MATH1131 Mathematics 1A – Algebra

Lecture 16: Row Echelon Form and Gaussian Elimination

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Based on slides by Jonathan Kress

Vector and matrix form

Consider the system of linear equations

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 1 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & -1 \\ 7x_1 & - & 5x_2 & - & 9x_3 & = & 0 \end{array}$$

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All of these presentations:

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are most simply represented by the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & -1 \\ 7 & -5 & -9 & 0 \end{array}\right).$$

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- Row 1 is a leading row with leading entry 5.
- Row 2 is a non-leading row (a row of zeros).
- Column 2 is the only leading column.

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not RREF   not RREF   RREF

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Reduced row echelon form

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A matrix is in reduced row echelon form (RREF) if:

- it is in row echelon form, and
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It can be useful to refer to special leading entries called pivots.

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Row 2 is the pivot row and column 2 is the pivot column.

To solve a system of linear equations, we apply Gaussian elimination to its augmented matrix to achieve row echelon form.

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To further reduce the matrix to reduced row echelon form:

5. divide each leading row through by its leading entry to ensure all leading entries are 1,
6. use row operations of the form $R_i \to R_i + \alpha R_j$ to create zero entries above each leading entry, working from the bottom row upwards.

Solutions to a matrix in row echelon form

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Gaussian elimination examples

Example

Solve the following system of linear equations:

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\begin{aligned}
x + 2y + 4z &= 2 \\
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We want to reduce this to row echelon form...

Gaussian elimination examples

Row-reducing the augmented matrix:

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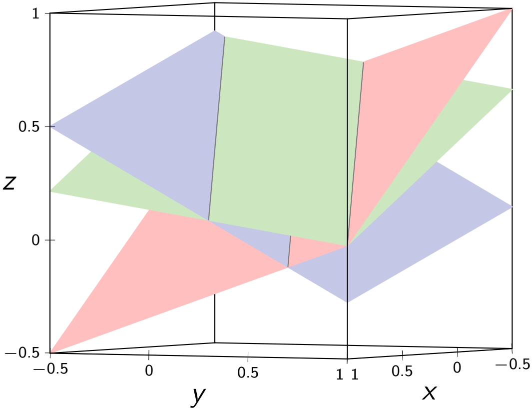
(Notice that R_3 means $0x + 0y + 0z = 1$, which is impossible. So this confirms the system is inconsistent.)

Gaussian elimination examples



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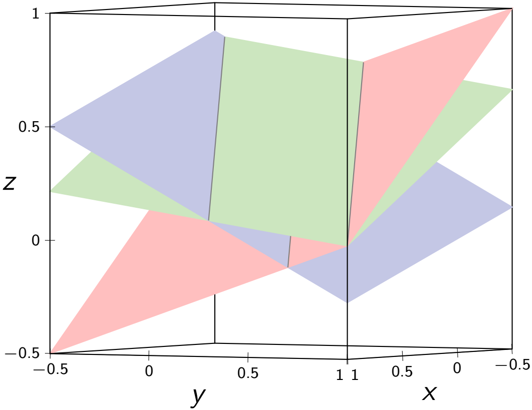

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Geometrically, there is no solution because the planes only ever meet in pairs. The three lines at which the pairs intersect are parallel.

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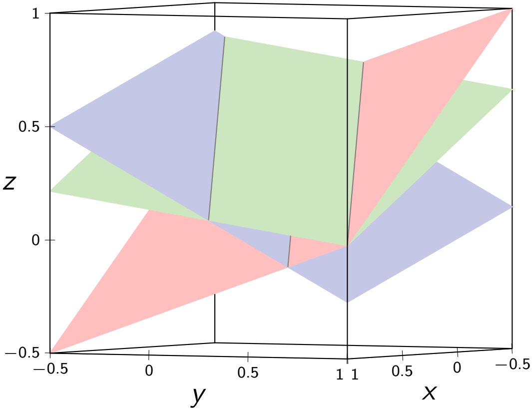

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In other cases with no solutions, it's possible that two or all three of the planes could be parallel.

Gaussian elimination examples

Example

Solve the following system of linear equations:

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\begin{aligned}
x + y + 3z &= 4 \\
2x + y + z &= 0 \\
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Gaussian elimination examples

Example

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The corresponding augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 2 & 1 & 1 & 0 \\ 1 & 3 & -1 & 6 \end{array}\right)$$

We again want to reduce this to row echelon form...

Row-reducing the augmented matrix:

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Using back-substitution, R_3 tells us $-14z = -14$, so $\boxed{z = 1}$.

Gaussian elimination examples

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From R_2 we know $-y - 5z = -8$, so $\boxed{y = 3}$.

Row-reducing the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 2 & 1 & 1 & 0 \\ 1 & 3 & -1 & 6 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & -1 & -5 & -8 \\ 0 & 2 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & -1 & -5 & -8 \\ 0 & 0 & -14 & -14 \end{array}\right)$$

Every column left of the vertical line contains a leading entry. So the system has a unique solution.

Using back-substitution, R_3 tells us $-14z = -14$, so $\boxed{z = 1}$.
From R_2 we know $-y - 5z = -8$, so $\boxed{y = 3}$.
From R_1 we know $x + y + 3z = 4$, so $\boxed{x = -2}$.

Gaussian elimination examples

Row-reducing the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 2 & 1 & 1 & 0 \\ 1 & 3 & -1 & 6 \end{array}\right) \xrightarrow[]{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1}} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & -1 & -5 & -8 \\ 0 & 2 & -4 & 2 \end{array}\right)$$

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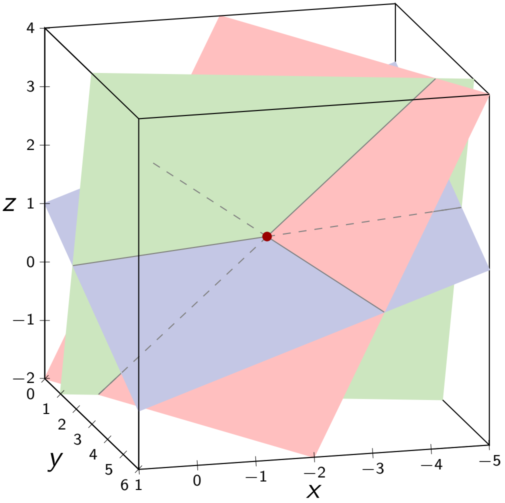
Using back-substitution, R_3 tells us $-14z = -14$, so $\boxed{z = 1}$.
From R_2 we know $-y - 5z = -8$, so $\boxed{y = 3}$.
From R_1 we know $x + y + 3z = 4$, so $\boxed{x = -2}$.

So the unique solution is $x = -2$, $y = 3$, and $z = 1$.

Gaussian elimination examples

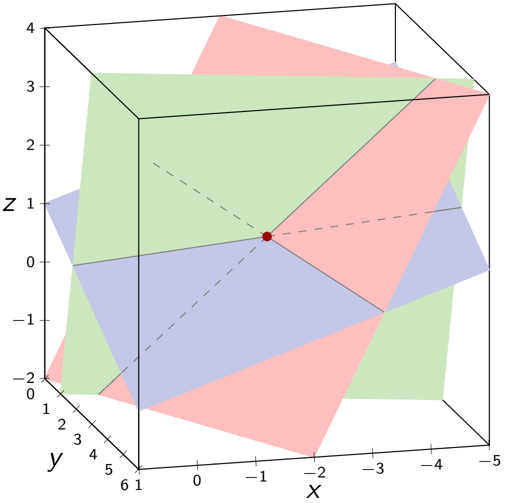


$$\begin{aligned} x + y + 3z &= 4 \\ 2x + y + z &= 0 \\ x + 3y - z &= 6 \end{aligned}$$

has solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}.$$

Gaussian elimination examples



$$\begin{aligned} x + y + 3z &= 4 \\ 2x + y + z &= 0 \\ x + 3y - z &= 6 \end{aligned}$$

has solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}.$$

Geometrically, there is a unique solution because all three planes meet at exactly one point.

Example

Solve the following system of linear equations:

$$\begin{aligned} x - 3y - 7z &= -17 \\ 2x - y - 4z &= -14 \\ 2x + 7y + 12z &= 18 \end{aligned}$$

Gaussian elimination examples

Example

Solve the following system of linear equations:

$$\begin{aligned} x - 3y - 7z &= -17 \\ 2x - y - 4z &= -14 \\ 2x + 7y + 12z &= 18 \end{aligned}$$

The corresponding augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & -3 & -7 & -17 \\ 2 & -1 & -4 & -14 \\ 2 & 7 & 12 & 18 \end{array}\right)$$

We again want to reduce this to row echelon form...

Gaussian elimination examples

Row-reducing the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -3 & -7 & -17 \\ 2 & -1 & -4 & -14 \\ 2 & 7 & 12 & 18 \end{array}\right)$$

Gaussian elimination examples

Row-reducing the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -3 & -7 & -17 \\ 2 & -1 & -4 & -14 \\ 2 & 7 & 12 & 18 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & -3 & -7 & -17 \\ 0 & 5 & 10 & 20 \\ 0 & 13 & 26 & 52 \end{array}\right)$$

Gaussian elimination examples

Row-reducing the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -3 & -7 & -17 \\ 2 & -1 & -4 & -14 \\ 2 & 7 & 12 & 18 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & -3 & -7 & -17 \\ 0 & 5 & 10 & 20 \\ 0 & 13 & 26 & 52 \end{array}\right)$$

$$\xrightarrow[R_3 \to \frac{1}{13} R_3]{R_2 \to \frac{1}{5} R_2} \left(\begin{array}{ccc|c} 1 & -3 & -7 & -17 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 2 & 4 \end{array}\right)$$

Gaussian elimination examples

Row-reducing the augmented matrix:

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$$\xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & -3 & -7 & -17 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

There is no leading entry in the rightmost column, so there is at least one solution.

Gaussian elimination examples

Row-reducing the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -3 & -7 & -17 \\ 2 & -1 & -4 & -14 \\ 2 & 7 & 12 & 18 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & -3 & -7 & -17 \\ 0 & 5 & 10 & 20 \\ 0 & 13 & 26 & 52 \end{array}\right)$$

$$\xrightarrow[R_3 \to \frac{1}{13} R_3]{R_2 \to \frac{1}{5} R_2} \left(\begin{array}{ccc|c} 1 & -3 & -7 & -17 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 2 & 4 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & -3 & -7 & -17 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

There is no leading entry in the rightmost column, so there is at least one solution.

Furthermore, the third column does not contain a leading entry. So the system has infinitely many solutions...

Gaussian elimination examples

We found:

$$\left(\begin{array}{ccc|c} 1 & -3 & -7 & -17 \\ 2 & -1 & -4 & -14 \\ 2 & 7 & 12 & 18 \end{array}\right)$$

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Since the third column does not contain a leading entry, we can set the corresponding variable z to be a parameter. So let $\boxed{z = \lambda}$.

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Since the third column does not contain a leading entry, we can set the corresponding variable z to be a parameter. So let $\boxed{z = \lambda}$.
From R_2 we know $y + 2z = 4$, so $\boxed{y = 4 - 2\lambda}$.

Gaussian elimination examples

We found:

$$\left(\begin{array}{ccc|c} 1 & -3 & -7 & -17 \\ 2 & -1 & -4 & -14 \\ 2 & 7 & 12 & 18 \end{array}\right) \xrightarrow{\cdots} \left(\begin{array}{ccc|c} 1 & -3 & -7 & -17 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Since the third column does not contain a leading entry, we can set the corresponding variable z to be a parameter. So let $\boxed{z = \lambda}$.

From R_2 we know $y + 2z = 4$, so $\boxed{y = 4 - 2\lambda}$.

From R_1 we know $x - 3y - 7z = -17$, so $\boxed{x = \lambda - 5}$.

Gaussian elimination examples

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So the set of infinite solutions is given by

$$x = \lambda - 5,\ y = 4 - 2\lambda, \text{ and } z = \lambda \text{ for any } \lambda \in \mathbb{R}.$$

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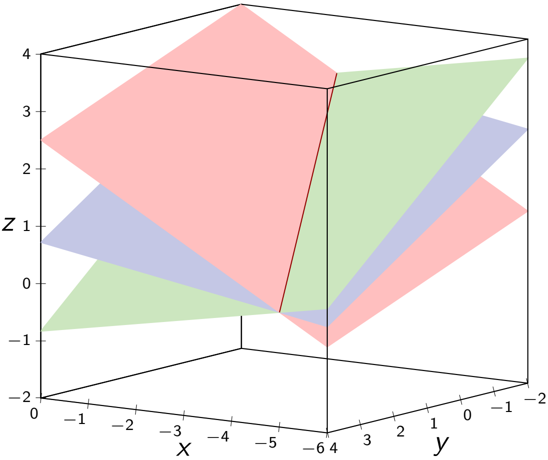
So the set of infinite solutions is given by

$$x = \lambda - 5,\ y = 4 - 2\lambda, \text{ and } z = \lambda \text{ for any } \lambda \in \mathbb{R}.$$

Notice that since this is a parametrised solution in one parameter, geometrically the solution will be a line.

Gaussian elimination examples



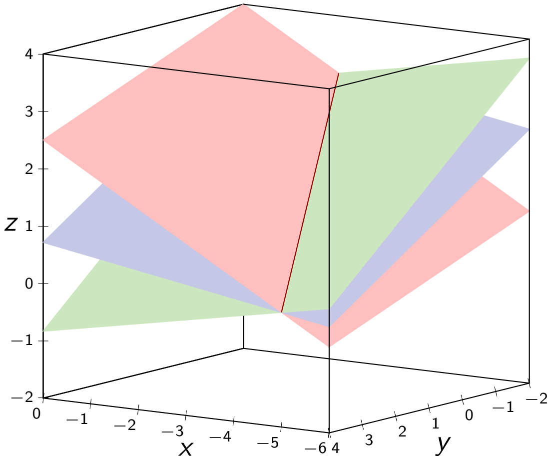
$$\begin{aligned} x - 3y - 7z &= -17 \\ 2x - y - 4z &= -14 \\ 2x + 7y + 12z &= 18 \end{aligned}$$

has solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},$$

$$\lambda \in \mathbb{R}.$$

Gaussian elimination examples



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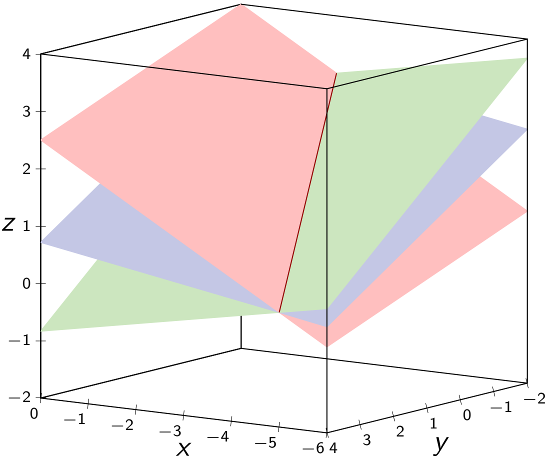
$$\lambda \in \mathbb{R}.$$

Geometrically, there are infinitely many solutions in one parameter because all three planes meet at a single line.

Gaussian elimination examples



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$$\lambda \in \mathbb{R}.$$

Geometrically, there are infinitely many solutions in one parameter because all three planes meet at a single line.

In other cases with infinitely many solutions, it's possible that two or all three of the planes could be identical.

Nature of solutions – examples

Example

For each of the following augmented matrices in REF, solve the corresponding linear system, and describe the nature of the solution.

a) $\left(\begin{array}{ccc|c} 1 & 4 & 7 & 4 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & 8 & 2 \end{array}\right)$

b) $\left(\begin{array}{ccc|c} 0 & 5 & 1 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 5 \end{array}\right)$

c) $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$

d) $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

e) $\left(\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$

f) $\left(\begin{array}{ccccc|c} 3 & 5 & 1 & 0 & 2 & 4 \\ 0 & 0 & -1 & 8 & 1 & 6 \end{array}\right)$

Nature of solutions – examples

a) $\left(\begin{array}{ccc|c} 1 & 4 & 7 & 4 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & 8 & 2 \end{array}\right)$

Nature of solutions – examples

a) $\left(\begin{array}{ccc|c} 1 & 4 & 7 & 4 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & 8 & 2 \end{array}\right)$

- There are no leading entries in the last column, so there is at least one solution.

Nature of solutions – examples

a) $\left(\begin{array}{ccc|c} 1 & 4 & 7 & 4 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & 8 & 2 \end{array}\right)$

- There are no leading entries in the last column, so there is at least one solution.
- There are leading entries in every column left of the vertical bar, so there is a unique solution.

Nature of solutions – examples

a) $\left(\begin{array}{ccc|c} 1 & 4 & 7 & 4 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & 8 & 2 \end{array}\right)$

- There are no leading entries in the last column, so there is at least one solution.
- There are leading entries in every column left of the vertical bar, so there is a unique solution.
- R_3 means $8z = 2$, so we know $\boxed{z = \frac{1}{4}}$.

Nature of solutions – examples

a) $\left(\begin{array}{ccc|c} 1 & 4 & 7 & 4 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & 8 & 2 \end{array}\right)$

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- There are leading entries in every column left of the vertical bar, so there is a unique solution.
- R_3 means $8z = 2$, so we know $\boxed{z = \frac{1}{4}}$.
- R_2 means $3y - z = 5$, so we know $\boxed{y = \frac{7}{4}}$.

Nature of solutions – examples

a) $\left(\begin{array}{ccc|c} 1 & 4 & 7 & 4 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & 8 & 2 \end{array}\right)$

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- There are leading entries in every column left of the vertical bar, so there is a unique solution.
- R_3 means $8z = 2$, so we know $\boxed{z = \frac{1}{4}}$.
- R_2 means $3y - z = 5$, so we know $\boxed{y = \frac{7}{4}}$.
- R_1 means $x + 4y + 7z = 4$, so we know $\boxed{x = -\frac{19}{4}}$.

Nature of solutions – examples

a) $\left(\begin{array}{ccc|c} 1 & 4 & 7 & 4 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & 8 & 2 \end{array}\right)$

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- R_2 means $3y - z = 5$, so we know $\boxed{y = \frac{7}{4}}$.
- R_1 means $x + 4y + 7z = 4$, so we know $\boxed{x = -\frac{19}{4}}$.

The unique solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \dfrac{1}{4} \begin{pmatrix} -19 \\ 7 \\ 1 \end{pmatrix}$.

Nature of solutions – examples

a) $\left(\begin{array}{ccc|c} 1 & 4 & 7 & 4 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & 8 & 2 \end{array}\right)$

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- There are leading entries in every column left of the vertical bar, so there is a unique solution.
- R_3 means $8z = 2$, so we know $\boxed{z = \frac{1}{4}}$.
- R_2 means $3y - z = 5$, so we know $\boxed{y = \frac{7}{4}}$.
- R_1 means $x + 4y + 7z = 4$, so we know $\boxed{x = -\frac{19}{4}}$.

The unique solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -19 \\ 7 \\ 1 \end{pmatrix}$.

The solution is a single point in $\mathbb{R}^3$, and geometrically represented by the unique intersection of three planes.

Nature of solutions – examples

b) $\left(\begin{array}{ccc|c} 0 & 5 & 1 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 5 \end{array}\right)$

Nature of solutions – examples

b) $\left(\begin{array}{ccc|c} 0 & 5 & 1 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 5 \end{array}\right)$

- There is a leading entry in the last column, so there are no solutions.

Nature of solutions – examples

b) $\left(\begin{array}{ccc|c} 0 & 5 & 1 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 5 \end{array}\right)$

- There is a leading entry in the last column, so there are no solutions.
- (Note that R_3 means $0x + 0y + 0z = 5$, which is impossible.)

b) $\left(\begin{array}{ccc|c} 0 & 5 & 1 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 5 \end{array}\right)$

- There is a leading entry in the last column, so there are no solutions.
- (Note that R_3 means $0x + 0y + 0z = 5$, which is impossible.)

The system is inconsistent.

Nature of solutions – examples

b) $\left(\begin{array}{ccc|c} 0 & 5 & 1 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 5 \end{array}\right)$

- There is a leading entry in the last column, so there are no solutions.
- (Note that R_3 means $0x + 0y + 0z = 5$, which is impossible.)

The system is inconsistent.

The solution could be geometrically represented by three parallel planes, three planes of which two are parallel, or three planes which intersect pairwise in three parallel lines.

Nature of solutions – examples

c) $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Nature of solutions – examples

c) $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$

- There are no leading entries in the last column, so there is at least one solution.

Nature of solutions – examples

c) $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$

- There are no leading entries in the last column, so there is at least one solution.
- Not every column left of the vertical bar contains a leading entry, so there are infinitely many solutions.

Nature of solutions – examples

c) $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$

- There are no leading entries in the last column, so there is at least one solution.
- Not every column left of the vertical bar contains a leading entry, so there are infinitely many solutions.
- The second column has no leading entry, so we can set its corresponding variable as a parameter. So let $\boxed{y = \lambda}$.

Nature of solutions – examples

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- Not every column left of the vertical bar contains a leading entry, so there are infinitely many solutions.
- The second column has no leading entry, so we can set its corresponding variable as a parameter. So let $\boxed{y = \lambda}$.
- R_2 means $-z = 6$, so we know $\boxed{z = -6}$.

Nature of solutions – examples

c) $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$

- There are no leading entries in the last column, so there is at least one solution.
- Not every column left of the vertical bar contains a leading entry, so there are infinitely many solutions.
- The second column has no leading entry, so we can set its corresponding variable as a parameter. So let $\boxed{y = \lambda}$.
- R_2 means $-z = 6$, so we know $\boxed{z = -6}$.
- R_1 means $x + 2y = 4$, so we know $\boxed{x = 4 - 2\lambda}$.

Nature of solutions – examples

c) $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$

- There are no leading entries in the last column, so there is at least one solution.
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- R_2 means $-z = 6$, so we know $\boxed{z = -6}$.
- R_1 means $x + 2y = 4$, so we know $\boxed{x = 4 - 2\lambda}$.

The infinite set of solutions is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \ \lambda \in \mathbb{R}$.

c) $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$

- There are no leading entries in the last column, so there is at least one solution.
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The solution is a line in $\mathbb{R}^3$, and geometrically represented by the intersection of two or three planes at a common line.

Nature of solutions – examples

d) $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Nature of solutions – examples

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Nature of solutions – examples

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Nature of solutions – examples

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- The third column has no leading entry, so we can set its corresponding variable as another parameter. So let $\boxed{z = \mu}$.

Nature of solutions – examples

d) $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

- There are no leading entries in the last column, so there is at least one solution.
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- R_1 means $x + 2y + 3z = 4$, so we know $\boxed{x = 4 - 2\lambda - 3\mu}$.

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The set of solutions is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}, \ \lambda, \mu \in \mathbb{R}$.

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The solution is a plane in $\mathbb{R}^3$, and geometrically represented by the intersection of up to three identical planes.

Nature of solutions – examples

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The solution is a single point in $\mathbb{R}^2$, and geometrically represented by the unique intersection of two lines.

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(Notice that the extra zero rows did not provide any additional information about the solution.)

Nature of solutions – examples

f) $\left(\begin{array}{ccccc|c} 3 & 5 & 1 & 0 & 2 & 4 \\ 0 & 0 & -1 & 8 & 1 & 6 \end{array}\right)$

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- The fifth column has no leading entry, so let $\boxed{x_5 = \lambda_3}$.
- R_2 means $-x_3 + 8x_4 + x_5 = 6$, so $\boxed{x_3 = -6 + 8\lambda_2 + \lambda_3}$.

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- Similarly, from R_1 we get $\boxed{x_1 = \frac{1}{3}(10 - 5\lambda_1 - 8\lambda_2 - 3\lambda_3)}$.

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Solution: $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 10/3 \\ 0 \\ -6 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -5/3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -8/3 \\ 0 \\ 8 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \ \lambda_i \in \mathbb{R}$$

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The solution is a 3-dimensional object in $\mathbb{R}^5$, and geometrically represented by the intersection of two 5-dimensional objects.