LECTURE 12
Inverse Trigonometric Functions

$$\sin^{-1} : [-1,1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\cos^{-1} : [-1,1] \longrightarrow [0,\pi]$$

$$\tan^{-1} : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\sin^{-1}(-x) = -\sin^{-1}(x)$$

$$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$$

$$\tan^{-1}(-x) = -\tan^{-1}(x)$$

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$$

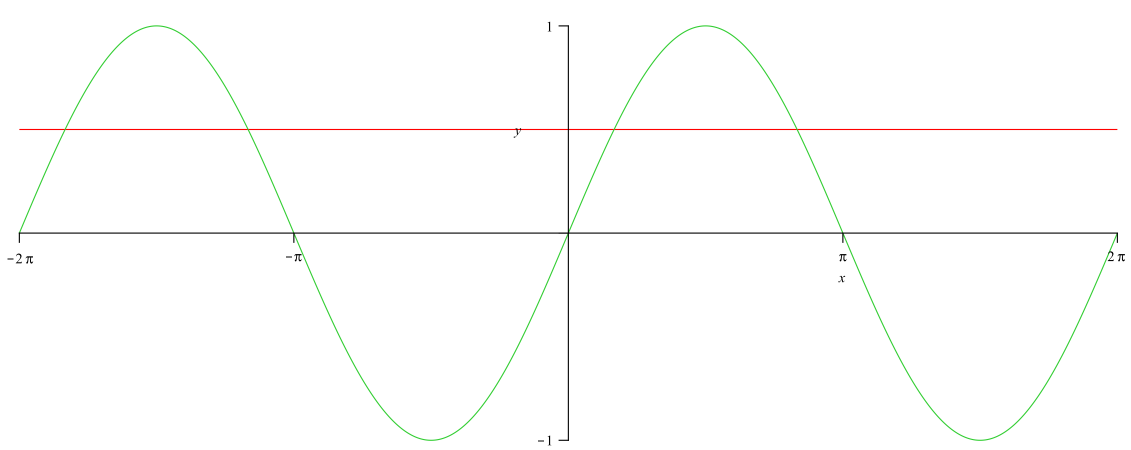
$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt}$$

We will now use the constructions of the previous lecture on the inverse trig functions. Let's analyse the sine curve with a view to constructing its inverse $\sin^{-1}(x)$.

We know that sin : angles$\rightarrow$ numbers and hence $\sin^{-1}$: numbers$\rightarrow$ angles. But the sine curve fails the horizontal line test dreadfully!

$y = \sin(x)$



We have
$$\frac{1}{2} = \sin(\frac{\pi}{6}) = \sin(\frac{5\pi}{6}) = \sin(\frac{-11\pi}{6}) = \sin(\frac{-7\pi}{6})\ldots$$
What do we mean by $\sin^{-1}(\frac{1}{2})$? Well it's the angle whose sine is $\frac{1}{2}$. But which one?

Lets trim up the graph:

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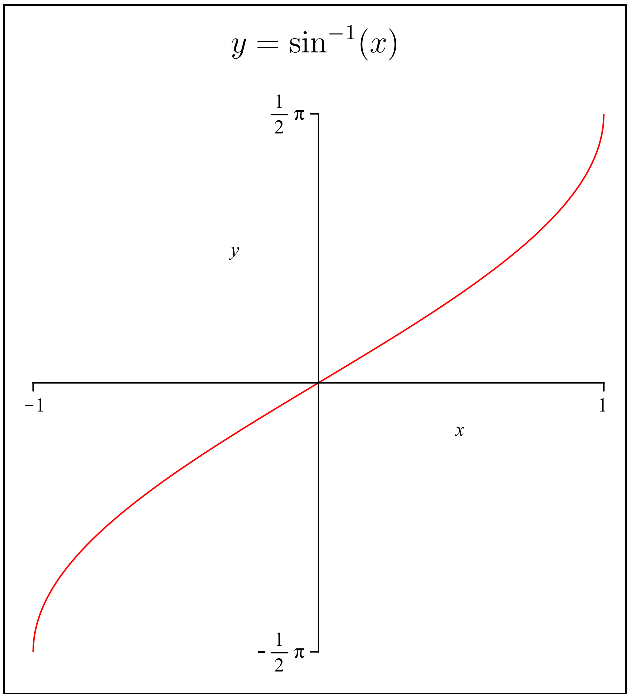
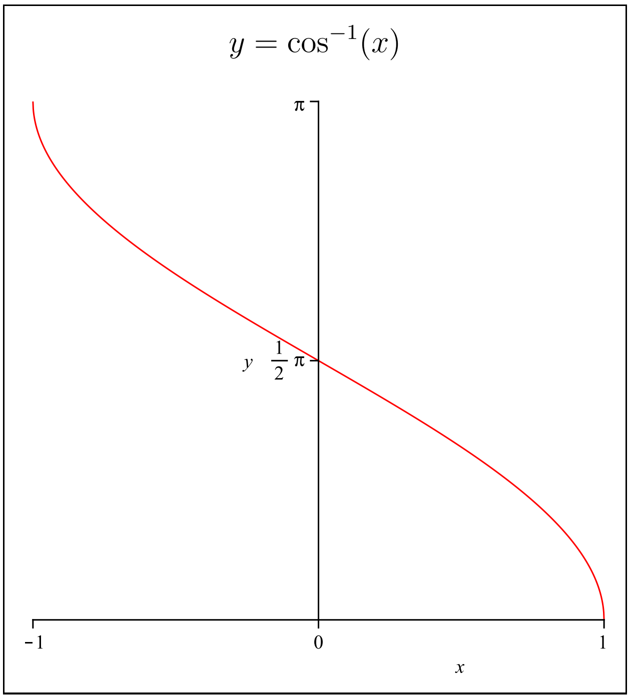
Hence the graph of $y = \sin^{-1}(x)$ is:

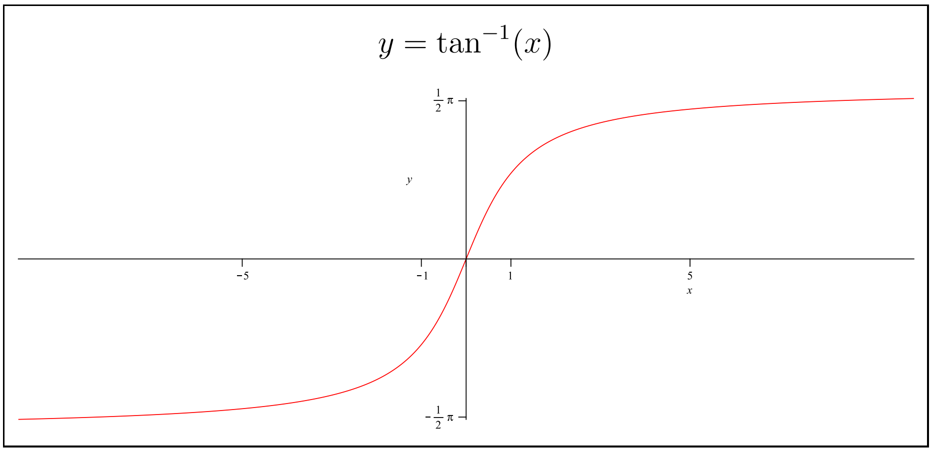


Always remember that $\sin^{-1} : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
Similarly we have:



Always remember that $\cos^{-1} : [-1, 1] \longrightarrow [0, \pi]$

Finally



Always remember that $\tan^{-1} : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

When dealing with the inverse trig functions always be very careful with domain and range! Some other facts of intetrest which may be used:

$$\sin^{-1}(-x) = -\sin^{-1}(x)$$

$$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$$

$$\tan^{-1}(-x) = -\tan^{-1}(x)$$

Example 1: Evaluate each of the following:

a) $\sin^{-1}(\frac{1}{\sqrt{2}}) =$

b) $\cos^{-1}(-\frac{1}{2}) =$

c) $\sin^{-1}(\sin(\frac{2\pi}{3})) =$

d) $\cos^{-1}(\cos(\frac{2\pi}{3})) =$

e) $\cos(\sin^{-1}(\frac{3}{7})) =$

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Observe that

$$\sin(\sin^{-1}(x)) = \cos(\cos^{-1}(x)) = \tan(\tan^{-1}(x)) = x \quad \textbf{always!!}$$

$$\sin^{-1}(\sin(x)) = \cos^{-1}(\cos(x)) = \tan^{-1}(\tan(x)) = x \quad \textbf{sometimes.}$$

Example 2: Sketch the graph of $y = 3\sin^{-1}(2x)$ and hence write down its domain and range.

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Despite their elaborate definitions the inverse trig functions are just functions! Hence we should be able to differentiate them.

Facts:

a) $$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

b) $$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$$

c) $$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

Discussion:

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Proof a:

Method 1:

Method 2: Using $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$

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Example 3: Find the derivative of each of the following:

a) $y = \sin^{-1}(x^7 + 5x)$.

b) $y = \ln(x)\cos^{-1}(x)$.

c) $y = \dfrac{\tan^{-1}(x)}{6x}$.

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Parametrically Defined Curves

We sometimes define relations between x and y in terms of a third party called a parameter. The advantage of this approach is that all concerns become focused on a single object, the parameter rather than a multiplicity of other variables. You have already seen the power of parameters in the algebra strand where lines and planes in space are defined in parametric vector form.

Example 4: Prove that the circle $x^2 + y^2 = 25$ can be written parametrically as

$$\begin{cases} x = 5\cos(\theta) \\ y = 5\sin(\theta) \end{cases}$$

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Other parametrically defines curves are:

Conic section	Cartesian equation	Parametric equation
Parabola	$4ay = x^2$	$x(t) = 2at$ $y(t) = at^2$
Circle	$x^2 + y^2 = a^2$	$x(t) = a\cos t$ $y(t) = a\sin t$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x(t) = a\cos t$ $y(t) = b\sin t$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x(t) = a\sec t$ $y(t) = b\tan t$

• Note that **any** function may be rewritten parametrically in many different ways.

Example 5: The function

$y = x^3 + 7$ may be expressed as $\begin{cases} x = t \\ y = t^3 + 7 \end{cases}$ or $\begin{cases} x = e^t \\ y = e^{3t} + 7 \end{cases}$

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• Note also that it is often (but not always) possible to recover the Cartesian equation of a parametrically defined curve.

Example 6: Find the Cartesian equation of $\begin{cases} x = 3t - 1 \\ y = 9t^2 - 6t + 8 \end{cases}$

★ $y = x^2 + 7$ ★

Even though we do not have a direct relationship, it is still possible to find $\dfrac{dy}{dx}$ through the use of parametric differentiation.

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt}\bigg/\frac{dx}{dt}}$$

Example 7: Suppose that a curve $\mathcal{C}$ is defined as $x = t^2 - 1$ and $y = \dfrac{3}{t}$.

a) Find a Cartesian relation between x and y.

b) Which point on the curve corresponds to $t = 6$?

c) Using parametric differentiation find $\dfrac{dy}{dx}$ at the point $(8, 1)$?

★ *a)* $y^2 = \dfrac{9}{x+1}$ *b)* $(35, \dfrac{1}{2})$ *c)* $-\dfrac{1}{18}$ ★

Example 8: Suppose that a curve is define parametrically by

$$x = t + \cos(t) \quad \text{and} \quad y = t^4 + 2t + 5.$$

a) Find a Cartesian relation between x and y.

b) Find the equation of the tangent to the curve at the point $(1, 5)$.

c) What is $\dfrac{d^2y}{dx^2}$?

$$\star \quad a) \quad Impossible \quad b) \quad y = 2x + 3 \quad c) \quad \frac{12t^2(1 - \sin(t)) + (4t^3 + 2)\cos(t)}{(1 - \sin(t))^3} \quad \star$$