MATH1131 Mathematics 1A – Algebra

Lecture 5: Lengths and the Dot Product

Lecturer: Sean Gardiner – sean.gardiner@unsw.edu.au

Based on slides by Jonathan Kress

Length in n dimensions

Recall the length of $\mathbf{a} \in \mathbb{R}^n$ with

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n$$

is defined to be

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}\ .$$

If $|\mathbf{a}| = 1$, we say that $\mathbf{a}$ is a unit vector.

Properties

1. $|\mathbf{a}|$ is a real number.
2. $|\mathbf{a}| \geq 0$.
3. $|\mathbf{a}| = 0$ if and only if $\mathbf{a} = \mathbf{0}$.
4. $|\lambda\mathbf{a}| = |\lambda||\mathbf{a}|$ for all $\lambda \in \mathbb{R}$.

Proof of properties

Property 1 follows from the definition of $\sqrt{\cdot}$. Since the components $a_1, \dots, a_n$ are in $\mathbb{R}$, we have $a_1^2 + a_2^2 + \cdots + a_n^2 \geq 0$. Hence $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$ is defined and a real number.

In fact, the definition of $\sqrt{\cdot}$ says that $\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \geq 0$. This means $|\mathbf{a}| \geq 0$, which is Property 2.

For Property 3 we use that $\sqrt{x} = 0$ if and only if $x = 0$. Hence

$$\begin{aligned} |\mathbf{a}| = 0 \quad &\iff \quad a_1^2 + a_2^2 + \cdots + a_n^2 = 0 \\ &\iff \quad a_1 = a_2 = \cdots = a_n = 0 \\ &\iff \quad \mathbf{a} = \mathbf{0} \end{aligned}$$

Proof of properties (continued)

For Property 4, take $\lambda \in \mathbb{R}$. Since

$$\lambda \mathbf{a} = \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \\ \vdots \\ \lambda a_n \end{pmatrix}$$

we have

$$\begin{aligned} |\lambda \mathbf{a}| &= \sqrt{(\lambda a_1)^2 + (\lambda a_2)^2 + \cdots + (\lambda a_n)^2} \\ &= \sqrt{\lambda^2 (a_1^2 + a_2^2 + \cdots + a_n^2)} \\ &= \sqrt{\lambda^2}\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \\ &= |\lambda||\mathbf{a}|. \end{aligned}$$

$\square$

Length in n dimensions - Examples

Example

Find the two unit vectors parallel to $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 6 \end{pmatrix}$.

Here $|\mathbf{b}| = \sqrt{1^2 + 3^2 + 2^2 + 6^2} = \sqrt{50} = 5\sqrt{2}$.

So the unit vector $\hat{\mathbf{b}} = \dfrac{1}{|\mathbf{b}|}\mathbf{b} = \dfrac{1}{5\sqrt{2}} \begin{pmatrix} 1 \\ 3 \\ 2 \\ 6 \end{pmatrix}$.

The second unit vector that is parallel to $\mathbf{b}$ is $-\hat{\mathbf{b}}$, that is,

$$-\frac{1}{5\sqrt{2}} \begin{pmatrix} 1 \\ 3 \\ 2 \\ 6 \end{pmatrix}.$$

Example

Find a vector of length 5 parallel to $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Here $|\mathbf{w}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$.

So the unit vector $\hat{\mathbf{w}} = \dfrac{1}{|\mathbf{w}|}\mathbf{w} = \dfrac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

A vector parallel to $\mathbf{w}$ with length 5 is therefore given by

$$5\hat{\mathbf{w}} = \frac{5}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Dot product

Definition

The dot product (or scalar product) of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ with

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{k=1}^{n} a_k b_k.$$

Dot product

Examples

$$\begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}3\\4\end{pmatrix} = 1\times 3+2\times 4 = 11$$

$$\begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\2\end{pmatrix} = 1\times 1+2\times 2 = 5$$

$$\begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}-2\\1\end{pmatrix} = 1\times(-2)+2\times 1 = 0$$

$$\begin{pmatrix}3\\0\\4\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} = 3\times 1+0\times 1+4\times 1 = 7$$

Properties

Properties of the dot product

For all vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^n$ and scalars $\lambda \in \mathbb{R}$,

- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$, so $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$
- $\mathbf{a} \cdot \mathbf{b} \in \mathbb{R}$
- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (commutative law)
- $\mathbf{a} \cdot (\lambda \mathbf{b}) = \lambda(\mathbf{a} \cdot \mathbf{b})$ (associative law of scalar multiplication)
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (distributive law)

Exercise

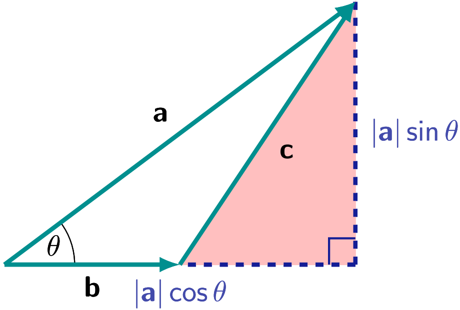
Prove these laws.

Note that the dot product is not itself associative, since an expression like $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ has no sensible meaning.

Cosine rule for triangles

Consider a triangle in $\mathbb{R}^n$ with sides $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c} = \mathbf{a} - \mathbf{b}$.

Let θ be the smaller angle between $\mathbf{a}$ and $\mathbf{b}$.



Applying Pythagoras' Theorem to the shaded triangle:

$$\begin{aligned}
|\mathbf{c}|^2 &= \Big(|\mathbf{a}|\sin\theta\Big)^2 + \Big(|\mathbf{a}|\cos\theta - |\mathbf{b}|\Big)^2 \\
&= |\mathbf{a}|^2\sin^2\theta + |\mathbf{a}|^2\cos^2\theta + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta \\
&= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta
\end{aligned}$$

This is called the cosine rule.

Geometric interpretation

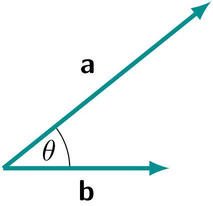
From the cosine rule for triangles, replacing **c** by **a** − **b** gives

$$
\begin{aligned}
&2|\mathbf{a}||\mathbf{b}|\cos\theta \\
&\quad = |\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2 \\
&\quad = |\mathbf{a}|^2 + |\mathbf{b}|^2 - \Big((a_1-b_1)^2 + \cdots + (a_n-b_n)^2\Big) \\
&\quad = |\mathbf{a}|^2 + |\mathbf{b}|^2 - \Big((a_1^2+b_1^2-2a_1b_1) + \cdots + (a_n^2+b_n^2-2a_nb_n)\Big) \\
&\quad = |\mathbf{a}|^2 + |\mathbf{b}|^2 - \Big(|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a}\cdot\mathbf{b}\Big) \\
&\quad = 2\mathbf{a}\cdot\mathbf{b}
\end{aligned}
$$

So $\quad \mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

and $\quad \cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

where θ is the smaller angle between vectors **a** and **b** joined tail-to-tail:



Dot product - Examples

Example

Use

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

to find the smaller angle θ between $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$\begin{aligned} \cos\theta &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \\ &= \frac{2 \times 1 + 0 \times 1}{\sqrt{2^2 + 0^2}\sqrt{1^2 + 1^2}} \\ &= \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}. \end{aligned}$$

So the angle θ is given by $\arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$.

Dot product - Examples

Example

Find the smaller angle θ between $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ 3 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$.

$$\begin{aligned}
\cos\theta &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \\
&= \frac{2\times 1 + 0\times 1 + 3\times 0 + (-1)\times 2}{\sqrt{2^2+0^2+3^2+(-1)^2}\sqrt{1^2+1^2+0^2+2^2}} \\
&= \frac{0}{\sqrt{14}\sqrt{6}} = 0.
\end{aligned}$$

So the angle θ is given by $\arccos(0) = \dfrac{\pi}{2}$.

Theorem

For any two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$

$$\mathbf{a} \perp \mathbf{b} \text{ if and only if } \mathbf{a} \cdot \mathbf{b} = 0.$$

Proof

Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero vectors in $\mathbb{R}^n$, and let θ be the smaller angle between $\mathbf{a}$ and $\mathbf{b}$.

If $\mathbf{a} \perp \mathbf{b}$, that is, if $\theta = \frac{\pi}{2}$, then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \frac{\pi}{2} = 0$.

Conversely, if $\mathbf{a} \cdot \mathbf{b} = 0$, then $\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = 0$. Hence $\theta = \frac{\pi}{2}$ and $\mathbf{a} \perp \mathbf{b}$. $\square$

This is where we need that $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors.

Theorem (Cauchy-Schwarz inequality)

For any two vectors **a** and **b** in $\mathbb{R}^n$

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$$

Proof

Let **a** and **b** be two vectors in $\mathbb{R}^n$, and let θ be the smaller angle between **a** and **b**.

Then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$.

Since $-1 \leq \cos\theta \leq 1$, it follows that

$$-|\mathbf{a}||\mathbf{b}| \leq \mathbf{a} \cdot \mathbf{b} \leq |\mathbf{a}||\mathbf{b}|,$$

which means $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$. $\square$

Triangle inequality

Triangle inequality (Minkowski's inequality)

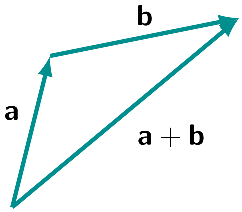
For any two vectors **a** and **b** in $\mathbb{R}^n$

$$|\mathbf{a}+\mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

This inequality is commonly known as the triangle inequality since we can illustrate it as follows:

When we add two vectors **a** and **b** in $\mathbb{R}^n$ we get this triangle.

Look at the lengths of the sides of the triangle.



The distance travelled along **a** and then **b** can never be shorter than the distance travelled along $\mathbf{a}+\mathbf{b}$, and will only be equal in distance when **a** and **b** point in the same direction.

Proofs with dot products

Triangle inequality

For any two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$

$$|\mathbf{a}+\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$$

Proof

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Using the Cauchy-Schwarz inequality,

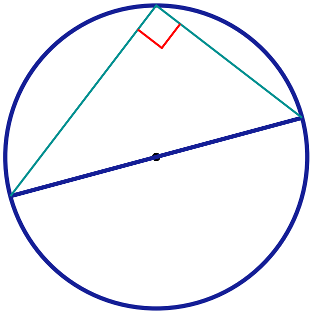
$$\begin{aligned}
|\mathbf{a}+\mathbf{b}|^2 &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) \\
&= \mathbf{a}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} \\
&= |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}\cdot\mathbf{b} \\
&\le |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2|\mathbf{a}||\mathbf{b}| \\
&= (|\mathbf{a}| + |\mathbf{b}|)^2.
\end{aligned}$$

Taking square roots of both sides gives the claim. $\square$

Proofs with dot products

Theorem

The angle subtended by a diameter at the circumference of a circle is a right angle.

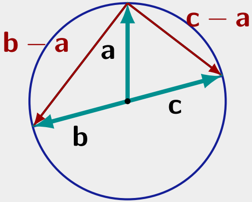


Proofs with dot products

Theorem

The angle subtended by a diameter at the circumference of a circle is a right angle.

Proof



Let **a**, **b** and **c** be the vectors as shown in the diagram. We need to show that

$$(\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a}) = 0$$

as this means $(\mathbf{b}-\mathbf{a}) \perp (\mathbf{c}-\mathbf{a})$.

Now, $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$ and $\mathbf{c} = -\mathbf{b}$. Hence

$$\begin{aligned}
(\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a}) &= \mathbf{b}\cdot\mathbf{c} - \mathbf{b}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{c} + \mathbf{a}\cdot\mathbf{a} \\
&= -\mathbf{b}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{a} \\
&= -|\mathbf{b}|^2 + |\mathbf{a}|^2 \\
&= 0,
\end{aligned}$$

which gives the claim. $\square$