LECTURE 11
Inverse Functions

> To find f^{-1} swap and solve.
>
> $$(f^{-1} \circ f)(x) = x \qquad \text{and} \qquad (f \circ f^{-1})(x) = x \ .$$
>
> Dom(f)=Range(f^{-1}) and Range(f)=Dom(f^{-1})
>
> The graph of f^{-1} is the graph of f reflected in the line $y = x$.
>
> $$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

The concept of a function is all about transformation. For example the incredibly simple function $y = f(x) = 2x + 5$ transforms $x = 1$ to $y = 7$ and $x = 4$ to $y = 13$. Whenever there is change however, we are also interested in undoing that change. This is accomplished through the use of the inverse function f^{-1} whose sole job is to undo whatever f did. That is $f^{-1}(7) = 1$ and $f^{-1}(13) = 4$. We can find the equation for f^{-1} by swapping y and x and solving for y. Note that in general $f^{-1} \neq \dfrac{1}{f}$!!.

Example 1: Find a formula for f^{-1} for the function $y = f(x) = 2x + 5$ defined above. Check that $f^{-1}(7) = 1$ and $f^{-1}(13) = 4$.

$f(1) = 7, \quad f(4) = 13.$

$f^{-1} \qquad f^{-1}$

$$y = 2x + 5$$
$$x = 2y + 5$$
$$2y = x - 5$$
$$y = \frac{x}{2} - 5/2 \Rightarrow f^{-1}(x) = \frac{x}{2} - 5/2$$

check

$$f^{-1}(7) = \frac{7}{2} - 5/2 = 2/2 = 1 \checkmark$$
$$f^{-1}(13) = \frac{13}{2} - 5/2 = 8/2 = 4 \checkmark$$

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Fact: $(f^{-1} \circ f)(x) = x$ for all $x \in \mathrm{Dom}(f)$.

Discussion:

$$(f^{-1} \circ f)(x) = f^{-1}(f(x))$$

$$(f \circ f^{-1})(x) \quad = x.$$

Example 2: Check that $(f^{-1} \circ f)(x) = x$ for f in example 1.

$$f(x) = 2x+5, \quad f^{-1}(x) = \frac{x}{2} - 5/2$$

$$f^{-1}(f(x)) = f^{-1}(2x+5)$$

$$= \frac{2x+5}{2} - 5/2 = x + 5/2 - 5/2 = \underline{x}$$

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Example 3: Find a formula for f^{-1} for the function $y = f(x) = 3e^{2x}$ and check that $(f^{-1} \circ f)(x) = x$.

$$y = 3e^{2x}.$$

$$x = 3e^{2y}$$

$$\frac{x}{3} = e^{2y}$$

$$\ln(\frac{x}{3}) = \ln(e^{2y})$$

$$2y = \ln(x/3) \Rightarrow y = \frac{1}{2}\ln(\frac{x}{3})$$

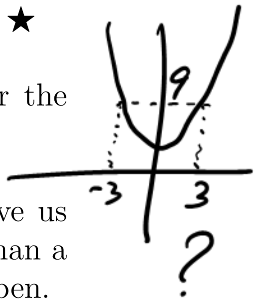
$$\therefore f^{-1}(x) = \frac{1}{2}\ln(x/3)$$

$$(f^{-1} \circ f)(x) = f^{-1}(f(x))$$

$$= f^{-1}(3e^{2x}) = \frac{1}{2}\ln(\frac{3e^{2x}}{3})$$

$$= \frac{1}{2}\ln(e^{2x}) = \frac{1}{2}(2x) = \underline{\underline{x}}$$

$$\star \quad f^{-1}(x) = \frac{1}{2}\ln(\frac{x}{3}) \quad \star$$



We do however have a technical problem when it comes to inverses. Consider the function $y = g(x) = x^2$. Then $g(-3) = 9$ and $g(3) = 9$.

What is $g^{-1}(9)$? Is it 3 or -3. Clearly its both and a function must never give us such a choice! What this means is that for $y = g(x) = x^2$, g^{-1} is a relation rather than a function. This is not the end of the world but we would prefer that this didn't happen.

Definition: A function f is said to be 1-1 if

$$f(a) = f(b) \rightarrow a = b.$$

We will see later that if f is 1-1 then it will always have a unique inverse function f^{-1}.

Example 4: Prove that $y = f(x) = 2x + 5$ is 1-1 and that $y = g(x) = x^2$ is not 1-1.

$f(a) = f(b)$
$\Rightarrow 2a + 5 = 2b + 5$
$\Rightarrow 2a = 2b$
$\Rightarrow a = b$
$\therefore f$ is 1-1

$g(a) = g(b)$
$\Rightarrow a^2 = b^2$
$\not\Rightarrow a = b.$
$\therefore g$ is <u>not</u> 1-1

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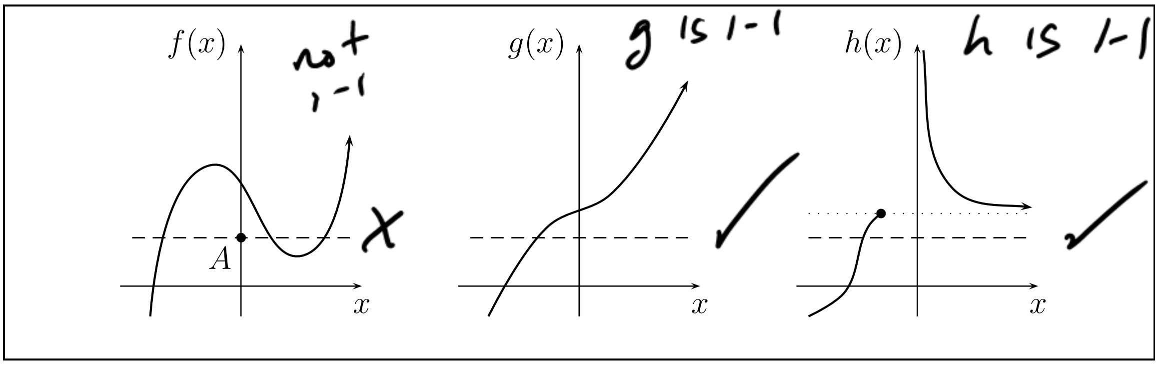
1-1 functions may also be identified graphically via the horizontal line test. Recall that the vertical line test established whether or not a relation was a function. The horizontal line test works in much the same way and tests whether or not a function has an inverse:

The Horizontal Line Test: A function f is 1-1 if and only if every horizontal line cuts the graph of f at most once.

Consider the functions graphed below.



f is not one-to-one because the dotted horizontal line passing through the point A cuts the graph of f more than once;

g is one-to-one (in fact, since g is increasing, every horizontal line can cut the graph of g graph no more than once);

h is also one-to-one (even though it is not always increasing).

Fact: A 1-1 function f (that is a function which passes the horizontal line test) will have a unique inverse function f^{-1}.

Some other facts regarding inverse functions:

- Dom(f)=Range(f^{-1}) and Range(f)=Dom(f^{-1})
- The graph of f^{-1} is the graph of f reflected in the line $y = x$.
- $(f^{-1} \circ f)(x) = x$ for all $x \in \text{Dom}(f)$.

Example 5: Consider the function $y = f(x) = x^2 + 5$.

a) Explain why the function f^{-1} does not exist.

b) Restrict Dom(f) so that f becomes a 1-1 function g which has an inverse.

c) If $f = g$? No (domains different!)

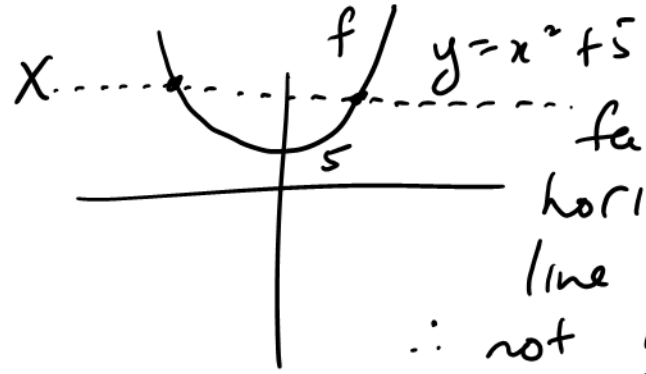
d) Sketch the restricted function g and its inverse g^{-1} on the same set of axes.

e) Write down Dom(g), Range(g), Dom(g^{-1}) and Range(g^{-1}) in interval notation.

f) Find g^{-1}.

g) Show that $(g^{-1} \circ g)(x) = x$.

a)



fails horizontal line test

$\therefore$ not 1-1

$\therefore$ no f^{-1}

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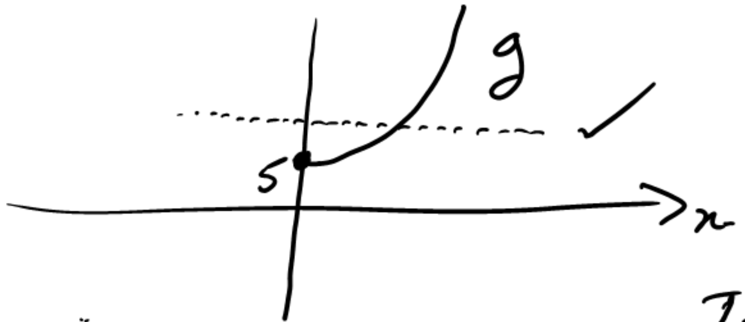
b) d)



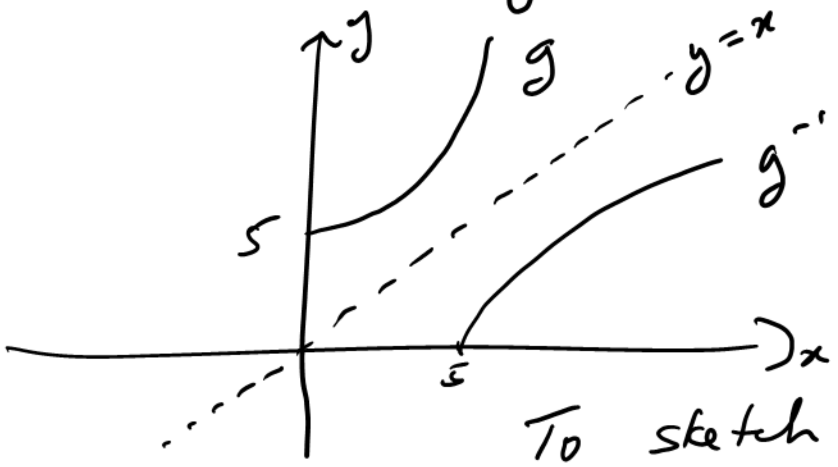
passes hor. line test

It is 1-1

$\therefore$ g^{-1} exists

e) $\mathrm{Dom}(g) = [0, \infty) = \mathrm{Range}(g^{-1})$

$\mathrm{Range}(g) = [5, \infty) = \mathrm{Dom}(g^{-1})$



To sketch g^{-1} reflect the graph of g in the line $y = x$

f) $y = x^2+5$

$x = y^2+5 \Rightarrow y^2 = x-5$

$\Rightarrow y = \pm\sqrt{x-5}$

$\therefore \ \underline{\underline{g^{-1}(x) = \sqrt{x-5}}}$ ✓

g) $\underline{(g^{-1} \circ g)(x)} = g^{-1}(g(x))$ $\qquad x \geq 0$

$= g^{-1}(x^2+5)$

$= \sqrt{(x^2+5)-5}$

$= \sqrt{x^2} = \underline{x}$ ✓

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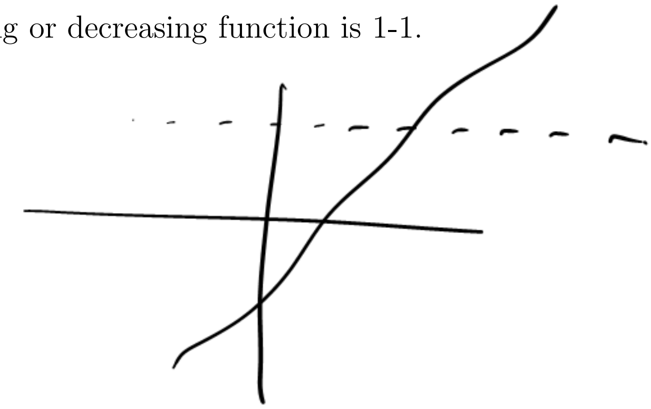
Example 6: Let $f(x) = 2x^5 + x^3 + x - 10$. Prove that f has an inverse function.

We have a problem here! The sketch is unclear and it is difficult to prove algebraically that f is 1-1. But we have one extra trick:

Fact: An increasing or decreasing function is 1-1.

Discussion:



✓ passes hor. line test.

Now $f'(x) =$

$10x^4 + 3x^2 + 1 \geq 0$

in fact ≥ 1

$\therefore f'(x) > 0 \Rightarrow f$ increasing

$\therefore f^{-1}$ exists

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We close with the an important result which helps us to find the derivative of the inverse:

Fact: If f is differentiable and has an inverse f^{-1} then the derivative of the inverse $(f^{-1})'$ is given by

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f' \text{ at } f^{-1}(x)}$$

In other words the derivative of the inverse is one over the derivative of the original function evaluated at the inverse.

Proof: We start with $f(f^{-1}(x)) = x$. Differentiating both sides and using the chain rule yields:

$$f(f^{-1}(x)) = x.$$

$\frac{d}{dx}$ both sides: $f'(f^{-1}(x))(f^{-1}(x)') = 1$

$$\Rightarrow [f^{-1}(x)]' = \frac{1}{f'(f^{-1}(x))}$$ ★

Example 7: Let $f(x) = 3x + \cos(x)$. Show that f^{-1} exists on $\mathbb{R}$ and without actually finding f^{-1} evaluate $(f^{-1})'(1)$.

f is increasing $\therefore f^{-1}$ exists

Since $f'(x) = 3 - \sin(x)$ we have $f'(x) > 0$ for all $x \in \mathbb{R}$ and hence f is an increasing function implying that f has an inverse.

Note also that the Range of f is $\mathbb{R}$ which is in turn the Domain of f^{-1}.

Now $(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))}$

What is $f^{-1}(1)$????

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$$

But

$$f(x) = 3x + \cos(x)$$

$$f(0) = 3(0) + \cos(0) = 1$$

$$\therefore f(0) = 1$$

$$\therefore f^{-1}(1) = 0$$

$$(f^{-1})'(1) = \frac{1}{f'(0)}$$

$$f(x) = 3x + \cos(x)$$

$$f'(x) = 3 - \sin(x) \Rightarrow f'(0) = 3.$$

$$\therefore \text{answer} = \frac{1}{f'(0)} = \frac{1}{3}$$

★ $\frac{1}{3}$ ★