THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Term 2 2019

MATH1131
MATHEMATICS 1A

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 3

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS IS APPENDED TO THE PAPER

(9) TO OBTAIN FULL MARKS, YOUR ANSWERS MUST NOT ONLY BE CORRECT, BUT ALSO ADEQUATELY EXPLAINED, CLEARLY WRITTEN AND LOGICALLY SET OUT.

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 1

1. i) Use integration by parts to evaluate $\displaystyle\int_0^{\sqrt{3}} \tan^{-1}(x)\,dx$.

ii) a) Write down the definitions of $\sinh(x)$ and $\cosh(x)$ in terms of the exponential function.

b) Evaluate

$$\int_0^{\ln(3)} \frac{1}{\cosh(x)+\sinh(x)}\,dx.$$

iii) Let $\mathbf{u} = \begin{pmatrix} 1 \\ 3 \\ -2 \\ 5 \\ -1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ 9 \\ -6 \\ \beta \\ -3 \end{pmatrix}$ be two vectors in $\mathbb{R}^5$.

a) Find β if $\mathbf{u}$ and $\mathbf{v}$ are parallel.

b) Find β if $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.

iv) Let $\mathbf{a} = \begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 0 \\ 10 \end{pmatrix}$ be two vectors in $\mathbb{R}^3$.

a) Calculate $\text{Proj}_{\mathbf{b}}(\mathbf{a})$, the projection of $\mathbf{a}$ onto $\mathbf{b}$.

b) Hence find a vector $\mathbf{c}$ (not equal to $\mathbf{a}$) in the plane spanned by $\mathbf{a}$ and $\mathbf{b}$ such that $|\mathbf{a}| = |\mathbf{c}|$ and $\text{Proj}_{\mathbf{b}}(\mathbf{a}) = \text{Proj}_{\mathbf{b}}(\mathbf{c})$.

v) For the following system of equations, use Gaussian elimination to determine which values of λ (if any) will yield:

a) no solution b) infinite solutions c) a unique solution.

$$\begin{aligned} x + y + z &= 4 \\ x + \lambda y + 2z &= 5 \\ 2x + (\lambda+1)y + (\lambda^2-1)z &= \lambda + 7. \end{aligned}$$

vi) Let $f(x) = \dfrac{1}{2+x}$ and consider the uniform partition $\mathcal{P}_n$ of the interval $[0, 1]$ given by

$$\mathcal{P}_n = \left\{0,\ \frac{1}{n}\ ,\ \frac{2}{n}\ ,\ \ldots\ ,\frac{n-1}{n}\ ,\ \frac{n}{n}\right\}.$$

a) Sketch a graph of f for $0 \le x \le 1$ and indicate the rectangles defining the lower Riemann sum for f on $\mathcal{P}_4$.

b) Write down an expression for the lower Riemann sum for f on $\mathcal{P}_n$.

c) Hence, or otherwise, show that

$$\lim_{n\to\infty}\left(\frac{1}{2n+1} + \frac{1}{2n+2} + \cdots + \frac{1}{3n}\right) = \ln 3 - \ln 2.$$

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 2

2. i) Let the region S be defined as

$$S = \{z \in \mathbb{C} : |z - 2 - 2i| \leq 2 \text{ and } \mathrm{Re}(z) > 1\}.$$

a) Sketch the region S on a carefully labelled Argand diagram.

b) State in $a + ib$ form, the complex number in S of maximum modulus.

ii) A table sits on horizontal ground. In Figure 1, a bird stands on the table directly above a cat sitting on the ground, and the distance between the tops of their heads is 105 cm as shown. In Figure 2, the cat and the bird have swapped places, and the distance between the tops of their heads is now 145 cm.

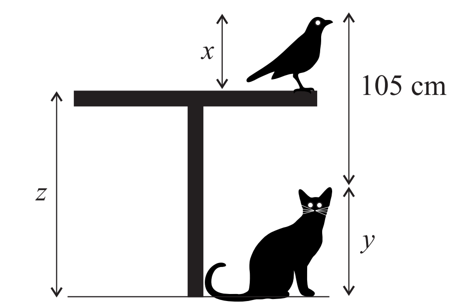


Figure 1

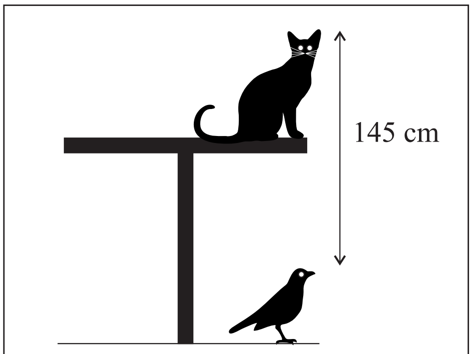


Figure 2

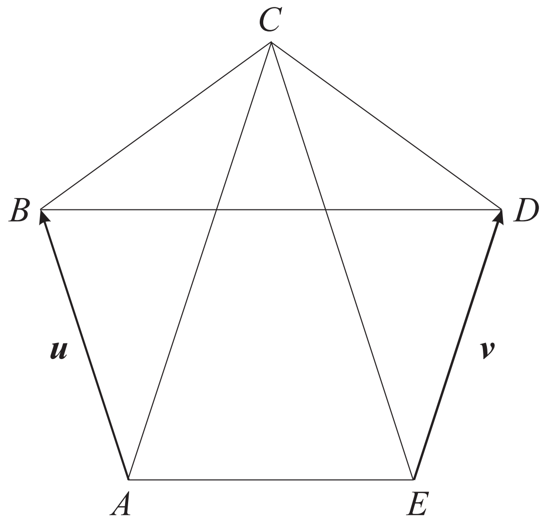
Let x be the height of the bird, y be the height of the cat and z be the height of the table, with all measurements made in cm.

a) It follows from Figure 1 that $x - y + z = 105$. By considering Figure 2, write down a second linear equation in x, y and z.

b) By reducing an appropriate augmented matrix to echelon form and back-substituting, find the height z of the table, and express the heights of the bird and cat in terms of a parameter.

c) Given that the sum of the heights of the bird and cat is 112 cm, find the height of the bird.

iii) Consider the following pentagon $ABCDE$ with five equal-length sides.



Let $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{ED}$.

Observe that $\overrightarrow{BD}$ is parallel to $\overrightarrow{AE}$ and hence

$$\overrightarrow{BD} = \lambda\left(\overrightarrow{AE}\right) \text{ for some } \lambda > 1.$$

Similarly $\overrightarrow{AC} = \lambda\left(\overrightarrow{ED}\right) = \lambda\mathbf{v}$ and $\overrightarrow{EC} = \lambda\left(\overrightarrow{AB}\right) = \lambda\mathbf{u}$.

a) By considering the triangle ACE prove that $\overrightarrow{AE} = \lambda(\mathbf{v} - \mathbf{u})$

b) Find another expression for $\overrightarrow{AE}$ in terms of λ, $\mathbf{v}$ and $\mathbf{u}$ and hence find the exact value of λ, expressing your answer in surd form.

iv) Use the following Maple output to explain why A is invertible and to calculate A^{-1} for the given matrix A. Give reasons for your answer.

```
> with(LinearAlgebra):
> A := <<2/3,-1/3,-2/3>|<-1/3,2/3,-2/3>|<-2/3,-2/3,-1/3>>;
```

$$A := \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

```
> Determinant(A);
```

$$-1$$

```
> A^3-A;
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

v) a) By considering seventh roots of unity, find a non-real solution to the equation

$$\left(\frac{1+w}{1-w}\right)^7 = 1.$$

b) Show that your solution in part a) is purely imaginary.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 3

3. i) Suppose that $y = h(x)$ is a continuous function over the real line with the property that

$$\lim_{x\to\infty} h(x) = 2.$$

a) Draw a possible sketch of the graph of h.

b) Using the formal definition of the limit, explain why there exists a real number M with the property that

$$x > M \quad \Rightarrow \quad h(x) > 1.$$

c) Hence prove that $\displaystyle\int_0^\infty h(x)\,dx$ is a divergent improper integral.

ii) The function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \frac{x}{e^x}\ .$$

a) Find and classify the stationary point of f.

b) Using L'Hopital's rule, evaluate $\displaystyle\lim_{x\to\infty} f(x)$.

c) What is the range of f?

d) Sketch the graph of f.

e) Define a function $g : [1,\infty) \to (0, e^{-1}]$ by the rule

$$g(x) = f(x)$$

Explain why the function g is invertible while f is not.

f) Let g^{-1} be the inverse function of g. Show that

$$g^{-1}(x) = xe^{g^{-1}(x)}$$

g) On what interval is g^{-1} differentiable?

h) Show that the derivative of the inverse of g satisfies

$$[g^{-1}(x)]' = \frac{g^{-1}(x)}{x(1-g^{-1}(x))}\ .$$

iii) Suppose that both the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), with $g(a) \neq g(b)$ and $g'(c) \neq 0$. Let

$$h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x).$$

By applying the Mean Value theorem to h prove that there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}\ .$$

BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln|k|$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$

$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$

$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$

$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$

$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$

$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$

$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$

$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\begin{aligned}\int \frac{dx}{a^2 - x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, \qquad |x| < a \\ &= \frac{1}{a}\coth^{-1}\frac{x}{a} + C, \qquad |x| > a > 0 \\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, \qquad x^2 \neq a^2\end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$

END OF EXAMINATION