§5 Matrices (2020T1: W9-Tu-We, W10-Tu)

- **Matrices.** An $m \times n$ (read "m by n") *matrix* is an array of m rows and n columns of numbers of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

  - We say that the *size* of A is $m \times n$.
  - The number a_{ij} is referred to as the (i, j)th *entry* or *element* of A; it is the number in the ith row and the jth column. We also use the notation $[A]_{ij}$.
  - Two matrices are *equal* if and only if they have the same size and their corresponding entries are equal.

- **Some special matrices.**
  - A *zero* matrix is a matrix in which every entry is zero.
  - A *square* matrix is a matrix in which the number of rows equals the number of columns.
  - An *identity* matrix is a square matrix with 1s on the diagonal and 0s off the diagonal.
  - An upper/lower *triangular* matrix is a square matrix with only 0s below/above the diagonal.

Example.

- The zero matrix of size 3×4:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- The identity matrix of size 3:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- An upper triangular matrix:

$$\begin{pmatrix} 5 & 1 & -2 & 0 \\ 0 & 0 & 4 & 11 \\ 0 & 0 & 3 & -5 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

- A lower triangular matrix:

$$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 0 & 11 & 3 & 0 \end{pmatrix}$$

- **Matrix addition.** If A and B are both $m \times n$ matrices, then their *sum* $C = A + B$ is an $m \times n$ matrix whose entries are

$$[C]_{ij} = [A]_{ij} + [B]_{ij} \qquad \text{for } i = 1, \ldots, m \text{ and } j = 1, \ldots, n.$$

  - The addition of matrices of different sizes is undefined.
  - Matrix subtraction is defined in a similar way.

- **Matrix scalar multiplication.** If A is an $m \times n$ matrix and λ is a scalar, then the *scalar multiple* $C = \lambda A$ is an $m \times n$ matrix whose entries are

$$[C]_{ij} = \lambda [A]_{ij} \qquad \text{for } i = 1, \ldots, m \text{ and } j = 1, \ldots, n.$$

Exercise.

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & -2 \\ 2 & 2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}$$

$$4 \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$$

- **Matrix multiplication.** If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the *product* $C = AB$ is an $m \times p$ matrix whose entries are

$$[C]_{ij} = \sum_{k=1}^{n} [A]_{ik} [B]_{kj} \qquad \text{for } i = 1, \ldots, m \text{ and } j = 1, \ldots, p.$$

  - The product AB is defined only when the number of columns of A equals the number of rows of B.
  - The (i, j)th entry of the product AB is obtained by multiplying the corresponding components from the ith row of A and the jth column of B and then summing the results.
  - In general, $AB \neq BA$.

Example.

$$\left(\begin{array}{ccc} 2 & 1 & 3 \\ \hline 1 & 0 & -1 \\ \hline 0 & -1 & 2 \\ \hline -2 & 2 & 0 \end{array} \right) \left(\begin{array}{c|c} 1 & 2 \\ 1 & 0 \\ 2 & -1 \end{array} \right) = \begin{pmatrix} 9 & 1 \\ -1 & 3 \\ 3 & -2 \\ 0 & -4 \end{pmatrix}$$

e.g. The $(2, 1)$th entry in the product matrix is $1 \times 1 + 0 \times 1 + (-1) \times 2 = -1$.

Exercise. Given the matrices

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 3 & -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 2 & 0 \\ 3 & -1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix},$$

evaluate if possible:

$$AB = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 3 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 3 & -1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & 2 & 0 \\ 3 & -1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 3 & -1 & 0 \end{pmatrix}$$

$$AC = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 3 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$CA = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 3 & -1 & 0 \end{pmatrix}$$

$$D^2 = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

$$C^2 = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

Properties of matrix arithmetic. Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

- $A + B = B + A$ (*Commutative law for addition*)
- $(A + B) + C = A + (B + C)$ (*Associative law for addition*)
- $(AB)C = A(BC)$ (*Associative law for multiplication*)
- $(A + B)C = AC + BC$ (*Left distributive law for multiplication*)
- $A(B + C) = AB + AC$ (*Right distributive law for multiplication*)
- $\lambda(A + B) = \lambda A + \lambda B$
- $(\lambda + \mu)A = \lambda A + \mu A$
- $\lambda(AB) = (\lambda A)B = A(\lambda B)$
- $A + O = A = O + A$, where O is the zero matrix
- $AI = A = IA$, where I is the identity matrix

Note. Matrix multiplication is not commutative: $AB \neq BA$.

Example. We prove the left distributive law: $(A + B)C = AC + BC$.

Proof: Suppose A and B are $m \times n$ matrices, and C is an $n \times p$ matrix. Then for $i = 1, \ldots, m$ and $j = 1, \ldots, p$, the (i, j)th entry of $(A + B)C$ is given by

$$\begin{aligned}
&[(A + B)C]_{ij} \\
&= \sum_{k=1}^{n} [A + B]_{ik}[C]_{kj} && \text{definition of matrix multiplication} \\
&= \sum_{k=1}^{n} ([A]_{ik} + [B]_{ik})[C]_{kj} && \text{definition of matrix addition} \\
&= \sum_{k=1}^{n} ([A]_{ik}[C]_{kj} + [B]_{ik}[C]_{kj}) && \text{distributive law of } \mathbb{R} \\
&= \sum_{k=1}^{n} [A]_{ik}[C]_{kj} + \sum_{k=1}^{n} [B]_{ik}[C]_{kj} && \text{property of sum of } \mathbb{R} \\
&= [AC]_{ij} + [BC]_{ij} && \text{definition of matrix multiplication}
\end{aligned}$$

Hence $(A + B)C = AC + BC$. $\square$

- **Transpose.** The *transpose* of an $m \times n$ matrix A is the $n \times m$ matrix $A^\top$ whose entries are

$$[A^\top]_{ij} = [A]_{ji} \qquad \text{for } i = 1, \ldots, n \text{ and } j = 1, \ldots, m.$$

  - Properties of transpose
    - $(A^\top)^\top = A$
    - $(\lambda A + \mu B)^\top = \lambda A^\top + \mu B^\top$
    - $\boxed{(AB)^\top = B^\top A^\top}$
  - A matrix A is said to be *symmetric* if $A^\top = A$. A symmetric matrix must be square!

Exercise.

$$A^\top = \begin{pmatrix} 2 & 1 & 1 & 3 \\ -1 & 0 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{pmatrix}^\top$$

$$B^\top = \begin{pmatrix} 0 & 2 & 3 \\ 2 & -1 & -2 \\ 3 & -2 & 1 \end{pmatrix}^\top$$

Exercise. For $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, find $\mathbf{a}^\top\mathbf{b}$, $\mathbf{b}^\top\mathbf{a}$, $\mathbf{a}\mathbf{b}^\top$, and $\mathbf{b}\mathbf{a}^\top$.

Example. We prove the property $(AB)^\top = B^\top A^\top$.

Proof: Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then for $i = 1, \ldots, p$ and $j = 1, \ldots, m$, the (i, j)th entry of $(AB)^\top$ is given by

$$\begin{aligned}
[(AB)^\top]_{ij} &= [AB]_{ji} && \text{definition of transpose} \\
&= \sum_{k=1}^{n} [A]_{jk}[B]_{ki} && \text{definition of matrix multiplication} \\
&= \sum_{k=1}^{n} [A^\top]_{kj}[B^\top]_{ik} && \text{definition of transpose} \\
&= \sum_{k=1}^{n} [B^\top]_{ik}[A^\top]_{kj} && \text{commutative law of } \mathbb{R} \\
&= [B^\top A^\top]_{ij} && \text{definition of matrix multiplication}
\end{aligned}$$

Hence $(AB)^\top = B^\top A^\top$. $\square$

Exercise. Suppose A is a 2×3 matrix and B is a 3×4 matrix. What are the sizes of the matrices (if defined) AB, BA, $(AB)^\top$, $(BA)^\top$, $A^\top B^\top$, $B^\top A^\top$?

- **Inverse.** An *inverse* of a matrix A is a matrix X which satisfies $AX = I$ and $XA = I$, where I is an identity matrix of the appropriate size.
  - If a matrix has an inverse, then it is said to be *invertible* or *non-singular*.
  - Properties of inverse
    - The inverse of an invertible matrix A is *unique*; we denote it by A^{-1}:

      $$\boxed{AA^{-1} = I = A^{-1}A}.$$

    - All invertible matrices are square. Not all square matrices are invertible!
    - For a square matrix A, if $AX = I$ or $XA = I$ then $A^{-1} = X$.
    - $(A^{-1})^{-1} = A$
    - $\boxed{(AB)^{-1} = B^{-1}A^{-1}}$
    - $(A+B)^{-1} \neq A^{-1} + B^{-1}$ in general

Example. Since

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}}_{B} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I_3},$$

we conclude that the inverse of A is B and the inverse of B is A.

Exercise. Verify that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, provided $ad - bc \neq 0$.

Exercise. If A, B and C are invertible matrices of the same size, simplify the expression $CB(ACB)^{-1}AB$.

Example. We prove the property $(AB)^{-1} = B^{-1}A^{-1}$.
Proof: We need to prove that the inverse of AB is $B^{-1}A^{-1}$. This can be achieved by showing that AB multiplied by $B^{-1}A^{-1}$ gives an identity matrix.

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I.$$

Hence $(AB)^{-1} = B^{-1}A^{-1}$. $\square$

Finding inverse. Row reduce the augmented matrix $(A|I)$ into $(I|B)$. If the procedure is successful, then A is invertible and B is the inverse of A.

Exercise. Find the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$ if it exists.

Exercise. Find the inverse of $A = \begin{pmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{pmatrix}$ if it exists.

Finding the solution to $A\mathbf{x} = \mathbf{b}$ using A^{-1}. A square matrix *A is invertible* if and only if the system *$A\mathbf{x} = \mathbf{b}$ has a unique solution*. In this case, the unique solution is $\mathbf{x} = A^{-1}\mathbf{b}$.

Notes:

(i) Finding the inverse A^{-1} and then forming $\mathbf{x} = A^{-1}\mathbf{b}$ requires more computation than Gaussian elimination followed by back substitution.

(ii) The formula $\mathbf{x} = A^{-1}\mathbf{b}$ only applies when the system has a unique solution. When A is not invertible, the system could have no solution or infinitely many solutions.

Exercise. Given

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix},$$

solve the system

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 5x_2 + 3x_3 = 2 \\ x_1 \qquad\quad + 8x_3 = -1 \end{cases}$$

Inverse and transpose.

Exercise. Prove that $(A^\top)^{-1} = (A^{-1})^\top$ for any invertible matrix A.

Exercise. If A and B are invertible matrices of the same size, simplify the expression $B^\top(A^\top B)^{-1}A^\top$.

Exercise. A square matrix A is said to be *orthogonal* if $A^\top = A^{-1}$. Show that the following matrix is orthogonal

$$A = \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ 1 & -2 & 2 \\ -2 & 1 & 2 \end{pmatrix}$$

Determinant. The *determinant* of a matrix A, denoted by $\det(A)$ or $|A|$, is a number. It is only defined for square matrices.

- $A = (a) \quad \Rightarrow \quad \det(A) = a$
- $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Rightarrow \quad \det(A) = ad - bc$
- $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \Rightarrow \quad \det(A) = aei + bfg + cdh - ceg - bdi - afh$

Note. Pattern to remember:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\text{“minus”} \quad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Exercise. Find the determinant of

$$A = (-1), \qquad B = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}, \qquad \text{and} \qquad C = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 3 & 1 & -2 \end{pmatrix}.$$

- **Finding the determinant by expanding along a row or a column.**
  - The (i, j)th *minor* of a matrix A, denoted by $|A_{ij}|$, is the determinant of the matrix obtained by deleting row i and column j from A.
  - If A is $n \times n$ matrix whose (i, j)th entry is a_{ij}, then

$$\begin{aligned} |A| &= \sum_{k=1}^{n} (-1)^{1+k}\, a_{1k}\, |A_{1k}| \qquad (\textit{expanding along the top row}) \\ &= a_{11}\,|A_{11}| - a_{12}\,|A_{12}| + a_{13}\,|A_{13}| - \cdots + (-1)^{1+n}\, a_{1n}\,|A_{1n}| \end{aligned}$$

or

$$|A| = \sum_{k=1}^{n} (-1)^{i+k}\, a_{ik}\, |A_{ik}| \qquad (\textit{expanding along the } i\textit{th row})$$

or

$$|A| = \sum_{k=1}^{n} (-1)^{k+j}\, a_{kj}\, |A_{kj}| \qquad (\textit{expanding along the } j\textit{th column})$$

Note. The signs in the expansion alternate as follows:

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Example. We find the determinant of

$$A = \begin{pmatrix} 2 & 1 & -2 & 3 \\ 4 & 0 & 3 & 11 \\ 1 & 0 & 2 & -2 \\ 0 & 0 & 3 & 0 \end{pmatrix}.$$

Expanding along the top row, we obtain

$$\begin{aligned} |A| &= 2\begin{vmatrix} 0 & 3 & 11 \\ 0 & 2 & -2 \\ 0 & 3 & 0 \end{vmatrix} - 1\begin{vmatrix} 4 & 3 & 11 \\ 1 & 2 & -2 \\ 0 & 3 & 0 \end{vmatrix} + (-2)\begin{vmatrix} 4 & 0 & 11 \\ 1 & 0 & -2 \\ 0 & 0 & 0 \end{vmatrix} - 3\begin{vmatrix} 4 & 0 & 3 \\ 1 & 0 & 2 \\ 0 & 0 & 3 \end{vmatrix} \\ &= 2 \times 0 - 1 \times 57 + (-2) \times 0 - 3 \times 0 \\ &= -57 \end{aligned}$$

Exercise. For the same matrix

$$A = \begin{pmatrix} 2 & 1 & -2 & 3 \\ 4 & 0 & 3 & 11 \\ 1 & 0 & 2 & -2 \\ 0 & 0 & 3 & 0 \end{pmatrix},$$

find the determinant by

(a) expanding along the 4th row;

(b) expanding along the 2nd column.

- **Properties of determinant.**
  - $\det(I) = 1$
  - $\det(A^\top) = \det(A)$
  - $\det(AB) = \det(A)\det(B)$
  - $\det(A^{-1}) = \dfrac{1}{\det(A)}$
  - In general, $\det(A + B) \neq \det(A) + \det(B)$ and $\det(\lambda A) \neq \lambda \det(A)$.
- **Determinants, inverses, and solutions of $A\mathbf{x} = \mathbf{b}$.** Let A be a square matrix. Then
  - $\det(A) \neq 0 \iff A$ is invertible $\iff A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$
  - $\det(A) = 0 \iff A$ is not invertible $\iff A\mathbf{x} = \mathbf{b}$ has no solution or infinitely many solutions

- **Finding the determinant by row reduction.**
  - If B is obtained from A by *interchanging two rows/columns*, then $\det(B) = -\det(A)$.
  - If B is obtained from A by *multiplying one row/column by a scalar λ*, then $\det(B) = \lambda \det(A)$.
  - If B is obtained from A by *adding/subtracting a multiple of one row/column to/from another*, then $\det(B) = \det(A)$.

  - If A contains *a row/column of zeros* then $\det(A) = 0$.
  - If any row/column of A is *a multiple of another row/column* then $\det(A) = 0$.
  - If A is *triangular* then $\det(A)$ is the product of its diagonal entries $a_{11}a_{22}\cdots a_{nn}$.

Exercise.

$$\begin{vmatrix} 6 & 1 & 0 & 3 \\ 0 & 1 & 0 & 11 \\ -3 & -5 & 0 & -6 \\ 2 & 4 & 0 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 1 & -2 & 3 \\ 4 & 0 & 3 & 11 \\ -4 & -2 & 4 & -6 \\ 0 & 0 & 3 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 1 & -2 & 3 \\ 0 & 1 & 3 & 11 \\ 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

Exercise. Find the determinant of $A = \begin{pmatrix} 2 & 1 & -2 & 3 \\ 4 & -1 & 3 & 11 \\ 3 & -1 & -2 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix}$ using row reduction.

Exercise. Given $A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$ and $\det(A) = 10$, find

$$\begin{vmatrix} 2a & 2b & 2c & 2d \\ e & f & g & h \\ -3i & -3j & -3k & -3l \\ m & n & o & p \end{vmatrix}$$

$$\begin{vmatrix} a & b & c & d \\ -e & -f & -g & -h \\ i-2a & j-2b & k-2c & l-2d \\ m+a & n+b & o+c & p+d \end{vmatrix}$$

$$\begin{vmatrix} e & f & g & h \\ i & j & k & l \\ m & n & o & p \\ a & b & c & d \end{vmatrix}$$

$\det(2A)$