Recall $\ln(x) = \int_1^x \frac{1}{t}\,dt$

LECTURE 21

The exponential function e^x

The inverse of $\ln(x)$ is e^x.

$$e^{\ln(x)} = \ln(e^x) = x.$$

$$\frac{d}{dx}e^x = e^x.$$

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C.$$

$$e^x e^y = e^{x+y}.$$

$$\frac{d}{dx}(a^x) = a^x \ln(a).$$

$$\int a^x\,dx = \frac{1}{\ln(a)}a^x.$$

You will recall from the previous lecture that the natural log function $y = \ln(x)$ is an increasing function. Hence it is $1-1$ and thus invertible. The inverse of $\ln(x)$ is without doubt the most important function in all of mathematics....the exponential function $y = e^x$.

The irrational real number e is approximately 2.71828 and can be defined in many ways:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

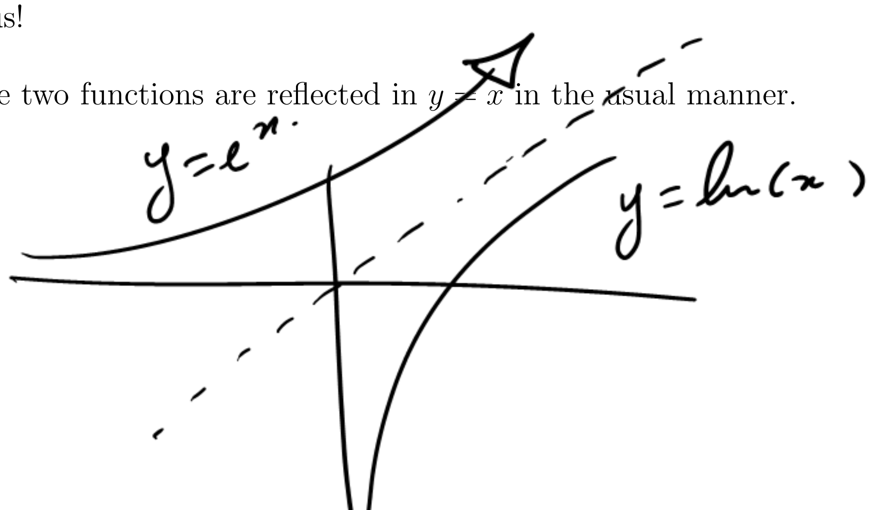
or

$$e = \lim_{n\longrightarrow\infty}(1 + \frac{1}{n})^n$$

What makes e^x such a fascinating function is the simple fact $\frac{d}{dx}e^x = e^x$. It is its own derivative! No other function has this remarkable property. The exponential function e^x is immune to calculus!

The graphs of the two functions are reflected in $y = x$ in the usual manner.

Sketch:



Observe that $\text{Dom}(\ln(x)) = (0, \infty) = \text{Range}(e^x)$ and $\text{Range}(\ln(x)) = \mathbb{R} = \text{Dom}(e^x)$.

Both functions are increasing however the exponential function grows with enormous strength while the natural log function increases very weakly.

Further properties of the two functions are:

a) $e^{\ln(x)} = x$. This is just $(f^{-1} \circ f)(x) = x$.

b) $\ln(e^x) = x$. This is just $(f \circ f^{-1})(x) = x$.

c) $\dfrac{d}{dx} e^x = e^x$.

d) $\dfrac{d}{dx} e^{f(x)} = e^{f(x)} f'(x)$.

e) $\displaystyle\int e^x \, dx = e^x$.

f) $\displaystyle\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C$.

g) $e^x e^y = e^{x+y}$.

Note that g) indicates that the inverse of $\ln(x)$ actually has something to do with exponentials!!

Proofs:

a) and b) This is just the definition of the inverse function!

c) We start with $\ln(e^x) = x$ and differentiate both sides with respect to x.

$$\ln(e^x) = x$$

$$\frac{1}{e^x} (e^x)' = 1$$

$$\Rightarrow (e^x)' = e^x.$$

$$\Rightarrow \frac{d}{dx}(e^x) = e^x$$

d) This is just the chain rule.

e) follows from c)

f) Exercise.

g) $e^{x+y} = e^{\ln(e^x) + \ln(e^y)} = e^{\ln(e^x e^y)} = e^x e^y$.

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Example 1:

a) Evaluate $\int_{\ln(2)}^{\ln(5)} e^{3x}\,dx$.

b) Solve $2^x = 9$.

c) Find $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

d) Find $\frac{d}{dx}(x^3 e^{5x})$.

$\ln(5)$

a) $\int_{\ln(2)}^{\ln(5)} e^{3x}\,dx = \left[\frac{1}{3}e^{3x}\right]_{\ln(2)}^{\ln(5)}$

$= \frac{1}{3}e^{3\ln 5} - \frac{1}{3}e^{3\ln(2)}$

$= \frac{1}{3}\cancel{e}^{\cancel{\ln}(5^3)} - \frac{1}{3}\cancel{e}^{\cancel{\ln}(2^3)}$

$= \frac{1}{3}(125) - \frac{1}{3}8 = \frac{117}{3} = \underline{\underline{39}}$

b) $2^x = 9 \Rightarrow \ln(2^x) = \ln(9)$

$\Rightarrow x\ln 2 = \ln 9 \Rightarrow x = \frac{\ln(9)}{\ln(2)}$

c) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ Let $u = x^{\frac{1}{2}} = \sqrt{x}$. $2du$ $\quad du = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}\,dx$.

$2du = \frac{dx}{\sqrt{x}}$.

$= \int e^u (2du)$

$= 2\int e^u\,du = 2e^u = \underline{2e^{\sqrt{x}} + C}$

d) $\frac{d}{dx}(x^3 e^{5x}) = 3x^2(e^{5x}) + (5e^{5x})(x^3)$

$(uv)' = u'v + vu'$

★ $a)$ 39 $b)$ $\frac{\ln(9)}{\ln(2)} \approx 3.17$ $c)$ $2e^{\sqrt{x}} + C$ $d)$ $e^{5x}\{3x^2 + 5x^3\}$ ★

The functions 2^x, 3^x and 7^x are also exponential functions. Why do we obsess about e^x ? Only e^x is equal to its own derivative! So what is the derivative of 3^x? To answer this question we use what is called logarithmic differentiation. This is simply taking the log of both sides before differentiating implicitly. This works well to eliminate troublesome exponentials.

Example 2: Find $\frac{d}{dx}(7^x)$.

$y = 7^x.$

$\Rightarrow \ln y = \ln 7^x = x\ln(7)$

$\underline{\frac{d}{dx} \text{ all}} \Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln(7)$

$\Rightarrow \frac{dy}{dx} = \ln(7)y = \ln(7)(7^x)$
$= 7^x\ln(7)$

★ $7^x \ln(7)$ ★

It follows from the same argument that

$$\frac{d}{dx}(a^x) = a^x \ln(a),$$

and hence after integrating both sides with respect to x

$$\int a^x \, dx = \frac{1}{\ln(a)} a^x.$$

Proof:

$\int \frac{d}{dx}(a^x) = \int a^x \ln(a) \, dx.$

$a^x = \ln(a) \int a^x \, dx.$

$\int a^x = \left(\frac{1}{\ln a}\right)(a^x)$ ★

Example 3: Use the above facts to find:

a) $\frac{d}{dx}(4^x) =$ $4^x \ln(4)$

b) $\frac{d}{dx}(e^x) =$ $e^x \ln(e) = e^x(1) = e^x$

c) $\frac{d}{dx}(e^5 + \ln 7) =$ 0

d) $\int 6^x \, dx =$ $\frac{1}{\ln(6)}(6^x) + C.$

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The process of logarithmic differentiation is a versatile tool, handy whenever exponents are blocking your path:

Example 4: Use logarithmic differentiation to find $\dfrac{dy}{dx}$ for $y = \dfrac{x\sqrt{x^2+1}}{x^2-1}$

$$\ln(y) = \ln\left(\frac{x\sqrt{x^2+1}}{x^2-1}\right)$$

$$= \ln x + \ln\sqrt{x^2+1} - \ln(x^2-1)$$

$$= \ln x + \ln(x^2+1)^{\frac{1}{2}} - \ln(x^2-1)$$

$$\ln y = \ln(x) + \tfrac{1}{2}\ln(x^2+1) - \ln(x^2-1)$$

$\frac{d}{dx}$ both sides

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2}\,\frac{1}{x^2+1}\cdot 2x - \frac{1}{x^2-1}\cdot 2x.$$

$$= \frac{1}{x} + \frac{x}{x^2+1} - \frac{2x}{x^2-1}$$

$$\frac{dy}{dx} = \left(\frac{1}{x} + \frac{x}{x^2+1}\ \frac{-2x}{x^2-1}\right)\left(\frac{x\sqrt{x^2+1}}{x^2-1}\right)$$

$$\bigstar \quad \frac{dy}{dx} = \frac{x\sqrt{x^2+1}}{x^2-1}\left\{\frac{1}{x} + \frac{x}{x^2+1} - \frac{2x}{x^2-1}\right\} \quad \bigstar$$

Example 5: Find $\dfrac{dy}{dx}$ for $y = x^x$

$$y = x^{x}.$$

$$\ln y = \ln(x^x) = x\ln(x)$$

$$\frac{1}{y}\frac{dy}{dx} = 1\ln(x) + \frac{1}{x}\cdot x.$$

$$= 1 + \ln(x)$$

$$\therefore \frac{dy}{dx} = (1+\ln(x))y$$

$$= (1+\ln(x))(x^x)$$

$$\bigstar \quad y = x^x\{1 + \ln(x)\} \quad \bigstar$$

The same log tricks can also be used on limits. Simply give the limit a name and then log both sides:

Example 6: Evaluate $\lim\limits_{x \longrightarrow \infty} x^{\frac{1}{x}}$

$$L = \lim_{x\to\infty} x^{\frac{1}{x}} = \text{"}\infty^0\text{"}$$

$$\ln(L) = \lim_{x\to\infty} \ln(x^{\frac{1}{x}})$$

$$= \lim_{x\to\infty} \frac{1}{x}\ln x = \lim_{x\to\infty} \frac{\ln x}{x}$$

$$= \text{"}\frac{\infty}{\infty}\text{"}$$

$$\stackrel{l'h}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \underline{\underline{0}}$$

$$\therefore \ln(L) = 0$$

$$\therefore L = e^0 = \underline{\underline{1}}$$

★ 1 ★

Example 7: Evaluate $\lim\limits_{x \longrightarrow \infty} (1+\frac{1}{x})^x$

$$L = \lim_{x\to\infty} (1+\tfrac{1}{x})^x \Rightarrow \ln(L) = \lim_{x\to\infty} \ln(1+\tfrac{1}{x})^x$$

$$= \lim_{x\to\infty} x\ln(1+\tfrac{1}{x}) = \infty . 0 .$$

$$= \lim_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{(\frac{1}{x})} = \text{"}\frac{0}{0}\text{"}$$

$$\stackrel{l'h}{=} \lim_{x\to\infty} \frac{\frac{1}{(1+\frac{1}{x})}\cancel{(-\frac{1}{x^2})}}{\cancel{(-\frac{1}{x^2})}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = \underline{\underline{1}}$$

$$\ln(L) = 1,$$

$$\Rightarrow L = e^1 = \underline{\underline{e}}$$

★ e ★