THE UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1131 Calculus

Section 6: - Inverse Functions.

We have intuitively thought of a function as a rule, which starts from one real number and produces another. We now ask the question as to when we can *reverse* the procedure.
For example, under the function $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = 2x + 3$, the real number 5 maps to 13. On the other hand what number maps on to 10? Answer 3.5. Indeed given the y value, the corresponding x-value it came from is $\frac{y-3}{2}$. This new rule, is itself a function, which can be written as $g(x) = \frac{x-3}{2}$. We say that these two functions are **inverses** of each other the write $g(x) = f^{-1}(x)$. (Note that the index does NOT mean 'one over').

Also note that if we compose f and g we obtain the identity function, i.e. $f \circ g(x) = f(g(x)) = f(\frac{x-3}{2}) = x$ and $g \circ f(x) = g(f(x)) = g(2x + 3) = x$. Hence:

Definition: Given a function $f : A \to B$, if there is a function $g : B \to A$ such that $f \circ g(x) = x$ and $g \circ f(x) = x$, then we say that g is the inverse of f and write $g = f^{-1}$.

Ex: Show that if $f(x) = e^x$ then $g(x) = \log x$ is the inverse of f.

$f(x) = e^x \quad g(x) = \log x.$

$f \circ g(x) = \exp(\log x) = x.$

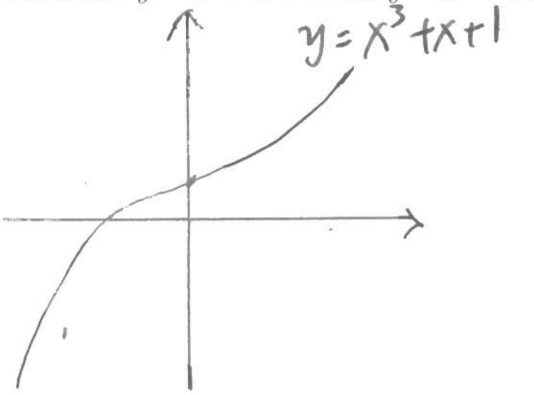
$g \circ f(x) = \log(\exp x) = x.$

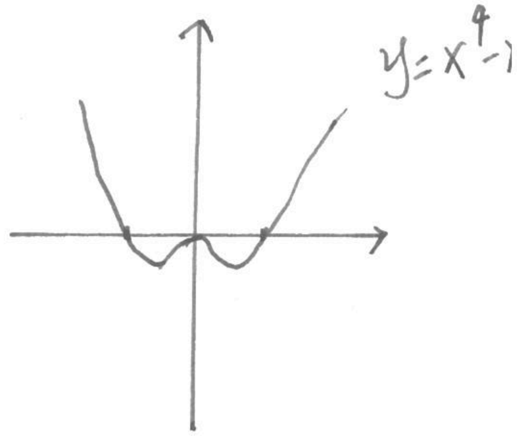
Clearly not all functions have inverses, for example $f(x) = x^2$. The y value 9 came from both 3 and -3.

When does a given function f defined on an interval $[a, b]$ have an inverse?

One simple test is known as the *horizontal line test.* It says that if we look at the graph of f with domain D and co-domain R and draw any horizontal line, $y = b$, where $b \in R$ then f will have an inverse if the line cuts the graph at **exactly one point**.

Ex: Draw $y = x^3 + x + 1$ and $y = x^4 - x^2$ to illustrate this.





Ex: $\lim_{x\to 0} \dfrac{e^x - 1}{\sin 2x}$. $\quad$ "$\dfrac{0}{0}$"

$$= \lim_{x\to 0} \frac{e^x}{2\cos 2x}$$

$$= \frac{1}{2}$$

Ex: $\lim_{x\to 1} \dfrac{1 - x + \log x}{1 + \cos \pi x}$. $\quad$ "$\dfrac{0}{0}$"

$$= \lim_{x\to 1} \frac{-1 + \frac{1}{x}}{-\pi \sin \pi x}, \quad \text{"}\frac{0}{0}\text{"}$$

$$= \lim_{x\to 1} \frac{-\frac{1}{x^2}}{-\pi^2 \cos \pi x}$$

$$= -\frac{1}{\pi^2}$$

When dealing with limits to infinity, we need the following version of L'Hôpital's rule.

Theorem: Suppose f and g are differentiable. Suppose further that $f(x) \to 0$ and $g(x) \to 0$ as $x \to \infty$ (or $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to \infty$).
If $\lim_{x\to\infty} \dfrac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

Ex: $\lim_{x\to\infty} \frac{\log x}{x}$. "$\frac{\infty}{\infty}$"

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

$$= 0.$$

Ex: $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$.

$$= \lim_{x\to\infty} \exp\left(\ln\left(1+\frac{1}{x}\right)^x\right)$$

$$= \exp\left(\lim_{x\to\infty} x \ln\left(1+\frac{1}{x}\right)\right). \qquad \text{"}0\cdot\infty\text{"}$$

$$\lim_{x\to\infty} x \ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \qquad \text{"}\frac{0}{0}\text{"}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1.$$

So $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$.

Theorem: Suppose f is differentiable on (a, b) and $f'(x) \neq 0$ for all $x \in (a, b)$ then f has an inverse on (a, b).

Ex: $y = x^3 + x + 1$. (Note that although this function has an inverse, it is not easy to explicitly write down the formula for the inverse.)

$y' = 3x^2 + 1$ which is always positive.

So y has an inverse.

Ex: $f(x) = 2x + \sin x$.

$f'(x) = 2 + \cos x$, always positive.

Hence $f(x)$ has an inverse.

We can sometimes restrict the domain of a function f so that although f does not have an inverse on its natural domain, it does on this restricted domain.

Ex: Find maximal regions on which the function $f(x) = x^3 - x$ has an inverse.

$f(x)$ is one-to-one on

$(-\infty, -\sqrt{\frac{1}{3}}]$, $[-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}]$, $[\sqrt{\frac{1}{3}}, \infty)$.

On these three intervals, $f(x)$ has inverses.

Suppose f has an inverse on (a, b) and f is differentiable on (a, b). How do we find the derivative of the inverse?

Theorem: Suppose f is diffble on (a, b) and has an inverse $g(x)$ on (a, b), then

$$g'(x) = \frac{1}{f'(g(x))}.$$

Proof: $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$.

But $f \circ g(x) = x$. So $(f \circ g)'(x) = 1$.

Hence $g'(x) = \dfrac{1}{f'(g(x))}$. $\boxed{\not{4}}$

Ex: Let $f(x) = x^3 + x + 1$. Find $g'(1)$. where $g = f^{-1}$.

$f(0) = 1$. $f'(x) = 3x^2 + 1$

$x^3 + x + 1 = 1 \Rightarrow x(x^2+1) = 0$.

Thus, $g(1) = 0$. and

$$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(0)} = 1.$$

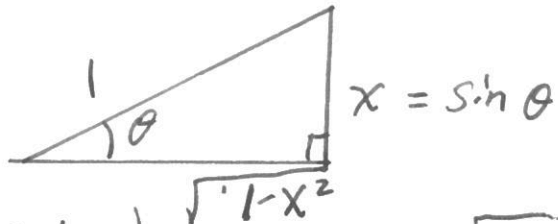
Inverse Trigonometric Functions:

We know

$$\frac{d}{dx}(\sin x) = \cos x.$$

So by the inverse function theorem,

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\cos(\arcsin x)}$$



So $\cos(\arcsin x) = \cos\theta = \sqrt{1-x^2}$

Hence $$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}.$$

Similarly, we can derive.

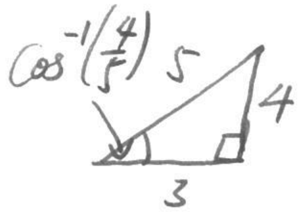
$$\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}, \quad \frac{d}{dx}(\operatorname{arccsc} x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\operatorname{arccot} x) = \frac{-1}{1+x^2}$$

Ex: Find a. $\sin^{-1}(\sin(\frac{5\pi}{3}))$ b. $\sin(\sin^{-1}(-\frac{1}{2}))$, c. $\sin(2\cos^{-1}(\frac{4}{5}))$.

a) $\sin^{-1}\left(\sin\left(\frac{5\pi}{3}\right)\right) = -\frac{\pi}{3}$



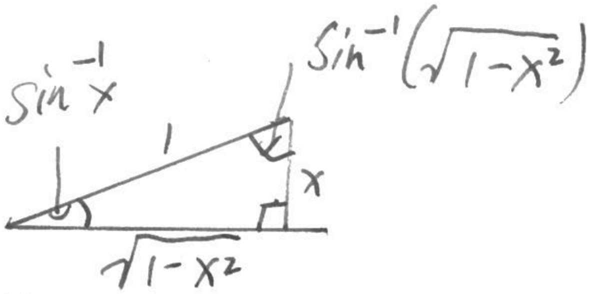
b) $\sin\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = -\frac{1}{2}$

c) Using the double angle formula.

$\sin\left(2\cos^{-1}\left(\frac{4}{5}\right)\right) = 2\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)\sin\left(\sin^{-1}\left(\frac{4}{5}\right)\right)$

$= 2 \cdot \frac{3}{5} \cdot \frac{4}{5} = \frac{24}{25}$

Ex: Prove that $\sin^{-1}(x) + \sin^{-1}\sqrt{1-x^2} = \frac{\pi}{2}$.



$\sin\left(\sin^{-1}x + \sin^{-1}\sqrt{1-x^2}\right)$

$= \sin(\sin^{-1}x)\cos(\sin^{-1}x)$

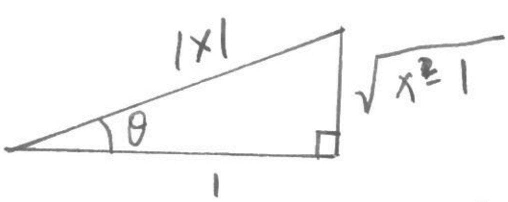
$\quad + \cos(\sin^{-1}x)\sin\left(\sin^{-1}\left(\sqrt{1-x^2}\right)\right)$

$= x^2 + \left(\sqrt{1-x^2}\right)^2 = 1$

So $\sin^{-1}x + \sin^{-1}\sqrt{1-x^2} = \sin^{-1}1 = \frac{\pi}{2}$.

Ex: Find $\frac{d}{dx}\csc^{-1}x$

$\frac{d}{dx}\operatorname{arccsc} x$



$\theta = \operatorname{arccsc} x$.

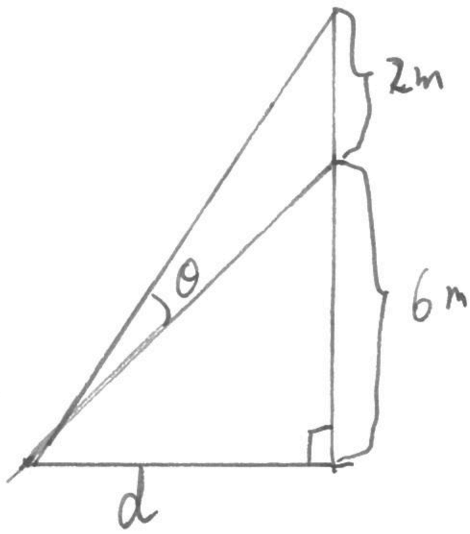
$= \dfrac{1}{\frac{d}{dx}\csc(\theta)\Big|_{\theta = \operatorname{arccsc} x}}$

$= \dfrac{-1}{\csc(\operatorname{arccsc} x)\cot(\operatorname{arccsc} x)}$

$= \dfrac{-1}{|x|\sqrt{x^2-1}}$

Ex: a. Find $\frac{d}{dx}(\cot^{-1}(x))$.
b. A statue 2 metres high is mounted on a pedestal. The base of the statute is 6m above the eye-level of an observer. How far from the base of the pedestal should the observer stand to get the 'best' view.

best view means the largest possible value of θ.



$$\theta(d) = \arctan\frac{8}{d} - \arctan\frac{6}{d}.$$

Now $\theta(0) = 0$ and $\lim_{d\to\infty} \theta(d) = 0$.

$$\theta'(d) = \frac{1}{1+(8/d)^2}\,\frac{-8}{d^2} - \frac{1}{1+(6/d)^2}\cdot\frac{-6}{d^2}$$

$$= \frac{-8}{d^2+64} + \frac{6}{d^2+36}$$

$$= \frac{-8d^2 - 288 + 6d + 384}{(d^2+64)(d^2+36)} = \frac{-2d^2+96}{(d^2+64)(d^2+36)}$$

So $\theta'(d) = 0$ if $d = \pm\sqrt{48} = \pm 4\sqrt{3}$.
Hence the distance for the best view is $4\sqrt{3}$ m.

Derivatives Involving Inverse Trigonometric Functions:

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\operatorname{arccot} x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\operatorname{arccsc} x) = \frac{-1}{|x|\sqrt{x^2-1}}$$