School of Mathematics and Statistics
Math1131-Algebra

Lec13: Fundamental Theorem of Algebra

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Complex polynomials

Complex polynomials = Polynomials with coefficients in $\mathbb{C}$

A *complex polynomial* $p(z)$ is a complex valued function of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

where $a_0, \ldots, a_n \in \mathbb{C}$ with $a_n \neq 0$ are the *coefficients* of $p(z)$.

- If the coefficients $a_0, \ldots, a_n$ are real numbers, then $p(z)$ is a *real polynomial*.
- $a_n x^n$ is the *leading term*; a_n is the *leading coefficient*. If $a_n = 1$ then $p(z)$ is *monic*.
- The *degree* of $p(z)$ is n.
- The degree of the zero polynomial $p(z) = 0$ is undefined.

Exercise 1. Which of the following functions are polynomials? " ✓" or " ✗ "

a) $f(z) = z^3 + z + 1$

b) $f(z) = z^5 + 6z^{-2} + 3z$

c) $f(z) = i\sqrt{3}z$

d) $f(z) = \sin z$

e) $f(z) = i$

f) $f(z) = e^z$

g) $f(z) = 2 + 3i$

h) $f(z) = 2z^3 - iz^2 + 4$

i) $f(z) = \dfrac{z+1}{z-1}$

Roots and factors

Roots and factors

- If $p(\alpha) = 0$ then α is called a *root* of $p(z)$.
- If $p(z) = q(z)g(z)$ then $q(z)$ and $g(z)$ are *factors* of $p(z)$.
- A factor of degree 1 is called a *linear* factor and a factor of degree 2 is called a *quadratic* factor.

Example 2. $p(z) = z^2 + 1$ has roots $\pm i$ because

$$\begin{aligned} p(i) &= i^2 + 1 = -1 + 1 = 0 \\ p(-i) &= (-i)^2 + 1 = -1 + 1 = 0. \end{aligned}$$

Example 3. $p(z) = z^2 - 1 = (z+1)(z-1)$ has linear factors $z + 1$ and $z - 1$.

Exercise 4. $p(z) = z^2 + 3 =$ ....

Therefore, its linear factors are .... and ....

Long Division and the Remainder Theorem

Long Division for polynomials.

Given polynomials $a(x), b(x)$, we can find polynomials $q(x)$ and $r(x)$ such that

$$a(x) = q(x)b(x) + r(x)$$

where the degree of the remainder $r(x)$ is less than the degree of the divisor $b(x)$.
(We are dividing a by b, so b is the *divisor*, q is the *quotient* and r is the *remainder*).

The remainder theorem

If the polynomial $p(z)$ is divided by the linear factor $z - \alpha$, then the remainder is simply the number $r = p(\alpha)$.

PROOF

Write $p(z) = q(z)(z - \alpha) + r(z)$.

The degree of $r(z)$ must be smaller than the degree of $z - \alpha$ which is 1, and so $r(z)$ must be constant.
Therefore, $r(z) = r(\alpha) = p(\alpha) - q(\alpha)(\alpha - \alpha) = p(\alpha)$.

The Remainder Theorem

Dividing $p(z)$ by $z - \alpha$ gives the remainder $r = p(\alpha)$.

Exercise 5. Find the remainder when $z^3 - 6z^2 + 11z - 7$ is divided by $z - 4$. Do this by both long division and using the Remainder Theorem.

Checking some of our answers with Maple

```
> # Long division for polynomials

  a := z -> z^3 -6*z^2 +11*z -7;
  b := z -> z-4;
```

$$a := z \mapsto z^3 - 6\,z^2 + 11\,z - 7$$

$$b := z \mapsto z - 4$$

```
> # quotient when we divide a by b, the variable being z

  quo(a(z), b(z),z);
```

$$z^2 - 2\,z + 3$$

```
> # remainder when we divide a by b, the variable being z

  rem(a(z), b(z),z);
```

$$5$$

```
> # Using the remainder theorem

  a(4);
```

$$5$$

The Factor Theorem

Example 6. We have seen that $p(z) = z^2 + 1 = (z - i)(z + i)$ has roots i and $-i$, and its factors are $z - i$ and $z + i$.

The Factor Theorem

$z - \alpha$ is a factor of $p(z)$ if and only if $P(\alpha) = 0$.

PROOF Let r be the remainder of $p(z)$ when divided by $z - \alpha$. Then

$$\begin{gathered} \alpha \text{ is a root of } p(z) \\ \Updownarrow \\ p(\alpha) = 0 \\ \Updownarrow \\ r = r(\alpha) = p(\alpha) = 0 \\ \Updownarrow \\ z - \alpha \text{ is a factor of } p(z) \end{gathered}$$

Factorisation

Fundamental Theorem of Algebra.

Each complex polynomial of degree $n \geqslant 1$ has at least one complex root.

This leads to...

The Factorisation Theorem

Each complex polynomial $p(z)$ of degree $n \geqslant 1$ has a factorisation

$$p(z) = a_n(z - \alpha_1)(z - \alpha_2)\dots(z - \alpha_n),$$

where the n complex numbers $\alpha_1, \alpha_2, \dots \alpha_n$ are roots of $p(z)$.
The roots are not necessarily distinct.

Exercise 7. Factorise the following polynomials into complex linear factors (= degree one factors).

a) $p_1(z) = z^2 - 4$ b) $p_2(z) = 2z^3 + 2z^2 - 4z$ c) $p_3(z) = z^3 - 8i$

Exercise 7, continued.

Factorise the following polynomials into linear factors (= degree one factors).

a) $p_1(z) = z^2 - 4$ b) $p_2(z) = 2z^3 + 2z^2 - 4z$ c) $p_3(z) = z^3 - 8i$

Factorising polynomials in $\mathbb{C}$ with Maple

```
>  # Example 7, part b
> p := z -> 2*z^3 +2*z^2 -4*z;
```

$$p := z \mapsto 2z^3 + 2z^2 - 4z$$

```
> factor(p(z));
```

$$2z(z+2)(z-1)$$

```
> # Example 7, part c
> p(z) := z^3 -8*I;
```

$$p(z) := z^3 - 8\,\mathrm{I}$$

```
> # Naive approach
  factor(p(z));
```

$$-(2\,\mathrm{I}z - z^2 + 4)(z + 2\,\mathrm{I})$$

```
> # The problem is that if the coefficients are all integers then 'factor'
  computes all irreducible factors with integer coefficients. Thus factor does
  not necessarily factor into linear factors.

> # All roots in a+ib form

  solve(p(z) = 0);
```

$$\mathrm{I} + \sqrt{3}, \mathrm{I} - \sqrt{3}, -2\,\mathrm{I}$$

```
> # In order to get linear factors, we force a factorisation using i and sqrt(3)
  factor(p(z), {I, sqrt(3)});
```

$$\left(-z + \mathrm{I} + \sqrt{3}\right)\left(-z + \mathrm{I} - \sqrt{3}\right)(z + 2\,\mathrm{I})$$

Real polynomials and conjugate roots

Real polynomials and conjugate roots

If α is a root of a real polynomial[a] $p(z)$, then its conjugate $\overline{\alpha}$ is also a root of this polynomial.

[a] which means all the coefficients are real numbers rather than complex numbers

PROOF Suppose that α is a root of a real polynomial

$$p(z) = a_n z^n + \ldots + a_1 z + a_0,$$

that is, $a_0, a_1, \ldots, a_n$ are real and $p(\alpha) = 0$. Then

$$\begin{aligned} p(\overline{\alpha}) &= a_n \overline{\alpha}^n + \ldots + a_1 \overline{\alpha} + a_0 \\ &= \overline{a_n}\,\overline{\alpha}^n + \ldots + \overline{a_1}\,\overline{\alpha} + \overline{a_0} \\ &= \overline{a_n \alpha^n + \ldots + a_1 \alpha + a_0} \\ &= \overline{p(\alpha)} \\ &= \overline{0} \\ &= 0. \end{aligned}$$

Hence $\overline{\alpha}$ is also a root of $p(z)$.

Real polynomials and conjugate roots

Suppose that the real polynomial $p(z)$ has non-real root α. Then $\overline{\alpha}$ is also a root and $z - \alpha$ and $z - \overline{\alpha}$ are factors of $p(z)$. Hence $p(z)$ has the quadratic factor

$$\begin{aligned}(z-\alpha)(z-\overline{\alpha}) &= z^2 - (\alpha + \overline{\alpha})z + \alpha\overline{\alpha} \\ &= z^2 - 2\text{Re}(\alpha)z + |\alpha|^2.\end{aligned}$$

Since $\text{Re}(\alpha)$ and $|\alpha|^2$ are real, this quadratic factor is real.

As a consequence, every real polynomial can be factored into real linear and quadratic factors.

Exercise 8.

a) Express $z^6 - 1$ as a product of linear factors.

b) Express $z^6 - 1$ as a product of real linear and quadratic factors.

Assessed

Exercise 8, continued.

a) Express $z^6 - 1$ as a product of linear factors.

b) Express $z^6 - 1$ as a product of real linear and quadratic factors.

Complex polynomials and Maple: Ex 8

```
> p := z -> z^6-1:
> evalc([solve(z^6 = 1)]);
  # Straight brackets to store the answers as a list so we can apply
  'map' next
```

$$\left[1, -1, \frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}, \frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}\right]$$

```
> map(polar, %);
  #  '%' means 'previous result'
  # 'map' is used to apply 'polar' to each term of the previous list
```

$$\left[\mathrm{polar}(1, 0), \mathrm{polar}(1, \pi), \mathrm{polar}\left(1, -\frac{\pi}{3}\right), \mathrm{polar}\left(1, \frac{2\pi}{3}\right), \mathrm{polar}\left(1, \frac{\pi}{3}\right), \mathrm{polar}\left(1, -\frac{2\pi}{3}\right)\right]$$

```
> factor(p(z));
```

$$(z-1)(z+1)(z^2+z+1)(z^2-z+1)$$

```
> # If the coefficients are all integers then 'factor' computes all
  irreducible factors with integer coefficients. Thus 'factor' does not
  necessarily factor into linear factors.

> # To get linear factors, we force a factorisation using i and sqrt(3)
  factor(p(z), {I, sqrt(3)});
```

$$\frac{(z-1)(\mathrm{I}\sqrt{3}-2z+1)(\mathrm{I}\sqrt{3}+2z-1)(z+1)(\mathrm{I}\sqrt{3}+2z+1)(\mathrm{I}\sqrt{3}-2z-1)}{16}$$