School of Mathematics and Statistics

Math1131 Mathematics 1A

CALCULUS LECTURE 6

PROPERTIES OF CONTINUOUS FUNCTIONS

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MATH1131 CALCULUS
Properties of Continuous Functions

The Intermediate Value Theorem: Suppose that f is a continuous function on the closed interval $[a, b]$. Then if z lies between $f(a)$ and $f(b)$ there will be at least one real number c in $[a, b]$ with the property that $z = f(c)$.

The Maximum-Minimum Theorem: A continuous function on a closed interval will always attain both a maximum and a minimum value over the interval.

Continuous functions are very well behaved. Most of the elementary functions including all polynomial functions, e^x, $\ln(x)$, $\sin(x)$ and $\cos(x)$ are continuous over their domains. In this lecture we will examine the features that make continuous functions so appealing.

Recall that a function f is said to be continuous at $x = a$ if $\lim_{x \longrightarrow a} f(x) = f(a)$. What this means is that a function is continuous at $x = a$ if its behaviour **near** a perfectly matches its behaviour **at** a. Note that for continuity we need two things! The limit must exist and also be equal to the function value. You also have the intuitive idea of the pen lifting off the paper to help you out.

First note that the sum, product and difference of continuous functions is clearly again continuous.

Example 1: Is the quotient of continuous functions over $\mathbb{R}$ always continuous over $\mathbb{R}$?

★ No ★

Piecemeal functions often have continuity problems, however it is also possible to carefully patch the pieces together in a continuous fashion.

Example 2: Let $f(x) = \begin{cases} x^2, & x \leq 3; \\ -5x + b, & x > 3 \end{cases}$

Sketch the situation and find the value of b which makes f continuous.

Note that f is clearly continuous everywhere except at $x = 3$ so we only need to consider what's happening at $x = 3$. Now

★ $b = 24$ ★

The Intermediate Value Theorem: Suppose that f is a continuous function on the closed interval $[a, b]$. Then if z lies between $f(a)$ and $f(b)$ there will be at least one real number c in $[a, b]$ with the property that $z = f(c)$.

The proof of this theorem is surprisingly difficult and will not be examined.

Lets draw a picture of the situation. Sketch a continuous graph over a random closed interval $[a, b]$ in the x axis.

Note that a closed interval $[a, b]$ takes the form $a \leq x \leq b$ and MUST include the endpoints. Thus $[a, b) \equiv \{x \in \mathbb{R} \,|\, a \leq x < b\}$, $(a, b] \equiv \{x \in \mathbb{R} \,|\, a < x \leq b\}$ and $(a, b) \equiv \{x \in \mathbb{R} \,|\, a < x < b\}$ are all NOT closed intervals.

The intermediate value theorem should make a lot of sense! All it says is that continuous functions map closed intervals on the x axis over to closed intervals on the y axis with no possibility of gaps or holes.

An immediate corollary (consequence) is that if a continuous function $f(x)$ changes sign over a closed interval $[a, b]$ then the interval will contain a solution to the equation $f(x) = 0$.

Example 3: Prove that the equation $e^x = x^2 + 4$ has a solution somewhere in the closed interval $1 \leq x \leq 3$.

First note that $e^x = x^2 + 4 \leftrightarrow e^x - x^2 - 4 = 0$. So we let $f(x) = e^x - x^2 - 4$ and note that f is clearly continuous since all of its components are continuous. (always mention continuity!!) Now:

$f(1) \approx -2.28$ and $f(3) = 7.08$.

Since $-2.28 \leq 0 \leq 7.08$ it follows from the intermediate value theorem that there is $c \in [1, 3]$ such that $f(c) = 0$, as required.

★

Question: Could there be more than one such c ?

★

Question: Does the intermediate value theorem help you to find c ?

★ Not at all ★

Note also that there is $c \in [1, 3]$ such that $f(c) = 6$, though this isn't quite as interesting.

Be careful to make sure that your calculator is in radian mode if you do an example like the one above involving any of the trig functions.

Example 4: Let $f(x) = \begin{cases} x, & 2 \le x \le 3; \\ x+5, & 3 < x \le 4 \end{cases}$ be a function defined over the closed interval $[2,4]$.

Sketch:

Then $f(2) = 2$ and $f(4) = 9$. Noting that $2 \le 7 \le 9$ is there a $c \in [2,4]$ such that $f(c) = 7$? Does this example violate the intermediate value theorem?

★ NO. The function is not continuous so the theorem does not apply. ★

We now turn to the Maximum Minimum Theorem for continuous functions. First a careful definition.

Definition: Suppose that f is defined on a closed interval $[a,b]$.

(a) We say that a point c in $[a,b]$ is an *absolute minimum point* for f on $[a,b]$ if $f(c) \le f(x)$ for all x in $[a,b]$. The corresponding value $f(c)$ is called the *absolute minimum value* of f on $[a,b]$. If f has an absolute minimum point on $[a,b]$ then we say that *f attains its minimum on* $[a,b]$.

(b) We say that a point d in $[a,b]$ is an *absolute maximum point* for f on $[a,b]$ if $f(x) \le f(d)$ for all x in $[a,b]$. The corresponding value $f(d)$ is called the *absolute maximum value* of f on $[a,b]$. If f has an absolute maximum point on $[a,b]$ then we say that f *attains its maximum on* $[a,b]$.

An absolute maximum point and an absolute minimum point are sometimes referred to as a *global maximum point* and a *global minimum point.*

All we are really saying above is that c is an *absolute maximum point* of f on $[a,b]$ if $f(c)$ is bigger than or equal to any other y value over the interval. Similarly for *absolute minimum point.*

There are a number of crucial facts to observe. Firstly the definitions of absolute maxima and minima have nothing to do with calculus! We use derivatives as a testing mechanism later but the definitions themselves do not require such sophisticated concepts.

Secondly absolute maxima or minima of a function may or may not exist over an interval. We see in the next theorem that if f is continuous then the function will always attain its max and min over any closed interval.

The Maximum-Minimum Theorem: A continuous function on a closed interval will always attain both a maximum and a minimum value over the interval.

Sketch:

Continuous functions over a closed interval will always achieve both a maximum value and a minimum value. Note that these do not have to be a turning points!!

Example 5: Consider $f(x) = |2x - 5|$ over the closed interval $[-1, 3]$.

a) Is f continuous over the interval?
b) How many stationary points does f have?
c) What is the maximum and minimum value of f over $[-1, 3]$?

★ *a)* Yes *b)* None *c)* Max=7 and Min=0 ★

Example 6: Let $f(x) = x^2 + 4$ over the interval $-1 < x < 3$.

a) Is f continuous over the interval?

b) What is the maximum and minimum value of f over $(-1, 3)$?

c) Does this violate the Max/Min Theorem?

★ *a)* Yes *b)* Max D.N.E. and Min=4 *c)* No, The interval is not closed ★

Example 7: Consider $f(x) = \begin{cases} -x, & -1 \leq x < 2; \\ 0, & 2 \leq x \leq 3 \end{cases}$ over the closed interval $[-1, 3]$.

a) Is f continuous over the interval?

b) What is the maximum and minimum value of f over $[-1, 3]$?

c) Does this violate the Max/Min Theorem?

★ *a)* No *b)* Max=1 and Min D.N.E *c)* No, The function is not continuous ★

Observe from the above example that we need both continuity and for the interval to be closed to draw the conclusions of The Max/Min theorem.

Example 8: Sketch the graph of a discontinuous function f defined over the open interval $(-1, 3)$ with the property that f attains both a maximum and a minimum value over the interval. Does this violate the max/min theorem ?

$\bigstar$ *No* $\bigstar$