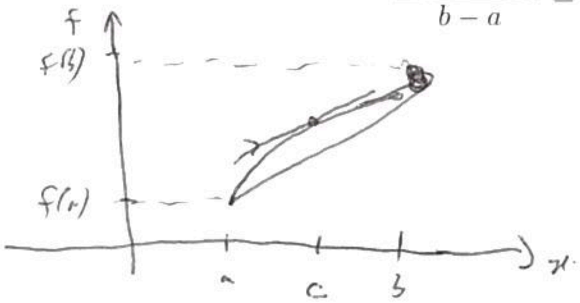
THE UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1131 Calculus

Section 5: - Mean Value Theorem.

Mean Value Theorem:

Suppose f is cts on $[a, b]$ and diffble on (a, b). Then there is a real number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$



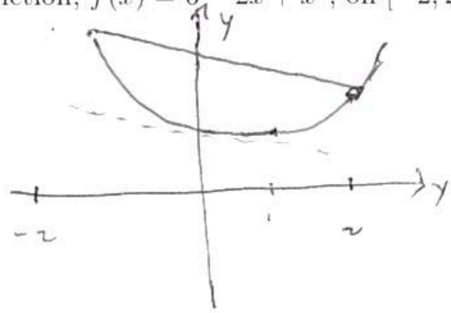
Ex: Demonstrate the Mean Value Theorem for the function, $f(x) = 6 - 2x + x^2$, on $[-2, 2]$.

f cts on $[-2,2]$, diff^{ble} on $(-2,2)$.

$$\frac{f(2) - f(-2)}{2 - -2} = f'(c), \quad c \in (-2,2)$$

$f'(c) = -2.$

Now $f'(c) = 2c - 2 = -2 \Rightarrow c = 0$



We can use the MVT to do a range of problems.

Ex: Use the MVT to find an approximate value of $\sqrt{17}$.

Let $f(x) = \sqrt{x}$ on $[16, 17]$

f cts on $[16,17]$, f diff^{ble} on $(16,17)$

By M.V.T.

$$f(17) - f(16) = \frac{\sqrt{17} - 4}{17 - 16} = \frac{1}{2\sqrt{c}},$$

for some $c \in (16, 17)$.

$\therefore \sqrt{17} - 4 = \frac{1}{2\sqrt{c}}$.

Now $c > 16 \Rightarrow \frac{1}{2\sqrt{16}} < \frac{1}{8}$

$\therefore \sqrt{17} - 4 < \frac{1}{8}$

$\sqrt{17} < 4\frac{1}{8} = 4.125$

$(\sqrt{17} \simeq 4.123)$

Ex: Use the MVT to prove that $\tan x \geq x$ for all $x \in [0, \frac{\pi}{2})$.

Let $f(x) = \tan x$ on $[0, x]$, for some $x \in [0, \frac{\pi}{2})$

f cts on $[0, x]$, diff^{ble} on $(0, x)$

By M.V.T.

$$\frac{\tan x - \tan 0}{x - 0} = \sec^2 c, \quad c \in (0, x)$$

$0 \quad c \quad x$

$$\therefore \frac{\tan x}{x} = \frac{1}{\cos^2 c} = \sec^2 c \geq 1$$

$\therefore \tan x \geq x$

Ex: Prove that $e^{\frac{1}{2}x} \geq \frac{1}{2}x + 1$ for all $x \geq 0$.

Let $f(t) = e^{\frac{1}{2}t}$, on $[0, x]$, x

f cts on $[0, x]$, f diff^{ble} on $(0, x)$

so by M.V.T. $\frac{f(x) - f(0)}{x - 0} = \frac{1}{2}e^{\frac{1}{2}c}$

$c \in (0, x)$

$0 \quad c \quad x$

$$\therefore \frac{e^{\frac{1}{2}x} - 1}{x} = \frac{1}{2}e^{\frac{1}{2}c} \geq \frac{1}{2}$$

$\therefore e^{\frac{1}{2}x} \geq \frac{1}{2}x + 1$

Ex: Prove that for all real x and y, $|\sin x - \sin y| \leq |x - y|$.

Consider $f(t) = \sin t$ on $[y, x]$, $y \leq x$.

f cts and diff^{ble} everywhere. By MVT,

$$\frac{f(x) - f(y)}{x - y} = f'(c), \quad c \in (y, x)$$

$$\frac{\sin x - \sin y}{x - y} = \cos c$$

$$\therefore \frac{|\sin x - \sin y|}{|x - y|} = |\cos c| \leq 1$$

$\Rightarrow |\sin x - \sin y| \leq |x - y|$

Error Estimates:

Suppose I measure an angle in radians to be 0.7^c and I take the sine of that angle. If the error involved in my measurement is approximately 0.01^c what is the worst error involved in taking the sine of this number?

That is, if $f(x) = \sin x$ and $\Delta x = \pm 0.01$, we want a bound on the size of

$$|\Delta f(x)| = |f(x + \Delta x) - f(x)|.$$

Theorem: If $f'(x)$ exists, then

$$|\Delta f(x)| = |f(x + \Delta x) - f(x)| \approx f'(x)\Delta x.$$

start √

~~Ex: In the above example,~~ $\cancel{\Delta f(x) \approx \cos 0.7 \times 0.01 \approx 7.65 \times 10^{-3}}$.

Ex: In an isosceles triangle with two equal sides a and included angle 60^o, the percentage change in a is 10%. Find the percentage change in the area.

% change in a is $\frac{\Delta a}{a}$ %

$\therefore \frac{\Delta a}{a} = 0.1 \Rightarrow \boxed{\Delta a = 0.1\,a}$

Area $= \frac{1}{2} a^2 \sin 60^\circ$

$= \frac{1}{2} a^2 \frac{\sqrt{3}}{2} = \frac{\sqrt{3}\,a^2}{4}$

Let $f(a) = \frac{\sqrt{3}\,a^2}{4}$

% change in f is

$\frac{\Delta f(a)}{f(a)} = \frac{f(a+\Delta a) - f(a)}{f(a)}$

$\approx \frac{f'(a)\,\Delta a}{f(a)}$

$= \frac{\frac{\sqrt{3}\,a}{2}\, 0.1\, a}{\frac{\sqrt{3}\,a^2}{4}}$

$= \underline{0.2}$

Here are some consequences of the MVT:

Definition: A function f defined on $[a, b]$ is said to be **increasing** if $f(x) > f(y)$ whenever $x > y$, and **decreasing** when $f(x) < f(y)$ whenever $x > y$.

Theorem: Suppose f is diffble on (a, b),

(i) If $f'(x) > 0$ for all $x \in (a, b)$ then f is increasing on (a, b).

(ii) If $f'(x) = 0$ for all $x \in (a, b)$ then f is constant on (a, b)

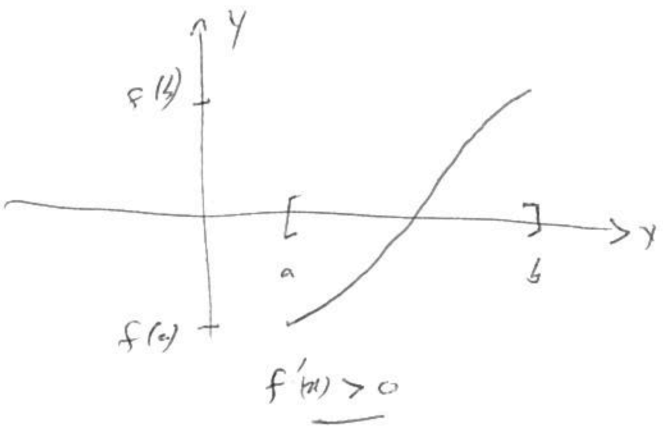
(iii) If $f'(x) < 0$ for all $x \in (a, b)$ then f is decreasing on (a, b).

Proof: The proof of all of these comes from applying the MVT to f on (x, y), any subset of (a, b) giving

$$\frac{f(y) - f(x)}{y - x} = f'(c).$$

In the first case we have $f(y) > f(x)$ whenever $y > x$ so f is increasing. Similarly for (iii). For (ii), we have $f(x) = f(y)$, for all x and y so f is a constant.

Theorem: Suppose that f is cts on $[a, b]$ and diffble on (a, b) and that $f(a)$ and $f(b)$ have opposite signs. If $f'(x) > 0$ for all $x \in (a, b)$ (or $f'(x) < 0$ for all $x \in (a, b)$), then f has **exactly** one real zero in (a, b).



Ex: $f(x) = x^3 + x + 1$ on $[-1, 1]$.

f cts & diff^{ble} everywhere.

$f(1) = 3 > 0$, $f(-1) = -1 < 0$

$\therefore$ By I.V.T. f has at least one real root in $[-1, 1]$.

$f'(c) = 3x^2 + 1 > 0$ for all x

$\therefore f$ has exactly one real root.

Ex: Show that $5x^5 + 2x + 1 = 0$ has exactly one real solution.

Let $f(x) = 5x^5 + 2x + 1$ on $[-1, 0]$

f cts & diff^{ble} everywhere.

$f(0) = 1 > 0$, $f(-1) = -6 < 0$

by I.V.T. f has at least one real root in $[-1, 0]$

Now consider f on $[a, b]$ for any $a, b \in \mathbb{R}$ with $a \leq -1$, $b \geq 0$

~~$f'(x) \geq x$~~

By above, f has at least one real root in $[a, b]$

$f'(x) = 25x^4 + 2 > 0$ for all x.

$\therefore f$ has exactly one real root on $[a, b]$

Since a, b are arbitrary, f has exactly 1 real root on $\mathbb{R}$.

So $5x^5 + 2x + 1 = 0$ has exactly one real solution

Theorem: Suppose that f, g are differentiable functions such that $f(a) = g(a)$ and for all $x > a$, we have $f'(x) > g'(x)$.
Then $f(x) > g(x)$ for all $x > a$.

Ex: Prove that $\sin x < x$ for all $x > 0$.

Let $f(x) = \sin x$
$g(x) = x$

For $x \in (0, \frac{\pi}{2})$

$f'(x) = \cos x$, $g'(x) = 1$

& $\cos x < 1$ for all $x \in (0, \frac{\pi}{2})$

$\therefore$ $\sin x < x$ for all $x \in (0, \pi/2)$

For $x \geq \frac{\pi}{2}$, $f(x)$ ~~sin x~~ ≤ 1

and $g(x) \geq \frac{\pi}{2} > 1$

$\therefore$ $f(x) \leq g(x)$ for all $x \geq 0$

Types of points:

We wish to classify all the sorts of interesting points a function can have.

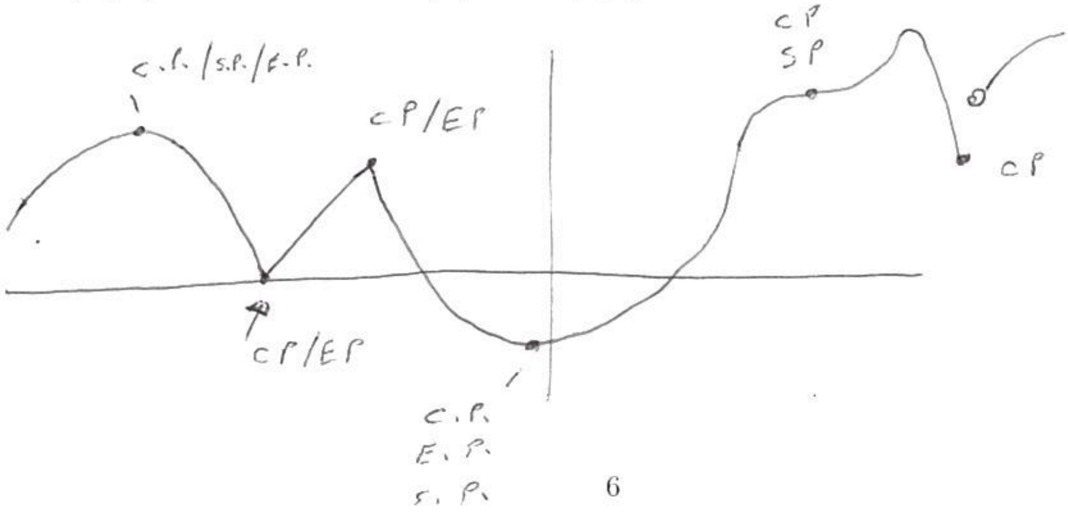
Definition:

Suppose that f is a function defined on an interval $[a, b]$ and let $x_0 \in [a, b]$.

(i) x_0 is called a **critical point** if $f'(x_0) = 0$ or if f is not differentiable at x_0.

(ii) x_0 is called an **extreme point** if x_0 is a local maximum or local minimum.

(iii) x_0 is called a **stationary point** if $f'(x_0) = 0$.



In practise, to find the (global) maximum and minimum, we need to find the stationary points and check their y values and also check the y values at the end points.

Ex: Find the global max and min of $f(x) = x^3 - 3x^2 + 1$ on the interval $[0, 4]$.

$f(0) = \underline{1}$

$f(4) = 64 - 48 + 1 = \underline{17}$

$f'(x) = 3x^2 - 6x = 0$ at S.P.

$\underline{x = 0}$, $\underline{x = 2}$

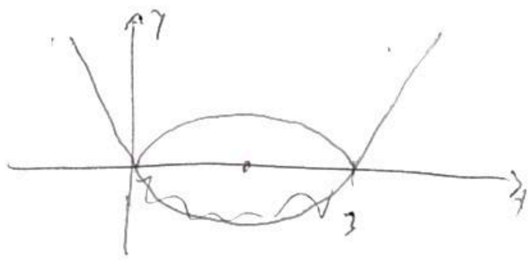
$f(0) = 1$

$f(2) = 8 - 12 + 1 = -3$.

$\therefore$ Global min. is -3

Global max is 17.

Ex: Find the local max and min of $f(x) = |x - 3||x|$

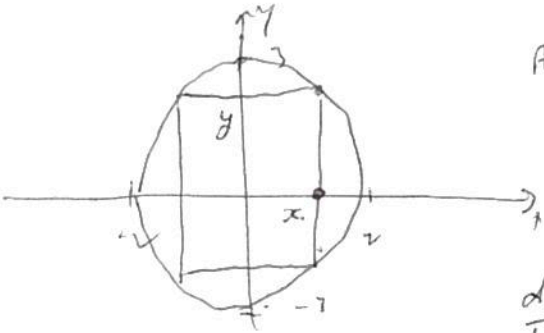


[f has no global max
f has global min 0.]

Local max at $x = 3/2$ of $\frac{9}{4}$

Local min of 0 at $x = 0$, $x = 3$.

Ex: Find the dimensions of the rectangle (with vertical and horizontal sides) of maximum area which can be inscribed in the ellipse, $\frac{x^2}{4} + \frac{y^2}{9} = 1$.



A : Area $= 2x \times 2y$

$= 4 \cdot x \sqrt{\left(1 - \frac{x^2}{4}\right)9}$

$= 6x \cdot \sqrt{4 - x^2}$

$\frac{dA}{dx} = 6\sqrt{4-x^2} - \frac{6x^2}{\sqrt{4-x^2}} = 0$ for max or mi

$\Rightarrow \; x^2 = 4 - x^2$

$x = \sqrt{2}, \quad y^2 = \left(1 - \frac{1}{2}\right)9 = \frac{9}{2}$

$y = 3/\sqrt{2}$.

L'Hôpital's Rule:

We return to the problem of calculating limits.

Theorem: (L'Hôpital's Rule)

Suppose that f and g are differentiable functions (except possibly at a) and that $f(a)$ and $g(a)$ are both equal to 0, or both tend to ∞ as $x \to a$.

If $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Proof: (Outline). Suppose we have the case $f(a) = g(a) = 0$. Apply the MVT to f and g on the interval (a, x), where $x > a$, so that for some $c, d \in (a, x)$ we have $\frac{f(x)-0}{x-a} = f'(c)$ and $\frac{g(x)-0}{x-a} = g'(d)$.

Hence

$$\frac{f(x)}{g(x)} = \frac{\frac{f(x)}{x-a}}{\frac{g(x)}{x-a}} = \frac{f'(c)}{g'(d)}.$$

Hence as $x \to a^+$ we have $c \to a^+$ and $d \to a^+$, so that if the limit of $\frac{f'(x)}{g'(x)}$ exists as $x \to a$, we have $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$.

Ex: $\lim_{x\to 0} \dfrac{e^x - 1}{\sin 2x}$.

$$= \lim_{x\to 0} \frac{e^x}{2\cos 2x} = \frac{1}{2}.$$

Ex: $\lim_{x\to 1} \dfrac{1 - x + \log x}{1 + \cos \pi x}$.

$$= \lim_{x\to 1} \frac{-1 + \frac{1}{x}}{-\pi \sin \pi x}$$

$$= \lim_{x\to 1} \frac{-\frac{1}{x^2}}{-\pi^2 \cos \pi x}$$

$$= \frac{1}{\pi^2}.$$

When dealing with limits to infinity, we need the following version of L'Hôpital's rule.

Theorem: Suppose f and g are differentiable. Suppose further that $f(x) \to 0$ and $g(x) \to 0$ as $x \to \infty$ (or $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to \infty$).
If $\lim_{x\to\infty} \dfrac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

Ex: $\lim_{x\to\infty} \frac{\log x}{x}$.

$$= \lim_{x\to\infty} \frac{1/x}{1}$$

$$= 0$$

Ex: $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x . = L$

$$\ln L = \lim_{x\to\infty} x \ln\left(1+\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

$$\stackrel{L}{=} \frac{\frac{-\frac{1}{x^2}}{1+\frac{1}{x}}}{-\frac{1}{x^2}}$$

$$= 1$$

$$\therefore \underline{L = e}$$

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Section 6: - Inverse Functions.

We have intuitively thought of a function as a rule, which starts from one real number and produces another. We now ask the question as to when we can *reverse* the procedure.
For example, under the function $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = 2x + 3$, the real number 5 maps to 13. On the other hand what number maps on to 10? Answer 3.5. Indeed given the y value, the corresponding x-value it came from is $\frac{y-3}{2}$. This new rule, is itself a function, which can be written as $g(x) = \frac{x-3}{2}$. We say that these two functions are **inverses** of each other the write $g(x) = f^{-1}(x)$. (Note that the index does NOT mean 'one over').

Also note that if we compose f and g we obtain the identity function, i.e. $f \circ g(x) = f(g(x)) = f(\frac{x-3}{2}) = x$ and $g \circ f(x) = g(f(x)) = g(2x + 3) = x$. Hence:

Definition: Given a function $f : A \to B$, if there is a function $g : B \to A$ such that $f \circ g(x) = x$ and $g \circ f(x) = x$, then we say that g is the inverse of f and write $g = f^{-1}$.

Ex: Show that if $f(x) = e^x$ then $g(x) = \log x$ is the inverse of f.

$f \circ g(x) = f(\ln x) = e^{\ln x} = x$

$g \circ f(x) = g(e^x) = \ln e^x = x.$

$\therefore f = g^{-1}$

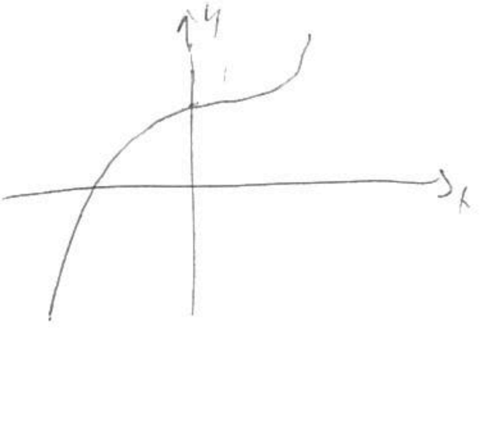
Clearly not all functions have inverses, for example $f(x) = x^2$. The y value 9 came from both 3 and -3.

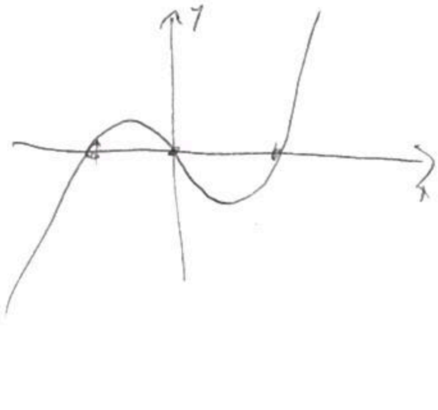
When does a given function f defined on an interval $[a, b]$ have an inverse?

One simple test is known as the *horizontal line test.* It says that if we look at the graph of f with domain D and co-domain R and draw any horizontal line, $y = b$, where $b \in R$ then f will have an inverse if the line cuts the graph at **exactly one point**.

Ex: Draw $y = x^3 + x + 1$ and $y = x^4 - x^2$ to illustrate this.

$= x(x^2 - 1)$





Theorem: Suppose f is differentiable on (a, b) and $f'(x) \neq 0$ for all $x \in (a, b)$ then f has an inverse on (a, b).

Ex: $y = x^3 + x + 1$. (Note that although this function has an inverse, it is not easy to explicitly write down the formula for the inverse.)

$y' = 3x^2 + 1 \neq 0$ for any real x.

$\therefore$ $f(x) = x^3 + x + 1$ has an inverse on $\mathbb{R}$

Ex: $f(x) = 2x + \sin x$.

$f'(x) = 2 + \cos x \neq 0$

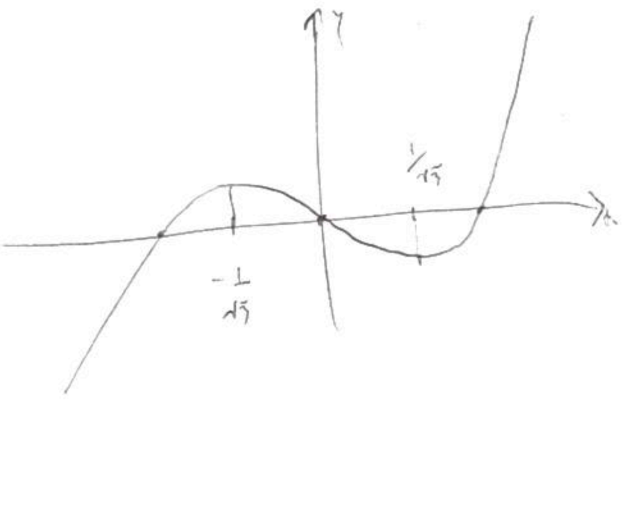
$\therefore$ f has an inverse.

[MAPLE]

We can sometimes restrict the domain of a function f so that although f does not have an inverse on its natural domain, it does on this restricted domain.

Ex: Find maximal regions on which the function $f(x) = x^3 - x$ has an inverse.

$f'(x) = 3x^2 - 1 = 0$ at $x = \pm \frac{1}{\sqrt{3}}$



f has inverse for $x \geq \frac{1}{\sqrt{3}}$

& for $-\frac{1}{\sqrt{3}} \leq x \leq \frac{1}{\sqrt{3}}$

& for $x \leq -\frac{1}{\sqrt{3}}$

Suppose f has an inverse on (a, b) and f is differentiable on (a, b). How do we find the derivative of the inverse?

Theorem: Suppose f is diffble on (a, b) and has an inverse $g(x)$ on (a, b), then

$$g'(x) = \frac{1}{f'(g(x))}.$$

Proof: $f \circ g(x) = x$

$\therefore \ f(g(x)) = x$

$\frac{d}{dx}$ $\quad f'(g(x)) \cdot g'(x) = 1$

$\therefore \ g'(x) = \dfrac{1}{f'(g(x))}$

Ex: Let $f(x) = x^3 + x + 1$, with inverse function g. Find $g'(1)$.

Note that $f(0) = 1$

$\Rightarrow \ g(1) = 0$

$g'(1) = \dfrac{1}{f'(g(1))} = \dfrac{1}{f'(0)} = 1$

$f' = 3x^2 + 1.$

Ex: Let $f(x) = 2x + \sin x$. Find $(f^{-1})'(\cancel{\pi})$. (2π) ✻

$f(\pi) = 2\pi$

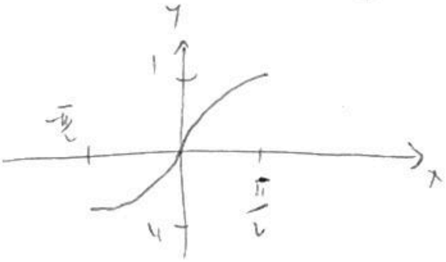
$\therefore \ g(2\pi) = \pi$

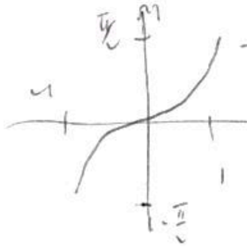
$g'(\pi) = \dfrac{1}{f'(g(2\pi))} = \dfrac{1}{f'(\pi)}$

$= \dfrac{1}{2 + \cos \pi} = 1$

Inverse Trigonometric Functions:

From $\underline{y = \sin x}$, restrict domain to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ to obtain an inverse.





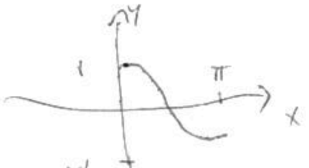
$\underline{\text{Dom:}}$ $-1 \le x \le 1$

$\underline{\text{Range:}}$ $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

$f(x) = \sin^{-1} x$ is odd,

i.e. $\sin^{-1}(-x) = -\sin^{-1}(x)$

For $\underline{y = \cos x}$, restrict dom to $\underline{0 \le x \le \pi}$ to obtain an inverse.



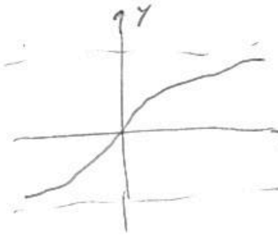
$y = \cos^{-1} x$.

$\underline{\text{Dom:}}$ $-1 \le x \le 1$

$\underline{\text{Range:}}$ $0 \le y \le \pi$

$\underline{\text{N.B.}}$ $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

$y = \tan x$, restrict dom to $-\frac{\pi}{2} < x < \frac{\pi}{2}$.



$\underline{\text{Dom:}}$ $-\infty < x < \infty$

$\underline{\text{Range:}}$ $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$\tan^{-1}(-x) = -\tan^{-1}(x)$ (odd f^n)

$\underline{\text{Derivatives}}$

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} (\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

Ex: Find $\frac{d}{dx}\csc^{-1}x$

Let $y = \operatorname{cosec}^{-1} x$ =

$x = \operatorname{cosec} y = \dfrac{1}{\sin y}$

$\therefore \sin y = \dfrac{1}{x}$

$\dfrac{dx}{dy} = -1(\sin y)^{-2} \cdot \cos y$

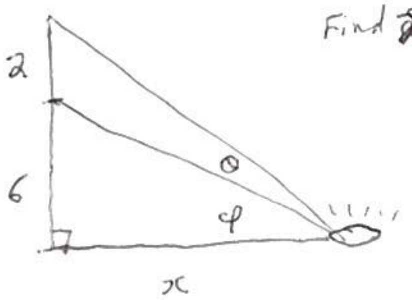
$= -\dfrac{\cos y}{\sin^2 y}$

$= \dfrac{\sqrt{x^2-1}}{x} \cdot x^2 = x\sqrt{x^2-1}$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{x\sqrt{x^2-1}}$

valid for $-1 \leq x \leq 1$, $x \neq 0$

Ex: a. Find $\frac{d}{dx}(\cot^{-1}(x))$.
b. A statue 2 metres high is mounted on a pedestal. The base of the statute is 6m above the eye-level of an observer. How far from the base of the pedestal should the observer stand to get the 'best' view.



Find x to maximise θ.

From the diagram

$\tan\varphi = \dfrac{6}{x}$

$\tan(\varphi+\theta) = \dfrac{8}{x}$.

$\therefore \varphi = \tan^{-1}\dfrac{6}{x}, \quad \varphi+\theta = \tan^{-1}\dfrac{8}{x}$

$\therefore \theta = \tan^{-1}\dfrac{8}{x} - \tan^{-1}\dfrac{6}{x}$

$\dfrac{d\theta}{dx} = \dfrac{-8/x^2}{1+\frac{64}{x^2}} + \dfrac{\frac{6}{x^2}}{1+\frac{36}{x^2}}$

$= \dfrac{-8}{x^2+64} + \dfrac{6}{x^2+36}$

Set $\dfrac{d\theta}{dx} = 0$ for a s.p.

$\Rightarrow \dfrac{8}{x^2+64} = \dfrac{6}{x^2+36} \Rightarrow 2x^2 = -8\times36+6\times64$

$\Rightarrow x^2 = 48 \Rightarrow \underline{x = 4\sqrt{3}\text{ m.}}$

[$\dfrac{d^2\theta}{dx^2} < 0$ at $x = 4\sqrt{3}$ so we have a max.]

Ex: Find a. $\sin^{-1}(\sin(\frac{5\pi}{3})$ b. $\sin(\sin^{-1}(-\frac{1}{2}))$, c. $\sin(2\cos^{-1}(\frac{4}{5}))$.

a) $\sin^{-1}\left(\sin\frac{5\pi}{3}\right)$

$= \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$= -\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$= -\frac{\pi}{3}$

b) $\sin\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

$= -\sin\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$

$= -\frac{1}{2}$.

c) Let $\alpha = \cos^{-1}\frac{4}{5}$

$\Rightarrow \cos\alpha = 4/5$

$\therefore \sin(2\alpha)$

$= 2\sin\alpha\cos\alpha$

$= 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25}$.

Theorem

For $-1 \le x \le 1$ we have

$$\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}.$$

Proof:

Let $f(x) = \cos^{-1}x + \sin^{-1}x$.

for $-1 \le x \le 1$.

$f'(x) = \frac{-1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1-x^2}} = 0$

$\therefore f$ is constant

$f(0) = \frac{\pi}{2}$

$\therefore f(x) = \frac{\pi}{2}$ for all $x \in [-1, 1]$

Ex: Prove that $\sin^{-1}(x) + \sin^{-1}\sqrt{1-x^2} = \frac{\pi}{2}$.

Let $f(x) = \sin^{-1}(x) + \sin^{-1}\sqrt{1-x^2}$

$f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-(1-x^2)}}\cdot\frac{-x}{\sqrt{1-x^2}}$

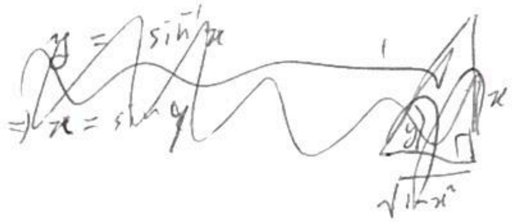
$= \frac{1}{\sqrt{1-x^2}} - \frac{\cancel{x}}{x\sqrt{1-x^2}} = 0$

$\therefore f$ is constant on $[-1, 1]$.

$f(0) = \frac{\pi}{2}$

$\therefore f(x) = \pi/2$ for all $x \in [-1, 1]$

Integrals Involving Inverse Trigonometric Functions:



since $\dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1-x^2}}$

we have $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$

Similarly
$$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + C$$

& $\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$