School of Mathematics and Statistics
Math1131-Algebra

Lec15: Matrix notations - Elementary row operations

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Vector and matrix form

Consider the system of linear equations

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 1 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & -1 \\ 7x_1 & - & 5x_2 & - & 9x_3 & = & 0 \end{array}$$

This is the same as the vector equation

$$x_1 \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 5 \\ -5 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 6 \\ -9 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

and we also write it as the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

where A is the coefficient matrix and

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -5 & -9 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Vector and matrix form

All of these,

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 1 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & -1 \\ 7x_1 & - & 5x_2 & - & 9x_3 & = & 0 \end{array}$$

$$x_1 \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 5 \\ -5 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 6 \\ -9 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -5 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

are represented by the augmented matrix.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & -1 \\ 7 & -5 & -9 & 0 \end{array}\right)$$

Leading rows and entries

We need some definitions

Leading Entry, Leading Row, Leading Column.

- A *leading row* is a nonzero row (one or more entries are not zero).
- A *leading entry* is the left most nonzero entry in a leading row.
- A *leading column* is a column containing a leading entry.

Example 1. For example, consider the following matrix.

$$\begin{pmatrix} 0 & 5 & 7 \\ 0 & 0 & 0 \end{pmatrix}$$

- Row 1 is a leading row with leading entry 5.
- Row 2 is a non-leading row (ie, a row of zeros).
- Column 2 is the only leading column.

Row Echelon Form

Row Echelon Form.

A matrix is in *Row Echelon Form* if

1. all rows of zeros are at the bottom and
2. each leading entries is further to the right than leading entries in the rows above.

Exercise 3.

In the following augmented matrices, indicate the leading entries (box them).
Which of theses augmented matrices are in Row Echelon Form (REF)?

$$\left(\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 2 & 2 \end{array}\right) \quad \left(\begin{array}{cc|c} 5 & 1 & 2 \\ 0 & 0 & 2 \end{array}\right) \quad \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 3 & 2 & 2 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 2 & 0 \end{array}\right) \quad \left(\begin{array}{cc|c} 0 & 2 & 1 \\ 0 & 0 & 2 \end{array}\right) \quad \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

REDUCED Row Echelon Form

REDUCED Row Echelon Form.

A matrix is in ***Reduced*** *Row Echelon Form* if

1. it is in Row Echelon Form and
2. each leading entry is 1 and
3. each leading entry is the only nonzero entry in its column.

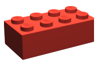
Exercise 4.

In the following augmented matrices, indicate the leading entries. Which are in Reduced Row Echelon Form (RREF)?

$$\left(\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 2 & 2 \end{array}\right) \quad \left(\begin{array}{cc|c} 5 & 1 & 2 \\ 0 & 0 & 2 \end{array}\right) \quad \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 2 & 0 \end{array}\right) \quad \left(\begin{array}{cc|c} 0 & 2 & 1 \\ 0 & 0 & 2 \end{array}\right) \quad \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Pivots



Pivot element, pivot row, pivot column.

- A *pivot element* is a nonzero entry in the first nonzero column.
- A *pivot row* is the row containing a pivot element.
- A *pivot column* is the column containing a pivot element.

Example 5.
Consider the following augmented matrix:

$$\left(\begin{array}{ccc|c} 0 & 0 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 2 & 0 & 4 \end{array}\right).$$

The highlighted entry is a pivot element.
Row 2 is a pivot row and column 2 is a pivot column.

Gaussian elimination (to REF)

To solve a system of linear equations apply Gaussian elimination to its Augmented matrix. That is,

The Gaussian Elimination Algorithm.

1. Find a pivot element. If one of these nonzero entry in the first non-zero column is 1, we pick this one to be the pivot. Otherwise we usually pick the first nonzero entry to be the pivot.

2. Swap row 1 with the pivot row (the pivot element is now in the first row.)

3. Add a multiple of the pivot row to the rows below it to get zero entries below the pivot element.

4. Repeat for the submatrix below and right of the pivot element.

Once in Row Echelon Form (triangular form), we can see the nature of the solutions and use back substitution to solve when solutions exist.

Gaussian elimination examples

Exercise 6. Solve, if possible, the system of linear equations

$$\begin{aligned} 3x - y &= 4 \\ x + y &= 8 \\ x - y &= -2 \\ 6x - 3y &= 3 \end{aligned}$$

and give a geometric interpretation.

Gaussian elimination examples

Exercise 6, continued.

What Row Operations exactly are allowed?

The *elementary row operations* are the ones we are allowed to use when we put a matrix in Row Echelon Form using Gaussian Elimination.
To ensure the reader can follow the process step by step, we record the operations used.

Elementary Row Operations.

1. **Interchange two rows**. Interchanging row i and row k is recorded by $Ri \leftrightarrow Rk$.
2. **Multiply a row by a nonzero number**. Multiplying row i by a nonzero number α is recorded by $R_i \leftarrow \alpha R_i$.
3. **Add a multiple of a row to another row**. Adding α times row k to row i is recorded by $R_i \leftarrow R_i + \alpha R_k$.

Using elementary row operations at each step ensures that we change our system of equations to an equivalent one, that is, one which has exactly the same solutions.

TIP!

"$R_2 \leftarrow R_2 - 2R_1$" can be read "*$R_2$ is assigned the value $R_2 - 2R_1$*" or "*R_2 is replaced by $R_2 - 2R_1$*" or "*We put* $R_2 - 2R_1$ into R_2".

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Gaussian elimination examples

Exercise 7. Solve, if possible, the system of linear equations

$$\begin{aligned} x + y + z &= 6 \\ x + y + 3z &= 14 \\ 2x + 3y + 4z &= 23 \\ -x + 2y + z &= 11 \end{aligned}$$

and give a geometric interpretation.

Exercise 7, continued

Checking our answers with Maple

```
> # Load the LinearAlgebra package

  with(LinearAlgebra):

> # Enter the matrix column by column

  A := < <1,1,2,-1> | <1,1,3,2> | <1,3,4,1> >;
  b := <6, 14, 23, 11>:
```

$$A := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix}$$

```
> # The augmented matrix is:

  Ab := <A|b>;
```

$$Ab := \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 1 & 3 & 14 \\ 2 & 3 & 4 & 23 \\ -1 & 2 & 1 & 11 \end{bmatrix}$$

```
> GaussianElimination(Ab);
```

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 11 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Gaussian elimination examples

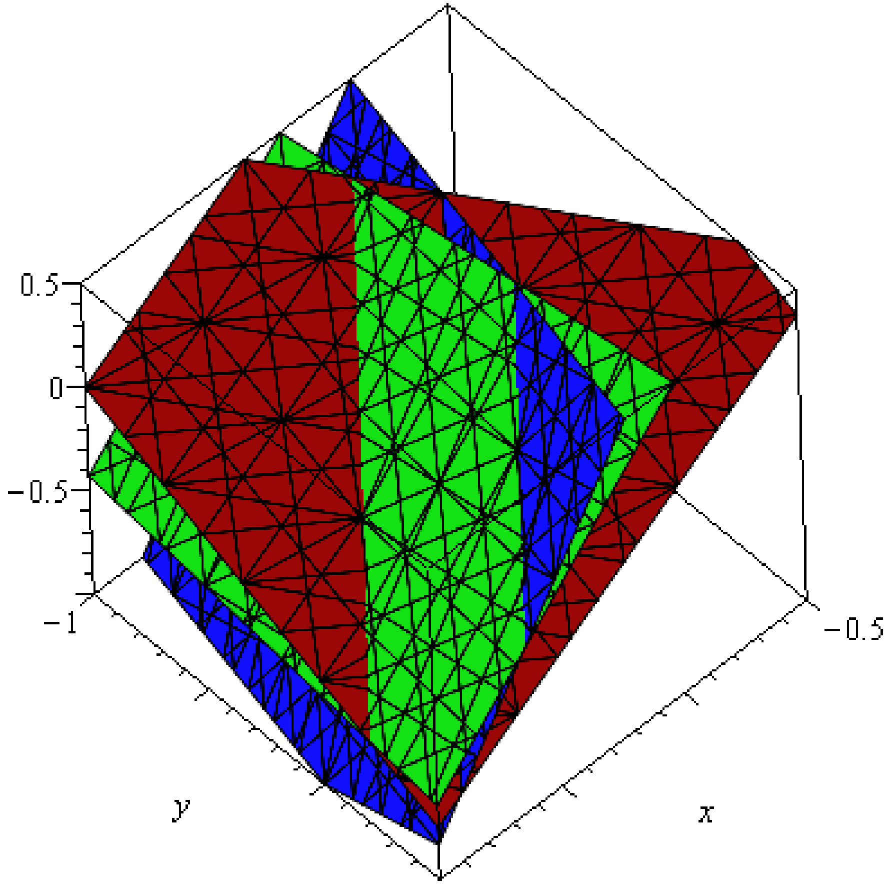
Exercise 8. Solve the system of linear equations

$$\begin{aligned} 3x + 2y + 4z &= 1 \\ 5x - y + 3z &= 2 \\ 8x + y + 7z &= 4 \end{aligned}$$

and give a geometric interpretation.

3 planes meeting only in pairs

Exercise 8, geometric interpretation.



$$3x + 2y + 4z = 1$$
$$5x - y + 3z = 2$$
$$8x + y + 7z = 4$$

Gaussian elimination examples

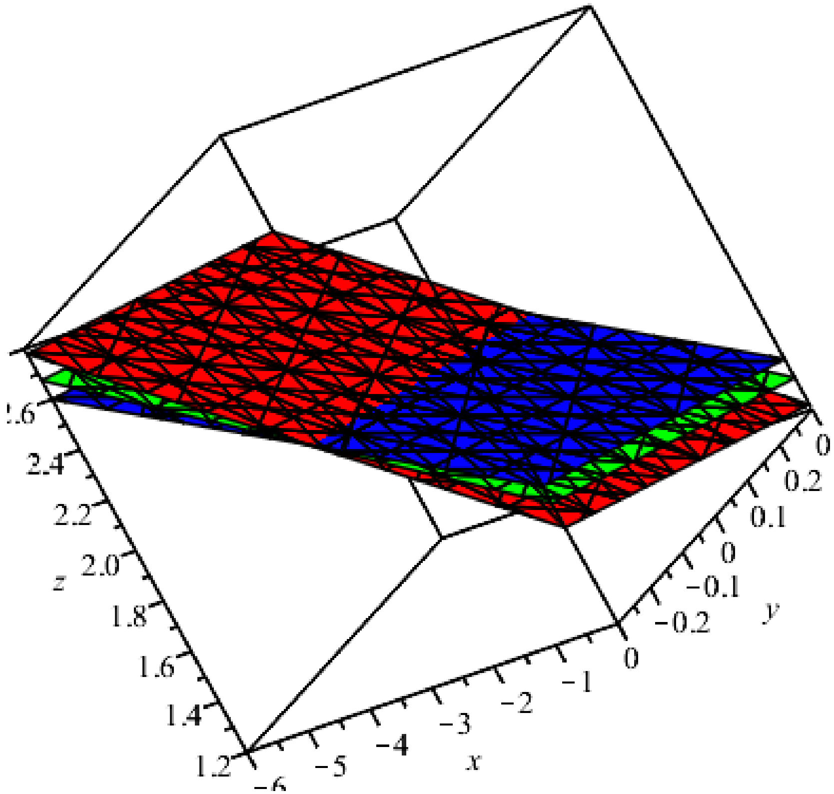
Exercise 9. ♡ Solve the system of linear equations

$$\begin{aligned} x + 3y + 5z &= 7 \\ x + 4y + 7z &= 11 \\ 2x + 7y + 12z &= 18 \end{aligned}$$

and give a geometric interpretation.

3 planes meeting in a line

Exercise 9, geometric interpretation.



$$x + 3y + 5z = 7$$
$$x + 4y + 7z = 11$$
$$2x + 7y + 12z = 18$$

Gaussian elimination examples

Exercise 10.

Use the Maple output given below to solve the following system of linear equations.

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 - x_5 &= 1 \\ 2x_1 + 2x_2 + 2x_4 - 6x_5 &= -4 \\ 6x_1 + 6x_2 + 4x_3 + 3x_4 - 10x_5 &= -3 \end{aligned}$$

```
> A := < <1,2,6> | <1,2,6> | <1,0,4> | <1,2,3> |
  <-1,-6,-10> >;
  b := <1,-4,-3>:
```

$$A := \begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 2 & 2 & 0 & 2 & -6 \\ 6 & 6 & 4 & 3 & -10 \end{bmatrix}$$

```
> Ab := <A|b>;
```

$$Ab := \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & 1 \\ 2 & 2 & 0 & 2 & -6 & -4 \\ 6 & 6 & 4 & 3 & -10 & -3 \end{bmatrix}$$

```
> GaussianElimination(Ab);
```

$$\begin{bmatrix} 1 & 1 & 1 & 1 & -1 & 1 \\ 0 & 0 & -2 & 0 & -4 & -6 \\ 0 & 0 & 0 & -3 & 0 & -3 \end{bmatrix}$$

Nature of solutions

Exercise 11. For each of the following augmented matrices, describe the nature of the solutions of the corresponding linear system. Write down the solutions if the system is consistent.

a) $\left(\begin{array}{ccc|c} 1 & 4 & 7 & 4 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & 8 & 2 \end{array}\right)$ b) $\left(\begin{array}{ccc|c} 5 & 0 & 0 & 14 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 8 & 7 \end{array}\right)$

Nature of solutions

Exercise 11, continued. For each of the following augmented matrices, describe the nature of the solutions of the corresponding linear system. Write down the solutions if the system is consistent.

c) $\left(\begin{array}{ccc|c} 0 & 5 & 1 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 5 \end{array}\right)$ d) $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Nature of solutions

Exercise 11, continued. For each of the following augmented matrices, describe the nature of the solutions of the corresponding linear system. Write down the solutions if the system is consistent.

e) $\left(\begin{array}{ccccc|c} 3 & 5 & 1 & 0 & 2 & 4 \\ 0 & 0 & -1 & 8 & 1 & 6 \end{array}\right)$ f) $\left(\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$

Nature of solutions

Investigation 12. If the augmented matrix

$$(A|\mathbf{b})$$

is reduced to Row Echelon Form

$$(U|\mathbf{y}),$$

what conditions on U and $\mathbf{y}$ signal the following?

- no solutions
- a unique solution
- infinitely many solutions

In the case of infinitely many solutions, how can you tell how many parameters are needed to describe the solutions?