MATH1131 Mathematics 1A – Algebra

Lecture 16: Row Echelon Form and Gaussian Elimination

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Based on slides by Jonathan Kress

Vector and matrix form

Consider the system of linear equations

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 1 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & -1 \\ 7x_1 & - & 5x_2 & - & 9x_3 & = & 0 \end{array}$$

This is the same as the vector equation

$$x_1 \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 5 \\ -5 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 6 \\ -9 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

We can also write this as the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

where A is called the coefficient matrix and

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -5 & -9 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Vector and matrix form

All of these presentations:

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 1 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & -1 \\ 7x_1 & - & 5x_2 & - & 9x_3 & = & 0, \end{array}$$

$$x_1 \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 5 \\ -5 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 6 \\ -9 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -5 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

are most simply represented by the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & -1 \\ 7 & -5 & -9 & 0 \end{array} \right).$$

Leading rows and entries

Our goal is to simplify the augmented matrix via elementary row operations. So we'd like to define what makes a matrix simpler.

Definitions

- A leading row is a non-zero row.
- A leading entry is the leftmost non-zero entry in a leading row.
- A leading column is a column containing a leading entry.

For example, consider the following matrix:

$$\begin{pmatrix} 0 & 5 & 7 \\ 0 & 0 & 0 \end{pmatrix}$$

- Row 1 is a leading row with leading entry 5.
- Row 2 is a non-leading row (a row of zeros).
- Column 2 is the only leading column.

Row echelon form

Definition

A matrix is in row echelon form (REF) if:

- all rows of zeros are at the bottom, and
- each leading entry is further to the right than all leading entries in the rows above it.

For example:

$$\left(\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 2 & 2 \end{array}\right) \quad \left(\begin{array}{cc|c} 5 & 1 & 2 \\ 0 & 0 & 2 \end{array}\right) \quad \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 3 & 2 & 2 \end{array}\right)$$

REF   REF   not REF

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 3 & 2 & 0 \end{array}\right) \quad \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

not REF   REF   REF

Reduced row echelon form

Definition

A matrix is in reduced row echelon form (RREF) if:

- it is in row echelon form, and
- each leading entry is 1, and
- each leading entry is the only non-zero entry in its column.

For example:

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 3 \end{array}\right) \quad \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 4 \end{array}\right) \quad \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right)$$

not RREF — not RREF — RREF

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{cc|c} 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

RREF — RREF — RREF

Pivots

It can be useful to refer to special leading entries called pivots.

Definitions

- The pivot element of a matrix is the first non-zero entry in the first non-zero column.
- The pivot row is the row containing the pivot element.
- The pivot column is the column containing the pivot element.

For example, consider the following augmented matrix:

$$\left(\begin{array}{ccc|c} 0 & 0 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 2 & 0 & 4 \end{array}\right)$$

The highlighted entry is its pivot element.

Row 2 is the pivot row and column 2 is the pivot column.

Gaussian elimination

To solve a system of linear equations, we apply Gaussian elimination to its augmented matrix to achieve row echelon form.

The steps to follow are:

1. find the pivot element,
2. swap row 1 with the pivot row,
3. use row operations of the form $R_i \to R_i + \alpha R_j$ to create zero entries below the pivot element,
4. repeat for the submatrix below and to the right of the pivot element.

To further reduce the matrix to reduced row echelon form:

5. divide each leading row through by its leading entry to ensure all leading entries are 1,
6. use row operations of the form $R_i \to R_i + \alpha R_j$ to create zero entries above each leading entry, working from the bottom row upwards.

Solutions to a matrix in row echelon form

A system of linear equations can be easily solved once its augmented matrix is reduced to REF.

- If the column right of the vertical line contains a leading entry, then the system has no solutions (it is an inconsistent system).
- Otherwise, if the column right of the vertical line does not contain a leading entry, the system has at least one solution:
  - If all of the columns left of the vertical line contain a leading entry, then the system has a unique solution.
  - If any of the columns left of the vertical line do not contain a leading entry, then the system has infinitely many solutions.

For example:

$$\left(\begin{array}{ccc|c} 1 & 4 & 7 & 2 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 3 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 4 & 7 & 2 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 4 & 3 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 4 & 7 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

has no solutions — has a unique solution — has infinitely many solutions

Gaussian elimination examples

Example

Solve the following system of linear equations:

$$
\begin{aligned}
x + 2y + 4z &= 2 \\
2x - y + 3z &= 1 \\
3x + y + 7z &= 4
\end{aligned}
$$

The corresponding augmented matrix is:

$$
\left(\begin{array}{ccc|c}
1 & 2 & 4 & 2 \\
2 & -1 & 3 & 1 \\
3 & 1 & 7 & 4
\end{array}\right)
$$

We want to reduce this to row echelon form...

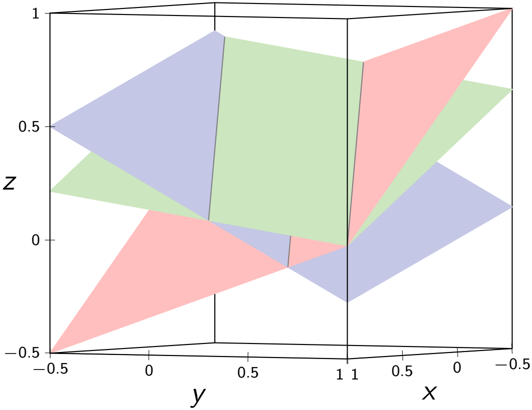
Row-reducing the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 2 \\ 2 & -1 & 3 & 1 \\ 3 & 1 & 7 & 4 \end{array}\right) \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 2 \\ 0 & -5 & -5 & -3 \\ 0 & -5 & -5 & -2 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 2 \\ 0 & -5 & -5 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

The column right of the vertical line contains a leading entry. So the system has no solutions.

(Notice that R_3 means $0x + 0y + 0z = 1$, which is impossible. So this confirms the system is inconsistent.)



$$\begin{aligned} x + 2y + 4z &= 2 \\ 2x - y + 3z &= 1 \\ 3x + y + 7z &= 4 \end{aligned}$$

has no solutions.

Geometrically, there is no solution because the planes only ever meet in pairs. The three lines at which the pairs intersect are parallel.

In other cases with no solutions, it's possible that two or all three of the planes could be parallel.

Gaussian elimination examples

Example

Solve the following system of linear equations:

$$\begin{aligned} x + y + 3z &= 4 \\ 2x + y + z &= 0 \\ x + 3y - z &= 6 \end{aligned}$$

The corresponding augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 2 & 1 & 1 & 0 \\ 1 & 3 & -1 & 6 \end{array}\right)$$

We again want to reduce this to row echelon form...

Gaussian elimination examples

Row-reducing the augmented matrix:

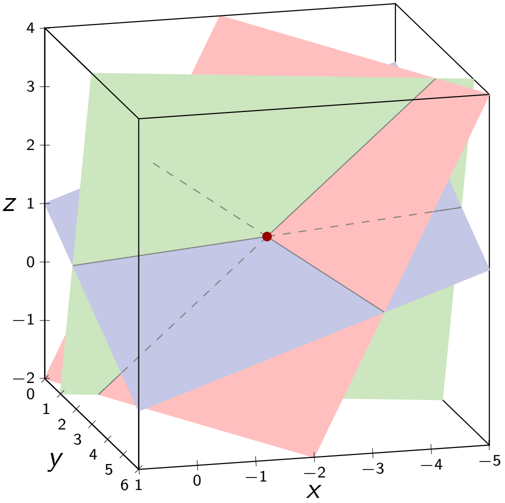
$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 2 & 1 & 1 & 0 \\ 1 & 3 & -1 & 6 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & -1 & -5 & -8 \\ 0 & 2 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & -1 & -5 & -8 \\ 0 & 0 & -14 & -14 \end{array}\right)$$

Every column left of the vertical line contains a leading entry. So the system has a unique solution.

Using back-substitution, R_3 tells us $-14z = -14$, so $\boxed{z = 1}$.
From R_2 we know $-y - 5z = -8$, so $\boxed{y = 3}$.
From R_1 we know $x + y + 3z = 4$, so $\boxed{x = -2}$.

So the unique solution is $x = -2$, $y = 3$, and $z = 1$.



$$\begin{aligned} x + y + 3z &= 4 \\ 2x + y + z &= 0 \\ x + 3y - z &= 6 \end{aligned}$$

has solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}.$$

Geometrically, there is a unique solution because all three planes meet at exactly one point.

Example

Solve the following system of linear equations:

$$\begin{aligned} x - 3y - 7z &= -17 \\ 2x - y - 4z &= -14 \\ 2x + 7y + 12z &= 18 \end{aligned}$$

The corresponding augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & -3 & -7 & -17 \\ 2 & -1 & -4 & -14 \\ 2 & 7 & 12 & 18 \end{array}\right)$$

We again want to reduce this to row echelon form...

Gaussian elimination examples

Row-reducing the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -3 & -7 & -17 \\ 2 & -1 & -4 & -14 \\ 2 & 7 & 12 & 18 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & -3 & -7 & -17 \\ 0 & 5 & 10 & 20 \\ 0 & 13 & 26 & 52 \end{array}\right)$$

$$\xrightarrow[R_3 \to \frac{1}{13} R_3]{R_2 \to \frac{1}{5} R_2} \left(\begin{array}{ccc|c} 1 & -3 & -7 & -17 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 2 & 4 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & -3 & -7 & -17 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

There is no leading entry in the rightmost column, so there is at least one solution.

Furthermore, the third column does not contain a leading entry. So the system has infinitely many solutions...

We found:

$$\left(\begin{array}{ccc|c} 1 & -3 & -7 & -17 \\ 2 & -1 & -4 & -14 \\ 2 & 7 & 12 & 18 \end{array}\right) \xrightarrow{\cdots} \left(\begin{array}{ccc|c} 1 & -3 & -7 & -17 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

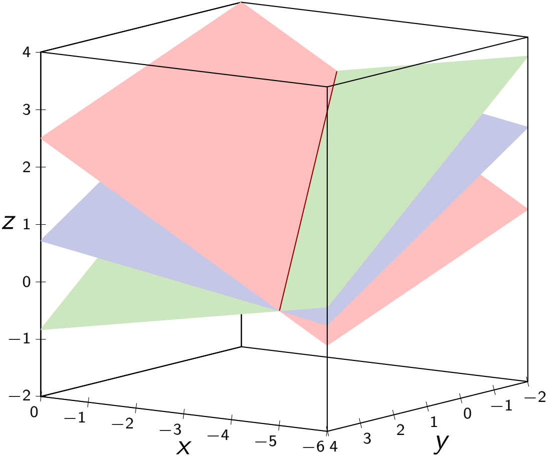
Since the third column does not contain a leading entry, we can set the corresponding variable z to be a parameter. So let $\boxed{z = \lambda}$.
From R_2 we know $y + 2z = 4$, so $\boxed{y = 4 - 2\lambda}$.
From R_1 we know $x - 3y - 7z = -17$, so $\boxed{x = \lambda - 5}$.

So the set of infinite solutions is given by

$$x = \lambda - 5, \; y = 4 - 2\lambda, \text{ and } z = \lambda \text{ for any } \lambda \in \mathbb{R}.$$

Notice that since this is a parametrised solution in one parameter, geometrically the solution will be a line.

Gaussian elimination examples



$$\begin{aligned} x - 3y - 7z &= -17 \\ 2x - y - 4z &= -14 \\ 2x + 7y + 12z &= 18 \end{aligned}$$

has solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},$$

$$\lambda \in \mathbb{R}.$$

Geometrically, there are infinitely many solutions in one parameter because all three planes meet at a single line.

In other cases with infinitely many solutions, it's possible that two or all three of the planes could be identical.

Nature of solutions – examples

Example

For each of the following augmented matrices in REF, solve the corresponding linear system, and describe the nature of the solution.

a) $\left(\begin{array}{ccc|c} 1 & 4 & 7 & 4 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & 8 & 2 \end{array}\right)$ b) $\left(\begin{array}{ccc|c} 0 & 5 & 1 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 5 \end{array}\right)$

c) $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$ d) $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

e) $\left(\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$ f) $\left(\begin{array}{ccccc|c} 3 & 5 & 1 & 0 & 2 & 4 \\ 0 & 0 & -1 & 8 & 1 & 6 \end{array}\right)$

Nature of solutions – examples

a) $\left(\begin{array}{ccc|c} 1 & 4 & 7 & 4 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & 8 & 2 \end{array}\right)$

- There are no leading entries in the last column, so there is at least one solution.
- There are leading entries in every column left of the vertical bar, so there is a unique solution.
- R_3 means $8z = 2$, so we know $\boxed{z = \frac{1}{4}}$.
- R_2 means $3y - z = 5$, so we know $\boxed{y = \frac{7}{4}}$.
- R_1 means $x + 4y + 7z = 4$, so we know $\boxed{x = -\frac{19}{4}}$.

The unique solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -19 \\ 7 \\ 1 \end{pmatrix}$.

The solution is a single point in $\mathbb{R}^3$, and geometrically represented by the unique intersection of three planes.

b) $\left(\begin{array}{ccc|c} 0 & 5 & 1 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 5 \end{array}\right)$

- There is a leading entry in the last column, so there are no solutions.
- (Note that R_3 means $0x + 0y + 0z = 5$, which is impossible.)

The system is inconsistent.

The solution could be geometrically represented by three parallel planes, three planes of which two are parallel, or three planes which intersect pairwise in three parallel lines.

c) $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$

- There are no leading entries in the last column, so there is at least one solution.
- Not every column left of the vertical bar contains a leading entry, so there are infinitely many solutions.
- The second column has no leading entry, so we can set its corresponding variable as a parameter. So let $\boxed{y = \lambda}$.
- R_2 means $-z = 6$, so we know $\boxed{z = -6}$.
- R_1 means $x + 2y = 4$, so we know $\boxed{x = 4 - 2\lambda}$.

The infinite set of solutions is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \ \lambda \in \mathbb{R}$.

The solution is a line in $\mathbb{R}^3$, and geometrically represented by the intersection of two or three planes at a common line.

Nature of solutions – examples

d) $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

- There are no leading entries in the last column, so there is at least one solution.
- Not every column left of the vertical bar contains a leading entry, so there are infinitely many solutions.
- The second column has no leading entry, so we can set its corresponding variable as a parameter. So let $\boxed{y = \lambda}$.
- The third column has no leading entry, so we can set its corresponding variable as another parameter. So let $\boxed{z = \mu}$.
- R_1 means $x + 2y + 3z = 4$, so we know $\boxed{x = 4 - 2\lambda - 3\mu}$.

The set of solutions is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}, \ \lambda, \mu \in \mathbb{R}$.

The solution is a plane in $\mathbb{R}^3$, and geometrically represented by the intersection of up to three identical planes.

Nature of solutions – examples

e) $\left(\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$

- There are no leading entries in the last column, so there is at least one solution.
- There are leading entries in every column left of the vertical bar, so there is a unique solution.
- R_2 means $\boxed{y = 5}$.
- R_1 means $x + y = 8$, so we know $\boxed{x = 3}$.

The unique solution is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

The solution is a single point in $\mathbb{R}^2$, and geometrically represented by the unique intersection of two lines.

(Notice that the extra zero rows did not provide any additional information about the solution.)

Nature of solutions – examples

f) $\left(\begin{array}{ccccc|c} 3 & 5 & 1 & 0 & 2 & 4 \\ 0 & 0 & -1 & 8 & 1 & 6 \end{array}\right)$

- There are only two leading entries, neither of which is in the last column, so there are infinitely many solutions.
- The second column has no leading entry, so let $\boxed{x_2 = \lambda_1}$.
- The fourth column has no leading entry, so let $\boxed{x_4 = \lambda_2}$.
- The fifth column has no leading entry, so let $\boxed{x_5 = \lambda_3}$.
- R_2 means $-x_3 + 8x_4 + x_5 = 6$, so $\boxed{x_3 = -6 + 8\lambda_2 + \lambda_3}$.
- Similarly, from R_1 we get $\boxed{x_1 = \frac{1}{3}(10 - 5\lambda_1 - 8\lambda_2 - 3\lambda_3)}$.

Solution: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 10/3 \\ 0 \\ -6 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -5/3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -8/3 \\ 0 \\ 8 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \ \lambda_i \in \mathbb{R}$

The solution is a 3-dimensional object in $\mathbb{R}^5$, and geometrically represented by the intersection of two 5-dimensional objects.