MATH1131 Mathematics 1A – Algebra

Lecture 12: Euler's Formula and De Moivre's Theorem

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Based on slides by Jonathan Kress

Euler's formula

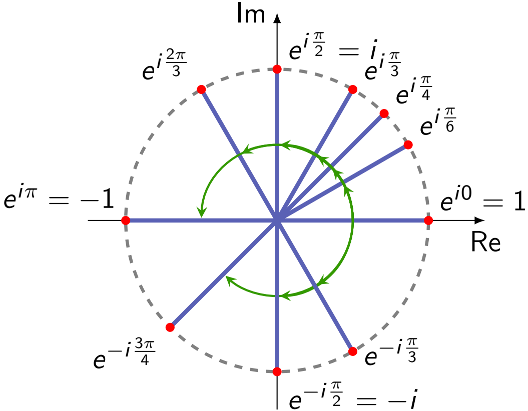
Euler's formula defines:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

The (exponential) polar form of any non-zero complex number $z = a + bi$ can therefore be written

$$z = re^{i\theta}$$

where $r = |z| = \sqrt{a^2 + b^2}$ and $\theta = \arg(z)$.



$$e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

Examples

Example

Find the exponential polar form of $-1-i$.

$$|-1-i| = \sqrt{2}, \quad \text{and} \quad \mathrm{Arg}(-1-i) = -\frac{3\pi}{4}.$$

So $-1-i = \sqrt{2}e^{-i\frac{3\pi}{4}}$.

Example

Find the Cartesian form of $6e^{i\frac{\pi}{3}}$.

$$6e^{i\frac{\pi}{3}} = 6\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) = 6 \times \frac{1}{2} + 6i \times \frac{\sqrt{3}}{2} = 3 + 3\sqrt{3}i.$$

Euler's formula

Does Euler's formula make sense?

$$e^{i0} = \cos 0 + i \sin 0 = 1 + i \times 0 = 1$$

$$\begin{aligned}
\frac{d}{d\theta} e^{i\theta} &= \frac{d}{d\theta}\Big(\cos\theta + i\sin\theta\Big) \\
&= -\sin\theta + i\cos\theta \\
&= i(\cos\theta + i\sin\theta) \\
&= ie^{i\theta}
\end{aligned}$$

So this has properties that behave as expected if we were to extend the definition of the real exponential.

Products in polar form

Also note the following:

$$\overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}.$$

and

$$\begin{aligned} e^{i\theta}e^{i\phi} &= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi) \\ &= \cos(\theta+\phi) + i\sin(\theta+\phi) \\ &= e^{i(\theta+\phi)} \end{aligned}$$

This gives us an easy way to multiply complex numbers in polar form:

For $r, s, \theta, \phi \in \mathbb{R}$,

$$z = re^{i\theta} \quad \text{and} \quad w = se^{i\phi} \quad \Longrightarrow \quad zw = rse^{i(\theta+\phi)}$$

That is, we take the product of the moduli, and the sum of the arguments.

Division in polar form

Next note that:

$$e^{-i\theta}e^{i\theta} = (\cos\theta - i\sin\theta)(\cos\theta + i\sin\theta) = \cos^2\theta + \sin^2\theta = 1$$

and hence

$$\frac{1}{e^{i\theta}} = e^{-i\theta}.$$

This gives us an easy way to divide complex numbers in polar form:

So for $r, s, \theta, \phi \in \mathbb{R}$, and $s \neq 0$,

$$z = re^{i\theta} \quad \text{and} \quad w = se^{i\phi} \quad \Longrightarrow \quad \frac{z}{w} = \frac{r}{s}e^{i(\theta-\phi)}$$

That is, we take the quotient of the moduli, and the difference of the arguments.

Multiplication and division in polar form

So for $z, w \in \mathbb{C}$,

- zw is the complex number with modulus

  $$|zw| = |z||w|$$

  and principal argument

  $$\operatorname{Arg}(zw) = \operatorname{Arg}(z) + \operatorname{Arg}(w) + 2k\pi \qquad \text{for suitable } k \in \mathbb{Z}.$$

- $\dfrac{z}{w}$ is the complex number with modulus

  $$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$$

  and principal argument

  $$\operatorname{Arg}\left(\frac{z}{w}\right) = \operatorname{Arg}(z) - \operatorname{Arg}(w) + 2k\pi \qquad \text{for suitable } k \in \mathbb{Z}.$$

Example

Let $z = 2e^{i\frac{2\pi}{3}}$ and $w = 5e^{i\frac{3\pi}{4}}$. Find each of the following in polar form:

(a) zw (b) $\dfrac{z}{w}$ (c) $\overline{z}$

$$zw = 2e^{i\frac{2\pi}{3}} \times 5e^{i\frac{3\pi}{4}} = 10e^{i\frac{17\pi}{12}} = 10e^{i(\frac{17\pi}{12}-2\pi)} = 10e^{-i\frac{7\pi}{12}}$$

$$\frac{z}{w} = 2e^{i\frac{2\pi}{3}} \div 5e^{i\frac{3\pi}{4}} = \frac{2}{5}e^{i(\frac{2\pi}{3}-\frac{3\pi}{4})} = \frac{2}{5}e^{-i\frac{\pi}{12}}$$

$$\overline{z} = \overline{2e^{i\frac{2\pi}{3}}} = 2e^{-i\frac{2\pi}{3}}$$

De Moivre's Theorem

So far we have seen that index laws for the product and quotients of real exponentials hold for complex exponentials. De Moivre's Theorem states that the index laws also hold for integer powers.

De Moivre's Theorem

For any real number θ,

$$(e^{i\theta})^n = e^{in\theta} \quad \text{for all } n \in \mathbb{Z}.$$

An alternative form of De Moivre's Theorem is:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad \text{for all } n \in \mathbb{Z}.$$

Proof of De Moivre's Theorem

Proof

Let $\theta \in \mathbb{R}$.

First note that $(e^{i\theta})^1 = e^{i\theta}$ and $(e^{i\theta})^0 = 1 = e^{i0\theta}$.

Suppose that for some positive integer n we have $(e^{i\theta})^n = e^{in\theta}$. Then,

$$(e^{i\theta})^{n+1} = (e^{i\theta})^n e^{i\theta} = e^{in\theta} e^{i\theta} = e^{i(n\theta+\theta)} = e^{i(n+1)\theta}.$$

So by induction, it follows that

$$(e^{i\theta})^n = e^{in\theta} \quad \text{for all } n \in \mathbb{N}.$$

Now suppose that n is negative. Then

$$(e^{i\theta})^n = (e^{i\theta})^{-(-n)} = \frac{1}{(e^{i\theta})^{-n}} = \frac{1}{e^{-ni\theta}} = e^{in\theta}.$$

So

$$(e^{i\theta})^n = e^{in\theta} \quad \text{for all } n \in \mathbb{Z}.$$

Example

Find $(-1+i)^{202}$.

In polar form, $-1+i = \sqrt{2}e^{i\frac{3\pi}{4}}$.

So $(-1+i)^{202} = (\sqrt{2}e^{i\frac{3\pi}{4}})^{202} = (\sqrt{2})^{202}\, e^{i\frac{3\pi}{4}\times 202} = 2^{101}e^{i\frac{606\pi}{4}}$.

To find the principal argument, note:

$$\frac{606\pi}{4} = 151\frac{1}{2}\times\pi = 76\times(2\pi) - \frac{1}{2}\pi.$$

So $(-1+i)^{202} = 2^{101}e^{-i\frac{\pi}{2}} = 2^{101}\times(-i) = -2^{101}i$.