THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 2 2016

MATH1131
MATHEMATICS 1A

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS IS APPENDED TO THE PAPER

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 1

1. i) Determine the value of each of the following limits or explain why the limit does not exist.

a) $\displaystyle\lim_{x\to\infty} \frac{3x^2+4}{6x^2-8x+3}$

b) $\displaystyle\lim_{x\to\infty} \frac{x^2}{e^{x^2}}$

c) $\displaystyle\lim_{x\to 3} \frac{|x-3|}{x-3}$

d) $\displaystyle\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x$

ii) Evaluate the following integrals:

a) $\displaystyle I_1 = \int_1^e x^3 \ln x \, dx,$

b) $\displaystyle I_2 = \int_0^{\frac{\pi}{4}} \cos x \, \sin^4 x \, dx.$

iii) Consider the equation $\ln(1+x) = \cos x$.

a) Show that this equation has at least one positive solution.

b) Can the equation have any solution for $x > 2$? Give reasons for your answer.

iv) Find all values of a and b (if any) such that the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \begin{cases} ax+b & \text{if } x < 0 \\ e^{2x} & \text{if } x \geq 0 \end{cases}$$

is both continuous and differentiable at 0.

v) Given that $y = (1-3x)^{\cos x}$, find $\dfrac{dy}{dx}$.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 2

2. i) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1 + \sinh x \cosh x$.

a) Using the definitions of $\sinh x$ and $\cosh x$ show that $\sinh 2x = 2 \sinh x \cosh x$.

b) State the range of f.

c) Explain why f has an inverse function $g : \mathbb{R} \to \mathbb{R}$.

d) Find the value of $g'(1)$.

ii) Determine, with reasons, whether or not the following improper integral converges.

$$\int_1^\infty \frac{x\cos(2x)}{x^3 + x + 1}\, dx$$

iii) a) State carefully the Mean Value Theorem.

b) Illustrate the Mean Value Theorem for the function $f(x) = x^2$ on the interval $[-a, a+4]$ where $a > 0$.

c) By applying the Mean Value Theore on the interval $[0, \pi/2]$ prove that the function

$$g(x) = x^2 \cos(5x) - (x - \frac{\pi}{2})^2 \sin(7x)$$

has a stationary point.

iv) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function on $\mathbb{R}$. Let

$$g(x) = \int_0^x f(u)(x-u)\, du \quad \text{and} \quad h(x) = \int_0^x \left(\int_0^u f(t)\, dt \right) du.$$

a) Show that $g'(x) = h'(x)$.

b) What are the values of $g(0)$ and $h(0)$?

c) Prove that $g(x) = h(x)$ for all $x \in \mathbb{R}$.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 3

3. i) Let $z = 4 + i$ and $w = 2 - 2i$.

a) Find zw in **Cartesian form**.

b) Find w/z in **Cartesian form**.

c) Find $|w^{15}|$.

d) Find $\text{Arg}(4w)$.

ii) Consider the line ℓ given by

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}, \text{ for } t \in \mathbb{R},$$

and the plane $\mathcal{P}$ with Cartesian equation $3x + 2y + z = 15$. Find the point of intersection of the line ℓ and the plane $\mathcal{P}$.

iii) The points A, B and C in $\mathbb{R}^3$ have position vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \text{ and } \mathbf{c} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}.$$

a) Write down the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b) Hence or otherwise, find the position vector of the point D such that $ABCD$ (in that order) is a parallelogram.

c) Find a vector equation of the line which passes through C and is parallel to $\overrightarrow{AB}$.

d) Write down a parametric vector equation of the plane which passes through the points A, B and C.

e) Find the Cartesian equation of the plane in part (d).

iv) Let A be the matrix
$$A = \begin{pmatrix} 1 & -3 & 3 \\ 2 & -2 & 1 \\ 3 & -2 & 1 \end{pmatrix}$$
and let I denote the 3×3 identity matrix.
Use the following Maple session to assist you in answering the questions below.

```
> with(LinearAlgebra):
> A := <<1,2,3>|<-3,-2,-2>|<3,1,1>>;
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$$A := \begin{bmatrix} 1 & -3 & 3 \\ 2 & -2 & 1 \\ 3 & -2 & 1 \end{bmatrix}$$

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> A2 := A.A;
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$$A2 := \begin{bmatrix} 4 & -3 & 3 \\ 1 & -4 & 5 \\ 2 & -7 & 8 \end{bmatrix}$$

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> A3 := A.A2;
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$$A3 := \begin{bmatrix} 7 & -12 & 12 \\ 8 & -5 & 4 \\ 12 & -8 & 7 \end{bmatrix}$$

a) Find $A^3 - 4A$ and express it as a scalar multiple of I.
b) Hence express A^6 in terms of A^2, A and I.
c) Using part (a), or otherwise, find A^{-1}.

v) Sketch the following region on an Argand diagram:
$$S = \left\{ z \in \mathbb{C} : -\frac{\pi}{4} \le \operatorname{Arg}(z - 1 - i) \le \frac{\pi}{4} \right\}.$$

vi) Let $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$ be two vectors in $\mathbb{R}^3$.

a) Prove that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.
b) State whether or not $\{\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v}\}$ is an orthonormal set.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 4

4. i) Find the three cube roots of -8, expressing your answers in $a + ib$ form.

ii) By solving an appropriate system of linear equations find a parametric vector equation of the line of intersection of the two planes with Cartesian equations $x + y - 5z = 5$ and $x + 2y - 7z = 6$.

iii) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

a) Calculate the determinant of A.

b) Is the matrix A invertible? Give reasons.

iv) Let $\mathcal{L}$ be the line in $\mathbb{R}^3$ with a parametric vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -13 \\ 11 \\ 2 \end{pmatrix} + t \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}, \qquad t \in \mathbb{R},$$

and Q be a point in $\mathbb{R}^3$ with position vector $\overrightarrow{OQ} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

a) Write down a vector $\mathbf{v}$ parallel to $\mathcal{L}$ and a point P on $\mathcal{L}$.

b) Using P and $\mathbf{v}$ from part (a), find the projection of $\overrightarrow{PQ}$ onto $\mathbf{v}$.

c) Hence determine the point on the line $\mathcal{L}$ which is closest to Q.

v) During a holiday Misty caught a total of x flathead, y mullet and z garfish. To catch each flathead she needed to use 1 worm, walk 2 kilometres and fish for 1 hour. To catch each mullet she needed to use 4 worms, walk 9 kilometres and fish for 5 hours. To catch each garfish she needed to use 2 worms, walk 5 kilometres and fish for 5 hours. Altogether she used 25 worms, walked for 59 kilometres and fished for 46 hours.

a) Explain why $x + 4y + 2z = 25$.

b) Write down a system of linear equations that determine x, y and z.

c) Reduce the system in (b) to echelon form and solve to find the number of fish of each type caught.

vi) Suppose that A is a 2×1 matrix and B is a 1×2 matrix. Given that $AB = \begin{pmatrix} 2 & 1 \\ -7 & -15 \end{pmatrix}$ find BA.

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BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$

$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$

$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$

$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$

$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$

$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$

$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$

$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\begin{aligned}\int \frac{dx}{a^2 - x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, \qquad |x| < a\\ &= \frac{1}{a}\coth^{-1}\frac{x}{a} + C, \qquad |x| > a > 0\\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, \qquad x^2 \neq a^2\end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$