LECTURE 9
The Mean Value Theorem

The Mean Value Theorem: Suppose that f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is at least one real number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

We now turn to one of the central theorems in calculus the Mean Value Theorem.

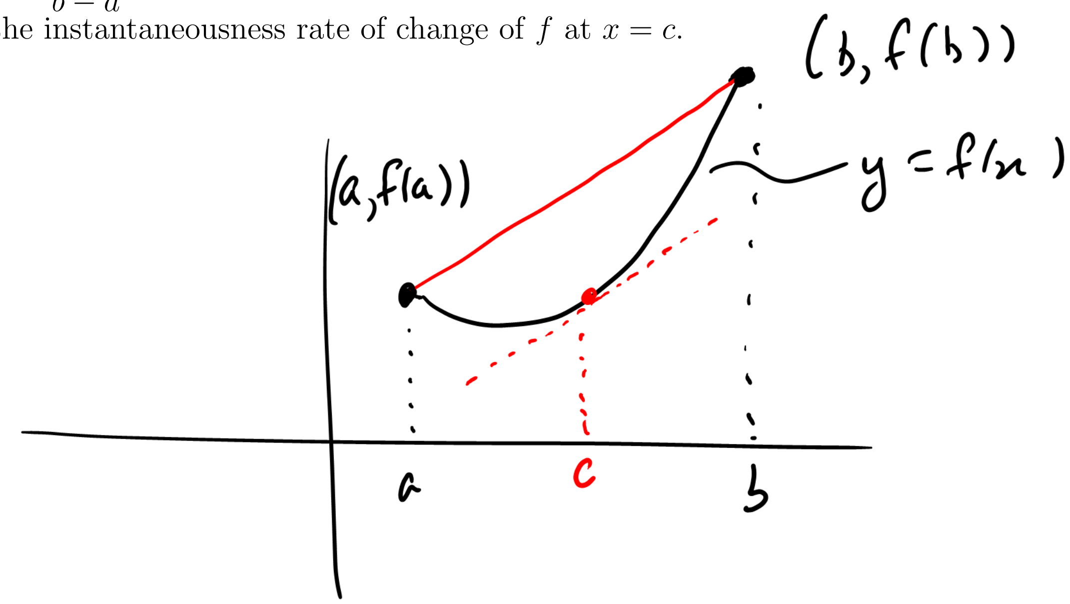
Suppose that you travel to Wollongong from Sydney and your average speed on the journey is 73 km/hr. The Mean Value Theorem guarantees that there will be at least on time on your journey where your speedometer reading is also 73. In other words you can't have an average speed of 73 km/hr without at least once traveling at 73 km/hr. Makes sense!

The Mean Value Theorem: Suppose that f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is at least one real number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Discussion and Sketch:

$\dfrac{f(b) - f(a)}{b - a}$ is the average rate of change of f over the entire interval $[a, b]$ while $f'(c)$ is the instantaneousness rate of change of f at $x = c$.



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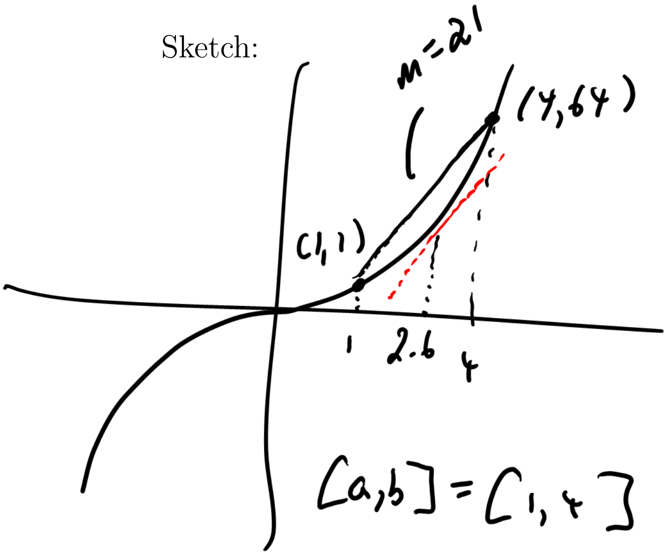
The jumping around from open to closed intervals is not really all that important but must be mentioned. It stems from the problem that the end of of an arc has no tangent.

You must be able to accurately state the mean value theorem for the exams ! The proof (available in your printed notes) is not examinable. Lets take a look at the theorem in action.

Example 1: Find c which satisfies the conclusions of the mean value theorem for $f(x) = x^3$ on the closed interval $[1, 4]$.

Sketch:



$$\frac{f(b)-f(a)}{b-a} = \frac{64-1}{4-1} = \frac{63}{3} = 21$$

$$f'(x) = 3x^2$$

$$\therefore \quad 3x^2 = 21$$

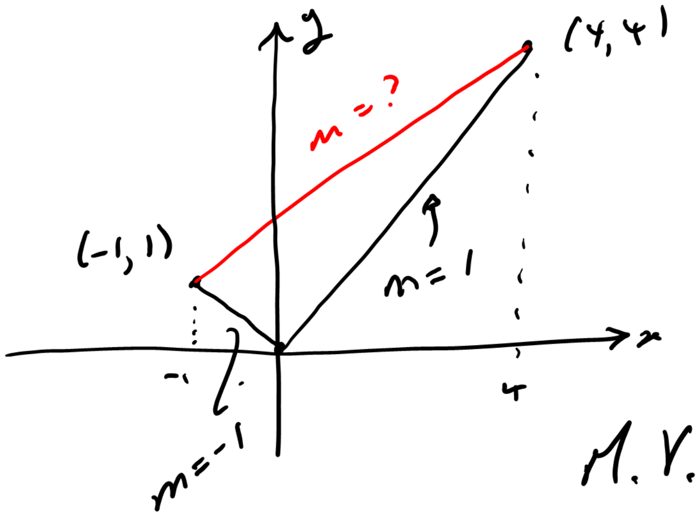
$$\Rightarrow x^2 = 7$$

$$\Rightarrow x = \sqrt{7} \cong 2.6$$

$$1 < 2.6 < 4$$

$\star \quad c = \sqrt{7} \approx 2.6 \quad \star$

Example 2: Show that $f(x) = |x|$ over the closed interval $[-1, 4]$ does not satisfy the conclusion of the mean value theorem. Explain.



$$\frac{f(b)-f(a)}{b-a} = \frac{4-1}{4--1} = \frac{3}{5}$$

<u>no</u> tangent to $y=|x|$ parallel to the red line!

M.V.T. does not apply since $y=|x|$ is continuous but <u>not</u> differential over $(-1,4)$

$\star$

One the the sneaky applications of the mean value theorem is the verification of certain inequalities. The next two examples are from your printed notes but are done slightly differently:

Example 3: Use the mean value theorem to prove that $\ln(x) < x - 1$ whenever $x > 1$.

We will show that $x - 1 - \ln(x) > 0$ whenever $x > 1$.

Note that it would also be OK to show that $\ln(x) - x + 1 < 0$ but it is a little easier to show that something is positive rather than negative.

Consider $f(t) = t - 1 - \ln(t)$ over the closed interval $[1, x]$.

Note that we have introduced the variable t for the function to avoid overloading on the $x's$.

Then f is continuous on the closed interval $[1, x]$ and differentiable on the open interval $(1, x)$ so the mean value theorem may be applied.

That is there is $c \in (1, x)$ such that $\dfrac{f(x) - f(1)}{x - 1} = f'(c)$
Now

$f(x) = x - 1 - \ln(x)$

$f(1) = 1 - 1 - 0 = 0$

$f'(t) = 1 - \dfrac{1}{t} \to f'(c) = 1 - \dfrac{1}{c}$.
Therefore $\dfrac{(x - 1 - \ln(x)) - 0}{x - 1} = 1 - \dfrac{1}{c} \to x - 1 - \ln(x) = \boxed{\overbrace{(1 - \frac{1}{c})}^{\text{positive}}\overbrace{(x - 1)}^{\text{positive}}}$.
So we have that $x - 1 - \ln(x) = (1 - \dfrac{1}{c})(x - 1)$.

Therefore all we need to do now is prove that $(1 - \dfrac{1}{c})(x - 1) > 0$.

Well $(x - 1)$ is clearly not negative since $x > 1$. Also

$c > 1 \longrightarrow \dfrac{1}{c} < 1 \longrightarrow -\dfrac{1}{c} > -1 \longrightarrow 1 - \dfrac{1}{c} > 0$.

So $(1 - \dfrac{1}{c})(x - 1) > 0$.

Thus $x - 1 - \ln(x) > 0$ whenever $x > 1$ as required.

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These are quite tricky. Let's do another one:

Example 4: Use the mean value theorem to prove that $\sqrt{x+4}-2<\frac{x}{4}$ whenever $x>0$.

We will show that $\frac{x}{4}-\sqrt{x+4}+2>0$ whenever $x>0$.

Consider $f(t)=\frac{t}{4}-\sqrt{t+4}+2$ over the closed interval $[0,x]$.

Then f is continuous on the closed interval $[0,x]$ and differentiable on the open interval $(0,x)$ so the mean value theorem may be applied.

That is there is $c\in(0,x)$ such that $\frac{f(x)-f(0)}{x-0}=f'(c)$. Now

$f(x)=\frac{x}{4}-\sqrt{x+4}+2$ so $f(0)=0$.

$f'(t)=\frac{1}{4}-\frac{1}{2}\frac{1}{\sqrt{t+4}}\longrightarrow f'(c)=\frac{1}{4}-\frac{1}{2}\frac{1}{\sqrt{c+4}}$.

Therefore via the M.V.T. there exists $c\in(0,x)$ such that:

$$\frac{(\frac{x}{4}-\sqrt{x+4}+2)-(0)}{x-0}=\frac{1}{4}-\frac{1}{2}\frac{1}{\sqrt{c+4}}\longrightarrow\frac{x}{4}-\sqrt{x+4}+2=x(\frac{1}{4}-\frac{1}{2}\frac{1}{\sqrt{c+4}}).$$

So all we need to show is that $x(\frac{1}{4}-\frac{1}{2}\frac{1}{\sqrt{c+4}})>0$.

Well $x>0$. So we only need to verify that

$\frac{1}{4}-\frac{1}{2}\frac{1}{\sqrt{c+4}}>0\quad$ if $\quad c>0$.

Now $\frac{1}{4}-\frac{1}{2}\frac{1}{\sqrt{c+4}}>0\iff\frac{1}{4}>\frac{1}{2}\frac{1}{\sqrt{c+4}}\iff 1>\frac{2}{\sqrt{c+4}}$

$\iff 1>\frac{4}{c+4}\iff c+4>4\iff c>0$.

Thus $\frac{x}{4}-\sqrt{x+4}+2>0$ whenever $x>0$ as required.

OR

$c>0$

$\Rightarrow \sqrt{c+4}>2$

$\Rightarrow \frac{1}{\sqrt{c+4}}<\frac{1}{2}$

$\Rightarrow \frac{1}{2}\frac{1}{\sqrt{c+4}}<\frac{1}{4}$

$\Rightarrow \frac{1}{4}-\frac{1}{2}\frac{1}{\sqrt{c+4}}>0$

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Example 5: By considering $f(t)=\sin(t)$ over the interval $[x,y]$ show that

$$|\sin(y)-\sin(x)|\leq y-x \text{ for all } x,y\in\mathbb{R}, x<y.$$

We begin by noting that $f(t)=\sin(t)$ is continuous over $[x,y]$ and differentiable over (x,y). Hence by the M.V.T. there exists $c\in(x,y)$ such that $\frac{f(y)-f(x)}{y-x}=f'(c)$.

Hence $\frac{\sin y-\sin(x)}{y-x}=\cos(c)$

$\sin y-\sin x=(y-x)\cos(c)\Rightarrow|\sin y-\sin x|=|y-x||\cos c|$

$\Rightarrow|\sin y-\sin x|\leq(y-x)$

$(y-x)$ since $y>x$

$0\leq|\cos c|\leq 1$

Your printed notes present a detailed revision of the theory of maxima and minima at this stage and you are encouraged to read the section carefully before the next lecture. An important note that I wish to stress however, is that the definitions of maxima and minima, increasing and decreasing **have nothing to do with calculus!!!** These are simple concepts that can be easily explained to a five year old child. We certainly use calculus as tools in this arena but the concepts themselves do not require the sledgehammer of the derivative. Lets have a look at the definition of an increasing function.

Definition: A function f is increasing on an interval I if for every two points a and b in I

$$a < b \longrightarrow f(a) < f(b)$$

In other words it's going up!

Example 6: Explain why $f(x) = x^3$ is increasing over every interval.

Assume $a<b$ RTP $f(a)<f(b)$

ie $a^3<b^3$

$b^3-a^3>0$

$b^3-a^3=(b-a)(b^2+ab+a^2)$

$=(b-a)(b^2+ab+\frac{a^2}{4}+\frac{3}{4}a^2)$

$=(b-a)((b+\frac{a}{2})^2+\frac{3}{4}a^2)$

$b^3-a^3>0 \Rightarrow b^3>a^3$

But doesn't $y = x^3$ have a stationary point at $x = 0$? That is a zero derivative?

It certainly has but that doesn't stop the function from increasing (going up) everywhere!

It is important to get the direction of the theorems correct.

For an open interval I we have

Theorem: If $f'(x) > 0$ for all $x \in I$ then f is increasing on I.

That is, a function with a positive derivative is increasing. This is **NOT** saying that an increasing function has a positive derivative! We have seen that $y = x^3$ is an increasing function without a positive derivative!

We close the lecture with a proof of this Theorem. In this proof we are trying to link the concept of increasing with the concept of a positive derivative. It is situations like this where the MVT saves the day.

Proof:
Let $a, b \in I$ be such that $a < b$. We must show that $f(a) < f(b)$.

From the fact that f is differentiable on I and Theorem 4.5.1, f is continuous on $[a, b]$ and differentiable on (a, b). Invoking the mean value theorem there is $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

That is

$$f(b) - f(a) = (b - a)f'(c)$$

But $(b - a) > 0$ and $f'(c) > 0$ so $f(b) - f(a) > 0$ and hence $f(a) < f(b)$ as required. ★

$\therefore f(a) < f(b)$