THE UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1131 Calculus

MATHEMATICS 1A CALCULUS.

Introduction.

This course has two main purposes. Firstly, it will continue and consolidate your High School Calculus (we assume NSW HSC Extension 1 or equivalent). Intertwined with this, is the need to look more deeply at some of the ideas you learnt at school and see precisely why and when they work. You may find this aspect the most challenging part of the course.

Section 1: - Functions and Graphs.

1. Numbers.

We will use the following notation:

1. The set of natural numbers, denoted by $\mathbb{N}$, consists of all the whole numbers $\{0, 1, 2, ...\}$.

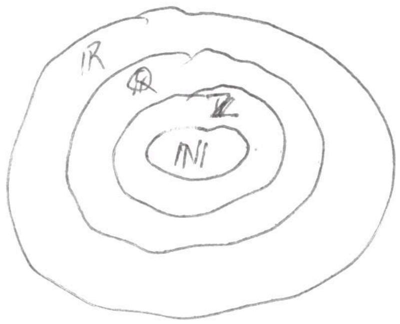
2. The set of integers, denoted by $\mathbb{Z}$, consists of all the whole numbers $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$.

3. The set of rational numbers, denoted by $\mathbb{Q}$, consists of all numbers of the form $\frac{p}{q}$, where p, q are integers and $q \neq 0$.

The ancient Greeks initially thought that this was all there was (they didn't believe in negative numbers and zero either), until they discovered that $\sqrt{2}$ could not be written as a rational number.

$\sqrt{2}$ and numbers such as π and e are examples of irrational numbers. It is easiest at this stage to think of the set of all real numbers as points which lie on the real line. Giving a formal definition of real numbers is difficult.

4. We denote the set of **real numbers** by $\mathbb{R}$. In this course we will be dealing (almost) exclusively with the calculus of functions defined over the real numbers.



Set Notation.

We will use the following set notation:

$\{x \in A : P(x)\}$ denotes the set of all elements x of A satisfying property P.
For example, $\{x \in \mathbb{R} : -1 \leq x \leq 1\}$ denotes all the real numbers between -1 and 1 (inclusive).

$A \cap B$ is the intersection of A and B and denotes all the elements that are in both A and B.

$A \cup B$ is the union of A and B and denotes all the elements that are in either A or B (or both).

$\emptyset$, called the *empty set*, is the set which has no elements, for example $\{x \in \mathbb{R} : x^2 < -1\} = \emptyset$.

Inequalities:

You are aware of the following facts about inequalities:

For $x, y, z \in \mathbb{R}$ we have
i. if $x > y$ then $x + z > y + z$
ii. if $x > y$ and $z > 0$ then $xz > yz$ and if $z < 0$ we have $xz < yz$.

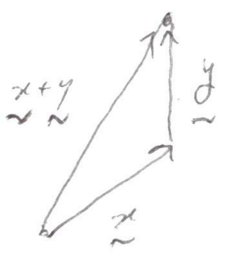
Note carefully the definition for $|x|$.

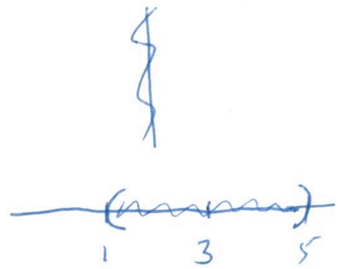
$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

So for example, $|a - 3|$ is equal to $a - 3$ if $a \geq 3$ and $-(a - 3) = 3 - a$ if $a < 3$.

Note then that $|x| < 3$ means $-3 < x < 3$ and that $|-x| = |x|$. Also note that $\{x : |x-3| < 2\}$ represents the set of all real numbers whose distance from 3 is less that 2.

Finally note that $|xy| = |x||y|$ and that $|x + y| \leq |x| + |y|$. This last result is called the **triangle inequality.** You will see a complex version of this in the algebra strand of the course.





Also of importance is:

Theorem: (AM-GM inequality).

If $x, y \geq 0$ are real numbers then

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

(This says that the arithmetic mean of two positive real numbers exceeds their geometric mean.)

Proof: We begin with the true statement

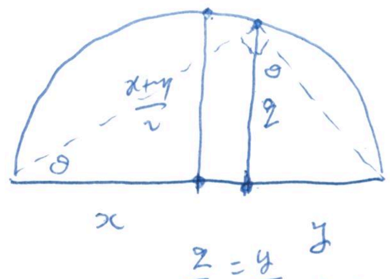
$$(a-b)^2 \geq 0, \text{ where } a, b \text{ are any real numbers.}$$

Expanding,

$$a^2 + b^2 - 2ab \geq 0 \Rightarrow \frac{a^2+b^2}{2} \geq ab$$

Put $a^2 = x$, $b^2 = y$ then

$$\frac{x+y}{2} \geq \sqrt{xy}$$



$$\frac{q}{x} = \frac{y}{q} \Rightarrow q = \sqrt{xy}$$

Ex: Prove that for $x > 0$, we have $x + \frac{1}{x} \geq 2$.

Write the AM-GM inequality as $x + y \geq 2\sqrt{xy}$

then $x + \frac{1}{x} \geq 2\sqrt{x \times \frac{1}{x}} = 2$

Ex: Suppose a, b, c are positive real numbers. Prove that $a^2 + b^2 + c^2 \geq ab + ac + bc$.

$$a^2 + b^2 \geq 2ab$$
$$b^2 + c^2 \geq 2bc$$
$$c^2 + a^2 \geq 2ac$$

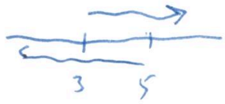
Adding $2(a^2+b^2+c^2) \geq 2(ab+bc+ac)$

$\therefore \; a^2+b^2+c^2 \geq ab+bc+ac$

Intervals: We will use the following notation when dealing with intervals. A round bracket means we do not include the endpoint while we do when a square bracket is used. For example $(3, 9]$ means the interval $3 < x \leq 9$. Note that since infinity is NOT a real number, if we wish to represent the interval from 3 onwards we write this as $[3, \infty)$ (never use a square bracket with infinity.). Here are some further examples:

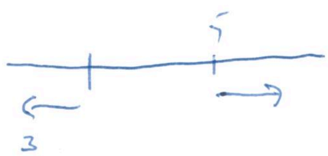
$$\{x \in \mathbb{R} : x > 3\} \cap \{x \in \mathbb{R} : x < 5\} = (3, 5)$$

$$\{x \in \mathbb{R} : x > 3\} \cup \{x \in \mathbb{R} : x < 5\} = \mathbb{R}$$



$$\{x \in \mathbb{R} : x > 5\} \cup \{x \in \mathbb{R} : x < 3\} = (-\infty, 3) \cup (5, \infty)$$

$$\{x \in \mathbb{R} : x > 5\} \cap \{x \in \mathbb{R} : x < 3\} = \emptyset$$



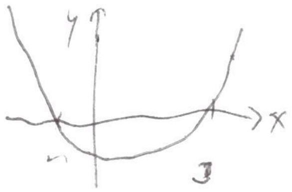
Solving Inequalities:

These are very similar to equations except that we must careful when multiplying by an unknown. You should be familiar with solving quadratic inequations such as
Ex: Find and sketch the set $\{x : x^2 - 2x - 3 > 0\}$.

$x^2 - 2x - 3 > 0$

$(x-3)(x+1) > 0$

Graph $y = (x-3)(x+1)$



$y > 0$ when

$x > 3$ or $x < -1$

$\therefore$ Solution is

$\{x \in \mathbb{R} : x > 3 \text{ or } x < -1\}$

$[= \{x \in \mathbb{R} : x > 3\} \cup \{x \in \mathbb{R} : x < -1$

For more difficult inequalities we use the following idea.

Ex: Solve $x > 1 + \frac{2}{x}$.

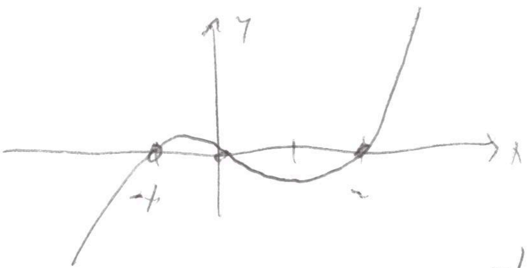
multiply by x^2 (which is always pos. if $x \neq 0$)

$x^3 > x^2 + 2x$

$\therefore x(x^2 - x - 2) > 0$

$x(x-2)(x+1) > 0$

Graph $y = x(x-2)(x+1)$



$y > 0$ when $x > 2$ or $-1 < x < 0$

Solution is

$\{x \in \mathbb{R} : x > 2 \text{ or } -1 < x < 0\}$

Ex: Solve $\frac{2}{x} \leq \frac{3}{x-1}$.

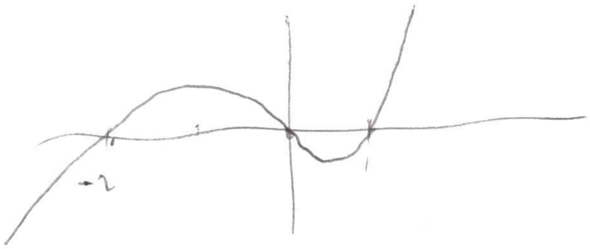
mult. by $x^2(x-1)^2$

$2x(x-1)^2 \leq 3x^2(x-1)$

$x(x-1)[3x-2(x-1)] \geq 0$

$x(x-1)(x+2) \geq 0$

Graph $y = x(x-1)(x+2)$



$y \geq 0$ when $x \geq 1$, $-2 \leq x \leq 0$

But $x \neq 1$ or 0

Soln set is

$\{x \in \mathbb{R} : x > 1$ or $-2 \leq x < 0\}$

Functions:

You should be familiar with the function concept from school. Roughly speaking, a function $f : A \to B$ is a rule or formula which associates to each element of a set A (called the domain) **exactly one** element from another set B (called the co-domain). For the most part, we will have $A = B = \mathbb{R}$. The range of the function is the set of values b in B for which there is an $a \in A$ with $f(a) = b$.

In less formal terms, the range consists of the output of the function.We will use the term *natural domain* to describe the set of allowable input values.
You will need to be able to find the (natural) domain and range of basic functions.

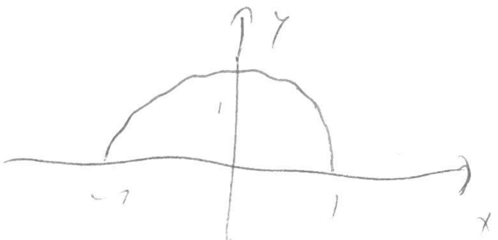
Ex: Find the natural domain and range of $f(x) = \sqrt{1-x^2}$.

Dom: $1 - x^2 \geq 0$

$x^2 \leq 1$

$-1 \leq x \leq +1$

Range: $0 \leq y \leq 1$

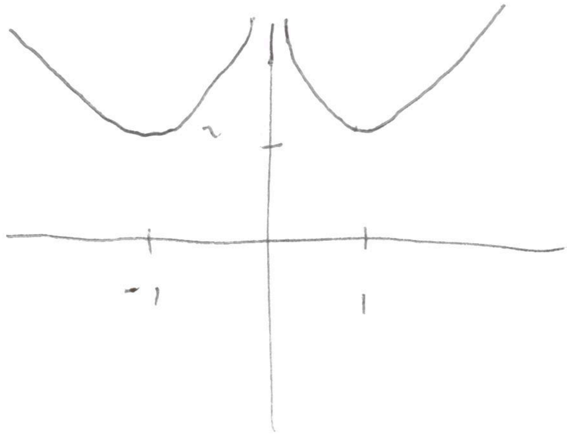


Ex: Use the AM-GM inequality to find the range of $y = x^2 + \frac{1}{x^2}$ and sketch the graph.

$x^2 + \frac{1}{x^2} \geq 2\sqrt{x^2 \times \frac{1}{x^2}} = 2$

& $x = 1$ gives $y = 2$

$\therefore$ Range is $y \geq 2$



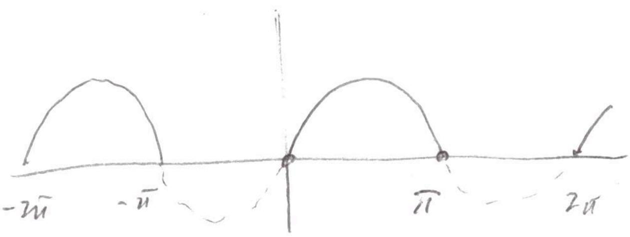
Ex: Find the domain of $f(x) = \sqrt{\sin x}$.

Dom: $[0, \pi] \cup [2\pi, 3\pi] \cdots$

$\cup [-2\pi, -\pi] \cup \cdots$

$= \bigcup_{k \in \mathbb{N}} [2k\pi, (2k+1)\pi] \cup$

$\bigcup_{k \in \mathbb{N}} [-2k\pi, (2k-1)\pi]$



It is often difficult to find the range of a function. For example, what is the range of $2 + \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4}$?

It is important to be able to draw the graph of a given function. In most Calculus problems this is crucial.

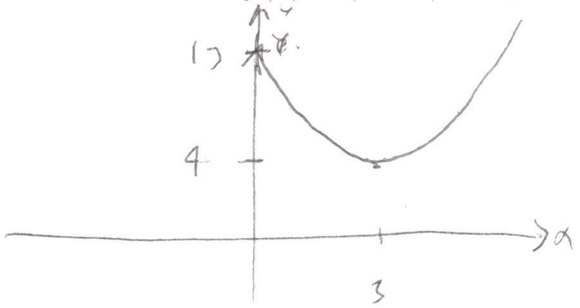
It often helps if the function is even or odd. You will recall that f is **even** if $f(x) = f(-x)$ and f is **odd** if $f(-x) = -f(x)$. Even functions are symmetric about the y axis and odd functions have a central symmetry with respect to the origin.

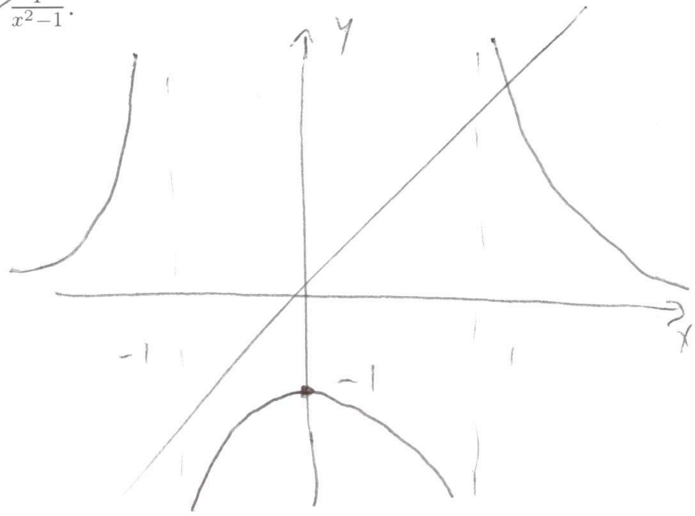
Thus, if we can draw such a function on the positive half plane we get the rest of the picture for free.
Note that if f is odd and has 0 in its domain, then $f(0) = 0$.

We say that f is periodic of period T if $f(x+T) = f(x)$ for all real x in the domain of f.

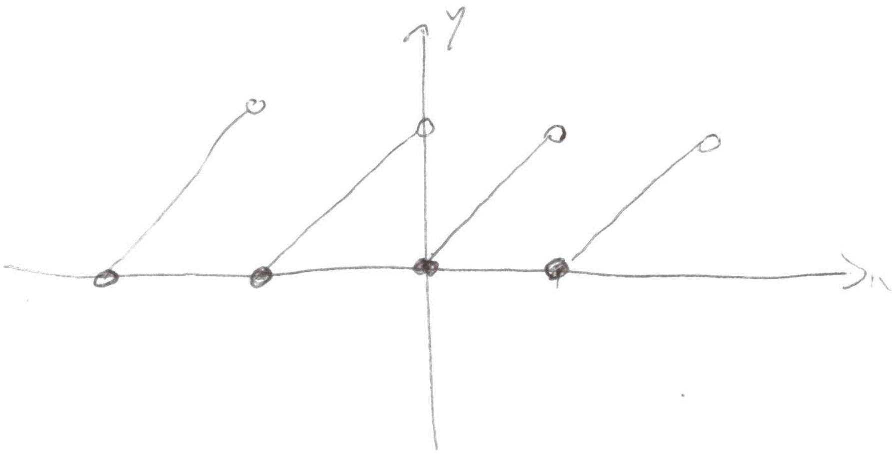
You have met the trig. functions which are periodic with period 2π.

Ex: Sketch: $f(x) = (x-3)^2 + 4$, and $f(x) = \frac{1}{x^2-1}$.

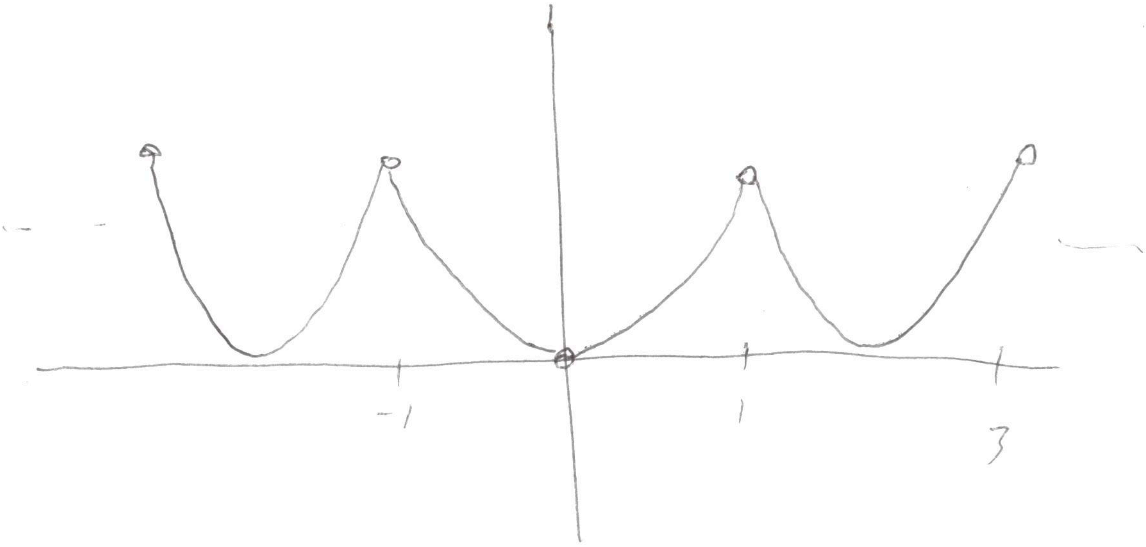




Ex: Sketch: $f(x) = x$ if $0 \leq x < 1$ and $f(x+1) = f(x)$ for all x.



Ex: Sketch $f(x) = x^2$ for $0 < x < 1$, f is periodic of period 2 and f is even.



~~Ex: Sketch $f(x) = x - \lfloor x \rfloor$.~~

Combining Functions:
If f and g are two functions, we can add, subtract and multiply them in the obvious way. We can also divide them provided g is not zero. If the range of g equals the domain of f we compose the two functions to form $f \circ g$ which we define to be

$$f \circ g(x) = f(g(x)).$$

$f \circ g$ is called the composite function of f and g.

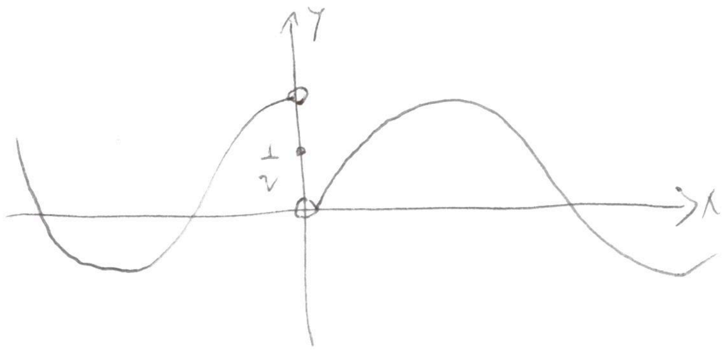
Ex: Find $f \circ g$ and $g \circ f$ if $f(x) = x^3$ and $g(x) = \sqrt{x^2+1}$.

$$f\circ g(x) = f(g(x)) = f(\sqrt{x^2+1}) = (x^2+1)^{3/2}$$

$$g\circ f(x) = g(x^3) = \sqrt{(x^3)^2+1} = \sqrt{x^6+1}$$

Note that some functions cannot be defined by one simple equation. Many functions which occur in the real world are defined piecewise.

Ex: $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ \frac{1}{2} & \text{if } x = 0 \\ \sin x & \text{if } x > 0 \end{cases}$

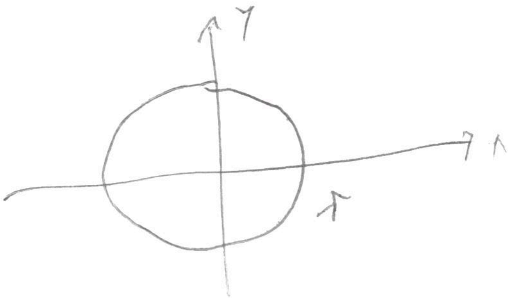


Conic Sections:

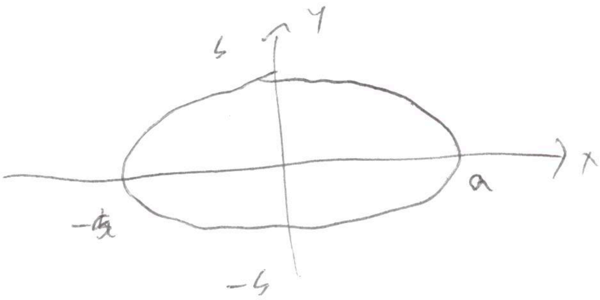
An important class of implicitly defined functions arises from the *conic sections* (so called because they are obtained by slicing a cone with various planes.)

You will need to recognise these:

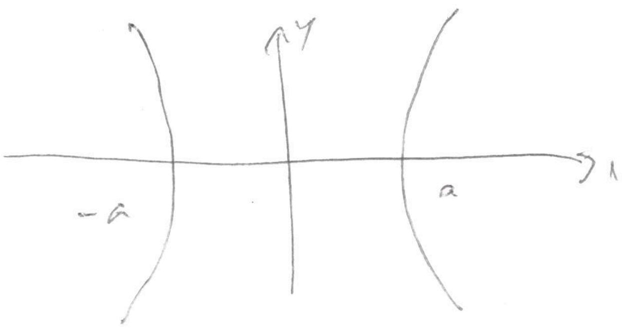
(i) **Circle** $x^2 + y^2 = r^2$



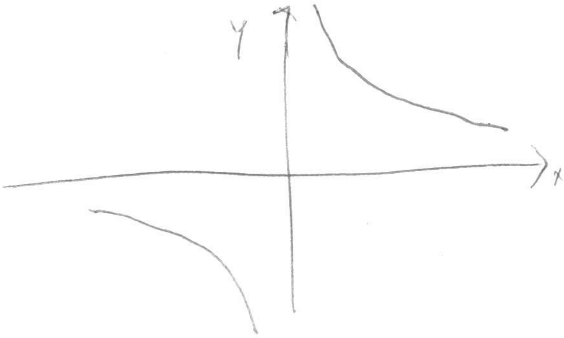
(ii) **Ellipse** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a, b \neq 0$.

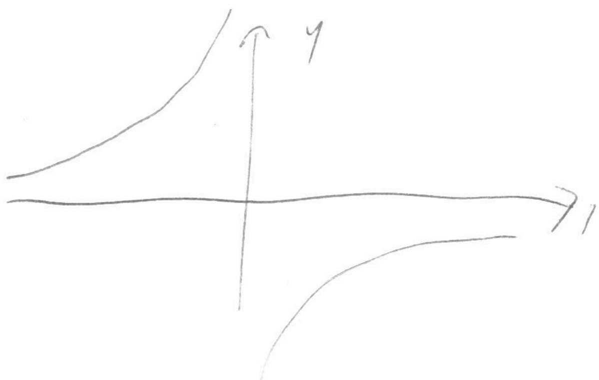


(iii) **Hyperbola** $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $a, b \neq 0$.



(iv) **Rectangular Hyperbola** $y = \frac{a}{x}$, $a \neq 0$.





$a > 0$

$a < 0$

Other Functions:

It is assumed that you are familiar with the basic properties of polynomial functions, rational functions, the trigonometric functions, the exponential and logarithmic functions.