

Lec16: Solving systems of equations by Gaussian elimination

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Number of solutions of a system of linear equations.

Example 1. How many solutions do the systems of equations represented by the augmented matrices below have? Note that they are already in REF.

$$M_1 = \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 4 \end{array} \right) \quad M_2 = \left(\begin{array}{ccc|c} 1 & -5 & 5 & 7 \\ 0 & 2 & 1 & -9 \\ 0 & 0 & 3 & 1 \end{array} \right) \quad M_3 = \left(\begin{array}{ccc|c} 1 & 2 & 5 & -7 \\ 0 & 3 & -12 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right)$$



Number of solutions of a system of linear equations.

As we have seen, every time we solve a linear system of equations we either obtain, **no solutions**, **exactly one solution** (unique solution) or **infinitely many solutions**. You can tell how many solutions the system has by following these steps:

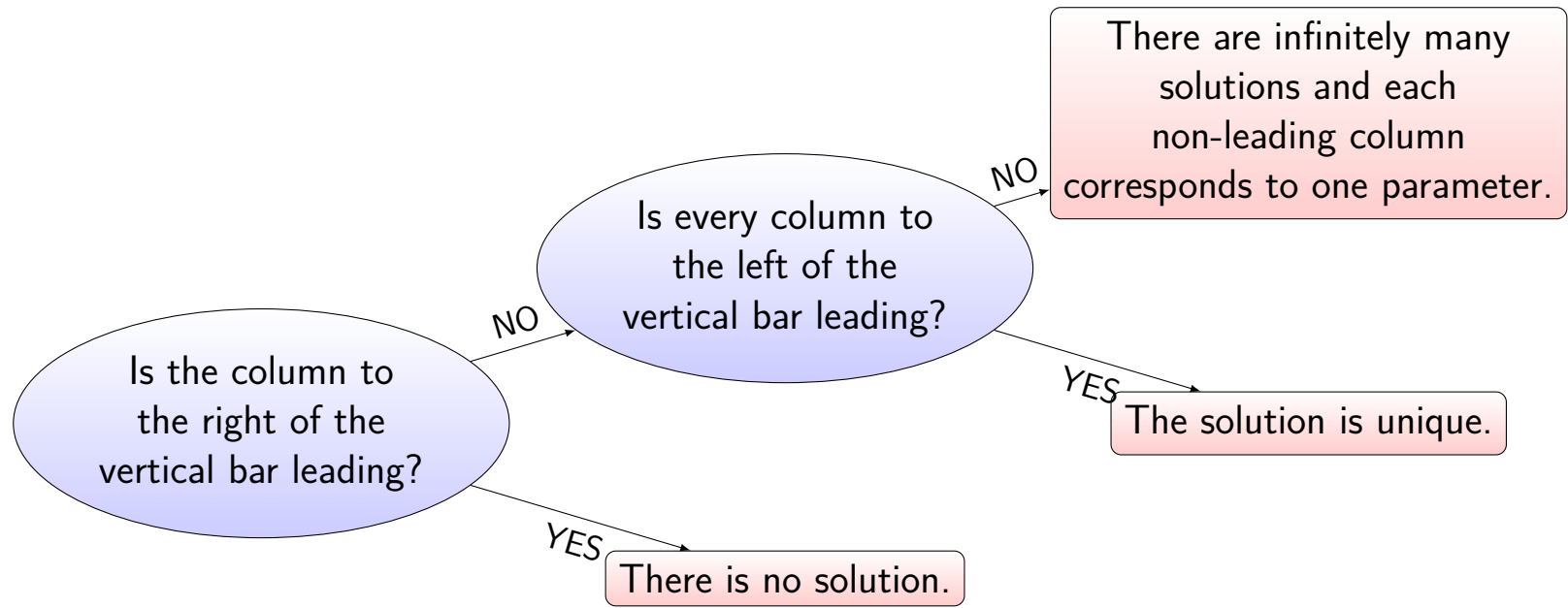
1. Rewrite your system in augmented matrix form $(A|\mathbf{b})$;
2. Row reduce, to obtain a row-echelon form $(U|\mathbf{y})$;
3. Circle or box the leading entries;
4. If **y is a leading** column, then the system has **no solutions**. If y is not a leading column then
 - If **every column of U is leading** then there is only **one** solution.
 - If **U has non-leading columns**, then the system has **infinitely many solutions**, and each the variable corresponding to the non-leading column will be treated as a parameter.

TIP!

N.B. Having (or not having) rows which are all zeros does not affect the number of solutions!!

Number of solutions of a system of linear equations.

The same method described in a flow chart. After reduction to Row Echelon Form:



Example 1, again. How many solutions do the systems of equations represented by the augmented matrices below have? Note that they are already in REF.

$$M_1 = \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 4 \end{array} \right) \quad M_2 = \left(\begin{array}{ccc|c} 1 & -5 & 5 & 7 \\ 0 & 2 & 1 & -9 \\ 0 & 0 & 3 & 1 \end{array} \right) \quad M_3 = \left(\begin{array}{ccc|c} 1 & 2 & 5 & -7 \\ 0 & 3 & -12 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Gaussian elimination examples

Exercise 2. For the following system of linear equations, find conditions on the vector

$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ such that the following system is consistent (ie has a solution)



$$\begin{aligned}x + 2y + 5z &= b_1 \\3x + 7y + 17z &= b_2 \\x + 3y + 7z &= b_3.\end{aligned}$$

Row-reduction with Maple

```
> # Load the LinearAlgebra package
with(LinearAlgebra):

> # Enter the matrix column by column
A := < <1,3,1> | <2,7,3> | <5,17,7> >:
b := <b1, b2,b3>:

> # The augmented matrix is:
Ab := <A|b>;

> # R2 <- R2 - 3*R1
RowOperation(Ab, [2, 1], -3);

> # R3 <- R3 - R1
RowOperation(%, [3, 1], -1);

> # To get the Row Echelon form at once
GaussianElimination(Ab);
```

$$Ab := \left[\begin{array}{ccc|c} 1 & 2 & 5 & b1 \\ 3 & 7 & 17 & b2 \\ 1 & 3 & 7 & b3 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & b1 \\ 0 & 1 & 2 & b2 - 3b1 \\ 1 & 3 & 7 & b3 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & b1 \\ 0 & 1 & 2 & b2 - 3b1 \\ 0 & 1 & 2 & b3 - b1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & b1 \\ 0 & 1 & 2 & b2 - 3b1 \\ 0 & 0 & 0 & b3 + 2b1 - b2 \end{array} \right]$$

Gaussian elimination examples

Exercise 3.

Use the Maple output given below to determine for which values of λ (if any) the following system of linear equations will have : (1) no solutions; (2) a unique solution; (3) infinitely many solutions.

$$x + y + z = 4$$

$$x + \lambda y + 2z = 5$$

$$2x + (\lambda + 1)y + (\lambda^2 - 1)z = \lambda + 7.$$

```
> with(LinearAlgebra):
```

```
> A := < <1,1,2> | <1,lambda,lambda + 1> | <1,2,lambda^2 - 1> >:
```

```
  b := <4, 5, lambda + 7>:
```

```
  Ab := <A|b>;
```

$$Ab := \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & \lambda & 2 & 5 \\ 2 & \lambda + 1 & \lambda^2 - 1 & \lambda + 7 \end{bmatrix}$$

```
> GaussianElimination(Ab);
```

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & \lambda - 1 & 1 & 1 \\ 0 & 0 & \lambda^2 - 4 & \lambda - 2 \end{bmatrix}$$

Gaussian elimination examples

Exercise 3, continued.

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & \lambda - 1 & 1 & 1 \\ 0 & 0 & \lambda^2 - 4 & \lambda - 2 \end{bmatrix}$$

Gaussian elimination examples

Exercise 3, continued.

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & \lambda - 1 & 1 & 1 \\ 0 & 0 & \lambda^2 - 4 & \lambda - 2 \end{bmatrix}$$

Gaussian elimination examples

Exercise 4. Determine whether the line

$$\ell : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad t \in \mathbb{R}$$

meets the plane

$$\Pi : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}$$

Solve this problem in two different ways.

Gaussian elimination examples

Exercise 4, continued. Determine whether the line ℓ meets the plane Π .

Gaussian elimination examples

Exercise 5. Consider the following system of linear equations.

$$x + 2y + z = 8$$

$$4x + 3y - z = 7$$

$$3x + y - 2z = -1.$$

1. Find the general solution.
2. Find the solution which has an x -value of 10.
3. Given x , y and z must all be non-negative, find the maximum value of y .

Gaussian elimination examples

Exercise 5, continued.

Gaussian elimination examples

Exercise 6. A forest contains 3 different types of gums trees: the red gum, the blue gum and the swamp gum. Each red gum tree houses 1 koala and 3 bandicoots. Each blue gum houses 2 koalas and 1 bandicoot and each swamp gum houses 3 koalas and 2 bandicoots. Conservationists are planning to occupy and protect the forest from logging. They have assigned 1 person for each red gum, 2 people for each blue gum and 4 people for each swamp gum. There are a total 1700 koalas and 1200 bandicoots in the forest and 1900 conservationists.

Set up the system of equations needed to determine how many of each type of gum tree does the forest contains.



Gaussian elimination examples

Exercise 6, extra space and ANSWERS.

ANSWER: 100 red gums, 500 blue gums and 200 swamp gums.

Gaussian elimination examples

Exercise 7. Find the equation of the parabola that passes through the points $(1, 2)$, $(-1, 4)$, and $(2, 4)$.

Checking our answers with Maple (Parabola)

```
> # Load the LinearAlgebra package
with(LinearAlgebra):

> # Enter the matrix column by column
A := < <1,1,4> | <1,-1,2> | <1,1,1> >:
b := <2,4,4>:
```

```
> # The augmented matrix is:
Ab := <A|b>;
```

$$Ab := \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 4 \\ 4 & 2 & 1 & 4 \end{bmatrix}$$

```
> # R2 <- R2 - R1
RowOperation(Ab, [2, 1], -1);
```

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & 2 \\ 4 & 2 & 1 & 4 \end{bmatrix}$$

```
> # R3 <- R3 - 4*R1
RowOperation(%, [3, 1], -4);
```

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & 2 \\ 0 & -2 & -3 & -4 \end{bmatrix}$$

```
> # R3 <- R3 - R2
RowOperation(%, [3, 2], -1);
```

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -3 & -6 \end{bmatrix}$$

```
> # Gaussian elimination followed by back substitution
REF := GaussianElimination(Ab);
BackwardSubstitute(REF);
```

$$REF := \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -3 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

```
> # Checking our answers
f := x -> x^2 - x + 2;
f(1); f(-1); f(2);
```

$$f := x \mapsto x^2 - x + 2$$

$$\begin{matrix} 2 \\ 4 \\ 4 \end{matrix}$$