§4 Linear Equations and Matrices (2020T1: W7-Tu-We, W8-Tu-We-Th)

- **A linear equation in one unknown.**

$$ax = b, \qquad a, b \in \mathbb{R}.$$

  - If $a \neq 0$ then there is *a unique solution* $x = \frac{b}{a}$.
  - If $a = 0$ and $b = 0$ then there is *an infinite number of solutions* $x \in \mathbb{R}$.
  - If $a = 0$ and $b \neq 0$ then there is *no solution*.

- **A linear equation in two unknowns.**

$$a_1x_1 + a_2x_2 = b, \qquad a_1, a_2, b \in \mathbb{R}.$$

  - If $a_1 \neq 0$ then there is *an infinite number of solutions*:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{b}{a_1} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -\frac{a_2}{a_1} \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R} \qquad \text{. . . a line in } \mathbb{R}^2.$$

  - If $a_2 \neq 0$ then there is *an infinite number of solutions*:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{b}{a_2} \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -\frac{a_1}{a_2} \end{pmatrix}, \quad \lambda \in \mathbb{R} \qquad \text{. . . a line in } \mathbb{R}^2.$$

  - If $a_1 = a_2 = 0$ and $b = 0$ then there is *an infinite number of solutions*:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R} \qquad \text{. . . the entire } \mathbb{R}^2.$$

  - If $a_1 = a_2 = 0$ and $b \neq 0$ then there is *no solution*.

- **A system of two linear equations in two unknowns.** Geometrically, this represents the *points of intersection of two lines in* $\mathbb{R}^2$.
  - No solution $\Rightarrow$ two lines are parallel and do not intersect
  - A unique solution $\Rightarrow$ two lines intersect at one point
  - An infinite number of solutions $\Rightarrow$ two lines coincide

Example.

$$\begin{cases} 2x_1 - 3x_2 = 7 \\ -4x_1 + 6x_2 = 3 \end{cases} \xrightarrow{(2)\leftarrow(2)+2\times(1)} \begin{cases} 2x_1 - 3x_2 = 7 \\ 0x_1 + 0x_2 = 17 \end{cases}$$

The system has no solution. The two equations are said to be *inconsistent*.

Example.

$$\begin{cases} x_1 + x_2 = 1 \\ 2x_1 - 3x_2 = 7 \end{cases} \xrightarrow{(2)\leftarrow(2)-2\times(1)} \begin{cases} x_1 + x_2 = 1 \\ \quad - 5x_2 = 5 \end{cases}$$

Thus

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Two lines intersect at one point.

Example.

$$\begin{cases} 2x_1 - 3x_2 = 7 \\ x_1 - \dfrac{3}{2}x_2 = \dfrac{7}{2} \end{cases} \xrightarrow{(2)\leftarrow(2)-\frac{1}{2}\times(1)} \begin{cases} 2x_1 - 3x_2 = 7 \\ 0x_1 + 0x_2 = 0 \end{cases}$$

Let $x_2 = \lambda$. Then

$$x_1 = \frac{7+3x_2}{2} = \frac{7}{2} + \frac{3}{2}\lambda.$$

Thus

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{7}{2} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Two lines coincide.

- **A system of two linear equations in three unknowns.** Geometrically, this represents the *points of intersection of two planes in* $\mathbb{R}^3$.
  - No solution $\Rightarrow$ two planes are parallel and do not intersect
  - An infinite number of solutions $\Rightarrow$ $\begin{cases} \text{two planes intersect in one line} \\ \text{two planes coincide} \end{cases}$

Example.

$$\begin{cases} 2x_1 - 3x_2 + x_3 = 20 \\ 4x_1 - 6x_2 + 2x_3 = 10 \end{cases} \xrightarrow{(2)\leftarrow(2)-2\times(1)} \begin{cases} 2x_1 - 3x_2 + x_3 = 20 \\ 0x_1 + 0x_2 + 0x_3 = -30 \end{cases}$$

There is no solution. The two planes are parallel and do not intersect.

Example.

$$\begin{cases} 2x_1 - 3x_2 + x_3 = 20 \\ 4x_1 + 2x_2 - 3x_3 = 34 \end{cases} \xrightarrow{(2)\leftarrow(2)-2\times(1)} \begin{cases} 2x_1 - 3x_2 + x_3 = 20 \\ 8x_2 - 5x_3 = -6 \end{cases}$$

Let $x_3 = \lambda$. Then

$$x_2 = \frac{-6+5x_3}{8} = -\frac{3}{4} + \frac{5}{8}\lambda,$$
$$x_1 = \frac{20+3x_2-x_3}{2} = 10 + \frac{3}{2}\left(-\frac{3}{4}+\frac{5}{8}\lambda\right) - \frac{1}{2}\lambda = \frac{71}{8} + \frac{7}{16}\lambda.$$

Thus

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{71}{8} \\ -\frac{3}{4} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{7}{16} \\ \frac{5}{8} \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Two planes intersect in one line.

Example.

$$\begin{cases} 3x_1 + x_2 - 2x_3 = 9 \\ -2x_1 - \frac{2}{3}x_2 + \frac{4}{3}x_3 = -6 \end{cases} \xrightarrow{(2)\leftarrow(2)+\frac{2}{3}\times(1)} \begin{cases} 3x_1 + x_2 - 2x_3 = 9 \\ 0x_1 + 0x_2 + 0x_3 = 0 \end{cases}$$

Let $x_2 = \lambda_1$ and $x_3 = \lambda_2$. Then

$$x_1 = \frac{9 - x_2 + 2x_3}{3} = 3 - \frac{1}{3}\lambda_1 + \frac{2}{3}\lambda_2.$$

Thus

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} \frac{2}{3} \\ 0 \\ 1 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

Two planes coincide.

Systems of linear equations.

- A *system of m linear equations in n variables* (or *unknowns*) is of the form

$$\left\{\begin{array}{ccccccccc} a_{11}\,x_1 & + & a_{12}\,x_2 & + & \cdots & + & a_{1n}\,x_n & = & b_1 \\ a_{21}\,x_1 & + & a_{22}\,x_2 & + & \cdots & + & a_{2n}\,x_n & = & b_2 \\ & & & & & & & \vdots & \\ a_{m1}\,x_1 & + & a_{m2}\,x_2 & + & \cdots & + & a_{mn}\,x_n & = & b_m \end{array}\right. \qquad (*)$$

  where the m linear equations must be simultaneously satisfied.

- There are three alternative ways to write the system $(*)$:
  - Vector equation form

$$x_1 \underbrace{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}}_{\mathbf{a}_1} + x_2 \underbrace{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}}_{\mathbf{a}_2} + \cdots + x_n \underbrace{\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}}_{\mathbf{a}_n} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{\mathbf{b}}$$

  - Matrix equation form

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{\mathbf{b}}$$

  - Augmented matrix form

$$\underbrace{\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right)}_{A|\mathbf{b}}$$

- The system $(*)$ is said to be *homogeneous* if $b_1 = b_2 = \cdots = b_m = 0$.

Notes:

(i) Note the subscript convention:
- a_{ij} is the coefficient of x_j in the ith equation in $(*)$;
- a_{ij} is the entry in the ith row and jth column of the matrix A.

(ii) The columns of the $m \times n$ matrix A are precisely the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$.

(iii) "Matrix-vector multiplication" (see Chapter 5):

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} a_{11}\,x_1 + a_{12}\,x_2 + \cdots + a_{1n}\,x_n \\ a_{21}\,x_1 + a_{22}\,x_2 + \cdots + a_{2n}\,x_n \\ \vdots \\ a_{m1}\,x_1 + a_{m2}\,x_2 + \cdots + a_{mn}\,x_n \end{pmatrix}}_{A\mathbf{x}}$$

Exercise. Write the system of linear equations

$$\left\{ \begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 5x_3 & = & 1 \\ 3x_1 & + & x_2 & - & 7x_3 & = & -3 \\ 6x_1 & + & 5x_2 & + & x_3 & = & 0 \end{array} \right.$$

in vector equation form, matrix equation form, and augmented matrix form.

Exercise. Write down the system of linear equations associated with the augmented matrix

$$\left(\begin{array}{ccccc|c} 3 & -2 & 5 & 1 & 0 & 6 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 1 & 0 & 0 & 2 & 0 & -4 \end{array}\right),$$

and express it in vector equation form and matrix equation form.

Elementary row operations. Each equation in a linear system is represented by one row of the augmented matrix. There are three permissible elementary row operations:

- *Interchanging two rows*;
- *Adding/subtracting a multiple of one row to/from another*;
- *Multiplying one row by a non-zero number*.

When applied appropriately (see example below), these elementary row operations preserve the solution to the original linear system.

Example. To solve the linear system

$$\left\{\begin{array}{rcrcrcr} 3x_1 & + & x_2 & - & 6x_3 & = & -4 \\ x_1 & - & 2x_2 & + & 5x_3 & = & 1 \\ -2x_1 & + & 5x_2 & + & x_3 & = & 4 \end{array}\right.$$

we carry out elementary row operations on the augmented matrix as follows, recording the operations as we go:

$$\left(\begin{array}{rrr|r} 3 & 1 & -6 & -4 \\ 1 & -2 & 5 & 1 \\ -2 & 5 & 1 & 4 \end{array}\right) \qquad R_1 \leftrightarrow R_2$$

$$\left(\begin{array}{rrr|r} 1 & -2 & 5 & 1 \\ 3 & 1 & -6 & -4 \\ -2 & 5 & 1 & 4 \end{array}\right) \qquad \begin{array}{l} R_2 \leftarrow R_2 - 3\,R_1 \\ R_3 \leftarrow R_3 + 2\,R_1 \end{array}$$

$$\left(\begin{array}{rrr|r} 1 & -2 & 5 & 1 \\ 0 & 7 & -21 & -7 \\ 0 & 1 & 11 & 6 \end{array}\right) \qquad R_2 \leftarrow \tfrac{1}{7}\,R_2$$

$$\left(\begin{array}{rrr|r} 1 & -2 & 5 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & 11 & 6 \end{array}\right) \qquad R_3 \leftarrow R_3 - R_2$$

$$(\clubsuit) \qquad \left(\begin{array}{rrr|r} 1 & -2 & 5 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 14 & 7 \end{array}\right) \qquad R_3 \leftarrow \tfrac{1}{14}\,R_3$$

$$\left(\begin{array}{rrr|r} 1 & -2 & 5 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right) \qquad \begin{array}{l} R_1 \leftarrow R_1 - 5\,R_3 \\ R_2 \leftarrow R_2 + 3\,R_3 \end{array}$$

$$\left(\begin{array}{rrr|r} 1 & -2 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right) \qquad R_1 \leftarrow R_1 + 2\,R_2$$

$$\left(\begin{array}{rrr|r} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$$

Thus

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

This process is known as *Gaussian elimination* (details to come). Alternatively, we can obtain the solution from $(\clubsuit)$ using *back substitution* (details to come):

$$x_3 = \frac{7}{14} = \frac{1}{2}, \qquad x_2 = -1 + 3x_3 = \frac{1}{2} \qquad x_1 = 1 + 2x_2 - 5x_3 = -\frac{1}{2}.$$

Row-echelon form and reduced row-echelon form.

- In any matrix,
  - a *leading row* is a row which is not entirely zero;
  - the *leading entry* in a leading row is the leftmost non-zero entry;
  - a *leading column* is a column containing the leading entry for some row.
- A matrix is said to be in *row-echelon form* if and only if

  1. all non-leading rows are at the bottom of the matrix, and
  2. the leading entry in every row is to the right of the leading entries in the rows above.

- A matrix is said to be in *reduced row-echelon form* if and only

  1 & 2. it is in row-echelon form, and

  3. every leading entry is 1, and
  4. every leading entry is the only non-zero entry in its column.

Note. These definitions vary from one book to another!

Exercise. For each matrix, circle the leading entry in every leading row and determine whether or not it is in row-echelon form or reduced row-echelon form.

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 4 & 3 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right) \quad \left(\begin{array}{cccc|c} 11 & 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 3 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{cccc|c} 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 3 & -2 & 4 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{array}\right) \quad \left(\begin{array}{ccc|c} 0 & 2 & 1 & 1 \\ 3 & 1 & 0 & 4 \end{array}\right) \quad \left(\begin{array}{cccc|c} 5 & 1 & 0 & 3 & 4 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 1 & 3 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \quad \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Gaussian elimination. The essence of *Gaussian elimination* is to apply elementary row operations to bring a matrix into row-echelon form.

1. SELECT A PIVOT ELEMENT: from the leftmost column which is not all zeros, select a non-zero entry. This is called a *pivot element*. The row and the column containing the pivot element are called the *pivot row* and the *pivot column*.
2. SWAP THE TOP ROW WITH THE PIVOT ROW.
3. ELIMINATE ALL ENTRIES IN THE PIVOT COLUMN BELOW THE PIVOT ELEMENT: subtract an appropriate multiple of the pivot row from each row below the pivot row.
4. Repeat steps 1 to 3 on the sub-matrix strictly to the right of and strictly below the pivot element. Stop when the matrix is in row-echelon form.

Notes:

(i) The process is also known as *row reduction*.

(ii) When we carry out Gaussian elimination by hand, it is best to have 1 as pivot elements to avoid working with ugly fractions. This can be achieved by row swapping or by dividing the entire pivot row by the pivot element.

(iii) When we implement Gaussian elimination on a computer, we follow other "*pivoting*" rules to reduce rounding errors and improve numerical stability.

Example.

$$\left(\begin{array}{rrr|r} 3 & 1 & -6 & -4 \\ 1 & -2 & 5 & 1 \\ -2 & 5 & 1 & 4 \end{array}\right) \qquad \begin{array}{l} R_1 \leftrightarrow R_2 \\ \text{Make the pivot element } = 1 \end{array}$$

$$\left(\begin{array}{rrr|r} \boxed{1} & -2 & 5 & 1 \\ 3 & 1 & -6 & -4 \\ -2 & 5 & 1 & 4 \end{array}\right) \qquad \begin{array}{l} \text{Make entries below the pivot element } = 0 \\ R_2 \leftarrow R_2 - 3\,R_1 \\ R_3 \leftarrow R_3 + 2\,R_1 \end{array}$$

$$\left(\begin{array}{rrr|r} \boxed{1} & -2 & 5 & 1 \\ \boxed{0} & 7 & -21 & -7 \\ \boxed{0} & 1 & 11 & 6 \end{array}\right) \qquad \begin{array}{l} R_2 \leftarrow \frac{1}{7}\,R_2 \\ \text{Make the pivot element } = 1 \end{array}$$

$$\left(\begin{array}{rrr|r} 1 & -2 & 5 & 1 \\ 0 & \boxed{1} & -3 & -1 \\ 0 & 1 & 11 & 6 \end{array}\right) \qquad \begin{array}{l} \text{Make entries below the pivot element } = 0 \\ R_3 \leftarrow R_3 - R_2 \end{array}$$

$$\left(\begin{array}{rrr|r} 1 & -2 & 5 & 1 \\ 0 & \boxed{1} & -3 & -1 \\ 0 & \boxed{0} & 14 & 7 \end{array}\right)$$

Exercise. Row reduce the following augmented matrix into row-echelon form.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 1 & 3 \\ 1 & 3 & 3 & 2 \end{array}\right)$$

Exercise. Row reduce the following augmented matrix into row-echelon form.

$$\left(\begin{array}{ccccc|c} 0 & 0 & 0 & 3 & 2 & -13 \\ 0 & 3 & -3 & 3 & 3 & 6 \\ 0 & 0 & 0 & 1 & 3 & -2 \\ 0 & 6 & -6 & 6 & 3 & 9 \end{array}\right)$$

- **Deducing solubility from row-echelon form.** Suppose that an augmented matrix $(A|\mathbf{b})$ associated with a linear system has been transformed into an equivalent row-echelon form $(U|\mathbf{y})$.
  - A variable x_i is a *leading variable* if the ith column of U is a leading column.
  - The system has *no solution* if and only if $\mathbf{y}$ is a leading column. In this case, the linear equations in the system are said to be *inconsistent*.
  - If $\mathbf{y}$ is not a leading column, then the system has
    - a *unique solution* if and only if every variable is a leading variable.
    - *infinitely many solutions* if and only if there is at least one non-leading variable.

      In the latter case, every non-leading variable becomes a free parameter.
  - A *homogeneous* system $A\mathbf{x} = \mathbf{0}$ always has at least one solution, that is, $\mathbf{x} = \mathbf{0}$.

Example. Consider the following augmented matrix in row-echelon form

$$\left(\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The leading variables are x_2, x_4 and x_5. The right-hand column is not a leading column. Thus we have infinitely many solutions. The non-leading variables x_1 and x_3 are free parameters.

Exercise. Discuss the number of solutions associated with each of the following augmented matrices.

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 3 & -2 & 4 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 11 & 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 3 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right) \quad \left(\begin{array}{ccccc|c} 1 & 3 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{array}\right)$$

- **Back substitution.**
  - Assign a parameter to each non-leading variable.
  - Obtain an expression for the last leading variable, and then substitute this expression into the previous leading row to obtain an expression for the second last leading variable, and so on.

Example.

$$\left(\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The non-leading variables are x_1 and x_3. Let $x_1 = \lambda_1$ and $x_3 = \lambda_2$. Then

$$\begin{aligned} x_5 &= 1 \\ x_4 &= -2 - 3x_5 = -5 \\ x_2 &= 2 + x_3 - x_4 - x_5 = 6 + \lambda_2 \end{aligned}$$

Thus

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 0 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \lambda_1, \lambda_2 \in \mathbb{R}.$$

Exercise. Find the solution associated with the augmented matrix

$$\left(\begin{array}{ccccc|c} -5 & 0 & 4 & 4 & 1 & 0 \\ 0 & 0 & 2 & 4 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- **From row-echelon form to reduced row-echelon form.**
  - Start with the lowest leading row. Divide the entire row by the leading entry so that the leading entry becomes 1. Then subtract multiples of this row from higher rows to eliminate all entries above the leading 1.
  - Repeat this procedure with the next leading row, and so on.

Example.

$$\left(\begin{array}{ccc|c} 1 & -2 & 5 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 14 & 7 \end{array}\right) \qquad \begin{array}{l} \text{Make the lowest leading entry} = 1 \\ R_3 \leftarrow \frac{1}{14} R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 5 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & \boxed{1} & \frac{1}{2} \end{array}\right) \qquad \begin{array}{l} R_1 \leftarrow R_1 - 5\,R_3 \\ R_2 \leftarrow R_2 + 3\,R_3 \\ \text{Make entries above the leading entry} = 0 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & \boxed{0} & -\frac{3}{2} \\ 0 & 1 & \boxed{0} & \frac{1}{2} \\ 0 & 0 & \boxed{1} & \frac{1}{2} \end{array}\right) \qquad \begin{array}{l} R_1 \leftarrow R_1 + 2\,R_2 \\ \text{Make the next leading entry} = 1 \text{ (it is already 1)} \\ \text{Make entries above the leading entry} = 0 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & \boxed{0} & 0 & -\frac{1}{2} \\ 0 & \boxed{1} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$$

Exercise. Reduce the following augmented matrix into reduced row-echelon form.

$$\left(\begin{array}{ccccc|c} 2 & 2 & 0 & 3 & 2 & -4 \\ 0 & 3 & -3 & 2 & 3 & 5 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{array}\right)$$

Solving $A\mathbf{x} = \mathbf{b}$ for indeterminate $\mathbf{b}$. Suppose that a matrix A has been transformed into an equivalent row-echelon form U. Then for each $\mathbf{b}$, the linear system $A\mathbf{x} = \mathbf{b}$ has

- *at least one solution* if and only if U has no non-leading rows.
- *at most one solution* if and only if U has no non-leading columns.
- *exactly one solution* if and only if U has no non-leading rows and no non-leading columns.

Example. Consider the linear system

$$\begin{cases} 2x_1 - 3x_2 = b_1 & = 1b_1 + 0b_2 \\ -3x_1 + 4x_2 = b_2 & = 0b_1 + 1b_2 \end{cases}$$

The solution for x_1 and x_2 will be expressed in terms of linear combinations of b_1 and b_2. To make our lives easier, we keep the coefficients of b_1 and b_2 separate in the augmented matrix:

$$\left(\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ -3 & 4 & 0 & 1 \end{array}\right) \qquad R_1 \leftarrow \tfrac{1}{2} R_1$$

$$\left(\begin{array}{cc|cc} 1 & -\frac{3}{2} & \frac{1}{2} & 0 \\ -3 & 4 & 0 & 1 \end{array}\right) \qquad R_2 \leftarrow R_2 + 3R_1$$

$$\left(\begin{array}{cc|cc} 1 & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{3}{2} & 1 \end{array}\right) \qquad R_2 \leftarrow -2\, R_2$$

$$\left(\begin{array}{cc|cc} 1 & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -3 & -2 \end{array}\right) \qquad R_1 \leftarrow R_1 + \tfrac{3}{2} R_2$$

$$\left(\begin{array}{cc|cc} 1 & 0 & -4 & -3 \\ 0 & 1 & -3 & -2 \end{array}\right)$$

The solution is

$$\begin{aligned} x_1 &= -4b_1 - 3b_2 \\ x_2 &= -3b_1 - 2b_2 \end{aligned}$$

Example. Suppose we have a linear system $A\boldsymbol{x} = \boldsymbol{b}$ with indeterminate $\boldsymbol{b}$ which reduces to the row-echelon form

$$\left(\begin{array}{cc|cc} 1 & 0 & -4 & -3 \\ 0 & 0 & -3 & -2 \end{array}\right)$$

If $-3b_1 - 2b_2 \neq 0$, then there is no solution.
If $-3b_1 - 2b_2 = 0$, then there are infinitely many solutions.

Exercise. When does the following system have a solution?

$$\left\{ \begin{array}{rcrcl} x_1 & + & 2x_2 & = & b_1 \\ 2x_1 & + & 3x_2 & = & b_2 \\ 3x_1 & + & 4x_2 & = & b_3 \\ 4x_1 & + & 5x_2 & = & b_4 \end{array} \right.$$

Exercise. Find conditions on μ such that the system

$$\left\{\begin{array}{rcrcrcr} x_1 & + & x_2 & + & \mu\, x_3 & = & 1 \\ x_1 & + & 2\mu\, x_2 & + & x_3 & = & 0 \\ 2\, x_1 & + & 4\mu\, x_2 & + & \mu\, x_3 & = & -1 \end{array}\right.$$

has unique solution, no solution, infinitely many solutions.

- **General properties of the solution of $A\mathbf{x} = \mathbf{b}$.**
  - Every linear system $A\mathbf{x} = \mathbf{b}$ has
    - *no solution*, or
    - *a unique solution*, or
    - *infinitely many solutions*.
  - In the latter case, the solutions can be written in parametric vector form as

    $$\mathbf{x} = \mathbf{x}_p + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_k\mathbf{v}_k, \quad \lambda_1, \ldots, \lambda_k \in \mathbb{R},$$

    where $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are non-zero vectors.
    - $\mathbf{x} = \mathbf{x}_p$ is a *particular solution* to $A\mathbf{x} = \mathbf{b}$,
    - $\mathbf{x} = \mathbf{v}_i$ for each i is a solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

    So the general solution to $A\mathbf{x} = \mathbf{b}$ is

    $$\begin{aligned}\mathbf{x} = {} & (\text{a particular solution to } A\mathbf{x} = \mathbf{b}) \\ & + (\text{a linear combination of the solutions to } A\mathbf{x} = \mathbf{0})\end{aligned}$$

Applications of linear systems.

Exercise. Does the vector $\begin{pmatrix} 3 \\ 0 \\ 5 \\ 6 \end{pmatrix}$ belong to span $\left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 1 \\ 2 \end{pmatrix} \right\}$?

Exercise. Does the point $(2, 5, 5)$ lie on the plane

$$\mathbf{x} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}\,?$$

Exercise. Is the vector $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ parallel to the plane

$$\mathbf{x} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}\,?$$

Exercise. Find the intersection of the line

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R},$$

and the plane

$$\mathbf{x} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \mu_1 \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} + \mu_2 \begin{pmatrix} -3 \\ 0 \\ -1 \end{pmatrix}, \quad \mu_1, \mu_2 \in \mathbb{R}.$$

Exercise. A farmer intends to sow 2000 hectares of land with oats, corn, wheat and rice. Because of the different requirements, it will take him 5 hour per hectare to plant each of oats and wheat, 7 hours per hectare to plant the corn and 9 hours per hectare to plant the rice. The cost of seed for each of the oats and corn is \$20 per hectare, for wheat \$24 per hectare and for rice it is \$28 per hectare. He has a total of 16000 hours and \$50400 available.

(a) Write down a system of equations to determine the number of hectares of each grain that he can sow.

(b) Solve the equations, showing any restrictions on the parameters.

(c) Because of market prices, he wishes to sow as much rice as possible. How much should he sow of each grain in order to achieve this?