School of Mathematics and Statistics
Math1131-Algebra

Lec18. Matrices: Transposes and Inverses

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Matrix multiplication to extract rows or columns

Exercise 1. Let

$$B = \begin{pmatrix} 3 & 1 & -5 & 4 \\ 8 & 7 & 2 & 2 \\ 9 & -1 & -2 & -7 \end{pmatrix}.$$

a) Find a (column) vector $\overrightarrow{\boldsymbol{u}}$ such that $B\overrightarrow{\boldsymbol{u}}$ is the third column of B.

b) Find a (row) vector $\overrightarrow{\boldsymbol{v}}$ such that $\overrightarrow{\boldsymbol{v}}B$ is the second row of B.

c) Find a vector $\overrightarrow{\boldsymbol{w}}$ such that $B\overrightarrow{\boldsymbol{w}}$ is 2 times the first column of B plus 5 times the third column of B.

Transpose of a matrix

Transpose of a matrix.

For any $m \times n$ matrix A, its transpose A^T is the $n \times m$ matrix whose columns are the rows of A.
That is,

$$[A^T]_{ij} = [A]_{ji}$$

Example. For example, if

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 4 & 7 & 8 \end{pmatrix}$$

then

$$A^T = \begin{pmatrix} 3 & 4 \\ 1 & 7 \\ 2 & 8 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \iff B^T = \begin{pmatrix} 3 & 6 & 9 \end{pmatrix}$$

Transpose of a matrix

Exercise 2. Let $A = \begin{pmatrix} 3 & -5 \\ -7 & 8 \\ 0 & -1 \end{pmatrix}$

a) A is a $\ldots \times \ldots$ matrix and its transpose is a $\ldots \times \ldots$ matrix.

b) The transpose of A is denoted $\ldots$ which we read "A *transpose*".

c) The transpose of A is:

Transpose of a product

Transpose of a product of matrices.

- If A and B are matrices for which the *sum* $A+B$ is defined then
$$(A+B)^T = A^T + B^T.$$
- If A and B are matrices for which the *product* AB is defined then
$$(AB)^T = B^T A^T.$$

TIP!

Note that when we take the transpose of a product, the matrices are swapped. (Nothing special happens when we take the transpose of a sum).

Exercise 3. Let A, B and C are matrices for which the product ABC is defined. Conjecture (= guess) what $(ABC)^T$ is, and then prove your conjecture.

(For fast students) Could you extend this result to a product of n matrices?

Symmetric matrices

The transpose of the transpose of a matrix is the original matrix.

For any matrix A, we have $(A^T)^T = A$.

An operation which when composed with itself gives the identity function (i.e. doing it twice amounts to doing nothing) is called an *involution*.
Taking the transpose of a matrix is an involution.
Complex conjugation is another example of an involution.

Definition (Symmetric matrix).

An $n \times n$ matrix A is said to be *symmetric* *if and only if* $A = A^T$.

For example, $B = \begin{pmatrix} 1 & 6 & 8 \\ 6 & 5 & 7 \\ 8 & 7 & 2 \end{pmatrix}$ is symmetric.

Note that a symmetric matrix must necessarily be square (necessary condition).
Symmetric matrices have remarkable properties which are studied in second year (MATH2501).

Symmetric matrices

Exercise 4. Let $A = \begin{pmatrix} 3 & -5 & \dots \\ \dots & 8 & \dots \\ 0 & \dots & \dots \end{pmatrix}$

Write values of the missing entries which make A symmetric.

These matrices are called symmetric because ....

Exercise 5. Show that $C = A^T A$ is symmetric for any matrix A.

Transpose and dot product

Dot product of two vectors written as a matrix product.

If $\overrightarrow{u}$ and $\overrightarrow{v}$ are column vectors, we can consider them as $n \times 1$ matrices.
The *dot product* of $\overrightarrow{u}$ and $\overrightarrow{v}$ is equal to the *matrix product* of the transpose of $\overrightarrow{u}$, which is a row vector, and the column vector $\overrightarrow{v}$:

$$\overrightarrow{u} \cdot \overrightarrow{v} = \overrightarrow{u}^T \overrightarrow{v}.$$

Example.
If

$$\overrightarrow{u} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{v} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

then

$$\overrightarrow{u} \cdot \overrightarrow{v} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = 3 \times 2 + 1 \times 0 + (-1) \times 1 = 5.$$

and

$$\overrightarrow{u}^T \overrightarrow{v} = \begin{pmatrix} 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = 3 \times 2 + 1 \times 0 + (-1) \times 1 = 5.$$

Inverses

Consider the equation

$$ax = b$$

for real numbers a, b, x with $a \neq 0$. How do we solve this for x?

We multiply both sides by a^{-1} to give

$$a^{-1}ax = a^{-1}b$$

and because $a^{-1}a = 1$, this simplifies to

$$x = a^{-1}b$$

which we often write as

$$x = \frac{b}{a}.$$

Could we do the same for matrices?

Inverses

The matrices that acts like the number 1 are the identity matrices I.

Suppose A, B and X are matrices and we want to solve the matrix equation

$$AX = B.$$

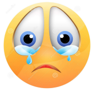
If we could find a matrix A^{-1} with the property that

$$A^{-1}A = I$$

then we could multiply both sides of the equation on the left by A^{-1} to give

$$\begin{aligned} A^{-1}AX &= A^{-1}B \\ \Longrightarrow \qquad IX &= A^{-1}B \\ \Longrightarrow \qquad X &= A^{-1}B. \end{aligned}$$

But can we find a suitable A^{-1}?



Sadly enough, not always. Some matrices have an inverse ... and others just do not! For instance if a matrix is not square (i.e. its number of rows is different from its number of columns), it cannot have an inverse.

Inverse of a matrix

Example 6. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -1 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$.

a) Find AB and BA.

b) Hence solve the matrix equation $\begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix} X = \begin{pmatrix} 5 & 7 & -1 \\ 19 & 21 & -5 \end{pmatrix}$.

Inverse of a matrix

Example 6, continued. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -1 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$.

a) Find AB and BA.

b) Hence solve the matrix equation $\begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix} X = \begin{pmatrix} 5 & 7 & -1 \\ 19 & 21 & -5 \end{pmatrix}$.

Checking our answers with Maple

```
> with(LinearAlgebra):
> # Enter the matrices column by column

  A := < <1,3>|<2,8> >;
  B := < <4,-3/2>|<-1,1/2> >;
```

$$A := \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$$

$$B := \begin{bmatrix} 4 & -1 \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

```
> # For matrix multiplication, use . not *

  AB := A.B;
  BA := B.A;
```

$$AB := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
> # We solve the equation

  C := < <5,19>|<7,21>|<-1,-5> >;
  X := B.C;
```

$$C := \begin{bmatrix} 5 & 7 & -1 \\ 19 & 21 & -5 \end{bmatrix}$$

$$X := \begin{bmatrix} 1 & 7 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

```
> # We check our answer
  A.X;
```

$$\begin{bmatrix} 5 & 7 & -1 \\ 19 & 21 & -5 \end{bmatrix}$$

Inverse of a matrix

Definitions and properties (Inverse of a matrix).

- For a matrix A,
  - B is a *left inverse* of A means $BA = I$.
  - B is a *right inverse* of A means $AB = I$.
  - If B is both a left and right inverse of A then B is an *inverse* of A.
- If A has an inverse then that inverse is unique and is denoted A^{-1} (say "A inverse"). In that case, we say that A is *invertible* or *non-singular*.
- Conditions for matrices to have an inverse :
  - Only *square* matrices can have an inverse (necessary condition).
  - If a square matrix has a left *or* right inverse then this inverse works on both sides so the matrix is invertible (sufficient condition).
- Inverse of a product :
  - If A and B are invertible matrices for which the *product* AB is defined then
$$(AB)^{-1} = B^{-1}A^{-1}.$$

TIP!

So just like with *transpose*, when we take the *inverse* of a *product*, the matrices are *swapped*
Note that no rule exists for $(A + B)^{-1}$.

Inverse of a 2×2 matrix

For 2×2 matrices, we have a formula for the inverse.

- The 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible *if and only if* $ad - bc \neq 0$.
- and in that case its inverse is : $A^{-1} = \dfrac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

  You should remember this!
- Note that if $ad - bc = 0$ then this matrix has no inverse and we say it is *not* invertible or singular.

Example 7. Use this to find $\begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}^{-1}$.

Exercise 8. PROOF Check that the formula given above works.

Determinant and inverse of a 2×2 matrix

Determinant and inverse of a 2×2 matrix.

- For the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the *number* $ad - bc$ (given by the "gamma rule") is called the *determinant* of A and is denoted $\det(A)$ or $|A|$.
- A^{-1} exists *if and only if* $\det(A) \neq 0$.
- If $det(A) \neq 0$, $A^{-1} = \dfrac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \dfrac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Example 9. For each of the following, determine if they are invertible, and if so, find their inverse.

$$A = \begin{pmatrix} 3 & 9 \\ 2 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} -3 & -1 \\ 2 & 6 \end{pmatrix}$$

How can you check that your inverse is correct?

Checking our answers with Maple

```
> with(LinearAlgebra):
> # Enter the matrices column by column

  B := < <-3,2>|<-1,6> >;
```

$$B := \begin{bmatrix} -3 & -1 \\ 2 & 6 \end{bmatrix}$$

```
> # We calculate the determinant of B

  Determinant(B);
```

$$-16$$

```
> # We find the inverse of B

  B^(-1);
  -16*B^(-1);
```

$$\begin{bmatrix} -\frac{3}{8} & -\frac{1}{16} \\ \frac{1}{8} & \frac{3}{16} \end{bmatrix}$$

$$\begin{bmatrix} 6 & 1 \\ -2 & -3 \end{bmatrix}$$

Inverses and transposes

Exercise 10 [Two important results]. When the given inverses exist, prove the following

1. $(AB)^{-1} = B^{-1}A^{-1}$

   Note that like for transpose, the matrices are swapped!

2. $(A^{-1})^T = (A^T)^{-1}$.

   You can take the inverse and the transpose in any order. This will be useful and is worth remembering!

Inverses and transposes

Example 11. Assuming all of the relevant inverses exist, simplify

$$M = B^2(AB)^{-1}A(B^{-1})^T(AB)^T$$

Examples

Exercise 12.

Given that $A^2 = 2A + 5I$, express A^4 and A^{-1} as linear combinations of A and I.