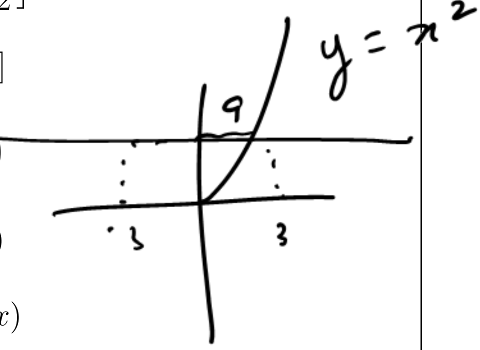
LECTURE 12
Inverse Trigonometric Functions

$$\sin^{-1} : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\cos^{-1} : [-1, 1] \longrightarrow [0, \pi]$$

$$\tan^{-1} : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$



$$\sin^{-1}(-x) = -\sin^{-1}(x)$$

$$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$$

$$\tan^{-1}(-x) = -\tan^{-1}(x)$$

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$$

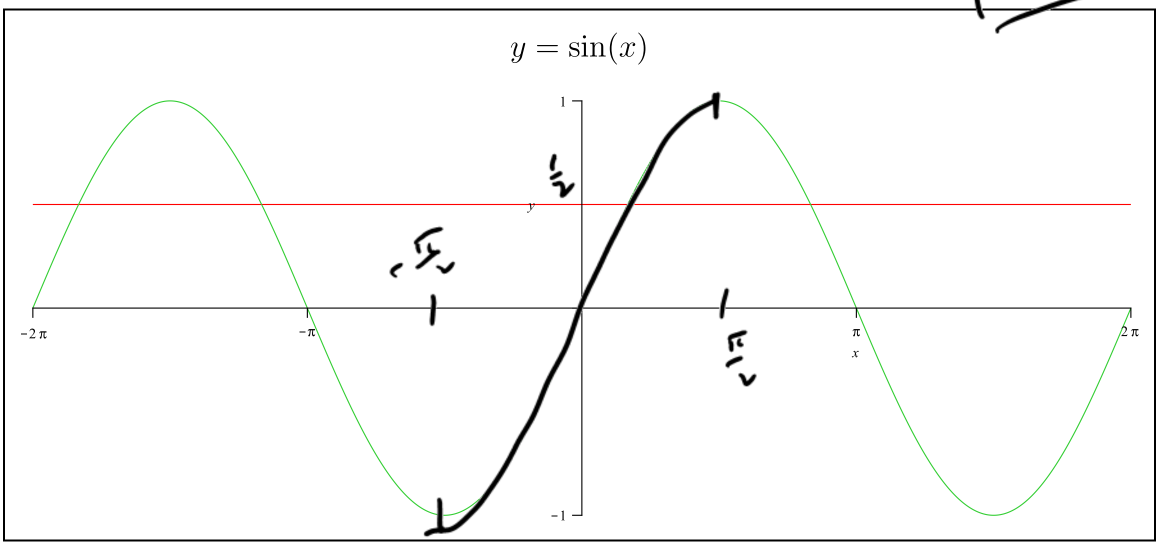
$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$$

We will now use the constructions of the previous lecture on the inverse trig functions. Let's analyse the sine curve with a view to constructing its inverse $\sin^{-1}(x)$.

We know that sin : angles→ numbers and hence $\sin^{-1}$: numbers→ angles. But the sine curve fails the horizontal line test dreadfully!

$\boxed{\sin^{-1}(x)}$

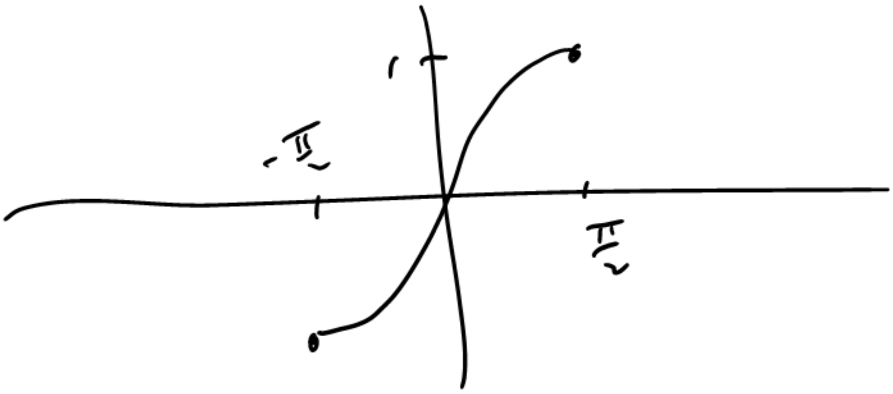


We have

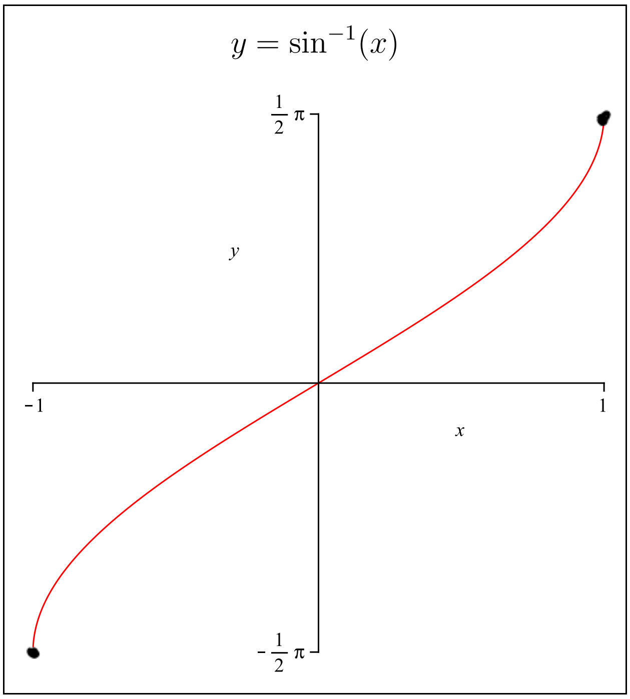
$$\frac{1}{2} = \sin(\frac{\pi}{6}) = \sin(\cancel{\frac{5\pi}{6}}) = \sin(\cancel{-\frac{11\pi}{6}}) = \sin(\cancel{-\frac{7\pi}{6}})\ldots$$

What do we mean by $\sin^{-1}(\frac{1}{2})$? Well it's the angle whose sine is $\frac{1}{2}$. But which one?

Lets trim up the graph:

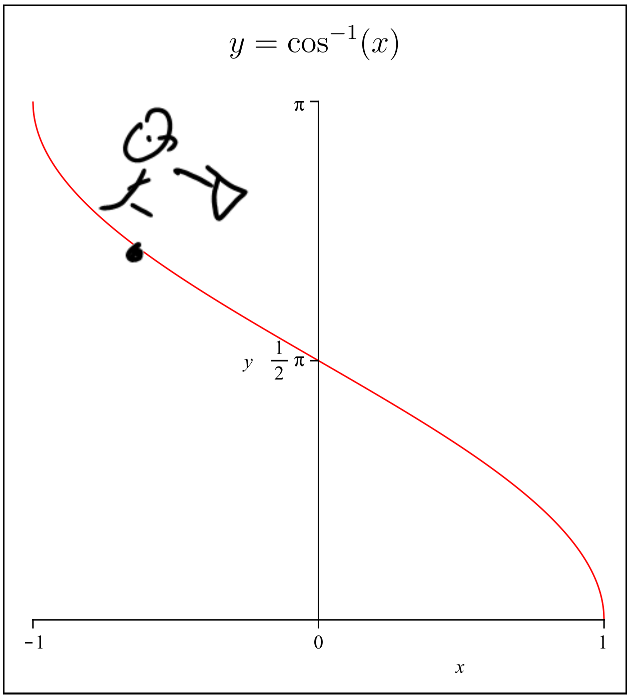


Hence the graph of $y = \sin^{-1}(x)$ is:



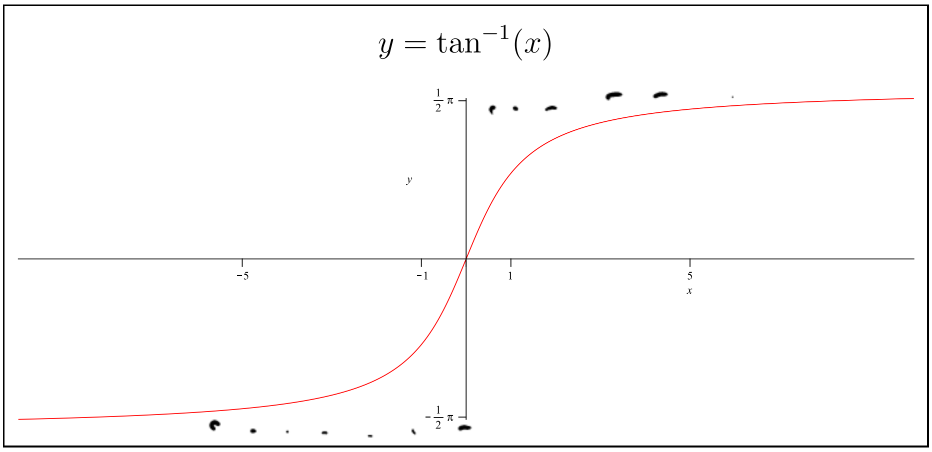
Always remember that $\sin^{-1} : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Similarly we have:



Always remember that $\cos^{-1} : [-1, 1] \longrightarrow [0, \pi]$

Finally



Always remember that $\tan^{-1} : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

When dealing with the inverse trig functions always be very careful with domain and range! Some other facts of intetrest which may be used:

$$\sin^{-1}(-x) = -\sin^{-1}(x)$$

$$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$$

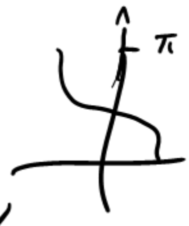
$$\tan^{-1}(-x) = -\tan^{-1}(x)$$

$[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Example 1: Evaluate each of the following:

a) $\sin^{-1}(\frac{1}{\sqrt{2}}) = \frac{\pi}{4}$ $\cancel{\frac{3\pi}{4}}$

b) $\cos^{-1}(-\frac{1}{2}) = \pi - \cos^{-1}(\frac{1}{2}) = \pi - \pi/3 = \frac{2\pi}{3}$

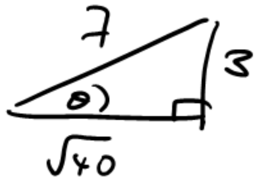


c) $\sin^{-1}(\sin(\frac{2\pi}{3})) =$ $\cancel{\frac{2\pi}{3}}$ $\sin^{-1}(\sin \pi/3) = \pi/3$

d) $\cos^{-1}(\cos(\frac{2\pi}{3})) = 2\pi/3$ ✓

e) $\cos(\underbrace{\sin^{-1}(\frac{3}{7})}_{\theta}) = \cos\theta = \frac{\sqrt{40}}{7}$ ✓

$\theta = \sin^{-1}(3/7) \Rightarrow \sin\theta = 3/7$



★

Observe that

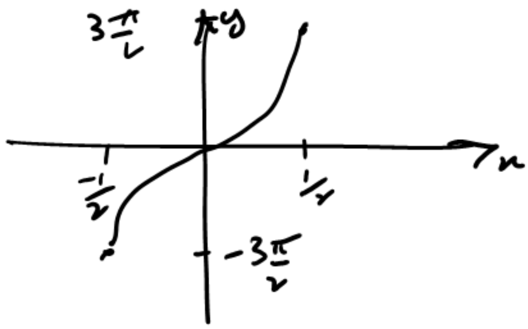
$$\cancel{\sin}(\sin^{-1}(x)) = \cos(\cos^{-1}(x)) = \tan(\tan^{-1}(x)) = x \quad \textbf{always!!}$$

✓ ✓ ✓

$$\sin^{-1}(\sin(x)) = \cos^{-1}(\cos(x)) = \tan^{-1}(\tan(x)) = x \quad \textbf{sometimes.}$$

×2

Example 2: Sketch the graph of $y = 3\sin^{-1}(2x)$ and hence write down its domain and range.



D: $[-\frac{1}{2}, \frac{1}{2}]$

R: $[-\frac{3\pi}{2}, \frac{3\pi}{2}]$.

★

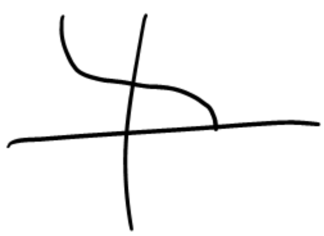
Despite their elaborate definitions the inverse trig functions are just functions! Hence we should be able to differentiate them.

Facts:

a) $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$

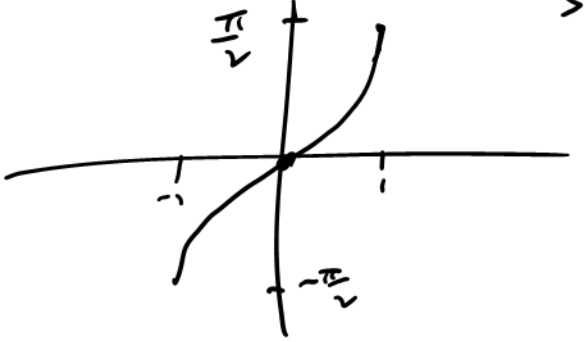
b) $\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$

c) $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$



Discussion:

$\sin^{-1}(x)$



$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

★

Proof a:

Method 1:

$$y = \sin^{-1}(x)$$

$c^2+s^2=1$

$$\Rightarrow \quad x = \sin y$$

$$\Rightarrow \frac{dx}{dy} = \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$f^{-1}(x) = \sin^{-1}(x)$

Method 2: Using $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$

$$(\sin^{-1})'(x) = \frac{1}{\cos(\sin^{-1}(x))}$$

$$= \frac{1}{\sqrt{1-\sin^2(\sin^{-1}(x))}} = \frac{1}{\sqrt{1-x^2}}$$

★

Example 3: Find the derivative of each of the following:

a) $y = \sin^{-1}(x^7 + 5x)$.

b) $y = \ln(x)\cos^{-1}(x)$.

c) $y = \dfrac{\tan^{-1}(x)}{6x}$.

a) $y' = \dfrac{1}{\sqrt{1-(x^7+5x)^2}} \; (7x^6+5)$

b) $(uv)' = u'v + v'u$
$= \frac{1}{x} \cdot \cos^{-1}(x) + \left(-\frac{1}{\sqrt{1-x^2}}\right)\ln x.$

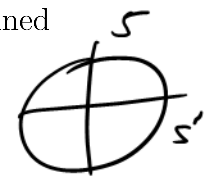
c) $\left(\frac{u}{v}\right)' = \dfrac{vu' - uv'}{v^2} = \dfrac{\text{both}}{\text{both}^2}$
$= \dfrac{(6x)\left(\frac{1}{1+x^2}\right) - \tan^{-1}(x)(6)}{(6x)^2}.$

★

Parametrically Defined Curves

We sometimes define relations between x and y in terms of a third party called a parameter. The advantage of this approach is that all concerns become focused on a single object, the parameter rather than a multiplicity of other variables. You have already seen the power of parameters in the algebra strand where lines and planes in space are defined in parametric vector form.

Example 4: Prove that the circle $x^2 + y^2 = 25$ can be written parametrically as



$$\begin{cases} x = 5\cos(\theta) \\ y = 5\sin(\theta) \end{cases}$$

$x^2 + y^2 = 25\cos^2\theta + 25\sin^2\theta$
$= 25(1) = 25$

★

Other parametrically defines curves are:

Conic section	Cartesian equation	Parametric equation
Parabola	$4ay = x^2$	$x(t) = 2at$ $y(t) = at^2$
Circle	$x^2 + y^2 = a^2$	$x(t) = a\cos t$ $y(t) = a\sin t$
Ellipse	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	$x(t) = a\cos t$ $y(t) = b\sin t$
Hyperbola	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$x(t) = a\sec t$ $y(t) = b\tan t$

• Note that **any** function may be rewritten parametrically in many different ways.

Example 5: The function

$y = x^3 + 7$ may be expressed as $\begin{cases} x = t \\ y = t^3 + 7 \end{cases}$ or $\begin{cases} x = e^t \\ y = e^{3t} + 7 \end{cases}$

★

• Note also that it is often (but not always) possible to recover the Cartesian equation of a parametrically defined curve.

Example 6: Find the Cartesian equation of $\begin{cases} x = 3t - 1 \\ y = 9t^2 - 6t + 8 \end{cases}$ $\Rightarrow 3t = x+1 \Rightarrow t = \dfrac{(x+1)}{3}$

$$y = 9\left(\frac{x+1}{3}\right)^2 - 6\left(\frac{x+1}{3}\right) + 8$$

$$= (x+1)^2 - 2(x+1) + 8$$

$$= x^2 + \cancel{2x} + 1 - \cancel{2x} - 2 + 8$$

$$\boxed{y = x^2 + 7}$$

★ $y = x^2 + 7$ ★

Even though we do not have a direct relationship, it is still possible to find $\frac{dy}{dx}$ through the use of parametric differentiation.

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

Example 7: Suppose that a curve $\mathcal{C}$ is defined as $x = t^2 - 1$ and $y = \frac{3}{t}$.

a) Find a Cartesian relation between x and y.

b) Which point on the curve corresponds to $t = 6$?

c) Using parametric differentiation find $\frac{dy}{dx}$ at the point $(8, 1)$?

a) $y = \frac{3}{t} \Rightarrow t = \frac{3}{y}$

$\Rightarrow x = \left(\frac{3}{y}\right)^2 - 1 = \frac{9}{y^2} - 1$

$\frac{9}{y^2} = x + 1 \Rightarrow \frac{y^2}{9} = \frac{1}{x+1} \Rightarrow \boxed{y^2 = \frac{9}{x+1}}$

b) $x = 6^2 - 1 = 35,\ y = \frac{3}{6} = \frac{1}{2} \Rightarrow (35, \frac{1}{2})$

$t = 6$

c) $(8, 1) \rightarrow ??$ $\frac{3}{t} = 1 \Rightarrow t = 3$ ✓

$t = 3$

$x = t^2 - 1 = 9 - 1 = 8$ ✓

$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \left(-\frac{3}{t^2}\right) \Big/ 2t = -\frac{3}{2}\frac{1}{t^3}$

$= -\frac{3}{2}\frac{1}{3^3} = \boxed{-\frac{1}{18}}$

★ $a)\ y^2 = \frac{9}{x+1}$ $b)\ (35, \frac{1}{2})$ $c)\ -\frac{1}{18}$ ★

Example 8: Suppose that a curve is define parametrically by

$$x = t + \cos(t) \quad \text{and} \quad y = t^4 + 2t + 5.$$

a) Find a Cartesian relation between x and y. $\times\times$

b) Find the equation of the tangent to the curve at the point $(1, 5)$.

c) What is $\dfrac{d^2y}{dx^2}$?

b) $(1,5) \Leftrightarrow t = 0$

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right) \Big/ \frac{dx}{dt} = \frac{4t^3+2}{1-\sin t} = \frac{2}{1} = 2 = m_{\text{tangent}}$$

$$y - y_1 = m(x - x_1) \Rightarrow y - 5 = 2(x-1)$$

$$y = 2x+3$$

c) $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{4t^3+2}{1-\sin t}\right)$$

$$= \frac{d}{dt}\left(\frac{4t^3+2}{1-\sin t}\right)\frac{dt}{dx} = \frac{d}{dt}\left(\frac{4t^3+2}{1-\sin t}\right) \Big/ \frac{dx}{dt}$$

$$= \left(\frac{(1-\sin t)(12t^2) - (4t^3+2)(-\cos t)}{(1-\sin t)^2}\right) \Big/ (1-\sin t)$$

$$= \frac{(1-\sin t)(12t^2) + \cos t\,(4t^3+2)}{(1-\sin t)^3}$$

★ *a)* *Impossible* *b)* $y = 2x + 3$ *c)* $\dfrac{12t^2(1 - \sin(t)) + (4t^3 + 2)\cos(t)}{(1 - \sin(t))^3}$ ★