MATH1131 Mathematics 1A – Algebra

# Lecture 7: Cross Product

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Based on slides by Jonathan Kress

# Cross product

**Definition**

Suppose that

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

are vectors in $\mathbb{R}^3$.
Then the cross product (or vector product) of $\mathbf{a}$ and $\mathbf{b}$ is:

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

Note that while $\mathbf{a} \cdot \mathbf{b}$ is defined for vectors in all dimensions, $\mathbf{a} \times \mathbf{b}$ is only defined for vectors in $\mathbb{R}^3$.

The cross product of **a** and **b** is:

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

This formula can be difficult to memorise. One way to remember the terms is to think of the subscripts cycling through in order:

$$1 \ \rightarrow \ 2 \ \rightarrow \ 3 \ \rightarrow \ 1.$$

Writing $\mathbf{v} = \mathbf{a} \times \mathbf{b}$, we have:

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

# Determinants

Later in the course we will encounter determinants. Interpreting the cross product as a determinant is another useful way to remember the formula. We can briefly see how this works now, though the motivation might not make sense until the matrix chapter has been covered.

**Definition: $2 \times 2$ determinant**

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For example:

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 2 \times 3 = -2$$

$$\begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix} = 4 \times 2 - 1 \times 5 = 3$$

# Cross product

The cross product can be written as a $3 \times 3$ determinant:

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\
&= \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}
\end{aligned}$$

Note the different signs!

# Cross product – Examples

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}2\\3\\5\end{pmatrix} = \begin{pmatrix}2\times 5 - 3\times 3\\3\times 2 - 1\times 5\\1\times 3 - 2\times 2\end{pmatrix} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$$

$$\begin{pmatrix}2\\3\\-1\end{pmatrix} \times \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}3\times 1 - (-1)\times 1\\(-1)\times 1 - 2\times 1\\2\times 1 - 3\times 1\end{pmatrix} = \begin{pmatrix}4\\-3\\-1\end{pmatrix}$$

$$\begin{pmatrix}1\\0\\2\end{pmatrix} \times \begin{pmatrix}3\\4\\5\end{pmatrix} = \begin{pmatrix}0\times 5 - 2\times 4\\2\times 3 - 1\times 5\\1\times 4 - 0\times 3\end{pmatrix} = \begin{pmatrix}-8\\1\\4\end{pmatrix}$$

# Properties

## Properties of the cross product

If $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^3$, then $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$ and to $\mathbf{b}$.

For example, we saw

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} -8 \\ 1 \\ 4 \end{pmatrix},$$

and can confirm that

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -8 \\ 1 \\ 4 \end{pmatrix} = 0 \qquad \text{and} \qquad \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -8 \\ 1 \\ 4 \end{pmatrix} = 0.$$

# Properties

## Properties of the cross product

If **a** and **b** are vectors in $\mathbb{R}^3$, then $\mathbf{a} \times \mathbf{b}$ is orthogonal to **a** and **b**.

## Exercise

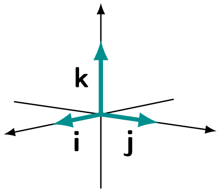
Show that for all $a_i, b_i \in \mathbb{R}$:

$$\begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

and

$$\begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 0.$$

# Examples




$$\mathbf{i} \times \mathbf{j} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \mathbf{j}$$

Similarly $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$, and $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

The direction of the cross product vector can be determined from the "right-hand rule": To find the direction of $\mathbf{a} \times \mathbf{b}$, join the vectors tail-to-tail, and point your right thumb in the direction of $\mathbf{a}$ and fingers in the direction of $\mathbf{b}$. Your palm will point in the direction of $\mathbf{a} \times \mathbf{b}$.

Notice also that $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$.

# Examples

We can also use this behaviour of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ as another way to memorise the cross product formula.

For example:

$$
\begin{aligned}
\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} &= (\mathbf{i} + 2\mathbf{k}) \times (3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}) \\
&= 3\mathbf{i} \times \mathbf{i} + 4\mathbf{i} \times \mathbf{j} + 5\mathbf{i} \times \mathbf{k} + 6\mathbf{k} \times \mathbf{i} + 8\mathbf{k} \times \mathbf{j} + 10\mathbf{k} \times \mathbf{k} \\
&= \mathbf{0} + 4\mathbf{k} + 5(-\mathbf{j}) + 6\mathbf{j} + 8(-\mathbf{i}) + \mathbf{0} \\
&= -8\mathbf{i} + \mathbf{j} + 4\mathbf{k} \\
&= \begin{pmatrix} -8 \\ 1 \\ 4 \end{pmatrix}
\end{aligned}
$$

Notice we have assumed some properties of the cross product, which we should now prove...

## Properties of the cross product

For all vectors **a**, **b** and **c** in $\mathbb{R}^3$ and scalars $\lambda \in \mathbb{R}$,

- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
- $\mathbf{0} \times \mathbf{a} = \mathbf{a} \times \mathbf{0} = \mathbf{0}$
- $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$ (anti-commutative law)
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (distributive law)
- $\mathbf{a} \times (\lambda \mathbf{b}) = (\lambda \mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b})$ (associative scalar multiplication)

## Proof

$$\mathbf{a} \times \mathbf{a} = \begin{pmatrix} a_2 a_3 - a_3 a_2 \\ a_3 a_1 - a_1 a_3 \\ a_1 a_2 - a_2 a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}$$

# Properties of the cross product

## Properties of the cross product

For all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$ and scalars $\lambda \in \mathbb{R}$,

- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
- $\mathbf{0} \times \mathbf{a} = \mathbf{a} \times \mathbf{0} = \mathbf{0}$
- $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$ (anti-commutative law)
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (distributive law)
- $\mathbf{a} \times (\lambda \mathbf{b}) = (\lambda \mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b})$ (associative scalar multiplication)

## Proof

$$\mathbf{0} \times \mathbf{a} = \begin{pmatrix} 0a_3 - 0a_2 \\ 0a_1 - 0a_3 \\ 0a_2 - 0a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \mathbf{a} \times \mathbf{0} = \begin{pmatrix} a_2 0 - a_3 0 \\ a_3 0 - a_1 0 \\ a_1 0 - a_2 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

## Properties of the cross product

For all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$ and scalars $\lambda \in \mathbb{R}$,

- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
- $\mathbf{0} \times \mathbf{a} = \mathbf{a} \times \mathbf{0} = \mathbf{0}$
- $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$ (anti-commutative law)
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (distributive law)
- $\mathbf{a} \times (\lambda\mathbf{b}) = (\lambda\mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b})$ (associative scalar multiplication)

### Proof

$$\mathbf{b} \times \mathbf{a} = \begin{pmatrix} b_2a_3 - b_3a_2 \\ b_3a_1 - b_1a_3 \\ b_1a_2 - b_2a_1 \end{pmatrix} = -\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} = -\mathbf{a} \times \mathbf{b}$$

## Properties of the cross product

For all vectors **a**, **b** and **c** in $\mathbb{R}^3$ and scalars $\lambda \in \mathbb{R}$,

- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
- $\mathbf{0} \times \mathbf{a} = \mathbf{a} \times \mathbf{0} = \mathbf{0}$
- $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$ (anti-commutative law)
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (distributive law)
- $\mathbf{a} \times (\lambda \mathbf{b}) = (\lambda \mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b})$ (associative scalar multiplication)

### Proof

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \begin{pmatrix} a_2(b_3 + c_3) - a_3(b_2 + c_2) \\ a_3(b_1 + c_1) - a_1(b_3 + c_3) \\ a_1(b_2 + c_2) - a_2(b_1 + c_1) \end{pmatrix}$$

$$= \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} + \begin{pmatrix} a_2 c_3 - a_3 c_2 \\ a_3 c_1 - a_1 c_3 \\ a_1 c_2 - a_2 c_1 \end{pmatrix} = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

# Properties of the cross product

## Properties of the cross product

For all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$ and scalars $\lambda \in \mathbb{R}$,

- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
- $\mathbf{0} \times \mathbf{a} = \mathbf{a} \times \mathbf{0} = \mathbf{0}$
- $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$ (anti-commutative law)
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (distributive law)
- $\mathbf{a} \times (\lambda \mathbf{b}) = (\lambda \mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b})$ (associative scalar multiplication)

## Proof

$$\mathbf{a} \times (\lambda \mathbf{b}) = \begin{pmatrix} a_2(\lambda b_3) - a_3(\lambda b_2) \\ a_3(\lambda b_1) - a_1(\lambda b_3) \\ a_1(\lambda b_2) - a_2(\lambda b_1) \end{pmatrix} = \lambda \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} = \lambda(\mathbf{a} \times \mathbf{b})$$

and

$$(\lambda \mathbf{a}) \times \mathbf{b} = \begin{pmatrix} (\lambda a_2) b_3 - (\lambda a_3) b_2 \\ (\lambda a_3) b_1 - (\lambda a_1) b_3 \\ (\lambda a_1) b_2 - (\lambda a_2) b_1 \end{pmatrix} = \lambda \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} = \lambda(\mathbf{a} \times \mathbf{b})$$

## Properties of the cross product

For all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$ and real numbers $\lambda$,

- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
- $\mathbf{0} \times \mathbf{a} = \mathbf{a} \times \mathbf{0} = \mathbf{0}$
- $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$ (anti-commutative law)
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (distributive law)
- $\mathbf{a} \times (\lambda \mathbf{b}) = (\lambda \mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b})$ (associative scalar multiplication)

Note that the cross product is not itself associative. In general, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ .

For example,

$$\begin{aligned} \mathbf{i} \times (\mathbf{i} \times \mathbf{j}) &= \mathbf{i} \times \mathbf{k} = -\mathbf{j} \\ (\mathbf{i} \times \mathbf{i}) \times \mathbf{j} &= \mathbf{0} \times \mathbf{j} = \mathbf{0} \neq -\mathbf{j} \end{aligned}$$

# Length of the cross product

**Theorem**

For all vectors $\mathbf{a}, \mathbf{b}$ in $\mathbb{R}^3$:

$$|\mathbf{a} \cdot \mathbf{b}|^2 + |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2.$$

**Proof**

For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$:

$$
\begin{aligned}
&|\mathbf{a} \cdot \mathbf{b}|^2 + |\mathbf{a} \times \mathbf{b}|^2 \\
&= (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 + (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2 \\
&= a_1^2 b_1^2 + a_1^2 b_2^2 + a_1^2 b_3^2 + a_2^2 b_1^2 + a_2^2 b_2^2 + a_2^2 b_3^2 + a_3^2 b_1^2 + a_3^2 b_2^2 + a_3^2 b_3^2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \\
&= |\mathbf{a}|^2 |\mathbf{b}|^2
\end{aligned}
$$

# Length of the cross product

**Theorem**

For all vectors $\mathbf{a}, \mathbf{b}$ in $\mathbb{R}^3$:

$$|\mathbf{a} \cdot \mathbf{b}|^2 + |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2.$$

So if $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}|^2 &= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a} \cdot \mathbf{b}|^2 \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2 \cos^2\theta \\ &= |\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2\theta) \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 \sin^2\theta \end{aligned}$$

Hence (since $\sin(\theta)$ is positive for all $0 < \theta < \pi$):

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

(Note that if $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$, then both sides are 0 although $\sin\theta$ is not defined.)

## Geometric interpretation of the cross product

We can now describe the cross product of two vectors geometrically:

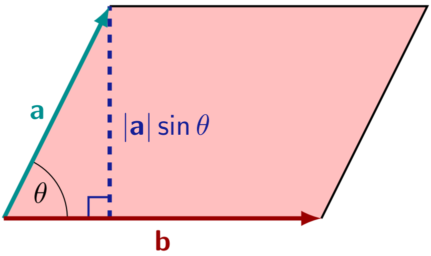
### Summary

For non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$, the cross product $\mathbf{a} \times \mathbf{b}$ is a vector of length

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

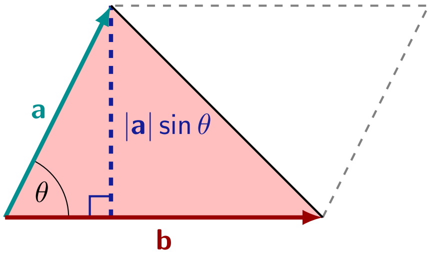
in the direction perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ as determined by the right-hand rule.

# Area of a parallelogram




$$\text{Area} \quad = \quad \text{base} \times \text{altitude} \quad = \quad |\mathbf{b}||\mathbf{a}|\sin\theta \quad = \quad |\mathbf{a} \times \mathbf{b}|$$

# Area of a triangle

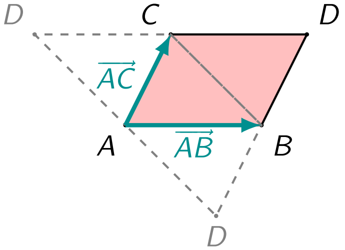



$$\text{Area} \;=\; \frac{1}{2}\text{base} \times \text{altitude} \;=\; \frac{1}{2}|\mathbf{b}||\mathbf{a}|\sin\theta \;=\; \frac{1}{2}|\mathbf{a}\times\mathbf{b}|$$

# Area of a parallelogram – Example

Find the area of a parallelogram with vertices at points $A(1,0,1)$, $B(-2,1,3)$, $C(3,1,4)$ and $D$.

What are the possibilities for $D$?




$$\text{Area} = |\overrightarrow{AB} \times \overrightarrow{AC}| = \left| \begin{pmatrix} -2-1 \\ 1-0 \\ 3-1 \end{pmatrix} \times \begin{pmatrix} 3-1 \\ 1-0 \\ 4-1 \end{pmatrix} \right| = \left| \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} 1 \times 3 - 2 \times 1 \\ 2 \times 2 - (-3) \times 3 \\ (-3) \times 1 - 1 \times 2 \end{pmatrix} \right| = \left| \begin{pmatrix} 1 \\ 13 \\ -5 \end{pmatrix} \right| = \sqrt{195}$$

**Exercise**

Show that the same area value arises for the other choices of $D$.