Chapter 10: The hyperbolic functions

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(Alina Ostafe's notes, based on Fedor Sukochev's notes)

MATH1131

UNSW

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Motivation

- Every function $f : \mathbb{R} \to \mathbb{R}$ can be written as a sum of an even and an odd function:

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{even function}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{odd function}}$$

- If $f(x) = e^x$ then

$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{\text{even function}} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\text{odd function}}$$

- These functions are useful in their own right: they describe motions of waves in elastic solid, shapes of hanging electric power cables, etc.

Hyperbolic sine and cosine functions

The hyperbolic cosine function $\cosh : \mathbb{R} \to \mathbb{R}$ is defined by

$$\cosh x := \frac{e^x + e^{-x}}{2}.$$

The hyperbolic sine function $\sinh : \mathbb{R} \to \mathbb{R}$ (pronounced 'shine') is defined by

$$\sinh x := \frac{e^x - e^{-x}}{2}.$$

Questions:

1. These are just simple combinations of the exponential function, so why bother with giving them names?
2. What have they got to do with $\cos$ and $\sin$?
3. Why 'hyperbolic'?

Although the graphs of these functions are nothing like those of $\cos$ and $\sin$, they have a fantastic range of identities that mimic those of the standard trig functions. We'll be able to use these to find antiderivatives for a whole range of new functions.

Remark. $\cosh$ and $\sinh$ are differentiable with

$$\frac{d}{dx}(\sinh x) = \cosh x, \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

so that $\cosh x$ and $\sinh x$ obey the differential equation

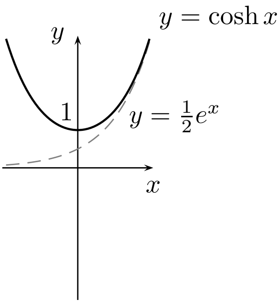
$$\frac{d^2y}{dx^2} = y.$$

Properties of the $\cosh$ function

The hyperbolic cosine function $\cosh : \mathbb{R} \to \mathbb{R}$ is defined by

$$\cosh x := \frac{e^x + e^{-x}}{2}.$$

- $\cosh$ is an even function.
- $\cosh 0 = 1$.
- $\cosh$ is decreasing on $(-\infty, 0)$, stationary at 0 and increasing on $(0, \infty)$.
- $\cosh x \geq 1$ for all x in $\mathbb{R}$.
- $\cosh x$ gets arbitrarily close to $\dfrac{1}{2}e^{\pm x}$ as $x \to \pm\infty$.
- $\frac{d}{dx}(\cosh x) = \sinh x$

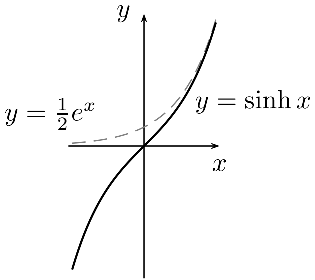


Properties of the $\sinh$ function

The hyperbolic sine function $\sinh : \mathbb{R} \to \mathbb{R}$ (pronounced 'shine') is defined by

$$\sinh x := \frac{e^x - e^{-x}}{2}.$$

- $\sinh$ is an odd function.
- $\sinh 0 = 0$.
- $\sinh$ is increasing on $(-\infty, \infty)$ and has a point of inflexion at 0.
- $\sinh x < 0$ for $x < 0$ and $\sinh x > 0$ for $x > 0$.
- $\sinh x$ gets arbitrarily close to $\pm\dfrac{1}{2}e^{\pm x}$ as $x \to \pm\infty$.
- $\frac{d}{dx}(\sinh x) = \cosh x$



Theorem

The hyperbolic functions are related by

$$\cosh^2 x - \sinh^2 x = 1.$$

Remark. The similarity to relations such as

$$\cos^2 x + \sin^2 x = 1, \quad \frac{d}{dx}\cos x = -\sin x, \quad \frac{d}{dx}\sin x = \cos x$$

explains the words cosine and sine in the hyperbolic functions.

Proof.

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Proof.

By definition,

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{1}{4}\left[(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})\right]$$

$$= 1.$$

The term hyperbolic is motivated in the following manner:

Example. Sketch the curve $\gamma(t)$ defined by

$$\gamma(t) = \big(x(t), y(t)\big) = (\cosh t, \sinh t), \quad t \in \mathbb{R}.$$

Elimination of the parameter t leads to

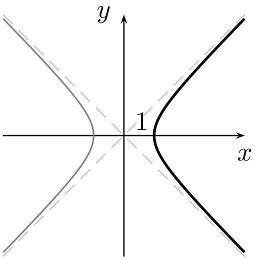
$$[x(t)]^2 - [y(t)]^2 = \cosh^2 t - \sinh^2 t = 1$$

so that γ parametrises the branch of the hyperbola

$$x^2 - y^2 = 1, \qquad x > 0.$$

The other branch of the hyperbola is parametrised by

$$\big(x(t), y(t)\big) = (-\cosh t, \sinh t).$$



Other hyperbolic functions

Other hyperbolic functions are defined in analogy with the trigonometric functions according to

$$\tanh x = \frac{\sinh x}{\cosh x}, \qquad \coth x = \frac{\cosh x}{\sinh x},$$

$$\operatorname{sech} x = \frac{1}{\cosh x}, \qquad \operatorname{cosech} x = \frac{1}{\sinh x}.$$

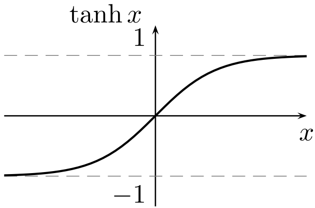
Properties of the $\tanh$ function

Recall that

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Properties of the $\tanh$ function.

- $\tanh$ is an odd function.
- $\tanh 0 = 0$.
- $\dfrac{d}{dx} \tanh x = \operatorname{sech}^2 x > 0$.
- $\tanh$ is increasing on $(-\infty, \infty)$ and has a point of inflexion at 0.
- $\tanh x < 0$ for $x < 0$ and $\tanh x > 0$ for $x > 0$.
- $\displaystyle\lim_{x \to \pm\infty} \tanh x = \pm 1$.



Proof of the derivative of $\tanh$**.**

Note. The slope at the point of inflexion is

$$\left.\frac{d}{dx}\tanh x\right|_{x=0} = \operatorname{sech}^2 0 = 1.$$

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By definition,

$$\begin{aligned}
\frac{d}{dx}(\tanh x) &= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) \\
&= \frac{\cosh x \,\frac{d}{dx}\sinh x - \sinh x \,\frac{d}{dx}\cosh x}{\cosh^2 x} \\
&= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \\
&= \frac{1}{\cosh^2 x} \\
&= \operatorname{sech}^2 x.
\end{aligned}$$

Hyperbolic identities

'Difference of squares' identities.

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\coth^2 x - 1 = \operatorname{cosech}^2 x$$

'Sum and difference' formulae.

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

'Double-angle' formulae.

$$\sinh(2x) = 2\sinh x\cosh x$$

$$\cosh(2x) = \cosh^2 x + \sinh^2 x$$

$$\tanh(2x) = \frac{2\tanh x}{1+\tanh^2 x}.$$

Exercise. Prove the first two 'sum and difference' formulae and, hence, derive the third.

Hyperbolic derivatives and integrals

The following derivatives may be readily verified, using definitions of $\sinh$ and $\cosh$:

$$\frac{d}{dx}\sinh x = \cosh x$$

$$\frac{d}{dx}\cosh x = \sinh x$$

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$$

Corresponding indefinite integrals are, for instance,

$$\int \sinh x \, dx = \cosh x + C, \qquad \int \operatorname{sech}^2 x \, dx = \tanh x + C.$$

Example. Determine the definite integral

$$I = \int_0^{(\ln 2)^2} \frac{\operatorname{sech}^2 \sqrt{x}}{\sqrt{x}} \, dx.$$

Solution. Make the substitution $u = \sqrt{x}$, $u \in [0, \ln 2]$. Then $du = \frac{1}{2\sqrt{x}} dx$, and therefore,

$$I = 2 \int_0^{\ln 2} \operatorname{sech}^2 u \, du = 2 \tanh u \Big|_0^{\ln 2} = \frac{6}{5}.$$

The inverse hyperbolic functions

Recall the graphs of $\sinh$ and $\tanh$ are increasing functions and hence are one-to-one.

$\cosh$ however is not one-to-one and so we need to restrict the domain to $[0, \infty)$.

For inverses then we are dealing with

$$\begin{aligned}
\cosh &: [0, \infty) \to [1, \infty), & \cosh^{-1} &: [1, \infty) \to [0, \infty) \\
\sinh &: \mathbb{R} \to \mathbb{R}, & \sinh^{-1} &: \mathbb{R} \to \mathbb{R} \\
\tanh &: \mathbb{R} \to (-1, 1), & \tanh^{-1} &: (-1, 1) \to \mathbb{R}.
\end{aligned}$$

Of course we can get the graphs of these functions by just reflecting the graphs of $\cosh$, $\sinh$ and $\tanh$ in the line $y = x$.

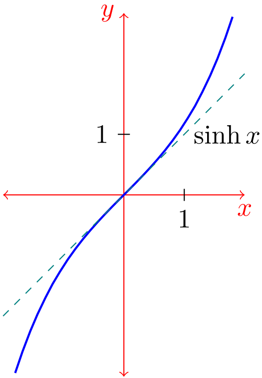
Inverse hyperbolic sine

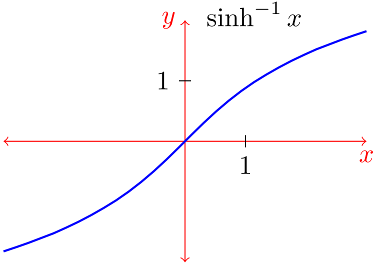
It turns out that inverse hyperbolic functions can be expressed in terms of logarithms.

$$\begin{aligned}
& y = \sinh x \\
\iff\ & y = \frac{e^x - e^{-x}}{2} \\
\iff\ & e^x - 2y - e^{-x} = 0 \\
\iff\ & \left(e^x\right)^2 - 2ye^x - 1 = 0 : \quad \text{quadratic equation for } e^x \\
\iff\ & e^x = \frac{2y \pm \sqrt{4y^2+4}}{2} = y \pm \sqrt{y^2+1} \\
\iff\ & e^x = y + \sqrt{y^2+1} \quad (\text{since } y - \sqrt{y^2+1} < 0) \\
\iff\ & x = \ln\left(y + \sqrt{y^2+1}\right) \\
\iff\ & \sinh^{-1} y = \ln\left(y + \sqrt{y^2+1}\right).
\end{aligned}$$

Hence we have:

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) \qquad \forall x \in \mathbb{R}$$

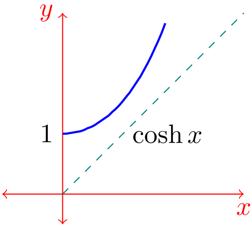


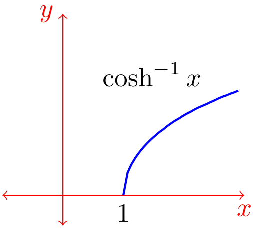


Inverse hyperbolic cosine

As on the last slide, you can show that

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad \forall x \in [1, \infty)$$

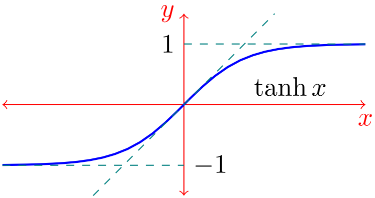


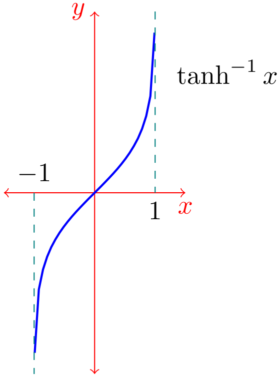


Inverse hyperbolic tangent

Here we have:

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad \forall x \in (-1,1)$$





Example. Evaluate

$$\sinh\left(\cosh^{-1}\tfrac{4}{3}\right).$$

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Solution. Evaluation of

$$\sinh^2 t = \cosh^2 t - 1$$

at $t = \cosh^{-1}\frac{4}{3}$ yields

$$\sinh^2\left(\cosh^{-1}\tfrac{4}{3}\right) = \left(\tfrac{4}{3}\right)^2 - 1 = \tfrac{7}{9}.$$

Accordingly,

$$\sinh\left(\cosh^{-1}\tfrac{4}{3}\right) = +\tfrac{\sqrt{7}}{3}$$

since $t > 0$ (recall that $\cosh^{-1}$ is a non-negative function).

The main interest in these inverse hyperbolic functions however is in their derivatives:

$$\begin{aligned}
\frac{d}{dx}\,\sinh^{-1} x &= \frac{1}{\sqrt{x^2+1}}, \quad x \in \mathbb{R},\\
\frac{d}{dx}\,\cosh^{-1} x &= \frac{1}{\sqrt{x^2-1}}, \quad x > 1,\\
\frac{d}{dx}\,\tanh^{-1} x &= \frac{1}{1-x^2}, \quad -1 < x < 1.
\end{aligned}$$

Thus, the inverse hyperbolic functions provide antiderivatives for some relatively simple functions which we otherwise can't integrate.

There are two ways to prove these:

1. use the formulae in terms of $\ln$ on the previous slide, and a bit of algebra.
2. use the Inverse Function Theorem

Example. Use the inverse function theorem to confirm that

$$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}.$$

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$$\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{x^2+1}}.$$

If we set $y = \sinh^{-1} x$ then the inverse function theorem implies that

$$\frac{dy}{dx} = \frac{1}{\sinh'(y)} = \frac{1}{\cosh y} = \frac{1}{\cosh(\sinh^{-1} x)}.$$

On the other hand, since

$$\cosh t = \sqrt{1+\sinh^2 t},$$

we obtain

$$\cosh(\sinh^{-1} x) = \sqrt{1+x^2}$$

so that

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}.$$

Integration leading to the inverse hyperbolic functions

From the previous considerations, it follows that

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + C$$
$$= \ln\left(x+\sqrt{x^2+a^2}\right) + \tilde{C}, \qquad a > 0$$

Proof: let $u = x/a$ and thus, $dx = a\ du$. We have

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{dx}{a\sqrt{(x/a)^2+1}} = \int \frac{du}{\sqrt{u^2+1}}$$
$$= \sinh^{-1} u + C = \ln\left(x+\sqrt{x^2+a^2}\right) + \tilde{C}.$$

Similarly,

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C$$
$$= \ln\left(x+\sqrt{x^2-a^2}\right) + \tilde{C}, \qquad x \geq a > 0$$

[These formulae are included in the table of standard integrals which is issued at the final examination.]

Example. Find $I = \int \frac{dx}{4x - 3 - x^2}$.

If we 'complete the square'

$$4x - 3 - x^2 = -(x-2)^2 + 1$$

and let $u = x - 2$, we obtain

$$I = \int \frac{du}{1 - u^2} = \tanh^{-1} u + C = \tanh^{-1}(x-2) + C.$$

Of course, we can do these ones without $\tanh^{-1}$:

$$\frac{1}{4x-3-x^2} = \frac{1}{(3-x)(-1+x)} = \frac{((3-x)+(-1+x))/2}{(3-x)(-1+x)} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x-3}\right)$$

Thus

$$\begin{aligned} I &= \frac{1}{2}\int \frac{1}{x-1} - \frac{1}{x-3}\,dx \\ &= \frac{1}{2}\big(\ln|x-1| - \ln|x-3|\big) + C \\ &= \frac{1}{2}\ln\left|\frac{x-1}{x-3}\right| + C. \end{aligned}$$

which (check this!) comes to the same thing!

Example. Determine the indefinite integral

$$\int \frac{dx}{\sqrt{x^2 - 2x + 10}}.$$

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Example. Determine the indefinite integral

$$\int \frac{dx}{\sqrt{x^2 - 2x + 10}}.$$

If we 'complete the square', that is,

$$x^2 - 2x + 10 = (x-1)^2 + 9 = u^2 + 3^2,$$

where $u = x - 1$, then

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2 - 2x + 10}} &= \int \frac{du}{\sqrt{u^2 + 3^2}} \\ &= \sinh^{-1} \frac{u}{3} + C \\ &= \sinh^{-1} \frac{x-1}{3} + C.\end{aligned}$$

Example. Find $\displaystyle\int \cosh^{-1} x dx$.

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Use integration by parts:

$$\begin{cases} u &= \cosh^{-1} x \\ dv &= dx \end{cases} \implies \begin{cases} du &= \dfrac{dx}{\sqrt{x^2-1}} \\ v &= x. \end{cases}$$

Hence

$$\int \cosh^{-1} x dx = x \cosh^{-1} x - \int \frac{x}{\sqrt{x^2-1}} dx.$$

Use substitution $u = x^2 - 1$. Then $du = 2xdx$ and thus

$$\begin{aligned} \int \cosh^{-1} x dx &= x \cosh^{-1} x - \frac{1}{2} \int \frac{du}{\sqrt{u}} \\ &= x \cosh^{-1} x - u^{1/2} + C \\ &= x \cosh^{-1} x - \sqrt{x^2-1} + C. \end{aligned}$$