# Chapter 7: Curve Sketching

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MATH1131

UNSW

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# Goal of the chapter

You all know how to sketch the graph of $y = x^2 - 4x$ or $y = \frac{1}{x}$.

In this section we will look at

- Additional information you can put into a sketch for complicated functions (Cartesian form) .
- Implicitly defined curves, such as $x^2 + \frac{y^2}{4} = 1$.
- Curves given by a parameter, such as

$$x(t) = \sin t \cos t \ln |t|, \quad y(t) = \sqrt{|t|}\,\cos t$$

$$t \in [-1, 1], \quad t \neq 0.$$

- Curves in polar coordinates, such as

$$x = r\cos\theta, \quad y = r\sin\theta, \qquad \text{where} \qquad r = \cos 4\theta.$$

# Curves defined by a Cartesian equation $y = f(x)$

Many high school students always start curve sketching by differentiating $f$.

Instead of doing this, use the following checklist:

- The domain of $f$.
- What are the $x$ and $y$-intercepts?
- Does $f$ have any symmetries?
- How does $f(x)$ behave as $x \to \infty$ or $x \to -\infty$?
- Are there any asymptotes?
- Identify critical points;
- Find where the function is increasing or decreasing.

Some of these may be irrelevant, or very hard, for some functions, but you should think about them all before you proceed too far.

# Asymptotic behaviour of a curve $y = f(x)$

An asymptote is a straight line which is approached closer and closer by the curve.

There are three types of asymptotes:

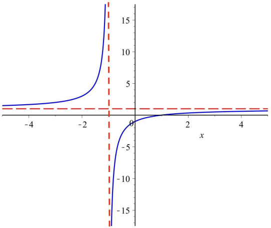
- The line $x = a$ is a vertical asymptote if

$$f(x) \to \pm\infty \quad \text{as } x \to a^+ \text{ or } x \to a^-.$$

- The line $y = b$ is a horizontal asymptote if

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$

**Example:** The asymptotes of the rational function $f(x) = \frac{x-1}{x+1}$



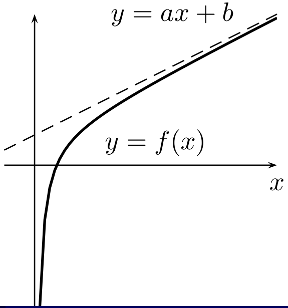
# Oblique asymptotes

The line $y = ax + b$ is an oblique asymptote if

$$\lim_{x\to\infty} (f(x) - (ax + b)) = 0 \quad \text{or} \quad \lim_{x\to-\infty} (f(x) - (ax + b)) = 0.$$

To find the oblique asymptotes we can write

$$f(x) = ax + b + \epsilon(x) \quad \text{with} \quad \lim_{x\to\pm\infty} \epsilon(x) = 0.$$



# Remarks:

- To find $a$ and $b$ when the oblique asymptotes exist we can compute

$$a = \lim_{x\to\infty} \frac{f(x)}{x} \quad \text{or} \quad a = \lim_{x\to-\infty} \frac{f(x)}{x}$$

and

$$b = \lim_{x\to\infty} (f(x) - ax) \quad \text{or} \quad b = \lim_{x\to-\infty} (f(x) - ax).$$

- A horizontal asymptote is a special case of an oblique asymptote with $a = 0$.
- If $f$ is a rational function with

$$f(x) = \frac{p(x)}{q(x)}, \quad \deg(p) = \deg(q) + 1$$

then the oblique asymptotes of $f$ may be determined by polynomial division.

# Example

Find the oblique asymptotes to the function $f$ defined by

$$f(x) = \frac{x^2 - x}{x - 2}, \quad \text{for all} \quad x \neq 2.$$

**Solution.** Dividing polynomials, we have

$$f(x) = \frac{x^2 - x}{x - 2} = \frac{(x-2)(x+1) + 2}{x - 2} = x + 1 + \frac{2}{x - 2}.$$

Since $\dfrac{2}{x-2} \to 0$ as $x \to \pm\infty$, the line $y = x + 1$ is an oblique asymptote as $x \to \infty$ and as $x \to -\infty$.

# Tips: asymptotes for rational functions

In general, if $f$ is a rational function then

- $x = c$ is a vertical asymptote for $f$ if $c$ is a zero of the denominator in the definition of $f$;
- $f$ has a horizontal asymptote if the degree of the numerator is the same as the degree of the denominator;
- $f$ has an oblique asymptote if the degree of the numerator is one higher than the degree of the denominator.

# Example

Find the oblique asymptotes for

$$f(x) = \frac{(x-2)|x| + \sin x}{x}.$$

Example

Find the oblique asymptotes for

$$f(x) = \frac{(x-2)|x| + \sin x}{x}.$$

**Solution.** If $x \to \infty$ we can assume $x > 0$ and thus

$$f(x) = \frac{(x-2)x + \sin x}{x} = (x-2) + \frac{\sin x}{x}.$$

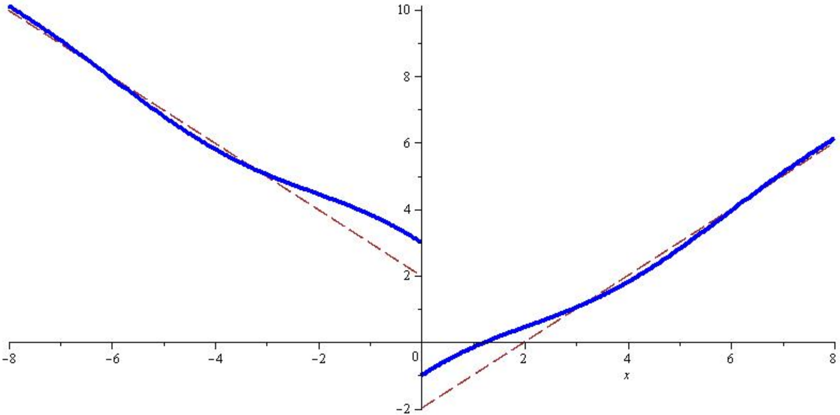
In this case we have

$$\lim_{x\to\infty} \big(f(x) - (x-2)\big) = \lim_{x\to\infty} \frac{\sin x}{x} = 0.$$

Thus $y = x - 2$ is an oblique asymptote as $x \to \infty$.
If $x \to -\infty$ we can assume $x < 0$ and thus

$$f(x) = \frac{-(x-2)x + \sin x}{x} = -(x-2) + \frac{\sin x}{x}.$$

In this case we obtain that $y = -x + 2$ is an oblique asymptote as $x \to -\infty$.


10
8
6
4
2
-2
-8
-6
-4
-2
0
2
4
6
8
x

# Symmetries

Identify any symmetries:

- $f$ is even if $f(-x) = f(x)$ for all $x \in \mathrm{Dom}(f)$.
  The graph of an even function has reflectional symmetry in the $y$-axis.
- $f$ is odd if $f(-x) = -f(x)$ for all $x \in \mathrm{Dom}(f)$.
  The graph of an odd function has rotational symmetry about the origin.

  **Remark.** If $f$ is an odd function and $0 \in \mathrm{Dom}(f)$, then $f(0) = 0$ (Why?)
- $f$ is periodic of period $T$ if $f(x+T) = f(x)$ for all $x \in \mathrm{Dom}(f)$.

**Example.** Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \frac{|\sin x|}{2 + \cos(2x)}.$$

- $f$ is even since $f(-x) = f(x)$ for all $x \in \mathbb{R}$.
- $f$ is of period $\pi$ since $f(x+\pi) = f(x)$ for all $x \in \mathbb{R}$. (since $\cos(kx)$, $k \neq 0$ has a period of $2\pi/k$ and $|\sin(x+\pi)| = |-\sin(x)| = |\sin x|$.)

# Curve Sketching Example

Sketch
$$f(x) = \frac{x^2 - 3}{2x - 4}.$$

**Solution.**

**Domain:** $\mathrm{Dom}(f) = \{x \in \mathbb{R} : x \neq 2\}$.

**Axes Intercepts:**

- $y$-intercept: when $x = 0$ we have $y = f(0) = 3/4$.
- $x$-intercept: when $y = f(x) = 0$ we have $x = \pm\sqrt{3}$.

**Symmetries:**

| | |
|---|---|
| $f(-x) = f(x)$? | No, |
| $f(-x) = -f(x)$? | No, |
| Periodic? | No. |

**Limits / Asymptotes:** We first look for vertical asymptotes. We note that

$$f(x) \to \infty \quad \text{as } x \to 2^+ \quad \text{and} \quad f(x) \to -\infty \quad \text{as } x \to 2^-$$

Thus, $x = 2$ is a vertical asymptote.

Oblique asymptotes:

$$f(x) \to \pm\infty \quad \text{when} \quad x \to \pm\infty.$$

Hence there may be oblique asymptotes. Use polynomial division to get:

$$\begin{aligned} f(x) &= \frac{1}{2}\ \frac{2x^2 - 6}{2x - 4} \\ &= \frac{1}{2}\frac{(2x-4)(x+2)+2}{2x-4} \\ &= \frac{1}{2}\left(x + 2 + \frac{2}{2x-4}\right) = \frac{1}{2}(x+2) + \frac{1}{2x-4}. \end{aligned}$$

Since $\displaystyle\lim_{x\to\pm\infty} \frac{1}{2x-4} = 0$, we have $\displaystyle\lim_{x\to\pm\infty} f(x) - \frac{1}{2}(x+2) = 0$,

Hence $y = \frac{1}{2}(x+2)$ is an oblique asymptote at $\pm\infty$.

**Critical points:**

$$f'(x) = \frac{2x(2x-4) - 2(x^2-3)}{(2x-4)^2} = \frac{2(x^2-4x+3)}{(2x-4)^2} = \frac{2(x-1)(x-3)}{(2x-4)^2}.$$

So $f'(x) = 0$ at $x = 1$ and $x = 3$ (stationary points). Besides $f$ is not differentiable at $x = 2$. Therefore, $x = 1, 2, 3$ are critical points.

The following table may be useful (but is not required).

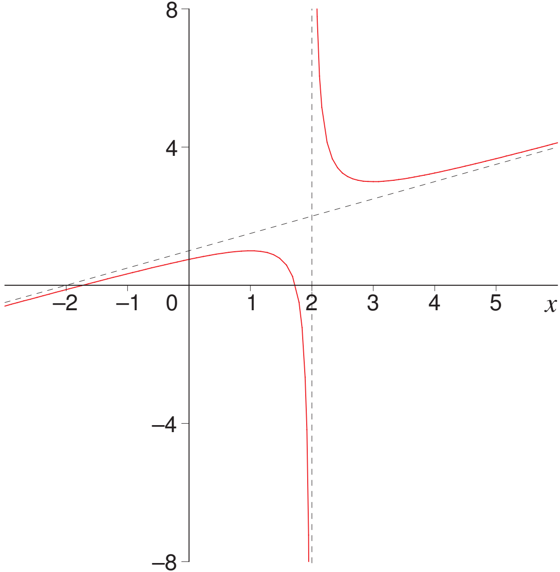
| $x$ | | $1$ | | $2$ | | $3$ | |
|---|---|---|---|---|---|---|---|
| $(x-1)$ | $-$ | $0$ | $+$ | | $+$ | | $+$ |
| $(x-3)$ | $-$ | | $-$ | | $-$ | $0$ | $+$ |
| $f'(x)$ | $+$ | $0$ | $-$ | $\Vert$ | $-$ | $0$ | $+$ |
| $f(x)$ | $\nearrow$ | $1$ | $\searrow$ | $\Vert$ | $\searrow$ | $3$ | $\nearrow$ |

$f$ is increasing on $(-\infty, 1)$ and on $(3, \infty)$ because $f' > 0$ on these intervals.

$f$ is decreasing on $(1, 2)$ and on $(2, 3)$ because $f' < 0$ on these intervals.

$x = 1$ is a local maximum and $x = 3$ is a local minimum, and $f$ is not defined at $x = 2$.

Hence the graph of $f$ is:



# Example

Sketch the graph of the function

$$f(x) = x\tan^{-1}(2x).$$

# Cont.

**Solution.**
**Domain:** $\mathrm{Dom}(f) = \mathbb{R}$.
**Axes intercepts:**

- $y$-intercept: when $x = 0$ we have $y = f(0) = 0$
- $x$-intercept: $y = f(x) = 0$ we have $x = 0$.

**Symmetries:** $f$ is even because

$$f(-x) = (-x)\tan^{-1}(-2x) = -(-x)\tan^{-1}(2x) = f(x).$$

Hence we consider only $x \geq 0$ and then take the reflection in the $y$-axis.

**Asymptotes:** Vertical: $f$ has no vertical asymptotes.
Oblique:

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \tan^{-1}(2x) = \frac{\pi}{2}$$

and

$$\begin{aligned} b &= \lim_{x\to\infty} [f(x) - ax] = \lim_{x\to\infty} \left( x \tan^{-1}(2x) - \frac{\pi x}{2} \right) \\ &= \lim_{x\to\infty} x \left( \tan^{-1}(2x) - \frac{\pi}{2} \right) \\ &= \lim_{x\to\infty} \frac{\tan^{-1}(2x) - \frac{\pi}{2}}{x^{-1}} \\ &= \lim_{x\to\infty} \frac{\frac{2}{1+(2x)^2}}{-x^{-2}} = -\lim_{x\to\infty} \frac{2x^2}{1+(2x)^2} = -\frac{1}{2}. \end{aligned}$$

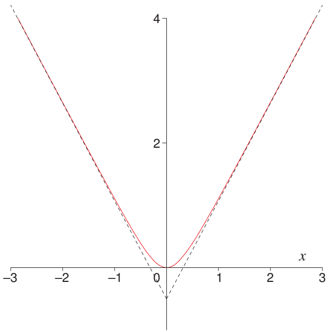
The oblique asymptote for $f$ is $y = \frac{\pi}{2}x - \frac{1}{2}$ for $x \to +\infty$.

**Critical points:**

$$f(x) = x\tan^{-1}(2x)$$
$$f'(x) = \tan^{-1}(2x) + x\frac{2}{1+(2x)^2} \geq 0 \quad \forall x \geq 0,$$

Thus, $f'(x) = 0$ if and only if $x = 0$. Hence the only critical point is $x = 0$.
We also note that $f$ is increasing on the interval $(0, \infty)$.
We sketch the graph of $f$ when $x > 0$ and get the rest by reflection in the $y$-axis.

# Cont.



# Parametrically defined curves

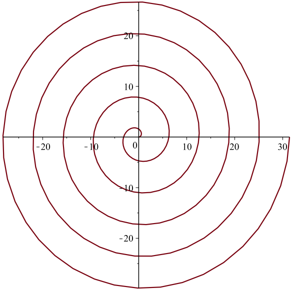
Parametrically defined curves in a plane are given by

$$\big(x(t), y(t)\big), \quad t \in A,$$

where $t$ is the parameter and $A$ is a given domain.

A parametrically defined curve may be interpreted as a path of the motion of a particle on a plane. At every moment $t$ you are given a position $(x(t), y(t))$ of the particle.

For example, $\gamma(t) = (t\cos(t), t\sin(t))$, $t \in [0, 10\pi]$.



A curve in Cartesian form $y = f(x)$ can always be written parametrically $(x(t), y(t)) = (t, f(t))$.

Given a curve $(x(t), y(t))$ sometimes you can see a relationship between $x$ and $y$.

- Sometimes you can write $y$ as a function of $x$ or vice versa. For example, if

$$(x(t), y(t)) = (t+1, t^2-1)$$

  then $t = x - 1$, hence $y = (x-1)^2 - 1$ which is obviously a parabola.
- Or, if

$$(x(t), y(t)) = (3\cos t, 2\sin t)$$

  then

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

  which is an ellipse.

But often you can't do this!

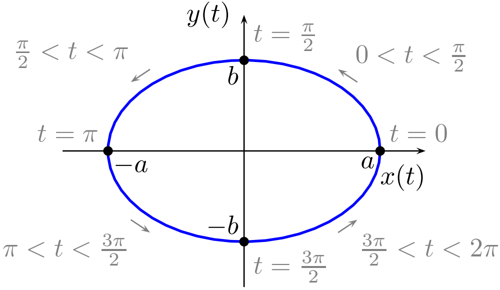
# Parametrisation of conic sections

The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

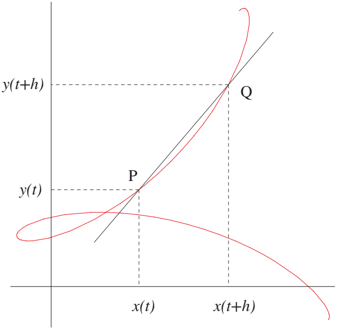
with semi-axes $a$ and $b$ admits the parametrisation

$$x(t) = a\cos t, \quad y(t) = b\sin t, \qquad 0 \le t < 2\pi.$$




Each point $(x, y)$ of the ellipse corresponds to a unique $t \in [0, 2\pi)$.

# Slope of a parametric curve




Produced using the MAPLE command:

```
plot([3*cos(t-Pi/4)+2*cos(2*t+Pi/4),
3*sin(t-Pi/4)-2*sin(2*t+Pi/4),
t=Pi/2..2*Pi])
```

The slope of the curve at the point P $(x(t), y(t))$ is given by

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{y(t+h)-y(t)}{x(t+h)-x(t)} = \lim_{h\to 0} \frac{\frac{y(t+h)-y(t)}{h}}{\frac{x(t+h)-x(t)}{h}} = \frac{\lim_{h\to 0} \frac{y(t+h)-y(t)}{h}}{\lim_{h\to 0} \frac{x(t+h)-x(t)}{h}} = \frac{y'(t)}{x'(t)}$$

provided that $x(t)$ and $y(t)$ are differentiable and $x'(t) \neq 0$.

**Remark.** Unlike curves defined in Cartesian form $y = f(x)$, a parametric curve can have more tangents at the same point: it is allowed to intersect itself!

# Tangents of a parametric curve

**Remark.** We know that a curve defined by the equation $y = f(x)$ has a horizontal tangent at $x = a$ if $\frac{dy}{dx}(a) = f'(a) = 0$ and a vertical tangent if at $x = a$ the slope $\frac{dy}{dx}$ is "infinite" (vertical asymptote).

For parametric curves,

- we detect horizontal tangents by determining where $\frac{dy}{dx} = 0$. This is the case when $y'(t) = 0$ but $x'(t) \neq 0$ (to avoid $\frac{0}{0}$).
- Similarly, vertical tangents are when $\frac{dx}{dy} = 0$, that is, when $x'(t) = 0$ but $y'(t) \neq 0$.

# Example

**Example.** A curve is described parametrically by

$$(x(t), y(t)) = (t^2 + t, t^2 - t), \quad t \in \mathbb{R}.$$

a) Find the equation of the tangent to the curve at the point $(0, 0)$.

b) Find the horizontal and vertical tangents.

**Solution.** We have

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2t - 1}{2t + 1}.$$

a) We want to compute the tangent at the point $(0, 0)$, and thus we need to solve

$$x(t) = t^2 + t = 0, \quad and \quad y(t) = t^2 - t = 0,$$

from where we get $t = 0$. From here we have $\frac{dy}{dx} = -1$.

The equation of the tangent to the curve at the point $(0, 0)$ is $y = -x$.

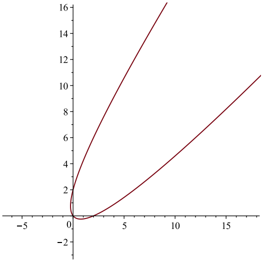
# Example

b) We have: $y'(t) = 2t - 1 = 0$ and $x'(t) \neq 0$ whenever $t = \dfrac{1}{2}$.

Thus, we have horizontal tangent at the point $(x(\dfrac{1}{2}), y(\dfrac{1}{2})) = (\dfrac{3}{4}, -\dfrac{1}{4})$.

Also $x'(t) = 2t + 1 = 0$ and $y'(t) \neq 0$ whenever $t = -\dfrac{1}{2}$.

Thus, we have a vertical tangent at the point $(x(-\dfrac{1}{2}), y(-\dfrac{1}{2})) = (-\dfrac{1}{4}, \dfrac{3}{4})$.



# Sketching parametrically defined curves

We shall use some of the same ideas as we did in sketching curves given explicitly.

1. The domain refers to $t$ and now range of $x(t)$ gives $x$ extent of the graph; range of $y(t)$ gives $y$ extent of graph;
2. The $x$-intercepts are given by $y(t) = 0$, and the $y$-intercepts by $x(t) = 0$;
3. Look for asymptotes;
4. Look for horizontal and vertical tangents;
5. Find where the graph has positive or negative slope.

**Exercise.** Sketch the curve given parametrically as

$$x(t) = e^t + e^{-t} \quad \text{and} \quad y(t) = e^t - e^{-t}.$$

Sketching parametrically defined curves

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**Exercise.** Sketch the curve given parametrically as

$$x(t) = e^t + e^{-t} \quad \text{and} \quad y(t) = e^t - e^{-t}.$$

**Solution.** Both $x(t)$ and $y(t)$ are defined for all $t \in \mathbb{R}$.
**Intercepts:** $x(t) \geq 2$, so there are no $y$-intercepts.
$y(t) = 0$ only when $t = 0$, so the $x$-intercept is the point $(x(0), y(0)) = (2, 0)$.
**Boundness:** Both $x(t)$ and $y(t)$ are unbounded as $t \to \infty$ and as $t \to -\infty$.
**Asymptotes:** When $t \to \infty$, write $y(t)$ as

$$y(t) = e^t + e^{-t} - 2e^{-t} = x(t) - 2e^{-t}$$

showing asymptote $y = x$.
Similarly, when $t \to -\infty$ write $y(t)$ as

$$y(t) = -e^t - e^{-t} + 2e^t = -x(t) + 2e^t$$

showing asymptote $y = -x$.

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- Look for horizontal and vertical tangents;
- Find where the graph has positive or negative slope.

**Exercise.** Sketch the curve given parametrically as

$$x(t) = e^t + e^{-t} \quad \text{and} \quad y(t) = e^t - e^{-t}.$$

Slope of curve is given by

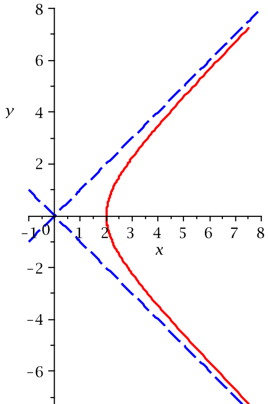
$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{e^t + e^{-t}}{e^t - e^{-t}} = \frac{x}{y}$$

As $x(t) \neq 0$ anywhere, $\frac{dy}{dx}$ is never zero.

As $y(0) = 0$, $\frac{dy}{dx}$ is undefined at $(2, 0)$.

In the first quadrant the slope is positive and tends to 1 as $t \to \infty$.

Finally, as $x(-t) = x(t)$ and $y(-t) = -y(t)$ the curve is symmetric about the $x$-axis.



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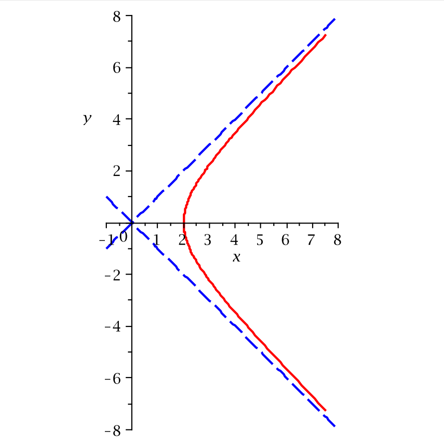
**Exercise.** Sketch the curve given parametrically as

$$x(t) = e^t + e^{-t} \quad \text{and} \quad y(t) = e^t - e^{-t}.$$

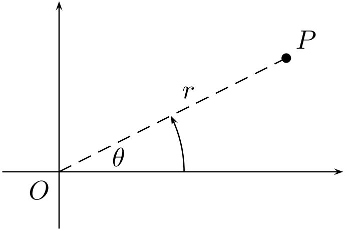


# Curves defined by polar coordinates

Many problems in mathematics are easier to solve if one chooses a suitable coordinate system. Usually we use Cartesian coordinates. Here, we focus on polar coordinates.

Every point $P$ in a plane can be specified by Cartesian coordinates $(x, y)$ or by polar coordinates, $(r, \theta)$, where

- $r \geq 0$ is the distance of $P$ from the origin; and
- $0 \leq \theta < 2\pi$ is the angle (taken in the anticlockwise direction) between $OP$ and the positive horizontal axis.




**Note.** If $P$ is the origin then $r = 0$ and $\theta$ is not defined.

Polar coordinates $(r, \theta)$ and Cartesian coordinates $(x, y)$ of a point $P$ are related by

$$x = r\cos\theta, \quad y = r\sin\theta$$

and

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}$$

provided that $x \neq 0$.

**Note.** Finding Cartesian coordinates of a point $P$ given in terms of polar coordinates is easy but care must be taken in the opposite case (we often need to use $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$ to find $\theta$ unless we think about the quadrant to expect).

# Example

Find the cartesian coordinates of the point with polar coordinates $(r, \theta) = (\sqrt{8}, 7\pi/12)$.

**Solution.** First

$$(x, y) = (r\cos\theta, r\sin\theta) = \left(\sqrt{8}\cos\left(\frac{7\pi}{12}\right), \sqrt{8}\sin\left(\frac{7\pi}{12}\right)\right).$$

Now

$$\begin{aligned}\cos(7\pi/12) &= \cos(3\pi/12 + 4\pi/12)\\ &= \cos(\pi/4)\cos(\pi/3) - \sin(\pi/4)\sin(\pi/3) = \frac{1-\sqrt{3}}{2\sqrt{2}}\\ \sin(7\pi/12) &= \sin(\pi/4)\cos(\pi/3) + \cos(\pi/4)\sin(\pi/3) = \frac{1+\sqrt{3}}{2\sqrt{2}}\end{aligned}$$

Thus,

$$(x, y) = \left(1-\sqrt{3}, 1+\sqrt{3}\right).$$

The $(x, y)$ form shows the point is in the 2nd quadrant, agreeing with interpretation of the polar coordinates.

# Example

Find all possible pairs of polar coordinates for the point $P$ with cartesian coordinates $(x, y) = (-3, -\sqrt{3})$.

Example

Find all possible pairs of polar coordinates for the point $P$ with cartesian coordinates $(x, y) = (-3, -\sqrt{3})$.

**Solution.**
We have $r = \sqrt{x^2 + y^2} = \sqrt{12}$, so $r = 2\sqrt{3}$.
Next find $\theta \in [0, 2\pi)$ satisfying

$$\cos\theta = \frac{x}{r} = \frac{-3}{2\sqrt{3}} = -\frac{\sqrt{3}}{2} \quad (1)$$

$$\sin\theta = \frac{y}{r} = \frac{-\sqrt{3}}{2\sqrt{3}} = -\frac{1}{2}. \quad (2)$$

From (1) we derive

$$\theta = \frac{5\pi}{6} \quad \text{or} \quad \frac{7\pi}{6},$$

and from (2)

$$\theta = \frac{7\pi}{6} \quad \text{or} \quad \frac{11\pi}{6}.$$

Example

Find all possible pairs of polar coordinates for the point $P$ with cartesian coordinates $(x, y) = (-3, -\sqrt{3})$.

Thus $\theta = \frac{7\pi}{6}$, agreeing with the given point $P$ being in the 3rd quadrant.
So the infinite set of possible polar coordinates for the cartesian coordinates $(x, y) = (-3, -\sqrt{3})$ is

$$\left(2\sqrt{3}, \frac{7\pi}{6} + 2k\pi\right), \quad \text{where } k \in \mathbb{Z}.$$

# Basic sketches of polar curves

Many curves can be described by equations of the form

$$r = f(\theta)$$

so that we obtain the parametrically defined curves

$$\gamma(\theta) = (r\cos\theta, r\sin\theta) = \big(f(\theta)\cos\theta, f(\theta)\sin\theta\big)$$

where $\theta$ or $r$ plays the role of the parameter.

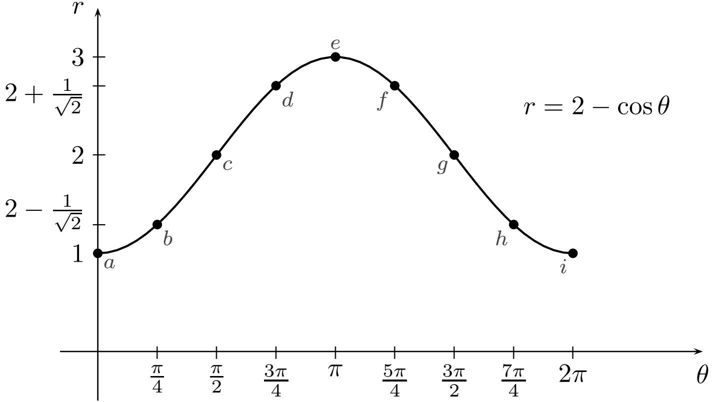
**Remark.** Polar forms of equations may be simpler or more involved compared to their Cartesian counterparts...

**Remark 2.** In order to sketch a curve represented by an equation in polar form, it is helpful to begin with an $r$ vs $\theta$ sketch.

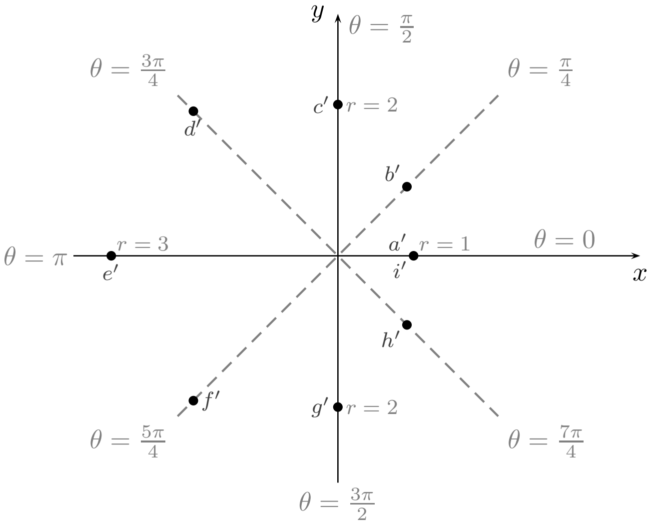
**Example.** Sketch the polar curve defined by

$$r = 2 - \cos\theta, \qquad 0 \le \theta \le 2\pi.$$

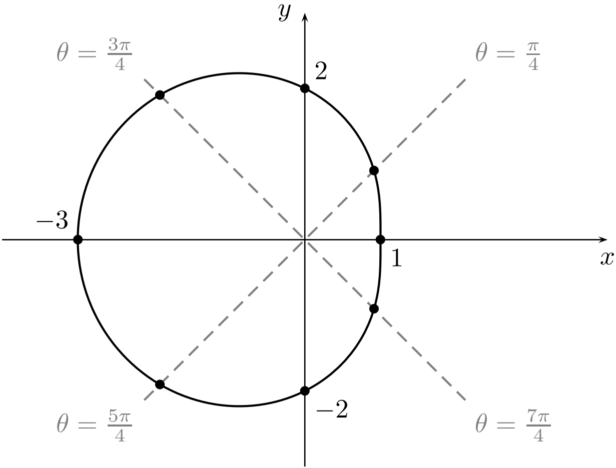
We first graph $r$ against $\theta$ ...



... and then mark the corresponding points on the $(x, y)$-plane: the $r$-value of each point $a, b, c, \ldots$ in the $r\theta$-plane corresponds to the distance from the origin to each point $a', b', c', \ldots$ in the $xy$-plane.



These considerations lead to the final sketch.




$r = 2 - \cos\theta$

# Symmetries

- If $f(-\theta) = f(\theta)$ then the polar curve is symmetric about the $x$-axis.
- If $f(\pi - \theta) = f(\theta)$ then the polar curve is symmetric about the $y$-axis .
- If $f$ is $2\pi$–periodic then it suffices to consider $\theta$ in the domain $0 \leq \theta < 2\pi$.

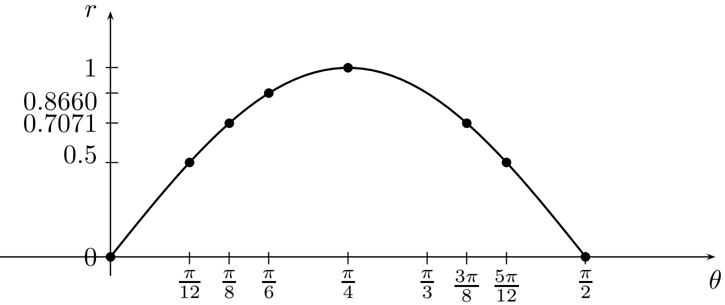
# Example

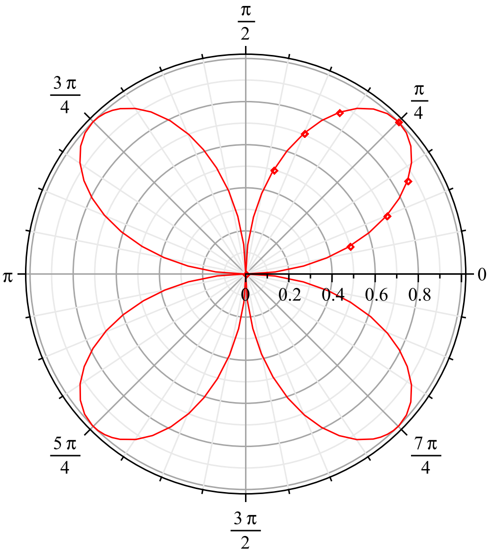
Sketch the curve described by the polar equation $r = |\sin(2\theta)|$.

**Solution.** Since $f(\theta) = |\sin(2\theta)|$ is an even function $\big(f(-\theta) = f(\theta)\big)$, the curve is symmetric about the $x$-axis.

Moreover, since $f(\pi - \theta) = f(\theta)$, the curve is symmetric about the $y$-axis.

Thus it suffices to sketch for $0 \le \theta \le \pi/2$ and then take the reflection about the two axes.






Produced using the MAPLE command:

```
with{plots};   polarplot(abs(sin(2*t),t=0..2*Pi));
```

# Sketching polar curves using calculus

Suppose that a curve can be expressed in polar form as $r = f(\theta)$.

The curve's parametric form is given by

$$\gamma(\theta) = \big(x(\theta), y(\theta)\big) = (r\cos\theta, r\sin\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta).$$

This form allows us to compute the slope of the tangent at the curve, which (as we have seen before) is given by (provided that $x'(\theta) \neq 0$)

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}.$$

Thus, horizontal tangents are obtained by solving

$$y'(\theta) = 0 \quad \text{but} \quad x'(\theta) \neq 0,$$

while vertical tangents correspond to

$$x'(\theta) = 0 \quad \text{but} \quad y'(\theta) \neq 0.$$

# Summary: What did we learn in this chapter?