School of Mathematics and Statistics
Math1131-Algebra

Lec09: Complex numbers: Introduction

Laure Helme-Guizon (Dr H)
Laure@unsw.edu.au
Jonathan Kress
j.kress@unsw.edu.au

Red-Centre, Rooms 3090 and 3073

2020 Term 1

Numbers and polynomial equations

When you first learnt to count you used the natural numbers

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$

Numbers and polynomial equations

When you first learnt to count you used the natural numbers

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$

but soon you found you couldn't solve equations like

$$x + 5 = 3.$$

Numbers and polynomial equations

When you first learnt to count you used the natural numbers

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$

but soon you found you couldn't solve equations like

$$x + 5 = 3.$$

So you learnt about negative numbers and the integers

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Numbers and polynomial equations

When you first learnt to count you used the natural numbers

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$

but soon you found you couldn't solve equations like

$$x + 5 = 3.$$

So you learnt about negative numbers and the integers

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}.$$

But, even though multiplication is defined for both $\mathbb{N}$ and $\mathbb{Z}$, these numbers can not be used to solve equations like

$$3x = 5.$$

Numbers and polynomial equations

When you first learnt to count you used the natural numbers

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$

but soon you found you couldn't solve equations like

$$x + 5 = 3.$$

So you learnt about negative numbers and the integers

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}.$$

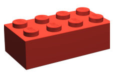
But, even though multiplication is defined for both $\mathbb{N}$ and $\mathbb{Z}$, these numbers can not be used to solve equations like

$$3x = 5.$$

So you graduated to the rational numbers

$$\mathbb{Q} = \left\{ \frac{p}{q} \;:\; \mathsf{gcd}(p, q) = 1,\; q \neq 0 \right\}.$$

Fields



Definition of a field

A field $\mathbb{F}$ is a set of with two operations, $+$ and $\times$, that satisfies the following properties for all $x, y, z \in \mathbb{F}$:

1. Associative laws: $(x + y) + z = x + (y + z)$ and $x(yz) = (xy)z$
2. Commutative laws: $x + y = y + x$ and $xy = yx$
3. Distributive law: $x(y + z) = xy + xz$
4. Existence of 0: There is a 0 such that $0 + x = x + 0 = x$
5. Existence of 1: There is a (non-zero) 1 such that $1x = x$
6. Existence of negatives: There is $-x$ such that $x + (-x) = 0$
7. Existence of inverses: If $x \neq 0$ there is x^{-1} such that $x^{-1}x = 1$

Example 1.

$\{\ldots, -2, -1, 0, 1, 2 \ldots\}$

a) The integers $\mathbb{Z}$ .... (are / are not) a field.

b) The rational numbers $\mathbb{Q}$ .... (are / are not) a field.

"closed under . . . " meaning

Note that for $+$ and $\times$ to be operations on $\mathbb{F}$, their result must be in $\mathbb{F}$. We say that $\mathbb{F}$ is *closed* under $+$ and $\times$.

Exercise 2.

a) Is the interval $I = [0, \infty)$ closed under $\overset{\text{yes}}{+}$ and $\overset{\text{yes}}{\times}$?

b) Is it a field? NO

$3 \in I$ but it has no negative in I

$3 + \underbrace{\boxed{}}_{\in I} \geq 3 > 0$

Exercise 3. What about $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$?

a) Is it closed under $+$ and $\times$?

$\times$: NO: $9 \times 9 = 81 \notin S$

$+$: NO: $9 + 8 = 17 \notin S$

b) Is it a field?

NO, it is not closed under $+$

Exercise 4. [Left to the reader] Show that $\mathbb{Q}$ is closed under $+$ and $\times$.

Real numbers

Can you solve

$$x^2 = 2?$$

Pythagoras knew this could not be solved using rational numbers but kept this fact secret!

Real numbers

Can you solve

$$x^2 = 2?$$

Pythagoras knew this could not be solved using rational numbers but kept this fact secret!

To solve this equation we need a field called the *real numbers* $\mathbb{R}$ which fills in the gaps, such as $\sqrt{2}$, in the rationals.

Real numbers

Can you solve

$$x^2 = 2?$$

Pythagoras knew this could not be solved using rational numbers but kept this fact secret!

To solve this equation we need a field called the *real numbers* $\mathbb{R}$ which fills in the gaps, such as $\sqrt{2}$, in the rationals.

Happy now? Can we solve all the equations we want to solve?

Real numbers

Can you solve

$$x^2 = 2?$$

Pythagoras knew this could not be solved using rational numbers but kept this fact secret!

To solve this equation we need a field called the *real numbers* $\mathbb{R}$ which fills in the gaps, such as $\sqrt{2}$, in the rationals.

Happy now? Can we solve all the equations we want to solve?

What about

$$x^2 = -1?$$

There is no real number solution!

Complex numbers

There is no real number solution to $\boldsymbol{x^2 = -1}$,
but we can extend out number system again by introducing the imaginary unit i and thinking of this as $i = \sqrt{-1}$ (although this notation should be avoided)
In other words, the square of i is -1, that is $\boldsymbol{i^2 = -1}$

Complex numbers

There is no real number solution to $\boldsymbol{x^2 = -1}$,
but we can extend out number system again by introducing the imaginary unit i and thinking of this as $i = \sqrt{-1}$ (although this notation should be avoided)
In other words, the square of i is -1, that is $\boldsymbol{i^2 = -1}$

What do we gain by extending our number system?

- We can now solve $x^2 = -1$, but also $x^2 = -24$ and even *any* quadratic equation.
- Using complex numbers, it will be possible to factorise any polynomial. Yes, any!

Complex numbers

There is no real number solution to $\boldsymbol{x^2 = -1}$,
but we can extend out number system again by introducing the imaginary unit i and thinking of this as $i = \sqrt{-1}$ (although this notation should be avoided)
In other words, the square of i is -1, that is $\boldsymbol{i^2 = -1}$

What do we gain by extending our number system?

- We can now solve $x^2 = -1$, but also $x^2 = -24$ and even *any* quadratic equation.
- Using complex numbers, it will be possible to factorise any polynomial. Yes, any!

What do we lose?

Complex numbers

There is no real number solution to $\boldsymbol{x^2 = -1}$,
but we can extend out number system again by introducing the imaginary unit i and thinking of this as $i = \sqrt{-1}$ (although this notation should be avoided)
In other words, the square of i is -1, that is $\boldsymbol{i^2 = -1}$

What do we gain by extending our number system?

- We can now solve $x^2 = -1$, but also $x^2 = -24$ and even *any* quadratic equation.
- Using complex numbers, it will be possible to factorise any polynomial. Yes, any!

What do we lose?

- sign
- ordering

Algebra with Complex numbers

Algebra with Complex numbers

THEORY PRACTICE

Calculation with i

We treat i like a variable but replace i^2 with -1 since $\boldsymbol{i^2 = -1}$.

Example 5.

a)

$$\begin{aligned}(3+2i)+(5-4i) &= 3+5+(2-4)i \\ &= 8-2i\end{aligned}$$

b)

$$\begin{aligned}(3+2i)(5-4i) &= 3\times 5+(3\times(-4)+2\times 5)i+2\times(-4)i^2 \\ &= 15+(-12+10)i-8\times(-1) \\ &= 15+8-2i \\ &= 23-2i.\end{aligned}$$

Algebra with Complex numbers

$\boldsymbol{i^2 = -1}$

Example 6. Expand and simplify $z = -i + (2+i)(1-3i) - 5$

$$z = -i + 2\times(-3)i + 2\times 1 + i\times 1 + i\times(-3i) - 5$$
$$= 2 - 5 + (-1 - 6 + 1)i - 3i^2$$
$$= -3 - 6i - 3\times(-1)$$
$$= -6i$$

Exercise 7. Simplify $1, i, i^2, i^3, i^4, i^5, i^6, i^7, i^8, i^9, \ldots$

1

i

$i^2 = -1$

$\times i$

$i^3 = -i$

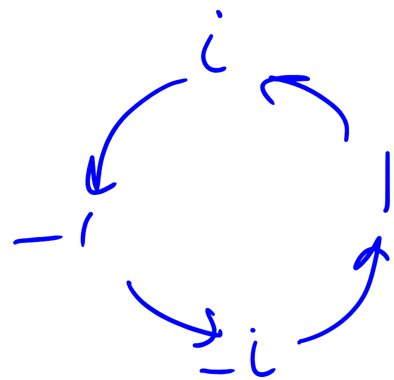
$\times i$

$i^4 = 1$

$i^5 = i$

$i^6 = -1$ $\times i$

i^7



With complex numbers, we can now solve ANY quadratic equation

We introduced i to solve the quadratic equation $x^2 = -1$.
Can we solve other quadratic equations?

Exercise 8. What about $x^2 = -9$?

$$x^2 = -9 = (3i)^2$$

$$\boxed{x = 3i \quad \text{or} \quad x = -3i}$$

Exercise 9.
Solve in $\mathbb{C}$ the equation $\quad z^2 + 2z + 3 = 0 \quad$ by "completing the square".

$$z^2 + 2z + 3 = (z+1)^2 + 2 = 0$$

$$\iff (z+1)^2 = -2$$

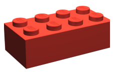
$$\iff z+1 = \sqrt{2}i \text{ or } -\sqrt{2}i$$

$$\iff \boxed{z} = -1+\sqrt{2}i \text{ or } -1-\sqrt{2}i$$

$$= \boxed{-1 \pm \sqrt{2}i}$$

Complex numbers

The field of complex numbers $\mathbb{C}$



The set of complex numbers

$$\mathbb{C} = \{a + bi \ : \ a, b \in \mathbb{R}\}$$

along with the operations of $+$ and $\times$ defined by

$$(a + bi) + (c + di) \stackrel{def}{=} (a + c) + (b + d)i$$

and

$$(a + bi) \times (c + di) \stackrel{def}{=} ac - bd + (ad + bc)i$$

forms a field.

Exercise 10. Check the field properties!

Therefore, subtraction is:

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

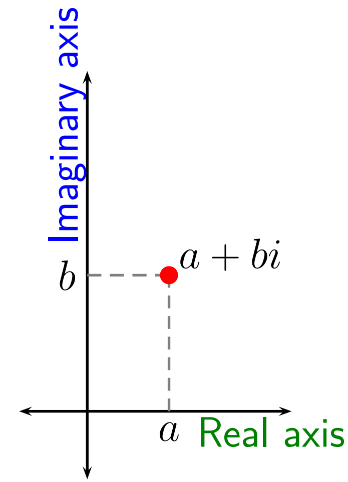
We will discuss division in the next lecture.

Representing complex numbers in the plane

- The set of complex numbers

$$\mathbb{C} = \{a + bi \ : \ a, b \in \mathbb{R}\}$$

is often represented as a plane called the Argand diagram or complex plane.



- If a and b are real numbers then the complex number $z = a + bi$ has real part and imaginary part

$$\text{Re}(z) = a \quad \text{and} \quad \text{Im}(z) = b.$$

The imaginary part is a *real* number, it is "b", *not* "bi".

Exercise 11. Find the real and imaginary parts of (a) $2 + 3i$, (b) $6i$, (c) -2.

(a) $\text{Re}(2+3i) = 2$ | (b) $\text{Re}(6i) = 0$ | (c) $\text{Re}(-2) = -2$

$\text{Im}(2+3i) = 3$ | $\text{Im}(6i) = 6$ | $\text{Im}(-2) = 0$

If a point in the plane has coordinates $(1, 3)$, we can store its coordinates in the complex number $1 + 3i$ and identify the point in the plane and the complex number.

Exercises

Exercise 12. For $z = 2 + 3i$ and $w = 4 - 7i$ evaluate:

(a) $z + w$

(b) $5z + 6w$

(c) $z - 2w$

(d) zw

(c) $z - 2w = (2+3i) - 2(4-7i)$

$= 2 - 8 + i(3+14)$

$= -6 + 17i$

UNSW SYDNEY

Exercises

Exercise 13. Solve $z^2 - 6z + 34 = 0.$

$$z^2 - 6z + 34 = 0$$

$$(z-3)^2 - 9 + 34 = 0$$

$$(z-3)^2 = -25 = (5i)^2$$

$$z - 3 = \pm 5i$$

$$\boxed{z = 3 \pm 5i}$$

Exercises

Exercise 14. Show that $z = 2 + i$ is a solution of the cubic equation

$$z^3 - 5z^2 + 9z - 5 = 0.$$

$(a+b)^2 = a^2 + 2ab + b^2$

$n=2 \quad 1 \quad 2 \quad 1$

$n=3 \quad 1 \quad 3 \quad 3 \quad 1$

Sub $z = 2+i$ in the LHS.

$z^3 = (2+i)^3$

$= 2^3 + 3 \times 2^2 \times i^1 + 3 \times 2^1 (i)^2 + i^3$

$= 8 + 12i - 6 - i$

$= 2 + 11i$

Maple

```
> # In Maple, the imaginary unit is I not i.

  z := 2 + 3*I;
  w := 4 - 7*I;

  z+w;
  5*z + 6*w;
  z - 2*w;
  z*w;
```

$$z := 2 + 3\,\mathrm{I}$$
$$w := 4 - 7\,\mathrm{I}$$
$$6 - 4\,\mathrm{I}$$
$$34 - 27\,\mathrm{I}$$
$$-6 + 17\,\mathrm{I}$$
$$29 - 2\,\mathrm{I}$$

```
> # Use restart to clear Maple's memory.

  restart:

  eqn := z^2 - 6*z + 34 = 0;
  solve(%);
```

$$eqn := z^2 - 6\,z + 34 = 0$$
$$3 + 5\,\mathrm{I},\ 3 - 5\,\mathrm{I}$$

```
> # Check that 2 + I is a root.

  p := z^3 - 5*z^2 + 9*z - 5;
  subs(z = 2 + I, p);
```

$$p := z^3 - 5\,z^2 + 9\,z - 5$$
$$0$$