School of Mathematics and Statistics
Math1131-Algebra

Lec17: Matrices

Laure Helme-Guizon (Dr H)
Laure@unsw.edu.au
Jonathan Kress
j.kress@unsw.edu.au

Red-Centre, Rooms 3090 and 3073

2020 Term 1

Matrices

- *Matrices* are rectangular arrays of numbers with parentheses (round brackets) around.

Example 1. Here are three matrices :

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 7 & 8 \\ -3 & 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{3} & \frac{3}{11} \\ -\frac{1}{4} & 4 \\ \frac{2}{9} & \frac{7}{11} \end{pmatrix} \qquad \begin{pmatrix} \pi & -1 \\ \sqrt{2} & e \end{pmatrix}$$

- The *sizes* (or *dimensions*) of these matrics are:

$$3 \times 3 \qquad 3 \times 2 \qquad 2 \times 2.$$

We always refer to the rows before the columns when giving the size or describing the position of an element.

Example 2. What is the size of the matrix below?

$$\begin{pmatrix} \frac{1}{3} & -\frac{1}{4} & \frac{2}{9} \\ \frac{3}{11} & 4 & \frac{7}{11} \end{pmatrix}$$

ANSWER : It is a $\ldots \times \ldots$ matrix.

Matrices

Entries of a matrix

1. The numbers in a matrix are called *entries*.
2. For the matrix A, the entry in row i and column j is written $[A]_{ij}$.

Exercise 3. For example, if $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ -4 & 7 & 8 & -2 \\ -3 & 5 & 0 & 10 \end{pmatrix}$,

then $[A]_{23} = 8$ and $[A]_{33} = 0$ $[A]_{32} = \dots$ and $[A]_{24} = \dots$.

$M_{mn}(\mathbb{R})$ denotes the set of $m \times n$ matrics with real entries.

Example. So for A above, $A \in M_{34}(\mathbb{R})$.

Sometimes we write $[A]_{ij}$ as a_{ij}.
We generally use capital letters for matrices.

Adding or scaling matrices

Adding and scaling (multiplying by a scalar) matrices

To add or scale matrices, we just add or scale the corresponding entries, just like vectors.

Sometimes, you cannot add matrices : When the size of the matrices do not match, their sum or difference is not defined.

Exercise 4. Let $A = \begin{pmatrix} 2 & 3 \\ 1 & -3 \\ 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 3 \\ 3 & 0 \\ 2 & 5 \end{pmatrix}$.

Find, if possible, $A + C$, $A + B$, $4B$ and $2A - C$.

Adding or scaling matrices

Exercise 4, continued. Let $A = \begin{pmatrix} 2 & 3 \\ 1 & -3 \\ 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 3 \\ 3 & 0 \\ 2 & 5 \end{pmatrix}$.

Find, if possible, $A + C$, $A + B$, $4B$ and $2A - C$.

UNSW SYDNEY

Linear equations in matrix form

The system of linear equations

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 1 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & -1 \\ 7x_1 & - & 5x_2 & - & 9x_3 & = & 0 \end{array}$$

can be written in matrix form as

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -5 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Let's look at the left hand side and see how the "multiplication" works.

$$\begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \\ \mathbf{7} & \mathbf{-5} & \mathbf{-9} \end{pmatrix} \begin{pmatrix} \mathbf{x_1} \\ \mathbf{x_2} \\ \mathbf{x_3} \end{pmatrix} = \begin{pmatrix} \mathbf{x_1+2x_2+3x_3} \\ \mathbf{4x_1+5x_2+6x_3} \\ \mathbf{7x_1-5x_2-9x_3} \end{pmatrix}$$

This is the basis of matrix multiplication.

Matrix multiplication

Example 5. If $A = \begin{pmatrix} 7 & 2 \\ 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 5 \\ 6 & 8 \end{pmatrix}$

then

$$AB = \begin{pmatrix} 7 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 7 \times 3 + 2 \times 6 & 7 \times 5 + 2 \times 8 \\ 1 \times 3 + 4 \times 6 & 1 \times 5 + 4 \times 8 \end{pmatrix} = \begin{pmatrix} 33 & 51 \\ 27 & 37 \end{pmatrix}.$$

What about

$$BA = \begin{pmatrix} 3 & 5 \\ 6 & 8 \end{pmatrix} \begin{pmatrix} 7 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 26 & 26 \\ 50 & 44 \end{pmatrix}.$$

Notice that $AB \neq BA$! Matrix multiplication is not commutative.

Matrix multiplication

Exercise 6. Given $A = \begin{pmatrix} 1 & 5 & 3 \\ -3 & 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 4 & -1 \end{pmatrix}$, find BA and AB.

What do you notice about the sizes of these matrices?

⚠ AB and BA have different sizes!

Matrix multiplication with Maple

```
> with(LinearAlgebra):
> # Enter the matrices column by column

  A := < <1,-3>|<-5,2>|<3,4> >;
  B := < <1,2,4>|<0,3,-1> >;
```

$$A := \begin{bmatrix} 1 & -5 & 3 \\ -3 & 2 & 4 \end{bmatrix}$$

$$B := \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 4 & -1 \end{bmatrix}$$

```
> # For matrix multiplication, use . not *

  BA := B.A;
  AB := A.B;
```

$$BA := \begin{bmatrix} 1 & -5 & 3 \\ -7 & -4 & 18 \\ 7 & -22 & 8 \end{bmatrix}$$

$$AB := \begin{bmatrix} 3 & -18 \\ 17 & 2 \end{bmatrix}$$

Matrix multiplication

Exercise 7. Let $C = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ -1 & 0 & -2 \end{pmatrix}$ and $D = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$.

Find, if possible, CD and DC.

Matrix multiplication with Maple

```
> with(LinearAlgebra):
> # Enter the matrices column by column

  C := < <1,1,-1>|<2,0,0>|<3,1,-2> >;
  M := < -1,-2,-3>;
```

$$C := \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ -1 & 0 & -2 \end{bmatrix}$$

$$M := \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$$

```
> # For matrix multiplication, use . not *

  CM := C.M;
```

$$CM := \begin{bmatrix} -14 \\ -4 \\ 7 \end{bmatrix}$$

```
> MC := M.C;
Error, (in LinearAlgebra:-Multiply) cannot multiply a column Vector and a Matrix
```

Zero matrices

Zero matrices : Defintion.

The $m \times n$ **zero** matrix is the $m \times n$ matrix with all entries equal to zero.

For example, $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the 2×2 zero matrix and

$0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is the 3×4 zero matrix.

Zero matrices : Property.

For any matrix A, $\quad A + 0 = 0 + A = A$,
where 0 denotes the zero matrix which is the same size as A.

Identity matrices

Diagonal entries of a matrix.

The diagonal entries of a matrix A are the entries $[A]_{11}, [A]_{22}, \ldots, [A]_{ii} \ldots$.

Examplee 8. Circle the diagonal entries of the following matrix $\begin{pmatrix} 7 & 2 & 3 \\ \sqrt{3} & 1 & \cos 9 \\ -1 & \pi & 0 \end{pmatrix}$.

Identity matrices : Definition.

The $n \times n$ identity matrix is the $n \times n$ matrix with all diagonal entries equal to 1 and zeros everywhere else.

*NB : All identity matrices are **square** matrices.*

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the 2×2 identity matrix and

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the 3×3 zero matrix.

Identity matrices : Property.

For any matrix A, $AI = IA = A$.

*In other words, multiplying a matrix by the identity matrix leaves it unchanged/**ident**ical.*

Zero and identity matrices

Example 9. For $A = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}$, use the properties seen earlier to find AI, IA, $A + 0$, $0 + A$ and check by performing the appropriate calculations.

Matrix multiplication: Definition and Properties

Multiplying two matrices is not always possible!

Suppose that A is an $m \times n$ matrix and B is a $p \times q$ matrix.

- Calculating AB is possible if and only $n = p$, i.e. iff the number of colums of A, which is on the left of the product, is equal to the number of row of B, which is on the right of the product.
- If $n = p$, the product exists and is a matrix with m rows and q columns, i.e. the same number of rows as A, which is on the left of the product, and the same number of columns as B, which is on the right of the product.

Formal definition of matrix multiplication

Suppose that A, B and C are matrices for which the relevant products exist. Then, if A is an $m \times n$ matrix and B is a $n \times q$ matrix then

$$[\boldsymbol{AB}]_{ij} = \Sigma_{k=1}^{n} [\boldsymbol{A}]_{ik} [\boldsymbol{B}]_{kj}$$

Eg, for $i = 2$, $j = 1$, $n = 3$, $[AB]_{21} = [A]_{21}[B]_{11} + [A]_{22}[B]_{21} + [A]_{23}[B]_{31}$.

Matrix multiplication: Definition and Properties

Properties of matrix multiplication.

Suppose that A, B and C are matrices for which the relevant products exist. Then,

1. $A(BC) = (AB)C$ (associativity)
2. $A(B + C) = AB + AC$ (distributivity)
3. $A(\lambda B) = \lambda AB$ for any $\lambda \in \mathbb{R}$
4. $AI = IA = A$
5. In general $AB \neq BA$

Matrix multiplication is not commutative

Exercise 10. Is it true that $(A + B)^2 = A^2 + 2AB + B^2$?