THE UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1131 Calculus

MATHEMATICS 1A CALCULUS.

Section 1: - Functions and Graphs.

1. Numbers.

We will use the following notation:
The set of natural numbers, denoted by $\mathbb{N}$, consists of all the whole numbers $\{0, 1, 2, ...\}$.
The set of integers, denoted by $\mathbb{Z}$, consists of all the whole numbers $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$.
The set of rational numbers, denoted by $\mathbb{Q}$, consists of all numbers of the form $\frac{p}{q}$ where p, q are integers and $q \neq 0$.

The ancient Greeks initially thought that this was all there was (they didn't believe in negative numbers and zero either), until they discovered that $\sqrt{2}$ could not be written as a rational number.

Theorem: $\sqrt{2}$ is irrational.

Proof: Suppose that $\sqrt{2} \in \mathbb{Q}$. Then $\exists\ a, b \in \mathbb{Z}$ s.t. $\sqrt{2} = \frac{a}{b}$ and $\gcd(a, b) = 1$.

So $2 = \frac{a^2}{b^2} \Rightarrow a^2 = 2b^2$. So a^2 is even.

Then a must be even too. So $a = 2l$ for some $l \in \mathbb{Z}$. Now $2 = \frac{(2l)^2}{b^2}$ or

$$b^2 = 2l^2.$$

b^2 is even and hence b is even. Now a, b are both even. $\rightarrow\leftarrow$. So $\sqrt{2} \notin \mathbb{Q}$. $\blacksquare$

$\sqrt{2}$ and numbers such as π and e are examples of irrational numbers. We think of the set of all real numbers are points which lie on the real line. Giving a formal definition of real numbers is difficult.

We will use the following set notation:

$\{x \in A : P(x)\}$ denotes the set of all elements x of A satisfying property P. For example, $\{x \in \mathbb{R} : -1 \leq x \leq 1\}$ denotes all the real numbers between -1 and 1 (inclusive).

$A \cap B$ is the intersection of A and B and denotes all the elements that are in both A and B.

$A \cup B$ is the union of A and B and denotes all the elements that are in either A or B (or both).

$\emptyset$ is the set which has no elements, for example $\{x \in \mathbb{R} : x^2 < -1\} = \emptyset$.

Inequalities:

You are aware of the following facts about inequalities:

For $x, y, z \in \mathbb{R}$ we have
i. if $x > y$ then $x + z > y + z$
ii. if $x > y$ and $z > 0$ then $xz > yz$ and if $z < 0$ we have $xz < yz$.

Note carefully the definition for $|x|$.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

So for example, $|a - 3|$ is equal to $a - 3$ if $a \geq 3$ and $-(a - 3) = 3 - a$ if $a < 3$.

Note then that $|x| < 3$ means $-3 < x < 3$ and that $|-x| = |x|$. Also note that $\{x : |x-3| < 2\}$ represents the set of all real numbers whose distance from 3 is less that 2.

Finally note that $|xy| = |x||y|$ and that $|x + y| \leq |x| + |y|$. This last result is called the **triangle inequality.** You will see a complex version of this in the algebra strand of the course.

Also of importance is:

Theorem: (AM-GM inequality).

If $x, y \geq 0$ are real numbers then
$$\frac{x+y}{2} \geq \sqrt{xy}.$$
(This says that the arithmetic mean of two positive real numbers exceeds their geometric mean.)

Proof: $(x-y)^2 \geq 0$

or $x^2 + y^2 - 2xy \geq 0$

$\Rightarrow x^2 + y^2 + 2xy \geq 4xy$

$\Rightarrow \dfrac{(x+y)^2}{4} \geq xy$

$\Rightarrow \dfrac{x+y}{2} \geq \sqrt{xy}$. $\boxtimes$

Ex: Prove that for $x > 0$, we have $x + \frac{1}{x} \geq 2$.

$x > 0$, so $\frac{1}{x} > 0$.

By the AM-GM inequality, we get that

$$\frac{x + \frac{1}{x}}{2} \geq \sqrt{x \cdot \frac{1}{x}}$$

or $x + \frac{1}{x} \geq 2$. $\boxtimes$

Ex: Suppose a, b, c are positive real numbers. Prove that $a^2 + b^2 + c^2 \geq ab + ac + bc$.

Using AM-GM, we have.

$$\frac{a^2+b^2}{2} \geq ab, \quad \frac{b^2+c^2}{2} \geq bc, \quad \frac{a^2+c^2}{2} \geq ac.$$

Adding the above three inequlities gives

$$a^2+b^2+c^2 \geq ab + bc + ac.$$

$\boxtimes$

Intervals: We will use the following notation when dealing with intervals. A round bracket means we do not include the endpoint while we do when a square bracket is used. For example $(3, 9]$ means the interval $3 < x \leq 9$. Note that since infinity is NOT a real number, if we wish to represent the interval from 3 onwards we write this as $[3, \infty)$ (never use a square bracket with infinity.). Here are some further examples:

$$\{x \in \mathbb{R} : x > 3\} \cap \{x \in \mathbb{R} : x < 5\} = (3, 5)$$

$$\{x \in \mathbb{R} : x > 3\} \cup \{x \in \mathbb{R} : x < 5\} = \mathbb{R}$$

$$\{x \in \mathbb{R} : x > 5\} \cup \{x \in \mathbb{R} : x < 3\} = (-\infty, 3) \cup (5, \infty)$$

$$\{x \in \mathbb{R} : x > 5\} \cap \{x \in \mathbb{R} : x < 3\} = \emptyset$$

Solving Inequalities:

These are very similar to equations except that we must careful when multiplying by an unknown. You should be familiar with solving quadratic inequations such as
Ex: Find $\{x : x^2 - 2x - 3 > 0\}$.

$$x^2 - 2x - 3 = 0 \iff (x-3)(x+1) = 0.$$

$$x^2 - 2x - 3 = (x-3)(x+1).$$

This is positive if and only if
$(x-3)$ and $(x+1)$ have the same sign.
So $x > 3$ or $x < -1$.
Thus $\{x : x^2 - 2x - 3 > 0\} = (-\infty, -1) \cup (3, \infty)$.

For more difficult inequalities we use the following idea.

Ex: Solve $x > 1 + \frac{2}{x}$.

Clearly $x = 0$ is not a solution. Hence
if $x > 0$,

$$x^2 > x + 2$$
$$x^2 - x - 2 > 0$$
$$(x-2)(x+1) > 0.$$
$$x > 2 \text{ or } x < -1.$$

But we must have $x > 0$. So $x > 2$.

Now if $x < 0$ then
$x^2 < x + 2$ or $(x-2)(x+1) < 0$.
So $-1 < x < 2$. Again x must be negative
so $-1 < x < 0$. So altogether, the sol. set is

$(2, \infty) \cup (-1, 0)$.

Ex: Solve $\frac{2}{x} \leq \frac{3}{x-1}$.

If $x > 1$ or $x < 0$, x and $x-1$ have the same sign. So

$$\frac{2}{x} \leq \frac{3}{x-1} \Rightarrow 2(x-1) \leq 3x$$

$$\Rightarrow -2 \leq x$$

So we get $(-2,0) \cup (1,\infty)$.

If $0 < x < 1$, then x and $x-1$ have the opposite signs, so

$$\frac{2}{x} \leq \frac{3}{x-1} \Rightarrow 2(x-1) \geq 3x$$

$\Rightarrow -2 \geq x$. So we have no sol. in this case.

Functions:

You should be familiar with the function concept from school. Roughly speaking, a function $f : A \to B$ is a rule or formula which associates to each element of a set A (called the domain) **exactly one** element from another set B (called the co-domain). For the most part, we will have $A = B = \mathbb{R}$. The range of the function is the set of values b in B for which there is an $a \in A$ with $f(a) = b$. In less formal terms, the range consists of the output of the function.You will need to be able to find the domain and range of basic functions.

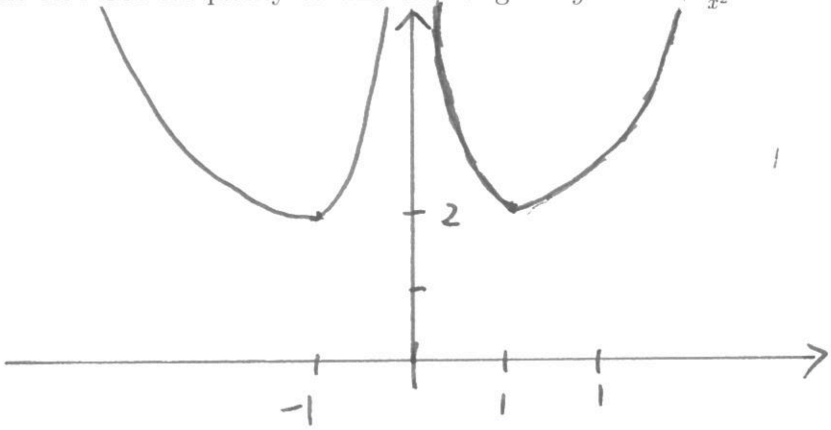
Ex: Find the domain and range of $f(x) = \sqrt{1 - x^2}$.

$$\{x : 1 - x^2 \geq 0\}$$

$$= \{x : 1 \geq x^2\}$$

$$= (-1, 1).$$

Ex: Use the AM-GM inequality to find the range of $y = x^2 + \frac{1}{x^2}$ and sketch the graph.



we know $x^2 + \frac{1}{x^2} \geqslant 2$ by AM-GM.

Ex: Find the domain of $f(x) = \sqrt{\cos x}$.

$\{x : \cos x \geq 0\}$.

~~On $[0, 2\pi]$, $\cos x \geq 0$ if~~

on $[-\pi, \pi]$, if $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, then $\cos x \geq 0$.

Now by the periodicity of $\cos x$, the domain in question is

$$\bigcup_{k=-\infty}^{\infty} [2k\pi - \tfrac{\pi}{2}, 2k\pi + \tfrac{\pi}{2}].$$

It is often difficult to find the range of a function. For example, what is the range of $2 + \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4}$?

It is important to be able to draw the graph of a given function. In most Calculus problems this is crucial.

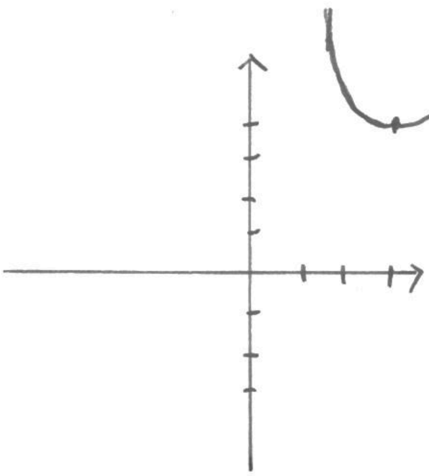
It often helps if the function is even or odd. You will recall that f is **even** if $f(x) = f(-x)$ and f is **odd** if $f(-x) = -f(x)$. Even functions are symmetric about the y axis and odd functions have a central symmetry with respect to the origin.

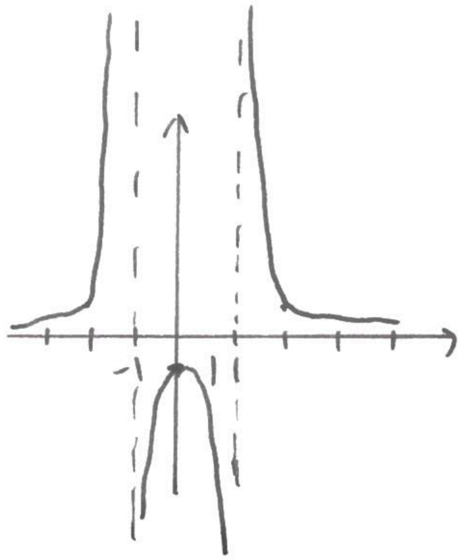
Thus, if we can draw such a function on the positive half plane we get the rest of the picture for free.
Note that if f is odd and has 0 in its domain, then $f(0) = 0$.

We say that f is periodic of period T if $f(x+T) = f(x)$ for all real x in the domain of f.

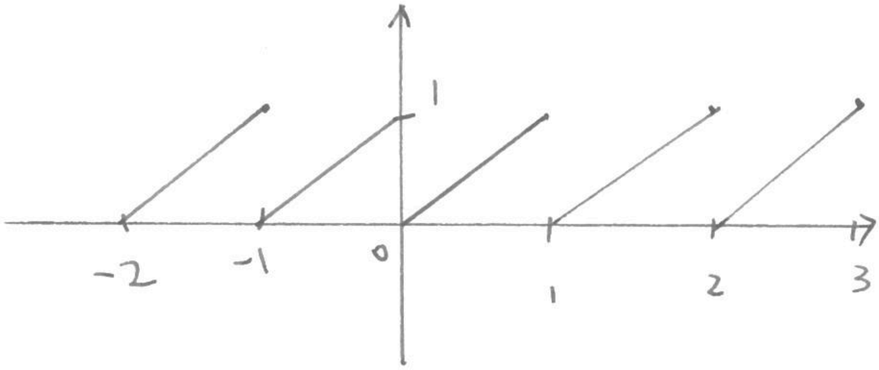
You have met the trig. functions which are periodic with period 2π.

Ex: Sketch: $f(x) = (x-3)^2 + 4$, and $f(x) = \frac{1}{x^2-1}$.

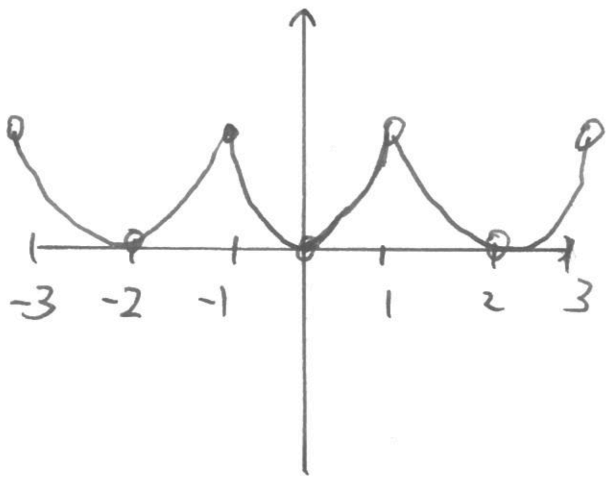




Ex: Sketch: $f(x) = x$ if $0 \le x < 1$ and $f(x+1) = f(x)$ for all x.



Ex: Sketch $f(x) = x^2$ for $0 < x < 1$, f is periodic of period 2 and f is even.



Floor and Ceiling Functions:

$x \in \mathbb{R}$

$[x]$ or $\lfloor x \rfloor$ is $\max\{k \in \mathbb{Z} : k \leq x\}$.

~~$\max_{k \in \mathbb{Z}} k \leq x$~~

$\lceil x \rceil = \min\{k \in \mathbb{Z} : k \geq x\}$.

Ex: Sketch $f(x) = x - \lfloor x \rfloor$.

$f(x) = x - \lfloor x \rfloor = \{x\}$.

This is the exact same fuction as that in the last example on page 8.

Combining Functions:
If f and g are two functions, we can add, subtract and multiply them in the obvious way. We can also divide them provided g is not zero. If the range of g equals the domain of f we compose the two functions to form $f \circ g$ which we define to be

$$f \circ g(x) = f(g(x)).$$

$f \circ g$ is called the composite function of f and g.

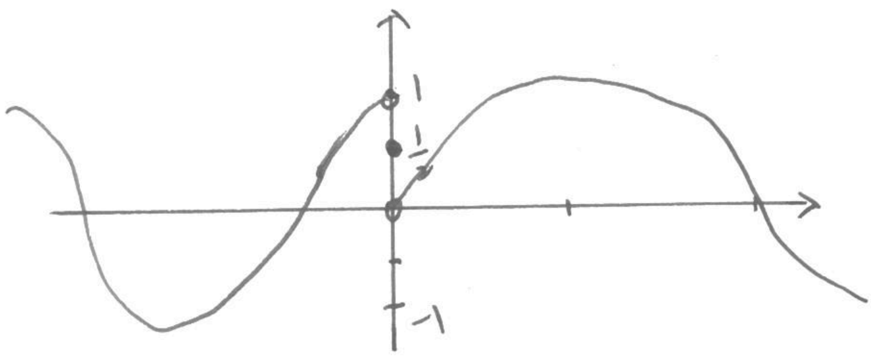
Ex: Find $f \circ g$ and $g \circ f$ if $f(x) = x^3$ and $g(x) = \sqrt{x^2 + 1}$.

$$f \circ g(x) = \left(\sqrt{x^2+1}\right)^3 = (x^2+1)^{\frac{3}{2}}$$

$$g \circ f(x) = \sqrt{(x^3)^2+1} = \sqrt{x^6+1}$$

Note that some functions cannot be defined by one simple equation. Many functions which occur in the real world are defined piecewise.

Ex: $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ \frac{1}{2} & \text{if } x = 0 \\ \sin x & \text{if } x > 0 \end{cases}$

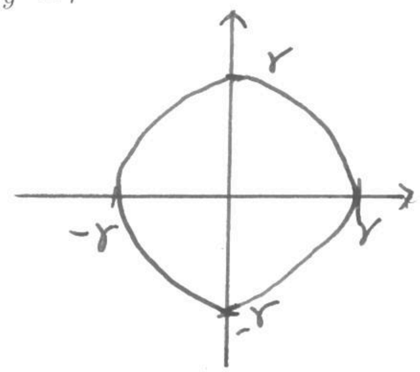


Conic Sections:

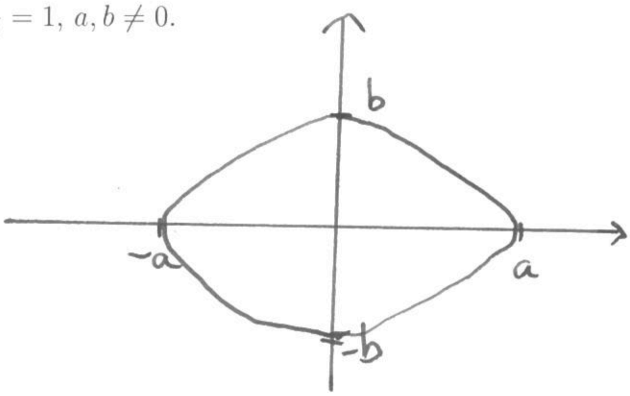
An important class of implicitly defined functions arises from the *conic sections* (so called because they are obtained by slicing a cone with various planes.)

You will need to recognise these:

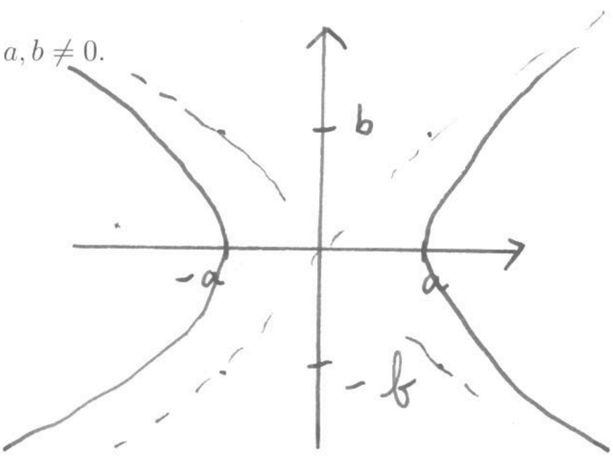
(i) **Circle** $x^2 + y^2 = r^2$



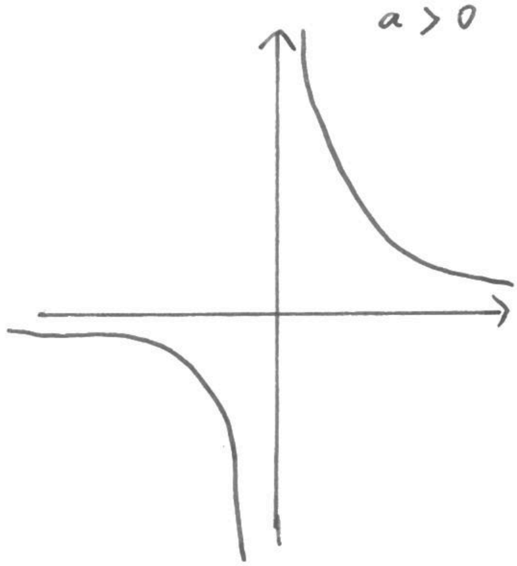
(ii) **Ellipse** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,\ a, b \neq 0.$

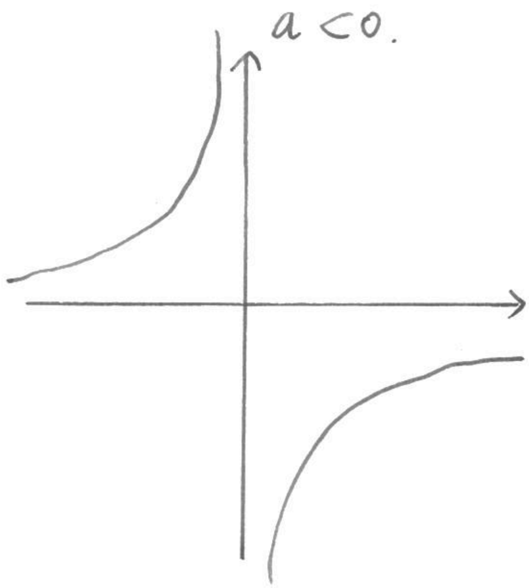


(iii) **Hyperbola** $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1,\ a, b \neq 0.$



(iv) **Rectangular Hyperbola** $y = \frac{a}{x},\ a \neq 0.$





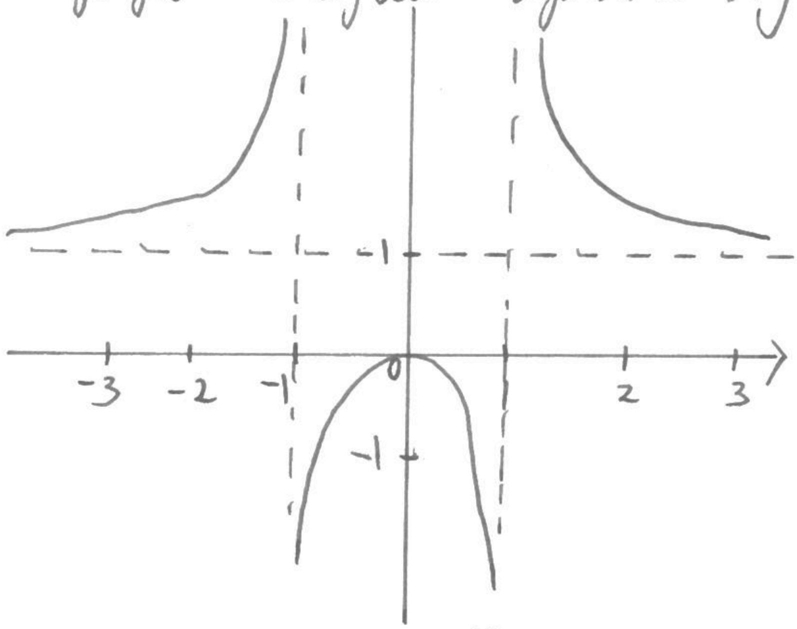
Other Functions:

It is assumed that you are familiar with the basic properties of polynomial functions, rational functions, the trigonometric functions, the exponential and logarithmic functions.

Graph $y = \frac{x^2}{x^2-1}$.

$$y = \frac{x^2 - 1 + 1}{x^2 - 1} = 1 + \frac{1}{x^2-1}.$$

We have seen the graph of $y = \frac{1}{x^2-1}$ before on page 8. So the graph here is just that graph shifted upward by 1 unit.



$$y = \frac{x^2}{x^2-1}.$$