Chapter 1: Sets, inequalities and functions

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MATH1131

UNSW

Term 1 2020

Goals of this chapter

In this chapter we will

- fix some basic terminology and notation;
- review some facts about inequalities and absolute values;
- review important facts about functions;
- discuss some important classes of functions:
  - polynomials and rational functions;
  - trigonometric functions;
  - elementary functions.

Sets of numbers

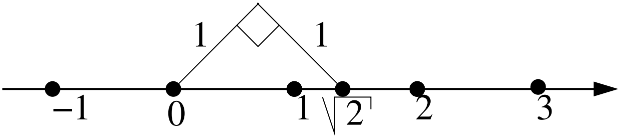
A set is a collection of distinct objects. The objects in a set are called the elements or members of the set. The empty set is denoted by $\emptyset$.

- The set $\mathbb{N}$ of natural numbers is given by
$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}.$$
- The set $\mathbb{Z}$ of integers is given by
$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$
- The set $\mathbb{Q}$ of rational numbers consists of numbers of the form $\frac{p}{q}$ where p, q are integers and $q \neq 0$, that is,
$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\}.$$
- There are numbers, such that $\sqrt{2}$, which are not rational numbers.

- The set

$$\{x \in \mathbb{R} : x \notin \mathbb{Q}\}$$

  is the set of all real numbers x such that (":") x is not an element of $\mathbb{Q}$, that is, the set of irrational numbers.
- $\sqrt{2}$ and numbers such that $\sqrt{3}$, π, e are examples of irrational numbers.
- The totality of all rational and irrational numbers is called the set of real numbers, $\mathbb{R}$, and is represented by the real line.
- The following figure gives us understanding where we should put the number $\sqrt{2}$ on a number line.



Notation

If x is a member of a set A, then we write $x \in A$.

If x is not a member of A then we write $x \notin A$.

Example.

$$2 \in \mathbb{N}, \quad -12 \notin \mathbb{N}, \quad \frac{22}{7} \notin \mathbb{Z}, \quad \sqrt{2} \notin \mathbb{Q}, \quad \sqrt{2} \in \mathbb{R}.$$

Exercise.

$$-\frac{1}{2} \square \mathbb{Q}, \quad -12 \square \mathbb{Q}, \quad 0 \square \mathbb{R}, \quad \sqrt{5} \square \mathbb{Q}, \quad 1 \square \mathbb{N}.$$

$$-\frac{1}{2} \in \mathbb{Q}, \quad -12 \in \mathbb{Q}, \quad 0 \in \mathbb{R}, \quad \sqrt{5} \notin \mathbb{Q}, \quad 1 \in \mathbb{N}.$$

Notation for intervals

Suppose that a and b are real numbers and that $a < b$. Then

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

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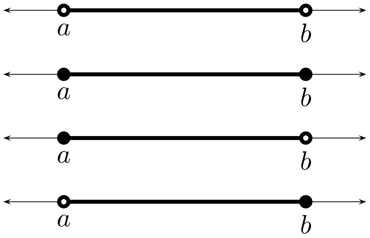


An interval $[a, b]$ that includes its endpoints a and b is called a closed interval, while an interval (a, b) that excludes its endpoints is called an open interval. The intervals $[a, b)$ and $(a, b]$ are neither open nor closed.

Rays of the real line using the symbol ∞

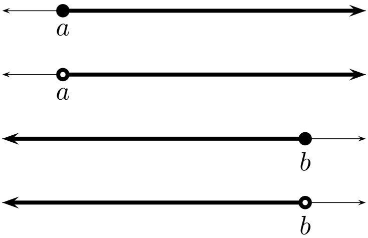
$$[a,\infty) = \{x \in \mathbb{R} : x \geq a\}$$

$$(a,\infty) = \{x \in \mathbb{R} : x > a\}$$

$$(-\infty,b] = \{x \in \mathbb{R} : x \leq b\}$$

$$(-\infty,b) = \{x \in \mathbb{R} : x < b\}$$

$$(-\infty,\infty) = \mathbb{R}$$



Definition

We say that a set A is a subset of a set B (and write $A \subseteq B$) if every element of A is an element of B. If A is a subset of B then we also say that B contains the set A.

Examples.

- $\mathbb{N}$ is a subset of $\mathbb{Z}$, and $\mathbb{Z}$ is a subset of $\mathbb{Q}$, and $\mathbb{Q}$ is a subset of $\mathbb{R}$.
- $\{0, 2, 3\}$ is a subset of $\{0, 1, 2, 3, 5\}$.
- $(-1, 2]$ is not a subset of $[0, \infty)$.
- $\{1\}$ is a subset of $[0, \infty)$.
- Any set is a subset of itself.
- $(1, 3)$ is a subset of $[1, 3)$.

Solving inequalities ($x < y$, $x > y$, $x \leq y$, $x \geq y$)

Remember:

- You can always add or subtract the same "thing" from both sides.
- You can always multiply or divide both sides by a positive quantity.
- You can't multiply or divide by zero!
- If you multiply by a negative quantity you need to swap the direction of the inequality.

Often solving an inequality turns into solving an equality.

Two types of inequalities deserve special attention: polynomial inequalities and rational inequalities.

Examples

(a) Find $T = \{u \in \mathbb{R} : u^3 - 3u > 2u^2\}$.

(b) Solve the rational inequality

$$\frac{1}{x+1} < \frac{1}{x-2}.$$

Absolute values

The absolute value, of a real number x is denoted by $|x|$ and defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

The number $|x|$ may be interpreted as the "size" or "magnitude" of the number x. It can be also viewed as a distance from x to the origin.

Properties

Suppose that x and y are real numbers. Then

- $|-x| = |x|$,
- $|xy| = |x||y|$,
- $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$ provided that $y \neq 0$.

More useful facts about absolute values

- For every real number x,

$$|x| = \sqrt{x^2}, \quad |x|^2 = x^2.$$

- For any positive real number a,

$$|x| < a \quad \Leftrightarrow \quad -a < x < a.$$

- For any positive real number a, $|x - x_0| < a$ is equivalent to

$$-a < x - x_0 < a \quad \Leftrightarrow \quad x_0 - a < x < x_0 + a.$$

  Geometrically, the number $|x - x_0|$ is interpreted as the *distance* from x to x_0 (or from x_0 to x).

- For any positive real number a,

$$|x| > a \quad \Leftrightarrow \quad x < -a \quad \text{or} \quad x > a.$$

Exercise. Find $a, b \in \mathbb{R}$ such that

$$[2,8] = \{x \in \mathbb{R} : |x - b| \leq a\}.$$

Solution. We know that $|x - b| \leq a$ is equivalent to $x \in [b - a, b + a]$. This implies that

$$b - a = 2, \quad b + a = 8 \quad \Longrightarrow \quad b = 5,\ a = 3.$$

The triangle inequality

Suppose that x and y are real numbers. Then

$$|x + y| \leq |x| + |y|.$$

Proof. We have the inequalities

$$-|x| \leq x \leq |x|, \quad -|y| \leq y \leq |y|.$$

Add them and obtain

$$-(|x| + |y|) \leq x + y \leq |x| + |y|,$$

which implies the result.

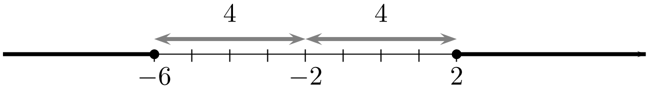
Exercise. Prove that $|x - y| \geq |x| - |y|$.

Examples

Solve the following inequalities:

$$(a) \qquad |x+2| \geq 4$$

Geom. solution: The distance from -2 to x is greater than or equal to 4



So the solution is $x \leq -6$ or $x \geq 2$.

Alg. solution: The inequality $|x+2| \geq 4$ is equivalent to $x+2 \geq 4$ or $x+2 \leq -4$, which is equivalent to $x \geq 2$ or $x \leq -6$.

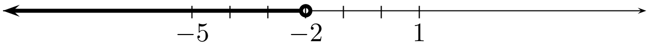
$$(b) \qquad \frac{|x+5|}{|x-1|} < 1$$

Alg. solution: Suppose that $x \neq 1$.

Multiply both sides of the inequality by the positive number $|x-1|$ and obtain $|x+5| < |x-1|$.

Squaring both (positive) sides, we have $(x+5)^2 < (x-1)^2$. Expanding and solving, we have $x < -2$.

Geom. solution: The distance from x to -5 is less than the distance from x to 1. In other words, x is closer to -5 than to 1.



More examples: proving inequalities

(a) Prove that for all $x, y \geq 0$,

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

(b) Prove that for $x > 0$,

$$x + \frac{1}{x} \geq 2.$$

Disproving inequalities

To prove that an inequality (or equality) does not hold, it is enough to give one example, for which the inequality (or equality) does not work.

Example.

Is it true or false (and why) that

$$\text{if} \;\; a > b, \;\; \text{then} \;\; |a| > |b|.$$

Solution: The claim is evidently true for positive a and b. Therefore, we will look for the example among negative numbers.

Take, for example, $a = -1$ and $b = -2$. Then $a > b$ and $|a| = 1$, $|b| = 2$.

So $|a| > |b|$ is false for this example. Thus the claim can not be true in general. So it is false.

Functions

A function $f : A \to B$ is a rule which assigns to every element x belonging to a set A exactly one element $f(x)$ belonging to a set B, that is $x \mapsto f(x)$.

Terminology.

- A is called the domain of the function f, that is,

$$A = \mathrm{Dom}(f) = \{\text{all allowable inputs}\}.$$

- B is called the codomain of f, that is,

$$B = \mathrm{Codom}(f) = \{\text{all allowable outputs}\}.$$

- The range of f is

$$\begin{aligned}\mathrm{Range}(f) &= \{f(x) : x \in A\} \\ &= \{\text{all outputs that actually occur}\} = f(A).\end{aligned}$$

Remark. $\mathrm{Range}(f) \subseteq \mathrm{Codom}(f)$

Example.

$$\begin{aligned} f : [1, \infty) &\to \mathbb{R} \\ x &\mapsto \sqrt{x-1}. \end{aligned}$$

$$\mathrm{Dom}(f) = [1, \infty), \quad \mathrm{Codom}(f) = \mathbb{R}, \quad \mathrm{Range}(f) = [0, \infty).$$

Remarks.

- In this course, the domain and codomain are always sets of real numbers.
- f denotes a function, while $f(x) \in B$ is a number, namely the value of f at the point $x \in A$.
- The codomain of f may be changed but it must contain all the outputs of f.
- The statement

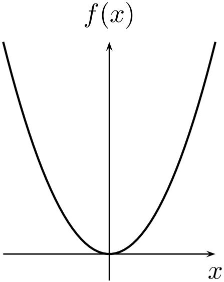
  $$\text{‘}f(x) = \sqrt{x} \text{ for all } x \text{ in } [0, \infty)\text{’}$$

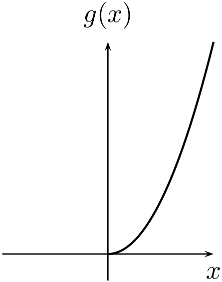
  may be abbreviated as

  $$f(x) = \sqrt{x} \qquad \forall x \in [0, \infty).$$

- Functions which are defined by the same rule but have different domains are not the same. For example, consider

$$\begin{aligned} f : \mathbb{R} \to \mathbb{R}, &\quad f(x) = x^2 \\ g : [0, \infty) \to \mathbb{R}, &\quad g(x) = x^2 \end{aligned}$$





Natural domain. If, for whatever reason, the domain of a function is not defined then we may choose the natural domain or maximal domain, that is the largest possible domain for which the rule makes sense (for real numbers).

Examples

(a) Find the range of the following functions:

- $f_1 : \mathbb{R} \to \mathbb{R}$, $f_1(x) = 1$ has $\mathrm{Range}(f_1) = \{1\}$;
- $f_2 : \mathbb{R} \to \mathbb{R}$, $f_2(x) = \cos(x\pi)$ has $\mathrm{Range}(f_2) = [-1, 1]$;
- $f_3 : \mathbb{Z} \to \mathbb{R}$, $f_3(n) = \cos(n\pi)$ has $\mathrm{Range}(f_3) = \{-1, 1\}$;
- $f_4 : (0, 1] \to \mathbb{R}$, $f_4(x) = 1/x$ has $\mathrm{Range}(f_4) = [1, \infty)$.

(b) Find the maximal domain and range for

$$f(x) = \sqrt{9 - |x|}.$$

(c) Find the maximal domain for

$$f(x) = \frac{1}{x^2 + x - 2}.$$

(d) Find the maximal domain and the range for

$$f(x) = \sqrt{\cos x}.$$

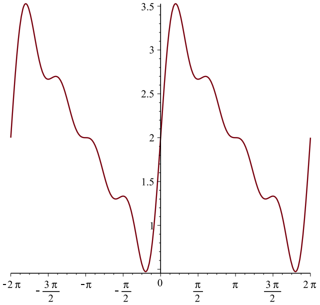
Remark. We distinguish between the range and the codomain of a function since it is often difficult to find the range of a function.

For example, what is the range of

$$2 + \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4} \ ?$$

Using MAPLE:

$>$plot(2+sin(x)+(1/2)*sin(2*x)+(1/3)*sin(3*x)+(1/4)*sin(4*x), x = -2*Pi..2*Pi)



Operations with functions

If f and g are two functions with the same domain, then one can combine f and g to form new functions.

Definition

Suppose that $f : A \to B$ and $g : A \to B$ are real-valued functions. Then, the functions $f + g$, $f - g$, $f \cdot g$ and f/g are defined by the rules

$$(f+g)(x) = f(x) + g(x) \qquad \forall x \in A$$

$$(f-g)(x) = f(x) - g(x) \qquad \forall x \in A$$

$$(f \cdot g)(x) = f(x)g(x) \qquad \forall x \in A$$

$$(f/g)(x) = \frac{f(x)}{g(x)} \qquad \forall x \in A \text{ such that } g(x) \neq 0,$$

Examples

(a) Find $(f+g)(0)$ and the maximal domain of f/g, where

$$f(x) = 1 + x^2, \quad g(x) = \cos(x).$$

Solution:

Since $\mathrm{Dom}(f) = \mathrm{Dom}(g) = \mathbb{R}$, the function $f+g$ is well-defined and $\mathrm{Dom}(f+g) = \mathbb{R}$.

$(f+g)(0) = f(0) + g(0) = 1 + 1 = 2.$

The function f/g is defined for all x such that $g(x) \neq 0$.

$\cos x \neq 0$ is equivalent to $x \neq \pi/2 + \pi n$, where $n \in \mathbb{Z}$. Thus,

$$\mathrm{Dom}(f/g) = \mathbb{R} \setminus \{\pi/2 + \pi n : n \in \mathbb{Z}\}.$$

(b) Let $$f(x) = \sqrt{x}, \quad g(x) = \frac{1}{x-1}.$$

Think how the function $f+g$ can be defined.

Solution:

Note that $\mathrm{Dom}(f) = \{x \in \mathbb{R} : x \geq 0\}$, and $\mathrm{Dom}(g) = \{x \in \mathbb{R} : x \neq 1\}$.

The function $f+g$ can be defined only for points x which belong to both domains $\mathrm{Dom}(f)$ and $\mathrm{Dom}(g)$.

That is $\mathrm{Dom}(f+g) = \{x \in \mathbb{R} : x \geq 0 \text{ and } x \neq 1\}$.

So $(f+g)(x) = \sqrt{x} + \frac{1}{x-1}$, for all $x \in \mathrm{Dom}(f+g)$.

Composition of functions

Definition

Suppose that

$$f : C \to D \qquad \text{and} \qquad g : A \to B$$

are functions such that $\mathrm{Range}(g)$ is a subset of $\mathrm{Dom}(f)$. Then the composition

$$f \circ g : A \to D$$

is defined by the rule

$$(f \circ g)(x) = f(g(x)) \qquad \forall x \in A.$$

Remark. The order of how we compose f and g is important!

Example

Let the functions f and g be given by the rules

$$f(x) = \sqrt{x}, \quad g(x) = \cos(x) - 2$$

Find if exist, $(f \circ g)$ and $(g \circ f)$.

Polynomials

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is called a polynomial if

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where $n \in \mathbb{N}$ is the degree and the coefficients $a_0, a_1, \ldots, a_n$ are real numbers with the leading coefficient $a_n \neq 0$.

Remarks.

- If $n = 1$ we say that f is a linear polynomial and its graph is a straight line;
- If $n = 2$ we say that f is a quadratic polynomial and its graph is a parabola.

Examples.

The function p defined by $p(x) = 2x^3 - 5x$, for $x \in \mathbb{R}$, is a polynomial of degree 3 and the leading coefficient 2.

The function p defined by $p(x) = 3$, for $x \in \mathbb{R}$, is also a polynomial of degree 0 and the leading coefficient is 3.

Rational functions

Definition

Let p and q be polynomials. The function f defined by the rule

$$f(x) = \frac{p(x)}{q(x)}, \quad \mathrm{Dom}(f) = \{x \in \mathbb{R} : q(x) \neq 0\}$$

is called a rational function.

Examples.

The function f defined by

$$f(x) = \frac{1}{x^2 + x - 2}, \qquad \mathrm{Dom}(f) = \mathbb{R} \setminus \{-2, 1\}$$

is rational.

The function f defined by

$$f(x) = x - 1 + \frac{3}{x^2 + 3} = \frac{(x^2 + 3)(x - 1) + 3}{x^2 + 3}$$

is also rational and $\mathrm{Dom}(f) = \mathbb{R}$.

Trigonometric functions

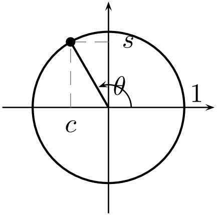
The trigonometric functions

$$\sin : \mathbb{R} \to \mathbb{R} \qquad \text{and} \qquad \cos : \mathbb{R} \to \mathbb{R}$$

are defined by

$$\sin\theta = s \qquad \text{and} \qquad \cos\theta = c,$$

where s and c are defined in a geometric manner:



In other words, $(\cos\theta, \sin\theta)$ are the coordinates on the unit circle if one travels distance θ anticlockwise around the unit circle.

Remark. In this course angles are always measured in radians. Recall that

$$2\pi \text{ radians } = 360 \text{ degrees.}$$

Properties of $\sin$ and $\cos$

The following properties are immediate from the definition.

- $\mathrm{Dom}(\sin) = \mathrm{Dom}(\cos) = \mathbb{R}$.
- $\mathrm{Range}(\sin) = \mathrm{Range}(\cos) = [-1, 1]$.
- $\sin$ and $\cos$ are periodic of period 2π, that is

$$\sin(x + 2\pi) = \sin x, \quad \cos(x + 2\pi) = \cos x.$$

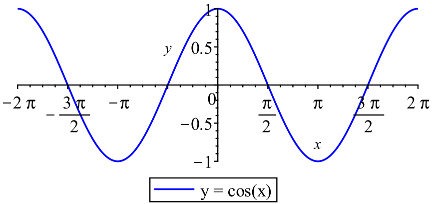
- $\cos$ is an even function, that is

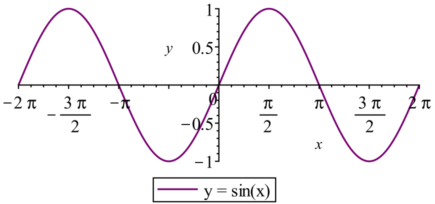
$$\cos(-x) = \cos x.$$

- $\sin$ is an odd function, that is

$$\sin(-x) = -\sin x.$$

- $\sin^2 x + \cos^2 x = 1$.





Other trigonometric functions with suitable domains are defined by

$$\tan x = \frac{\sin x}{\cos x}, \quad \text{provided that } \cos x \neq 0,$$

$$\cot x = \frac{\cos x}{\sin x}, \quad \text{provided that } \sin x \neq 0,$$

$$\sec x = \frac{1}{\cos x}, \quad \text{provided that } \cos x \neq 0,$$

$$\operatorname{cosec} x = \frac{1}{\sin x}, \quad \text{provided that } \sin x \neq 0.$$

Relations between trigonometric functions

The six trigonometric functions are related by various identities and formulae (which you are supposed to know):

- complementary identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x,$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

- Pythagorean identities

$$\cos^2 x + \sin^2 x = 1,$$

$$1 + \tan^2 x = \sec^2 x,$$

$$\cot^2 x + 1 = \operatorname{cosec}^2 x$$

- the sum and difference formulae

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}.$$

- double-angle formulae

$$\sin(2x) = 2 \sin x \cos x,$$

$$\cos(2x) = \cos^2 x - \sin^2 x,$$

$$\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}.$$

Elementary functions

The elementary functions are all those functions that can be constructed by combining (a finite number of) polynomials, exponentials, logarithms, roots (that is, functions of the form $f(x) = x^{1/n}$, $n \in \mathbb{N}$), trigonometric functions (including the inverse trigonometric functions) and absolute value function via function composition, addition, subtraction, multiplication and division.

Example.

$$f(x) = e^{\sin x} + x^2,$$

$$g(x) = \frac{\ln x - \tan x}{\sqrt{x}},$$

$$h(x) = \sqrt[3]{x^4 - 2x^2 + 5},$$

$$k(x) = |x| = \sqrt{x^2}.$$

Every rational function is an elementary function.

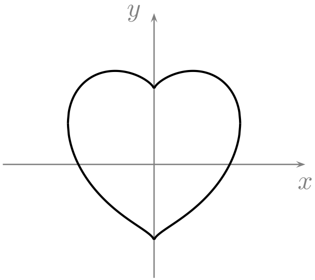
However, there exist important functions which are not elementary!

Implicitly defined functions

Many curves on the plane can be described as all those points (x, y) on the plane that satisfy some equation involving x and y. For example, consider the equation

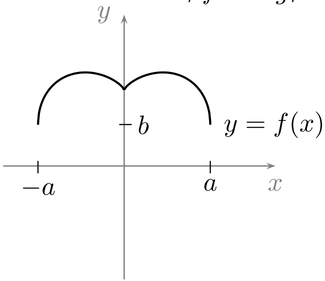
$$(x^2 + y^2 - 1)^3 - x^2y^3 = 0. \qquad (\heartsuit)$$

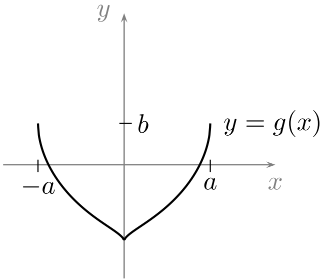
The set of points (x, y) satisfying this equation are shown on the graph below.



Properties

- There exist several y-values for some/all x-values. Hence, the curve cannot be the graph of one function of x.
- The curve may be decomposed into two curves which may be regarded as the graphs of two functions, f and g, say

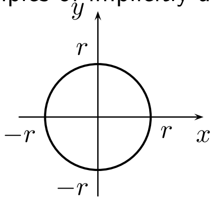




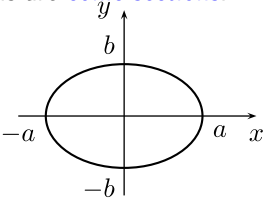
- We say that the functions f and g are implicitly defined by the relation (♡).

Other examples

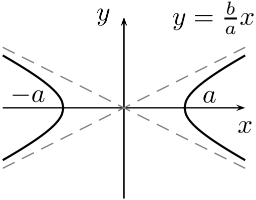
Other examples of implicitly defined functions are conic sections:



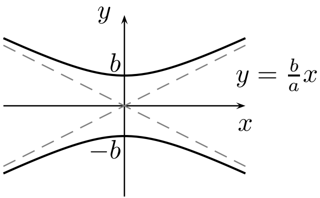
Circle: $x^2 + y^2 = r^2$



Ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



Hyperbola: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



Hyperbola: $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$

Sometimes it is better to leave things in the implicit form. For example, for

$$x^4 + \sin(y^4) - x^2 + 200xy - y^2 = 95,$$

you have no choice!

Continuous functions

Question. How would you define continuity?

'Intuitive' (incorrect) answer. The function is continuous if 'its graph can be drawn without lifting the pencil off the page'.

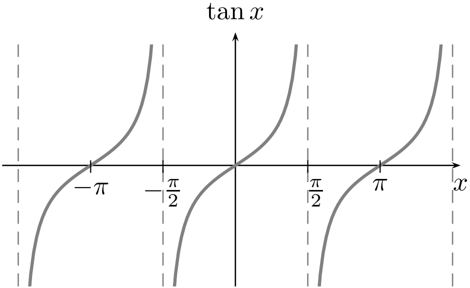
Rigorous (correct) answer. Via limits at a point! (see Chapter 2)

'Counterexample'

Consider the function

$$\tan : A \to \mathbb{R}$$

with $A = \mathrm{Dom}(f) = \{x \in \mathbb{R} : x \neq \frac{\pi}{2} + n\pi,\ n \in \mathbb{Z}\}$.



The function $\tan$ is continuous on its domain! The break in the graph is merely due to the 'missing' points in the domain A.