MATH1131 Mathematics 1A – Algebra

Lecture 7: Cross Product

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Based on slides by Jonathan Kress

Cross product

Definition

Suppose that

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

are vectors in $\mathbb{R}^3$.

Then the cross product (or vector product) of **a** and **b** is:

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

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Note that while $\mathbf{a} \cdot \mathbf{b}$ is defined for vectors in all dimensions, $\mathbf{a} \times \mathbf{b}$ is only defined for vectors in $\mathbb{R}^3$.

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This formula can be difficult to memorise. One way to remember the terms is to think of the subscripts cycling through in order:

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Cross product

The cross product can be written as a 3×3 determinant:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

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&= \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\
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\end{aligned}$$

Note the different signs!

Cross product – Examples

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}2\\3\\5\end{pmatrix} \qquad =$$

$$\begin{pmatrix}2\\3\\-1\end{pmatrix} \times \begin{pmatrix}1\\1\\1\end{pmatrix} \quad =$$

$$\begin{pmatrix}1\\0\\2\end{pmatrix} \times \begin{pmatrix}3\\4\\5\end{pmatrix} \qquad =$$

Cross product – Examples

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}2\\3\\5\end{pmatrix} = \begin{pmatrix}2\times 5 - 3\times 3\\3\times 2 - 1\times 5\\1\times 3 - 2\times 2\end{pmatrix}$$

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$$\begin{pmatrix}1\\0\\2\end{pmatrix} \times \begin{pmatrix}3\\4\\5\end{pmatrix} = \begin{pmatrix}0\times 5 - 2\times 4\\2\times 3 - 1\times 5\\1\times 4 - 0\times 3\end{pmatrix}$$

Cross product – Examples

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$$\begin{pmatrix}1\\0\\2\end{pmatrix}\times\begin{pmatrix}3\\4\\5\end{pmatrix} = \begin{pmatrix}0\times 5-2\times 4\\2\times 3-1\times 5\\1\times 4-0\times 3\end{pmatrix} = \begin{pmatrix}-8\\1\\4\end{pmatrix}$$

Properties

Properties of the cross product

If $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^3$, then $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$ and to $\mathbf{b}$.

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For example, we saw

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} -8 \\ 1 \\ 4 \end{pmatrix},$$

and can confirm that

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -8 \\ 1 \\ 4 \end{pmatrix} = 0 \qquad \text{and} \qquad \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -8 \\ 1 \\ 4 \end{pmatrix} = 0.$$

Properties

Properties of the cross product

If **a** and **b** are vectors in $\mathbb{R}^3$, then $\mathbf{a} \times \mathbf{b}$ is orthogonal to **a** and **b**.

Exercise

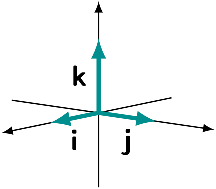
Show that for all $a_i, b_i \in \mathbb{R}$:

$$\begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

and

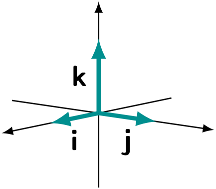
$$\begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 0.$$

Examples



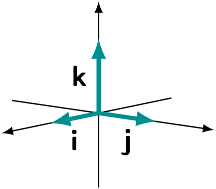
$\mathbf{i} \times \mathbf{j}$

Examples



$$\mathbf{i} \times \mathbf{j} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Examples



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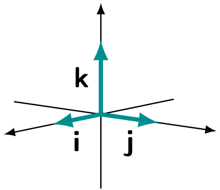
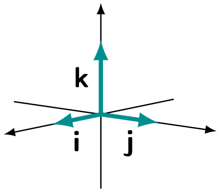


$$\mathbf{i} \times \mathbf{j} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{k}$$

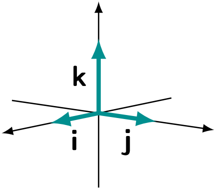
Examples



$$\mathbf{i} \times \mathbf{j} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{i}$$

Examples



$$\mathbf{i} \times \mathbf{j} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{k}$$

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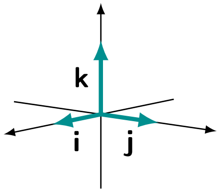

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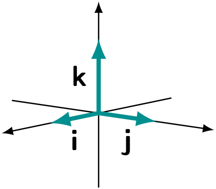
Similarly $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$, and $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

Examples



$$\mathbf{i}\times\mathbf{j} = \begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} = \mathbf{k}$$

$$\mathbf{j}\times\mathbf{k} = \begin{pmatrix}0\\1\\0\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} = \mathbf{i}$$

$$\mathbf{k}\times\mathbf{i} = \begin{pmatrix}0\\0\\1\end{pmatrix} \times \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} = \mathbf{j}$$

Similarly $\mathbf{j}\times\mathbf{i} = -\mathbf{k}$, $\mathbf{k}\times\mathbf{j} = -\mathbf{i}$, and $\mathbf{i}\times\mathbf{k} = -\mathbf{j}$.

The direction of the cross product vector can be determined from the "right-hand rule": To find the direction of $\mathbf{a}\times\mathbf{b}$, join the vectors tail-to-tail, and point your right thumb in the direction of $\mathbf{a}$ and fingers in the direction of $\mathbf{b}$. Your palm will point in the direction of $\mathbf{a}\times\mathbf{b}$.

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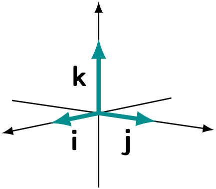

$$\mathbf{i} \times \mathbf{j} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{k}$$

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Notice also that $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$.

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$$\begin{aligned}\begin{pmatrix}1\\0\\2\end{pmatrix} \times \begin{pmatrix}3\\4\\5\end{pmatrix} &= (\mathbf{i} + 2\mathbf{k}) \times (3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}) \\ &= 3\mathbf{i} \times \mathbf{i} + 4\mathbf{i} \times \mathbf{j} + 5\mathbf{i} \times \mathbf{k} + 6\mathbf{k} \times \mathbf{i} + 8\mathbf{k} \times \mathbf{j} + 10\mathbf{k} \times \mathbf{k}\end{aligned}$$

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&= \mathbf{0} + 4\mathbf{k} + 5(-\mathbf{j}) + 6\mathbf{j} + 8(-\mathbf{i}) + \mathbf{0}
\end{aligned}
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&= \mathbf{0} + 4\mathbf{k} + 5(-\mathbf{j}) + 6\mathbf{j} + 8(-\mathbf{i}) + \mathbf{0} \\
&= -8\mathbf{i} + \mathbf{j} + 4\mathbf{k}
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&= \begin{pmatrix}-8\\1\\4\end{pmatrix}
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\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} &= (\mathbf{i} + 2\mathbf{k}) \times (3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}) \\
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&= \begin{pmatrix} -8 \\ 1 \\ 4 \end{pmatrix}
\end{aligned}$$

Notice we have assumed some properties of the cross product, which we should now prove...

Properties of the cross product

Properties of the cross product

For all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$ and scalars $\lambda \in \mathbb{R}$,

- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
- $\mathbf{0} \times \mathbf{a} = \mathbf{a} \times \mathbf{0} = \mathbf{0}$
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Proof

$$\mathbf{a} \times \mathbf{a} = \begin{pmatrix} a_2 a_3 - a_3 a_2 \\ a_3 a_1 - a_1 a_3 \\ a_1 a_2 - a_2 a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}$$

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$$\mathbf{0} \times \mathbf{a} = \begin{pmatrix} 0a_3 - 0a_2 \\ 0a_1 - 0a_3 \\ 0a_2 - 0a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \mathbf{a} \times \mathbf{0} = \begin{pmatrix} a_2 0 - a_3 0 \\ a_3 0 - a_1 0 \\ a_1 0 - a_2 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

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Proof

$$\mathbf{b} \times \mathbf{a} = \begin{pmatrix} b_2a_3 - b_3a_2 \\ b_3a_1 - b_1a_3 \\ b_1a_2 - b_2a_1 \end{pmatrix} = - \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} = -\mathbf{a} \times \mathbf{b}$$

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Proof

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \begin{pmatrix} a_2(b_3 + c_3) - a_3(b_2 + c_2) \\ a_3(b_1 + c_1) - a_1(b_3 + c_3) \\ a_1(b_2 + c_2) - a_2(b_1 + c_1) \end{pmatrix}$$

$$= \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} + \begin{pmatrix} a_2 c_3 - a_3 c_2 \\ a_3 c_1 - a_1 c_3 \\ a_1 c_2 - a_2 c_1 \end{pmatrix} = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

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Proof

$$\mathbf{a} \times (\lambda\mathbf{b}) = \begin{pmatrix} a_2(\lambda b_3) - a_3(\lambda b_2) \\ a_3(\lambda b_1) - a_1(\lambda b_3) \\ a_1(\lambda b_2) - a_2(\lambda b_1) \end{pmatrix} = \lambda \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} = \lambda(\mathbf{a} \times \mathbf{b})$$

and

$$(\lambda\mathbf{a}) \times \mathbf{b} = \begin{pmatrix} (\lambda a_2) b_3 - (\lambda a_3) b_2 \\ (\lambda a_3) b_1 - (\lambda a_1) b_3 \\ (\lambda a_1) b_2 - (\lambda a_2) b_1 \end{pmatrix} = \lambda \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} = \lambda(\mathbf{a} \times \mathbf{b})$$

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Length of the cross product

Theorem

For all vectors $\mathbf{a}, \mathbf{b}$ in $\mathbb{R}^3$:

$$|\mathbf{a} \cdot \mathbf{b}|^2 + |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2.$$

Length of the cross product

Theorem

For all vectors $\mathbf{a}, \mathbf{b}$ in $\mathbb{R}^3$:

$$|\mathbf{a} \cdot \mathbf{b}|^2 + |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2.$$

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$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$

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$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

$$= |\mathbf{a}|^2 |\mathbf{b}|^2$$

Length of the cross product

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So if $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then

$$|\mathbf{a} \times \mathbf{b}|^2 \quad =$$

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So if $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then

$$|\mathbf{a} \times \mathbf{b}|^2 \quad = \quad |\mathbf{a}|^2 |\mathbf{b}|^2 - |\mathbf{a} \cdot \mathbf{b}|^2$$

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&= |\mathbf{a}|^2 |\mathbf{b}|^2 \sin^2\theta
\end{aligned}$$

Length of the cross product

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&= |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta
\end{aligned}$$

Hence (since $\sin(\theta)$ is positive for all $0 < \theta < \pi$):

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

Length of the cross product

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Hence (since $\sin(\theta)$ is positive for all $0 < \theta < \pi$):

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

(Note that if $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$, then both sides are 0 although $\sin\theta$ is not defined.)

Geometric interpretation of the cross product

We can now describe the cross product of two vectors geometrically:

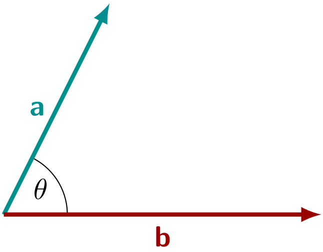
Summary

For non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$, the cross product $\mathbf{a} \times \mathbf{b}$ is a vector of length

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

in the direction perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ as determined by the right-hand rule.

Area of a parallelogram



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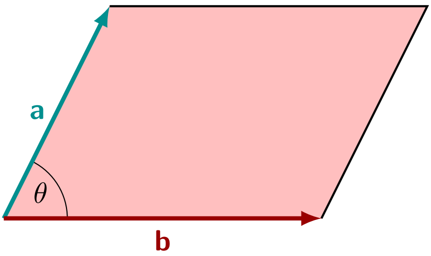

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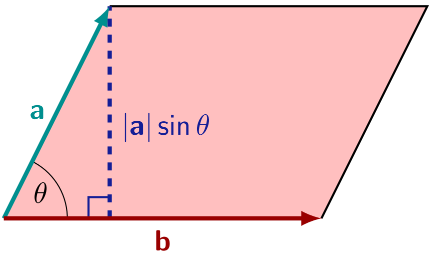
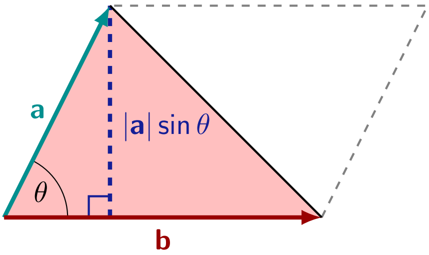


$$\text{Area} = \text{base} \times \text{altitude} = |\mathbf{b}||\mathbf{a}| \sin\theta = |\mathbf{a} \times \mathbf{b}|$$

Area of a triangle



$$\text{Area} \;=\; \frac{1}{2}\text{base} \times \text{altitude} \;=\; \frac{1}{2}|\mathbf{b}||\mathbf{a}|\sin\theta \;=\; \frac{1}{2}|\mathbf{a}\times\mathbf{b}|$$

Area of a parallelogram – Example

Find the area of a parallelogram with vertices at points $A(1,0,1)$, $B(-2,1,3)$, $C(3,1,4)$ and D.

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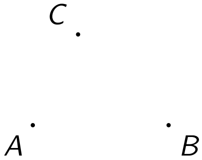

What are the possibilities for D?

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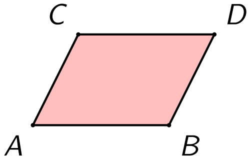

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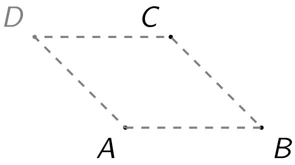

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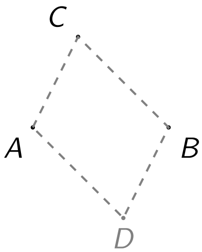

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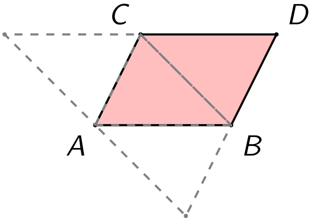

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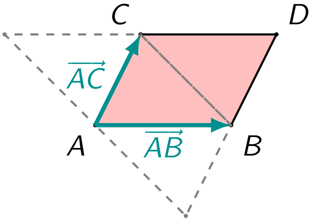

Area $= |\overrightarrow{AB} \times \overrightarrow{AC}|$

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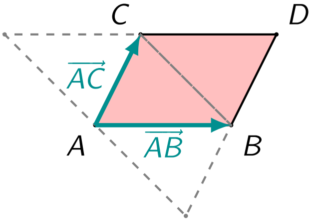

$$\text{Area} = |\overrightarrow{AB} \times \overrightarrow{AC}| = \left| \begin{pmatrix} -2-1 \\ 1-0 \\ 3-1 \end{pmatrix} \times \begin{pmatrix} 3-1 \\ 1-0 \\ 4-1 \end{pmatrix} \right|$$

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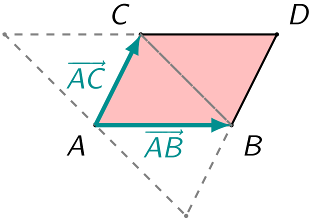

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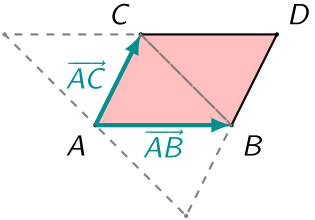

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&= \left| \begin{pmatrix} 1 \times 3 - 2 \times 1 \\ 2 \times 2 - (-3) \times 3 \\ (-3) \times 1 - 1 \times 2 \end{pmatrix} \right|
\end{aligned}$$

Area of a parallelogram – Example

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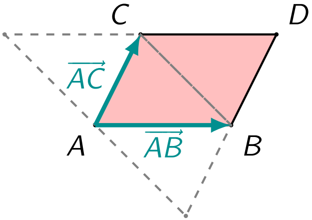

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$$= \left| \begin{pmatrix} 1 \times 3 - 2 \times 1 \\ 2 \times 2 - (-3) \times 3 \\ (-3) \times 1 - 1 \times 2 \end{pmatrix} \right| = \left| \begin{pmatrix} 1 \\ 13 \\ -5 \end{pmatrix} \right| = \sqrt{195}$$

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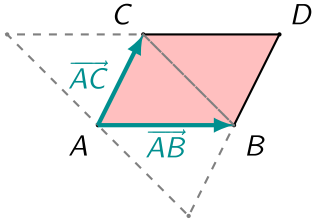

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Exercise

Show that the same area value arises for the other choices of D.