LECTURE 18
Integration by Parts

$$\int u dv = uv - \int v du$$

SOME BASIC INTEGRALS

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int \frac{1}{x}\, dx = \ln|x| + C$$

$$\int e^{ax} dx = \frac{e^{ax}}{a} + C$$

$$\int \sin\, ax\, dx = -\frac{\cos ax}{a} + C$$

$$\int \cos ax\, dx = \frac{\sin ax}{a} + C$$

$$\int \sec^2 ax\, dx = \frac{\tan ax}{a} + C$$

One of your fundamental differentiation techniques is the product rule, which enables you to differentiate the product of two or more standard functions. Integration is a much trickier process and with integration we do not have the luxury of global processes such as the product, quotient and chain rules. However under certain circumstances we can integrate a product of two functions using a technique called integration by parts.

Parts does not have the wide ranging application of the product rule but it is a very effective technique for integrals of the type $\int$ (little polynomial) $\times$ (nice function). Parts can also be tweaked to knock off other types of integrals.

If faced with an integral of a product we assign one of the functions to be u and the other to be dv. Then the integration by parts formula is:

$$\int u dv = uv - \int v du$$

Proof

$$\int \frac{d}{dx}(uv) = \int u dv + \int v du$$

$$uv = \int u dv + \int v du \quad \star$$

$$\int u dv = uv - \int v du.$$

Example 1: Find $\displaystyle\int 3xe^{4x}\,dx$.

This is a typical parts question. Observe the little poly $3x$ and the nice function e^{4x}. Note that, unlike the technique of substitution the increment dx plays no effective role in the parts process.

$$\int \underbrace{3x}_{u}\, \underbrace{e^{4x}\,dx}_{dv}.$$

u = little poly

$u = 3x \to du = 3$

$dv = e^{4x} \to v = \frac{1}{4}e^{4x}.$

$$\int u\,dv = uv - \int v\,du$$

$$\int 3xe^{4x}\,dx = (3x)\left(\tfrac{1}{4}e^{4x}\right) - \int \tfrac{1}{4}e^{4x}(3)\,dx$$

$$= \tfrac{3}{4}xe^{4x} - \tfrac{3}{4}\int e^{4x}\,dx. = \tfrac{3}{4}xe^{4x} - \tfrac{3}{4}\cdot\tfrac{1}{4}e^{4x} + C$$

$$\bigstar \quad \frac{3}{4}xe^{4x} - \frac{3}{16}e^{4x} + C \quad \bigstar$$

Note that, depending upon how you finished up, you may have produced

$$\pm\frac{3}{4}C \quad \text{or} \quad \pm\frac{3}{16}C \quad \text{or} \quad \pm 3C. \quad \text{All of these are equivalent to} \quad +C.$$

Let us check that the solution $\displaystyle\int 3xe^{4x}\,dx = \frac{3}{4}xe^{4x} - \frac{3}{16}e^{4x} + C$ is correct!

$$\frac{d}{dx}\left(\frac{3}{4}xe^{4x} - \frac{3}{16}e^{4x} + C\right) = \frac{3}{4}e^{4x} + 4e^{4x}\left(\tfrac{3}{4}x\right) - \tfrac{3}{16}e^{4x}(4)$$

$(uv)' = u'v + v'u$

$$= \cancel{\tfrac{3}{4}e^{4x}} + 3xe^{4x} - \cancel{\tfrac{3}{4}e^{4x}}$$

$$= 3xe^{4x} \quad \checkmark$$

$\bigstar$

As a general rule we let u be the little poly and dv be the other function although you will soon see that there are many exceptions to this rule. What if we get them around the wrong way?

$$\int \underbrace{3x}_{dv} \underbrace{e^{4x}}_{u}\, dx.$$

$u = e^{4x} \to du = 4e^{4x}$

$dv = 3x \to v = \frac{3}{2}x^2$

$\int u\,dv = uv - \int v\,du = \frac{3}{2}x^2 e^{4x} - \int \frac{3}{2}x^2 4e^{4x}\,dx$

$= \frac{3}{2}x^2 e^{4x} - 6\int x^2 e^{4x}\,dx.$

★

Let's have a go at another one:

Example 2: Find $\displaystyle\int 5t\cos(9t)\,dt$.

Do not be put off by the t variable. All variables in an integral are dummy variables.

Note also that when calculating v from dv don't fuss with $+C$. The constant comes much later.

$u = 5t \Rightarrow du = 5$

$dv = \cos(9t) \to v = \frac{1}{9}\sin(9t)$

$\int u\,dv = uv - \int v\,du = (5t)(\frac{1}{9}\sin 9t) - \int \frac{1}{9}\sin(9t)(5)$

$= \frac{5}{9}t\sin 9t - \frac{5}{9}\int \sin 9t\,dt$

$= \frac{5}{9}t\sin(9t) - \frac{5}{9}\cdot(-\frac{1}{9}\cos 9t)$

$= \frac{5}{9}t\sin(9t) + \frac{5}{81}\cos(9t) + C$

★ $\displaystyle\frac{5}{9}t\sin(9t) + \frac{5}{81}\cos(9t) + C$ ★

Sometimes we need to use parts multiple times!

Example 3: Find $\int x^2 \cos(x)\, dx$.

$u = x^2 \to du = 2x$

$dv = \cos x \to v = \sin x$

$\int u\,dv = uv - \int v\,du = x^2 \sin x - \int \sin(x)(2x)\, dx.$

$= x^2 \sin x - 2 \underbrace{\boxed{\int x \sin x\, dx}}_{I}.$

$I = \int x \sin(x)\, dx.$

$u = x \to du = 1$

$dv = \sin(x) \to v = -\cos(x)$

$\int u\,dv = uv - \int v\,du$

$= -x\cos x - \int (-\cos(x))(1)$

$= -x\cos x + \int \cos x\, dx$

$= -x\cos x + \sin x$

$\therefore \int = x^2 \sin x - 2\{-x\cos x + \sin x\}$

$= x^2 \sin x + 2x\cos x - 2\sin x + C$

$\int u\,dv = uv - \int v\,du$

★ $x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C$ ★

If the integral is definite (that is it has limits) then you must keep in mind that the first half of the parts equation has already been integrated so it has square brackets and limits while the second half still needs to be done.

Example 4: Find $\int_0^1 2xe^{3x}\,dx$.

$u = 2x \to du = 2$
$dv = e^{3x} \to v = \frac{1}{3}e^{3x}$

$$\int u\,dv = uv - \int v\,du$$
$$= \left[(2x)\left(\tfrac{1}{3}e^{3x}\right)\right]_0^1 - \int_0^1 \tfrac{1}{3}e^{3x}(2)\,dx.$$
$$= \tfrac{2}{3}e^3 - 0 - \tfrac{2}{3}\int_0^1 e^{3x}\,dx.$$
$$= \tfrac{2}{3}e^3 - \tfrac{2}{3}\cdot\tfrac{1}{3}\left[e^{3x}\right]_0^1$$
$$= \tfrac{2}{3}e^3 - \tfrac{2}{9}\{e^3 - 1\} = \underline{\tfrac{4}{9}e^3 + \tfrac{2}{9}}$$

★ $\frac{4}{9}e^3 + \frac{2}{9}$ ★

When the product involves the $\ln(x)$ function we often assign u to be $\ln(x)$.

Example 5: Find $\int x^2 \ln(x)\,dx$.

$u = \ln(x)$ for parts.

$u = \ln(x) \to du = \frac{1}{x}$.
$dv = x^2 \to v = \frac{1}{3}x^3$

$$\int u\,dv = uv - \int v\,du = \tfrac{1}{3}x^3\ln x - \int \tfrac{1}{3}x^3 \cdot \tfrac{1}{x}\,dx$$
$$= \tfrac{1}{3}x^3\ln x - \tfrac{1}{3}\int x^2\,dx.$$
$$= \tfrac{1}{3}x^3\ln x - \tfrac{1}{3}\cdot\tfrac{1}{3}x^3 + C$$

★ $\frac{x^3}{3}\ln(x) - \frac{1}{9}x^3 + C$ ★

Curiously we don't always need two functions to use parts. We can artificially multiply by 1 to get parts going. . . . . . but this rarely works!

Example 6: Find $\int \ln(x)\, dx$.

$$\int \ln(x)\,dx = \int \ln(x) \cdot 1\, dx.$$

$u = \ln(x) \to du = \frac{1}{x}$.

$dv = 1 \to v = x$

$$\int u\,dv = uv - \int v\,du$$
$$= x\ln x - \int x \cdot \frac{1}{x}\, dx$$
$$= x\ln x - \int 1\, dx$$
$$= x\ln x - x + C$$

★ $x\ln(x) - x + C$ ★

Example 7: Find $\int_0^{\sqrt{3}} \tan^{-1}(x)\, dx$.

$u = \tan^{-1}(x) \to du = \frac{1}{1+x^2}$

$dv = 1 \to v = x$.

$$\int_0^{\sqrt{3}} \tan^{-1}(x) \cdot 1\, dx$$

$$\int u\,dv = uv - \int v\,du = \left[x\tan^{-1}(x)\right]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} x \cdot \frac{1}{1+x^2}\, dx$$

$$= \sqrt{3}\tan^{-1}(\sqrt{3}) - 0 - \frac{1}{2}\int_0^{\sqrt{3}} \frac{2x}{1+x^2}\, dx.$$

$$= \sqrt{3}\,\frac{\pi}{3} - \frac{1}{2}\left[\ln(1+x^2)\right]_0^{\sqrt{3}}$$

$$= \frac{\sqrt{3}\pi}{3} - \frac{1}{2}\{(\ln(4)) - (0)\} = \frac{\sqrt{3}\pi}{3} - \frac{1}{2}\ln 4$$

$$= \frac{\sqrt{3}\pi}{3} - \frac{1}{2}\ln(2^2) \longrightarrow$$ ★ $\frac{\sqrt{3}\pi}{3} - \ln(2)$ ★

So we can now integrate a wider class of functions. Keep in mind however that most integrals cannot be done!! We finish off with a very strange trick.

Example 8: Find $\displaystyle\int e^x \cos(x)\, dx$.

Let $I = \displaystyle\int e^x \cos(x)\, dx$. It will be soon clear why we need to name the integral. Then

$u = e^x \longrightarrow du = e^x$.

$dv = \cos(x) \longrightarrow v = \sin(x)$.

$$I = \int udv = uv - \int vdu = e^x \sin(x) - \int \sin(x) e^x dx = e^x \sin(x) - J \qquad (1)$$

where $J = \displaystyle\int \sin(x) e^x dx$.

In this example it surprisingly doesn't matter which function you call u and which dv but you have to stick to your choice for e^x over the entire solution.

Let us now consider the second integral $J = \displaystyle\int \sin(x) e^x dx$.

$u = e^x \longrightarrow du = e^x$.

$dv = \sin(x) \longrightarrow v = -\cos(x)$.

$$\int udv = uv - \int vdu$$

$$J = -e^x \cos(x) - \int -\cos(x) e^x dx = -e^x \cos(x) + \int \cos(x) e^x dx = -e^x \cos(x) + I.$$

It is a little disturbing to see our question reappear in the answer but this is good news!

Substituting back into (1) yields

$I = e^x \sin(x) - (-e^x \cos(x) + I) \longrightarrow I = e^x \sin(x) + e^x \cos(x) - I$.

$\longrightarrow 2I = e^x \sin(x) + e^x \cos(x) \longrightarrow I = \dfrac{1}{2} e^x (\sin(x) + \cos(x)) + C$.

$$\bigstar \quad \frac{1}{2} e^x (\sin(x) + \cos(x)) + C \quad \bigstar$$

Integrals of the type: $\displaystyle\int \text{exponential} \times \{\sin \ \text{ or } \ \cos\}$ can be done in the above manner.

Homework Investigation:
What happens if we swap u and dv when calculating J? Try it and see.