Chapter 8: Integration

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MATH1131

UNSW

Term 1 2020

Motivation

In this chapter we want to take a much more careful approach to integration than you (probably) did at school. We want to understand why various results work, instead of just applying rules.

Main Problem

How does one find/measure/define areas (of regions with curved boundaries)?

Calculating the area of a region in the plane has a long history that can be traced back to

- Archimedes (the great Greek mathematician of antiquity) and then
- Isaac Barrow (Isaac Newton's mentor)
- Isaac Newton (one of the greatest mathematicians and physicists of all time)
- Gottfried Leibniz (Newton's contemporary)
- Bernhard Riemann (a German mathematician of the 19^{th} century)
- Henri Lebesgue (a French mathematician of the early 20^{th} century).

One (but not the only) method for calculating the area of regions with curved boundaries is known as Riemann integration.

Note. A priori, calculating areas (integration) and antidifferentiation are two separate problems but the remarkable **fundamental theorem of calculus** shows that these are essentially the same!

This theorem was actually known to Barrow but its implications were developed by Newton, Leibniz and their disciples.

Here, we start by confining ourselves to the determination of *the area under the graph of a function* via the Riemann integral.

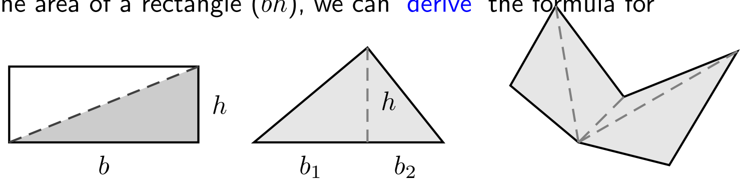
Area of polygons

We all (think that we) know how to calculate the area of a rectangle, a triangle or even general polygons by partitioning the polygon into triangles.

But what are the rules that we take for granted?

- The area of a rectangle is the product of its length and height.
- Areas of congruent regions (same shape and size) are equal.
- The area of a whole region is the sum of the areas of its 'parts'.

Thus, from the area of a rectangle (bh), we can 'derive' the formula for



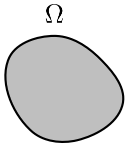
- the area of a right-angled triangle ($\frac{1}{2}bh$), then
- the area of an arbitrary triangle ($\frac{1}{2}(b_1 + b_2)h$) and then
- the area of an arbitrary polygon.

Areas of regions with curved boundaries

Formally, we demand that any definition of an area satisfies the following axioms:

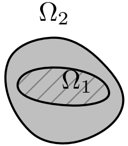
(A1) If Ω is a region of the plane then

$$\operatorname{area}(\Omega) \geq 0.$$



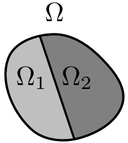
(A2) If one region Ω_1 is contained in another region Ω_2, then

$$\operatorname{area}(\Omega_1) \leq \operatorname{area}(\Omega_2).$$



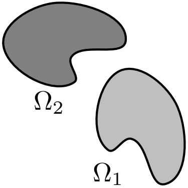
(A3) If the area of a region Ω is partitioned into two smaller disjoint regions Ω_1 and Ω_2, then

$$\operatorname{area}(\Omega) = \operatorname{area}(\Omega_1) + \operatorname{area}(\Omega_2).$$



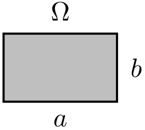
(A4) If Ω_1 and Ω_2 are congruent regions then

$$\text{area}(\Omega_1) = \text{area}(\Omega_2).$$



(A5) If Ω is a rectangle of length a and height b then

$$\text{area}(\Omega) = ab.$$

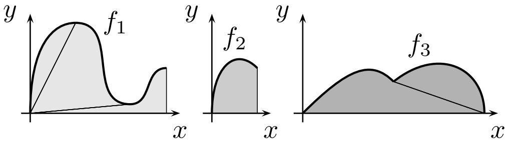


Question. How could one calculate the area of the region below?

Answer. By partitioning Ω with straight lines and applying axiom (A3) we conclude that

$$\text{area}(\Omega) = \text{area}(\Omega_1) + \text{area}(\Omega_2) + \text{area}(\Omega_3).$$

Then, rotate and translate each subregion, so that application of axiom (A4) implies that each of $\text{area}(\Omega_1)$, $\text{area}(\Omega_2)$ and $\text{area}(\Omega_3)$ is equal to the area under the graph of a function.

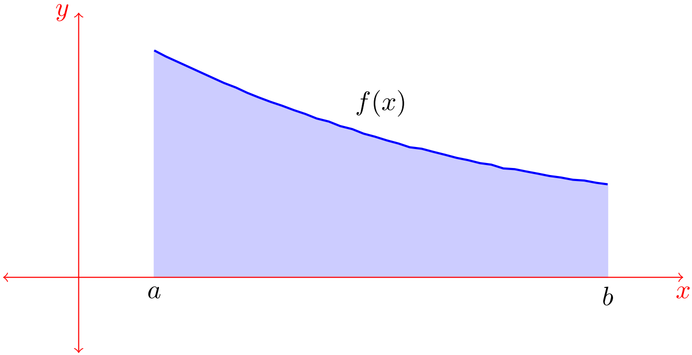


This procedure can be done for any region in the plane with a 'reasonable' boundary.

Thus, we reduce the problem of defining the area of a region with a curved boundary to defining the area under the graph of a function, that is, the area of a region of the form

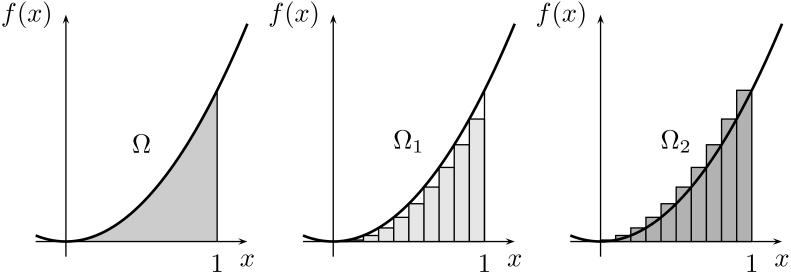
$$S = \{(x, y) \in \mathbb{R}^2 \,:\, a \le x \le b,\ 0 \le y \le f(x)\}$$

for some bounded positive function f.



Approximations of area using Riemann sums

Example of Riemann sums. Suppose that $f : [0, 1] \to \mathbb{R}$, $f(x) = x^2$. Let Ω denote the region bounded by the graph of f, the x-axis and the lines $x = 0$ and $x = 1$.



Idea. Find lower and upper bounds for $\operatorname{area}(\Omega)$ by choosing appropriate 'approximations' Ω_1 and Ω_2 of the region Ω in terms of n rectangles.

It is evident that

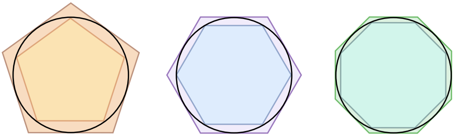
$$\operatorname{area}(\Omega_1) \le \operatorname{area}(\Omega) \le \operatorname{area}(\Omega_2).$$

If $\lim_{n\to\infty} \operatorname{area}(\Omega_1) = \lim_{n\to\infty} \operatorname{area}(\Omega_2)$ then

$$A = \operatorname{area}(\Omega) = \lim_{n\to\infty} \operatorname{area}(\Omega_1) = \lim_{n\to\infty} \operatorname{area}(\Omega_2),$$

by the pinching theorem.

Remark. One of Archimedes' best ideas was to estimate areas in circles and parabolas by approximating them by 'inner' and 'outer' polygons!

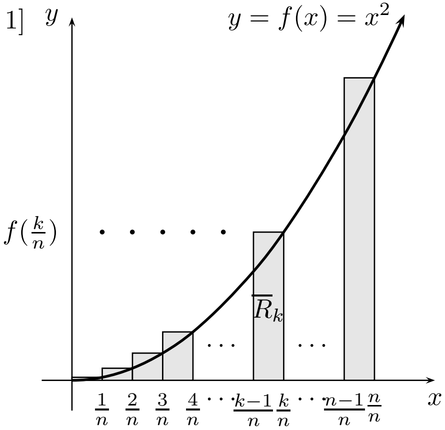


https://en.wikipedia.org/wiki/Method_of_exhaustion

Explicit evaluation of the bounds

We begin by subdividing the interval $[0, 1]$ into n subintervals

$$[0, \tfrac{1}{n}], [\tfrac{1}{n}, \tfrac{2}{n}], [\tfrac{2}{n}, \tfrac{3}{n}], \ldots, [\tfrac{n-1}{n}, 1].$$



The set $\mathcal{P}_n$ given by

$$\mathcal{P}_n = \{0, \tfrac{1}{n}, \tfrac{2}{n}, \tfrac{3}{n}, \ldots, \tfrac{n-1}{n}, 1\}$$

which divides the interval $[0, 1]$ into these subintervals is called a partition of $[0, 1]$.

Let $\overline{R}_k$ denote the area of the kth rectangle. Then

$$\overline{R}_k = \text{width} \times \text{height} = \frac{1}{n} \times f\left(\frac{k}{n}\right) = \frac{1}{n} \times \left(\frac{k}{n}\right)^2 = \frac{k^2}{n^3}.$$

If $\overline{S}_{\mathcal{P}_n}(f)$ denotes the total area of the shaded region (sum of rectangles) then

$$\overline{S}_{\mathcal{P}_n}(f) = \sum_{k=1}^{n} \overline{R}_k = \sum_{k=1}^{n} \frac{k^2}{n^3} = \frac{1}{n^3} \sum_{k=1}^{n} k^2.$$

One may show by induction (exercise!) that

$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1) = \frac{1}{6} n(2n^2 + 3n + 1)$$

and hence

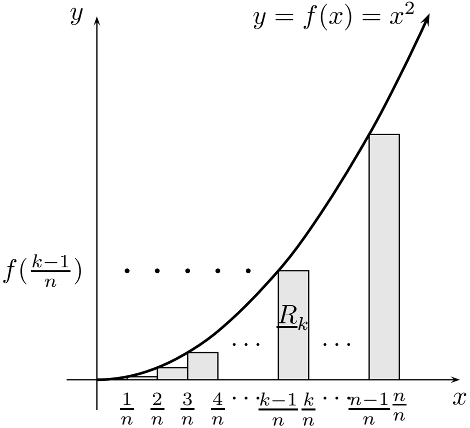
$$\begin{aligned} \overline{S}_{\mathcal{P}_n}(f) &= \frac{1}{6} \frac{n}{n^3} (2n^2 + 3n + 1) \\ &= \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}. \end{aligned}$$

The quantity $\overline{S}_{\mathcal{P}_n}(f)$ is called the upper Riemann sum of f with respect to the partition $\mathcal{P}_n$.

Axiom (A2) now implies that

$$A \leq \overline{S}_{\mathcal{P}_n}(f) = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}.$$

In a similar manner, a lower bound is obtained:



The area $\underline{R}_k$ of the kth rectangle is given by

$$\underline{R}_k = \frac{1}{n} \times f\left(\frac{k-1}{n}\right) = \frac{1}{n} \times \left(\frac{k-1}{n}\right)^2 = \frac{(k-1)^2}{n^3}.$$

The sum of all the areas of these rectangles is called the lower Riemann sum for the function f over the partition $\mathcal{P}_n$ and is denoted by $\underline{S}_{\mathcal{P}_n}(f)$. We obtain

$$\underline{S}_{\mathcal{P}_n}(f) = \sum_{k=1}^{n} \underline{R}_k = \sum_{k=1}^{n} \frac{(k-1)^2}{n^3} = \frac{1}{n^3} \sum_{k=1}^{n} (k-1)^2$$

so that

$$\underline{S}_{\mathcal{P}_n}(f) = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}.$$

Axiom (2) therefore implies that

$$A \geq \underline{S}_{\mathcal{P}_n}(f) = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}.$$

Hence, for every positive integer n (number of subintervals, hence rectangles), the inequality

$$\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2} \leq A \leq \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

gives an upper and a lower bound for A.

Conclusion. In the limit $n \to \infty$, we obtain

$$A = \frac{1}{3},$$

regardless of the actual definition of A as long as it is compatible with the axioms (A1)-(A5)!

Remark. The process of calculating upper and lower Riemann sums and taking a limit of the above type (provided it exists) is called **integration**.

The definition of area under the graph of a function and the Riemann integral

Now, we generalise the approach above to an arbitrary positive bounded function f (we shall frequently assume, that f is **continuous**, although the construction works for **some** discontinuous functions, too).

Suppose that f is a bounded function on $[a, b]$ and that $f(x) \geq 0$ for all x in $[a, b]$.

Definition

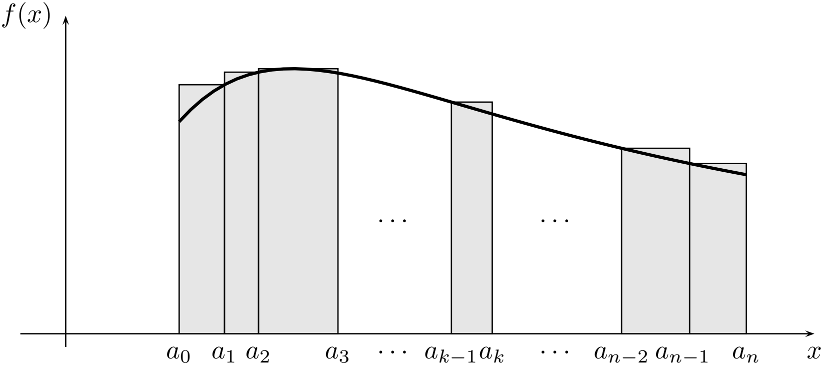
A finite set $\mathcal{P}$ of points in $\mathbb{R}$ is said to be a partition of $[a, b]$ if

$$\mathcal{P} = \{a_0, a_1, a_2, \ldots, a_n\}$$

and

$$a = a_0 < a_1 < a_2 < \ldots < a_n = b.$$

Suppose that $\mathcal{P}$ is a partition of $[a, b]$:



Note. The points of $\mathcal{P}$ need not be evenly spaced.

The area of the kth rectangle in the above figure is

$$\text{width} \times \text{height} = (a_k - a_{k-1}) \times \overline{f}_k,$$

where

$$\overline{f}_k = \text{the maximum value of } f \text{ on the subinterval } [a_{k-1}, a_k].$$

The upper Riemann sum $\overline{S}_{\mathcal{P}}(f)$ for f with respect to the partition $\mathcal{P}$ is defined by

$$\overline{S}_{\mathcal{P}}(f) = \sum_{k=1}^{n}(a_k - a_{k-1})\overline{f}_k$$

which is the total area of the rectangles in the above figure.

Likewise, the lower Riemann sum $\underline{S}_{\mathcal{P}}(f)$ for f with respect to the partition $\mathcal{P}$ is defined by

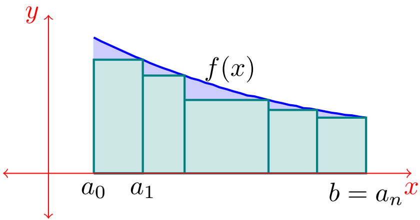
$$\underline{S}_{\mathcal{P}}(f) = \sum_{k=1}^{n}(a_k - a_{k-1})\underline{f}_k, \tag{1}$$

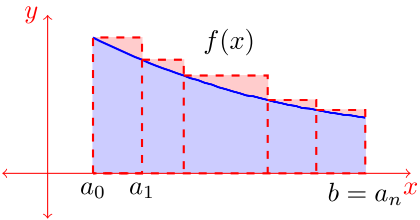
where

$$\underline{f}_k = \text{the minimum value of } f \text{ on the subinterval } [a_{k-1}, a_k].$$

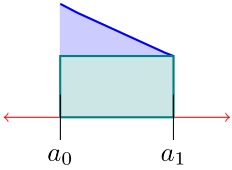
Sticking these all together gives that

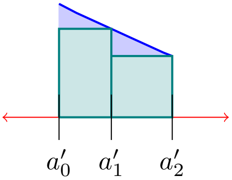
$$\underline{S}_{\mathcal{P}}(f) \leq \mathrm{Area}(\Omega) \leq \overline{S}_{\mathcal{P}}(f).$$





Note: As you add more points to $\mathcal{P}$, $\underline{S}_{\mathcal{P}}$ gets bigger and bigger.





Similarly, as you add more points, $\overline{S}_{\mathcal{P}}$ decreases.

Definition

Suppose that a function f is bounded on $[a, b]$ and that $f(x) \geq 0$ for all x in $[a, b]$.

If there exists a unique real number A such that

$$\underline{S}_{\mathcal{P}}(f) \leq A \leq \overline{S}_{\mathcal{P}}(f)$$

for every partition $\mathcal{P}$ of $[a, b]$ then we say that A is the area under the graph of f from a to b.

The following definition gives a special name to those which have a well-defined area under their graph. Also, we remove the condition that $f(x) \geq 0$ for all $x \in [a, b]$.

Definition

Suppose that a function f is bounded on $[a, b]$. If there exists a unique real number I such that

$$\underline{S}_{\mathcal{P}}(f) \leq I \leq \overline{S}_{\mathcal{P}}(f)$$

for every partition $\mathcal{P}$ of $[a, b]$ then we say that f is Riemann integrable on the interval $[a, b]$.

The unique real number I is called the definite integral of f from a to b and we write

$$I = \int_a^b f(x)\, dx.$$

The function f is called the integrand of the definite integral, while the points a and b are called the limits of the definite integral.

Remark. The notation

$$\int_a^b f(x)\,dx$$

is due to Leibniz. It evolved from a slightly different way of writing down lower and upper Riemann sums. For example, $\overline{S}_{\mathcal{P}}(f)$ may be written as

$$\overline{S}_{\mathcal{P}}(f) = \sum_{k=1}^{n} f(\overline{x}_k)\Delta x_k,$$

where $\Delta x_k = a_k - a_{k-1}$ and f attains its maximum value on $[a_{k-1}, a_k]$ at the point $\overline{x}_k$.

When taking a limit as before, Δx_k was replaced with dx and the symbol $\sum$ was replaced with an elongated 'S' ('S' stands for 'sum').

Note: The use of the variable x is only tradition. You can of course use any other sensible variable, such as

$$I = \int_a^b f(u)\,du.$$

Geometric interpretation of the Riemann integral

If f is a Riemann integrable function on $[a, c]$, then $\int_a^c f(x)dx$ is the area under the graph of f from a to c provided that $f(x) \geq 0$ for all $x \in [a, c]$.

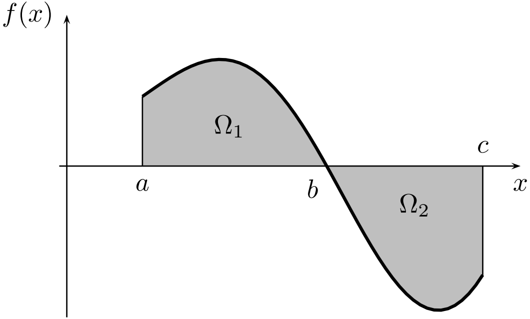
If f is Riemann integrable but not necessarily non-negative then

- $\int_a^c f(x)\,dx$: area of regions above the x-axis - area of regions below the x-axis.
  We call this quantity the signed area under the graph of f from a to c.
- The area of the region formed by the graph of f, the x-axis and the lines $x = a$ and $x = c$, also called the unsigned area, is

$$\mathrm{area}(\Omega) = \int_a^c |f(x)|\,dx,$$

provided that the latter integral exists.

Consider the following example:



In this case, the "unsigned" area between the x-axis and the curve of the function (positive) is

$$\text{area}(\Omega) = \Omega 1 + \Omega 2 = \int_a^c |f(x)|\, dx = \int_a^b f(x)\, dx + \int_b^c -f(x)\, dx.$$

Integration with Riemann sums

Even in the case $f(x) = C$, where C is a positive constant, the actual determination of the Riemann integral is somewhat tedious.

Question. Can we find simple sufficient conditions which guarantee that a Riemann integral exists?

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Indeed, let $\mathcal{P} = \{a_0 = a, a_1, \ldots, a_n = b\}$ be a partition of $[a, b]$. Since f is constant, we have

$$\overline{f}_k = \underline{f}_k = C$$

for every k between 1 and n and hence

$$\overline{S}_{\mathcal{P}}(f) = \sum_{k=1}^{n} \overline{f}_k (a_k - a_{k-1}) = C \sum_{k=1}^{n} (a_k - a_{k-1}) = C(b - a).$$

A similar calculation leads to

$$\underline{S}_{\mathcal{P}}(f) = C(b - a).$$

Accordingly,

$$\underline{S}_{\mathcal{P}}(f) = C(b - a) = \overline{S}_{\mathcal{P}}(f)$$

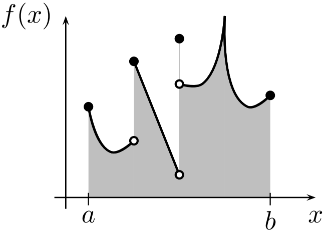
for every partition $\mathcal{P}$ of $[a, b]$ so that we conclude that f is Riemann integrable and

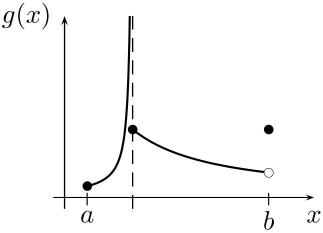
$$\int_a^b f(x)\, dx = C(b - a).$$

Definition

A function $f : [a, b] \to \mathbb{R}$ is said to be piecewise continuous if it is continuous on $[a, b]$ at all except perhaps a finite number of points.

Examples.





Both functions f and g are piecewise continuous but f is bounded while g is not!

Theorem

If f is bounded and piecewise continuous on $[a, b]$ then f is Riemann integrable on $[a, b]$.

Proof. Difficult!

Remark. If f is Riemann integrable on $[a, b]$ then it is sufficient to show that

$$\lim_{n\to\infty} \overline{S}_{\mathcal{P}_n}(f) = \lim_{n\to\infty} \underline{S}_{\mathcal{P}_n}(f) = I$$

for some sequence of partitions $\mathcal{P}_n$ of $[a, b]$, not necessarily all!. Then

$$\int_a^b f(x)\,dx = I.$$

Exercise. Use the formula

$$\sum_{k=1}^{N} k^3 = \frac{1}{4} N^2 (N+1)^2$$

to show that

$$\int_0^1 x^3 dx = \frac{1}{4}.$$

Hint: Use the uniform partition $\mathcal{P}_n = \{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n}, 1\}$, and check that

$$\underline{\mathcal{S}}_{\mathcal{P}_n} = \frac{(n-1)^2}{4n^2} \quad \text{and} \quad \overline{\mathcal{S}}_{\mathcal{P}_n} = \frac{(n+1)^2}{4n^2}.$$

Then note that

$$\lim_{n\to\infty} \frac{(n-1)^2}{4n^2} = \lim_{n\to\infty} \frac{(n+1)^2}{4n^2} = \frac{1}{4}.$$

Basic properties of the Riemann integral

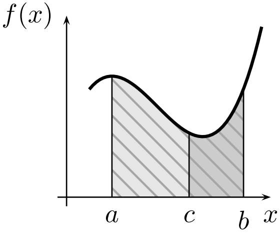
Theorem. Suppose that $f, g : [a, b] \to \mathbb{R}$ are Riemann integrable. Then,

(i) (Linearity) $\alpha f + \beta g$ is integrable for any $\alpha, \beta \in \mathbb{R}$ with

$$\int_a^b (\alpha f + \beta g)(x)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$

(ii) If $a < c < b$ then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

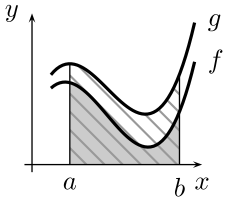


(iii) If $f(x) \geq 0$ for all x in $[a, b]$ then

$$\int_a^b f(x)\,dx \geq 0.$$

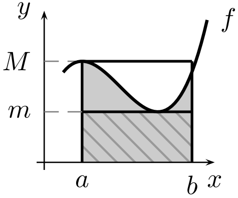
(iv) If $f(x) \leq g(x)$ for all x in $[a, b]$ then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$



(v) If $m \leq f(x) \leq M$ for all x in $[a, b]$ then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

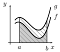


(vi) If $|f|$ is integrable on $[a, b]$ then

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

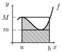
(iv) If $f(x) \le g(x)$ for all x in $[a,b]$ then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$



(v) If $m \le f(x) \le M$ for all x in $[a,b]$ then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$



(vi) If $|f|$ is integrable on $[a,b]$ then

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Sketch of proof. We omit the proof of (i) and (ii).

(iii) If $f(x) \geq 0$ then $0 \leq \underline{S}_{\mathcal{P}}(f) \leq \displaystyle\int_a^b f(x)\,dx$.

(iv) Apply (iii) to $h = g - f$.

(v) From (iv), it follows that $\displaystyle\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx$.

(vi) From

$$-|f(x)| \leq f(x) \leq |f(x)| \qquad \forall x \in [a,b]$$

and (iv), we deduce that

$$-\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx.$$

Hence, $\displaystyle\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$.

It is convenient to introduce the following definition ...

Definition

Suppose that $b < a$ and that f is integrable on $[b, a]$. Then,

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

and

$$\int_a^a f(x)\,dx = 0.$$

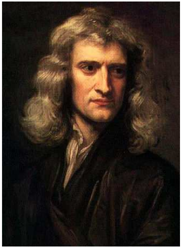
This definition gives a more general version of (ii) above.

The definition of the Riemann integral is fine, but

- Calculating $\underline{S}_{\mathcal{P}}$ and $\overline{S}_{\mathcal{P}}$ is hard.
- Using the definition finding $\displaystyle\int_a^b f(x)\,dx$ looks awful!

Terminology: From now on, we refer to Riemann integrable functions as merely 'integrable'.

Before Newton and Leibniz, each area calculation required the sort of hard work we did earlier to show that $\displaystyle\int_0^1 x^2\,dx = \frac{1}{3}$, and was a major feat of computational skill....

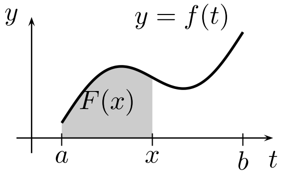


The remarkable insights of Newton and Leibniz were

1. Finding areas is a sort of inverse process to finding tangents, and
2. As one can find tangents by using simple 'symbolic' differentiation rules instead of actually taking limits, you can find areas by applying the differentiation rules backwards.

The first fundamental theorem of calculus

Idea (Leibniz, Newton). How does the integral (area) change as a boundary changes?



Suppose that a function f is continuous and therefore integrable on an interval $[a, b]$.

We define the signed area function F on $[a, x]$ by

$$F(x) = \int_a^x f(t)\, dt, \quad x \in [a, b].$$

The First Fundamental Theorem of Calculus (FTC1)

Suppose that $f : [a,b] \to \mathbb{R}$ is a continuous function. Then, the function $F : [a,b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f(t)\,dt \tag{2}$$

is continuous on $[a,b]$ and differentiable on (a,b) with

$$F'(x) = f(x)$$

for all x in (a,b).

We write

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

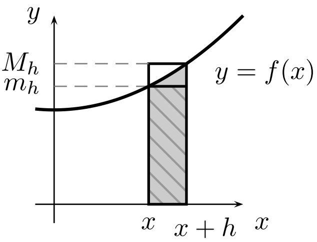
for any real number a.

Implications

- Every continuous function $f : [a, b] \to \mathbb{R}$ has an antiderivative F on (a, b) given by
$$F(x) = \int_a^x f(t)\,dt.$$
- Since two antiderivatives of f differ by a constant, every antiderivative of f is given by $F + \mathrm{const}$. Thus, integration and antidifferentiation are essentially the same procedures!
- Differentiation undoes what integration does to f since
$$f(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right).$$

  Is the converse true?

Proof of FTC1



1. Proof of Differentiability.

- Suppose that $x \in (a,b)$ and choose an $h > 0$ such that $x+h \in (a,b)$. Then,

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt.$$

- Since f is continuous on $[a,b]$, it attains a minimum value m_h and a maximum value M_h on $[x, x+h]$, that is,

$$m_h \le f(t) \le M_h$$

for all $t \in [x, x+h]$.

- We therefore conclude (from the areas) that

$$m_h h \leq \int_x^{x+h} f(t)\,dt \leq M_h h$$

and hence

$$m_h h \leq F(x+h) - F(x) \leq M_h h$$

so that

$$m_h \leq \frac{F(x+h) - F(x)}{h} \leq M_h$$

since $h > 0$.

- Since m_h is the minimum value of the function f on the interval $[x, x+h]$ and since f is continuous, there exists $x_h \in [x, x+h]$ such that $f(x_h) = m_h$.

Using continuity of the function f, we obtain: $\lim\limits_{h \to 0^+} m_h = \lim\limits_{h \to 0^+} f(x_h) = f(x)$.

Arguing similarly, we have $\lim\limits_{h \to 0^+} M_h = f(x)$.

- Hence, by the pinching theorem,

$$\lim_{h\to 0^+} \frac{F(x+h)-F(x)}{h} = f(x).$$

In a similar manner, one can show that

$$\lim_{h\to 0^-} \frac{F(x+h)-F(x)}{h} = f(x).$$

Hence, F is differentiable on (a,b) and for all x in (a,b): $F'(x) = f(x)$.

2. Proof of Continuity.

F is differentiable on (a,b) and is therefore continuous on (a,b), so we are left to show continuity only at the end points a and b

(i.e. show that $\lim_{x\to a^+} F(x) = F(a)$ and $\lim_{x\to b^-} F(x) = F(b)$).

- Since f is continuous on $[a,b]$, it is bounded on $[a,b]$ (from min-max theorem), that is, there exists an $M > 0$ such that

$$|f(x)| \leq M$$

for all x in $[a,b]$.

- For $x \in (a,b)$, we obtain:

$$\begin{aligned} |F(x) - F(a)| &= \left| \int_a^x f(t)\,dt - \int_a^a f(t)\,dt \right| \\ &= \left| \int_a^x f(t)\,dt \right| \\ &\le \int_a^x |f(t)|\,dt \\ &\le \int_a^x M\,dt \\ &= M|x-a|. \end{aligned}$$

Accordingly,

$$|F(x) - F(a)| \le M|x-a| \to 0 \quad \text{as } x \to a^+$$

Thus, $\lim_{x \to a^+} F(x) = F(a)$, and so F is continuous at a.

Similarly, F is also continuous at the endpoint b. We conclude thus the proof:)

Remark

For the moment, this result is pretty, but not very helpful if you want to find areas!

It will be very important soon as many functions are **defined as area functions**, e.g.

$$\ln(x) := \int_1^x \frac{1}{t}\,dt$$

$$\mathrm{Si}(x) := \int_0^x \frac{\sin t}{t}\,dt.$$

FTC1 tells us that these functions are continuous and differentiable.

Example. Find the following derivatives.

(1) $\dfrac{d}{dx}\displaystyle\int_0^x \exp(t^2)dt$

(2) $\dfrac{d}{dx}\displaystyle\int_x^0 \exp(t^2)dt$.

(3) $\dfrac{d}{dx}\displaystyle\int_{x^2}^x \exp(t^2)dt$.

Solution. Using FTC1, we obtain:
(1) $\frac{d}{dx}\int_0^x \exp(t^2)dt = \exp(x^2)$.
(2) $\frac{d}{dx}\int_x^0 \exp(t^2)dt = \frac{d}{dx}\left(-\int_0^x \exp(t^2)dt\right) = -\exp(x^2)$.

(3) Let

$$F(x) = \int_0^x \exp(t^2)dt.$$

Then

$$\begin{aligned}\int_{x^2}^x \exp(t^2)dt &= \int_{x^2}^0 \exp(t^2)dt + \int_0^x \exp(t^2)dt \\ &= \int_0^x \exp(t^2)dt - \int_0^{x^2} \exp(t^2)dt \\ &= F(x) - F(x^2).\end{aligned}$$

Hence

$$\begin{aligned}\frac{d}{dx}\int_{x^2}^x \exp(t^2)dt &= \frac{d}{dx}F(x) - \frac{d}{dx}F(x^2) \\ &= F'(x) - 2xF'(x^2) \\ &= \exp(x^2) - 2x\exp(x^4).\end{aligned}$$

Second fundamental theorem of calculus

There exists a fast way of calculating an integral if an explicit antiderivative is known.

The second fundamental theorem of calculus (FTC2)

Suppose that f is a continuous function on $[a, b]$. If F is an antiderivative of f on $[a, b]$, that is,

$$F'(x) = f(x)$$

for all $x \in [a, b]$, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Notation. We frequently use the notation

$$F(x)\Big|_a^b \qquad \text{or} \qquad \Big[F(x)\Big]_a^b$$

for the expression $F(b) - F(a)$.

Proof. By the Mean Value Theorem, we know that two antiderivatives can only differ by a constant and hence

$$F(x) = \int_a^x f(t)\,dt + C$$

for some constant C. Accordingly,

$$F(b) - F(a) = \int_a^b f(t)\,dt + C - (0 + C) = \int_a^b f(t)\,dt.$$

Example. Compute the area of the region bounded by the x-axis, the lines $x = 0$ and $x = 2$, and the function

$$f(x) = x(x-1).$$

Example. Compute the area of the region bounded by the x-axis, the lines $x = 0$ and $x = 2$, and the function

$$f(x) = x(x-1).$$

Solution. Note that $f(x) \leq 0$ on $[0,1]$, and thus the area of the region is

$$\begin{aligned}\int_0^2 |f(t)|dt &= -\int_0^1 f(t)dt + \int_1^2 f(t)dt \\ &= -\Big[\frac{t^3}{3} - \frac{t^2}{2}\Big]_0^1 + \Big[\frac{t^3}{3} - \frac{t^2}{2}\Big]_1^2 \\ &= 1.\end{aligned}$$

We come back to the question of whether integration and differentiation are inverse processes. If f is continuous on $[a, b]$, then FTC1 tells us that

$$f(x) = \frac{d}{dx}\left(\int_a^x f(t)\, dt\right).$$

Is the converse true, that is, if one differentiates f and then integrates, do we obtain f again?

This is **not** true (unless $f(a) = 0$), as the next consequence of FTC2 shows:

Corollary

Let f be continuous on $[a, b]$ with a continuous derivative on (a, b). Then

$$\int_a^x f'(t)dt = f(x) - f(a)$$

for all $x \in [a, b]$.

Proof. Follows from FTC2 applied to f' on the interval $[a, x]$.

Example. Find the continuous function f and the constant c satisfying

$$\int_0^x f(t)dt = \int_x^1 t^2 f(t)dt + \frac{x^{16}}{8} + \frac{x^{18}}{9} + c.$$

Example. Find the continuous function f and the constant c satisfying

$$\int_0^x f(t)dt = \int_x^1 t^2 f(t)dt + \frac{x^{16}}{8} + \frac{x^{18}}{9} + c.$$

Solution. By differentiating both sides with respect to x we obtain

$$f(x) = -x^2 f(x) + 2x^{15} + 2x^{17}.$$

Hence

$$(1+x^2)f(x) = 2x^{15}(1+x^2) \quad \text{or} \quad f(x) = 2x^{15}.$$

Let $x = 0$ and substitute $f(t) = 2t^{15}$ into the given equation. Then

$$0 = \int_0^1 2t^{17} dt + 0 + 0 + c.$$

It follows that

$$c = -\frac{1}{9}.$$

Indefinite integrals

If F is an antiderivative of f, then by the FTCs we have

$$\int_a^x f(t)dt = F(x) + C$$

for some suitable constant C.

If we are not interested in the interval $[a, x]$, we usually write

$$\int f(t)dt = F(t) + C$$

to denote a general antiderivative of f. This is called the indefinite integral of f.

The constant C is called the constant of integration.

Some familiar formulas

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \qquad n \neq -1$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C, \qquad a \neq 0$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C.$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C; \quad \int \frac{-dx}{\sqrt{1-x^2}} = \cos^{-1} x + C;$$

Integration by substitution

Question. What is the analogue of the chain rule in the context of integration?

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x)$$

Formally, if F is an antiderivative of a function f: $F'(g(x)) = f(g(x))$ and

$$\frac{d}{dx}F(g(x)) = f(g(x))g'(x),$$

hence

$$F(g(x)) + C = \int f(g(x))g'(x)\,dx.$$

Sometimes it is easy to recognise an integrand being of the form $F'(g(x))g'(x)$.

Example. Find $\int 3x^2e^{x^3}\,dx$ by inspection. We apply the above with $f(x) = e^x$ and $g(x) = x^3$. We obtain

$$\int 3x^2e^{x^3}\,dx = \int e^{x^3}\left(\frac{d}{dx}x^3\right)dx = \int \frac{d}{dx}\left(e^{x^3}\right)dx = e^{x^3} + C.$$

It is not always easy to recognise this form, in which case, the 'mechanical' procedure which might lead to a successful integration is called integration by substitution.

Thus, if we set

$$u = g(x)$$

then

$$du = g'(x)\, dx$$

and the integral becomes

$$\begin{aligned} \int f(g(x))g'(x)\, dx &= \int f(u)\, du \\ &= F(u) + C. \end{aligned}$$

Example. Evaluate $\displaystyle\int \frac{1}{x^2}\sqrt{1-\frac{4}{x}}\,dx$.

Let $u = 1 - \frac{4}{x}$. Then

$$\frac{du}{dx} = \frac{4}{x^2}, \quad \implies \frac{1}{x^2}dx = \frac{1}{4}du.$$

$$\begin{aligned}
\int \frac{1}{x^2}\sqrt{1-\frac{4}{x}}\,dx &= \int \sqrt{u}\frac{1}{4}\,du \\
&= \frac{1}{4}\int \sqrt{u}\,du \\
&= \frac{1}{4}\frac{2}{3}u^{3/2} + C \\
&= \frac{1}{6}\left(1-\frac{4}{x}\right)^{3/2} + C
\end{aligned}$$

for some constant C.

Note. The last step always consists of rewriting everything in terms of the original variable!

Example. Evaluate $\int \frac{1}{x^2}\sqrt{1-\frac{4}{x}}\,dx$.

Let $u = 1 - \frac{4}{x}$. Then

$$\frac{du}{dx} = \frac{4}{x^2}, \quad \Longrightarrow \frac{1}{x^2}dx = \frac{1}{4}du.$$

$$\begin{aligned}\int \frac{1}{x^2}\sqrt{1-\frac{4}{x}}\,dx &= \int \sqrt{u}\frac{1}{4}\,du \\ &= \frac{1}{4}\int \sqrt{u}\,du \\ &= \frac{1}{4}\frac{2}{3}u^{3/2} + C \\ &= \frac{1}{6}\left(1-\frac{4}{x}\right)^{3/2} + C\end{aligned}$$

for some constant C.

Note. The last step always consists of rewriting everything in terms of the original variable!

Example. Find $I = \displaystyle\int \frac{2x+1}{\sqrt{3x+2}}\,dx$. Set: $u = \sqrt{3x+2} \Rightarrow du = \dfrac{3}{2\sqrt{3x+2}}\,dx = \dfrac{3}{2u}\,dx, \qquad x = \dfrac{u^2-2}{3}$.
Hence,

$$I = \int \frac{1}{u}\left(\frac{2(u^2-2)}{3} + 1\right)\frac{2}{3}u\,du = \int \left(\frac{2}{3}u^2 - \frac{4}{3} + 1\right)\frac{2}{3}\,du$$

$$= \int \left(\frac{4}{9}u^2 - \frac{2}{9}\right)\,du = \frac{4}{27}u^3 - \frac{2}{9}u + C$$

$$= \frac{4}{27}(3x+2)^{3/2} - \frac{2}{9}\sqrt{3x+2} + C.$$

The precise statement in the case of definite integrals is the following:

Theorem

Suppose that g is a differentiable function such that g' is continuous on $[a, b]$. If f is continuous on any interval I containing $g(a)$ and $g(b)$ then the change of variables formula

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

holds.

Proof. Since f is continuous on I, it has an antiderivative $F : I \to \mathbb{R}$ by FTC1. By the chain rule we have

$$(F \circ g)'(x) = F'(g(x))\,g'(x) = f(g(x))g'(x).$$

Thus, applying FTC2 two times, we obtain

$$\begin{aligned}\int_a^b f(g(x))g'(x)\,dx &= (F \circ g)(b) - (F \circ g)(a)\\ &= F(g(b)) - F(g(a))\\ &= \int_{g(a)}^{g(b)} f(u)\,du.\end{aligned}$$

Example. Evaluate $\displaystyle\int_1^4 \frac{\sin(\pi\sqrt{x})}{\sqrt{x}}\,dx$.

Example. Evaluate $\int_1^4 \frac{\sin(\pi\sqrt{x})}{\sqrt{x}}\,dx$.

Solution. Let $u = \pi\sqrt{x}$. Then

$$\frac{du}{dx} = \frac{\pi}{2}\frac{1}{\sqrt{x}} \qquad \Longrightarrow \quad \frac{1}{\sqrt{x}}dx = \frac{2}{\pi}du$$

$$u(1) = \pi\sqrt{1} = \pi$$

$$u(4) = \pi\sqrt{4} = 2\pi$$

$$\begin{aligned}\int_1^4 \frac{\sin(\pi\sqrt{x})}{\sqrt{x}}\,dx &= \int_\pi^{2\pi} \sin(u)\frac{2}{\pi}\,du \\ &= \frac{2}{\pi}\int_\pi^{2\pi} \sin(u)\,du = \frac{2}{\pi}\left[-\cos(u)\right]_\pi^{2\pi} \\ &= \frac{2}{\pi}\left[-\cos(2\pi) + \cos(\pi)\right] \\ &= -\frac{4}{\pi}\end{aligned}$$

Example. Find $I = \displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$.

Note: Seeing a good substitution is partly art and partly science. The more you do, the easier it will become!

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Solution. Let $x = 2\sin u$, then

$$dx = 2\cos u\ du.$$

Now

$$4 - x^2 = 4 - 4\sin^2 u = (2\cos u)^2.$$

When $x = 0$, $u = 0$ and when $x = 1$, $\sin u = \frac{1}{2}$ and so $u = \frac{\pi}{6}$.
Thus

$$I = \int_0^{\pi/6} \frac{1}{2\cos u}\, 2\cos u\, du = \int_0^{\pi/6} 1\, du = \frac{\pi}{6}.$$

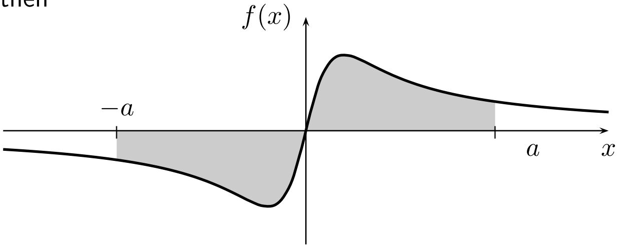
Note that here it is much harder to see this as integrating $f(g(x))\, g'(x)$.
(Check: here $f(x) = 1$ and $g(x) = \sin^{-1} \frac{x}{2}$)

Useful observations. Suppose that f is a continuous function and a is a real number.

(i) If f is even then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

(ii) If f is odd then



$$\int_{-a}^{a} f(x)\,dx = 0.$$

(iii) If f is periodic with period T then

$$\int_{a}^{a+T} f(x)\,dx = \int_{0}^{T} f(x)\,dx.$$

Proof. We prove only (ii) for odd functions, the rest follows similar lines. We have

$$\begin{aligned}
\int_{-a}^{a} f(x)dx &= \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx \\
&= \int_{a}^{0} f(-u)(-1)du + \int_{0}^{a} f(x)dx \qquad (u = -x) \\
&= \int_{0}^{a} f(-u)du + \int_{0}^{a} f(x)dx \\
&= -\int_{0}^{a} f(u)du + \int_{0}^{a} f(x)dx \qquad (f(-u) = -f(u)) \\
&= 0.
\end{aligned}$$

Integration by parts

Question. What is the analogue of the product rule

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

in the context of integration?

Integration of both sides leads to

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

Hence, we obtain the integration by parts formula

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

The corresponding formula for definite integrals reads

$$\int_a^b f(x)g'(x)\,dx = \Big[f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx.$$

Sometimes, it may be relatively easy to apply the above formulae...

Example. Find the indefinite integral

$$I = \int x \cos x \, dx.$$

Sometimes, it may be relatively easy to apply the above formulae...

Example. Find the indefinite integral

$$I = \int x \cos x \, dx.$$

Solution. It is evident that

$$\begin{aligned} I &= \int x \left(\frac{d}{dx} \sin x \right) dx \\ &= x \sin x - \int \left(\frac{d}{dx} x \right) \sin x \, dx \\ &= x \sin x + \cos x + C. \end{aligned}$$

... on other occasions, it may not be so easy:

Example. Find the definite integral

$$I = \int_1^e \ln x \, dx.$$

... on other occasions, it may not be so easy:

Example. Find the definite integral

$$I = \int_1^e \ln x \, dx.$$

Solution. Here we choose $f(x) = \ln x$ and $g(x) = x$. We obtain

$$\begin{aligned} I &= \int_1^e \ln x \cdot \left(\frac{d}{dx} x \right) dx \\ &= \left[x \ln x \right]_1^e - \int_1^e x \left(\frac{d}{dx} \ln x \right) dx \\ &= e - 0 - \int_1^e 1 dx = e - (e - 1) = 1. \end{aligned}$$

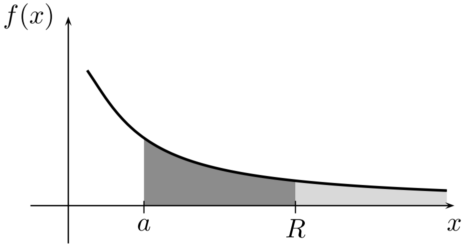
Improper integrals

So far we have discussed about integrals of bounded functions over **finite** intervals. However, for many applications (e.g. probability theory, quantum mechanics etc.), one needs to consider the improper integral

$$\int_a^\infty f(x)\,dx$$

over an infinite interval. But what is its exact definition?

To define this integral, we examine the behaviour of $\displaystyle\int_a^R f(x)dx$ as $R \to \infty$.



Definition

(a) Suppose that there exists a real number L such that

$$\int_a^R f(x)\,dx \xrightarrow[R\to\infty]{} L$$

Then, f is said to be integrable over $[a,\infty)$.

We say that the improper integral $\int_a^\infty f(x)\,dx$ is convergent and write

$$\int_a^\infty f(x)\,dx = L.$$

Definition (Cont.)

(b) Suppose that

$$\int_a^R f(x)\,dx$$

does not have a limit as $R \to \infty$.

Then, we say that f is not integrable over $[a, \infty)$

and the improper integral $\displaystyle\int_a^{\infty} f(x)\,dx$ is said to be divergent.

Remark. A similar definition for improper integrals of the form $\displaystyle\int_{-\infty}^{b} f(x)\,dx$ applies, that is,

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{R \to -\infty} \int_R^b f(x)\,dx.$$

Example. Evaluate the following improper integrals or show that they diverge.

1. $\displaystyle\int_0^{\infty} e^{-x}\,dx$;
2. $\displaystyle\int_0^{\infty} e^{3x}\,dx$.

Example. Evaluate the following improper integrals or show that they diverge.

- $\int_0^\infty e^{-x}dx$;
- $\int_0^\infty e^{3x}dx$.

Solution. (1) Let $R > 0$, then

$$\int_0^R e^{-x}dx = \left[-e^{-x}\right]_0^R = -e^{-R} + 1 \to 1 \quad \text{as} \quad R \to \infty.$$

Thus,

$$\int_0^\infty e^{-x}dx = 1.$$

(2) Let $R > 0$, then

$$\int_0^R e^{3x}dx = \left[\frac{1}{3}e^{3x}\right]_0^R = \frac{1}{3}e^{3R} - \frac{1}{3} \to \infty \quad \text{as} \quad R \to \infty.$$

Thus, $\int_0^\infty e^{3x}dx$ diverges.

What happens if the interval of integration is infinite in both directions?

Definition

We say that f is integrable over $(-\infty, \infty)$ if f is integrable over both $(-\infty, 0]$ and $[0, \infty)$. In this case, we write

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx.$$

If f is not integrable on either of the intervals $(-\infty, 0]$ or $[0, \infty)$ then we say that the improper integral

$$\int_{-\infty}^{\infty} f(x)\,dx$$

diverges.

Remark. The zeros on the right hand side could be replaced by any real number.

Example. Does the improper integral $\displaystyle\int_{-\infty}^{\infty} x\,dx$ converge?

Since

$$\int_0^R x\,dx = \left[\frac{1}{2}x^2\right]_0^R = \frac{R^2}{2} \xrightarrow[R\to\infty]{} \infty$$

the improper integral $\displaystyle\int_{-\infty}^{\infty} x\,dx$ diverges by definition.

Warning. Note that

$$\lim_{R\to\infty}\int_{-R}^{R} x\,dx = \lim_{R\to\infty} 0 = 0$$

and thus one might be tempted to say that $\int_{-\infty}^{\infty} x\,dx = 0$. However, this is wrong, this is just a geometric way of trying to calculate "$\infty - \infty$".

Example. Find $I = \displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

Example. Find $I = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

Solution. We have

$$\int_0^R \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right]_0^R = \tan^{-1} R \to \frac{\pi}{2},$$

as $R \to \infty$, while for $S < 0$

$$\int_S^0 \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right]_S^0 = -\tan^{-1} S \to \frac{\pi}{2},$$

as $S \to -\infty$.
Thus

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

converges and

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Theorem

Theorem (Convergence and divergence of p-integrals: "p-test")

The improper integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

is convergent if $p > 1$ and divergent if $p \leq 1$.

Convergence and divergence of p-integrals

Proof.

If $p \neq 1$ and $R > 1$ then

$$\int_1^R x^{-p}\,dx = \left[\frac{x^{1-p}}{1-p}\right]_1^R = \frac{R^{1-p}-1}{1-p}$$

$$\to \begin{cases} \frac{1}{p-1} & \text{when } 1-p<0 \\ \infty & \text{when } 1-p>0 \end{cases} \quad \text{as } R \to \infty,$$

and thus the integral converges if $p > 1$ and diverges if $p < 1$. If $p = 1$ then

$$\int_1^R \frac{1}{x}\,dx = \Big[\ln x\Big]_1^R = \ln R - \ln 1 \to \infty \quad \text{as } R \to \infty,$$

hence the integral diverges when $p = 1$.

Comparison tests for improper integrals

Some important improper integrals such as

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

do not admit 'elementary' antiderivatives, and thus deciding convergence via

$$\int_a^{\infty} f(x)dx = \lim_{R\to\infty}\int_a^R f(x)dx = \lim_{R\to\infty} F(R) - F(a),$$

where F is an antiderivative of f, is not working.

One may still be able to discuss convergence/divergence by comparing them with known improper integrals.

Theorem

Theorem (Comparison test)

Suppose that f and g are integrable functions and that for $x \geq a$:

$$0 \leq f(x) \leq g(x)$$

(i) If $\int_a^\infty g(x)\,dx$ converges then $\int_a^\infty f(x)\,dx$ converges.

(ii) If $\int_a^\infty f(x)\,dx$ diverges then $\int_a^\infty g(x)\,dx$ diverges.

Remark. When applying the comparison test, we often compare an improper integral I with the p-integral

$$\int_1^\infty \frac{1}{x^p}\,dx.$$

Note. If $\int_a^\infty g(x)\,dx$ diverges or $\int_a^\infty f(x)\,dx$ converges then the comparison test is useless.

Examples

Determine if the following improper integrals converge.

1) $\displaystyle\int_2^{\infty} \frac{x^2}{\sqrt{x^7+1}}dx;$

Examples

Determine if the following improper integrals converge.

1) $\int_2^\infty \frac{x^2}{\sqrt{x^7+1}}dx$;

Solution. (1) Since

$$\frac{x^2}{\sqrt{x^7+1}} \leq \frac{x^2}{\sqrt{x^7}} = \frac{1}{x^{3/2}}$$

and

$$\int_2^\infty \frac{1}{x^{3/2}}dx$$

converges (by the p-test), the improper integral $\int_2^\infty \frac{x^2}{\sqrt{x^7+1}}dx$ converges by the Comparison Test.

2) $\displaystyle\int_1^\infty \frac{1}{x^3+\cos^2 x}dx;$

3) $\displaystyle\int_2^\infty \frac{1}{x^{1/2}+1}dx;$

2) $\int_1^\infty \frac{1}{x^3+\cos^2 x} dx$;

3) $\int_2^\infty \frac{1}{x^{1/2}+1} dx$;

(2) At large values of x, $\cos^2 x$ is very small in comparison with x^3. So if $x \geq 1$ then

$$\cos^2 x \geq 0 \text{ and so } 0 \leq \frac{1}{x^3+\cos^2 x} \leq \frac{1}{x^3}.$$

We know $\int_1^\infty \frac{1}{x^3}\, dx$ converges (by the p-test), so by the Comparison Test $\int_1^\infty \frac{1}{x^3+\cos^2 x}\, dx$ converges.

(3) Since

$$\frac{1}{x^{1/2}+1} = \frac{1}{2x^{1/2}+(1-x^{1/2})} \geq \frac{1}{2x^{1/2}}, \quad x \geq 2,$$

and $\int_2^\infty \frac{1}{2x^{1/2}} dx$ diverges, the improper integral $\int_2^\infty \frac{1}{x^{1/2}+1} dx$ diverges.

Instead of using inequalities to estimate integrands, one often uses a 'dominant term analysis' such as

$$f(x) = \frac{\sqrt{\sin x + x^2}}{2x^4 - 1}$$

'behaves like'

$$g(x) = \frac{1}{2x^3}$$

for large x and hence one expects the convergence of the two associated improper integrals to be the same.

The precise formulation of this idea is as follows:

Theorem

Theorem (Limit form of the comparison test)

Suppose that f and g are nonnegative and bounded on $[a, \infty)$. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

and $0 < L < \infty$ then either

$$\text{both} \quad \int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx \quad \text{converge}$$

or

$$\text{both} \quad \int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx \quad \text{diverge.}$$

Example. Discuss the convergence of $\displaystyle\int_2^\infty \frac{\sqrt{3x^2+\sin x}}{x^4-2}\,dx$.

Here

$$f(x) = \frac{\sqrt{3x^2+\sin x}}{x^4-2}$$

behaves like $g(x) = \dfrac{1}{x^3}$ for large x, in the sense that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \sqrt{3}.$$

By the p-test $\displaystyle\int_2^\infty g(x)\,dx$ converges and so, by the limit form of the comparison test,

$$\int_2^\infty \frac{\sqrt{3x^2+\sin x}}{x^4-2}\,dx$$

converges.

Examples

Discuss the convergence of $\displaystyle\int_1^\infty \frac{3x^2 - x + 1}{\sqrt{4x^5 + 7}} dx$.

Examples

Discuss the convergence of $\int_1^\infty \frac{3x^2-x+1}{\sqrt{4x^5+7}}dx$.

Let $f(x) = \dfrac{3x^2 - x + 1}{\sqrt{4x^5+7}}$. Considering the highest powers of x in the numerator and denominator suggests we compare $f(x)$ with $g(x) = \dfrac{x^2}{\sqrt{x^5}} = \dfrac{1}{x^{1/2}}$.

$$\begin{aligned}\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{3x^2 - x + 1}{\sqrt{4x^5+7}} \Big/ \frac{1}{x^{1/2}} \\ &= \lim_{x\to\infty} \frac{(3x^2 - x + 1)\sqrt{x}}{\sqrt{4x^5+7}} \\ &= \lim_{x\to\infty} \frac{3 - x^{-1} + x^{-2}}{\sqrt{4 + 7x^{-5}}} = \frac{3}{2},\end{aligned}$$

which is not zero. We know that $\displaystyle\int_1^\infty \frac{1}{x^{1/2}}\,dx$ diverges, so by the limit form of the comparison test, $\displaystyle\int_1^\infty \frac{3x^2 - x + 1}{\sqrt{4x^5+7}}\,dx$ diverges too.

Summary 1: What did we learn in this chapter?

- Approximations of area using Riemann sums (p. 9)
- Partition of an interval (p. 17)
- Riemann sums: upper and lower (p. 19)
- Riemann sums and area (p. 21)
- Signed and unsigned area (p. 24)
- Piecewise continuous functions (p. 27)
- Properties of Riemann integrals (p. 30 and p. 31)
- First fundamental theorem of calculus (p. 35)
- Second fundamental theorem of calculus (p. 45)
- Second fundamental theorem of calculus, corollary (p. 48)

Summary 2: What did we learn in this chapter?

- Indefinite integrals (p. 50)
- Familiar formulas for indefinite integrals (p. 51)
- Integration by substitution (p. 53 and p. 55)
- Integration by part (p. 60)
- Improper integrals, convergent and divergent (p. 64 and p. 65)
- Integrable functions over $(-\infty, \infty)$ (p. 67)
- P-test (p. 70)
- Comparison tests for improper integrals (p. 73)
- Limit form of the Comparison tests (p. 77)