# LECTURE 15
## Riemann Sums

Suppose that a function $f$ is bounded on $[a, b]$. If there exists a *unique* real number $I$ such that $\underline{S}_{\mathcal{P}_n}(f) \leq I \leq \overline{S}_{\mathcal{P}_n}(f)$ for *every* partition $\mathcal{P}_n$ of $[a, b]$, then we say that $f$ is *Riemann integrable* on the interval $[a, b]$. If $f$ is Riemann integrable, then this unique real number $I$ is called the *definite integral of $f$ from $a$ to $b$* and we write

$$I = \int_a^b f(x)\, dx.$$

The function $f$ is called the *integrand* of the definite integral, while the points $a$ and $b$ are called the *limits* of the definite integral.

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If $f \geq 0$ then $I$ is the area bounded by $f$ and the $x$ axis from $x = a$ to $x = b$.

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If $f$ is bounded and piecewise continuous on the interval $[a, b]$ then $f$ is Reimann integrable on $[a, b]$.

We turn now to the other half of Calculus, the central problem of calculating areas under curves. You have already seen plenty of integration techniques in your high school studies. Here however, we are are interested in the structural underpinnings of the theory of integration. In this lecture we will formally define the Reimann Integral. In the following lecture we will examine how the Reimann integral may be efficiently calculated. To motivate discussions we will take a very careful look at the area of the region bounded by $y = x^2$ and the $x$ axis from $x = 0$ to $x = 1$.

**Sketch:**

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The evaluation of this area is not an easy task! It is not a standard object such as a circle or a triangle so we cannot just use a formula. Our strategy is to approximate the area by partitioning the $x$ interval from 0 to 1 into $n$ equal segments.

This partition will be denoted by

$$\mathcal{P}_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}\right\}$$

.

To get a feeling for the process let us partition the interval up into 5 pieces

$$\mathcal{P}_5 = \left\{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\right\}$$

We then estimate the area by building rectangles off each of the subintervals. We start by deliberately underestimating the area by choosing as the height of each rectangle the smallest $y$-value over each subinterval to produce the lower Reimann sum $\underline{S}_{\mathcal{P}_5}$

**Sketch (the lower Reimann sum $\underline{S}_{\mathcal{P}_5}$):**

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Clearly this lower Reimann sum is given by:

$\underline{S}_{\mathcal{P}_5} =$

$$\frac{1}{5}\times 0^2+\frac{1}{5}\times(\frac{1}{5})^2+\frac{1}{5}\times(\frac{2}{5})^2+\frac{1}{5}\times(\frac{3}{5})^2+\frac{1}{5}\times(\frac{4}{5})^2 = (\frac{1}{5})^3\left\{0^2+1^2+2^2+3^2+4^2\right\} = 0.24.$$

We can also deliberately overestimate the area by choosing the largest $y$ value over each subinterval to produce the upper Reimann sum $\overline{S}_{\mathcal{P}_5}$

**Sketch (the upper Reimann sum $\overline{S}_{\mathcal{P}_5}$):**

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Clearly this upper Reimann sum is given by:

$$\overline{S}_{\mathcal{P}_5} =$$

$$\frac{1}{5}\times(\frac{1}{5})^2+\frac{1}{5}\times(\frac{2}{5})^2+\frac{1}{5}\times(\frac{3}{5})^2+\frac{1}{5}\times(\frac{4}{5})^2+\frac{1}{5}\times(\frac{5}{5})^2 = (\frac{1}{5})^3\left\{1^2+2^2+3^2+4^2+5^2\right\} = 0.44.$$

It follows that the true area $A$ is such that $0.24 \leq A \leq 0.44$. This is pretty much useless! We can refine the approximation by taking more rectangles and making them thinner. If we partition $[0,1]$ into 50 equal subintervals then we obtain $0.32 \leq A \leq 0.34$, which is an improvement. But to appreciate the full force of Reimann integration we take an arbitrary number of rectangles, say $n$. Then

$$\mathcal{P}_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}\right\}$$

We then underestimate the true area:

**Sketch (the lower Reimann sum $\underline{S}_{\mathcal{P}_n}$):**

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Clearly this lower Reimann sum is given by:

$\underline{S}_{\mathcal{P}_n} =$

$\frac{1}{n} \times 0^2 + \frac{1}{n} \times (\frac{1}{n})^2 + \frac{1}{n} \times (\frac{2}{n})^2 + \frac{1}{n} \times (\frac{3}{n})^2 + \cdots + \frac{1}{n} \times (\frac{n-2}{n})^2 + \frac{1}{n} \times (\frac{n-1}{n})^2 =$
$(\frac{1}{n})^3 \left\{0^2 + 1^2 + 2^2 + \cdots + (n-2)^2 + (n-1)^2\right\}.$

It can be shown by induction (you will be given these formulae in an examination) that

$$0^2 + 1^2 + 2^2 + \cdots + (n-2)^2 + (n-1)^2 = \frac{1}{6}n(n-1)(2n-1) = \frac{1}{6}(2n^3 - 3n^2 + n).$$

Thus $\underline{S}_{\mathcal{P}_n} = (\frac{1}{n})^3 \frac{1}{6}(2n^3 - 3n^2 + n) = \frac{2n^3 - 3n^2 + n}{6n^3}$

We also overestimate the true area:

**Sketch (the upper Reimann sum $\overline{S}_{\mathcal{P}_n}$):**

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Clearly this upper Reimann sum is given by:

$\overline{S}_{\mathcal{P}_n} =$

$\frac{1}{n} \times (\frac{1}{n})^2 + \frac{1}{n} \times (\frac{2}{n})^2 + \frac{1}{n} \times (\frac{3}{n})^2 + \cdots + \frac{1}{n}(\frac{n-1}{n})^2 + \frac{1}{n} \times (\frac{n}{n})^2 =$
$(\frac{1}{n})^3 \left\{1^2 + 2^2 + \cdots + (n-1)^2 + (n)^2\right\}.$

It can be shown by induction that

$$1^2 + 2^2 + \cdots + (n-1)^2 + (n)^2 = \frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}(2n^3 + 3n^2 + n).$$

Thus $\overline{S}_{\mathcal{P}_n} = (\frac{1}{n})^3 \frac{1}{6}(2n^3 + 3n^2 + n) = \dfrac{2n^3 + 3n^2 + n}{6n^3}$.

Therefore $\dfrac{2n^3 - 3n^2 + n}{6n^3} \leq A \leq \dfrac{2n^3 + 3n^2 + n}{6n^3}$

As a check letting $n = 5$ yields $0.24 \leq A \leq 0.44$ which agrees with the previous analysis.

Letting $n = 1000$ (that's one thousand thin little rectangles each of width $\frac{1}{1000}$) we get the true area $A$ satisfying

$$0.3328335 \leq A \leq 0.3338335$$

But why not take infinitely many rectangles?! That is let $n \to \infty$.

We have $\lim_{n \longrightarrow \infty} \dfrac{2n^3 + 3n^2 + n}{6n^3} = \dfrac{2}{6} = \dfrac{1}{3}$ and $\lim_{n \longrightarrow \infty} \dfrac{2n^3 - 3n^2 + n}{6n^3} = \dfrac{2}{6} = \dfrac{1}{3}$. Thus

$$\frac{1}{3} \leq A \leq \frac{1}{3}$$

That is $A = \dfrac{1}{3}$.

So that's Reimann integration. We partition the interval into $n$ subintervals and pin the true area between the upper and lower Reimann sums. We let $n \to \infty$ and if the upper and lower sums both converge to a common value we denote that value by $\int_a^b f(x)dx$ and refer to it as the definite integral. The area of a positive function $f$ is then given by $A = \int_a^b f(x)dx$.

**Discussion:** Why do we use the notation $\int_a^b f(x)dx$?

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More formally

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| Suppose that a function $f$ is bounded on $[a, b]$. If there exists a **UNIQUE** real number $I$ such that $\underline{S}_{\mathcal{P}_n}(f) \leq I \leq \overline{S}_{\mathcal{P}_n}(f)$ for *every* partition $\mathcal{P}_n$ of $[a, b]$, then we say that $f$ is *Riemann integrable* on the interval $[a, b]$. If $f$ is Riemann integrable, then this unique real number $I$ is called the *definite integral* of $f$ from $a$ to $b$ and we write $$I = \int_a^b f(x)\,dx.$$ The function $f$ is called the *integrand* of the definite integral, while the points $a$ and $b$ are called the *limits* of the definite integral. |
| If $f \geq 0$ then $I$ is the area bounded by $f$ and the $x$ axis from $x = a$ to $x = b$. |

The clear problem here is that the evaluation of the Reimann integral is a tortuous application of summation and limits. In the next lecture, we will use the Fundamental Theorem of Calculus to establish a dramatic shortcut. But first some technicalities.

Not all functions are Reimann integrable!

**Example 1**: Let $f(x) = \begin{cases} 2, & x \in \mathbb{Q}; \\ 7, & x \notin \mathbb{Q}. \end{cases}$

Sketch the function and by considering upper and lower sums for any partition $\mathcal{P}_n$ explain why the Reimann integral $\displaystyle\int_0^3 f(x)\,dx$ does not exist.

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Fortunately most reasonable functions are integrable.

**Fact:** If $f$ is bounded and piecewise continuous (that is discontinuous at only a finite number of points) on the interval $[a, b]$ then $f$ is Reimann integrable on $[a, b]$.

So all bounded continuous functions are integrable and we can also tolerate a few discontinuities as well.

**Question:** Is $\displaystyle\int_{-1}^{1} \frac{1}{x}\,dx$ well defined?

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**Properties of the Reimann Integral**

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad \text{for } a \leq c \leq b$$

$$f \geq 0 \Rightarrow \int_a^b f(x)\,dx \geq 0$$

$$f \geq g \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx \quad \text{for } \alpha \in \mathbb{R}$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

An important thing to understand is that the definition of the Reimann integral has **NOTHING** to do with calculus! If you look at this entire lecture I have not once mentioned the derivative, the primitive or the anti-derivative. But clearly from your previous studies you know that the calculation the integral is all about calculus? The crucial link between the definition and its implementation will be made in the next lecture through the Fundamental Theorem of Calculus, one of the most amazing results in mathematics.