LECTURE 16
The Fundamental Theorems of Calculus

The First Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$$

The Second Fundamental Theorem of Calculus:

$$\int_a^b f(x)dx = F(b) - F(a) \quad \text{where } F \text{ is a primitive of } f.$$

SOME BASIC INTEGRALS

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^{ax} dx = \frac{e^{ax}}{a} + C$$

$$\int \sin ax\,dx = -\frac{\cos ax}{a} + C$$

$$\int \cos ax\,dx = \frac{\sin ax}{a} + C$$

$$\int \sec^2 ax\,dx = \frac{\tan ax}{a} + C$$

In the last lecture we defined the Reimann integral and then evaluated a few simple integrals using an excruciating limiting process of upper and lower sums of rectangles. In this lecture we will show that the machinery of calculus may be used to establish a dramatic short cut to the evaluation of integrals.

We first define the primitive of a function. Given a function $y = f(x)$ a primitive of f is another function F with the property that $F' = f$.

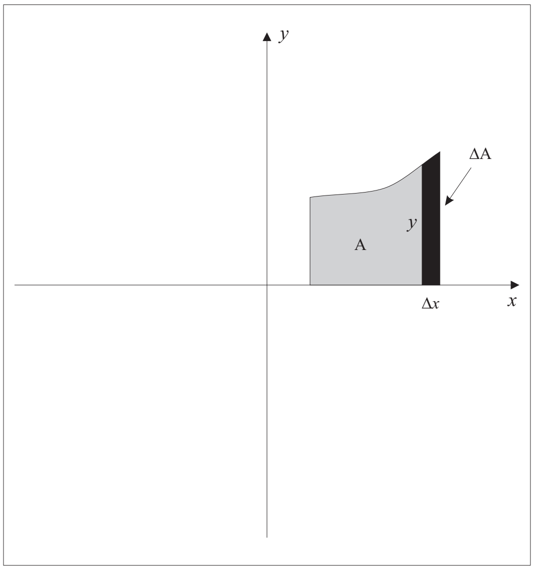
Thus a primitive of $3x^2$ is x^3 **because** $\frac{d}{dx}(x^3) = 3x^2$. The process of finding primitives is literally differentiation in reverse! We do of course have a slight technical problem in that $\frac{d}{dx}(x^3+7) = 3x^2$ as well. The most general form of the primitive of $3x^2$ is then $x^3 + C$ since the constant of integration C disappears when differentiated. What do primitives have to do with integration? There is certainly no obvious connection until we have the two Fundamental Theorems of Calculus:

The First Fundamental Theorem of Calculus: Let f be a continuous function defined on $[a, b]$ and define a new function F by

$$F(x) = \int_a^x f(t)\,dt$$

Then F is both continuous and differentiable on $[a, b]$ and $F' = f$.

Remember that (initially at least) the Reimann integral has nothing to do with the calculus. What the first fundamental theorem says is that if you differentiate an integral you get back to where you started. In other words integration is the opposite of differentiation and hence the process of integration can be managed by simply using primitives. A formal proof of the theorem is in your notes. Lets us have a look at an intuitive argument.



Let A be the area accumulated underneath the curve $y = f(x)$ from a to x, that is $A = \int_a^x f(t)\,dt$. Consider the extra increment of area ΔA. This region is approximately a rectangle so $\Delta A = y\Delta x = f(x)\Delta x$. Hence $\dfrac{\Delta A}{\Delta x} = f(x)$. Letting $\Delta x \to 0$ we have $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ which is the first fundamental theorem. It follows that we need to 'antidifferentiate' $f(x)$ to find A.

Just a small note on the equation $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ before we continue. Students are often confused by the presence of the variable t in this equation. Keep in mind however that t is really only a dummy variable and can be replaced with anything (except x). We can't use $\dfrac{d}{dx}\displaystyle\int_a^x f(x)\,dx = f(x)$ as the variable x will then be both a variable of integration and a limit, leading to all sorts of notational problems.

So the first fundamental theorem says that integration and antidifferentiation are essentially equivalent. Indeed we write the primitive of $f(x)$ as the indefinite integral $\int f(x)\,dx$. The second fundamental theorem provides us with a way of exploiting this identification to actually calculate areas under curves.

The Second Fundamental Theorem of Calculus: Let f be a continuous function defined on $[a, b]$ and let F be a primitive of f. Then

$$\int_a^b f(x)dx = \Big[F(x)\Big]_a^b = F(b) - F(a).$$

This is an amazing result. Calculation of the area under a curve over an interval $[a, b]$ depends only upon the value of the primitive at the endpoints of the interval! Lets take a look at how we use these theorems:

Example 1: Show that $\dfrac{d}{dx}\displaystyle\int_0^x t^2\,dt = x^2$ without using the first fundamental theorem of calculus.

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The example above shows that the first fundamental theorem says something very simple! If you differentiate an integral you go around in a big circle with the only difference being a change of variable at the end.

Recall how hard it was to find the area under $y = x^2$ from $x = 0$ to $x = 1$ in the last lecture. The second fundamental theorem now provides a wonderful shortcut! But you still need to be a little careful.

Example 2: Evaluate the area of the region bounded by $y = x^2$ and the x axis from $x = 0$ to $x = 1$.

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So much faster!

The above theorems put the processes of integration on a firm footing. You have however already seen a lot of integration theory in high school. Let's recall that theory before moving on to some more abstract questions.

Example 3: Evaluate $\int_0^2 1 - x^2\, dx$.

$\bigstar \quad -\dfrac{2}{3} \quad \bigstar$

Example 4: Find the area bounded by $y = 1 - x^2$ and the x axis from $x = 0$ to $x = 2$.

This is a different question! Whenever you are asked for area you must sketch the curve! The integral gets a little confused if the region is below the x axis and will assign a negative value to the area. You need to separate this section out and take its absolute value.

$\bigstar \quad 2 \quad u^2 \quad \bigstar$

We will not spend too much time on the evaluation of integrals here as this is a major topic extensively covered in the high school courses. An important point to keep in mind however is that integration is a touchy process. We have no general rules such as the product rule or the quotient rule for integration and many many functions cannot be integrated at all! Our main technique is to use a table of integrals (see page 1). Such a table will be available to you in your final exam. Integrals without limits are referred to as indefinite integrals and need a $+C$. Integrals with limits are called definite integrals and do not need a constant of integration.

Example 5: Evaluate

a) $\displaystyle\int_{\ln(5)}^{\ln(7)} e^{2x}\ dx$

b) $\displaystyle\int_{0}^{\frac{\pi}{6}} \sin(3x)\ dx$

$\bigstar \quad a)\ 12 \quad b)\ \dfrac{1}{3} \quad \bigstar$

We ask some fairly tricky questions involving the fundamental theorems of calculus. These always take the shape of the derivative of an integral:

Example 6: Evaluate

$$\frac{d}{dx}\int_3^x t^4\,dt.$$

This is trivial! By the first fundamental theorem the answer is x^4. Let's actually evaluate the integral to see what happens:

Note that the 3 can be replaced with any number and the result will be the same.

$\star \quad x^4 \quad \star$

Example 7: Use the first fundamental theorem of calculus to evaluate

$$\frac{d}{dx}\int_7^x \cos(t^3)\,dt.$$

We cannot do this integral! It is impossible. But that doesn't stop us:

$\star \quad \cos(x^3) \quad \star$

Example 8: Use the first fundamental theorem of calculus to evaluate

$$\frac{d}{dx}\int_7^{x^5} \cos(t^3)\,dt.$$

This is now complicated by the strange upper limit! Our approach is to admit defeat on the integral and hope that we can still wriggle out of the problem.

Let a primitive of $f(t) = \cos(t^3)$ be $F(t)$.

We can't work out what $F(t)$ is, but we do know that $F'(t) = f(t) = \cos(t^3)$. So:

$$\int_7^{x^5} \cos(t^3)\,dt =$$

$$\frac{d}{dx}\int_7^{x^5} \cos(t^3)\,dt =$$

$\star \quad \cos(x^{15})5x^4 \quad \star$

It is clear from the above example that we have the following structure:

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = (\text{sub in upper limit} \times \text{upper}') - (\text{sub in lower limit} \times \text{lower}').$$

Example 9: Use the first fundamental theorem of calculus to evaluate

$$\frac{d}{dx}\int_{x^2}^{x^4} \sec\left(\frac{7}{1+t}\right)\,dt.$$

$$\bigstar \quad 4x^3\sec(\frac{7}{1+x^4}) - 2x\sec(\frac{7}{1+x^2}) \quad \bigstar$$

If you are the sort of person that likes to memorize and use formulas you may use:

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v)v' - f(u)u'$$

Note also that if the examiner doesn't mention the first fundamental theorem in the question then you should mention it in your answer.

Example 10: Evaluate

$$\frac{d}{dx}\int_{\sin(x)}^{e^{3x}} \ln(1-z^2)\,dz.$$

$$\bigstar \quad 3\ln(1-e^{6x})e^{3x} - \ln(\cos^2(x))\cos(x) \quad \bigstar$$

Example 11: Evaluate

$$\frac{d}{d\alpha}\int_{\alpha^{30}}^{\ln\alpha} \sin(1+\beta^4)\, d\beta.$$

$$\bigstar \quad \sin(1+(\ln\alpha)^4)(\frac{1}{\alpha}) - \sin(1+\alpha^{120})(30\alpha^{19}) \quad \bigstar$$

When doing these sort of questions you must not fall into the trap of actually trying to evaluate the integral. Even simple little integrals can turn out to be impossible. For example it can be shown that

$$\int_0^x e^{-t^2}\, dt$$

cannot be expressed in terms of standard functions. Our response to these integrals is to admit defeat and just give them a name. Thus for example

$$\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt$$

is called $\mathrm{erf}(x)$ and the exotic erf function sits quite happily with all of its boring mates sin, cos, ln etc. There are many functions whose definition takes the form of an unachievable integral. Examples are

$$\mathrm{li}(x) = \int_2^x \frac{1}{\ln(t)}\, dt$$

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t}\, dt$$

$$\mathrm{FresnelC}(x) = \int_0^x \cos(\frac{\pi}{2}t^2)\, dt$$

You do not need to know these definition for Math1131 but you may come across these functions in your future studies. There are lots of strange functions in mathematics.