MATH1131 Mathematics 1A – Algebra

Lecture 8: Triple Scalar Product and the Point-Normal Form

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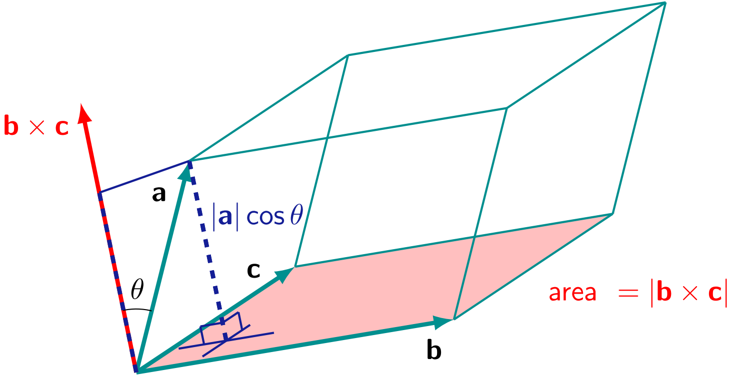
Based on slides by Jonathan Kress

Parallelepiped

The 3D version of a parallelogram is called the parallelepiped.



$$
\begin{aligned}
\text{Volume} &= \text{area of base} \times \text{altitude} \\
&= |\mathbf{b} \times \mathbf{c}||\mathbf{a}||\cos\theta| \\
&= |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|
\end{aligned}
$$

Scalar triple product

Definition

The triple scalar product of vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$ is

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

The scalar triple product can also be written as a 3×3 determinant:

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_2 c_3 - b_3 c_2 \\ b_3 c_1 - b_1 c_3 \\ b_1 c_2 - b_2 c_1 \end{pmatrix} \\
&= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}
\end{aligned}$$

Scalar triple product

Properties

For all $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$

- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$
- $\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = -\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
- $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0.$

Proof

By expanding and rearranging terms,

$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$

$= b_1(c_2a_3 - c_3a_2) + b_2(c_3a_1 - c_1a_3) + b_3(c_1a_2 - c_2a_1) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$

$= c_1(a_2b_3 - a_3b_2) + c_2(a_3b_1 - a_1b_3) + c_3(a_1b_2 - a_2b_1) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$

Scalar triple product

Properties

For all $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$

- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$
- $\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = -\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
- $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0.$

Proof

Using properties of the cross product (anti-commutativity) and the dot product (associative law of scalar multiplication),

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) &= \mathbf{a} \cdot (-1)(\mathbf{b} \times \mathbf{c}) \\ &= -\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \end{aligned}$$

Scalar triple product

Properties

For all $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$

- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$
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- $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0.$

Proof

Using the first property above and properties of the dot and cross products,

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) &= \mathbf{b} \cdot (\mathbf{a} \times \mathbf{a}) \\ &= \mathbf{b} \cdot \mathbf{0} \\ &= 0 \end{aligned}$$

Scalar triple product – Examples

Example

Find the volume of the parallelepiped with edges given by the vectors

$$\begin{pmatrix}1\\2\\3\end{pmatrix}, \quad \begin{pmatrix}-3\\-2\\7\end{pmatrix}, \quad \begin{pmatrix}-4\\3\\7\end{pmatrix}.$$

$$\begin{aligned}
\text{Volume} = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| &= \left|\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\left(\begin{pmatrix}-3\\-2\\7\end{pmatrix}\times\begin{pmatrix}-4\\3\\7\end{pmatrix}\right)\right| \\
&= \left|\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}-35\\-7\\-17\end{pmatrix}\right| \\
&= |-100| \\
&= 100
\end{aligned}$$

Scalar triple product – Examples

Example

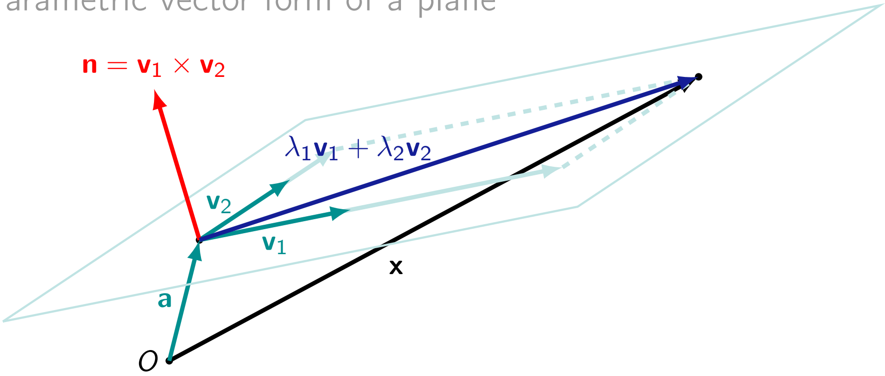
Show that the points $A(3,3,5)$, $B(1,0,1)$, $C(2,2,4)$ and $D(2,1,2)$ are coplanar.

Three vectors all lie in the same plane if an only if the parallelepiped spanned by them has zero volume.

$$\begin{aligned}
\text{Volume} = |\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD})| &= \left| \begin{pmatrix} -2 \\ -3 \\ -4 \end{pmatrix} \cdot \left(\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \times \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} \right) \right| \\
&= \left| \begin{pmatrix} -2 \\ -3 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right| \\
&= 0
\end{aligned}$$

So the four points are coplanar.

Parametric vector form of a plane

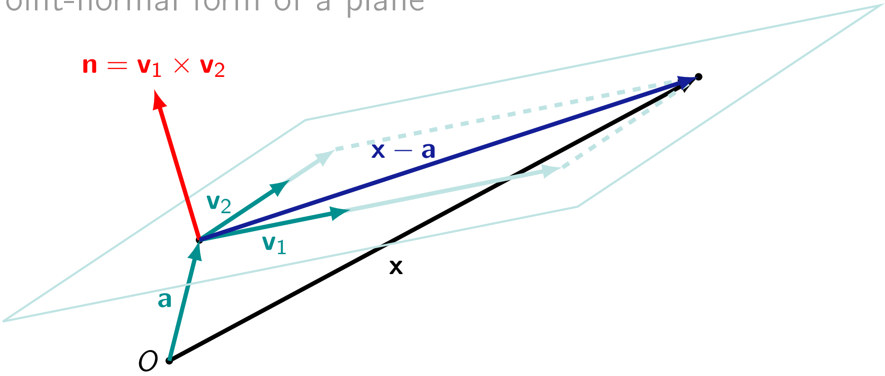


Recall: A plane parallel to two (non-parallel) vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, and containing the point with position vector $\mathbf{a}$, is described by:

$$\mathbf{x} = \mathbf{a} + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2, \qquad \lambda_1, \lambda_2 \in \mathbb{R}.$$

We called this the parametric vector form of the plane.
The linear combinations $\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2$ are all parallel to the plane.
The normal vector $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$ is perpendicular to all of these.

Point-normal form of a plane



The normal vector $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$ is perpendicular to the plane.

In particular, $\mathbf{x} - \mathbf{a}$ is perpendicular to $\mathbf{n}$ for all $\mathbf{x}$ in the plane, so

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = 0.$$

This is called the point-normal form of the plane.

It can also be written in the form $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{a}$.

Cartesian and point-normal forms

Suppose $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ is a vector normal to some plane passing through the point $A(a_1, a_2, a_3)$.

The point-normal form of this plane is:

$$\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{a}$$

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$n_1 x + n_2 y + n_3 z = n_1 a_1 + n_2 a_2 + n_3 a_3$$

or we could write:

$$n_1 x + n_2 y + n_3 z = d, \quad \text{where } d = \mathbf{n} \cdot \mathbf{a}$$

This is recognisable as a Cartesian form of the plane ($ax + by + cz = d$ for some $a, b, c, d \in \mathbb{R}$ with at least one of a, b, and c non-zero).

Cartesian and point-normal forms – Examples

Example

Find a Cartesian equation of the plane with normal $\begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$ that passes through the point $(1, 2, 1)$.

The Cartesian form will be $4x + (-2)y + 3z = d$, where:

$$d = \mathbf{n} \cdot \mathbf{a} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 3.$$

That is, a Cartesian form for the plane is $4x - 2y + 3z = 3$.

Cartesian and point-normal forms – Examples

Example

Write the plane

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R},$$

in point-normal form and in Cartesian form.

To find a normal vector $\mathbf{n}$ to the plane, we can take the cross product of the two parallel vectors $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \\ -3 \end{pmatrix}.$$

Cartesian and point-normal forms – Examples

Example

Write the plane

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R},$$

in point-normal form and in Cartesian form.

Since we know the plane contains the point $(1, 2, 3)$, we can write a point-normal form as:

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = \begin{pmatrix} 11 \\ 13 \\ -3 \end{pmatrix} \cdot \left(\mathbf{x} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right) = 0.$$

Similarly we can write a Cartesian form as:

$$11x + 13y - 3z = \mathbf{n} \cdot \mathbf{a} = 28.$$

Cartesian and point-normal forms – Examples

Example

Write the plane $2x + 9y - 7z = 21$ in point-normal form.

One normal vector $\mathbf{n}$ to the plane must be $\mathbf{n} = \begin{pmatrix} 2 \\ 9 \\ -7 \end{pmatrix}$.

To find a point on the plane, set any two of the unknowns to be 0 and solve for the third unknown.

For example, setting $x = y = 0$ gives $-7z = 21 \implies z = -3$.

So $(0, 0, -3)$ is a point on the plane.

We can therefore write a point-normal form of the plane as:

$$\begin{pmatrix} 2 \\ 9 \\ -7 \end{pmatrix} \cdot \left(\mathbf{x} - \begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix} \right) = 0.$$

Summary of plane equations – Vector parametric form

Given a plane in vector parametric form:

$$\mathbf{x} = \mathbf{a} + \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2, \quad \lambda_1, \lambda_2 \in \mathbb{R}$$

- Vectors parallel to the plane include $\mathbf{v}_1$, $\mathbf{v}_2$, $2\mathbf{v}_1 - 5\mathbf{v}_2$, etc.
- Position vectors for points on the plane include $\mathbf{a}$, $\mathbf{a} + \mathbf{v}_1$, $\mathbf{a} + \mathbf{v}_2$, $\mathbf{a} + 2\mathbf{v}_1 - 5\mathbf{v}_2$, etc.
- A normal vector to the plane is $\mathbf{v}_1 \times \mathbf{v}_2$.

Summary of plane equations – Cartesian form

Given a plane in Cartesian form:

$$ax + by + cz = d, \quad a, b, c, d \in \mathbb{R}$$

- A normal vector to the plane is $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.
- Points on the plane include:
  - $(\frac{d}{a}, 0, 0)$ (if $a \neq 0$)
  - $(0, \frac{d}{b}, 0)$ (if $b \neq 0$)
  - $(0, 0, \frac{d}{c})$ (if $c \neq 0$)
- Vectors parallel to the plane include $\begin{pmatrix} b \\ -a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ c \\ -b \end{pmatrix}, \begin{pmatrix} -c \\ 0 \\ a \end{pmatrix}$,

  and any linear combinations of these.

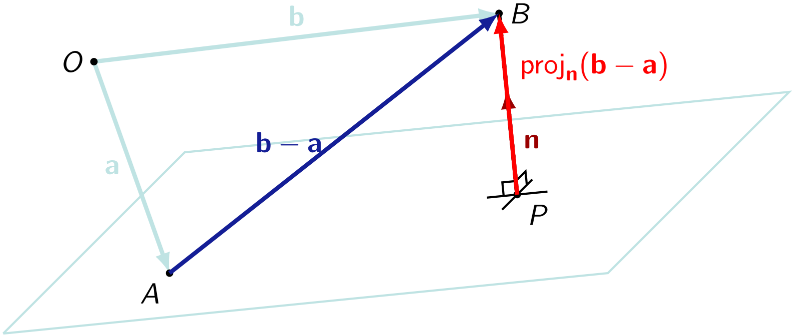
Summary of plane equations – Point-normal form

Given a plane in point-normal form:

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = 0$$

- A normal vector to the plane is $\mathbf{n}$.
- The position vector for a point on the plane is $\mathbf{a}$. To find others, it is easiest to first convert to Cartesian form.
- Vectors parallel to the plane are any vectors $\mathbf{v}$ such that $\mathbf{n} \cdot \mathbf{v} = 0$. It is easiest to find these by first converting to Cartesian form.

How might we find the shortest distance from a point B to the plane given by $\mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = 0$, or find the point P on the plane that is closest to B?



Position vector of P: $\mathbf{b} - \text{proj}_{\mathbf{n}}(\mathbf{b} - \mathbf{a}) = \mathbf{b} - \dfrac{\mathbf{n} \cdot (\mathbf{b} - \mathbf{a})}{|\mathbf{n}|^2}\mathbf{n}$

Shortest distance: $|\text{proj}_{\mathbf{n}}(\mathbf{b} - \mathbf{a})| = \dfrac{|\mathbf{n} \cdot (\mathbf{b} - \mathbf{a})|}{|\mathbf{n}|}$

Example

Find the shortest distance between the point $B(4, -2, 3)$ and the plane passing through the points $P(1, 2, 3)$, $Q(-3, 2, 1)$, and $R(4, 5, 6)$. Also find the point X on the plane that is closest to B.

First find a vector normal to the plane. As before, we can take the cross product of two vectors parallel to the plane. For example,

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{pmatrix} -4 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = 2 \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix} \times 3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

Since only the direction of the normal vector is important, we can disregard the length of $\overrightarrow{PQ} \times \overrightarrow{PR}$ and just choose $\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

Example

Find the shortest distance between the point $B(4,-2,3)$ and the plane passing through the points $P(1,2,3)$, $Q(-3,2,1)$, and $R(4,5,6)$. Also find the point X on the plane that is closest to B.

By choosing $\mathbf{a}$ to be the position vector for the point P, say, we can find the required projected vector as follows:

$$\begin{aligned}\text{proj}_{\mathbf{n}}(\mathbf{b}-\mathbf{a}) &= \frac{(\mathbf{b}-\mathbf{a})\cdot\mathbf{n}}{|\mathbf{n}|^2}\,\mathbf{n} \\ &= \frac{(4-1)\times 1+(-2-2)\times 1+(3-3)\times(-2)}{1^2+1^2+(-2)^2}\begin{pmatrix}1\\1\\-2\end{pmatrix} \\ &= -\frac{1}{6}\begin{pmatrix}1\\1\\-2\end{pmatrix}\end{aligned}$$

Example

Find the shortest distance between the point $B(4,-2,3)$ and the plane passing through the points $P(1,2,3)$, $Q(-3,2,1)$, and $R(4,5,6)$. Also find the point X on the plane that is closest to B.

So the position vector of the closest point X is given by:

$$\mathbf{b} - \text{proj}_{\mathbf{n}}(\mathbf{b}-\mathbf{a}) = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} - \frac{-1}{6}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 25/6 \\ -11/6 \\ 8/3 \end{pmatrix}$$

That is, X is the point $(\frac{25}{6}, -\frac{11}{6}, \frac{8}{3})$.

The shortest distance is given by:

$$|\text{proj}_{\mathbf{n}}(\mathbf{b}-\mathbf{a})| = \left| -\frac{1}{6}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right| = \frac{1}{6}\sqrt{1^2+1^2+(-2)^2} = \frac{1}{\sqrt{6}}$$