School of Mathematics and Statistics
Math1131-Algebra

Lec02:Algebraic Vectors and $\mathbb{R}^n$

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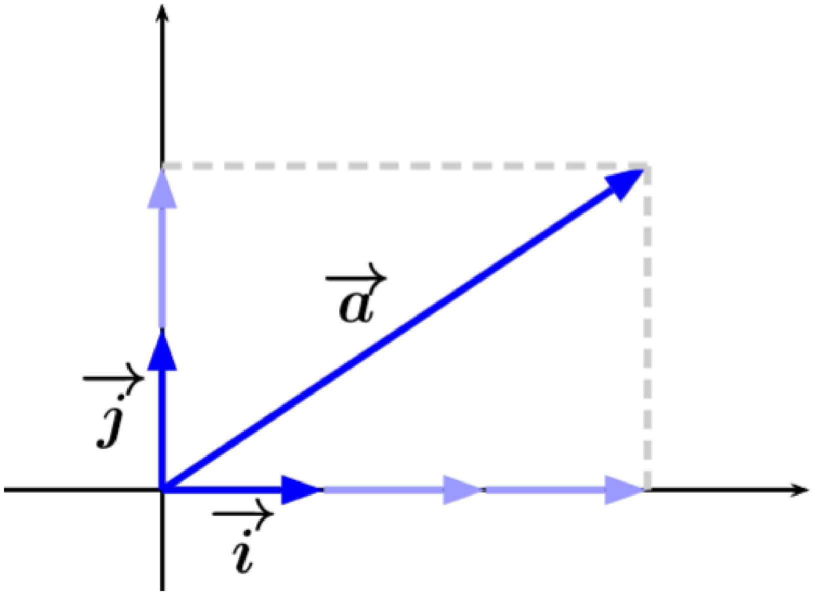
Algebraic vectors in two dimensions



$$\overrightarrow{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\overrightarrow{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\overrightarrow{a} = 3\overrightarrow{i} + 2\overrightarrow{j} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \qquad |\overrightarrow{a}| = \sqrt{3^2 + 2^2} = \sqrt{13}.$$

$\overrightarrow{i}$ and $\overrightarrow{j}$ are the standard basis vectors in $\mathbb{R}^2$.

Algebraic vectors in two dimensions

An *algebraic vector* $\overrightarrow{\boldsymbol{x}} \in \mathbb{R}^2$ is an ordered pair of real numbers, called components or coordinates, x_1 and x_2 written

$$\overrightarrow{\boldsymbol{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Suppose also $\overrightarrow{\boldsymbol{y}} \in \mathbb{R}^2$ with

$$\overrightarrow{\boldsymbol{y}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

and $\lambda \in \mathbb{R}$ is a scalar. Then

$$\overrightarrow{\boldsymbol{x}} + \overrightarrow{\boldsymbol{y}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$$

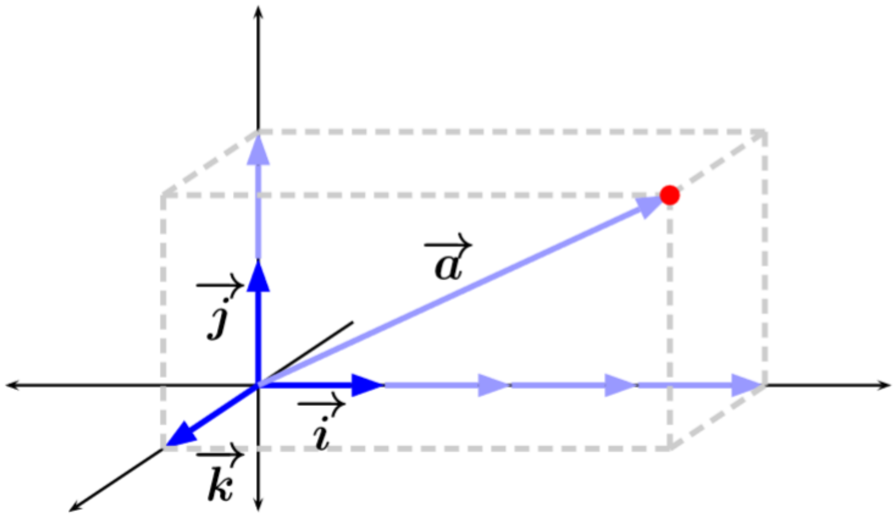
and

$$\lambda \overrightarrow{\boldsymbol{x}} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}.$$

Algebraic vectors in two dimensions

- We add and scale component by component
- Two vectors are equal if their components are equal.

Algebraic vectors in three dimensions



$$\overrightarrow{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\overrightarrow{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\overrightarrow{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\overrightarrow{a} = 4\overrightarrow{i} + 2\overrightarrow{j} + \overrightarrow{k} = \begin{pmatrix}4\\2\\1\end{pmatrix}$$

$$|\overrightarrow{a}| = \sqrt{4^2 + 2^2 + 1^2} = \sqrt{21}.$$

$\overrightarrow{i}$, $\overrightarrow{j}$ and $\overrightarrow{k}$ are the standard basis vectors in $\mathbb{R}^3$.

Algebraic vectors in three dimensions

Consider $\overrightarrow{\boldsymbol{x}} \in \mathbb{R}^3$ and $\overrightarrow{\boldsymbol{y}} \in \mathbb{R}^3$ written in components

$$\overrightarrow{\boldsymbol{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \text{and} \qquad \overrightarrow{\boldsymbol{y}} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

and $\lambda \in \mathbb{R}$ is a scalar. Then

$$\overrightarrow{\boldsymbol{x}} + \overrightarrow{\boldsymbol{y}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}$$

and

$$\lambda \overrightarrow{\boldsymbol{x}} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{pmatrix}.$$

Algebraic vectors in three dimensions

- We add and scale component by component
- Two vectors are equal if their components are equal.

Algebraic vectors in four dimensions

Consider $\overrightarrow{\boldsymbol{x}} \in \mathbb{R}^4$ and $\overrightarrow{\boldsymbol{y}} \in \mathbb{R}^4$ written in components

$$\overrightarrow{\boldsymbol{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad \text{and} \qquad \overrightarrow{\boldsymbol{y}} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

and $\lambda \in \mathbb{R}$ is a scalar. Then

$$\overrightarrow{\boldsymbol{x}} + \overrightarrow{\boldsymbol{y}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 \end{pmatrix}$$

and

$$\lambda \overrightarrow{\boldsymbol{x}} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \\ \lambda x_4 \end{pmatrix}.$$

Algebraic vectors in n dimensions

Consider $\overrightarrow{\boldsymbol{x}} \in \mathbb{R}^n$ and $\overrightarrow{\boldsymbol{y}} \in \mathbb{R}^n$ written in components

$$\overrightarrow{\boldsymbol{x}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \text{and} \qquad \overrightarrow{\boldsymbol{y}} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

and $\lambda \in \mathbb{R}$ is a scalar. Then

$$\overrightarrow{\boldsymbol{x}} + \overrightarrow{\boldsymbol{y}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

and

$$\lambda \overrightarrow{\boldsymbol{x}} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}.$$

Algebraic vectors in n dimensions

- We add and scale component by component
- Two vectors are equal if their components are equal.

Standard basis vectors in n dimensions

The standard basis vectors in $\mathbb{R}^n$ is

$$\overrightarrow{\boldsymbol{e_1}} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \overrightarrow{\boldsymbol{e_2}} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \quad \overrightarrow{\boldsymbol{e_n}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\overrightarrow{\boldsymbol{e_1}} + x_2\overrightarrow{\boldsymbol{e_2}} + \cdots + x_n\overrightarrow{\boldsymbol{e_n}}.$$

Eg, in three dimensions, $\overrightarrow{\boldsymbol{e_1}} = \overrightarrow{\boldsymbol{i}}$, $\overrightarrow{\boldsymbol{e_2}} = \overrightarrow{\boldsymbol{j}}$ and $\overrightarrow{\boldsymbol{e_3}} = \overrightarrow{\boldsymbol{k}}$.

Length in n dimensions

- The length of $\overrightarrow{\boldsymbol{a}} \in \mathbb{R}^n$ with

$$\overrightarrow{\boldsymbol{a}} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1 \overrightarrow{\boldsymbol{e_1}} + a_2 \overrightarrow{\boldsymbol{e_2}} + \cdots + a_n \overrightarrow{\boldsymbol{e_n}}$$

  is defined to be

$$|\overrightarrow{\boldsymbol{a}}| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

- If $|\overrightarrow{\boldsymbol{a}}| = 1$ we say that $\overrightarrow{\boldsymbol{a}}$ is a unit vector.

  For any nonzero vector $\overrightarrow{\boldsymbol{a}} \in \mathbb{R}^n$,

$$\hat{\boldsymbol{a}} = \frac{1}{|\overrightarrow{\boldsymbol{a}}|} \overrightarrow{\boldsymbol{a}}$$

  is a unit vector in the same direction as $\overrightarrow{\boldsymbol{a}}$.

Vectors in $\mathbb{R}^4$

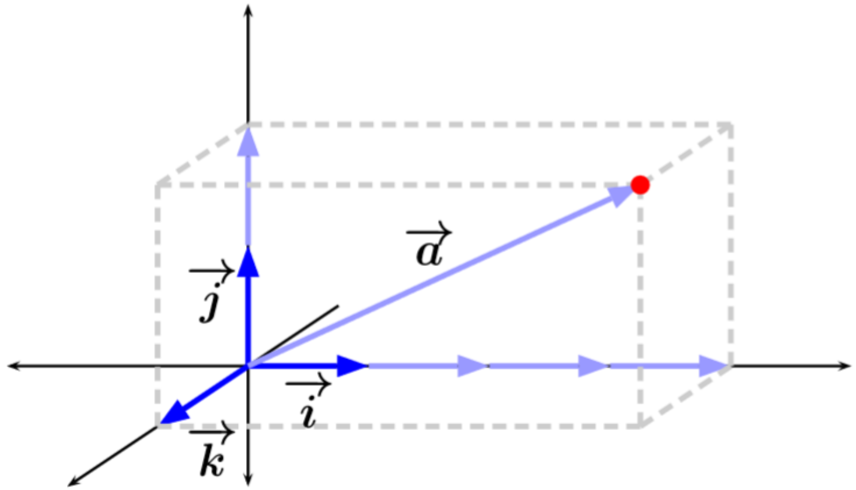
Example 1. Let $\overrightarrow{\boldsymbol{u}}$ be the vector defined by $\overrightarrow{\boldsymbol{u}} = \begin{pmatrix} -1 \\ 5 \\ -3 \\ 1 \end{pmatrix}$

1. Find the length of $\overrightarrow{\boldsymbol{u}}$.
2. Find two unit vectors parallel to $\overrightarrow{\boldsymbol{u}}$.

Points and Vectors

We write vectors as columns and points as rows (with commas).

If A is the point $(4, 2, 1)$ in $\mathbb{R}^3$, we write is $A(4, 2, 1)$ and its position vector is
$\overrightarrow{\boldsymbol{a}} = \overrightarrow{OA} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$.



Points and vectors in Maple

The notation for the vector $\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$ in Maple is

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<4,2,1>
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This is also used in the online tutorials.

The notation for a Maple list of numbers $4, 2, 1$ is

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[4,2,1]
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This is used in the online tutorials to represent the point $(4, 2, 1)$.

Although both points and vectors can be represented by elements of $\mathbb{R}^n$, they have different purposes and properties and so we use different notation.

Algebraic vector examples

Example 2. Let $\overrightarrow{\boldsymbol{v}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\overrightarrow{\boldsymbol{w}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \in \mathbb{R}^2$. Find $\overrightarrow{\boldsymbol{v}} + \overrightarrow{\boldsymbol{w}}$ and $3\overrightarrow{\boldsymbol{w}}$.

Algebraic Vector Examples

Example 3. Let $A(2, 0, -3)$ and $B(6, 7, 1)$ be two points in $\mathbb{R}^3$ and let M be their midpoint. Find $\overrightarrow{OM}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ and also by just taking the average of their components.

Collinear points

How to tell if points are collinear (= on the same line)

The points A, B and C are collinear if and only if $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel.

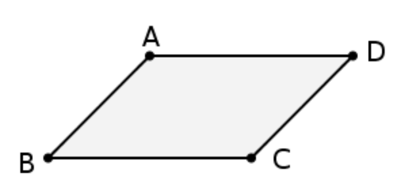
Example 4.

Are the points $A(1, 2, 3, 1)$, $B(1, -2, 3, 2)$ and $C(1, -10, 3, 4)$ collinear?

Parallelograms

Vectors and Parallelograms

$ABCD$ is a parallelogram
if and only if $\overrightarrow{AB} = \overrightarrow{DC}$



Example 5. Suppose that $A(2, 3, -1, 2)$, $B(2, 4, -1, -2)$ and $C(-1, -2, 1, 0)$ are 3 points in $\mathbb{R}^4$. Find the coordinates of the point D such that $ABCD$ is a parallelogram.