Revision questions

Inequalities and Absolute Values

1. Sketch the set of points (x, y) which satisfy the following relations.

   a) $0 \leq y \leq 2x$ and $0 \leq x \leq 2$ b) $y/2 \leq x \leq 2$ and $0 \leq y \leq 4$

2. Solve

   a) $x(x-1) > 0$ b) $(x-1)(x-2) < 0$ c) $\frac{1}{x} > -\frac{1}{2}$ d) $\frac{1}{1-x} > \frac{1}{2}$

3. Solve

   a) $|x+1| < 3$ b) $|x+2| > 3$ c) $|3x+2| < 1$ d) $\left|\frac{x-1}{x+1}\right| < 1$

Trigonometry

4. Find the **exact** value of each of the following:

   a) $\cos\left(\frac{\pi}{12}\right)$ b) $\sin\left(\frac{5\pi}{12}\right)$ c) $\tan\left(\frac{7\pi}{12}\right)$ d) $\sec\left(\frac{11\pi}{12}\right)$

5. If A and B are acute with $\sin(A) = \dfrac{3}{5}$ and $\tan(B) = \dfrac{12}{5}$ find (without the use of a calculator):

   a) $\cos(A)$ b) $\tan(A)$ c) $\sin(B)$ d) $\cos(B)$

   e) $\sin(A+B)$ f) $\cos(A-B)$ g) $\sin(2A)$ h) $\tan(2B)$

6. If A and B are acute with $\sin(A) = \dfrac{24}{25}$ and $\cos(B) = \dfrac{8}{17}$ find (without finding A and B):

   a) $\cos(2A)$ b) $\sin(A-B)$ c) $\tan(A+B)$

7. Find the period and amplitude for each of the following functions.

   a) $y = 3\sin\left(2x - \frac{\pi}{4}\right)$ b) $y = -2\cos\left(\frac{x}{3} + \frac{\pi}{2}\right)$

8. Express each of the following in terms of a single sine function in the form $R\sin(x \pm \alpha)$, where $R > 0$ and α is acute.

   a) $\sin(x) + \cos(x)$ b) $2\sin(x) + 2\sqrt{3}\cos(x)$

   c) $\sqrt{3}\sin(x) - \cos(x)$ d) $\sqrt{8}\sin(x) - \sqrt{8}\cos(x)$

Functions

9. What is the (maximal) domain and range of the following functions?

   a) $f(x) = \sqrt{5 - x^2}$ b) $f(x) = \sqrt{x^2 - 5}$

   c) $f(x) = \sqrt{1 - 2\sin x}$ d) $f(x) = (x - 8)^{-1/3}$

   e) $f(x) = \sqrt{x - 1}$ f) $f(x) = \frac{1}{\sqrt{x-1}}$

   g) $f(x) = \sqrt{\sin x}$ h) $f(x) = \begin{cases} \cos x & \text{if} \quad x < 0 \\ \sqrt{1-x} & \text{if} \quad 0 \le x \le 1 \\ |x| & \text{if} \quad x > 1 \end{cases}$

   i) $f(x) = 1 + \tan^2 x$

10. Sketch the graph of each of the functions in Problem 9.

11. Sketch each of the following functions without using calculus.

    a) An odd function, $f(x)$, defined on $[-2, 2]$ such that
    $$f(x) = x^2(1 - x) \quad \text{when} \quad 0 \le x \le 2.$$

    b) An even function, $f(x)$, defined on $[-3, 3]$ such that
    $$f(x) = (x - 1)^2(x - 2) \quad \text{when} \quad 0 \le x \le 3.$$

12. If $f(x) = x + 5$ and $g(x) = x^2 - 3$ find

    a) $g(f(0))$ b) $g(f(x))$ c) $f(g(2))$ d) $f(g(x))$

13. If $f(x) = x - 1$ and $g(x) = \dfrac{1}{\sqrt{x-1}}$, give the explicit forms of

    a) $f(x) + g(x)$ b) $f(x)g(x)$ c) $\frac{f(x)}{g(x)}$ d) $f(g(x))$

Limits of some Rational Functions

14. Find

a) $\lim_{x\to 2} \dfrac{x-2}{x^2-5x+6}$ b) $\lim_{x\to 2} \dfrac{x^2-5x+6}{2x^2-3x-2}$ c) $\lim_{\lambda\to 1} \dfrac{\lambda^2-0.8\lambda-0.2}{\lambda-1}$

d) $\lim_{x\to 1} \dfrac{1-x^4}{1-x}$ e) $\lim_{x\to\infty} \dfrac{2x^2-3x+7}{3x^2+x-1}$ f) $\lim_{x\to\infty} \dfrac{2x^3+3x+2}{-5x^3+4x-1}$

Simple Differentiation

15. Find the derivative of each of the following functions.

a) $f(x) = (2x+5)^3$ b) $g(t) = \sqrt{t^2-4}$ c) $h(x) = \dfrac{1}{(2x+3)^{3/2}}$

d) $f(x) = \sin^3 x$ e) $g(x) = \cos(x^3)$ f) $h(x) = \sec(2x^2+3)$

g) $f(x) = e^{-x^2/2}$ h) $g(x) = x^2(2x-1)^4$ i) $h(\theta) = \theta\tan\theta$

j) $f(x) = x\cos 2x$ k) $g(x) = x^3\sin x$ l) $h(x) = x\ln x$

m) $f(x) = \dfrac{x+e}{x+\pi}$ n) $g(x) = \dfrac{2x^2+3}{3x-2}$ o) $h(t) = \dfrac{t}{\sqrt{t^2-4}}$

p) $f(x) = \dfrac{\sin x}{2x+5}$

Tangents and Normals

16. Find the equation of the tangent and the equation of the normal to each of the following curves.

a) $y = 4x + \dfrac{1}{x}$ at the point $(1,5)$

b) $y = x^3 - 1 + \dfrac{1}{x^2}$ at the point $(1,1)$

c) $y = \dfrac{\cos x}{1-\sin x}$ at the point where $x = \dfrac{\pi}{6}$

Stationary Points

17. Locate and identify the stationary points for

a) $y = 2x^3 - 9x^2 + 12x - 3$ b) $y = \frac{x}{1+x^2}$

c) $y = e^{2x}(1-x)$ d) $y = xe^{-x}$

e) $y = x^n e^{-x}$ for $n \in \mathbb{Z},\ n \geq 2$ f) $y = \frac{\ln x}{x}$

g) $y = 4x^3 - x^4$ h) $y = x + \cos x$

18. The slope of the curve $y = f(x)$ is given by

$$\frac{dy}{dx} = x^2(2x-1)(x-1)$$

Determine the nature of the stationary points.

19. The slope of the curve $y = f(x)$ is

$$\frac{dy}{dx} = 3(x-1)^2(x-2)^3(x-3)^4(x-4)$$

For what value or values of x does y have

a) a local maximum? b) a local mimimum?

Integration

20. a) Use your answer to 15(i) to find a primitive function (indefinite integral) of
$g(\theta) = \theta \sec^2\theta$ [Hint: from tables $\int \tan\theta d\theta = \ln|\sec\theta| + C$]

b) Use your answer to 15(j) to find a primitive function (indefinite integral) of
$h(x) = x \sin 2x$

c) Use your answer to 15(l) to find a primitive function (indefinite integral) of
$f(x) = \ln x$

21. The curve $y = f(x)$ has $\frac{dy}{dx} = 3x^2 - 2x + 1$ and passes through the point $(2, 3)$. Find $f(x)$.

22. Find y where

a) $\frac{dy}{dx} = \sqrt{x} + \frac{1}{\sqrt{x}}$ for $x > 0$ b) $\frac{dy}{dx} = \frac{x^2+1}{x^2}$ for $x \neq 0$

23. Without recourse to tables find

a) $\int e^x \, dx$ b) $\int_0^1 e^{3x} \, dx$

c) $\int_0^{\pi} \sin(2x) \, dx$ d) $\int \cos(3x) \, dx$

e) $\int (2x^3 + 3x^2 + 4x + 5) dx$ f) $\int \frac{1}{3x+1} \, dx$

g) $\int_{-2}^{-1} \frac{1}{2x-3} \, dx$ h) $\int (2x-3)^5 \, dx$

For all the above *indefinite* integrals, check your answers by differentiating.

Integration by Substitution

24. Evaluate each of the following indefinite integrals by using the suggested substitution:

a) $\int x^2 \left(x^3+1\right)^5 dx;\ u = x^3+1$

b) $\int (t-1)\sqrt{t^2-2t+4}\, dt;\ u = t^2-2t+4$

c) $\int (x+1)\, e^{x^2+2x+3}\ dx;\ u = x^2+2x+3$

d) $\int x \sin\left(x^2+1\right) dx;\ u = x^2+1$

e) $\int e^{\sin 2x} \cos 2x\, dx;\ u = \sin 2x$

f) $\int e^{2x} \cos\left(e^{2x}\right) dx;\ u = e^{2x}$

g) $\int \dfrac{dz}{z \ln z};\ u = \ln z$

h) $\int \dfrac{x+1}{x^2+2x-1}\ dx;\ u = x^2+2x-1$

i) $\int \dfrac{e^x}{1+e^x}\ dx;\ u = 1+e^x$

j) $\int \dfrac{x+1}{(x^2+2x-1)^5}\ dx;\ u = x^2+2x-1$

k) $\int \dfrac{\sin(\ln x)\, dx}{x};\ u = \ln x$

25. Evaluate each of the following definite integrals by using the suggested substitution:

a) $\int_0^4 x e^{x^2+1}\ dx;\ u = x^2+1$

b) $\int_{\pi/6}^{\pi/4} \dfrac{\sec^2 x}{\tan x}\ dx;\ u = \tan x$

c) $\int_0^1 \dfrac{3x}{(3x+1)^2}\ dx;\ u = 3x+1$

d) $\int_5^{20} \dfrac{t}{\sqrt{t-4}}\ dt;\ u = t-4$

Area and Volume

26. For each of the following functions, find the area between the curve $y = f(x)$ and the x-axis over the given range of x values.

a) $f(x) = 2x^2 - 1$ from $x = 1$ to 2

b) $f(x) = x^3 - 3x^2 + 4x$ from $x = 0$ to 2

c) $f(x) = 2x^2 + \dfrac{1}{x^2}$ from $x = 1$ to 2

d) $f(x) = e^{-x/3}$ from $x = 0$ to 3

e) $f(x) = 2\cos x + 3$ from $x = 0$ to π

f) $f(x) = \dfrac{1}{x+1}$ from $x = 0$ to 2

27. For each of the following functions, find the volume of the solid formed when the curve $y = f(x)$ over the given range of x is rotated about the x-axis.

a) $f(x) = x^2 + 1$ from $x = 0$ to 1 b) $f(x) = x + \dfrac{2}{x}$ from $x = 1$ to 2

c) $f(x) = e^{-x/4}$ from $x = 0$ to 2 d) $f(x) = \sec x$ from $x = 0$ to $\dfrac{\pi}{4}$

e) $f(x) = \dfrac{1}{x+1}$ from $x = 0$ to 1

Logarithms

28. Simplify:

a) $\log_4 12 - \log_4 3$ b) $\dfrac{\log_2 16}{\log_2 8}$ c) $\log_{1/3} 729$

29. Solve for x:

a) $2^{2x+1} - (17)2^x + 8 = 0$ b) $\ln x = 3\ln 2 + 2\ln 3$ c) $\log_x 125 = -3$

Remainder Theorem

30. Without division find the remainder when $p(x) = x^3 - 5x^2 + 10x - 6$ is divided by

a) $x - 2$ b) $x - 1$ c) $x + 2$ d) $x + 1$

which (if any) of these is a factor of $p(x)$?

Binomial Theorem

31. Use Pascal's triangle to expand the following:

a) $(x + y)^5$ b) $(3x - 2y)^4$ c) $(2x + 3)^6$

32. Use the Binomial Theorem to find the following.

a) The coefficient of x^{12} in the expansion of $(2x^3 - 3)^7$.

b) The coefficient of x^3 in the expansion of $\left(x^2 - \dfrac{2}{x}\right)^3$.

c) The term independent of x in the expansion of $\left(2x^2 + \dfrac{1}{x}\right)^9$.

Answers for Revision Questions

1. Answer for both: the interior and boundary of the triangle with vertices at $(0, 0)$,
$(2, 0)$, and $(2, 4)$.

2. a) $x < 0$ or $x > 1$ b) $1 < x < 2$ c) $x < -2$ or $x > 0$ d) $-1 < x < 1$ e) $-1 < x < 1$ f) $x < 1$ or $x \geq 5$

3. a) $-4 < x < 2$ b) $x < -5$ or $x > 1$ c) $-1 < x < -1/3$ d) $0 < x$

4. a) $\frac{1}{4}\sqrt{2}\left(1+\sqrt{3}\right)$ b) $\frac{1}{4}\sqrt{2}\left(1+\sqrt{3}\right)$ c) $-\left(2+\sqrt{3}\right)$ d) $-\sqrt{2}\left(\sqrt{3}-1\right)$

5. a) $\frac{4}{5}$ b) $\frac{3}{4}$ c) $\frac{12}{13}$ d) $\frac{5}{13}$ e) $\frac{63}{65}$ f) $\frac{56}{65}$ g) $\frac{6}{13}$ h) $-\frac{120}{119}$

6. a) $-\frac{527}{625}$ b) $\frac{87}{425}$ c) $-\frac{297}{304}$

7. a) amplitude $= 3$, period $= \pi$ b) amplitude $= 2$, period $= 6\pi$

8. a) $\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$ b) $4\sin\left(x+\frac{\pi}{3}\right)$ c) $2\sin\left(x-\frac{\pi}{6}\right)$ d) $4\sin\left(x-\frac{\pi}{4}\right)$

9. a) $-\sqrt{5} \leq x \leq \sqrt{5}$; $0 \leq y \leq \sqrt{5}$ b) $x \leq -\sqrt{5}$ or $x \geq \sqrt{5}$; $y \geq 0$
c) $x \geq 1$; $y \geq 0$ d) $x > 1$; $y > 0$ e) $x \neq 8$; $y \neq 0$
f) $\{x : 2n\pi \leq x \leq (2n+1)\pi,\ n \in \mathbb{Z}\}$; $0 \leq y \leq 1$ g) $\{x : x \neq (2n+1)\pi/2,\ n \in \mathbb{Z}\}$; $y \geq 1$ h) $\mathbb{R}$; $y \geq -1$

12. a) 22 b) $x^2+10x+22$ c) 6 d) x^2+2

13. a) $x-1+1/\sqrt{x-1}$ b) $\sqrt{x-1}$ c) $(x-1)^{3/2}$ d) $(1/\sqrt{x-1})-1$

14. a) -1 b) $-1/5$ c) 1.2 d) 4 e) $\frac{2}{3}$ f) $-\frac{2}{5}$.

15. a) $6(2x+5)^2$ b) $\frac{t}{\sqrt{t^2-4}}$ c) $-\frac{3}{(2x+3)^{5/2}}$ d) $3\sin^2 x\cos x$
e) $-3x^2\sin(x^3)$ f) $4x\sec(2x^2+3)\tan(2x^2+3)$ g) $-xe^{-x^2/2}$
h) $2x(6x-1)(2x-1)^3$ i) $\theta\sec^2\theta+\tan\theta$ j) $-2x\sin 2x+\cos 2x$ k) $x^2(x\cos x+3\sin x)$ l) $1+\ln x$ m) $\frac{\pi-e}{(x+\pi)^2}$
n) $\frac{6x^2-8x-9}{(3x-2)^2}$ o) $-\frac{4}{(t^2-4)^{3/2}}$ p) $\frac{(2x+5)\cos x-2\sin x}{(2x+5)^2}$

16. a) $y = 3x+2,\ x+3y = 16$
b) $y-\sqrt{3} = 2(x-\frac{\pi}{6}),\ y-\sqrt{3} = -\frac{1}{2}(x-\frac{\pi}{6})$

17. a) $(1,2)$ is a local maximum and $(2,1)$ is a local minimum
    b) $(1,1/2)$ is a local maximum and $(-1,-1/2)$ is a local minimum
    c) $(1/2, e/2)$ is a local maximum
    d) $(1, e^{-1})$ is a local maximum
    e) $(n, n^n/e^n)$ is a local maximum and $(0,0)$ is a local minimum if n is even and a point of inflection if n is odd
    f) (e, e^{-1}) is a local maximum
    g) $(3,27)$ is a local maximum and $(0,0)$ is a point of inflection
    h) $(\pi/2 + 2k\pi, \pi/2 + 2k\pi)$ $k \in \mathbb{Z}$ are points of inflection

18. There is a point of inflection for $x = 0$, a local maximum for $x = 1/2$, and a local minimum for $x = 1$

19. a) $x = 2$ b) $x = 4$

20. a) $\theta \tan\theta - \ln|\sec\theta|$ b) $-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x$ c) $x\ln x - x$

21. $f(x) = x^3 - x^2 + x - 3$

22. a) $y = \frac{2}{3}x^{3/2} + 2\sqrt{x} + C$ b) $y = x - \frac{1}{x} + C$

23. a) $e^x + C$ b) $\frac{1}{3}(e^3 - 1)$ c) 0 d) $\frac{1}{3}\sin(3x) + C$ e) $\frac{1}{2}x^4 + x^3 + 2x^2 + 5x + C$ f) $\frac{1}{3}\ln|3x+1| + C$ g) $\frac{1}{2}\ln\left(\frac{5}{7}\right)$ h) $\frac{1}{12}(2x-3)^6 + C$

24. a) $\dfrac{1}{18}\left(x^3+1\right)^6 + C$ b) $\dfrac{1}{3}\left(t^2 - 2t + 4\right)^{\frac{3}{2}} + C$ c) $\dfrac{1}{2}e^{x^2+2x+3} + C$
    d) $-\dfrac{1}{2}\cos\left(x^2+1\right) + C$ e) $\dfrac{1}{2}e^{\sin 2x} + C$ f) $\dfrac{1}{2}\sin\left(e^{2x}\right) + C$
    g) $\ln|\ln z| + C$ h) $\dfrac{1}{2}\ln\left|x^2 + 2x - 1\right| + C$ i) $\ln\left(1 + e^x\right) + C$
    j) $-\dfrac{1}{8\left(x^2+2x-1\right)^4} + C$
    k) $-\cos(\ln x) + C$

25. a) $\dfrac{1}{2}\left(e^{17} - e\right)$ b) $\dfrac{1}{2}\ln 3$ c) $\dfrac{2}{3}\ln 2 - \dfrac{1}{4}$ d) 66

26. a) $\dfrac{11}{3}$ b) 4 c) $\dfrac{31}{6}$ d) $3 - \dfrac{3}{e}$ e) 3π f) $\ln 3$

27. a) $\dfrac{28\pi}{15}$ b) $\dfrac{25\pi}{3}$ c) $2\pi\left(1 - \dfrac{1}{e}\right)$ d) π e) $\dfrac{\pi}{2}$

28. a) 1 b) 4/3 c) -6

29. a) $-1, 3$ b) 72 c) 1/5

30. a) 2 b) 0 c) -54 d) -22; $x - 1$ is a factor.

31. a) $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$
    b) $81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4$
    c) $64x^6 + 576x^5 + 2160x^4 + 4320x^3 + 4860x^2 + 2916x + 729$

32. a) -15120 b) -6 c) 672