LECTURE 8

Split Functions, Implicit Differentiation and Related Rates

Implicit Differentiation $\leftrightarrow \dfrac{d}{dx}(f(y)) = \dfrac{d}{dy}(f(y))\dfrac{dy}{dx}$.

Differentiating a relation between x and y implicitly with respect to t will produce a new relation between the rates of change $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

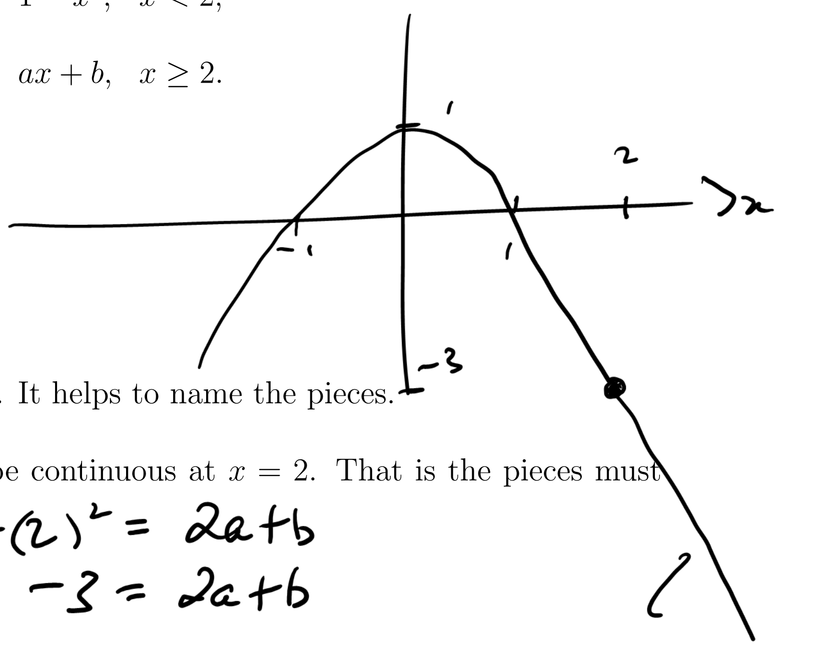
A split function is usually constructed from two or more differentiable component functions. To verify (or force) the differentiability of such a split function we simply need to first verify that the pieces join up (continuity) and then that they *join smoothly* (differentiability) by showing that the derivatives match up properly.

Example 1: Find all real values of a and b such that the function defined by

$$f(x) = \begin{cases} 1 - x^2, & x < 2; \\ ax + b, & x \geq 2. \end{cases}$$

is differentiable at $x = 2$.

A sketch:



Let $p(x) = 1 - x^2$ and $q(x) = ax + b$. It helps to name the pieces.

We first demand that the function be continuous at $x = 2$. That is the pieces must join, and hence $p(2) = q(2)$:

$1-(2)^2 = 2a+b$

$-3 = 2a+b$

We next force the weld to be smooth! Thus we require that $p'(2) = q'(2)$. That is:

$p'(x) = -2x \rightarrow p'(2) = -2(2) = -4$

$q'(x) = a \rightarrow q'(2) = a$

$\therefore \underline{a=-4} \longrightarrow 2a+b=-3$

$-8+b=-3 \Rightarrow b=5$.

$\therefore y = -4x+5$

★ $a = -4$ $\quad b = 5$ ★

Implicit Differentiation

Usually when you differentiate, your starting point is a nice clean function $y = f(x)$. But sometimes you need to start with a horrible messy relation instead, for example $x^2 + y^3 + 4y^2 = 3$. It can be difficult or even impossible to write y in terms of x. We can still find the derivative $\frac{dy}{dx}$ but need to use **implicit differentiation**. First a simple skill.

Example 2: If $\frac{3}{7} = \frac{3}{11} \times \frac{*}{*}$ what is $\frac{*}{*}$?

$$\frac{3}{7} = \frac{3}{\cancel{11}} \times \frac{11}{7}$$

★ $\frac{11}{7}$ ★

Implicit differentiation is little more than the above trick!

Example 3: Find $\frac{dy}{dx}$ if $x^2 + y^3 + 4y^2 = 3$.

$\frac{d}{dx}$ all $\Rightarrow$ $\underbrace{\frac{d}{dx}(x^2)}_{2x} + \frac{d}{dx}(y^3) + \frac{d}{dx}(4y^2) = \frac{d}{dx}(3)$

$= 0$

$$\frac{d}{dx}(y^3) = \frac{d}{dy}(y^3)\left(\frac{dy}{dx}\right) = (3y^2)\left(\frac{dy}{dx}\right)$$

$$\frac{d}{dx}(4y^2) = \frac{d}{dy}(4y^2)\frac{dy}{dx} = 8y\left(\frac{dy}{dx}\right)$$

$$2x + 3y^2\frac{dy}{dx} + 8y\frac{dy}{dx} = 0$$

$$3y^2\frac{dy}{dx} + 8y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx}\left\{3y^2 + 8y\right\} = -2x \Rightarrow \frac{dy}{dx} = \frac{-2x}{3y^2 + 8y}$$

★ $\frac{-2x}{3y^2 + 8y}$ ★

Example 4: Find $\dfrac{dy}{dx}$ if $\sin(x) + e^y = \ln(y) + x^3$

$$\frac{d}{dx}\sin x + \frac{d}{dx}e^y = \frac{d}{dx}\ln y + \frac{d}{dx}x^3$$

$$\cos x \qquad\qquad 3x^2$$

$$\frac{d}{dx}\ln y = \frac{d}{dy}\ln y \frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$$

$$\frac{d}{dx}e^y = \frac{d}{dy}e^y\frac{dy}{dx} = e^y\frac{dy}{dx}.$$

So $\cos x + e^y\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx} + 3x^2$

$$e^y\frac{dy}{dx} - \frac{1}{y}\frac{dy}{dx} = 3x^2 - \cos x \Rightarrow \frac{dy}{dx}\left\{e^y - \frac{1}{y}\right\} = 3x^2 - \cos x$$

$$\therefore \frac{dy}{dx} = \frac{3x^2 - \cos(x)}{e^y - \frac{1}{y}} = \frac{3x^2y - y\cos x}{ye^y - 1}$$

$\bigstar$ $\dfrac{3x^2y - y\cos(x)}{ye^y - 1}$ $\bigstar$

Example 5: Find the equation of the tangent to $x^2y^5 + 3y - 2x = 3$ at the point $(0, 1)$.

$$\frac{d}{dx}(x^2y^5) + \frac{d}{dx}(3y) - 2 = 0$$

2 1

1 $\frac{d}{dx}(3y) = \frac{d}{dy}(3y)\frac{dy}{dx} = 3\frac{dy}{dx}.$ (expected)

2 $\frac{d}{dx}(x^2y^5) = 2xy^5 + \frac{d}{dx}(y^5)(x^2)$ (product rule!!)

$$= 2xy^5 + \frac{d}{dy}(y^5)\left(\frac{dy}{dx}\right)(x^2)$$

$$= 2xy^5 + (5y^4)\left(\frac{dy}{dx}\right)(x^2) = 2xy^5 + 5x^2y^4\left(\frac{dy}{dx}\right)$$

So $\cancel{2xy^5} + \cancel{5x^2y^4\frac{dy}{dx}} + 3\frac{dy}{dx} - 2 = 0$ sub in $(0,1)$

$$3\frac{dy}{dx} - 2 = 0 \Rightarrow \frac{dy}{dx} = 2/3 = m_{\text{tangent}} \qquad (0,1)$$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 2/3(x - 0) \Rightarrow y = \frac{2}{3}x + 1$$

$\bigstar$ $2x - 3y + 3 = 0$ $\bigstar$

Related Rates

Differentiating a relation between x and y implicitly with respect to t will produce a new relation between the rates of change $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

Example 6: Suppose that the surface area S (in m^2) of a human body is related to its weight W (in kg) by

$$S^3 = \frac{W^2}{512}$$

a) Bob weighs 64 kg. What is the surface area of his body?

b) Find a relation between $\frac{dS}{dt}$ and $\frac{dW}{dt}$.

c) Prove that if Bob's weight were to change in any way, the rate of change of his surface area would be $\frac{1}{48}$ the rate of change of his weight.

a) $S^3 = \frac{(64)^2}{512} = 8 \Rightarrow S = 2 \ m^2$

b) $S^3 = \frac{W^2}{512} \Rightarrow \frac{d}{dt}(S^3) = \frac{d}{dt}\left(\frac{W^2}{512}\right)$

$$\frac{d}{dt}(S^3) = \frac{d}{dS}(S^3)\frac{dS}{dt} = 3S^2\frac{dS}{dt}$$

$$\frac{d}{dt}\left(\frac{W^2}{512}\right) = \frac{d}{dW}\left(\frac{W^2}{512}\right)\frac{dW}{dt} = \frac{W}{256}\frac{dW}{dt}$$

$$\Rightarrow 3S^2\frac{dS}{dt} = \frac{W}{256}\frac{dW}{dt}$$

c) Bob: $S = 2$, $W = 64$

$$3(2)^2\frac{dS}{dt} = \frac{64}{256}\frac{dW}{dt} \Rightarrow 12\frac{dS}{dt} = \frac{1}{4}\frac{dW}{dt}$$

$$\Rightarrow \frac{dS}{dt} = \frac{1}{48}\frac{dW}{dt}$$

★ *a)* S=2 *b)* $3S^2\frac{dS}{dt} = \frac{W}{256}\frac{dW}{dt}$ *c) Proof* ★

Example 7: A spherical balloon is inflated at a rate of 100 m^3/sec. Determine the rate at which the radius is increasing when

a) $r = 5$m.

b) $V = 36\pi$ m^3.

Our first task is to find a relationship between the central variables which remains fixed throughout the entire process. This is of course the volume formula for a sphere:

$$V = \frac{4}{3}\pi r^3$$

$\frac{d}{dt}$ both sides

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$100 = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{100}{4\pi r^2} = \frac{25}{\pi r^2}$$

a) $\underline{r=5} \Rightarrow \frac{dr}{dt} = \frac{25}{\pi(25)} = \frac{1}{\pi}$ m/sec.

b) $V = 36\cancel{\pi} = \frac{4}{3}\cancel{\pi} r^3 \Rightarrow r^3 = \frac{36 \times 3}{4} = 27$

$\Rightarrow r = 3$

$$\therefore \frac{dr}{dt} = \frac{100}{4\pi r^2} = \frac{100}{4\pi(9)} = \frac{100}{36\pi} \text{ m/sec}$$

$$= \frac{25}{9\pi} \text{ m/sec}$$

★ $a)\ \frac{1}{\pi}\ m/sec \quad b)\ \frac{25}{9\pi}\ m/sec$ ★

Error Estimates (Homework)

This topic was of enormous importance before the advent of calculators but is now a bit dated.

Recall that $\dfrac{\Delta y}{\Delta x} \approx \dfrac{dy}{dx}$ and hence $\Delta y \approx \dfrac{dy}{dx}\Delta x$. This gives us a way of estimating errors.

Example 8: Find an error estimate when approximating $\sqrt{9.001}$ by $\sqrt{9}$.

We have $x = 9$ and $\Delta x = 0.001$.

Let $y = \sqrt{x}$. Then $\dfrac{dy}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$.

Now $\Delta y \approx \dfrac{dy}{dx}\Delta x \to \Delta y \approx \dfrac{1}{2\sqrt{x}}\Delta x = \dfrac{1}{2\sqrt{9}}(0.001) = \dfrac{1}{6000}$.

$$\Delta y \simeq \frac{1}{2\sqrt{x}}\Delta x. = \frac{1}{2}\cdot\frac{1}{\sqrt{9}}(0.001) \quad \star$$
$$\doteqdot \frac{1}{6000}$$

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