MATH1131 Mathematics 1A – Algebra

Lecture 12: Euler's Formula and De Moivre's Theorem

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Based on slides by Jonathan Kress

Euler's formula

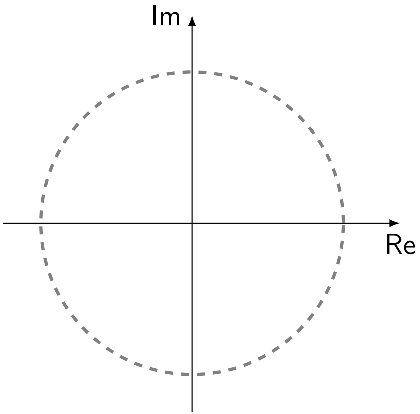
Euler's formula defines:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

The (exponential) polar form of any non-zero complex number $z = a + bi$ can therefore be written

$$z = re^{i\theta}$$

where $r = |z| = \sqrt{a^2 + b^2}$ and $\theta = \arg(z)$.



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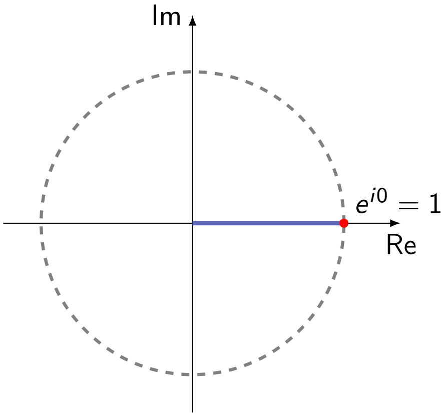

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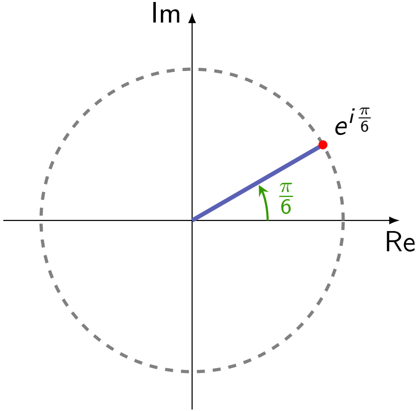

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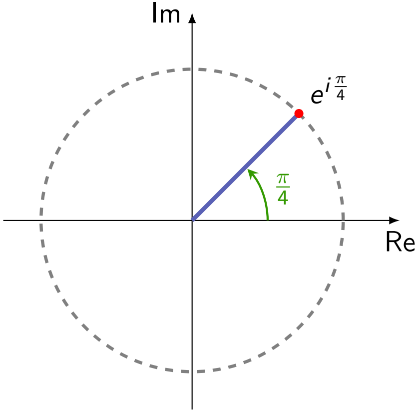

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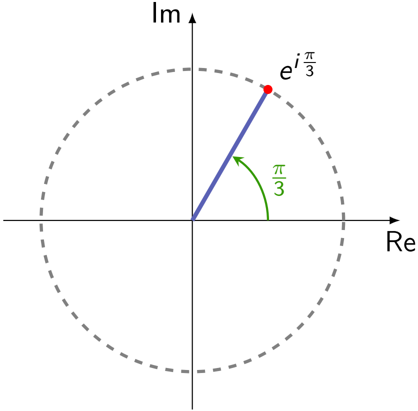

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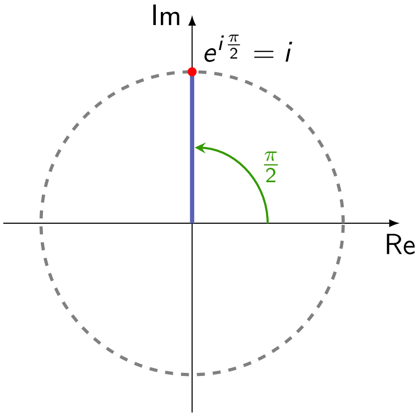

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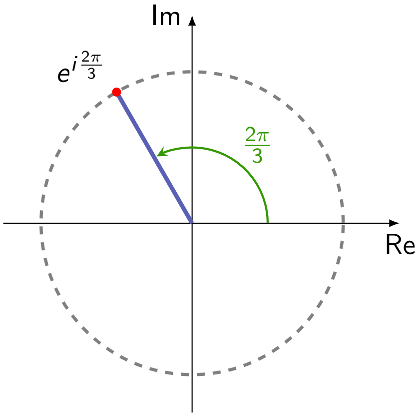

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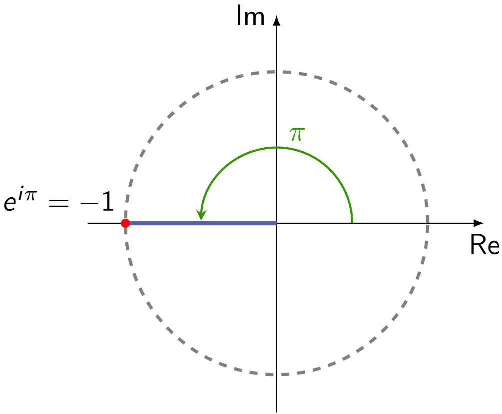

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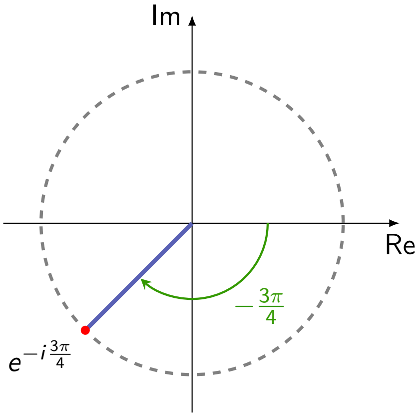

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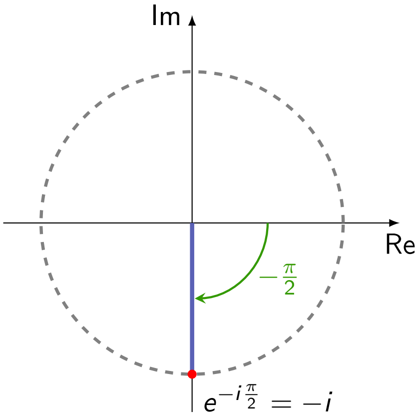

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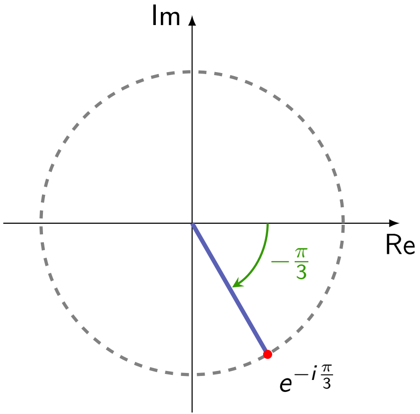

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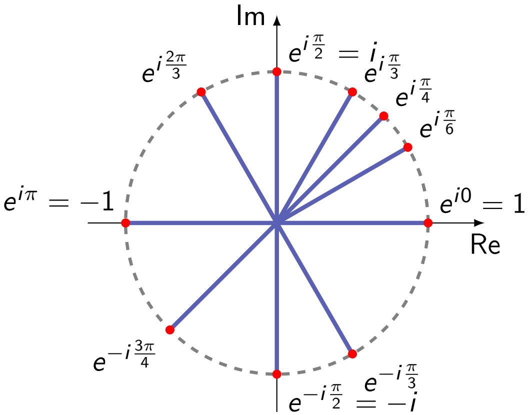

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$$e^{i0}$$

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So this has properties that behave as expected if we were to extend the definition of the real exponential.

Products in polar form

Also note the following:

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That is, we take the product of the moduli, and the sum of the arguments.

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Let $z = 2e^{i\frac{2\pi}{3}}$ and $w = 5e^{i\frac{3\pi}{4}}$. Find each of the following in polar form:

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An alternative form of De Moivre's Theorem is:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad \text{for all } n \in \mathbb{Z}.$$

Proof of De Moivre's Theorem

Proof

Let $\theta \in \mathbb{R}$.

First note that $(e^{i\theta})^1 = e^{i\theta}$ and $(e^{i\theta})^0 = 1 = e^{i0\theta}$.

Suppose that for some positive integer n we have $(e^{i\theta})^n = e^{in\theta}$. Then,

$$(e^{i\theta})^{n+1} = (e^{i\theta})^n e^{i\theta} = e^{in\theta} e^{i\theta} = e^{i(n\theta+\theta)} = e^{i(n+1)\theta}.$$

So by induction, it follows that

$$(e^{i\theta})^n = e^{in\theta} \quad \text{for all } n \in \mathbb{N}.$$

Now suppose that n is negative. Then

$$(e^{i\theta})^n = (e^{i\theta})^{-(-n)} = \frac{1}{(e^{i\theta})^{-n}} = \frac{1}{e^{-ni\theta}} = e^{in\theta}.$$

So

$$(e^{i\theta})^n = e^{in\theta} \quad \text{for all } n \in \mathbb{Z}.$$

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