§3 Complex Numbers (2020T1: W4-Tu-We, W5-Tu-We-Th)

- **Review of number systems.**
  - *Natural numbers* (or counting numbers)

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

  - *Integers*

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

  - *Rational numbers* (which include all fractions and recurring decimals)

$$\mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z},\ q \neq 0\right\}$$

  - *Real numbers* (which include *irrational numbers* such as π, e, and $\sqrt{2}$)

$$\mathbb{R}$$

Note. The set of natural numbers $\mathbb{N}$ is said to be *"closed" under addition* because the sum of any two natural numbers is also a natural number.

Exercise. Complete the following table.

closed?	$+$	$-$	$\times$	$\div$ (division by zero excluded)
$\mathbb{N}$	yes			
$\mathbb{Z}$				
$\mathbb{Q}$				
$\mathbb{R}$				

Which number systems are closed under all four standard arithmetic operations?

Note. We *"extend"* a number system by introducing new numbers so that certain operations become possible in the new number system.

- **Field.** Let $\mathbb{F}$ be a non-empty set of elements for which a rule of addition and a rule of multiplication are defined. Then the system is a *field* if the following twelve axioms (or fundamental number laws) are satisfied:
  - *Closure under addition*: if $x, y \in \mathbb{F}$ then $x + y \in \mathbb{F}$.
  - *Closure under multiplication*: if $x, y \in \mathbb{F}$ then $xy \in \mathbb{F}$.
  - *Commutative law of addition*: $x + y = y + x$ for all $x, y \in \mathbb{F}$.
  - *Commutative law of multiplication*: $xy = yx$ for all $x, y \in \mathbb{F}$.
  - *Associative law of addition*: $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{F}$.
  - *Associative law of multiplication*: $(xy)z = x(yz)$ for all $x, y, z \in \mathbb{F}$.
  - *Distributive law*: $x(y + z) = xy + xz$ for all $x, y, z \in \mathbb{F}$.
  - *Distributive law*: $(x + y)z = xz + yz$ for all $x, y, z \in \mathbb{F}$.
  - *Existence of a zero*: there exists an element $0 \in \mathbb{F}$ such that $0+x = x+0 = 0$ for all $x \in \mathbb{F}$.
  - *Existence of a one*: there exists a non-zero element $1 \in \mathbb{F}$ such that $1x = x1 = x$ for all $x \in \mathbb{F}$.
  - *Existence of a negative*: for each $x \in \mathbb{F}$ there exists an element $w \in \mathbb{F}$ (usually denoted by $-x$) such that $x + w = w + x = 0$.
  - *Existence of a multiplicative inverse*: for each $x \in \mathbb{F}$ there exists an element $w \in \mathbb{F}$ (usually denoted by $1/x$ or x^{-1}) such that $xw = wx = 1$.

Note. The set of rational numbers $\mathbb{Q}$ is the first and primary example of a *field*.

Exercise. Explain why $\mathbb{Z}$ is not a field. Explain why $\mathbb{N}$ is not a field.

Exercise. Is the set $\{1, -1\}$ closed under addition, multiplication, subtraction, or division?

Exercise. Solve the following equations under each number system.

	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$
$x + 3 = 5$	$x = 2$	$x = 2$	$x = 2$	$x = 2$
$x + 7 = 5$	no solution	$x = -2$	$x = -2$	$x = -2$
$2x = 5$	no solution	no solution	$x = \frac{5}{2}$	$x = \frac{5}{2}$
$x^2 = 2$	no solution	no solution	no solution	$x = \pm\sqrt{2}$
$x^2 = -1$	no solution	no solution	no solution	no solution
$2x^2 + 5x - 3 = 0$				
$\sin(\pi x) = 0$				
$\sin x = 0$				
$x^2 - x + 1 = 0$				

- **Cartesian form of complex numbers.** The set of *complex numbers* $\mathbb{C}$ contains numbers of the form (known as the *Cartesian form*)

$$z = a + b\,i\,, \qquad \text{where} \quad a, b \in \mathbb{R}, \qquad i := \sqrt{-1}\,.$$

  - The *real part* of z is $\mathrm{Re}(z) = a$.
  - The *imaginary part* of z is $\mathrm{Im}(z) = b$.
  - The *complex conjugate* of z is $\overline{z} = a - b\,i$.
  - A complex number is said to be *purely imaginary* if and only if its real part is 0.
  - Two complex numbers are *equal* if and only if their real parts are equal and their imaginary parts are equal.

Example. If $z = 4 - 3\,i$ then

$$\mathrm{Re}(z) = 4, \quad \mathrm{Im}(z) = -3, \quad \text{and} \quad \overline{z} = 4 + 3\,i\,.$$

The number $w = 2\,i$ is purely imaginary, and we have

$$\mathrm{Re}(w) = 0, \quad \mathrm{Im}(w) = 2, \quad \text{and} \quad \overline{w} = -2\,i\,.$$

- **Arithmetic of complex numbers.** Suppose that

$$z = a + b\,i \quad \text{and} \quad w = c + d\,i\,, \qquad a, b, c, d \in \mathbb{R}$$

  - *Addition* and *subtraction*:

$$z + w = (a + b\,i\,) + (c + d\,i\,) = (a + c) + (b + d)\,i$$

$$z - w = (a + b\,i\,) - (c + d\,i\,) = (a - c) + (b - d)\,i$$

  - *Multiplication*:

$$zw = (a+b\,i\,)(c+d\,i\,) = ac+bc\,i+ad\,i+bd\,i^{\,2} = (ac-bd)+(bc+ad)\,i$$

  - *Division*:

$$\frac{z}{w} = \frac{a + b\,i}{c + d\,i} = \frac{(a + b\,i\,)(c - d\,i\,)}{(c + d\,i\,)(c - d\,i\,)} = \frac{(ac + bd) + (bc - ad)\,i}{c^2 + d^2}$$

. . . rationalizing the denominator

Notes:

(i) The set of complex numbers $\mathbb{C}$ is closed under all four standard arithmetic operations.

(ii) The set of complex numbers $\mathbb{C}$ is a field.

(iii) Unlike the real numbers, it does not make sense to say that a complex number is positive (or negative), or that one complex number is greater than (or less than) another.

Exercise. Repeat the exercises on Page 1 and Page 3 for the number system $\mathbb{C}$.

Exercise. Given $z = 2 - 4\,i$ and $w = 3 + \,i$, find $\overline{z}$, $\overline{w}$, $z + w$, $z - w$, zw, z/w and w/z.

Exercise. Simplify $\dfrac{1}{(1+\,i\,)^2}$ and $(1+\,i\,)^8$.

Properties of complex conjugates.

(a) $\overline{\overline{z}} = z$

(b) $\overline{z + w} = \overline{z} + \overline{w}$

(c) $\overline{z - w} = \overline{z} - \overline{w}$

(d) $\overline{zw} = \overline{z}\,\overline{w}$

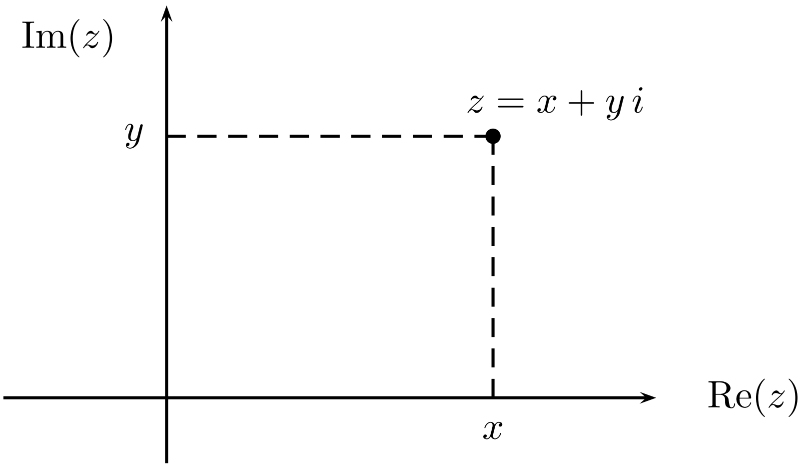
(e) $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$

(f) $\mathrm{Re}(z) = \dfrac{z + \overline{z}}{2}$

(g) $\mathrm{Im}(z) = \dfrac{z - \overline{z}}{2\,i}$

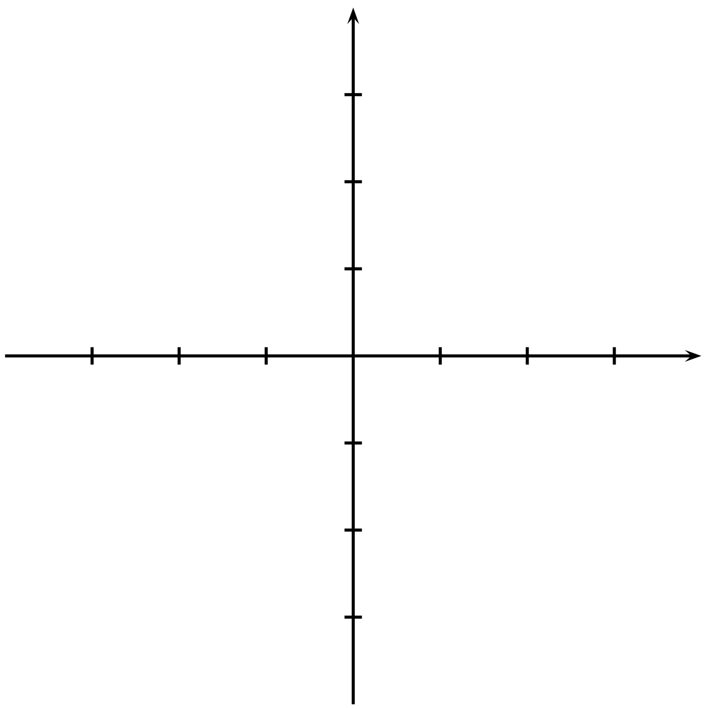
Exercise. Prove properties (b), (d), and (f).

The Argand diagram. A useful geometric picture of complex numbers can be obtained by identifying a complex number $z = x + y\,i$ with the point (x, y) in the *Cartesian plane* (or the xy-plane).



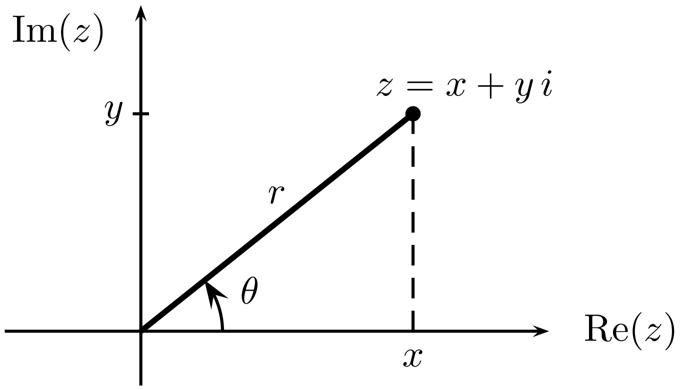
- This plot is called an *Argand diagram*.
- The horizontal axis (or the x-axis) is called the *real axis*.
- The vertical axis (or the y-axis) is called the *imaginary axis*.

Exercise. Plot the numbers 2, -1, $3\,i$, $-\,i$, $3 - 2\,i$, $-3 + \,i$, $-3 - \,i$ on an Argand diagram.



Exercise. What can you say about the relative position of a complex number z and its complex conjugate $\overline{z}$ on an Argand diagram?

- **Polar form of complex numbers.**
  - We refer to the representation $z = x + y\,i$ as the *Cartesian form* of z.



  - Using polar coordinates (r, θ) instead of the Cartesian coordinates (x, y), we obtain the *polar form* of z:

$$z = r(\cos\theta + i\,\sin\theta)\,,$$

where

r is the *distance from the origin*,
θ is the *anti-clockwise angle from the positive real axis.*

  - Using trigonometry, we see that for $z \neq 0$,

$$r = \sqrt{x^2 + y^2}, \quad \cos\theta = \frac{x}{r}, \quad \text{and} \quad \sin\theta = \frac{y}{r}.$$

Thus

$$\mathrm{Re}(z) = x = r\cos\theta \qquad \text{and} \qquad \mathrm{Im}(z) = y = r\sin\theta\,.$$

  - Two complex numbers $z_1 = r_1(\cos\theta_1 + i\,\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\,\sin\theta_2)$ are *equal* if and only if

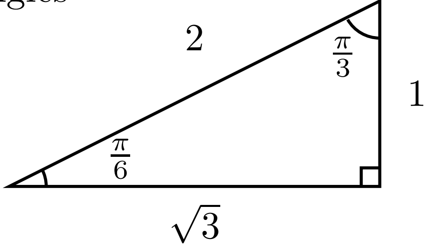
$$r_1 = r_2 \quad \text{and} \quad \theta_1 = \theta_2 + 2\pi k\,,\ k \in \mathbb{Z}.$$

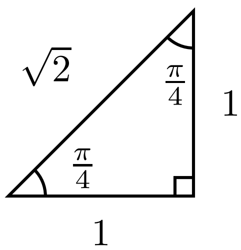
Notes:

(i) Do NOT use the notation $r\mathrm{cis}\theta$.

(ii) Use radians instead of degrees (π radians $= 180°$).

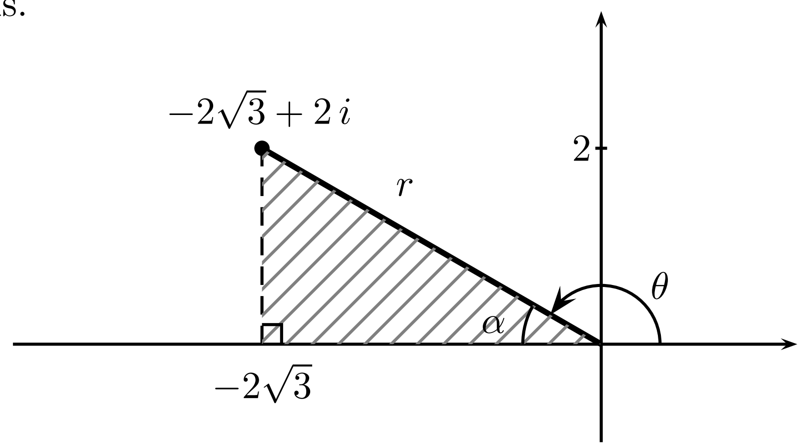
(iii) Memorise the special triangles



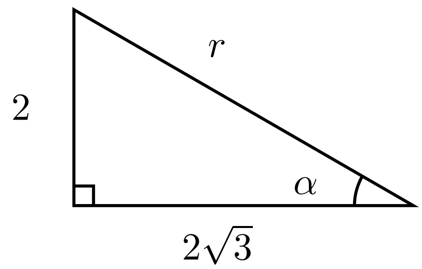


Example. We convert the complex number $-2\sqrt{3}+2\,i$ into polar form.

Step 1: Plot $z=-2\sqrt{3}+2\,i$ on an Argand diagram and form a right-angle triangle with the real axis.



Step 2: Use trigonometry to find the hypotenuse r and the acute angle α with the real axis.



$$\begin{aligned} r &= \sqrt{2^2+(2\sqrt{3})^2} = 4 \\ \tan\alpha &= \frac{2}{2\sqrt{3}} \quad \Rightarrow \quad \alpha = \frac{\pi}{6}. \end{aligned}$$

Step 3: Obtain θ.

$$\theta = \pi-\alpha = \pi-\frac{\pi}{6} = \frac{5\pi}{6}.$$

Thus the polar form of z is

$$z = 4\left(\cos\frac{5\pi}{6} + i\,\sin\frac{5\pi}{6}\right).$$

Note that due to the periodicity of sine and cosine, we can write

$$z = 4\left(\cos\left(\frac{5\pi}{6}+2\pi k\right) + i\,\sin\left(\frac{5\pi}{6}+2\pi k\right)\right), \quad k\in\mathbb{Z}.$$

- **Modulus and argument.** Let $z = x + y\,i\; = r(\cos\theta + \,i\,\sin\theta)$.
  - The *modulus* of z is
$$|z| \;=\; \sqrt{x^2 + y^2} \;=\; r\,.$$
  - The *principal argument* of $z \neq 0$, denoted by $\mathrm{Arg}(z)$, is the angle θ such that
$$\cos\theta \;=\; \frac{x}{\sqrt{x^2+y^2}}\,, \quad \sin\theta \;=\; \frac{y}{\sqrt{x^2+y^2}}\,,$$
and
$$-\pi < \theta \leq \pi\,.$$

Example. For $z = -2\sqrt{3} + 2\,i$ from the previous example, we have
$$|z| \;=\; 4 \qquad \text{and} \qquad \mathrm{Arg}(z) \;=\; \frac{5\pi}{6}.$$

Exercise. Find the modulus, principal argument, and polar form of each of the following numbers, and plot them on an Argand diagram.

(a) $3 + 3\,i$

(b) $2\,i$

(c) -5

(d) $\dfrac{1-\sqrt{3}\,i}{3}$

(e) $-3-4\,i$

Exercise. Find the modulus and principal argument of $\overline{z}$ if $|z| = r$ and $\operatorname{Arg}(z) = \theta$. What can you say about $\overline{z}$ on an Argand diagram?

- **Properties of polar form.**
  - *Simple lemma*:
    For any real numbers θ_1 and θ_2, we have
    $$(\cos\theta_1 + i\,\sin\theta_1)(\cos\theta_2 + i\,\sin\theta_2) \;=\; \cos(\theta_1+\theta_2) + i\,\sin(\theta_1+\theta_2)\,.$$
  - *De Moivre's Theorem*:
    For any real number θ and integer n, we have
    $$\boxed{(\cos\theta + i\,\sin\theta)^n \;=\; \cos(n\theta) + i\,\sin(n\theta)}\,.$$

Proof of the simple lemma. Expanding the left-hand side, we obtain

$$(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i\,(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)$$

The result then follows the trigonometric formulae

$$\begin{aligned}\cos(\theta_1+\theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\\ \sin(\theta_1+\theta_2) &= \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2.\end{aligned}$$

$\square$

Proof of De Moivre's Theorem. We prove this result by cases.

- For $n > 0$, the proof is by *mathematical induction*.

  Base step: The result is clearly true for $n = 1$.

  Induction step: Suppose that the result holds for some integer $k \geq 1$, i.e.,

  $$(\cos\theta + i\,\sin\theta)^k \;=\; \cos(k\theta) + i\,\sin(k\theta)\,.$$

  We now show that it also holds for $n = k+1$. We have

  $$\begin{aligned}&(\cos\theta + i\,\sin\theta)^{k+1}\\ &\quad= (\cos\theta + i\,\sin\theta)\,(\cos\theta + i\,\sin\theta)^k\\ &\quad= (\cos\theta + i\,\sin\theta)\,[\cos(k\theta) + i\,\sin(k\theta)] \qquad \text{by the induction hypothesis}\\ &\quad= \cos((k+1)\theta) + i\,\sin((k+1)\theta) \qquad \text{by the simple lemma.}\end{aligned}$$

  Thus the result also holds for $n = k+1$.

  Conclusion: Hence, by induction, the result holds for all integers $n \geq 1$.

- For $n = 0$, the result holds following the convention that $z^0 = 1$ for any complex number z.
- For $n = -1$, by definition, $z^{-1} = 1/z$, so we apply the division rule for complex numbers

$$\begin{aligned}(\cos\theta + i\,\sin\theta)^{-1} &= \frac{1}{\cos\theta + i\,\sin\theta} \\ &= \frac{1}{\cos\theta + i\,\sin\theta} \times \frac{\cos\theta - i\,\sin\theta}{\cos\theta - i\,\sin\theta} \\ &= \frac{\cos\theta - i\,\sin\theta}{\cos^2\theta + \sin^2\theta} \\ &= \cos(-\theta) + i\,\sin(-\theta),\end{aligned}$$

where we used the trigonometric identities

$$\cos(-\theta) = \cos\theta, \qquad \sin(-\theta) = -\sin\theta, \qquad \text{and} \qquad \sin^2\theta + \cos^2\theta = 1.$$

- For $n < -1$, we have

$$\begin{aligned}(\cos\theta + i\,\sin\theta)^{-n} &= [(\cos\theta + i\,\sin\theta)^{-1}]^n \\ &= [\cos(-\theta) + i\,\sin(-\theta)]^n \quad \text{by the result for } n = -1 \\ &= \cos(-n\theta) + i\,\sin(-n\theta) \quad \text{by the result for } n > 0.\end{aligned}$$

□

- **Euler's formula and polar form.**
  - For any real number θ, *Euler's formula* defines

$$\boxed{e^{i\theta} := \cos\theta + i\sin\theta}\,.$$

  - Henceforth, the *polar form* of a non-zero complex number is written as

$$z = re^{i\theta},$$

    where $r = |z|$ and $\theta = \operatorname{Arg}(z) + 2\pi k$, $k \in \mathbb{Z}$.
    - It follows from the periodicity of sine and cosine that

$$e^{i\theta} = e^{i(\theta+2\pi k)} \quad \text{for all} \quad k \in \mathbb{Z}.$$

    - Two complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ are *equal* if and only if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2\pi k$, $k \in \mathbb{Z}$.
    - The complex conjugate of $z = re^{i\theta}$ is $\overline{z} = re^{-i\theta}$.

Notes:

(i) We prefer this new "exponential" representation $z = re^{i\theta}$ of polar form to the "trigonometric" representation $z = r(\cos\theta + i\sin\theta)$.

(ii) Since cosine is an even function and sine is an odd function, we have

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta,$$

which is the complex conjugate of $e^{i\theta}$.

(iii) Euler's formula is a definition. It may seem somewhat arbitrary at first, but there are multiple reason why it is reasonable:

For any real numbers a, θ, ϕ and integer n, we have

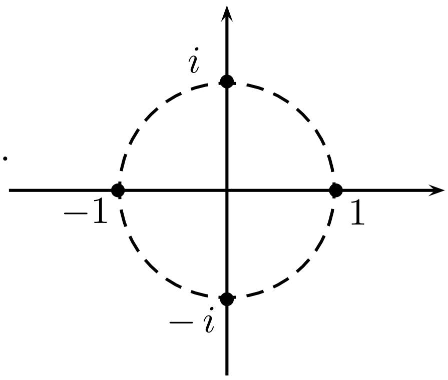
$$(e^{a\theta})^n = e^{an\theta}, \qquad e^{a\theta}\,e^{a\phi} = e^{a(\theta+\phi)}, \qquad e^0 = 1, \qquad \frac{\mathrm{d}}{\mathrm{d}\theta}e^{a\theta} = ae^{a\theta}\,.$$

If a is replaced by i and if $e^{a\theta}$ is replaced by $\cos\theta + i\sin\theta$, then all four formulae are still satisfied. So the definition for $e^{i\theta}$ is consistent with our experience with other exponential functions.

(iv) In more serious treatment of complex numbers, $e^{i\theta}$ will be defined by other means and then Euler's formula becomes a theorem.

Example. Important special cases:

$$1 = e^{i\,0}, \quad i = e^{i\,\pi/2}, \quad -1 = e^{i\,\pi}, \quad -i = e^{i\,(-\pi/2)}.$$



Exercise. Plot the following numbers on an Argand diagram and convert them into Cartesian form.

(a) $2\,e^{i\,\pi/6}$

(b) $e^{-i\,\pi/3}$

(c) $3\,e^{i\,(3\pi/4)}$

- **Arithmetic of polar form.**
  - *Multiplication* and *division*: if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then
$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} \qquad \text{and} \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$$
  - *Power*: for any positive integer n, if $z = re^{i\theta}$ then
$$z^n = r^n e^{in\theta}.$$

Notes:

(i) For multiplication, we have
$$|z_1 z_2| = |z_1||z_2| \qquad \text{and} \qquad \operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) + 2\pi k,$$
where k is an integer chosen so that $-\pi < \operatorname{Arg}(z_1 z_2) \leq \pi$.

(ii) For division, we have
$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad \text{and} \qquad \operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg}(z_1) - \operatorname{Arg}(z_2) + 2\pi k,$$
where k is an integer chosen so that $-\pi < \operatorname{Arg}(z_1/z_2) \leq \pi$.

(iii) For powers, we have
$$|z^n| = |z|^n \qquad \text{and} \qquad \operatorname{Arg}(z^n) = n\operatorname{Arg}(z) + 2\pi k,$$
where k is an integer chosen so that $-\pi < \operatorname{Arg}(z^n) \leq \pi$.

(iv) Multiplying a complex number z by $w = re^{i\theta}$ has the geometric interpretation of rotating z anti-clockwise about the origin by the angle θ and stretching its distance from the origin by a factor of r.

(v) For addition and subtraction of complex numbers, it is easier to work with Cartesian form; for multiplication, division, powers and roots (see later), it is easier to work with polar form.

Example. If $z = 2\,e^{i(4\pi/7)}$ and $w = 6\,e^{i\pi/3}$, then
$$\begin{aligned}
zw &= 2 \times 6\,e^{i(4\pi/7+\pi/3)} = 12\,e^{i(19\pi/21)},\\
\frac{z}{w} &= \frac{2}{6}\,e^{i(4\pi/7-\pi/3)} = \frac{1}{3}\,e^{i(5\pi/21)},\\
z^5 &= 2^5\,e^{i(5\times 4\pi/7)} = 32\,e^{i(20\pi/7)} = 32\,e^{i(6\pi/7)},\\
w^4 &= 6^4\,e^{i(4\times\pi/3)} = 1296\,e^{i(4\pi/3)} = 1296\,e^{i(-2\pi/3)}.
\end{aligned}$$

Exercise. Find the number obtained by rotating $z = 2\sqrt{3} - 2\,i$ anti-clockwise about the origin by an angle of $\pi/3$.

Exercise. Let $z = (-1 - i\,)^{1111}$. Find $|z|$ and $\mathrm{Arg}(z)$.

Exercise. Let $z = 1 + \sqrt{3}\,i$ and $w = 1 - i\,$. Find the modulus and principal argument of zw. Hence obtain $\sin(\pi/12)$ and $\cos(\pi/12)$.

Exercise. Prove that $z\bar{z} = |z|^2$.

Exercise. Let $w = (-1 - i)^3(1 - i\sqrt{3})^4$. Find $|w|$ and $\mathrm{Arg}(w)$.

Exercise. Let $z = \dfrac{-1 + i}{(\sqrt{3} - i)^4}$. Find $|z|$ and $\mathrm{Arg}(z)$.

Square roots of complex numbers. If $a + b\,i$ is a square root of z, then (by working with the Cartesian form)

$$\begin{cases} (a+b\,i)^2 = z \\ |a+b\,i|^2 = |z| \end{cases} \implies \begin{cases} (a^2-b^2)+2ab\,i = z \\ a^2+b^2 = |z| \end{cases}$$

$$\implies \begin{cases} a^2-b^2 = \mathrm{Re}(z) \\ 2ab = \mathrm{Im}(z) \\ a^2+b^2 = |z| \end{cases}$$

Example. To obtain the square roots of $1-\sqrt{3}\,i$, we consider

$$\begin{cases} (a+b\,i)^2 = 1-\sqrt{3}\,i \\ |a+b\,i|^2 = |1-\sqrt{3}\,i\,| \end{cases} \implies \begin{cases} a^2-b^2 = 1 \\ 2ab = -\sqrt{3} \\ a^2+b^2 = 2 \end{cases}$$

The first and the third equations lead to $a^2 = 3/2$ and $b^2 = 1/2$, which means $a = \pm\sqrt{3}/\sqrt{2}$ and $b = \pm 1/\sqrt{2}$. The second equation indicates that a and b have opposite signs. Thus the square roots of $1-\sqrt{3}\,i$ are

$$\frac{\sqrt{3}-i}{\sqrt{2}} \quad \text{and} \quad \frac{-\sqrt{3}+i}{\sqrt{2}}.$$

Exercise. Use the quadratic formula to solve the equation

$$z^2 - (4+\,i\,)z + (5+5\,i\,) = 0.$$

- **Roots of complex numbers.** Let n be a positive integer.
  - A complex number w is an *nth root* of a number z if z is the nth power of w, i.e., $w^n = z$.
  - If $r_{\text{root}}\, e^{i\,\theta_{\text{root}}}$ is an nth root of $r\, e^{i\,\theta}$, then

$$(r_{\text{root}}\, e^{i\,\theta_{\text{root}}})^n = r\, e^{i\,\theta} \implies \begin{cases} r_{\text{root}}^n &= r \\ n\,\theta_{\text{root}} &= \theta + 2\pi k \quad \text{for all } k \in \mathbb{Z} \end{cases}$$
$$\implies \begin{cases} r_{\text{root}} &= r^{1/n} \\ \theta_{\text{root}} &= \dfrac{\theta + 2\pi k}{n} \quad \text{for all } k \in \mathbb{Z} \end{cases}$$

  Thus the nth roots of $r\, e^{i\,\theta}$ satisfy

$$\boxed{r^{1/n}\, e^{i\,(\theta+2\pi k)/n}\,, k \in \mathbb{Z}}\,.$$

  - Every complex number has n distinct nth roots.
  - These n roots lie equally spaced on a circle on an Argand diagram, with adjacent roots at an angle of $2\pi/n$ apart.

Example. The fifth roots of unity are the numbers $r_{\text{root}}\, e^{i\,\theta_{\text{root}}}$ satisfying

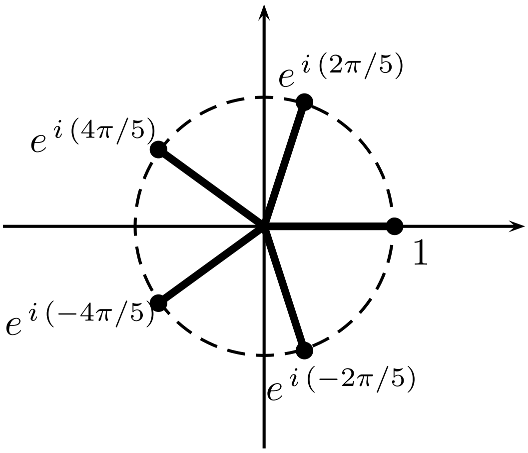
$$(r_{\text{root}}\, e^{i\,\theta_{\text{root}}})^5 = 1 = 1\, e^{i\,0}.$$

Thus

$$\begin{cases} r_{\text{root}}^5 = 1 \\ 5\,\theta_{\text{root}} = 0 + 2\pi k, \quad k \in \mathbb{Z} \end{cases} \implies \begin{cases} r_{\text{root}} &= 1 \\ \theta_{\text{root}} &= \dfrac{2\pi k}{5}, \quad k \in \mathbb{Z} \end{cases}$$

We obtain five distinct roots:

$$\begin{aligned} k &= -2 \;\Rightarrow\; e^{i\,(-4\pi/5)}\,, \\ k &= -1 \;\Rightarrow\; e^{i\,(-2\pi/5)}\,, \\ k &= 0 \;\Rightarrow\; e^{i\,0} = 1\,, \\ k &= 1 \;\Rightarrow\; e^{i\,(2\pi/5)}\,, \\ k &= 2 \;\Rightarrow\; e^{i\,(4\pi/5)}\,. \end{aligned}$$



The roots lie on a circle of radius 1, with adjacent roots at an angle of $2\pi/5$ apart. With e.g., $k = 4$, we get $e^{i\,(8\pi/5)} = e^{i\,(-2\pi/5+2\pi)}$, which is the same as $e^{i\,(-2\pi/5)}$. Apart from the root 1, we have 4 roots come in conjugate pairs.

Exercise. Find all sixth roots of -3 and plot them on an Argand diagram.

Exercise. Find all fifth roots of $\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}\,i$ and plot them on an Argand diagram.

Binomial Theorem and Pascal's triangle.

- *Binomial Theorem*: for any natural number n and complex numbers z and w, we have

$$\begin{aligned}(z+w)^n &= z^n + nz^{n-1}w + \frac{n(n-1)}{2!}z^{n-2}w^2 + \frac{n(n-1)(n-2)}{3!}z^{n-3}w^3 \\ &\quad + \cdots + w^n \\ &= \sum_{k=0}^{n} \binom{n}{k} z^{n-k}w^k,\end{aligned}$$

with *binomial coefficients*

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

- For small values of n, the binomial coefficients can be easily calculated using *Pascal's triangle*:

$$\begin{array}{lccccccccccc}
n=0 & & & & & & 1 & & & & & \\
n=1 & & & & & 1 & & 1 & & & & \\
n=2 & & & & 1 & & 2 & & 1 & & & \\
n=3 & & & 1 & & 3 & & 3 & & 1 & & \\
n=4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
n=5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & & \vdots & & & & &
\end{array}$$

Example. For $n = 5$, the binomial coefficients are 1, 5, 10, 10, 5, 1. Thus

$$\begin{aligned}(z+w)^5 &= z^5 + 5z^4w + 10z^3w^2 + 10z^2w^3 + 5zw^4 + w^5, \\ (z-w)^5 &= z^5 - 5z^4w + 10z^3w^2 - 10z^2w^3 + 5zw^4 - w^5.\end{aligned}$$

Exercise. Expand $(2x-y)^6$.

Trigonometric applications of complex numbers. It follows from Euler's formula that

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}.$$

Using these together with the *Binomial Theorem* and *De Moivre's Theorem*, we can derive trigonometric formulae which relate powers of $\sin\theta$ or $\cos\theta$ to sines or cosines of multiples of θ.

- Powers of sine and cosine, e.g., $\cos^5\theta$
- Sine and cosine of multiple angles, e.g., $\cos(5\theta)$

Example. We obtain a formula for $\cos^5\theta$ in terms of cosines of multiples of θ as follows:

$$\begin{aligned}
\cos^5\theta &= \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^5 \\
&= \frac{1}{2^5}\left(e^{i\theta}+e^{-i\theta}\right)^5 \\
&= \frac{1}{2^5}\left(e^{5i\theta}+5e^{3i\theta}+10e^{i\theta}+10e^{-i\theta}+5e^{-3i\theta}+e^{-5i\theta}\right) \\
&= \frac{1}{2^4}\left(\frac{e^{5i\theta}+e^{-5i\theta}}{2}+5\,\frac{e^{3i\theta}+e^{-3i\theta}}{2}+10\,\frac{e^{i\theta}+e^{-i\theta}}{2}\right) \\
&= \frac{1}{16}\Big(\cos(5\theta)+5\cos(3\theta)+10\cos\theta\Big).
\end{aligned}$$

Exercise. Find a formula for $\sin^5\theta$ in terms of multiples of θ.

Example. To obtain a formula for $\cos(5\theta)$, we expand

$$\begin{aligned}
\cos(5\theta) + i\,\sin(5\theta) &= (\cos\theta + i\,\sin\theta)^5 \\
&= \cos^5\theta + 5\cos^4\theta(i\,\sin\theta) + 10\cos^3\theta(i\,\sin\theta)^2 \\
&\quad + 10\cos^2\theta(i\,\sin\theta)^3 + 5\cos\theta(i\,\sin\theta)^4 + (i\,\sin\theta)^5 \\
&= \left(\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta\right) \\
&\quad + i\left(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta\right).
\end{aligned}$$

Separating the real and imaginary parts of the last expression leads to

$$\begin{aligned}
\cos(5\theta) &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta, \\
\sin(5\theta) &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta.
\end{aligned}$$

Using the identity $\sin^2\theta + \cos^2\theta = 1$, we can write

$$\begin{aligned}
\cos(5\theta) &= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2 \\
&= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta,
\end{aligned}$$

which is expressed purely in terms of powers of $\cos\theta$.

Exercise. Express $\sin(6\theta)$ as a product of $\cos\theta$ and a polynomial in $\sin\theta$.

Complex polynomials.

- A *complex polynomial* is a function $p : \mathbb{C} \to \mathbb{C}$ of the form

$$p(z) \;=\; a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0\,,$$

  where n is a natural number, and $a_n, a_{n-1}, \ldots, a_1, a_0$ are complex numbers called the *coefficients* of the polynomial.

- Two polynomials p and q satisfy $p(z) = q(z)$ for all z if and only if the corresponding coefficients of p and q are equal.

- A polynomial p satisfies $p(z) = 0$ for all z if and only if all of its coefficients are zero. This unique polynomial is called the *zero polynomial*.

- The *degree* of the polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$, denoted by $\deg(p)$, is the largest integer k such that $a_k \neq 0$.

- A number α is a *root* (or *zero*) of a polynomial p if $p(\alpha) = 0$.

- The polynomials p_1 and p_2 are called *factors* of a polynomial p if $p(z) = p_1(z)\,p_2(z)$ for all complex numbers z.

Example. Consider

$$\begin{aligned}
p_1(z) &= z^3 - 1 && \text{for all } z \in \mathbb{R},\\
p_2(z) &= z^3 - 1 && \text{for all } z \in \mathbb{C},\\
p_3(z) &= 2\,i\,z^5 - z^4 + i\,z^2 - 2 && \text{for all } z \in \mathbb{C},\\
p_4(z) &= z^3 + 1 && \text{for all } z \in \mathbb{C}.
\end{aligned}$$

Then

(a) $p_1(z)$ for all $z \in \mathbb{R}$ is a real polynomial of degree 3.

(b) $p_2(z)$ for all $z \in \mathbb{C}$ is a complex polynomial of degree 3 with real coefficients.

(c) $p_3(z)$ for all $z \in \mathbb{C}$ is a complex polynomial of degree 5.
The coefficient of z^5 is $2\,i$.
The coefficient of z^4 is -1.
The coefficient of z^3 is 0.

(d) Since $p_4(-1) = 0$, we know that $z = -1$ is a root of the polynomial p_4.

(e) We can write $p_4(z) = (z+1)\,(z^2 - z + 1)$ for all $z \in \mathbb{C}$. Thus $z + 1$ is a *linear* factor of p_4 and $z^2 - z + 1$ is a *quadratic* factor of p_4.

- **Roots and factors of complex polynomials.**
  - *Remainder Theorem*:
    When a polynomial $p(z)$ is divided by $z-\alpha$, the remainder is given by $p(\alpha)$.
  - *Factor Theorem*:
    A number α is a root of a polynomial $p(z)$ if and only if $z-\alpha$ is a factor of $p(z)$. Equivalently, $p(\alpha)=0$ if and only if $z-\alpha$ is a factor of $p(z)$.
  - *The Fundamental Theorem of Algebra*:
    Every complex polynomial of degree $n \geq 1$ has at least one root.
  - *Factorisation Theorem*:
    Every complex polynomial of degree $n \geq 1$ can be written as a product of linear factors

    $$p(z) \;=\; a_n(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)\,,$$

    where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of $p(z)$, and a_n is the coefficient of z^n.

Example. Let $p(z) = z^5 - 2z^4 + 2z - 1$.

(a) Since $p(1) = 1 - 2 + 2 - 1 = 0$, we know that 1 is a root of $p(z)$, and $z-1$ is a factor of $p(z)$. So when $p(z)$ is divided by $z-1$, the remainder is 0.

$$\begin{array}{rrrrrrr}
 & & z^4 & -z^3 & -z^2 & -z & +1 \\
\cline{2-7}
z-1\,) & z^5 & -2z^4 & & & +2z & -1 \\
 & z^5 & -z^4 & & & & \\
\cline{2-7}
 & & -z^4 & & & & \\
 & & -z^4 & +z^3 & & & \\
\cline{3-7}
 & & & -z^3 & & & \\
 & & & -z^3 & +z^2 & & \\
\cline{4-7}
 & & & & -z^2 & +2z & \\
 & & & & -z^2 & +z & \\
\cline{5-7}
 & & & & & z & -1 \\
 & & & & & z & -1 \\
\cline{6-7}
 & & & & & & 0
\end{array}$$

(b) The remainder when $p(z)$ is divided by $z-2$ is $p(2) = 32 - 32 + 4 - 1 = 3$.

(c) The remainder when $p(z)$ is divided by $2z+1$ is

$$p\Big(-\frac{1}{2}\Big) \;=\; -\frac{1}{32} - \frac{1}{8} - 1 - 1 \;=\; -\frac{69}{32}.$$

Exercise. Factorise $p(z) = z^4 + 2z^3 - 9z^2 - 2z + 8$.

Notes:

(i) The Factorisation Theorem guarantees that a complex polynomial of degree n always has n roots, but it does not tell us how to actually find these roots. In general, this can be a very difficult task. One easy case is

$$p(z) \;=\; z^n - a_0\,,$$

for which the roots are exactly the nth roots of a_0.

(ii) If α is a root of a complex polynomial with real coefficients, then so is $\overline{\alpha}$. In other words, the roots come in conjugate pairs. Note that

$$\begin{aligned}(z-\alpha)\,(z-\overline{\alpha}) \;&=\; z^2 - (\alpha+\overline{\alpha})z + \alpha\,\overline{\alpha}\\ &=\; z^2 - 2\,\mathrm{Re}(\alpha)\,z + |\alpha|^2\\ &=\; z^2 - 2\,r\cos\theta\,z + r^2 \quad \text{when } \alpha = r\,e^{\,i\,\theta}.\end{aligned}$$

(iii) A complex polynomial with real coefficients can be factorised into linear and/or quadratic factors all of which have real coefficients.

Example. Let $p(z) = z^6 + 1$.

- To factorise $p(z)$, we solve the equation
$$z^6 = -1.$$
Let $z = r\,e^{i\,\theta}$. We have
$$(r\,e^{i\,\theta})^6 = e^{i\,\pi},$$
which leads to
$$\begin{cases} r^6 = 1 \\ 6\,\theta = \pi + 2\pi k, \quad k \in \mathbb{Z} \end{cases} \implies \begin{cases} r = 1 \\ \theta = \dfrac{\pi + 2\pi k}{6}, \quad k \in \mathbb{Z} \end{cases}$$
Hence
$$z = e^{i\,(\pi + 2\pi k)/6}, \quad k \in \mathbb{Z}.$$
Taking $k = -3, -2, -1, 0, 1, 2$, we obtain six distinct roots
$$e^{i\,(-5\pi/6)}, \quad e^{i\,(-3\pi/6)}, \quad e^{i\,(-\pi/6)}, \quad e^{i\,(\pi/6)}, \quad e^{i\,(3\pi/6)}, \quad e^{i\,(5\pi/6)}.$$

- Expressing as *a product of complex linear factors*, we have in polar form
$$z^6 + 1 = \left(z - e^{i\,(\pi/2)}\right)\left(z - e^{i\,(-\pi/2)}\right)\left(z - e^{i\,(\pi/6)}\right)\left(z - e^{i\,(-\pi/6)}\right) \times \left(z - e^{i\,(5\pi/6)}\right)\left(z - e^{i\,(-5\pi/6)}\right),$$
or in Cartesian form
$$z^6 + 1 = (z - i)(z + i)\left(z - \frac{\sqrt{3} + i}{2}\right)\left(z - \frac{\sqrt{3} - i}{2}\right) \times \left(z - \frac{-\sqrt{3} + i}{2}\right)\left(z - \frac{-\sqrt{3} - i}{2}\right).$$

- We can replace the conjugate pairs by quadratic factors with real coefficients and obtain *a product of real linear and/or real irreducible quadratic factors*
$$z^6 + 1 = (z^2 + 1)\left(z^2 - \sqrt{3}z + 1\right)\left(z^2 + \sqrt{3}z + 1\right).$$

- If we want a factorisation of $z^6 + 1$ with only *rational coefficients*, we can multiply the last two factors together and obtain
$$z^6 + 1 = (z^2 + 1)\,(z^4 - z^2 + 1)\,.$$

Exercise. Express $z^5 + 32$ as

(a) a product of complex linear factors;

(b) a product of real linear and/or real irreducible quadratic factors;

(c) a product of factors with rational coefficients;

(d) hence obtain $\cos(\pi/5)$ and $\cos(3\pi/5)$.

Exercise. Let $p(z) = 2z^6 + 8z^4 + z^2 + 4$.

(a) By making use of $p(2\,i\,) = 0$, express $p(z)$ as a product of a quadratic polynomial and a quartic polynomial.

(b) Find all solutions to $p(z) = 0$.

(c) Express $p(z)$ as a product of real linear and real irreducible quadratic factors.