THE UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1131 Calculus

Section 9: - Logarithms and Exponentials.

In this section we assume you are familiar with the rules for logarithms and exponentials and concentrate on some of the underlying theory.

We begin with the problem of finding the area under the curve $y = \frac{1}{t}$ from 1 up to some point x.

Define $F(x) = \displaystyle\int_1^x \frac{1}{t}\, dt$.

Theorem: F has the following properties:

(i) $F(1) = 0$

(ii) If $0 < x < 1$ then $F(x) < 0$.

(iii) $F(xy) = F(x) + F(y)$ and $F(\frac{x}{y}) = F(x) - F(y)$.

(iv) $F(x^n) = nF(x)$ for all integers n.

Proof: (i) and (ii) are obvious from the definition. For (iii), write

$$F(xy) = \int_1^{xy} \frac{1}{t}\, dt = \int_1^{x} \frac{1}{t}\, dt + \int_x^{xy} \frac{1}{t}\, dt.$$

$F(x) = F\left(\frac{xy}{y}\right)$
$= F\left(\frac{x}{y}\right) + F(y)$

Make the change of variable $t = ux$ in the second integral to obtain

$$F(xy) = \int_1^{x} \frac{1}{t}\, dt + \int_1^{y} \frac{1}{t}\, dt = F(x) + F(y).$$

The remaining results now follow easily from (iii).

The function F behaves like a *logarithm* and so we rename the function F by $F(x) = \log_e(x) = \log x = \ln x$.

By the Fundamental Theorem of Calculus we have $\frac{d}{dx}(\log x) = \frac{1}{x}$, for all real $x > 0$.

Ex: Find $\frac{d}{dx} \log\log(x)$.

$$\frac{d}{dx} \frac{1}{\log\log x} = \frac{1}{\log x} \cdot \frac{1}{x}$$

Note that at some point x_0, the area under the curve $\frac{1}{t}$ from 1 to x_0 must reach 1, (by continuity), so we denote that point by the number e (after Euler). e has the value 2.7182818284... and is irrational.

We showed earlier that $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ and next Session we shall show that $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ...$

Exponential Function

Since the logarithmic function is continous and increasing for $x > 0$, it has an inverse. Also, since log is a logarithmic function with base e, its inverse is an exponential function which we write as e^x. This is sometimes written as $\exp(x)$.

Hence we know that $\log e^x = x$ for all real x and $e^{\log x} = x$, for all real $x > 0$.

Corollary If $y = e^x$ then $\frac{dy}{dx} = e^x$.

Proof:

$$\log e^x = x \quad \text{gives}$$

$$\frac{d}{dx} \log e^x = 1 \quad \text{or}$$

$$\frac{1}{e^x} \cdot \frac{d}{dx} e^x. \text{ and the result follows.}$$

General Exponentials:

If $a \in \mathbb{R}^+$, can we make sense of a^x?

Definition: If $a \in \mathbb{R}^+$ define a^x to be $e^{x \log a}$.

Ex: Find $2^{\sqrt{3}}$.

$$2^{\sqrt{3}} = e^{\sqrt{3} \log 2}$$

Ex: $5^x = e^{x \log 5}$

Theorem: $\dfrac{d}{dx} a^x = a^x \log a$.

Proof:

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{\log a \cdot x}$$

$$= e^{\log a \cdot x} \cdot \log a$$

$$= a^x \cdot \log a.$$

Ex: Find the derivative of 5^x.

$$\frac{d}{dx} 5^x = 5^x \cdot \log 5.$$

Ex: Find $\lim_{x \to 0^+} x^x$ and hence sketch the curve $y = x^x$ by finding any stationary points.

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} \exp(x \log x)$$

$$= \exp\left(\lim_{x \to 0^+} x \log x\right)$$

$$= \exp\left(\lim_{x \to 0^+} \frac{\log x}{\frac{1}{x}}\right)$$

$$= \exp\left(\lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}\right)$$

$$= \exp\left(\lim_{x \to 0^+} -x\right) = 1.$$

$$y' = \left(e^{x \log x}\right)'$$

$$= e^{x \log x}(\log x + 1$$

Integrals:

Ex: Find a. $\int x e^{5x^2}\, dx = \int \frac{1}{10} u' e^u \, dx = \frac{1}{10}\int e^u du = \frac{1}{10} e^{5x^2} + C$

b. $\int \tan\theta \, d\theta, = \int \frac{\sin\theta}{\cos\theta} d\theta = -\int \frac{u'}{u} d\theta = -\int \frac{1}{u} d\theta = -\log|\cos\theta| + C$

c. $\int_0^\infty e^{-x} \cos x \, dx$ Integrate by parts twice, we get

$$\int e^{-x} \cos x \, dx = \frac{1}{2} e^{-x}(\sin x - \cos x) + C$$

$$= -\frac{1}{2}$$

d. $\int \sec\theta \, d\theta$.

$$\frac{d}{dx}(\sec\theta + \tan\theta) = \sec\theta \tan\theta + \sec^2\theta$$
$$= \sec\theta(\tan\theta + \sec\theta)$$

So if $u = \sec\theta + \tan\theta$, the above gives

$$u' = \sec\theta \, u \quad \text{or} \quad \sec\theta = \frac{u'}{u}$$

$$\int \sec\theta \, d\theta = \int \frac{u'}{u} d\theta = \int \frac{1}{u} du = \ln|u| + C$$
$$= \ln|\sec\theta + \tan\theta| + C.$$

Logarithmic Differentiation:

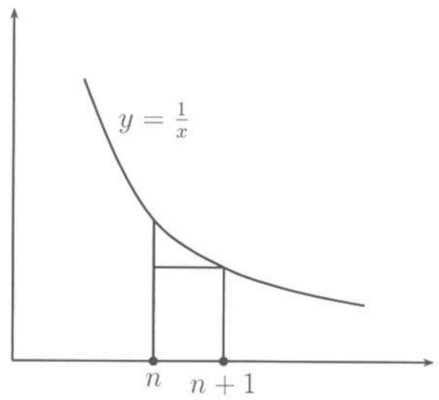
Ex: Differentiate $y = \dfrac{\sqrt{x^2 + x - 1}}{\sqrt[3]{x^4 + 1}}$.

$$\log y = \frac{1}{2}\log(x^2 + x - 1) - \frac{1}{3}\log(x^4 + 1)$$

$$\frac{y'}{y} = \frac{1}{2}\frac{2x+1}{x^2+x-1} - \frac{1}{3}\frac{4x^3}{x^4+1}.$$

An Interesting Limit: (If time).

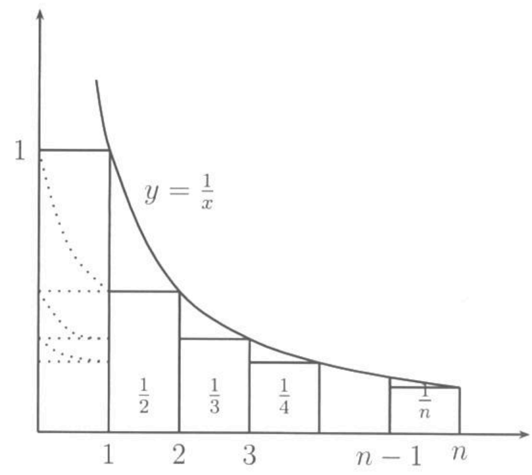
Let $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$, then $a_{n+1} - a_n = \frac{1}{n+1} - \log(n+1) + \log n$.
Consider the area under the curve $y = \frac{1}{x}$ from $x = n$ to $x = n+1$.



Clearly,

$$\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x}\, dx = \log(n+1) - \log n$$

hence $a_{n+1} < a_n$, and so $\{a_n\}$ is a decreasing sequence.
In the diagram, we can see that



$$\int_1^2 \frac{1}{x}\, dx - \frac{1}{2} + \int_2^3 \frac{1}{x}\, dx - \frac{1}{3} + \cdots + \int_{n-1}^n \frac{1}{x}\, dx - \frac{1}{n}$$
$$= \int_1^n \frac{1}{x}\, dx - \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

gives the sum of the shaded areas, but this is clearly less than 1. Subtracting 1 from both sides we have

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n > 0$$

and so the sequence is bounded below by 0. We can thus conclude that a_n converges to a limit. This number is written as γ and is called Euler's constant. It is approximately, 0.5772156649.... Little is known about it. It is not even known whether or not it is irrational!