THE UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1131 Calculus

Section 6: - Inverse Functions.

We have intuitively thought of a function as a rule, which starts from one real number and produces another. We now ask the question as to when we can *reverse* the procedure.
For example, under the function $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = 2x + 3$, the real number 5 maps to 13. On the other hand what number maps on to 10? Answer 3.5. Indeed given the y value, the corresponding x-value it came from is $\frac{y-3}{2}$. This new rule, is itself a function, which can be written as $g(x) = \frac{x-3}{2}$. We say that these two functions are **inverses** of each other the write $g(x) = f^{-1}(x)$. (Note that the index does NOT mean 'one over').

Also note that if we compose f and g we obtain the identity function, i.e. $f \circ g(x) = f(g(x)) = f(\frac{x-3}{2}) = x$ and $g \circ f(x) = g(f(x)) = g(2x + 3) = x$. Hence:

Definition: Given a function $f : A \to B$, if there is a function $g : B \to A$ such that $f \circ g(x) = x$ and $g \circ f(x) = x$, then we say that g is the inverse of f and write $g = f^{-1}$.

Ex: Show that if $f(x) = e^x$ then $g(x) = \log x$ is the inverse of f.

Clearly not all functions have inverses, for example $f(x) = x^2$. The y value 9 came from both 3 and -3.

When does a given function f defined on an interval $[a, b]$ have an inverse?

One simple test is known as the *horizontal line test.* It says that if we look at the graph of f with domain D and co-domain R and draw any horizontal line, $y = b$, where $b \in R$ then f will have an inverse if the line cuts the graph at **exactly one point**.

Ex: Draw $y = x^3 + x + 1$ and $y = x^4 - x^2$ to illustrate this.

Theorem: Suppose f is differentiable on (a, b) and $f'(x) \neq 0$ for all $x \in (a, b)$ then f has an inverse on (a, b).

Ex: $y = x^3 + x + 1$. (Note that although this function has an inverse, it is not easy to explicitly write down the formula for the inverse.)

Ex: $f(x) = 2x + \sin x$.

We can sometimes restrict the domain of a function f so that although f does not have an inverse on its natural domain, it does on this restricted domain.

Ex: Find maximal regions on which the function $f(x) = x^3 - x$ has an inverse.

Suppose f has an inverse on (a, b) and f is differentiable on (a, b). How do we find the derivative of the inverse?

Theorem: Suppose f is diffble on (a, b) and has an inverse $g(x)$ on (a, b), then

$$g'(x) = \frac{1}{f'(g(x))}.$$

Proof:

Ex: Let $f(x) = x^3 + x + 1$, with inverse function g. Find $g'(1)$.

Ex: Let $f(x) = 2x + \sin x$. Find $(f^{-1})'(\pi)$.

Inverse Trigonometric Functions:

Ex: Find a. $\sin^{-1}(\sin(\frac{5\pi}{3})$ b. $\sin(\sin^{-1}(-\frac{1}{2}))$, c. $\sin(2\cos^{-1}(\frac{4}{5}))$.

Theorem
For $-1 \leq x \leq 1$ we have
$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}.$$

Proof:

Ex: Prove that $\sin^{-1}(x) + \sin^{-1}\sqrt{1-x^2} = \frac{\pi}{2}$.

Ex: Find $\frac{d}{dx}\csc^{-1} x$

Ex: a. Find $\frac{d}{dx}(\cot^{-1}(x))$.
b. A statue 2 metres high is mounted on a pedestal. The base of the statute is 6m above the eye-level of an observer. How far from the base of the pedestal should the observer stand to get the 'best' view.

Integrals Involving Inverse Trigonometric Functions: