THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 1 2016

MATH1131
MATHEMATICS 1A

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS IS APPENDED TO THE PAPER

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Use a separate book clearly marked Question 1

1. i) For each of the following, either evaluate the limit or explain why it does not exist.

a) $\displaystyle\lim_{x\to 2} \frac{|x^2-4|}{x-2}$

b) $\displaystyle\lim_{x\to\infty} \frac{2\sin x + x}{3x-1}$

ii) Let

$$f(x) = \begin{cases} x^2 & x \geqslant 0 \\ 0 & x < 0. \end{cases}$$

a) Show that $f(x)$ is differentiable at $x = 0$ and find $f'(0)$.

b) Determine $f'(x)$ for all x.

iii) Let $g(x) = x^7 + 4x + 2$, defined for all real x.

a) Use the Intermediate Value Theorem to show that $g(x) = 0$ has at least one real solution.

b) Show that g has an inverse function with domain $\mathbb{R}$.

iv) Let $z = 1 + 3i$ and $w = 2 - 4i$. Find, in $a + ib$ form:

a) $\overline{w} + 2z$.

b) z/w.

v) Let $u = -\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i$.

a) Calculate $|u|$ and $\mathrm{Arg}(u)$.

b) Hence, or otherwise, find u^{30} in its simplest form.

vi) Let $P = \begin{pmatrix} 1 & 4 \\ 3 & 5 \\ 0 & 7 \end{pmatrix}$ and $Q = \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$.

a) Evaluate PQ^T.

b) What is the size of PQP^T?

Use a separate book clearly marked Question 2

2. i) a) State the Mean Value Theorem.

b) By using the Mean Value Theorem show that

$$\sin x \leq x \quad \text{for all} \quad x \geq 0.$$

ii) Sketch the polar curve

$$r = 1 - \cos\theta \quad \text{for} \quad 0 \leqslant \theta < 2\pi.$$

iii) Use logarithmic differentiation to find $\dfrac{dy}{dx}$ if $y = 2^{\sin x}$.

iv) The volume of a spherical balloon is increasing at a constant rate of 4 cm^3/min. How fast is the radius increasing when the radius is 12 cm? You are given that a sphere of radius r has volume $V = \dfrac{4}{3}\pi r^3$.

v) Let the set S in the complex plane be defined by

$$S = \left\{ z \in \mathbb{C} : -\frac{\pi}{4} \leq \text{Arg}(z) \leq \frac{\pi}{4} \text{ and } \text{Re}(z) \leq 3 \right\}.$$

a) Sketch the set S on a labelled Argand diagram.

b) Let w be the complex number in S with greatest imaginary part. By considering your sketch or otherwise find w in $a + ib$ form.

vi) The points A and B in $\mathbb{R}^3$ have position vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}.$$

a) Find a parametric vector equation of the line l passing through A and B.

b) Hence find a point P on the line such that the sum of the x, y and z coordinates of P is equal to 62.

vii) Let $A = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 3 & 8 \\ 0 & 2 & 6 \end{pmatrix}$.

a) Calculate the determinant of A.

b) Does A have an inverse?

c) Write down the determinant of $5A$.

Use a separate book clearly marked Question 3

3. i) Sketch, on one set of axes, the graphs of $y = \cosh x$ and $y = \cosh^{-1} x$.

ii) Find:

a) $\displaystyle\int x^2\sqrt{3+x^3}\,dx$,

b) $\displaystyle\int xe^{3x}\,dx$.

iii) Find $\dfrac{dy}{dx}$ for $y = \displaystyle\int_0^{x^3} e^{t^2}\,dt$.

iv) Determine which, if any, of the following improper integrals converge (give reasons for your answers):

a) $\displaystyle\int_0^{\infty} xe^{-x^2}\,dx$,

b) $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{1+x^6}}\,dx$.

v) Consider the curve $x = t^2$, $y = t^3$, for $t \in \mathbb{R}$.

a) Sketch the curve for $0 \le t \le 2$.

b) Find the tangent line to the curve at $(1, 1)$.

vi) Prove that $\displaystyle\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x = e^3$.

Use a separate book clearly marked Question ~~3~~ 4

4. i) Given that $p(z) = z^4 - 2z^3 + 7z^2 - 10z + 10$ has $z = 1 + i$ as a root, express $p(z)$ as a product of two real quadratic factors.

ii) A matrix A is defined in the the Maple session below. Use the Maple below to find $(A^{-1})^{17}$.

```
> with(LinearAlgebra):
> A := <<0,1,0,0>|<0,0,1,0>|<0,0,0,1>|<-1,-1,-1,-1>>;
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$$A := \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

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> A^2;
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$$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

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> A^3;
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$$\begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

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> A^4;
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$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

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> A^5;
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$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

iii) The following system of equations has an infinite number of solutions:

$$\begin{aligned} x + 2y + 4z &= 1 \\ x + 3y + 5z &= 2 \\ 2x + 5y + 9z &= 3 \end{aligned}$$

a) Find the general solution.

b) Hence or otherwise find the general solution to the following system of equations.

$$\begin{aligned} x + 2y + 4z &= 0 \\ x + 3y + 5z &= 0 \\ 2x + 5y + 9z &= 0 \end{aligned}$$

iv) Consider the plane Π with Cartesian equation $x + 2y + 9z = 3$.

a) Find a parametric vector equation for the plane Π.

b) Hence or otherwise find a vector $\mathbf{v}$ with a zero z coordinate which is parallel to the plane.

v) Three friends, Donald, Bernie and Hillary entered a specialist coffee chop for some beans. They bought the same three brands Brazilian, Zinger and Wings in varying quantities:

Donald paid \$13 in total for 1 kilogram of Brazilian, 2 kilograms of Zinger and 3 kilograms of Wings.

Bernie paid \$29 in total for 2 kilograms of Brazilian, 2 kilograms of Zinger and 8 kilograms of Wings.

Hillary paid \$52 in total for 4 kilograms of Brazilian, 10 kilograms of Zinger and 11 kilograms of Wings.

Let x be the price per kilogram for the Brazilian brand, y be the price per kilogram for the Zinger brand and z be the price per kilogram for the Wings brand.

a) Write down a system of equations to describe this situation.

b) Solve the system using Gaussian elimination and back-substitution to obtain the price per kilogram of each of the three brands of coffee.

vi) Find the area of the parallelogram with vertices $A(-2,1,3)$, $B(-1,3,0)$, $C(1,3,1)$ and $D(0,1,4)$.

vii) The plane Π_0 has Cartesian equation $ax+by+cz=0$ and the plane Π_1 has Cartesian equation $ax+by+cz=d$. The two planes are parallel and Π_0 passes through the origin.

   a) Find a parametric vector equation for the line which passes through the origin and is perpendicular to both planes.

   b) Hence or otherwise find the distance between the two planes in terms of a, b, c and d.

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BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$

$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$

$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$

$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$

$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$

$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$

$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$

$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\begin{aligned}\int \frac{dx}{a^2 - x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, \qquad |x| < a \\ &= \frac{1}{a}\coth^{-1}\frac{x}{a} + C, \qquad |x| > a > 0 \\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, \qquad x^2 \neq a^2\end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$