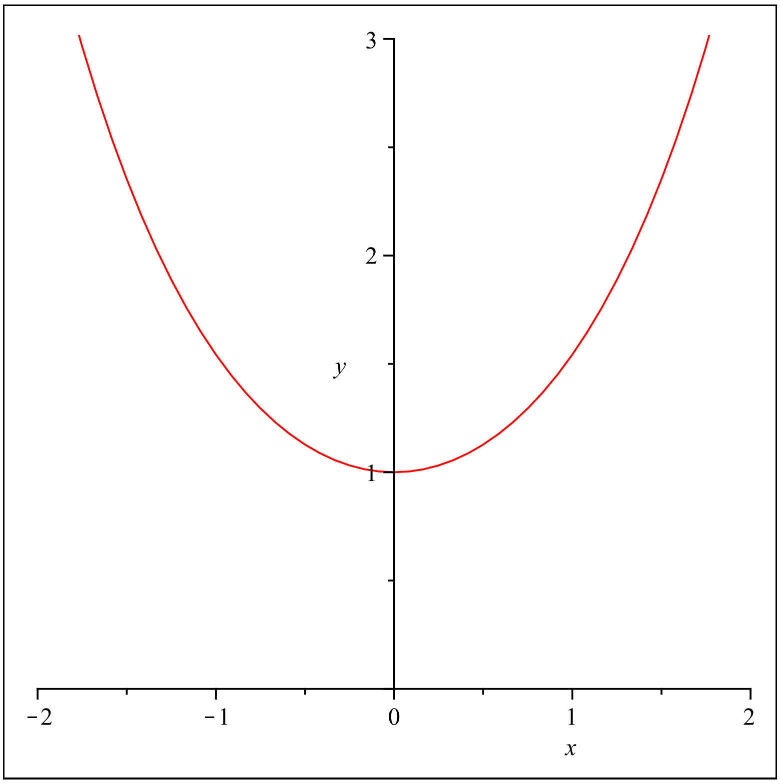
Dom($\sinh(x)$)=$\mathbb{R}$.

Range($\sinh(x)$)=$\mathbb{R}$.

For large x $\quad \sinh(x) \approx \dfrac{e^x}{2}.$

$f(x) = \cosh(x)$

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$



$\cosh(x)$ is an even function.

$\cosh(0) = 1.$

Dom($\cosh(x)$)=$\mathbb{R}$.

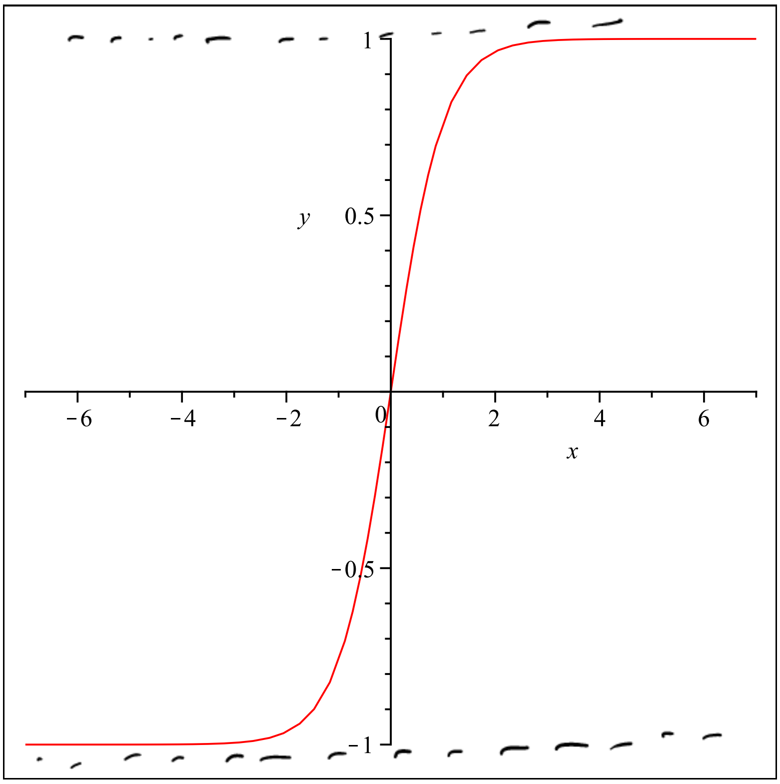
Range($\cosh(x)$)=$[1, \infty]$.

For large x $\quad \cosh(x) \approx \dfrac{e^x}{2}$.

The cosh curve is the shape of a hanging rope. Look at the hanging telegraph cables next time you are out on the road, they are all cosh curves!

$f(x) = \tanh(x)$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$



$\tanh(x)$ is an odd function.

$\tanh(x)$ is an increasing function

$\tanh(0) = 0.$

Dom($\tanh(x)$)=$\mathbb{R}$.

Range($\tanh(x)$)=$(-1, 1)$.

For large $x \quad \tanh(x) \approx 1.$

$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$
$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$

definitions

We also have some minor hyperbolic trig functions defined as:

Rare

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\operatorname{cosech}(x) = \frac{1}{\sinh(x)}$$

and

$$\coth(x) = \frac{1}{\tanh(x)}$$

Let's now take a look at the properties of these functions. They mimic almost exactly those of the trig functions. In the list below the corresponding trig property follows the hyperbolic result. You should not only know all of the results, you should also be able to prove them. The proofs are generally trivial.

a) $\frac{d}{dx}\sinh(x) = \cosh(x)$ ★ $\frac{d}{dx}\sin(x) = \cos(x)$ ★

b) $\frac{d}{dx}\cosh(x) = \sinh(x)$ ★ $\frac{d}{dx}\cos(x) = -\sin(x)$ ★

c) $\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$ ★ $\frac{d}{dx}\tan(x) = \sec^2(x)$ ★

d) $\cosh^2(x) - \sinh^2(x) = 1$ ★ $\cos^2(x) + \sin^2(x) = 1$ ★

e) $1 - \tanh^2(x) = \operatorname{sech}^2(x)$ ★ $1 + \tan^2(x) = \sec^2(x)$ ★

f) $\sinh(2x) = 2\sinh(x)\cosh(x)$ ★ $\sin(2x) = 2\sin(x)\cos(x)$ ★

g) $\cosh(2x) = \cosh^2(x) + \sinh^2(x)$ ★ $\cos(2x) = \cos^2(x) - \sin^2(x)$ ★

h) $\tanh(2x) = \frac{2\tanh(x)}{1 + \tanh^2(x)}$ ★ $\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$ ★

i) $\int \cosh(ax+b)\,dx = \frac{1}{a}\sinh(ax+b) + C$ ★ $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$ ★

j) $\int \sinh(ax+b)\,dx = \frac{1}{a}\cosh(ax+b) + C$ ★ $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$ ★

k) $\int \operatorname{sech}^2(ax+b)\,dx = \frac{1}{a}\tanh(ax+b) + C$ ★ $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$ ★

It is clear from the above properties that the hyperbolic trig functions do indeed behave very much like the real trig functions! But keep your eye on those negatives!

Let's prove some of the results. The proofs will involve the exponential definitions and a little algebra.

Proof b): $\dfrac{d}{dx}\cosh(x) = \sinh(x)$

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x + e^{-x}(-1)}{2}$$

$$= \frac{e^x - e^{-x}}{2} = \underline{\sinh(x)}$$

★

Proof d): $\cosh^2(x) - \sinh^2(x) = 1$

$$\text{LHS} = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4}$$

$$= \frac{(\cancel{e^{2x}} + 2 + \cancel{e^{-2x}}) - (\cancel{e^{2x}} - 2 + \cancel{e^{-2x}})}{4} = \frac{4}{4} = 1 = \underline{\text{RHS}}$$

★

Proof f): $\sinh(2x) = 2\sinh(x)\cosh(x)$

$$\text{RHS} = \cancel{2}\left(\frac{e^x - e^{-x}}{\cancel{2}}\right)\left(\frac{e^x + e^{-x}}{2}\right) = \frac{1}{2}\left((e^x)^2 - (e^{-x})^2\right)$$

$$= \frac{1}{2}\left(e^{2x} - e^{-2x}\right) = \sinh(2x) = \underline{\text{LHS}}$$

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Proof i): $\displaystyle\int \cosh(ax+b)\,dx = \frac{1}{a}\sinh(ax+b) + C$

$$\int \cosh(ax+b)\,dx = \int \frac{e^{ax+b} + e^{-(ax+b)}}{2}\,dx.$$

$$= \frac{1}{2}\left[\frac{1}{a}e^{ax+b} - \frac{1}{a}e^{-(ax+b)}\right]$$

$$= \frac{1}{a}\,\frac{1}{2}\left\{e^{ax+b} - e^{-(ax+b)}\right\} = \frac{1}{a}\sinh(ax+b) + C$$

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We finish the lecture with some typical questions on the hyperbolic trig functions. Note that some calculators allow you to evaluate the hyperbolic trig functions by using a HYP button.

Example 1: Find cosh(2) correct to 3 decimal places.

Method 1: $\cosh(2) = \dfrac{e^2 + e^{-2}}{2} = 3.762$ or

Method 2: HYP cos $2 = 3.762$

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Note that, despite the names, the hyperbolic trig functions don't actually have anything to do with the real trig functions. So when using a calculator to evaluate them, it doesn't matter in the slightest whether your calculator is in radian or degree mode.

Example 2: Find the derivative of $f(x) = x^3 \cosh(x)$.

$f'(x) = 3x^2 \cosh(x) + \sinh(x)\,(x^3)$

★ $3x^2 \cosh(x) + x^3 \sinh(x)$ ★

Example 3: Find the derivative of $y = \sinh(x^7 + 1)$.

$y' = \cosh(x^7+1)\,(7x^6)$

★ $7x^6 \cosh(x^7 + 1)$ ★

Example 4: Find the gradient of $y = \dfrac{\tanh(x)}{3x - 2}$ at $x = 0$.

$\left(\dfrac{u}{v}\right)' = \dfrac{vu' - uv'}{v^2}$

$\dfrac{dy}{dx} = \dfrac{(3x-2)\,\text{sech}^2(x) - \tanh(x)(3)}{(3x-2)^2}$

★ $-\dfrac{1}{2}$ ★

at $x=0$: $\dfrac{(-2)(\frac{1}{1}) - 0}{4} = \dfrac{-2}{4} = -\dfrac{1}{2}$

Example 5: Evaluate $\displaystyle\int_0^{\frac{\ln(3)}{5}} \sinh(5x)\,dx$ $= \left[\frac{1}{5}\cosh(5x)\right]_0^{\frac{\ln(3)}{5}}$

$$= \frac{1}{5}\left\{ (\cosh(\ln 3)) - (1) \right\}$$

$$= \frac{1}{5}\left\{ \frac{e^{\ln 3} + e^{-\ln 3}}{2} - 1 \right\} = \frac{1}{5}\left\{ \frac{3 + e^{\ln(3^{-1})}}{2} - 1 \right\}$$

$$= \frac{1}{5}\left\{ \frac{3+\frac{1}{3}}{2} - 1 \right\} = \frac{1}{5}\left\{ \frac{9+1}{6} - \frac{6}{6} \right\} = \frac{1}{5}\left(\frac{4}{6}\right) = \frac{4}{30} = \underline{\underline{\frac{2}{15}}}$$

★ $\dfrac{2}{15}$ ★

Example 6: Find $\displaystyle\int x\cosh(x)\,dx$

$u = x \to du = 1$

$dv = \cosh x \to v = \sinh x$

$$\int u\,dv = uv - \int v\,du$$

$$= x\sinh x - \int \sinh x\,(1)$$

$$= x\sinh x - \cosh x \underline{+C}$$

★ $x\sinh(x) - \cosh(x) + C$ ★

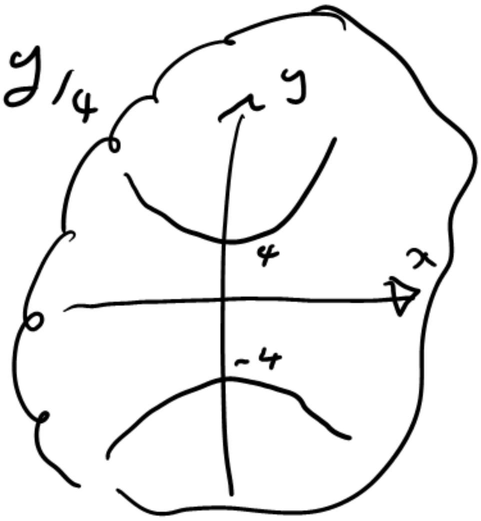
Example 7: Suppose that $x = 3\sinh(t)$ and $y = 4\cosh(t)$. Find a Cartesian relation between x and y.

$\sinh t = x/3$, $\cosh t = y/4$

$$\boxed{\cosh^2 t - \sinh^2 t = 1}$$

$$(y/4)^2 - \left(\frac{x}{3}\right)^2 = 1$$

$$y^2/16 - \frac{x^2}{9} = 1$$



★ $\dfrac{y^2}{16} - \dfrac{x^2}{9} = 1.$ A hyperbola. ★

wrong in video!