School of Mathematics and Statistics
Math1131-Algebra

Lec12: Powers and roots of complex numbers

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2020 Term 1

Powers

Exercise 1. Calculate $(1+i\sqrt{3})^{10}$.

$$(1+i\sqrt{3})^{10} = \left(2e^{i\pi/3}\right)^{10}$$

$$= 2^{10}e^{i10\pi/3} = 2^{10}e^{-2\pi i/3}$$

$$= 2^{10}\left(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}\right)$$

$$= 2^{10}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= 2^{9}\left(-1 - i\sqrt{3}\right)$$

512

$$\frac{10\pi}{3} = \frac{9\pi + \pi}{3} = 3\pi + \frac{\pi}{3}$$

$$3\pi + \frac{\pi}{3} = -\frac{2\pi}{3} + 2\pi$$

```
> z := (1 + sqrt(3)*I)^10;
```

$$z := \left(1 + I\sqrt{3}\right)^{10}$$

```
> evalc(z);
  # to get the Cartesian = rectangular form
```

$$-512 - 512\,I\sqrt{3}$$

Any non-zero complex number has n n^{th} roots

n^{th} root of a complex number

A complex number α is an nth root of z if

$$\alpha^n = z.$$

TIP! To find the nth roots, write all the complex numbers in polar form.

Exercise 2. Find the 5th roots of 1, that is, find all the complex numbers α such that

$$\alpha^5 = 1.$$

$\alpha = re^{i\theta}$

$(re^{i\theta})^5 = 1$

$r^5 e^{5i\theta} = 1 \times e^{i0}$

$\begin{cases} r^5 = 1 \\ 5\theta = 0 + 2k\pi \end{cases}$

$\begin{cases} r = 1 \\ \theta = \frac{2k\pi}{5} \quad k \in \mathbb{Z} \end{cases}$

$\boxed{\text{Sols} = \left\{ \underset{k=0}{1}, \underset{k=1}{1e^{\frac{2i\pi}{5}}}, \underset{k=2}{1e^{\frac{4i\pi}{5}}}, \underset{k=3}{e^{\frac{6i\pi}{5}}}, \underset{k=4}{e^{\frac{8i\pi}{5}}} \right\}}$

$\alpha = re^{i\theta}$

$\underset{k=5}{e^{\frac{10i\pi}{5}} = 1}$

Checking our answers with Maple

```
> # Naive approach

  solve(z^5 = 1);
```

$$1, \frac{\sqrt{5}}{4} - \frac{1}{4} + \frac{I\sqrt{2}\sqrt{5+\sqrt{5}}}{4}, -\frac{\sqrt{5}}{4} - \frac{1}{4} + \frac{I\sqrt{2}\sqrt{5-\sqrt{5}}}{4}, -\frac{\sqrt{5}}{4} - \frac{1}{4} - \frac{I\sqrt{2}\sqrt{5-\sqrt{5}}}{4},$$
$$\frac{\sqrt{5}}{4} - \frac{1}{4} - \frac{I\sqrt{2}\sqrt{5+\sqrt{5}}}{4}$$

```
> # Let's convert those to polar coordinates

  SolsPolar := map(polar, [solve(z^5 = 1)]);
  #  we use straight brackets to store the answers as a list
```

$$SolsPolar := \left[\mathrm{polar}(1, 0), \mathrm{polar}\left(\frac{\sqrt{(\sqrt{5}-1)^2 + 10 + 2\sqrt{5}}}{4}, \arctan\left(\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4\left(\frac{\sqrt{5}}{4} - \frac{1}{4}\right)}\right)\right),\right.$$
$$\mathrm{polar}\left(\frac{\sqrt{(-\sqrt{5}-1)^2 + 10 - 2\sqrt{5}}}{4}, \arctan\left(\frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4\left(-\frac{\sqrt{5}}{4} - \frac{1}{4}\right)}\right) + \pi\right),$$
$$\mathrm{polar}\left(\frac{\sqrt{(-\sqrt{5}-1)^2 + 10 - 2\sqrt{5}}}{4}, -\arctan\left(\frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4\left(-\frac{\sqrt{5}}{4} - \frac{1}{4}\right)}\right) - \pi\right),$$
$$\left.\mathrm{polar}\left(\frac{\sqrt{(\sqrt{5}-1)^2 + 10 + 2\sqrt{5}}}{4}, -\arctan\left(\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4\left(\frac{\sqrt{5}}{4} - \frac{1}{4}\right)}\right)\right)\right]$$

```
> map(simplify, SolsPolar);
  # 'Map' is used to apply 'simplify' to each term of the list 'SolsPolar'
```

$$\left[1, \mathrm{polar}\left(1, \frac{2\pi}{5}\right), \mathrm{polar}\left(1, \frac{4\pi}{5}\right), \mathrm{polar}\left(1, -\frac{4\pi}{5}\right), \mathrm{polar}\left(1, -\frac{2\pi}{5}\right)\right]$$

Roots

Exercise 3. Find the 6th roots of $z = 1 + \sqrt{3}i$.

Some facts about the nth roots of a complex number

Some facts about the nth roots of a complex number

$\alpha^n = z$ $\quad \alpha ?$

- There are n of them.
- They all have the same modulus, $|z|^{1/n}$, and so lie on a circle centred at 0.
- They are evenly spaced around that circle.

Exercise 4. Find the 3rd roots of -1.

We want to solve

$$z^3 = -1$$

$$\iff (re^{i\theta})^3 = 1.e^{i\pi}$$

$$\iff r^3 e^{3i\theta} = 1e^{i\pi}$$

$$\iff \begin{cases} r^3 = 1 \\ 3\theta = \pi + 2k\pi, \quad k \in \mathbb{Z} \end{cases}$$

$$\iff r = 1, \quad \theta = \frac{\pi}{3} + \frac{2k\pi}{3}$$

Checking our answers with Maple

```
> evalc([solve(z^3 = -1)]);
  # Straight brackets to store the answers as a list so we can apply 'map'
  next
```

$$\left[-1, \frac{1}{2} - \frac{I\sqrt{3}}{2}, \frac{1}{2} + \frac{I\sqrt{3}}{2}\right]$$

```
> map(polar, %);
  #  '%' means 'previous result'
  # 'map' is used to apply 'polar' to each term of the previous list
```

$$\left[\mathrm{polar}(1, \pi), \mathrm{polar}\left(1, -\frac{\pi}{3}\right), \mathrm{polar}\left(1, \frac{\pi}{3}\right)\right]$$

The Binomial theorem also works for complex numbers

The Binomial Theorem

For small powers, we can calculate powers using the *Binomial Theorem*.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \qquad \text{where} \qquad \binom{n}{k} = {}^nC_k = \frac{n!}{k!(n-k)!}.$$

TIP!

- Pascal's triangle can help you find the binomial coefficients.
- ${}^nC_k$ is read "*n choose k*".

Exercise 5. For example, calculate a) $(a+b)^3$ b) $(2-i)^5$

a) $(a+b)^3$

$= a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^2 = a^2 + 2ab + b^2$

$n=2$ 1 2 1

$n=3$ 1 3 3 1

Checking some of our answers with Maple

Pascal's triangle

Exercise 5, continued. Calculate a) $(a+b)^3$ b) $(2-i)^5$

$i^4 = 1$

$n=2$: 1 2 1

1 3 3 1

$n=4$: 1 4 6 4 1

$n=5$: 1 5 10 10 5 1

$$(2-i)^5 = 1\times 2^5 + 5\times 2^4(-i)^1 + 10\times 2^3(-i)^2 + 10\times 2^2(-i)^3 + 5\times 2^1\times(-i)^4 + (-i)^5$$

$$= 32 - 80i - 80 + 40i + 10 - i$$

$$= \boxed{-38 - 41i}$$

```
> expand((a + b)^3 );
```

$$a^3 + 3a^2b + 3ab^2 + b^3$$

```
> expand((2 - I)^5 );
```

$$-38 - 41\,\mathrm{I}$$

$\cos(n\theta)$ and $\sin(n\theta)$ in terms of $\cos^k\theta$ and $\sin^k\theta$

From sines and cosines of multiples of θ to powers of $\sin\theta$ and $\cos\theta$

1 3 3 1
1 4 6 4 1

TIP! For this, use De Moivre's Theorem together with the Binomial theorem.

Exercise 6.

a) Find formulas for $\cos(4\theta)$ and $\sin(4\theta)$ in terms of powers of $\sin\theta$ and $\cos\theta$. ✓

b) Hence or otherwise prove that $\cos 4\theta + 8\cos^2\theta \geqslant 1$ for all $\theta \in \mathbb{R}$.

$$(e^{i\theta})^4 = e^{4i\theta}$$

$$\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$$

$$= \cos^4\theta + 4\cos^3\theta\, i\sin\theta + 6\cos^2\theta\, i^2\sin^2\theta + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4$$

$$= \underbrace{(\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta)}_{\cos 4\theta} + i\underbrace{(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)}_{\sin 4\theta}$$

by equating the real and imaginary part

$\cos(n\theta)$ and $\sin(n\theta)$ in terms of $\cos^k\theta$ and $\sin^k\theta$

From sines and cosines of multiples of θ to powers of $\sin\theta$ and $\cos\theta$

Exercise 6, continued.

a) Find formulas for $\cos(4\theta)$ and $\sin(4\theta)$ in terms of powers of $\sin\theta$ and $\cos\theta$.

b) Hence or otherwise prove that $\boxed{\cos 4\theta + 8\cos^2\theta} \geqslant 1$ for all $\theta \in \mathbb{R}$.

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$

$\sin^2\theta = 1 - \cos^2\theta$

$$= 8\cos^4\theta - 8\cos^2\theta + 1$$

$$\boxed{\cos 4\theta + 8\cos^2\theta}$$

$$= 8\cos^4\theta - \cancel{8\cos^2\theta} + 1 - \cancel{8\cos^2\theta}$$

$$= \underbrace{8\cos^4\theta}_{\oplus} + 1 \geqslant 1$$

$\cos^n\theta$ and $\sin^n\theta$ in terms of $\cos(k\theta)$ and $\sin(k\theta)$

From *powers* of $\sin\theta$ and $\cos\theta$ to sines and cosines of *multiples* of θ

TIP!

For this, ◇ start from $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and/or $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$,

- ◇ use the binomial theorem,
- ◇ and then group the terms in pairs to get some sines and cosines back using $\cos k\theta = \frac{1}{2}(e^{ik\theta} + e^{-ik\theta})$ and $\sin k\theta = \frac{1}{2i}(e^{ik\theta} - e^{-ik\theta})$

Exercise 7. Write $\sin^3\theta$ in terms of sines of multiples of θ and hence evaluate

$$I = \int_0^{\pi/2} \sin^3\theta \, d\theta.$$

$\cos^n \theta$ and $\sin^n \theta$ in terms of $\cos(k\theta)$ and $\sin(k\theta)$

Exercise 7, continued. Write $\sin^3 \theta$ in terms of sines of multiples of θ and hence evaluate

$$I = \int_0^{\pi/2} \sin^3 \theta \, d\theta.$$

Checking some of our answers with Maple

```
> combine((sin(x))^3 , trig);

  # Powers to multiples
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$$-\frac{\sin(3x)}{4} + \frac{3\sin(x)}{4}$$

```
> int((sin(x))^3, x = 0 .. Pi/2);

  # which makes it easy to integrate
```

$$\frac{2}{3}$$