MATH1131 Mathematics 1A – Algebra

Lecture 14: Complex Polynomials

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Based on slides by Jonathan Kress

Definition

Suppose $n \in \mathbb{N}$. A complex polynomial of degree n is a complex-valued function p of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \qquad \text{for all } z \in \mathbb{C},$$

where $a_0, \ldots, a_n \in \mathbb{C}$ are the coefficients of p (with with $a_n \neq 0$).

- If $a_0, \ldots, a_n \in \mathbb{R}$ then p is a real polynomial.
- If $a_n = 1$ then p is called monic.
- The zero polynomial is the function given by

$$p(z) = 0 \qquad \text{for all } z \in \mathbb{C}.$$

- The degree of the zero polynomial is undefined.

These are polynomials:

$$p(z) = z^3 + z + 1 \qquad q(z) = 2z^3 - iz^2 + 4$$

$$r(z) = 4z^5 - z^2 + 3i \qquad s(z) = i$$

$$f(z) = z + 1 \qquad g(z) = z^2 + 1$$

These are not polynomials:

$$p(z) = \sin z \qquad q(z) = e^z$$

$$r(z) = \frac{z+1}{z-1} \qquad s(z) = z^2 + z - 1 + \sqrt{z}$$

Roots and factors

- If $p(\alpha) = 0$, then α is called a root (or zero) of p.

  For example, $p(z) = z^2 + 1$ has roots $\pm i$ because

$$\begin{aligned} p(i) &= i^2 + 1 = -1 + 1 = 0 \quad \text{and} \\ p(-i) &= (-i)^2 + 1 = -1 + 1 = 0. \end{aligned}$$

- If $p(z) = q(z)g(z)$, then $q(z)$ and $g(z)$ are factors of $p(z)$.

  For example, $p(z) = z^2 + 1 = (z + i)(z - i)$ has factors $z + i$ and $z - i$.

Remainder Theorem

When $p(z)$ is divided by $z-\alpha$, the remainder is $r = p(\alpha)$.

Proof

Write $p(z) = q(z)(z-\alpha) + r(z)$.

The degree of the remainder $r(z)$ must be smaller than the degree of $z-\alpha$ and so $r(z)$ must be constant.

Therefore, in particular when $z = \alpha$,

$$\begin{aligned} r(z) &= r(\alpha) \\ &= p(\alpha) - q(\alpha)(\alpha - \alpha) \\ &= p(\alpha). \end{aligned}$$

Example

Find the remainder when $z^3 + 5z^2 - 6z + 3$ is divided by $z - 4$.

Writing $p(z) = z^3 + 5z^2 - 6z + 3$, the remainder after division by $z - 4$ will be $p(4)$ by the Remainder Theorem.

So the remainder is $p(4) = 4^3 + 5 \times 4^2 - 6 \times 4 + 3 = 123$.

Example

Find the remainder when $z^3 + 5z^2 - 6z + 3$ is divided by $z + 4$.

Here dividing by $z + 4$ is the same as dividing by $z - (-4)$. So the remainder will be $p(-4)$ by the Remainder Theorem.

So the remainder is $p(-4) = (-4)^3 + 5 \times (-4)^2 - 6 \times (-4) + 3 = 43$.

The Factor Theorem

Factor Theorem

α is a root of p if and only if $z - \alpha$ is a factor of $p(z)$.

Proof

Let r be the remainder of $p(z)$ when divided by $z - \alpha$. Then

$$\begin{gathered} \alpha \text{ is a root of } p \\ \Updownarrow \\ p(\alpha) = 0 \\ \Updownarrow \\ r = p(\alpha) = 0 \\ \Updownarrow \\ z - \alpha \text{ is a factor of } p(z) \end{gathered}$$

Factorisation

The Fundamental Theorem of Algebra says that every complex polynomial of degree $n \geq 1$ has at least one (complex) root.

This leads to...

The Factorisation Theorem

Every complex polynomial p of degree $n \geq 1$ has a factorisation

$$p(z) = a(z - \alpha_1)(z - \alpha_2)\ldots(z - \alpha_n),$$

where the n complex numbers $\alpha_1, \alpha_2, \ldots \alpha_n$ are the roots of p, and $a \in \mathbb{C}$.

Note that this means every complex polynomial of degree n has exactly n complex roots, counting with multiplicity (i.e. counting repeated roots separately).

Example

Factorise $z^2 - 4$ into linear factors.

$z^2 - 4 = (z-2)(z+2)$

Example

Factorise $2z^3 + 2z^2 - 4z$ into linear factors.

$2z^3 + 2z^2 - 4z = 2z(z^2 + z - 2) = 2z(z-1)(z+2)$

Example

Factorise $z^3 - 8i$ into linear factors.

$z^3 - 8i = 0$ when $z^3 = 8i = 8e^{i\frac{\pi}{2}} = 8e^{i\left(\frac{\pi}{2}+2k\pi\right)}$ for $k \in \mathbb{Z}$.

So the three roots are $z = 2e^{i\frac{\pi}{6}}, 2e^{i\frac{5\pi}{6}}, 2e^{-i\frac{\pi}{2}}$.

$$\begin{aligned} \text{So } z^3 - 8i &= (z - 2e^{i\frac{\pi}{6}})(z - 2e^{i\frac{5\pi}{6}})(z - 2e^{-i\frac{\pi}{2}}) \\ &= (z - (\sqrt{3} + i))(z - (i - \sqrt{3}))(z + 2i). \end{aligned}$$

Real polynomials and conjugate roots

Theorem

If $\alpha \in \mathbb{C}$ is a root of a real polynomial p, then $\overline{\alpha}$ is also a root of p.

Proof

Suppose p is a real polynomial and $\alpha \in \mathbb{C}$ is a root of p, that is,

$$p(z) = a_n z^n + \ldots + a_1 z + a_0 \qquad \text{for all } z \in \mathbb{C},$$

where $a_0, a_1, \ldots, a_n$ are real, and $p(\alpha) = 0$. Then

$$\begin{aligned}
p(\overline{\alpha}) &= a_n \overline{\alpha}^n + \ldots + a_1 \overline{\alpha} + a_0 \\
&= \overline{a_n}\,\overline{\alpha}^n + \ldots + \overline{a_1}\,\overline{\alpha} + \overline{a_0} \\
&= \overline{a_n \alpha^n + \ldots + a_1 \alpha + a_0} \\
&= \overline{p(\alpha)} \\
&= \overline{0} \\
&= 0.
\end{aligned}$$

Hence $\overline{\alpha}$ is also a root of p.

Real polynomials and conjugate roots

Suppose that the real polynomial p has a non-real root α.

Then $\overline{\alpha}$ is also a root and $z-\alpha$ and $z-\overline{\alpha}$ are factors of $p(z)$.

So a quadratic factor of $p(z)$ is given by:

$$\begin{aligned}(z-\alpha)(z-\overline{\alpha}) &= z^2-(\alpha+\overline{\alpha})z+\alpha\overline{\alpha} \\ &= z^2-2\mathrm{Re}(\alpha)z+|\alpha|^2\end{aligned}$$

Since $\mathrm{Re}(\alpha)$ and $|\alpha|^2$ are real, this quadratic factor is real.

Using this method, every real polynomial can be factored into real linear and quadratic factors.

Real polynomials and conjugate roots – Example

Example

Express $z^6 - 1$ as a product of linear factors, and again as a product of **real** linear and quadratic factors.

$z^6 - 1 = 0$ when $z^6 = 1 = e^{i \times 2k\pi}$ for $k \in \mathbb{Z}$.

So the six (complex) roots are $z = 1\,,\ e^{i\frac{\pi}{3}}\,,\ e^{i\frac{2\pi}{3}}\,,\ -1\,,\ e^{-i\frac{2\pi}{3}}\,,\ e^{-i\frac{\pi}{3}}$.

So as a product of linear factors,

$$z^6 - 1 = (z-1)(z - e^{i\frac{\pi}{3}})(z - e^{i\frac{2\pi}{3}})(z+1)(z - e^{-i\frac{2\pi}{3}})(z - e^{-i\frac{\pi}{3}}).$$

Real polynomials and conjugate roots – Example

Example

Express $z^6 - 1$ as a product of linear factors, and again as a product of **real** linear and quadratic factors.

The six (complex) roots are $z = 1\,,\ e^{i\frac{\pi}{3}}\,,\ e^{i\frac{2\pi}{3}}\,,\ -1\,,\ e^{-i\frac{2\pi}{3}}\,,\ e^{-i\frac{\pi}{3}}$.

To find the real quadratic factors, consider the non-real roots in pairs of conjugates: $e^{i\frac{\pi}{3}}$ with $e^{-i\frac{\pi}{3}}$, and $e^{i\frac{2\pi}{3}}$ with $e^{-i\frac{2\pi}{3}}$.

$$\begin{aligned}(z - e^{i\frac{\pi}{3}})(z - e^{-i\frac{\pi}{3}}) &= z^2 - (e^{i\frac{\pi}{3}} + e^{-i\frac{\pi}{3}})z + e^{i\frac{\pi}{3}}e^{-i\frac{\pi}{3}} \\ &= z^2 - 2\mathrm{Re}(e^{i\frac{\pi}{3}})z + |e^{i\frac{\pi}{3}}|^2 \\ &= z^2 - 2\cos\left(\tfrac{\pi}{3}\right)z + 1 \\ &= z^2 - z + 1\end{aligned}$$

Similarly, $(z - e^{i\frac{2\pi}{3}})(z - e^{-i\frac{2\pi}{3}}) = z^2 - 2\cos\left(\frac{2\pi}{3}\right)z + 1 = z^2 + z + 1.$

So $z^6 - 1 = (z-1)(z+1)(z^2 - z + 1)(z^2 + z + 1).$