THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 1 2017

MATH1131
MATHEMATICS 1A

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS IS APPENDED TO THE PAPER

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 1

1. i) Evaluate each limit, giving brief reasons for your answer.

a) $\displaystyle\lim_{x\to\infty} \frac{2e^x + \cos x}{6e^x - \sin x}$

b) $\displaystyle\lim_{x\to 0} \frac{e^{2x} - 1}{x^2 - 3x}$

ii) Evaluate each of the following integrals:

a) $\displaystyle\int \cos x \sin^5 x \, dx$,

b) $\displaystyle\int \frac{1}{\sqrt{25 + 9x^2}} \, dx$.

iii) The floor of x is denoted $\lfloor x \rfloor$ and gives the greatest integer less than or equal to x. For example, $\lfloor 3.1 \rfloor = 3$.
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \lfloor 3x - 1 \rfloor.$$

a) Write down the value of $f(\frac{1}{2})$.

b) By considering left and right limits, state whether or not f is continuous at $x = \frac{1}{3}$.

iv) Let $z = \dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}}$.

a) Write z in polar form.

b) Find the smallest positive integer n such that $z^n = 1$.

c) Find all possible positive integers m such that $z^m = 1$.

v) Consider the following Maple output

```
> with(LinearAlgebra):
> A := <<607,207,75>|<-2286,-783,-288>|<1414,483,176>>;
```

$$A := \begin{bmatrix} 607 & -2286 & 1414 \\ 207 & -783 & 483 \\ 75 & -288 & 176 \end{bmatrix}$$

```
> A^2;
```

$$\begin{bmatrix} 1297 & -4896 & 3024 \\ -207 & 783 & -483 \\ -891 & 3366 & -2078 \end{bmatrix}$$

```
> A^3;
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$$\begin{bmatrix} 607 & -2286 & 1414 \\ 207 & -783 & 483 \\ 75 & -288 & 176 \end{bmatrix}$$

a) Find A^{100}.

b) Is A invertible? Give reasons for your answer.

vi) a) Express $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$.

b) Find constants a, b and c such that

$$\sin^4\theta = a\cos 4\theta + b\cos 2\theta + c.$$

c) Hence or otherwise evaluate

$$\int \sin^4\theta \, d\theta.$$

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 2

2. i) Find the gradient $\dfrac{dy}{dx}$ of the curve defined by $xy + y^2 = 4$ at the point where $y = 1$.

ii) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 3x - \cos x$.

a) Show that f has a zero in the interval $\left[0, \dfrac{\pi}{2}\right]$.

b) What is the minimum value of f'?

c) Explain why f has an inverse on $\mathbb{R}$.

d) Find the value of $g'(-1)$, where g is the inverse of f.

iii) Sketch the graph of the polar curve $r = 2 - \cos 2\theta$ for $0 \le \theta \le \pi$.

iv) For some values of the real parameters a, b, c and d, the curve

$$ax^2 + by^2 + cx + dy = 1$$

passes through the points $A(1, 1), B(2, 3)$ and $C(0, 1)$.

a) Explain why the following equations can be used to determine the values of a, b, c and d for which the curve passes through the points.

$$\begin{array}{rcrcrcrcl} a & + & b & + & c & + & d & = & 1 \\ 4a & + & 9b & + & 2c & + & 3d & = & 1 \\ & & b & & & + & d & = & 1. \end{array}$$

b) Use Gaussian Elimination to solve the system of linear equations in part (a).

c) Are there zero, one, or infintely many curves of the form $ax^2 + by^2 + cx + dy = 1$ which pass through the points A, B and C?

d) Using your answer from part (b), find the parabola of the form $y = \alpha x^2 + \beta x + \gamma$ which passes through A, B and C.

v) a) Write $\begin{pmatrix} 34 \\ -9 \\ -23 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 4 \\ 1 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

b) What is the volume of the parallelepiped spanned by the vectors $\begin{pmatrix} 34 \\ -9 \\ -23 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 1 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$?

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 3

3. i) a) Clearly state the Mean Value Theorem.

b) Apply the Mean Value Theorem to $f : \mathbb{R} \to \mathbb{R}$ given by $f(t) = e^{2t} - t$ on the interval $[0, x]$, $x \geq 0$, to prove that for all positive real numbers x,

$$e^{2x} \geq 2x + 1.$$

ii) a) Define the hyperbolic function $\cosh x$ in terms of exponentials.

b) Use the definition to prove that

$$\cosh 2x = 2\cosh^2 x - 1.$$

iii) Evaluate the integral $\displaystyle\int x \ln x \, dx$.

iv) The point $(x(t), y(t))$ is moving clockwise on the circle $x^2 + y^2 = 100$.

a) Prove that $\dfrac{dy}{dt} = -\dfrac{x}{y}\dfrac{dx}{dt}$.

b) If the velocity in the x direction at $x = 6$ is 2 units/sec, find the velocity in the y direction.

v) Consider the function f defined for all real x by

$$f(x) = \int_0^x \frac{t}{t^4 + 1} \, dt.$$

a) Find $\dfrac{d}{dx}\big(f(x^3)\big)$.

b) Explain why f is an even function.

c) By considering the improper integral, $\displaystyle\int_1^\infty \frac{t}{t^4 + 1} \, dt$, explain why $\displaystyle\lim_{x \to \infty} f(x)$ exists.

vi) The Maple output below shows a calculation and then the **upper** Riemann sum for the function $f : (0, \infty) \to \mathbb{R}$, given by $f(x) = \ln x$, on the interval $[1, 6]$ using the partition shown. The value of the upper Riemann sum is approximately 6.181297752.

Find, to two decimal places, the value of the corresponding **lower** Riemann sum.

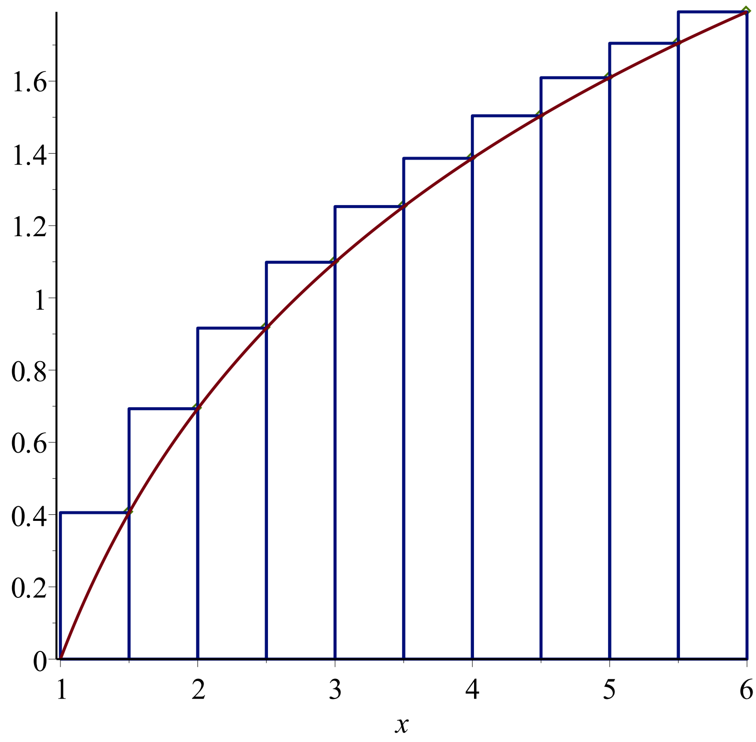
```
> [seq(i*0.5, i = 2..12)];
```

$$[1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 5.5, 6.0]$$

```
> map(ln,%);
```

$$[0., .4054651081, .6931471806, .9162907319, 1.098612289, 1.252762968,$$
$$1.386294361, 1.504077397, 1.609437912, 1.704748092, 1.791759469]$$

```
> with(Student[Calculus1]):
> RiemannSum(ln(x), x = 1..6, method = upper, output = plot);
```



USE A SEPARATE BOOK CLEARLY MARKED QUESTION 4

4. i) Let $\mathbf{a} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$ be two vectors in $\mathbb{R}^2$.

a) Calculate $\mathrm{proj}_{\mathbf{b}}(\mathbf{a})$.

b) Sketch the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathrm{proj}_{\mathbf{b}}(\mathbf{a})$ in the plane.

ii) Calculate the inverse of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

iii) Let $C = \begin{pmatrix} 1 & 2 \\ 4 & 0 \\ 6 & 3 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 5 & 0 \\ 5 & 0 & 5 \end{pmatrix}$.

a) Find DC.

b) What is the size of CD?

iv) Let $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

a) Calculate $\mathbf{u} \times \mathbf{v}$.

b) Hence or otherwise find $\begin{pmatrix} 300 \\ 200 \\ 100 \end{pmatrix} \times \begin{pmatrix} 600 \\ 0 \\ 600 \end{pmatrix}$.

v) Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the planes in three dimensional space with Cartesian equations $12x + 9y + 3z = 75$ and $z = 51 - 4x - 3y$ respectively.

a) Explain why $\mathcal{P}_1$ and $\mathcal{P}_2$ are parallel.

b) Show that $(5, 1, 2)$ is a point on $\mathcal{P}_1$.

c) It is given that the line $\mathcal{L}$ with parametric vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} \quad ; \quad \lambda \in \mathbb{R},$$

passes through the point $(5, 1, 2)$ and is perpendicular to both $\mathcal{P}_1$ and $\mathcal{P}_2$. Determine where the line $\mathcal{L}$ meets the plane $\mathcal{P}_2$.

d) Hence, or otherwise, find the shortest distance between $\mathcal{P}_1$ and $\mathcal{P}_2$.

e) Find a point Q with the property that the set of all points equidistant from $(5, 1, 2)$ and Q is the plane $\mathcal{P}_2$.

vi) Suppose that A and B are two $n \times n$ matrices and that $AB - A$ is invertible. Prove that $BA - A$ is also invertible.

Please see over ...

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BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$

$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$

$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$

$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$

$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$

$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$

$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$

$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\begin{aligned}\int \frac{dx}{a^2-x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, &\qquad |x| < a\\ &= \frac{1}{a}\coth^{-1}\frac{x}{a} + C, &\qquad |x| > a > 0\\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, &\qquad x^2 \neq a^2\end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$