MATH1131 Mathematics 1A – Algebra

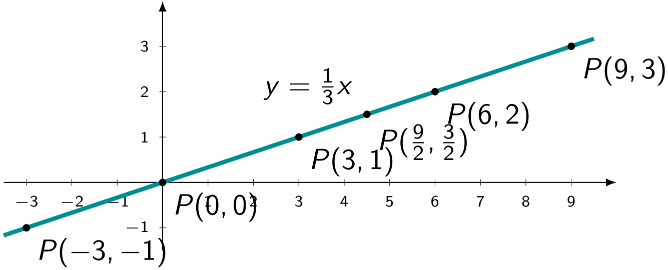
Lecture 3: Lines

Lecturer: Sean Gardiner – sean.gardiner@unsw.edu.au

Based on slides by Jonathan Kress

Lines in 2D through the origin

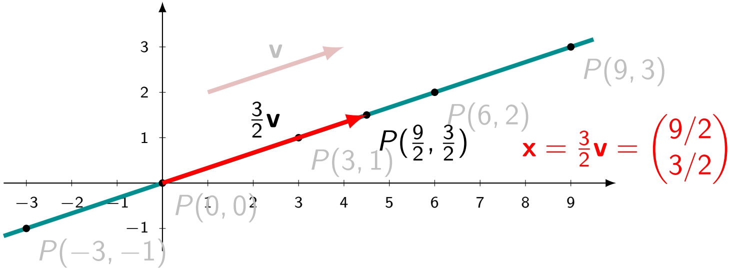
Consider the line $y = \frac{1}{3}x$:



We can find points on the line by picking values for x and substituting.

Lines in 2D through the origin

Let's describe this line using vectors.



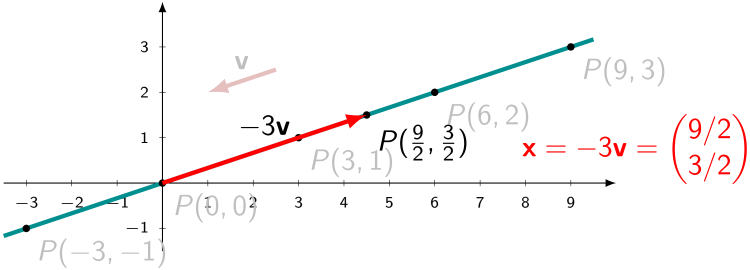
Pick a vector parallel to the line, e.g. $\mathbf{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

The position vector $\mathbf{x}$ for any point on the line can be expressed in terms of $\mathbf{v}$.

In general, any point on the line has position vector $\mathbf{x} = \lambda\mathbf{v}$ for some scalar $\lambda \in \mathbb{R}$.

Lines in 2D through the origin

What if we had picked a different $\mathbf{v}$?



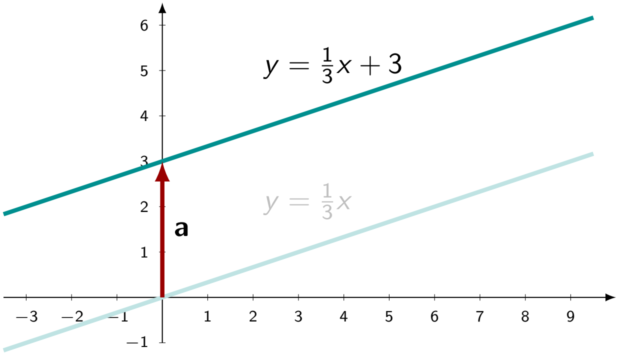
Pick a different vector parallel to the line, e.g. $\mathbf{v} = \begin{pmatrix} -3/2 \\ -1/2 \end{pmatrix}$.

We can still say any point on the line has position vector $\mathbf{x} = \lambda\mathbf{v}$ for some scalar $\lambda \in \mathbb{R}$.

So the choice of $\mathbf{v}$ doesn't matter, so long as it is parallel to the line.

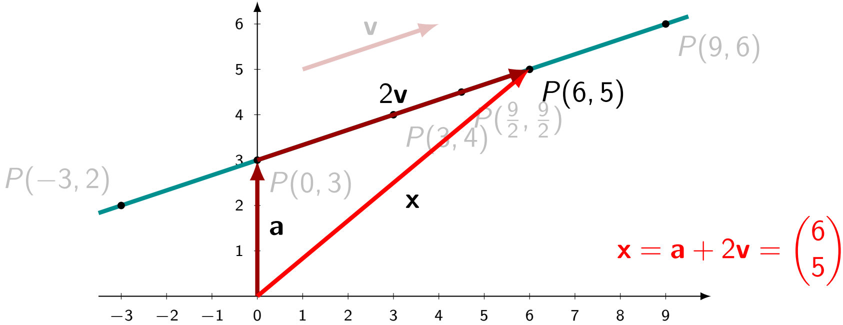
What if the line doesn't go through the origin?

Consider the line $y = \frac{1}{3}x + 3$:



We can think of this as the line $y = \frac{1}{3}x$ shifted upwards by 3 units.

... or the line $y = \frac{1}{3}x$ shifted by the vector $\mathbf{a} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

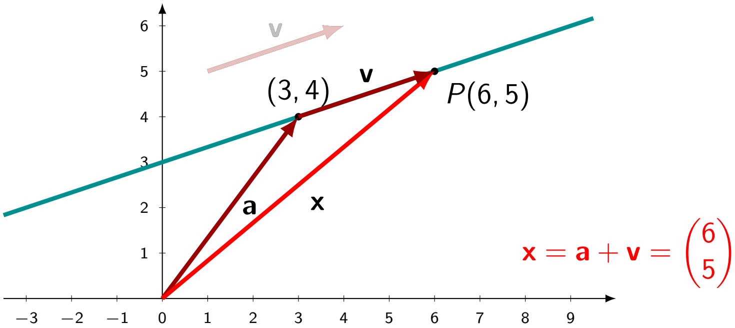


Pick a vector parallel to the line, e.g. $\mathbf{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Pick a vector pointing from the origin to the line, e.g. $\mathbf{a} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

The position vector $\mathbf{x}$ for any point on the line can be expressed in terms of $\mathbf{v}$ and $\mathbf{a}$.
In general, any point on the line has position vector $\mathbf{x} = \mathbf{a} + \lambda\mathbf{v}$ for some scalar $\lambda \in \mathbb{R}$.

What if we had picked a different $\mathbf{a}$?



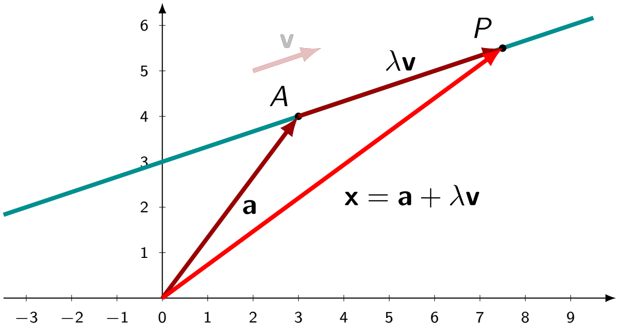
Pick a vector parallel to the line, e.g. $\mathbf{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Pick a different vector pointing from the origin to the line, e.g. $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

We can still say any point on the line has position vector $\mathbf{x} = \mathbf{a} + \lambda \mathbf{v}$ for some scalar $\lambda \in \mathbb{R}$. So the choice of $\mathbf{a}$ doesn't matter, so long as it points from the origin to the line.

Parametric vector form of a line

We can describe all the points on a given line using the position vector $\mathbf{a}$ of any point A on the line, and any vector $\mathbf{v}$ parallel to the line.



The expression $\mathbf{x} = \mathbf{a} + \lambda\mathbf{v}$ where $\lambda \in \mathbb{R}$, is called a parametric vector form of the line through $\mathbf{a}$ parallel to $\mathbf{v}$.

The scalar λ is called a parameter, and each distinct parameter value corresponds to a unique point on the line.

Parametric vector form of a line in n dimensions

Given a line in $\mathbb{R}^n$ which

- passes through a point A with position vector $\mathbf{a} \in \mathbb{R}^n$, and
- is parallel to the nonzero vector $\mathbf{v} \in \mathbb{R}^n$,

a parametric vector form for the position vectors of all points on the line is given by:

$$\mathbf{x} = \mathbf{a} + \lambda \mathbf{v} \quad \text{where } \lambda \in \mathbb{R}.$$

Example

Find the parametric vector form of the line in $\mathbb{R}^3$ which goes through the point $(1, 2, 3)$ and is parallel to the vector $\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$.

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \quad \text{where } \lambda \in \mathbb{R}.$$

Parallel lines

Definition

Two lines

$$\begin{aligned} \mathbf{x} &= \mathbf{a}_1 + \lambda \mathbf{v}_1 \quad \text{and} \\ \mathbf{x} &= \mathbf{a}_2 + \lambda \mathbf{v}_2 \end{aligned}$$

are parallel if their direction vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel.

Example

Consider the line $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$

Write down a parametric vector equation of the line through $(0, 1, 2)$ that is parallel to the given line.

$$\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Finding the equation of a line

Example

Find the equation of the line in $\mathbb{R}^4$ which passes through the points $A(2, -3, -1, 2)$ and $B(-1, 2, 2, 7)$.

Since both A and B lie on the line, a vector with the same direction as the line is $\overrightarrow{AB}$.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -1 \\ 2 \\ 2 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 3 \\ 5 \end{pmatrix}$$

So an equation for the line is:

$$\mathbf{x} = \begin{pmatrix} 2 \\ -3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 5 \\ 3 \\ 5 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Points on a parametric line

Example

Consider the line in $\mathbb{R}^3$ given parametrically as

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

- Does $(9, -2, -5)$ lie on the line?
- Does $(-3, 4, 0)$ lie on the line?

Points on a parametric line

Does $(9, -2, -5)$ lie on the line $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R}$?

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 + 2\lambda \\ 2 - \lambda \\ -1 - \lambda \end{pmatrix}$$

If $(9, -2, -5)$ lies on the line, there must be a value for λ in the above expression that makes $\mathbf{x} = \begin{pmatrix} 9 \\ -2 \\ -5 \end{pmatrix}$.

Equating components, we find $\lambda = 4$ in all cases.

So $(9, -2, -5)$ does lie on the line, because we can write

$$\begin{pmatrix} 9 \\ -2 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 4 \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}.$$

Points on a parametric line

Does $(-3, 4, 0)$ lie on the line $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R}$?

If $(-3, 4, 0)$ lies on the line, there must be a value for λ in the expression $\mathbf{x} = \begin{pmatrix} 1+2\lambda \\ 2-\lambda \\ -1-\lambda \end{pmatrix}$ that makes $\mathbf{x} = \begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix}$.

Equating components, we need $\lambda = -2$ for the first and second components, but $\lambda = -1$ for the third component.

So $(-3, 4, 0)$ does not lie on the line, because we cannot write

$$\begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

for any fixed real value of λ.

Cartesian form of a line

Consider the general line $\mathbf{x} = \mathbf{a} + \lambda \mathbf{v}$ in $\mathbb{R}^n$ in terms of its components:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \lambda \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Equating corresponding components gives:

$$\begin{aligned} x_1 &= a_1 + \lambda v_1 & \qquad \lambda &= \frac{x_1 - a_1}{v_1}, \quad \text{if } v_1 \neq 0 \\ x_2 &= a_2 + \lambda v_2 & \lambda &= \frac{x_2 - a_2}{v_2}, \quad \text{if } v_2 \neq 0 \\ &\vdots & &\vdots \\ x_n &= a_n + \lambda v_n & \lambda &= \frac{x_n - a_n}{v_n}, \quad \text{if } v_n \neq 0 \end{aligned}$$

Cartesian form of a line

Consider the general line $\mathbf{x} = \mathbf{a} + \lambda \mathbf{v}$ in $\mathbb{R}^n$ in terms of its components:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \lambda \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

We can eliminate the parameter λ to find the Cartesian form of the line:

- $\dfrac{x_1 - a_1}{v_1} = \dfrac{x_2 - a_2}{v_2} = \cdots = \dfrac{x_n - a_n}{v_n}$ for all non-zero v_i, and
- $x_i = a_i$ whenever $v_i = 0$.

Parametric form to Cartesian form

Example

Find the Cartesian form for the line given parametrically as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Here $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$.

So the Cartesian form of the line is:

$$\frac{x_1 - 1}{1} = \frac{x_2 - 2}{3} = \frac{x_3 - (-5)}{-1}$$

or

$$x_1 - 1 = \frac{x_2 - 2}{3} = -x_3 - 5.$$

Parametric form to Cartesian form

Example

Find the Cartesian form for the line given parametrically as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Here $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$.

So the Cartesian form of the line is:

$$\frac{x_1 - 1}{1} = \frac{x_2 - 2}{3}, \text{ and } x_3 = -5$$

or

$$x_1 - 1 = \frac{x_2 - 2}{3}, \text{ and } x_3 = -5.$$

Parametric form to Cartesian form

Example

Find the Cartesian form for the line given parametrically as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

The Cartesian form of the line is:

$$\frac{x_1 - 3}{3} = \frac{x_2 - 4}{1}$$

Often in $\mathbb{R}^2$ and $\mathbb{R}^3$, we use x, y, and z in place of x_1, x_2, and x_3.

Substituting $x_1 = x$ and $x_2 = y$, and rearranging yields a familiar equation for a line in two dimensions:

$$y = \frac{1}{3}x + 3$$

Parametric form to Cartesian form

Example

Find the Cartesian form for the line given parametrically as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

x_1 is the only component in terms of λ (that is, v_1 is the only non-zero component of $\mathbf{v}$), so there is only one expression in the $\frac{x_1 - a_1}{v_1} = \frac{x_2 - a_2}{v_2} = \cdots = \frac{x_n - a_n}{v_n}$ chain.

This means we can only write $\frac{x_1 - 3}{3} = \lambda$, or indeed $x_1 = 3\lambda + 3$.

Since λ can take on all real values, so too can x_1. So there is in fact no restriction on x_1.

Therefore the Cartesian form of the line is simply:

$$x_2 = 4 \quad (\text{and } x_1 \in \mathbb{R})$$

Cartesian form to parametric form

Example

Find a parametric vector form for the line

$$\frac{x_1-3}{3}=\frac{x_2+1}{2}=x_3-8 \text{ in } \mathbb{R}^3.$$

We can rewrite this as:

$$\frac{x_1-3}{3}=\frac{x_2-(-1)}{2}=\frac{x_3-8}{1}$$

This implies $\mathbf{a}=\begin{pmatrix}3\\-1\\8\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}3\\2\\1\end{pmatrix}$.

So a parametric vector form is:

$$\mathbf{x}=\begin{pmatrix}3\\-1\\8\end{pmatrix}+\lambda\begin{pmatrix}3\\2\\1\end{pmatrix},\quad \lambda\in\mathbb{R}.$$

Cartesian form to parametric form

Example

Find a parametric vector form for the line

$$\frac{1-x_1}{3} = \frac{2x_2+1}{3}, \text{ and } x_3 = 8 \text{ in } \mathbb{R}^3.$$

We can rewrite this as:

$$\frac{x_1 - 1}{-3} = \frac{x_2 - \left(-\frac{1}{2}\right)}{\frac{3}{2}}, \text{ and } x_3 = 8$$

This implies $\mathbf{a} = \begin{pmatrix} 1 \\ -1/2 \\ 8 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -3 \\ 3/2 \\ 0 \end{pmatrix}$.

So a parametric vector form is:

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1/2 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 3/2 \\ 0 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Cartesian form to parametric form

Example

Find a parametric vector form for the line

$$y = 3x + 2 \text{ in } \mathbb{R}^2.$$

We can rewrite this as:

$$\frac{x - (-\frac{2}{3})}{\frac{1}{3}} = \frac{y - 0}{1}$$

So a parametric vector form is:

$$\mathbf{x} = \begin{pmatrix} -2/3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1/3 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Alternatively, noting $(0, 2)$ is another point on the line (when $\lambda = 2$) and that $3\mathbf{v}$ is of course parallel to $\mathbf{v}$, we could instead write:

$$\mathbf{x} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Cartesian form to parametric form

Example

Find a parametric vector form for the line

$$x_1 = 8 \text{ in } \mathbb{R}^2.$$

Since x_2 has no restrictions, it can take all values in $\mathbb{R}$. So we can instead write:

$$x_1 = 8 \text{ and } x_2 = \lambda, \quad \lambda \in \mathbb{R}.$$

This corresponds with a parametric form of:

$$\mathbf{x} = \begin{pmatrix} 8 + 0\lambda \\ 0 + 1\lambda \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$