THE UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1131 Calculus

Section 5: - Mean Value Theorem.

Mean Value Theorem:

Suppose f is cts on $[a, b]$ and diffble on (a, b). Then there is a real number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

There must be a point $c \in (a, b)$ at which the tangent line has the same slope ~~as~~ the secant line through $(a, f(a))$ and $(b, f(b))$.

Ex: Demonstrate the Mean Value Theorem for the function, $f(x) = 6 - 2x + x^2$, on $[-2, 2]$.

$$\frac{f(2) - f(-2)}{2 - (-2)} = \frac{6 - 14}{4} = -2.$$

$$f'(x) = -2 + 2x \quad . \quad f'(0) = -2.$$

We can use the MVT to do a range of problems.

Ex: Use the MVT to find an approximate value of $\sqrt{17}$.

$f(x) = \sqrt{x}$. $\quad f(17) - f(16) = (17 - 16) \cdot f'(c)$

for some $c \in (16, 17)$.

$f'(x) = \frac{1}{2\sqrt{x}}$. So $\sqrt{17} \approx 4 + 1 \cdot \frac{1}{2 \cdot \sqrt{16}} = \frac{33}{8}$

Ex: Give a precise estimate of $\log 1.001$.

By MVT, with $f(x) = \log x$, on $[1, 1.001]$ we have

$$\frac{\log(1.001) - \log 1}{1.001 - 1} = f'(c)$$

for some $c \in [1, 1.001]$.

Hence $\frac{1}{1.001} < f'(c) < 1$ so $\frac{1}{1.001} < \frac{\log 1.001}{.001} < 1$. Thus $0.00099 < \log 1.001 < 0.001$ so $\log 1.001 = 0.000995 \pm 0.000005$.

Ex: Use the MVT to prove that $\tan x \geq x$ for all $x \in [0, \frac{\pi}{2})$.

Let $g(x) = \tan x - x$. $g(0) = 0$

$g'(x) = \sec^2 x - 1$.

So $g'(x) \geq 0$ if $x \in [0, \frac{\pi}{2})$.

If $x \in [0, \frac{\pi}{2})$, then

$g(x) = g(x) - g(0) = (x-0) g'(c)$, for some $c \in [0, \frac{\pi}{2})$. So $g(x) \geq 0$ for all $x \in [0, \frac{\pi}{2})$.

Ex: Prove that for all real x and y, $|\sin x - \sin y| \leq |x - y|$.

If $x = y$, then this inequality is trivial.

If $x \neq y$, then by the MVT,

$\left| \frac{\sin x - \sin y}{x - y} \right| = |\cos c|$, for some c between x and y. Now $|\cos x| \leq 1$. for all x. So $|\sin x - \sin y| \leq |x - y|$.

Error Estimates:
Suppose I measure an angle in radians to be 0.7^c and I take the sine of that angle. If the error involved in my measurement is approximately 0.01^c what is the worst error involved in taking the sine of this number?

That is, if $f(x) = \sin x$ and $\Delta x = \pm 0.01$, we want a bound on the size of

$$|\Delta f(x)| = |f(x + \Delta x) - f(x)|.$$

Theorem: If $f'(x)$ exists, then

$$|\Delta f(x)| = |f(x + \Delta x) - f(x)| \approx f'(x)\Delta x.$$

Ex: In the above example, $\Delta f(x) \approx \cos 0.7 \times 0.01 \approx 7.65 \times 10^{-3}$.

Here are some consequences of the MVT:

Definition: A function f defined on $[a, b]$ is said to be **increasing** if $f(x) > f(y)$ whenever $x > y$, and **decreasing** when $f(x) < f(y)$ whenever $x > y$.

Theorem: Suppose f is diffble on (a, b),

(i) If $f'(x) > 0$ for all $x \in (a, b)$ then f is increasing on (a, b).

(ii) If $f'(x) = 0$ for all $x \in (a, b)$ then f is constant on (a, b)

(iii) If $f'(x) < 0$ for all $x \in (a, b)$ then f is decreasing on (a, b).

Proof: The proof of all of these comes from applying the MVT to f on (x, y), any subset of (a, b) giving

$$\frac{f(y) - f(x)}{y - x} = f'(c).$$

In the first case we have $f(y) > f(x)$ whenever $y > x$ so f is increasing. Similarly for (iii). For (ii), we have $f(x) = f(y)$, for all x and y so f is a constant.

Theorem: Suppose that f is cts on $[a, b]$ and diffble on (a, b) and that $f(a)$ and $f(b)$ have opposite signs. If $f'(x) > 0$ for all $x \in (a, b)$ (or $f'(x) < 0$ for all $x \in (a, b)$), then f has **exactly** one real zero in (a, b).

proof: IVT tells us that $f(x)$ has a zero in (a, b). If there are two zeros in (a, b), say c_1 and c_2. Then by the MVT,

$$0 = f(c_1) - f(c_2) = (c_1 - c_2) f'(d),$$

for some d between c_1 and c_2. Since $c_1 \neq c_2$, we must have $f'(d) = 0$. But this contradicts the positivity (or negativity) of $f'(x)$. So there cannot be two zeros in (a, b). Thus, there is a unique zero

$\boxplus$

Ex: $f(x) = x^3 + x + 1$ on $[-1, 1]$.

$f(1) = 3, \quad f(-1) = -1.$

Moreover, $f'(x) = 3x^2 + 1$ which is alway non-negative. So by the theorem on the previous page. $f(x)$ has a unique zero on $[-1, 1]$.

Ex: Show that $5x^5 + 2x + 1 = 0$ has exactly one real solution.

$f(x) = 5x^5 + 2x + 1$

$f'(x) = 25x^4 + 2$. So $f'(x) > 0$.

$f(x)$ is increasing and can have at most one zero.

$f(0) = 1, \quad f(-1) = -6.$

Hence $f(x)$ has at least one zero.

Thus $f(x)$ has exactly one zero.

Theorem: Suppose that f, g are differentiable functions such that $f(a) = g(a)$ and for all $x > a$, we have $f'(x) > g'(x)$.
Then $f(x) > g(x)$ for all $x > a$.

Ex: Prove that $\sin x < x$ for all $x > 0$.

Let $f(x) = \sin x$ and $g(x) = x$. So

$$f(0) = g(0) = 0.$$

$$f'(x) = \cos x, \quad g'(x) = 1.$$

So $f'(x) < g'(x)$, for all $x \in (0, 2\pi)$.

if $x \geq 2\pi$, then $g(x) \geq 2\pi > 1 \geq f(x)$.

Types of points:

We wish to classify all the sorts of interesting points a function can have.

Definition:

Suppose that f is a function defined on an interval $[a, b]$ and let $x_0 \in [a, b]$.

(i) x_0 is called a **critical point** if $f'(x_0) = 0$ or if f is not differentiable at x_0.

(ii) x_0 is called an **extreme point** if x_0 is a local maximum or local minimum.

(iii) x_0 is called a **stationary point** if $f'(x_0) = 0$.

In practise, to find the (global) maximum and minimum, we need to find the stationary points and check their y values and also check the y values at the end points.

Ex: Find the global max and min of $f(x) = x^3 - 3x^2 + 1$ on the interval $[0, 4]$.

$$f'(x) = 3x^2 - 6x = 3x(x-2).$$

So the stationary pt's are $x = 0$, $x = 2$.

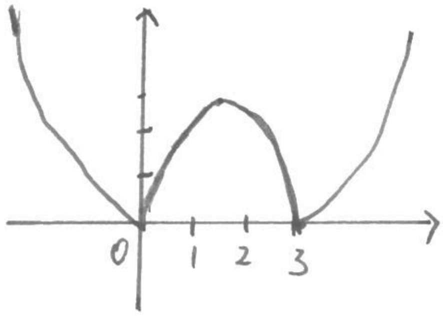
$f(0) = 1$, $f(2) = -3$, $f(4) = 17$.

Global max : $f(4)$.

Global min : $f(2)$.

Ex: Find the local max and min of $f(x) = |x-3||x|$

The critical points are at $x = 0$, $x = \frac{3}{2}$ and $x = 3$.



Thus the local min's are at

$$f(0) = f(3) = 0$$

The local max is at $f(\frac{3}{2}) = \frac{9}{4}$.

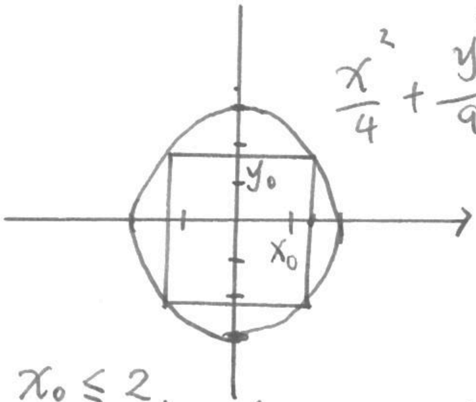
Ex: Find the dimensions of the rectangle (with vertical and horizontal sides) of maximum area which can be inscribed in the ellipse, $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

The area is $4x_0 y_0$.

and $y_0 = 3\sqrt{1 - \frac{x_0^2}{4}}$

So the area is

$$A = 12 x_0 \sqrt{1 - \frac{x_0^2}{4}}, \quad 0 \leq x_0 \leq 2.$$



$\frac{x^2}{4} + \frac{y^2}{9} = 1$

$$\frac{dA}{dx_0} = 12\sqrt{1 - \frac{x_0^2}{4}} + 12x_0 \frac{-\frac{x_0}{2}}{2\sqrt{1 - \frac{x_0^2}{4}}}$$

The area is zero if $x_0 = 0$ or $x_0 = 2$.

$\frac{dA}{dx_0} = 0$ if $x_0 = \sqrt{2}$.

So Max area is

$12 \cdot \sqrt{2} \cdot \sqrt{1 - \frac{2}{4}} = 12.$

L'Hôpital's Rule:

We return to the problem of calculating limits.

Theorem: (L'Hôpital's Rule)

Suppose that f and g are differentiable functions (except possibly at a) and that $f(a)$ and $g(a)$ are both equal to 0, or both tend to ∞ as $x \to a$.

If $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Proof: (Outline). Suppose we have the case $f(a) = g(a) = 0$. Apply the MVT to f and g on the interval (a, x), where $x > a$, so that for some $c, d \in (a, x)$ we have $\frac{f(x)-0}{x-a} = f'(c)$ and $\frac{g(x)-0}{x-a} = g'(d)$.

Hence

$$\frac{f(x)}{g(x)} = \frac{\frac{f(x)}{x-a}}{\frac{g(x)}{x-a}} = \frac{f'(c)}{g'(d)}.$$

Hence as $x \to a^+$ we have $c \to a^+$ and $d \to a^+$, so that if the limit of $\frac{f'(x)}{g'(x)}$ exists as $x \to a$, we have $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$.

Ex: $\lim_{x\to 0} \frac{e^x - 1}{\sin 2x}$. $\quad$ "$\frac{0}{0}$"

$$= \lim_{x\to 0} \frac{e^x}{2\cos 2x}$$

$$= \frac{1}{2}$$

Ex: $\lim_{x\to 1} \frac{1 - x + \log x}{1 + \cos \pi x}$. $\quad$ "$\frac{0}{0}$"

$$= \lim_{x\to 1} \frac{-1 + \frac{1}{x}}{-\pi \sin \pi x}, \quad \text{"}\frac{0}{0}\text{"}$$

$$= \lim_{x\to 1} \frac{\frac{-1}{x^2}}{-\pi^2 \cos \pi x}$$

$$= -\frac{1}{\pi^2}$$

When dealing with limits to infinity, we need the following version of L'Hôpital's rule.

Theorem: Suppose f and g are differentiable. Suppose further that $f(x) \to 0$ and $g(x) \to 0$ as $x \to \infty$ (or $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to \infty$).
If $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

Ex: $\lim_{x\to\infty} \frac{\log x}{x}$. "$\frac{\infty}{\infty}$"

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

$$= 0.$$

Ex: $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$.

$$= \lim_{x\to\infty} \exp\left(\ln\left(1+\frac{1}{x}\right)^x\right)$$

$$= \exp\left(\lim_{x\to\infty} x \ln\left(1+\frac{1}{x}\right)\right). \qquad \text{"}0\cdot\infty\text{"}$$

$$\lim_{x\to\infty} x \ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \qquad \text{"}\frac{0}{0}\text{"}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1.$$

So $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$.