LECTURE 11
Inverse Functions

To find f^{-1} swap and solve.

$$(f^{-1} \circ f)(x) = x \qquad \text{and} \qquad (f \circ f^{-1})(x) = x \ .$$

Dom(f)=Range(f^{-1}) and Range(f)=Dom(f^{-1})

The graph of f^{-1} is the graph of f reflected in the line $y = x$.

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

The concept of a function is all about transformation. For example the incredibly simple function $y = f(x) = 2x + 5$ transforms $x = 1$ to $y = 7$ and $x = 4$ to $y = 13$. Whenever there is change however, we are also interested in undoing that change. This is accomplished through the use of the inverse function f^{-1} whose sole job is to undo whatever f did. That is $f^{-1}(7) = 1$ and $f^{-1}(13) = 4$. We can find the equation for f^{-1} by swapping y and x and solving for y. Note that in general $f^{-1} \neq \dfrac{1}{f}$!!.

Example 1: Find a formula for f^{-1} for the function $y = f(x) = 2x + 5$ defined above. Check that $f^{-1}(7) = 1$ and $f^{-1}(13) = 4$.

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Fact: $(f^{-1} \circ f)(x) = x$ for all $x \in \text{Dom}(f)$.

Discussion:

Example 2: Check that $(f^{-1} \circ f)(x) = x$ for f in example 1.

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Example 3: Find a formula for f^{-1} for the function $y = f(x) = 3e^{2x}$ and check that $(f^{-1} \circ f)(x) = x$.

★ $f^{-1}(x) = \frac{1}{2}\ln(\frac{x}{3})$ ★

We do however have a technical problem when it comes to inverses. Consider the function $y = g(x) = x^2$. Then $g(-3) = 9$ and $g(3) = 9$.

What is $g^{-1}(9)$? Is it 3 or -3. Clearly its both and a function must never give us such a choice! What this means is that for $y = g(x) = x^2$, g^{-1} is a relation rather than a function. This is not the end of the world but we would prefer that this didn't happen.

Definition: A function f is said to be 1-1 if

$$f(a) = f(b) \to a = b.$$

We will see later that if f is 1-1 then it will always have a unique inverse function f^{-1}.

Example 4: Prove that $y = f(x) = 2x + 5$ is 1-1 and that $y = g(x) = x^2$ is not 1-1.

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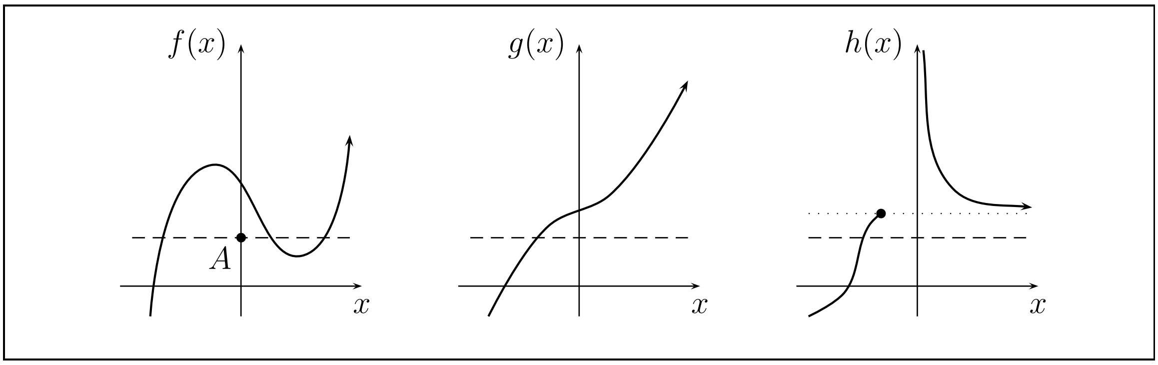
1-1 functions may also be identified graphically via the horizontal line test. Recall that the vertical line test established whether or not a relation was a function. The horizontal line test works in much the same way and tests whether or not a function has an inverse:

The Horizontal Line Test: A function f is 1-1 if and only if every horizontal line cuts the graph of f at most once.

Consider the functions graphed below.



f is not one-to-one because the dotted horizontal line passing through the point A cuts the graph of f more than once;

g is one-to-one (in fact, since g is increasing, every horizontal line can cut the graph of g graph no more than once);

h is also one-to-one (even though it is not always increasing).

Fact: A 1-1 function f (that is a function which passes the horizontal line test) will have a unique inverse function f^{-1}.

Some other facts regarding inverse functions:

- Dom(f)=Range(f^{-1}) and Range(f)=Dom(f^{-1})

- The graph of f^{-1} is the graph of f reflected in the line $y = x$.

- $(f^{-1} \circ f)(x) = x$ for all $x \in$ Dom(f).

Example 5: Consider the function $y = f(x) = x^2 + 5$.

a) Explain why the function f^{-1} does not exist.

b) Restrict Dom(f) so that f becomes a 1-1 function g which has an inverse.

c) If $f = g$?

d) Sketch the restricted function g and its inverse g^{-1} on the same set of axes.

e) Write down Dom(g), Range(g), Dom(g^{-1}) and Range(g^{-1}) in interval notation.

f) Find g^{-1}.

g) Show that $(g^{-1} \circ g)(x) = x$.

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Example 6: Let $f(x) = 2x^5 + x^3 + x - 10$. Prove that f has an inverse function.

We have a problem here! The sketch is unclear and it is difficult to prove algebraically that f is 1-1. But we have one extra trick:

Fact: An increasing or decreasing function is 1-1.

Discussion:

Now $f'(x) =$

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We close with the an important result which helps us to find the derivative of the inverse:

Fact: If f is differentiable and has an inverse f^{-1} then the derivative of the inverse $(f^{-1})'$ is given by

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

In other words the derivative of the inverse is one over the derivative of the original function evaluated at the inverse.

Proof: We start with $f(f^{-1}(x)) = x$. Differentiating both sides and using the chain rule yields:

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Example 7: Let $f(x) = 3x + \cos(x)$. Show that f^{-1} exists on $\mathbb{R}$ and without actually finding f^{-1} evaluate $(f^{-1})'(1)$.

Since $f'(x) = 3 - \sin(x)$ we have $f'(x) > 0$ for all $x \in \mathbb{R}$ and hence f is an increasing function implying that f has an inverse.

Note also that the Range of f is $\mathbb{R}$ which is in turn the Domain of f^{-1}.

Now $(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))}$

What is $f^{-1}(1)$????

$\bigstar \quad \frac{1}{3} \quad \bigstar$