THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Term 1 2019

MATH1131
MATHEMATICS 1A

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 3

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS IS APPENDED TO THE PAPER

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 1

1. i) Let $z = -\sqrt{3} + i$ and $w = -1 - i = \sqrt{2}e^{-3\pi i/4}$.

a) Find zw in $a + ib$ form.

b) Find $|z|$ and $\text{Arg}(z)$, and write z in polar form.

c) Find zw in polar form and hence obtain $\sin\left(\frac{\pi}{12}\right)$.

ii) Let $p(z) = z^5 + 1$.

a) Find all the roots of $p(z)$.

b) Write $p(z)$ as a product of complex linear factors.

c) Write $p(z)$ as a product of real linear or quadratic factors.

iii) By considering the Maple output below, or otherwise, sketch in the xy-plane the graph of the polar curve

$$r = 1 + \sin\theta, \qquad 0 \le \theta < 2\pi.$$

Your sketch should include the coordinates of any intersections of the graph with the x-axis and y-axis. There is no need to calculate the slope of the curve. You are given that the curve has a cusp at the origin.

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> plot(1+sin(theta), theta=0..2*Pi);
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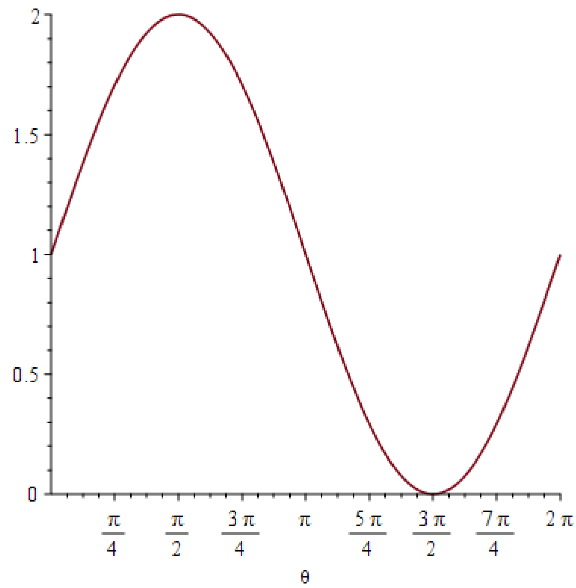


iv) By making an appropriate comparison, show that the improper integral

$$\int_0^\infty \frac{x^3}{1+x^8}\,dx$$

converges.

v) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with $f(2) = 1$ and $f'(2) = 3$. Find the tangent line to $y = f(x)$ at $x = 2$ and use it to find an approximation to $f(2.01)$.

vi) Find the following integral I by making an appropriate substitution.

$$I = \int \frac{\ln x \cos(\ln x)}{x}\,dx$$

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 2

2. i) Find a formula for $\sin^5\theta$ in terms of $\sin\theta$, $\sin 3\theta$ and $\sin 5\theta$.

ii) A *median* of a triangle is a line passing through one vertex and the midpoint of the opposite side.

Consider a triangle ABC with vertices $A(2,0,-3)$, $B(0,2,1)$ and $C(4,2,5)$. Let X be the midpoint of the side BC.

a) Find $\overrightarrow{AX}$.

b) Let ℓ_1 be the line through the points A and X, that is, the median of triangle ABC passing through A. Find a parametric vector form of the line ℓ_1.

c) Let ℓ_2 be the median of the triangle ABC through the point B. You are given that it has parametric vector form

$$\mathbf{x} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \qquad \mu \in \mathbb{R}.$$

Show that the two medians through A and B intersect at the point $G(2, \frac{4}{3}, 1)$.

d) Find a Cartesian equation for the plane Π which passes through the point $D(1,2,3)$ and is parallel to the plane containing the triangle ABC.

e) Find the distance between the point A and the plane Π defined in (d).

iii) Consider the following system of linear equations where k is a real number.

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & 2k\,x_3 & = & 3 \\ x_1 & + & 3k\,x_2 & + & x_3 & = & -2 \\ 2x_1 & + & 6k\,x_2 & + & k\,x_3 & = & 1 \end{array}$$

Find for which values of k the system has (I) no solutions, (II) a unique solution or (III) infinitely many solutions.

iv) Let

$$R = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & \sqrt{3} \end{pmatrix}.$$

a) Find $\det(R)$.

b) Find RR^T.

c) If P and Q are 3×3 invertible matrices, fully simplify

$$QR(PQR)^T(RP^{-1})^T.$$

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 3

3. i) Let y be a function of the real variable x defined implicitly by

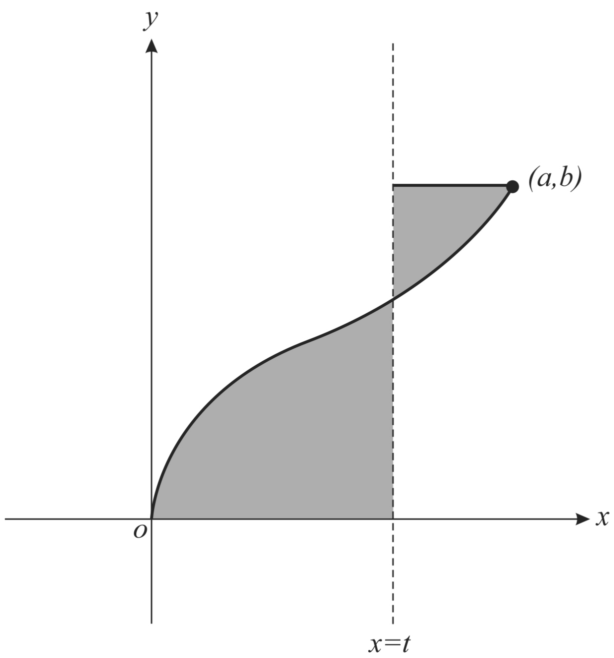
$$y^4 + x^3 - x^2 y = 8.$$

Find $\dfrac{dy}{dx}$ at the point where $y = 0$.

ii) State the Mean Value Theorem and use it to prove that

$$\sinh x > x \quad \text{for} \quad x > 0.$$

iii) Consider the following graph of an increasing differentiable function f, rising from the origin $(0,0)$ to the point (a,b).



The graph of f is cut by a vertical line $x = t$ where $0 < t < a$. The shaded region R is made of the region bounded by $y = f(x)$, $x = t$ and the x axis, together with the region bounded by $y = f(x)$, $x = t$ and $y = b$, as shown above.

The area $A(t)$ of R depends upon the choice of t.

a) Show that

$$A(t) = \int_0^t f(x)\,dx + \int_a^t \left(f(x) - b\right)\,dx.$$

b) Using the First Fundamental Theorem of Calculus, show that $A(t)$ has a local minimum at $t = f^{-1}\left(\dfrac{b}{2}\right)$.

Please see over ...

iv) The functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are defined and plotted in the Maple session below.

```
> f := x -> piecewise(x <= 0, 0, exp(-1/x));
```

$$f := x \mapsto \begin{cases} 0 & x \le 0 \\ e^{-\frac{1}{x}} & x > 0 \end{cases}$$

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> g := x -> f(x)*f(2-x);
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$$g := x \mapsto f(x)f(2-x)$$

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> plot(g(x), x=-1..3);
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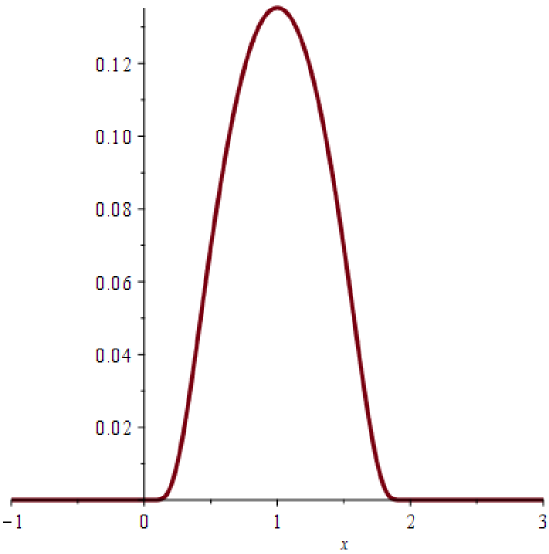

a) Show that f is continuous at 0.

b) Show that f is differentiable at 0 and find $f'(0)$.

c) Explain why g is continuous and differentiable everywhere.

d) Calculate $g'(1)$.

BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln|k|$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$

$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$

$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$

$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$

$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$

$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$

$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$

$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\begin{aligned}\int \frac{dx}{a^2 - x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, \qquad |x| < a\\ &= \frac{1}{a}\coth^{-1}\frac{x}{a} + C, \qquad |x| > a > 0\\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, \qquad x^2 \neq a^2\end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$

END OF EXAMINATION