MATH1131 Mathematics 1A

and

MATH1141 Higher Mathematics 1A

ALGEBRA NOTES

CRICOS Provider No: 00098G

Preface

Please read carefully.

These Notes form the basis for the algebra strand of MATH1131 and MATH1141. However, not all of the material in these Notes is included in the MATH1131 or MATH1141 algebra syllabuses. A detailed syllabus will be uploaded to Moodle.

In using these Notes, you should remember the following points:

1. Most courses at university present new material at a faster pace than you will have been accustomed to in high school, so it is essential that you start working right from the beginning of the session and continue to work steadily throughout the session. Make every effort to keep up with the lectures and to do problems relevant to the current lectures.

2. These Notes are **not** intended to be a substitute for attending lectures or tutorials. The lectures will expand on the material in the notes and help you to understand it.

3. These Notes may seem to contain a lot of material but not all of this material is equally important. One aim of the lectures will be to give you a clearer idea of the relative importance of the topics covered in the Notes.

4. Use the tutorials for the purpose for which they are intended, that is, to ask questions about both the theory and the problems being covered in the current lectures.

5. Some of the material in these Notes is more difficult than the rest. This extra material is marked with the symbol [**H**]. Material marked with an [**X**] is intended for students in MATH1141.

6. Problems marked with [**V**] have a video solution available from Moodle.

7. It is **essential** for you to do **problems** which are given at the end of each chapter. If you find that you do not have time to attempt all of the problems, you should at least attempt a representative selection of them.

8. You will be expected to use the computer algebra package Maple in tests and understand Maple syntax and output for the end of semester examination.

Note.

These notes have been prepared by many members of the School of Mathematics and Statistics. The main contributors include Peter Blennerhassett, Peter Brown, Shaun Disney, Peter Donovan, Ian Doust, David Hunt, Chi Mak, Elvin Moore and Colin Sutherland.

Contents

ALGEBRA SYLLABUS AND LECTURE TIMETABLE

The algebra course for both MATH1131 and MATH1141 is based on the MATH1131/MATH1141 Algebra Notes that are included in the Course Pack.

A detailed syllabus and lecture schedule will be posted on Moodle. Please note that the order of the syllabus changed in 2014, in accordance with requests from the Engineering Faculty and the School of Physics. It is important to note this in regard to the class tests from previous years.

The computer package Maple will be used in the algebra course. An introduction to Maple is included in the booklet titled *First Year Maple Notes*.

ALGEBRA PROBLEM SETS

The Algebra problems are located at the end of each chapter of the Algebra Notes booklet. They are also available from the course module on the UNSW Moodle server. The problems marked [**R**] form a basic set of problems which you should try first. Problems marked [**H**] are harder and can be left until you have done the problems marked [**R**]. You *do* need to make an attempt at the [**H**] problems because problems of this type will occur on tests and in the exam. If you have difficulty with the [**H**] problems, ask for help in your tutorial. Questions marked with a [**V**] have a video solution available from the course page for this subject on Moodle. The problems marked [**X**] are intended for students in MATH1141 – they relate to topics which are only covered in MATH1141. Extra problem sheets for MATH1141 may be issued in lectures.

There are a number of questions marked [**M**], indicating that Maple is required in the solution of the problem.

ALGEBRA PROBLEM SCHEDULE

Solving problems and writing mathematics clearly are two separate skills that need to be developed through practice. We recommend that you keep a workbook to practice *writing* solutions to mathematical problems. The range of questions suitable for each week will be provided on Moodle along with a suggestion of specific recommended problems to do before your classroom tutorials.

The Online Tutorials will develop your problem solving skills, and give you examples of mathematical writing. Online Tutorials help build your understanding from lectures towards solving problems on your own.

Chapter 1

INTRODUCTION TO VECTORS

"You see, the earth takes twenty-four hours to turn round on its axis —"
"Talking of axes," said the Duchess, "chop off her head."
Lewis Carroll, Alice in Wonderland.

The aims of this chapter are to introduce the idea of "vector" and in a relatively informal and intuitive manner, and to illustrate applications of these ideas to the geometry of lines and planes.

Until quite recently, the main applications of vectors had been in the physical and engineering sciences. However the study of vectors has now become an important branch of modern pure and applied mathematics, and vectors are now being used in such diverse fields as economics and management science, psychology and the social sciences, chemistry and chemical engineering, mechanical and electrical engineering, computer science, numerical analysis and computational mathematics.

The definition of vectors used in mathematics courses is essentially algebraic in nature whereas the one use by physicists is geometric. We shall begin with the geometric approach then we shall introduce the algebraic definition and show how they relate to one another. As we shall see however, the algebraic definition of a vector is not limited to describing quantities that arise in physics and engineering.

1.1 Vector quantities

Vector quantities, as opposed to scalar quantities, are very important in an understanding of the laws of physics and engineering.

A **scalar** quantity is anything that can be specified by a single number. Examples of scalar quantities are temperature, distance, mass and speed. For example, specifying the speed of a car as 60 km per hour only involves the single real number 60.

A **vector** quantity is one which is specified by both a magnitude and a direction. Examples of vector quantities are displacement, velocity, force and electric field. For example, specifying the velocity of a car as 60 km per hour northeast involves a **vector** of magnitude 60 and direction northeast.

The usual notational convention in books is to differentiate vector quantities from scalar ones by denoting them by boldface symbols such as **a**, and we shall do this in these notes. In handwriting, one usually signifies that a quantity is a vector by using a tilde sign under the letter (as in $\underset{\sim}{a}$), or by writing an arrow above the letter (as in $\vec{a}$). Because the properties of scalar and vector quantities are quite different, it is vital that you distinguish them, especially in solutions to problems.

The magnitude of the vector $\mathbf{a}$ is usually denoted by $|\mathbf{a}|$. Note that this is always a non-negative real number, and that $|\mathbf{a}| = 0$ only when $\mathbf{a}$ is the zero vector, usually denoted by $\mathbf{0}$.

> **Definition 1.** *The* ***zero vector*** *is the vector* $\mathbf{0}$ *of magnitude zero, and undefined direction.*

1.1.1 Geometric vectors

To represent a vector on a diagram we draw an arrow (i.e. a directed line segment) where the length of the arrow is the magnitude of the vector, and the direction of the arrow is the direction of the vector. An arrow can be specified by its *initial point* (the tail) and its *terminal point* (the head). In figure 1, a vector $\mathbf{a}$ is represented by an arrow with initial point P and terminal point Q. We denote this arrow by $\overrightarrow{PQ}$.

Two vectors are said to be equal if they have the same magnitude and direction. As the arrows $\overrightarrow{AB}$ and $\overrightarrow{EF}$ have the same length and direction as $\overrightarrow{PQ}$, these arrows all represent the same vector. We can write

$$\overrightarrow{PQ} = \overrightarrow{AB} = \overrightarrow{EF}.$$

Each vector may be represented by many arrows, but each arrow only represents one vector. Nonetheless, we shall sometimes find it convenient to blur the distinction and write expressions like $\mathbf{a} = \overrightarrow{PQ}$, when we really mean that $\overrightarrow{PQ}$ represents $\mathbf{a}$.

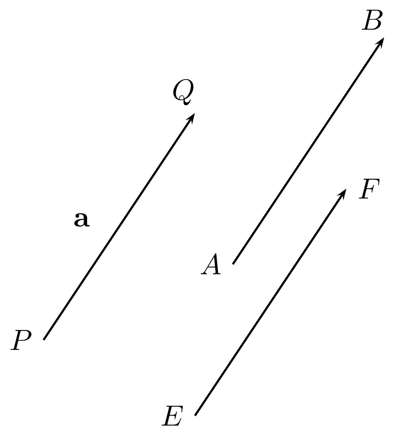
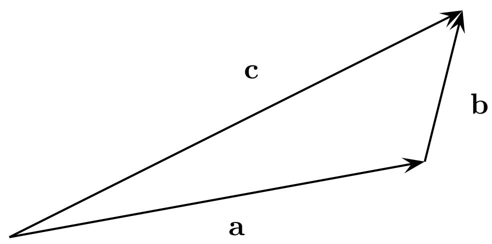


Figure 1.

NOTE. Since we use the intuitive notion of direction in our physical world, apparently we can only have two dimensional or three dimensional geometric vectors. We shall introduce the algebraic definition of vectors including the higher dimensional ones in the next section. Though we can still talk about higher dimensional geometric vectors, the notions of length and direction will depend on the algebraic nature of the vectors.

There are two equivalent ways to add two vectors together. The first addition rule is often known as the triangle law for vector addition.

> **Definition 2. (The addition of vectors).** *On a diagram drawn to scale, draw an arrow representing the vector* $\mathbf{a}$. *Now draw an arrow representing* $\mathbf{b}$ *whose initial point lies at the terminal point of the arrow representing* $\mathbf{a}$. *The arrow which goes from the initial point of* $\mathbf{a}$ *to the terminal point of* $\mathbf{b}$ *represents the vector* $\mathbf{c} = \mathbf{a}+\mathbf{b}$.

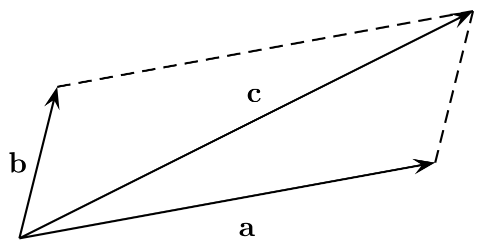


$\mathbf{c} = \mathbf{a} + \mathbf{b}$

Figure 2: Addition of Vectors.

The second addition rule is known as the parallelogram law for vector addition.

Definition 3. (Alternate definition of vector addition). *On a diagram drawn to scale, draw an arrow representing the vector* $\mathbf{a}$. *Now draw an arrow representing* $\mathbf{b}$ *whose initial point lies at the initial point of the arrow representing* $\mathbf{a}$. *Draw the parallelogram with the two arrows as adjacent sides. The initial point and the two terminal points are vertices of the parallelogram. The arrow which goes from the initial point of* $\mathbf{a}$ *to the fourth vertex of the parallelogram represents the vector* $\mathbf{c} = \mathbf{a} + \mathbf{b}$.



$\mathbf{c} = \mathbf{a} + \mathbf{b}$

Figure 3: Addition of Vectors.

Obviously, these two definitions are equivalent.

Vector addition is of course quite different from usual addition of numbers, but they do share some important properties. In particular, for any vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$,

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}, \qquad (\textbf{Commutative law of vector addition})$$

$$\text{and} \qquad (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}). \qquad (\textbf{Associative law of vector addition})$$

These laws, which follow from basic geometry (see Figures 4 and 5), assert that it makes no difference in what order, or in what grouping we add vectors.

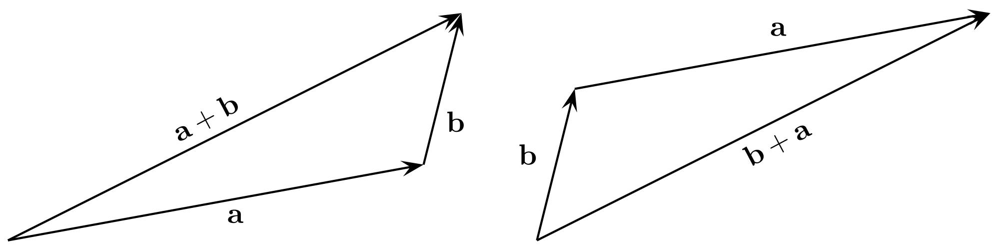


Figure 4: Commutative Law of Vector Addition.

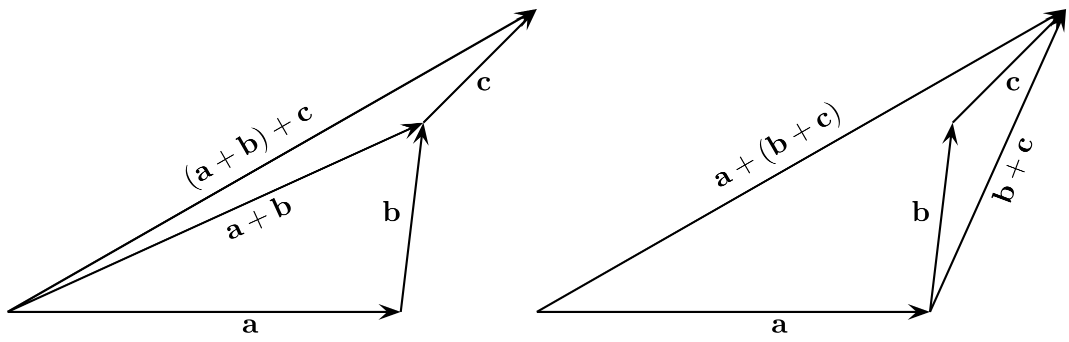


Figure 5: Associative Law of Vector Addition.

The law of addition can easily be extended to cover the zero vector by the natural condition that $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$ for all vectors $\mathbf{a}$. We then can introduce the negative of a vector and the subtraction of vectors.

Definition 4. (Negative of a vector and subtraction). *The* ***negative*** *of* $\mathbf{a}$*, written* $-\mathbf{a}$ *is a vector such that*

$$\mathbf{a} + (-\mathbf{a}) = -\mathbf{a} + \mathbf{a} = \mathbf{0}.$$

If $\mathbf{a}$ *and* $\mathbf{b}$ *are vectors, we define the subtraction by*

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$

From the definition, the vector $-\mathbf{a}$ is the unique vector which has the same magnitude as $\mathbf{a}$, but the direction is opposite to that of $\mathbf{a}$. As shown in Figure 6, the arrow representing $\mathbf{a} - \mathbf{b}$ can be drawn by first reversing the arrow representing $\mathbf{b}$, and then adding the vectors $\mathbf{a}$ and $-\mathbf{b}$.

On the other hand if we represent the vectors $\mathbf{a}$ and $\mathbf{b}$ by arrows with the same initial point O. Let P and Q be the terminal points of the two vectors, respectively. Suppose that $OPRQ$ is a parallelogram. From definition, we have

$$\begin{aligned} \overrightarrow{OQ} + \overrightarrow{QP} &= \overrightarrow{OP} \\ \mathbf{b} + \overrightarrow{QP} &= \mathbf{a} \\ \overrightarrow{QP} &= \mathbf{a} - \mathbf{b} \end{aligned}$$

Hence the diagonal $\overrightarrow{QP}$ represents the difference $\mathbf{a} - \mathbf{b}$. Moreover, $\mathbf{a} - \mathbf{b}$ is the vector which can be represented by the arrow from the terminal point of $\mathbf{b}$ to the terminal point of $\mathbf{a}$.

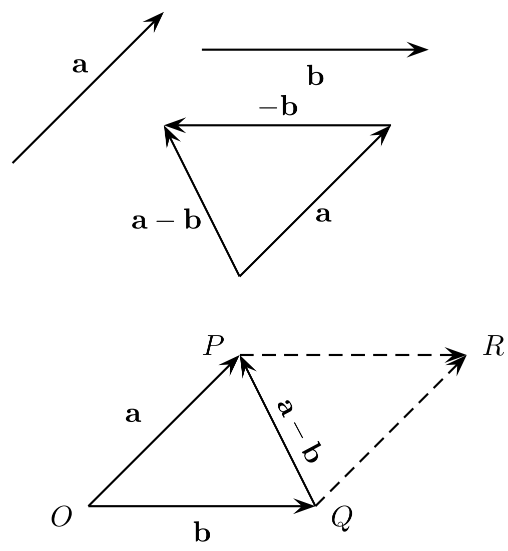


Figure 6: Subtraction of Vectors.

We now define the operation of multiplying a vector by a real number. Roughly speaking, to multiply a vector $\mathbf{a}$ by a real number λ, all we do is stretch the vector by a factor of λ, whilst keeping its direction unchanged. We need to be careful if λ is not positive.

Definition 5. (Multiplication of a vector by a scalar). *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be vectors and let* $\lambda \in \mathbb{R}$.

1. *If* $\lambda > 0$, *then* $\lambda\mathbf{a}$ *is the vector whose magnitude is* $\lambda|\mathbf{a}|$ *and whose direction is the same as that of* $\mathbf{a}$.
2. *If* $\lambda = 0$, *then* $\lambda\mathbf{a} = \mathbf{0}$.
3. *If* $\lambda < 0$, *then* $\lambda\mathbf{a}$ *is the vector whose length is* $|\lambda||\mathbf{a}|$ *and whose direction is the opposite of the direction of* $\mathbf{a}$.

From the above, the negative of $\mathbf{a}$ and the product of the scalar -1 and the vector $\mathbf{a}$ are both the vector which has the same magnitude as $\mathbf{a}$, but the direction is the opposite of that of $\mathbf{a}$. Hence we have

$$-\mathbf{a} = (-1)\mathbf{a}.$$

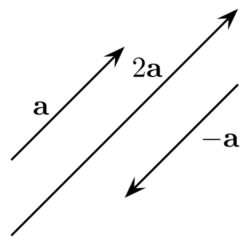


Figure 7: Scalar Multiplication.

We have already seen the commutative and associative laws of vector addition. There are some other important properties of scalar multiplication and vector addition. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors, λ and μ be real numbers, then:

$$\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a}, \qquad (\textbf{Associative law of multiplication by a scalar})$$
$$(\lambda+\mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}, \qquad (\textbf{Scalar distributive law})$$
$$\lambda(\mathbf{a}+\mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}. \qquad (\textbf{Vector distributive law})$$

The vector distributive law for the case that $\lambda > 0$ follows from properties of similar triangles and Figure 8. The proof for the other cases and the proofs of the other two laws are left as exercises for the 1141 students.

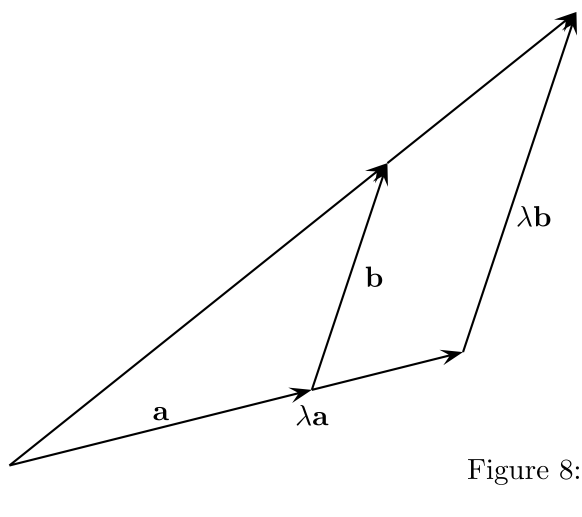


Figure 8: Vector Distributive Law.

We can use these rules to simplify vector expressions.

Example 1. Simplify $3(2\mathbf{a}-\mathbf{b})+(\mathbf{a}-2\mathbf{b})$.

SOLUTION. In this example, we shall quote all the rules that we used.

$$\begin{array}{rll}
 & 3(2\mathbf{a}-\mathbf{b})+(\mathbf{a}-2\mathbf{b}) & \\
= & 3[2\mathbf{a}+(-1)\mathbf{b}]+[\mathbf{a}+(-1)(2\mathbf{b})] & \text{(Definition of subtraction; } -\mathbf{a}=(-1)\mathbf{a}\text{)}\\
= & [3(2\mathbf{a})+3((-1)\mathbf{b})]+[\mathbf{a}+(-1)(2\mathbf{b})] & \text{(Vector distributive law)}\\
= & [6\mathbf{a}+(-3)\mathbf{b}]+[\mathbf{a}+(-2)\mathbf{b}] & \text{(Associative law of multiplication by a scalar)}\\
= & (6\mathbf{a}+\mathbf{a})+[(-3)\mathbf{b}+(-2)\mathbf{b}] & \text{(Associative law and commutative law)}\\
= & (6+1)\mathbf{a}+[(-3)+(-2)]\mathbf{b} & \text{(Scalar distributive law)}\\
= & 7\mathbf{a}-5\mathbf{b} & \text{(Definition of subtraction; } -\mathbf{a}=(-1)\mathbf{a}\text{)}
\end{array}$$

In practice, we simply write

$$3(2\mathbf{a}-\mathbf{b})+(\mathbf{a}-2\mathbf{b})=6\mathbf{a}-3\mathbf{b}+\mathbf{a}-2\mathbf{b}=7\mathbf{a}-5\mathbf{b}. \qquad \diamondsuit$$

Example 2. Simplify $2\overrightarrow{AC}-\overrightarrow{OC}+\overrightarrow{OA}$.

SOLUTION. Let $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OC}=\mathbf{c}$.

$$2\overrightarrow{AC}-\overrightarrow{OC}+\overrightarrow{OA}=2(\mathbf{c}-\mathbf{a})-\mathbf{c}+\mathbf{a}=2\mathbf{c}-2\mathbf{a}-\mathbf{c}+\mathbf{a}=\mathbf{c}-\mathbf{a}=\overrightarrow{AC}.$$

$\diamondsuit$

1.1.2 Two dimensional vector quantities

At first, we can apply geometric vectors to prove some geometry theorems. In some aspects, vectors do have some advantages. For instance, we can prove a quadrilateral $OABC$ to be a parallelogram simply by proving that $\overrightarrow{OA}=\overrightarrow{BC}$. To prove two lines PQ and RS are parallel, we need to show that there exists a real number λ such that $\overrightarrow{PQ}=\lambda\overrightarrow{RS}$.

Example 3. In a triangle OAB, take D and E such that $OD:DA=OE:EB=1:2$. Prove that DE is parallel to AB and the length of DE is $\frac{1}{3}$ times that of that of AB.

Proof. Since D divides OA in the ratio $1:2$, the vector $\overrightarrow{OD}$ has the same direction as $\overrightarrow{OA}$ and its length is $\frac{1}{3}$ times that of $\overrightarrow{OA}$. Hence $\overrightarrow{OD}=\frac{1}{3}\overrightarrow{OA}$. Similarly, we also have $\overrightarrow{OE}=\frac{1}{3}\overrightarrow{OB}$. Thus

$$\begin{aligned}
\overrightarrow{DE} &= \overrightarrow{OE}-\overrightarrow{OD} = \frac{1}{3}\overrightarrow{OB}-\frac{1}{3}\overrightarrow{OA}\\
&= \frac{1}{3}(\overrightarrow{OB}-\overrightarrow{OA}) = \frac{1}{3}\overrightarrow{AB}.
\end{aligned}$$

Hence DE is parallel to AB and the length of DE is $\frac{1}{3}$ times that of that of AB. $\square$

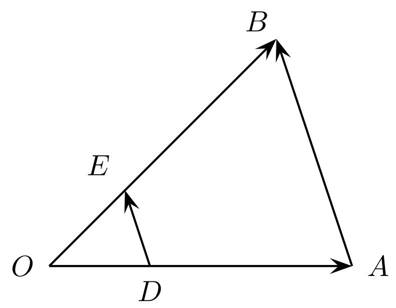


Figure 9.

[X] **Example 4.** Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. Prove that P is a point on the line AB between A and B if and only if $\overrightarrow{OP} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$ for some real number $0 < \lambda < 1$.

Proof. If P is a point on AB between A and B, there exists a real number λ between 0 and 1 such that $\overrightarrow{AP} = \lambda\overrightarrow{AB}$.

$$\begin{aligned} \text{Hence, } \overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AP} = \overrightarrow{OA} + \lambda\overrightarrow{AB} \\ &= \mathbf{a} + \lambda(\mathbf{b}-\mathbf{a}) = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}. \end{aligned}$$

Conversely, if $\overrightarrow{OP} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$ and $0 < \lambda < 1$, we have

$$\overrightarrow{AP} = [(1-\lambda)\mathbf{a} + \lambda\mathbf{b}] - \mathbf{a} = \lambda(\mathbf{b}-\mathbf{a}) = \lambda\overrightarrow{AB}.$$

Since $0 < \lambda < 1$, the point P lies on AB between A and B. ☐

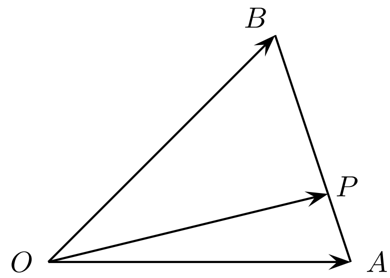


Figure 10.

[X] **Example 5.** Prove that the three medians of a triangle are concurrent.

Proof. Name the vertices of a triangle by O, A and B. Let D, E, F be the midpoints of OB, OA, AB respectively. Suppose that AD and BE intersect at G.

Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. Hence

$$\overrightarrow{OE} = \frac{1}{2}\mathbf{a} \quad \text{and} \quad \overrightarrow{OD} = \frac{1}{2}\mathbf{b}.$$

Since G lies on both AD and BE and inside the triangle, from Example 4 there exist real numbers λ and μ such that

$$\overrightarrow{OG} = (1-\lambda)\,\mathbf{a} + \lambda\left(\frac{1}{2}\mathbf{b}\right) = (1-\mu)\,\mathbf{b} + \mu\left(\frac{1}{2}\mathbf{a}\right).$$

By rearranging terms, we get

$$(1-\lambda)\,\mathbf{a} - \frac{1}{2}\mu\,\mathbf{a} = (1-\mu)\,\mathbf{b} - \frac{1}{2}\lambda\,\mathbf{b}.$$

Since $\mathbf{a}$ cannot be a non-zero scalar multiple of $\mathbf{b}$, we have

$$(1-\lambda) - \frac{1}{2}\mu = 0 \quad \text{and} \quad (1-\mu) - \frac{1}{2}\lambda = 0.$$

By solving the above simultaneous equations, we have $\lambda = \frac{2}{3}$ and $\mu = \frac{2}{3}$. So we have $\overrightarrow{OG} = \frac{1}{3}(\mathbf{a}+\mathbf{b})$.

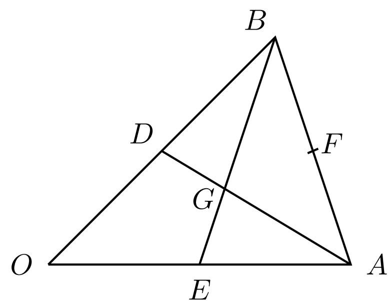


Figure 11.

Since F is the midpoint of AB, so $\overrightarrow{OF} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$. Thus $\overrightarrow{OG}$ and $\overrightarrow{OF}$ are in the same direction. Hence G lies on OF and therefore the three medians of the triangle OAB are concurrent. ☐

Many calculations with vector quantities in the plane can be done geometrically using scale diagrams.

Example 6. A yacht sails from a pier in the direction N 60° E for 15 km, then turns to N 45° W for 10 km. What are the distance and the bearing of the yacht from the pier?

SOLUTION. As shown in Figure 12, after the first leg, the yacht is at P which is $15\cos 30°$ km east and $15\sin 30°$ km north of the pier, O. The yacht then moves, $10\cos 45°$ km west and $10\sin 45°$ km north, to Q. The yacht is then

$$\begin{aligned} & 15\cos 30° - 10\cos 45° \approx 5.919\,\text{km east of } O \\ \text{and} \quad & 15\sin 30° + 10\sin 45° \approx 14.571\,\text{km north of } O. \end{aligned}$$

$$\begin{aligned} \text{Hence } OQ &\approx \sqrt{(5.919)^2 + (14.571)^2} \approx 15.73, \\ \theta &\approx \tan^{-1}\left(\frac{14.571}{5.919}\right) \approx 67°54'. \end{aligned}$$

Here, θ is angle of OQ measured from the east. The distance and the bearing of the yacht from the pier are then 15.73 km and N 22°6′ E. ◊

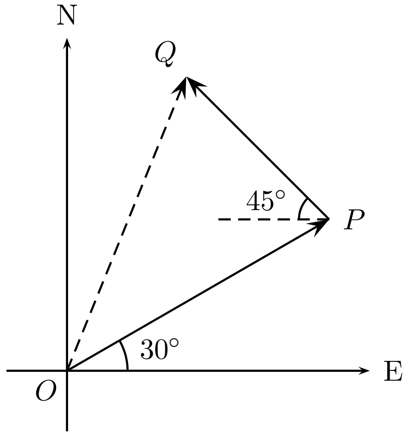


Figure 12.

In adding the two displacement vectors $\overrightarrow{OP}$ and $\overrightarrow{PQ}$, we add and subtract displacements in the east-west and the north-south direction. We then specified the sum $\overrightarrow{OQ}$ by its length and direction. We shall see how this relates to the algebraic definition of vector in the next section.

NOTE. For many of the problems in physical world, the vector quantities are in three dimensions rather than two. At least in theory, the same methods in this section could be used to solve such problems. Each vector could be represented as an arrow in space, and then we can add two vectors and multiply a vector by a scalar.

1.2 Vector quantities and $\mathbb{R}^n$

In Example 6 of Section 1.1.2, if we denote the vector of length 1 km towards the east by **i** and the vector of length 1 km towards the north by **j**, by the definition of geometric vectors we can write

$$\begin{aligned} \overrightarrow{OP} &= 15\cos 30°\ \mathbf{i} + 15\sin 30°\ \mathbf{j}, \\ \overrightarrow{PQ} &= -10\cos 45°\ \mathbf{i} + 10\sin 45°\ \mathbf{j}, \quad \text{and} \end{aligned}$$

By the associative, commutative and distributive laws of geometric vectors, we have

$$\begin{aligned} \overrightarrow{OQ} &= \overrightarrow{OP} + \overrightarrow{PQ} \\ &= (15\cos 30° - 10\cos 45°)\mathbf{i} + (15\sin 30° + 10\sin 45°))\mathbf{j}. \end{aligned}$$

In a more convenient way, we can write

$$\begin{aligned}\overrightarrow{OP} &= \begin{pmatrix}15\cos 30^\circ\\ 15\sin 30^\circ\end{pmatrix},\\ \overrightarrow{PQ} &= \begin{pmatrix}-10\cos 45^\circ\\ 10\sin 45^\circ\end{pmatrix},\\ \overrightarrow{OQ} &= \begin{pmatrix}15\cos 30^\circ - 10\cos 45^\circ\\ 15\sin 30^\circ + 10\sin 45^\circ\end{pmatrix}.\end{aligned}$$

There is no reason why we cannot generalise this to all two dimensional vectors, three dimensional vectors and beyond.

1.2.1 Vectors in $\mathbb{R}^2$

We first choose two vectors, conventionally denoted by $\mathbf{i}$ and $\mathbf{j}$, of unit length, and at right angles to each other so that $\mathbf{j}$ is pointing at an angle of $\frac{\pi}{2}$ anticlockwise from $\mathbf{i}$. The vectors $\mathbf{i}$ and $\mathbf{j}$ are known as the standard **basis vectors** for $\mathbb{R}^2$. See Figure 13.

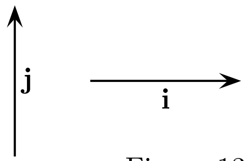


Figure 13.

As shown in Figure 14, every vector $\mathbf{a}$ can be 'resolved' (in a unique way) into the sum of a scalar multiple of $\mathbf{i}$ plus a scalar multiple of $\mathbf{j}$. That is there are unique real numbers a_1 and a_2 such that $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$. If the direction θ of $\mathbf{a}$ is measured from the direction of $\mathbf{i}$, then these scalars can be easily found using the formulae

$$a_1 = |\mathbf{a}|\cos\theta, \quad \text{and} \quad a_2 = |\mathbf{a}|\sin\theta.$$

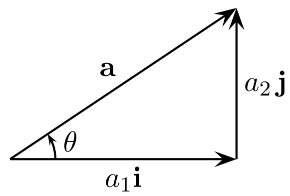


Figure 14.

We call a_1, a_2 the **components** of $\mathbf{a}$.

Now, every vector $\mathbf{a}$ in the plane can be specified by these two unique real numbers. We can write $\mathbf{a}$ in form of a **column vector** or a **2-vector** $\begin{pmatrix}a_1\\ a_2\end{pmatrix}$. We call the numbers a_1, a_2 the **components** of $\begin{pmatrix}a_1\\ a_2\end{pmatrix}$. In this case, $\mathbf{i} = \begin{pmatrix}1\\ 0\end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix}0\\ 1\end{pmatrix}$.

The column vector $\begin{pmatrix}a_1\\ a_2\end{pmatrix}$ is also called the **coordinate vector** with respect to the basis vectors $\left\{\begin{pmatrix}1\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 1\end{pmatrix}\right\}$ and a_1, a_2 are also called the **coordinates** of the vector.

Theorem 1. *Let $\mathbf{a}$ and $\mathbf{b}$ be (geometric) vectors, and let $\lambda \in \mathbb{R}$. Suppose that $\mathbf{a} = \begin{pmatrix}a_1\\ a_2\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}b_1\\ b_2\end{pmatrix}$. Then*

1. *the coordinate vector for $\mathbf{a}+\mathbf{b}$ is $\begin{pmatrix}a_1+b_1\\ a_2+b_2\end{pmatrix}$;*

2. *the coordinate vector for* $\lambda\mathbf{a}$ *is* $\begin{pmatrix}\lambda a_1\\ \lambda a_2\end{pmatrix}$.

Proof. The basis vectors are $\mathbf{i}$ and $\mathbf{j}$. We first give a detailed proof of the second part.

$$\begin{aligned}\lambda\mathbf{a} &= \lambda(a_1\mathbf{i}+a_2\mathbf{j}) && \text{(by definition of coordinate vectors)}\\ &= \lambda a_1\mathbf{i}+\lambda a_2\mathbf{j} && \text{(by vector distributive law)}\end{aligned}$$

Hence the coordinate vector for $\lambda\mathbf{a}$ is $\begin{pmatrix}\lambda a_1\\ \lambda a_2\end{pmatrix}$.

For the first part, by associative, commutative and distributive laws, we have

$$\begin{aligned}\mathbf{a}+\mathbf{b} &= (a_1\mathbf{i}+a_2\mathbf{j})+(b_1\mathbf{i}+b_2\mathbf{j})\\ &= (a_1\mathbf{i}+b_1\mathbf{i})+(a_2\mathbf{j}+b_2\mathbf{j})\\ &= (a_1+b_1)\mathbf{i}+(a_2+b_2)\mathbf{j}\end{aligned}$$

Hence the coordinate vector for $\mathbf{a}+\mathbf{b}$ is $\begin{pmatrix}a_1+b_1\\ a_2+b_2\end{pmatrix}$. □

We then can define the mathematics structure $\mathbb{R}^2$, which is the set of 2-vectors, by

$$\mathbb{R}^2 = \left\{\begin{pmatrix}a_1\\ a_2\end{pmatrix} : a_1, a_2 \in \mathbb{R}\right\},$$

with addition and multiplication by a scalar defined by — for any $\begin{pmatrix}a_1\\ a_2\end{pmatrix}, \begin{pmatrix}b_1\\ b_2\end{pmatrix} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$,

$$\begin{pmatrix}a_1\\ a_2\end{pmatrix}+\begin{pmatrix}b_1\\ b_2\end{pmatrix}=\begin{pmatrix}a_1+b_1\\ a_2+b_2\end{pmatrix} \qquad \text{and} \qquad \lambda\begin{pmatrix}a_1\\ a_2\end{pmatrix}=\begin{pmatrix}\lambda a_1\\ \lambda a_2\end{pmatrix}.$$

The elements in $\mathbb{R}^2$ are called **vectors** and sometimes **column vectors**. It is obvious that the set $\mathbb{R}^2$ is closed under addition and scalar multiplication. Like geometric vectors, the vectors in $\mathbb{R}^2$ also obey the commutative, associative, and distributive laws. There is a zero vector $\mathbf{0} = \begin{pmatrix}0\\ 0\end{pmatrix}$ and a negative $\begin{pmatrix}-a_1\\ -a_2\end{pmatrix}$ for any vector $\mathbf{a} = \begin{pmatrix}a_1\\ a_2\end{pmatrix}$.

1.2.2 Vectors in $\mathbb{R}^n$

The concept of components can easily be generalised from two dimensions to any number of dimensions.

Definition 1. *Let* n *be a positive integer. The set* $\mathbb{R}^n$ *is defined by*

$$\mathbb{R}^n = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} : a_1, a_2, \dots, a_n \in \mathbb{R} \right\}.$$

An element $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ *in* $\mathbb{R}^n$ *is called an* n*-vector or simply a vector; and* $a_1, a_2, \dots, a_n$ *are called the components of the vector.*

Note. We say that two vectors in $\mathbb{R}^n$ are equal if the corresponding components are equal. In other words $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ if and only if $a_1 = b_1, \dots, a_n = b_n$.

Example 1. 1. Clearly $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3$; $\begin{pmatrix} 0.5 \\ 1.4 \\ 2.3 \\ -4.1 \end{pmatrix} \in \mathbb{R}^4$.

2. $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ are different elements of $\mathbb{R}^3$. $\diamond$

As in $\mathbb{R}^2$, we can define addition of two vectors in $\mathbb{R}^n$ and multiplication of a vector in $\mathbb{R}^n$ by a scalar in $\mathbb{R}$.

Definition 2. *Let* $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ *be vectors in* $\mathbb{R}^n$ *and* λ *be a real number.*

We define the sum of $\mathbf{a}$ *and* $\mathbf{b}$ *by* $\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}$.

We define the scalar multiplication of $\mathbf{a}$ *by* λ *by* $\lambda\,\mathbf{a} = \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \\ \vdots \\ \lambda a_n \end{pmatrix}$.

NOTE. The addition rule tells us how to add two vectors with the same number of components. The sum of vectors with different numbers of components is not defined.

Example 2. 1. $\begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 4+2 \\ -3+4 \\ 5+(-2) \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 3 \end{pmatrix}.$

2. $\begin{pmatrix} 1.3 \\ 2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 3.2 \\ -2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4.5 \\ 0 \\ 2 \\ 2 \end{pmatrix}.$

3. $3 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}.$

4. $\pi \begin{pmatrix} -1 \\ 2 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} -\pi \\ 2\pi \\ 0 \\ 5\pi \end{pmatrix}.$

5. $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 4 \\ -1 \end{pmatrix}$ is *not defined!* ◊

Proposition 2. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. (Commutative Law of Addition)

2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$. (Associative Law of Addition)

[**X**] *Proof.* We prove the commutative law, while proving the associative law will be left as an exercise.

Let $\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$.

$$
\begin{aligned}
\mathbf{u}+\mathbf{v} &= \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \\
&= \begin{pmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{pmatrix} && \text{(definition of addition in } \mathbb{R}^n\text{)} \\
&= \begin{pmatrix} v_1+u_1 \\ \vdots \\ v_n+u_n \end{pmatrix} && \text{(commutative law of real numbers)} \\
&= \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} && \text{(definition of addition in } \mathbb{R}^n\text{)} \\
&= \mathbf{v}+\mathbf{u}
\end{aligned}
$$

□

Definition 3. *Let n be a positive integer.*

1. *The **zero vector** in $\mathbb{R}^n$, denoted by $\mathbf{0}$, is the vector with all n components 0.*
2. *Let $\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$. The **negative** of $\mathbf{a}$, denoted by $-\mathbf{a}$ is the vector $\begin{pmatrix} -a_1 \\ \vdots \\ -a_n \end{pmatrix}$.*
3. *Let $\mathbf{a}$, $\mathbf{b}$ be vectors in $\mathbb{R}^n$. We define the difference, $\mathbf{a}-\mathbf{b}$, by*
$$\mathbf{a}-\mathbf{b} = \mathbf{a}+(-\mathbf{b}).$$

NOTE. Let $\mathbf{a}$, $\mathbf{b}$ be vectors in $\mathbb{R}^n$ and $\mathbf{0}$ be the zero vector in $\mathbb{R}^n$.

1. $\mathbf{a}+\mathbf{0} = \mathbf{0}+\mathbf{a} = \mathbf{a}$.
2. $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} - \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1-b_1 \\ \vdots \\ a_n-b_n \end{pmatrix}$.
3. $\mathbf{a}+(-\mathbf{a}) = (-\mathbf{a})+\mathbf{a} = \mathbf{0}$.

Example 3. 1. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is the zero vector in $\mathbb{R}^3$, while $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is the zero vector in $\mathbb{R}^4$

2. $\begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 4-2 \\ -3-4 \\ 5-(-2) \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \\ 7 \end{pmatrix}.$

Proposition 3. Let λ, μ be scalars and $\mathbf{u}$, $\mathbf{v}$ be vectors in $\mathbb{R}^n$.

1. $\lambda(\mu\mathbf{v}) = (\lambda\mu)\mathbf{v}$ (Associative Law of Scalar Multiplication)
2. $(\lambda+\mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$ (Scalar Distributive Law)
3. $\lambda(\mathbf{u}+\mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$ (Vector distributive Law)

1.3 $\mathbb{R}^n$ and analytic geometry

1.3.1 Two dimensions

From an algebraic point of view, $\mathbb{R}^2$ behaves very much the same as $\mathbb{R}^{43}$ — the rules for addition and scalar multiplication are more or less the same. However, the different sets $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^4, \ldots$ tend to appear in different types of applications.

The sets $\mathbb{R}^2$ and $\mathbb{R}^3$ are particularly important for solving problems which lie in a plane or in "space". You are probably quite familiar with working with the coordinates of points in a plane. In this section we shall look more closely at the relationship between points in the plane and their coordinates.

Let's suppose that we have a plane with several objects in it, as in Figure 15.

Figure 15.

To solve problems such as "what is the distance between A and B?", it is often convenient to work with the coordinates of these points. In real life, points don't come with (x, y) coordinates attached.

We have to

(i) specify an origin point O somewhere in the plane;

(ii) specify a unit of length;

(iii) specify a direction (usually called the x-direction); and

(iv) specify a y-direction (usually at angle $\frac{\pi}{2}$ anti-clockwise to the x-direction).

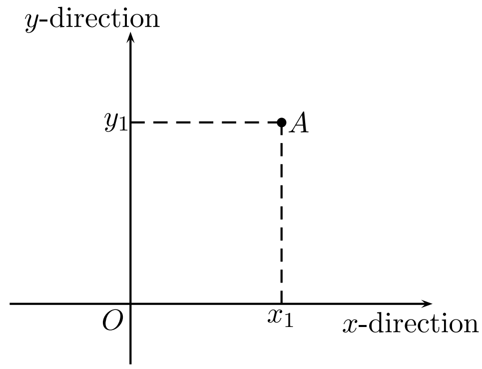


Figure 16.

Having done this we can then find the **coordinates** (x_1, y_1) **of** A (*with respect to our chosen coordinate system*) by requiring that A is x_1 units from the origin in the x-direction and y_1 units from the origin in the y-direction. In this case we represent A by the **2-tuple**, or ordered pair, (x_1, y_1).

The line through O in the x-direction is called the x-axis. The y-axis is similarly defined.

With the choice of the origin O, the point A is naturally associated with a vector $\overrightarrow{OA}$ which is called the **position vector** of A with respect to the origin. Then we choose $\mathbf{i}$ to be the unit vector in the x-direction and $\mathbf{j}$ to be the unit vector in the y-direction. The scalars x_1 and y_1 are often known as the **components** of $\mathbf{a} = \overrightarrow{OA}$ in the x and y-directions respectively. As in section 2.2.1, the position vector of A is represented by the coordinate vector $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ with respect to the basis vectors $\{\mathbf{i}, \mathbf{j}\}$.

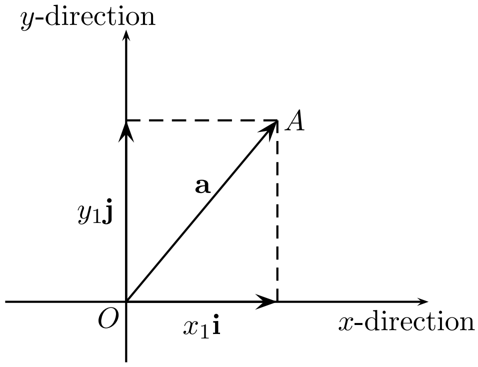


Figure 17.

When a coordinate system is specified, each point in the plane corresponds to a unique 2-tuple (x, y), and equivalently a unique column vector $\begin{pmatrix} x \\ y \end{pmatrix}$. Conversely, once the coordinate system has been fixed, each 2-tuple (x, y) or equivalently each $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ corresponds to a unique point in the plane. We shall often denote the position vector of A by $\mathbf{a}$.

It is a consequence of Pythagoras' Theorem that if the points A and B in the plane have position vectors $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ respectively, then the distance between A and B is given by

$$\text{dist}(A, B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Now, we can naturally define the length of a 2-vector $\mathbf{a}$. Choose the point A such that $\overrightarrow{OA}$ represents $\mathbf{a}$. The length of $\mathbf{a}$ is defined to be $\text{dist}(O, A)$. It is not difficult to see that as displacements in a plane, two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^2$ is parallel if and only if there is a non-zero real number λ such that $\mathbf{a} = \lambda\mathbf{b}$.

NOTE. The use of coordinates to solve problems in geometry is a relatively recent introduction. Although geometry has been studied for thousands of years, this "analytic geometry" was only introduced by R. Descartes (from whose name the term 'Cartesian' is derived) in the early 17th century.

1.3.2 Three dimensions

You can give coordinates to points in space just as you give coordinates to points in the plane. Again however, we need to fix our coordinate system. Now, let us

(i) specify an origin point O somewhere in space;

(ii) specify a unit of length;

(iii) specify three directions, usually called the x, y and z directions, each at right angles to the others.

We choose unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, respectively, in the x-direction, y-direction, z-direction. The **coordinates** of a point A with respect to the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ are given by a 3-**tuple** (x_1, y_1, z_1) such that

A is x_1 units from the origin in the x-direction, y_1 units from the origin in the y-direction, and z_1 units from the origin in the z-direction. The **position vector** of A, $\overrightarrow{OA}$, is $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$, which can also be represented by $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \in \mathbb{R}^3$.

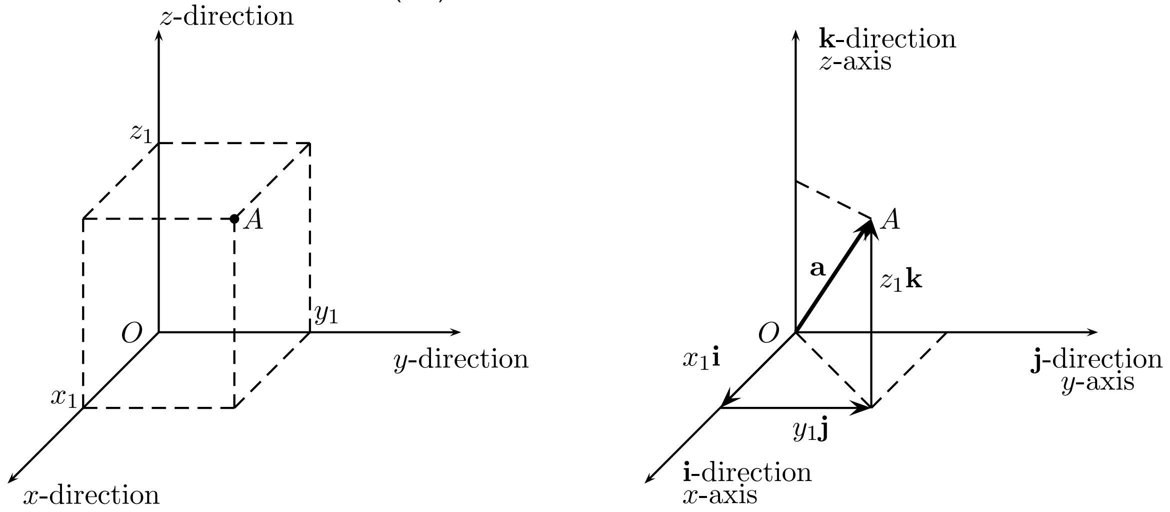


Figure 18: Right-handed Coordinate System.

The coordinate system in Figure 18 is what is called **right-handed Cartesian coordinate system**. Using your right hand, if you point your index finger in the x-direction, your middle finger in the y-direction, then your extended thumb will point in the z-direction. This is the orientation that is conventionally used in mathematics and physics.

Again once we have set up our coordinate system, each point in space corresponds uniquely to a 3-tuple and uniquely to a 3-vector (and vice-versa). It is rather hard to draw pictures of objects in three dimensions (on two dimensional paper!). Nevertheless, this identification will provide us with an important way of visualising relationships between vectors in $\mathbb{R}^3$.

Since the basis vectors chosen are of unit length and are perpendicular to each other, by Pythagoras' theorem the length of $\overrightarrow{OA}$ is $\sqrt{x_1^2 + y_1^2 + z_1^2}$. Let B be another point with position vector $\mathbf{b} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$, the distance between A and B is the same as the length of the vector $\mathbf{b} - \mathbf{a} = \overrightarrow{AB}$, i.e.

$$\operatorname{dist}(A, B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

Similar to $\mathbb{R}^2$, if $\mathbf{a} = \overrightarrow{OA}$, the length of $\mathbf{a}$ is $\operatorname{dist}(O, A)$. Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$ are parallel if and only if there is a non-zero real number λ such that $\mathbf{a} = \lambda\mathbf{b}$.

1.3.3 n-dimensions

We have used elements in $\mathbb{R}^2$ and $\mathbb{R}^3$ to represent the position vectors of points in the plane and in the three-dimensional space, respectively. How do we interpret n-vectors in $\mathbb{R}^n$ as position vectors of points in an n-dimensional space which generalises the 2-dimensional and the 3-dimensional spaces? To study the geometry in the n-dimensional space, we need the notions of length and direction.

In $\mathbb{R}^2$ or $\mathbb{R}^3$, two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if there exists a non-zero real number λ such that $\mathbf{b} = \lambda\mathbf{a}$. We use this idea as definition.

Definition 1. *Two non-zero vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *in* $\mathbb{R}^n$ *are said to be* ***parallel*** *if* $\mathbf{b} = \lambda\,\mathbf{a}$ *for some non-zero real number* λ.
They are said to be in the same ***direction*** *if* $\lambda > 0$.

Example 1. The vectors $\begin{pmatrix} 3 \\ -1 \\ 2 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ -2 \\ 4 \\ 8 \end{pmatrix}$ are parallel because $\begin{pmatrix} 6 \\ -2 \\ 4 \\ 8 \end{pmatrix} = 2\begin{pmatrix} 3 \\ -1 \\ 2 \\ 4 \end{pmatrix}$.

The vectors $\begin{pmatrix} 3 \\ 0 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 1 \\ 4 \\ 2 \end{pmatrix}$ are not parallel because we cannot find a scalar λ such that $\begin{pmatrix} 6 \\ 1 \\ 4 \\ 2 \end{pmatrix} = \lambda \begin{pmatrix} 3 \\ 0 \\ 2 \\ 1 \end{pmatrix}$. ◊

Definition 2. *Let* $\mathbf{e}_j$ *be the vector in* $\mathbb{R}^n$ *with a 1 in the* j*th component and 0 for all the other components. Then the* n *vectors* $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ *are called the* ***standard basis vectors*** *for* $\mathbb{R}^n$.

Example 2. The standard basis vectors for $\mathbb{R}^3$ are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The standard basis vectors for $\mathbb{R}^4$ are $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. ◊

We imagine that we can choose a point in the n-dimensional space as the origin O and n axes in the directions of these n standard basis vectors respectively. Then we can define the **coordinates** of a point A with respect to this coordinate system in the same way as in 3-dimensional space. For instance, if A is a_i units from O in the direction of $\mathbf{e}_i$ for each $1 \leqslant i \leqslant n$, the coordinates of A are given by an n-**tuple** $(a_1, \dots, a_n)$. Thus the **c**oordinate vector, or the **position vector**, of the point A is $\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$. We can also discuss n-dimensional geometric vectors. For example, the vector $\overrightarrow{OA}$, which has the initial point O and the terminal point A, represents $\mathbf{a}$.

Example 3. i) The coordinate vector for the displacement from the point A, with coordinates $(2,3)$ to the point B with coordinates $(-1,2)$ is

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}.$$

ii) The coordinate vector for the line segment joining the point A with coordinates $(2,4,-5,3)$ to the point B with coordinates $(-3,4,0,4)$ is

$$\overrightarrow{AB} = \begin{pmatrix} -3 \\ 4 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ 5 \\ 1 \end{pmatrix}.$$

iii) If the position vector of the point A is $\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$ and the displacement from A to B has coordinate vector $\begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$, then the position vector for B is

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}.$$

Thus B has coordinates $(3,4,-1)$. ◊

We can also define the length of a vector and the distance between two points which have coordinate vectors in $\mathbb{R}^n$.

Definition 3. *The* ***length*** *of a vector* $\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$ *is defined by*

$$|\mathbf{a}| = \sqrt{a_1^2 + \cdots + a_n^2}.$$

Remark 1. The quantity $|\mathbf{a}|$ is sometimes called the **magnitude** or **norm** of the vector $\mathbf{a}$.

Example 4. Find the lengths of $\mathbf{a} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -3 \\ 2 \\ -6 \\ 4 \end{pmatrix}$.

SOLUTION. $|\mathbf{a}| = \sqrt{4+16} = 2\sqrt{5}$; $|\mathbf{b}| = \sqrt{1+9+4+36+16} = \sqrt{66}$. ◊

> **Definition 4.** *The **distance** between two points A and B with position vectors in $\mathbb{R}^n$ is the length of the vector $\overrightarrow{AB}$.*

Then, if $\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ is the coordinate vector of A and $\mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ is the coordinate vector of B, we have

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} b_1 - a_1 \\ \vdots \\ b_n - a_n \end{pmatrix},$$

so the distance between A and B is

$$\left|\overrightarrow{AB}\right| = |\mathbf{b} - \mathbf{a}| = \sqrt{(b_1 - a_1)^2 + \ldots + (b_n - a_n)^2}.$$

Example 5. Find the distance between the point A with coordinates $(3, -2, 5, 1)$ and the point B with coordinates $(-1, 2, -8, 4)$.

Solution. As $\overrightarrow{AB} = \begin{pmatrix} -4 \\ 4 \\ -13 \\ 3 \end{pmatrix}$, the distance is $\left|\overrightarrow{AB}\right| = \sqrt{16 + 16 + 169 + 9} = \sqrt{210}$. ◊

Note. Until now, we distinguish an n-dimensional space from $\mathbb{R}^n$ which contains all the position vectors of the points in the space. From now on, we shall blur the difference. So we often refer to $\mathbb{R}^2$ as the plane and the $\mathbb{R}^3$ as the three dimensional space. In our algebra notes, a point in an n-space is usually referred as an n-tuple of coordinates $(a_1, \ldots, a_n)$ and the corresponding position vector is denoted by an n-vector $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$.

1.4 Lines

Suppose now that we wish to express a line in the plane in terms of coordinate vectors. One form for the equation of a line is $y = mx + b$, where m is the slope of the line and b is the y-intercept.

Since a line is really just a set of points, each point on the line will have a corresponding coordinate vector in $\mathbb{R}^2$. There is a simple way to describe the coordinate vector of each point on a line in the plane.

Let us look firstly at a line though the origin, $y = mx$. One way of reading this equation is to say that the line is the set of all points whose coordinates are of the form (x, mx). In other words, the set

$$S = \left\{ \mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = x \begin{pmatrix} 1 \\ m \end{pmatrix}, \text{ for some } x \in \mathbb{R} \right\}$$

is the set of all coordinate vectors for points on the line.

There is another way of thinking about this. Let $\mathbf{v}$ be some non-zero displacement in the plane. Let Q be a point in the plane such that $\overrightarrow{OQ} = \mathbf{v}$. Consider the set S of all points P in the plane whose displacement from the origin is $\lambda\mathbf{v}$ for some real number λ. It is not difficult to see that S is the set of all points on the line through O parallel to $\mathbf{v}$ or equivalently S is the line through O and Q. See Figure 19.

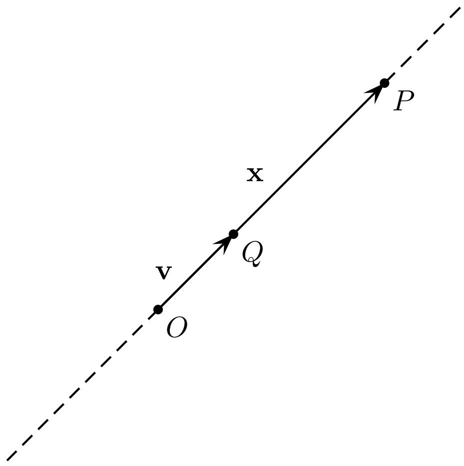


Figure 19: A Line Through O.

For each value of the 'parameter' λ, we obtain a point, in the form of a coordinate vector, on the line. It is for this reason that such a description is often called a **parametric vector form** for the line.

This is how we are going to define a 'line' spanned by a vector in $\mathbb{R}^n$.

Definition 1. *Let* $\mathbf{v}$ *be any non-zero vector in* $\mathbb{R}^n$. *The set*

$$S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \lambda\mathbf{v}, \text{ for some } \lambda \in \mathbb{R}\}.$$

is the ***line*** *in* $\mathbb{R}^n$ ***spanned*** *by* $\mathbf{v}$, *and we call the expression* $\mathbf{x} = \lambda\mathbf{v}$, *a* ***parametric vector form*** *for this line.*

Example 1. Draw a diagram of the line in $\mathbb{R}^2$ spanned by $(1, 2)$, and find a parametric vector form for this line.

Solution. This is the set of all points of the form $\lambda(1, 2) = (\lambda, 2\lambda)$ for some $\lambda \in \mathbb{R}$. In the plane, it is the line going though the origin and the point with coordinates $(1, 2)$. (See Figure 20.)

A parametric vector form is

$$\mathbf{x} = \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

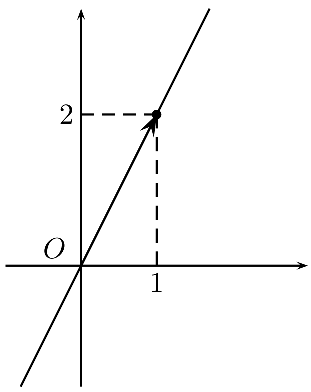


Figure 20.

◊

Example 2. Write a parametric vector form for the line in $\mathbb{R}^4$ spanned by the vector $\begin{pmatrix} 3 \\ 5 \\ -6 \\ 8 \end{pmatrix}$.

A solution is $\mathbf{x} = \lambda \begin{pmatrix} 3 \\ 5 \\ -6 \\ 8 \end{pmatrix}$ for $\lambda \in \mathbb{R}$. $\diamondsuit$

NOTE. A given line does not have a unique parametric vector form. The following are also parametric vector forms of the same line.

$$\mathbf{x} = \lambda \begin{pmatrix} -3 \\ -5 \\ 6 \\ -8 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \lambda \begin{pmatrix} 9 \\ 15 \\ -18 \\ 24 \end{pmatrix}.$$

On the other hand, the name of the parameter in the expression is not important. For example the forms

$$\mathbf{x} = \mu \begin{pmatrix} 3 \\ 5 \\ -6 \\ 8 \end{pmatrix} \text{ for } \mu \in \mathbb{R} \quad \text{and} \quad \mathbf{x} = s \begin{pmatrix} 3 \\ 5 \\ -6 \\ 8 \end{pmatrix} \text{ for } s \in \mathbb{R}$$

also represent the same line.

Most lines in the plane, or in three dimensional space do not go through the origin of the coordinate system being used. Consider the line in the plane going through the points A and B. To find the coordinate vector of some point P on this line, we shall calculate the displacement from O to P.

Clearly

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}.$$

Because A, P and B are collinear, thus $\overrightarrow{AP}$ is parallel to $\overrightarrow{AB}$. That is, $\overrightarrow{AP} = \lambda\overrightarrow{AB}$, for some real number λ. Let $\mathbf{a}$ denote the position vector of A and $\mathbf{x}$ the position vector of P and let $\mathbf{v}$ denote the displacement from A to B. In terms of these vectors, what we have just said is

$$\mathbf{x} = \mathbf{a} + \lambda\mathbf{v}, \quad \text{for some } \lambda \in \mathbb{R}.$$

Again, each value of the parameter λ gives the position vector of a point on the line, and each point on the line can be found by choosing an appropriate value of λ.

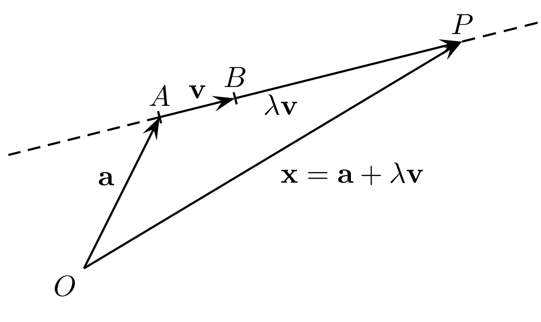


Figure 21.

Definition 2. *A **line** in $\mathbb{R}^n$ is any set of vectors of the form*

$$S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{a} + \lambda\mathbf{v},\ \text{for some } \lambda \in \mathbb{R}\},$$

where $\mathbf{a}$ *and* $\mathbf{v} \neq \mathbf{0}$ *are fixed vectors in* $\mathbb{R}^n$*. The expression*

$$\mathbf{x} = \mathbf{a} + \lambda\mathbf{v},\ \text{for some } \lambda \in \mathbb{R},$$

*is a **parametric vector form** of the line through* $\mathbf{a}$ *parallel to* $\mathbf{v}$.

Example 3. Find a parametric vector form of the line through the point $(2, -3, 1, 6)$ parallel to the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 4 \\ -6 \\ 2 \end{pmatrix}$.

SOLUTION. Each point on the line has coordinate vector of the form

$$\mathbf{x} = \begin{pmatrix} 2 \\ -3 \\ 1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -6 \\ 2 \end{pmatrix}, \quad \text{for some } \lambda \in \mathbb{R}. \tag{*}$$

$\diamondsuit$

NOTE. Once again, a line may have different parametric vector forms. For instance, the following line

$$\mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ -5 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -6 \\ 2 \end{pmatrix}, \quad \text{for some } \lambda \in \mathbb{R}. \tag{**}$$

is one which passes through $(3, 1, -5, 8)$ parallel to $\mathbf{v}$. Note that $\begin{pmatrix} 2 \\ -3 \\ 1 \\ 6 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \\ -6 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -5 \\ 8 \end{pmatrix}$, so $(3, 1, -5, 8)$ is another point on the line with equation (*). Thus both (*) and (**) are parametric vector forms of the same line.

1.4.1 Lines in $\mathbb{R}^2$

We can write the equation of a line in Cartesian form $y = mx + d$ and also in parametric vector form $\mathbf{x} = \mathbf{a} + \lambda\mathbf{v}$. It is often necessary to convert between these two forms.

Conversion from Cartesian form to parametric vector form.
The equation $y = mx + d$ can be converted to parametric vector form as follows:

Suppose $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ is the position vector of a point on the line and set $x = \lambda$. Then, since $y = m\lambda + d$, we obtain

$$\mathbf{x} = \begin{pmatrix} \lambda \\ m\lambda + d \end{pmatrix} = \begin{pmatrix} 0 \\ d \end{pmatrix} + \begin{pmatrix} \lambda \\ m\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ d \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ m \end{pmatrix}.$$

Thus a parametric vector form for the line is

$$\mathbf{x} = \begin{pmatrix} 0 \\ d \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ m \end{pmatrix}, \quad \text{for } \lambda \in \mathbb{R}.$$

You should think of this as saying that the equation $y = mx + d$ represents a line through the point $(0, d)$ parallel to the vector $\begin{pmatrix} 1 \\ m \end{pmatrix}$.

Example 4. The equation $y = 3x + 2$ has a parametric vector form

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \text{for } \lambda \in \mathbb{R}.$$

The line passes through $(0, 2)$ and is parallel to $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$. ◊

The equation $y = mx + d$ is a special case of the more general linear equation in two variables x and y, which is given by

$$ax + by = c,$$

where a, b, and c are fixed numbers. If $b \neq 0$, this linear equation can easily be converted to the $y = mx + d$ form by dividing by b. Alternatively, the equation $ax + by = c$ can be converted directly to parametric vector form by setting either x or y as a parameter. The following example illustrates this technique.

Example 5. Find parametric vector forms for the lines in $\mathbb{R}^2$ given by

$$\text{i) } 2x - 4y = 6, \qquad \text{ii) } 2x = 8, \qquad \text{and} \qquad \text{iii) } 4y = 8.$$

Solution. i) On setting $y = \lambda$, we have $2x - 4\lambda = 6$, and hence $x = 3 + 2\lambda$. Hence a parametric vector form for the line is

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 + 2\lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \text{for } \lambda \in \mathbb{R}.$$

ii) Note that $2x = 8$ means $x = 4$. So the x has a fixed value but the y value varies. We need to set $y = \lambda$. Hence a parametric vector form for the line is

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ \lambda \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{for } \lambda \in \mathbb{R}.$$

iii) In this case the equation fixes $y = 2$, whereas x can have any real value. We therefore set $x = \lambda$ as the parameter and obtain

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{for } \lambda \in \mathbb{R}.$$

as a parametric vector form. ◊

Conversion from parametric vector form to Cartesian form.
Let us start from a line in parametric vector form

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Obviously, we can find two different points (x_1, y_1) and (x_2, y_2) on the line by choosing two different values for the parameter, then we can find the Cartesian equation of the line by the two-point form $\dfrac{y-y_1}{x-x_1} = \dfrac{y_2-y_1}{x_2-x_1}$. However, we would like to use a method which can be easily generalised to the higher dimension cases.

By comparing the components of the vectors on the left side and right side of the parametric vector form, we can express the line as a pair of parametric equations:

$$\begin{cases} x = a_1 + \lambda v_1, \\ y = a_2 + \lambda v_2. \end{cases}$$

Then we can eliminate the parameter λ to get the line in Cartesian form.

Example 6. Find the Cartesian form of each of the following lines which are given in parametric vector form.

$$\text{i) } \mathbf{x} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \text{ for } \lambda \in \mathbb{R}, \qquad \text{ii) } \mathbf{x} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \text{ for } \lambda \in \mathbb{R}.$$

Solution. i) By comparing the components, we get the parametric equations $x = 3 + 2\lambda$ and $y = 1 + \lambda$. If we eliminate the parameter λ, we find

$$\lambda = \frac{x-3}{2} = y - 1,$$

which on rearranging gives

$$y = \frac{x-1}{2}, \quad \text{or} \quad x - 2y = 1.$$

ii) For the second line, the parametric equations are $x = 3 + 2\lambda$ and $y = 1$. Since y has a fixed values 1 and x can be any real number, the Cartesian form of the line is $y = 1$. ♢

1.4.2 Lines in $\mathbb{R}^3$

A line in $\mathbb{R}^3$ is still of the form $\mathbf{x} = \mathbf{a} + \lambda\mathbf{v}$, but now the vectors $\mathbf{x}, \mathbf{a}$ and $\mathbf{v}$ each have three components or three coordinates.

An alternative Cartesian (or symmetric) form for the equation of a line is sometimes used in engineering. This form can be obtained as follows:

Let $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ be vectors in $\mathbb{R}^3$. Then the parametric vector equation of the line through (a_1, a_2, a_3) parallel to $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ is

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

Thus the parametric equations of the line are

$$x = a_1 + \lambda v_1, \qquad y = a_2 + \lambda v_2, \qquad z = a_3 + \lambda v_3.$$

Eliminating the parameter λ yields (if all $v_i \neq 0$), the **Cartesian form**

$$\frac{x - a_1}{v_1} = \frac{y - a_2}{v_2} = \frac{z - a_3}{v_3} \ (= \lambda).$$

If v_1, v_2 or v_3 is 0, then x, y or z will, respectively, be constant.

Example 7. Find the Cartesian form for the lines

$$\text{i) } \mathbf{x} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}, \text{ for } \lambda \in \mathbb{R}, \qquad \text{ii) } \mathbf{x} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 6 \end{pmatrix}, \text{ for } \lambda \in \mathbb{R}.$$

Solution. i) Here $x = 2 + 3\lambda$, $y = -3 + 5\lambda$, $z = 1 + 6\lambda$. Thus, eliminating λ gives

$$\frac{x - 2}{3} = \frac{y + 3}{5} = \frac{z - 1}{6}.$$

ii) In this case, $x = 2 + 3\lambda$, $y = -3$, $z = 1 + 6\lambda$. By eliminating λ, the Cartesian form of the line is

$$\frac{x - 2}{3} = \frac{z - 1}{6} \quad \text{and} \quad y = -3.$$

$\diamondsuit$

We have seen how to convert a parametric vector form of a line in $\mathbb{R}^3$ to the Cartesian form. To convert a line from Cartesian to the parametric vector form, we find a point on the line and a vector parallel to the line. Alternatively, we can introduce a parameter.

Example 8. Find a parametric vector form for the line

$$\frac{x - 4}{7} = \frac{2y + 3}{2} = -\frac{z}{6}.$$

Solution. Method 1. Besides forming parametric equations and then eliminating the parameter, we can rewrite the Cartesian equations as

$$\frac{x - 4}{7} = \frac{y + \frac{3}{2}}{1} = \frac{z}{-6}.$$

Obviously, this is a line through $(4, -\frac{3}{2}, 0)$ parallel to $\begin{pmatrix} 7 \\ 1 \\ -6 \end{pmatrix}$. Hence a parametric vector form of the line is

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -\frac{3}{2} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 1 \\ -6 \end{pmatrix}, \quad \text{for } \lambda \in \mathbb{R}.$$

Method 2. Set each term to λ. That is

$$\frac{x-4}{7} = \frac{2y+3}{2} = -\frac{z}{6} = \lambda.$$

By rearranging terms, we get the parametric equations

$$x = 7\lambda + 4, \quad y = \lambda - \frac{3}{2}, \quad z = -6\lambda.$$

In vector form, we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7\lambda + 4 \\ \lambda - \frac{3}{2} \\ -6\lambda \end{pmatrix} = \begin{pmatrix} 4 \\ -\frac{3}{2} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 1 \\ -6 \end{pmatrix}.$$

Then we shall get the same parametric vector form as the one we have obtained by Method 1. $\Diamond$

Example 9. Find a parametric vector form of the line defined by

$$x - 2y = 4 \quad \text{and} \quad z = 1.$$

SOLUTION. Set y to be a parameter λ. Then $x = 4 + 2\lambda$. In vector form, we have

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 + 2\lambda \\ \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

Hence a parametric vector form of the line is:

$$\mathbf{x} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \text{for } \lambda \in \mathbb{R}.$$

$\Diamond$

Example 10. Find a parametric vector form of the line through the points $A\ (1, 3, -4)$ and $B\ (2, 0, 6)$.

SOLUTION. The line passes through the point $A\ (1, 3, -4)$ and it is parallel to the vector

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 10 \end{pmatrix}.$$

Hence a parametric vector form of the line is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ 10 \end{pmatrix}, \quad \text{for } \lambda \in \mathbb{R}.$$

$\Diamond$

1.4.3 Lines through two given points (in $\mathbb{R}^n$)

The methods used in handling lines in $\mathbb{R}^3$ can be generalised to deal with lines in $\mathbb{R}^n$. We have seen the definition of a parametric vector form of a line in $\mathbb{R}^n$ (Definition 2 on page 22) — $\mathbf{x} = \mathbf{a} + \lambda\mathbf{v}$. We can also use the symmetric form of a line in $\mathbb{R}^n$. In particular, if none of the components of $\mathbf{v}$ is 0, the **symmetric form**, or the **Cartesian form** of a line through $(a_1, \dots, a_n)$ parallel $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ is

$$\frac{x_1 - a_1}{v_1} = \frac{x_2 - a_2}{v_2} = \cdots = \frac{x_n - a_n}{v_n}, \qquad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Example 11. Find the equation of the line in $\mathbb{R}^4$ through the points $A\,(1, 0, 3, -2)$ and $B\,(2, 1, 0, -1)$ in parametric vector form and Cartesian form.

SOLUTION. The line is parallel to the vector $\begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix}$. Thus the line in parametric vector form is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix}, \quad \text{for } \lambda \in \mathbb{R}.$$

The Cartesian form of the line is

$$\frac{x_1 - 1}{1} = \frac{x_2}{1} = \frac{x_3 - 3}{-3} = \frac{x_4 + 2}{1}.$$

$\Diamond$

More generally, a vector equation of the line joining A to B, with position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, is

$$\mathbf{x} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}, \quad \lambda \in \mathbb{R}.$$

Different values of λ correspond to different points on this line. For example, the value $\lambda = \frac{1}{5}$, gives the position vector

$$\mathbf{x} = \frac{4}{5}\mathbf{a} + \frac{1}{5}\mathbf{b} = \mathbf{a} + \frac{1}{5}(\mathbf{b} - \mathbf{a}).$$

This vector can be written as

$$\mathbf{x} = \overrightarrow{OA} + \frac{1}{5}\overrightarrow{AB}.$$

In $\mathbb{R}^2$ or $\mathbb{R}^3$, it is the position vector of the point P which divides AB in the ratio $1 : 4$ (see Figure 22).

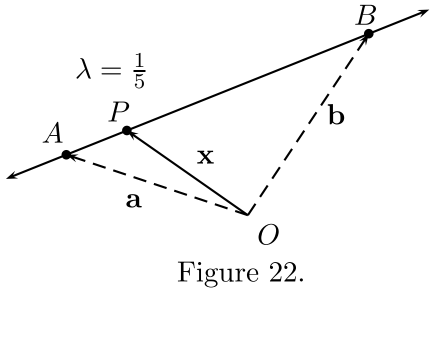


Figure 22.

Generally, in $\mathbb{R}^n$, if P lies between A and B, then $AP : PB = \lambda : 1-\lambda$, for some λ such that $0<\lambda<1$. Therefore, the position vector of P is

$$(1-\lambda)\mathbf{a}+\lambda\mathbf{b},$$

and the set all the points on the line segment AB in $\mathbb{R}^n$ is

$$\{\mathbf{x}\in\mathbb{R}^n : \mathbf{x}=(1-\lambda)\mathbf{a}+\lambda\mathbf{b},\ \text{for } 0\leqslant\lambda\leqslant 1\}\,.$$

Example 12. Find the medians of the triangle with vertices $A\ (1,2,3)$, $B\ (-2,1,-4)$ and $C\ (4,-1,2)$.

Solution. A median of a triangle is a line through a vertex and the midpoint of the opposite side. The median through A is therefore the line which passes through the point A and the midpoint P of the line segment BC, as shown in Figure 23.

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be the position vectors of A, B, C, respectively.

Since P is the midpoint of BC, the coordinate vector of P is

$$\frac{1}{2}(\mathbf{b}+\mathbf{c})=\frac{1}{2}\left[\begin{pmatrix}-2\\1\\-4\end{pmatrix}+\begin{pmatrix}4\\-1\\2\end{pmatrix}\right]=\begin{pmatrix}1\\0\\-1\end{pmatrix}.$$

and hence

$$\overrightarrow{AP}=\begin{pmatrix}1\\0\\-1\end{pmatrix}-\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}0\\-2\\-4\end{pmatrix}.$$

Since A lies on the median and the vector $\overrightarrow{AP}$ is parallel to the median, we can write

$$\mathbf{x}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda_1\begin{pmatrix}0\\-2\\-4\end{pmatrix},\qquad \lambda_1\in\mathbb{R}$$

as a parametric vector form for the median through A.

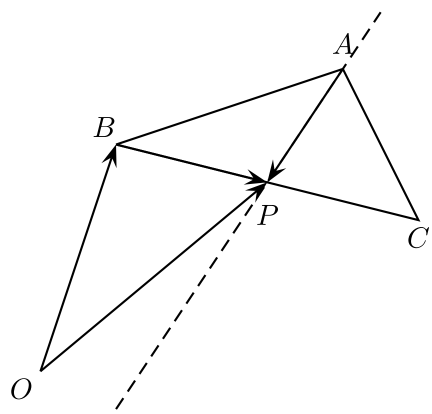


Figure 23.

Similarly, we can obtain the following parametric vector forms for the medians through B and C:

$$\mathbf{x}=\begin{pmatrix}-2\\1\\-4\end{pmatrix}+\lambda_2\left[\begin{pmatrix}\frac{5}{2}\\\frac{1}{2}\\\frac{5}{2}\end{pmatrix}-\begin{pmatrix}-2\\1\\-4\end{pmatrix}\right]=\begin{pmatrix}-2\\1\\-4\end{pmatrix}+\lambda_2\begin{pmatrix}\frac{9}{2}\\-\frac{1}{2}\\\frac{13}{2}\end{pmatrix},\qquad \lambda_2\in\mathbb{R};$$

and

$$\mathbf{x}=\begin{pmatrix}4\\-1\\2\end{pmatrix}+\lambda_3\begin{pmatrix}-\frac{9}{2}\\\frac{5}{2}\\-\frac{5}{2}\end{pmatrix},\qquad \lambda_3\in\mathbb{R}.$$

$\Diamond$

Example 13. Are the lines $\mathbf{x} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ \frac{9}{2} \\ -6 \end{pmatrix}$ parallel?

SOLUTION. Since two vectors are parallel if and only if one is a non-zero scalar multiple of the other, and $\begin{pmatrix} 3 \\ \frac{9}{2} \\ -6 \end{pmatrix} = -\frac{3}{2} \begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix}$, we can conclude that the two lines are parallel. ♢

1.5 Planes

We have seen that the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ which are scalar multiples of a fixed non-zero vector $\mathbf{v} \in \mathbb{R}^n$, i.e. the set $\{\mathbf{x} \in \mathbb{R}^n\colon\ \mathbf{x} = \lambda\mathbf{v}, \text{ for } \lambda \in \mathbb{R}\}$, has a simple geometric interpretation as a line through the origin. In this section we shall show that the set of all sums of scalar multiples of two non-zero non-parallel vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ has a geometric interpretation as a plane through the origin. Furthermore, we shall look at how to find equations for planes.

1.5.1 Linear combination and span

The set of all sums of scalar multiples of some collection of vectors is very important in linear algebra, and therefore has a special name. In the following definitions the word vector is used to mean either a geometric vector or an element of $\mathbb{R}^n$.

Definition 1. *A* ***linear combination*** *of two vectors* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *is a sum of scalar multiples of* $\mathbf{v}_1$ *and* $\mathbf{v}_2$*. That is, it is a vector of the form*

$$\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2,$$

where λ_1 *and* λ_2 *are scalars.*

Example 1. i) The vector $\begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix}$ is a linear combination of the vectors $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ since

$$\begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + (-2) \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}.$$

ii) The vector $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ is not a linear combination of $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ since there are no real numbers λ_1 and λ_2 such that

$$\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}.$$

(Can you prove this?) ♢

Definition 2. *The **span** of two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, written span$(\mathbf{v}_1, \mathbf{v}_2)$, is the set of all linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$. That is, it is the set*

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2) = \{\mathbf{x} : \ \mathbf{x} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2, \text{ for some } \lambda_1,\ \lambda_2 \in \mathbb{R}\ \}.$$

Example 2. Given that $\mathbf{v}_1$ and $\mathbf{v}_2$ are non-zero, non-parallel vectors in $\mathbb{R}^3$. What does span$(\mathbf{v}_1, \mathbf{v}_2)$ look like?

Take $\mathbf{x}$ in span$(\mathbf{v}_1, \mathbf{v}_2)$. So $\mathbf{x} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2$, for some $\lambda_1,\ \lambda_2 \in \mathbb{R}$. By the parallelogram law of vector addition, $\mathbf{x}$ lies in the plane passing through O and contains $\mathbf{v}_1$ and $\mathbf{v}_2$.

Conversely, suppose that X is a point with position vector $\mathbf{x}$ on this plane and OX is neither parallel to $\mathbf{v}_1$ nor parallel to $\mathbf{v}_2$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are non-zero and non-parallel, there exists a parallelogram $OAXB$, such that OA and OB are parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. Note also that, in this case, $\overrightarrow{OA} = \lambda_1\mathbf{v}_1$ and $\overrightarrow{OB} = \lambda_2\mathbf{v}_2$, for some real numbers λ_1, λ_2. By the parallelogram law, we have $\mathbf{x} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2$ which is in span$(\mathbf{v}_1, \mathbf{v}_2)$.

When OX is parallel to $\mathbf{v}_1$, we have $\mathbf{x} = \lambda_1\mathbf{v}_1$ which is also in the span. Similarly, $\mathbf{x}$ is also in the span when OX is parallel to $\mathbf{v}_2$.

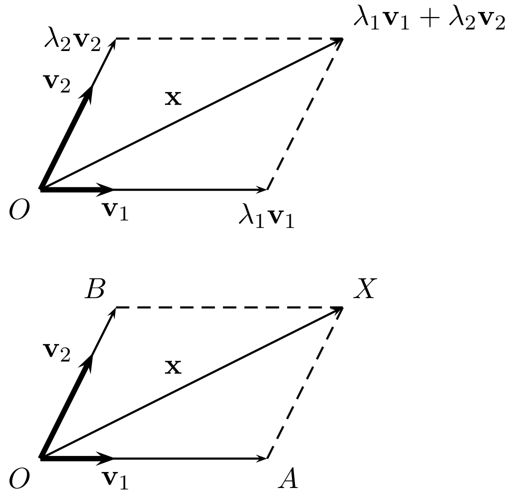


Figure 24.

Hence, span$(\mathbf{v}_1, \mathbf{v}_2)$ is the plane through the origin and parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$ with equation

$$\mathbf{x} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2, \quad \text{for some } \lambda_1,\ \lambda_2 \in \mathbb{R}.$$

$\diamondsuit$

Note that The above construction does not work if $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel or one of them is $\mathbf{0}$.

Theorem 1. *In $\mathbb{R}^3$, the span of any two non-zero non-parallel vectors is a plane through the origin.*

We can extend this to $\mathbb{R}^n$ using:

Definition 3. *A **plane through the origin** is the span of any two (non-zero) non-parallel vectors.*

Example 3. Describe geometrically span$\left(\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix}\right)$, and decide if it is a plane or a line.

Solution. By definition, the span is the set of all vectors $\mathbf{x} \in \mathbb{R}^3$ such that

$$\mathbf{x} = \lambda_1 \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix} + \lambda_2 \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix} \quad \text{for some } \lambda_1, \lambda_2 \in \mathbb{R}.$$

As $\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$ is not a scalar multiple of $\begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix}$, the span is a plane through the origin parallel to the vectors $\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix}$. ♢

Example 4. Describe geometrically span $\left(\begin{pmatrix} 3 \\ 5 \\ 2 \\ 7 \\ 8 \end{pmatrix}, \begin{pmatrix} 6 \\ 10 \\ 4 \\ 14 \\ 16 \end{pmatrix}\right)$ and decide whether it is a plane or a line.

SOLUTION. By definition, the span is the set of all vectors $\mathbf{x} \in \mathbb{R}^5$ satisfying

$$\mathbf{x} = \lambda_1 \begin{pmatrix} 3 \\ 5 \\ 2 \\ 7 \\ 8 \end{pmatrix} + \lambda_2 \begin{pmatrix} 6 \\ 10 \\ 4 \\ 14 \\ 16 \end{pmatrix} \quad \text{for some } \lambda_1, \lambda_2 \in \mathbb{R}.$$

As $\begin{pmatrix} 6 \\ 10 \\ 4 \\ 14 \\ 16 \end{pmatrix} = 2 \begin{pmatrix} 3 \\ 5 \\ 2 \\ 7 \\ 8 \end{pmatrix}$, the span is the line $\mathbf{x} = \lambda \begin{pmatrix} 3 \\ 5 \\ 2 \\ 7 \\ 8 \end{pmatrix}$ for some $\lambda \in \mathbb{R}$. ♢

1.5.2 Parametric vector form of a plane

The span of two non-zero non-parallel vectors is a plane through the origin. For planes that do not pass through the origin we use a similar approach to that we use for lines.

Consider first three vectors $\mathbf{a}$, $\mathbf{v}_1$, $\mathbf{v}_2 \in \mathbb{R}^3$. We shall assume that $\mathbf{v}_1$ and $\mathbf{v}_2$ are non-zero non-parallel. Let

$$S = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \mathbf{a} + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 \text{ for } \lambda_1, \lambda_2 \in \mathbb{R}\}.$$

This set consists of all the vectors in $\mathbb{R}^3$ for which $\mathbf{x} - \mathbf{a} \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$. That is, S contains all the points $\mathbf{x}$ for which $\mathbf{x} - \mathbf{a}$ lies in the plane through the origin spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. Thus S is the plane in $\mathbb{R}^3$ which passes through the point with position vector $\mathbf{a}$, and is parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$. The picture is shown in Figure 25.

So in $\mathbb{R}^3$,

$$\mathbf{x} = \mathbf{a} + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2, \quad \text{for } \lambda_1,\ \lambda_2 \in \mathbb{R}$$

is an equation of the plane passing through $\mathbf{a}$ and parallel to the non-zero non-parallel vectors $\mathbf{v}_1, \mathbf{v}_2$. This is called a parametric vector form of the plane.

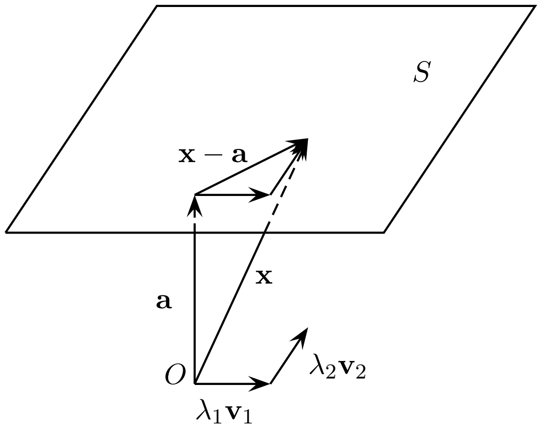


Figure 25: $\mathbf{x} = \mathbf{a} + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2$.

As before we shall use this as a way of defining what we mean by a plane in $\mathbb{R}^n$.

Definition 4. *Let $\mathbf{a}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ be fixed vectors in $\mathbb{R}^n$, and suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel. Then the set*

$$S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{a} + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2, \text{ for some } \lambda_1, \lambda_2 \in \mathbb{R}\},$$

*is the **plane** through the point with position vector $\mathbf{a}$, parallel to the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. The expression*

$$\mathbf{x} = \mathbf{a} + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2, \text{ for } \lambda_1, \lambda_2 \in \mathbb{R},$$

*is called a **parametric vector form** of the plane.*

Note. For a given plane, there is not a unique parametric vector form. For example, a plane parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$ is also parallel to $\mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_1 - \mathbf{v}_2$.

From the definition, a point P with position vector $\mathbf{p} \in \mathbb{R}^n$ is said to lie on the plane through $\mathbf{a}$ parallel to $\mathbf{v}_1$, $\mathbf{v}_2$, if there exist real numbers λ_1, λ_2 such that $\mathbf{p} = \mathbf{a} + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2$.

Example 5. Find a parametric vector form for the plane through the point $(1, 3, 0)$, parallel to the vectors $\begin{pmatrix} 4 \\ 2 \\ 9 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$.

Solution. One answer (there are many possible!) is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 4 \\ 2 \\ 9 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}, \quad \text{for } \lambda_1, \lambda_2 \in \mathbb{R}.$$

$\Diamond$

Example 6. Describe geometrically the set

$$\left\{ \mathbf{x} \in \mathbb{R}^4 : \mathbf{x} = \begin{pmatrix} 1 \\ 4 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ 2 \\ -4 \\ 1 \end{pmatrix}, \quad \text{for } \lambda_1, \lambda_2 \in \mathbb{R} \right\}.$$

Solution. Since $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 2 \\ -4 \\ 1 \end{pmatrix}$ are not parallel (i.e. they are not scalar multiples of each other), this set is the plane in $\mathbb{R}^4$ through the point $(1, 4, 2, 1)$ parallel to the vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 2 \\ -4 \\ 1 \end{pmatrix}$. ♢

Example 7. Let A, B and C be three points in $\mathbb{R}^n$ whose coordinate vectors with respect to some coordinate system are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Find a parametric vector form for the plane which passes through A, B and C.

Solution. Consider the plane through A parallel to $\overrightarrow{AB}$ and $\overrightarrow{AC}$. This has a parametric vector form:

$$\mathbf{x} = \mathbf{a} + \lambda_1 \overrightarrow{AB} + \lambda_2 \overrightarrow{AC} = \mathbf{a} + \lambda_1(\mathbf{b} - \mathbf{a}) + \lambda_2(\mathbf{c} - \mathbf{a}), \quad \text{for } \lambda_1, \lambda_2 \in \mathbb{R}.$$

When $\lambda_1 = \lambda_2 = 0$ we have $\mathbf{x} = \mathbf{a}$. When $\lambda_1 = 1,\ \lambda_2 = 0$ we have $\mathbf{x} = \mathbf{b}$, and when $\lambda_1 = 0,\ \lambda_2 = 1$ we have $\mathbf{x} = \mathbf{c}$. Hence it is the plane through A, B and C.

♢

Example 8. Find a parametric vector form for the plane through the three points $A\ (2, -1, 3)$, $B\ (-1, 4, 4)$ and $C\ (3, -1, 2)$.

Solution. The plane is parallel to $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 4 \\ -5 \\ -2 \end{pmatrix}$, and passes though the point $A\ (2, -1, 3)$. Hence a parametric vector form for the plane is

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 4 \\ -5 \\ -2 \end{pmatrix}, \quad \text{for } \lambda_1, \lambda_2 \in \mathbb{R}.$$

♢

1.5.3 Cartesian form of a plane in $\mathbb{R}^3$

In $\mathbb{R}^3$, any equation of the form

$$ax_1 + bx_2 + cx_3 = d$$

represents plane when not all a, b, c are zero. In $\mathbb{R}^n$, $n > 3$, if not all $a_1, a_2, \ldots, a_n$ are zero, an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = d$$

is called a hyperplane.

Example 9. Find a parametric vector form for the plane $x_1 - 3x_2 + 4x_3 = 4$, and hence find a point on the plane and two vectors parallel to the plane.

Solution. We set $x_2 = \lambda_1$ and $x_3 = \lambda_2$. Then solving for x_1 gives $x_1 = 4 + 3\lambda_1 - 4\lambda_2$. Hence

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 + 3\lambda_1 - 4\lambda_2 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix},$$

for $\lambda_1, \lambda_2 \in \mathbb{R}$. Thus the plane passes through the point $(4, 0, 0)$ and is parallel to $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$. ♢

As we did for lines, we can find a Cartesian equation for a plane by eliminating the parameters λ_1, λ_2 from the parametric vector form.

Example 10. Find the Cartesian form of the plane $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ for λ, $\mu \in \mathbb{R}$.

Solution. Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. By comparing the components, we get the following parametric equations

$$x_1 = 1 + \lambda - \mu, \qquad x_2 = 1 + \lambda \qquad \text{and} \qquad x_3 = 2 + \lambda + 2\mu.$$

Hence from the second equation, we get $\lambda = x_2 - 1$. Substituting this into the first equation, we have $x_1 = 1 + (x_2 - 1) - \mu$, so $\mu = x_2 - x_1$. Now substitute these values of λ and μ into the third parametric equation. The Cartesian form of the plane is, therefore,

$$x_3 = 2 + (x_2 - 1) + 2(x_2 - x_1) \qquad \text{or} \qquad 2x_1 - 3x_2 + x_3 = 1.$$

♢

1.6 Vectors and Maple

This section shows how to handle vectors in Maple. The capacity to plot simple curves in the plane is illustrated. The 3-dimensional plot facility needs an x-terminal.
The following instruction loads the linear algebra package:

```
with(LinearAlgebra);
```

Before attempting anything involving vectors you may wish to plot a piece of a parabola parametrically with:

```
plot([2*t,t∧2,t=-4..4]);
```

To enter the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 9 \end{pmatrix}$, we type:

```
a:=<1,2,3>;
b:=<2,6,9>;
```

The command

```
a[2];
```

selects the second component of $\mathbf{a}$.
Since $\mathbf{a}$ and $\mathbf{b}$ have the same number of components, they may be added with:

```
a+b;
```

Multiplication by a scalar is handled analogously:

```
4*a;
```

Now start another example with

```
a:=<2,3>; b:=<1,4>;
```

If $\mathbf{a}$ and $\mathbf{b}$ are two vectors, a parametric vector form for the line through $\mathbf{a}$ and $\mathbf{b}$ is given by:

```
v:=b-a;     x:=a+t*v;
```

To plot the line segment from $t = -2$ to $t = 3$, use:

```
plot([x[1],x[2],t=-2..3]);
```

Now let `a`, `v1` and `v2` be 3-vectors. The plane $\mathbf{x} = \mathbf{a} + s\mathbf{v}_1 + t\mathbf{v}_2$ can be entered as:

```
x:=a+s*v1+t*v2;
```

To find out about the three-dimensional plot facility, just type:

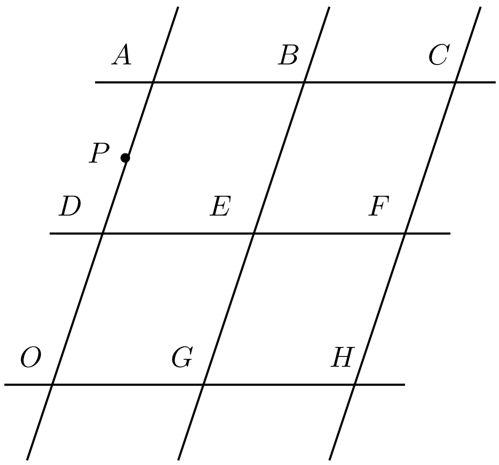
```
?plot3d
```

Problems for Chapter 1

Questions marked with [**R**] are routine, [**H**] harder, [**M**] Maple and [**X**] are for MATH1141 only. You should make sure that you can do the easier questions before you tackle the more difficult questions. Questions marks with a [**V**] have video solutions available on Moodle.

Problems 1.1 : Vector quantities

1. [**R**][**V**] Given that ABC, DEF, and OGH are equally spaced parallel lines, as are ADO, BEG and CFH. P is the mid point of AD.



If $\overrightarrow{OH} = \mathbf{h}$ and $\overrightarrow{OA} = \mathbf{a}$, express the following in terms of $\mathbf{a}$ and $\mathbf{h}$.

a) $\overrightarrow{OC}$, b) $\overrightarrow{HA}$, c) $\overrightarrow{GC}$, d) $\overrightarrow{OP}$, e) $\overrightarrow{GP}$.

2. [**R**] Simplify

a) $\overrightarrow{AB} - \overrightarrow{OB} + \overrightarrow{OA}$, b) $\overrightarrow{AB} - \overrightarrow{CB} + 3\overrightarrow{DA} + 3\overrightarrow{CD}$.

3. [**R**] Express each of the following in terms of $\mathbf{a}$ and $\mathbf{b}$.

a) $3(2\mathbf{a} + \mathbf{b}) - 2(5\mathbf{a} - \mathbf{b})$,
b) $2(p\,\mathbf{a} + q\,\mathbf{b}) + 3(r\,\mathbf{a} - s\,\mathbf{b})$ where $p,\ q,\ r,\ s \in \mathbb{R}$.

4. [**R**] Let ABC be a triangle with $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$ where O is the origin.

a) If M is the midpoint of the line segment AB and P is the midpoint of the line segment CB express the vectors $\overrightarrow{OM}$ and $\overrightarrow{OP}$ in terms of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.
b) Show that $\overrightarrow{MP}$ is parallel to $\overrightarrow{AC}$ and has half its length.

5. [**H**][**V**] Given a convex quadrilateral $ABCD$, prove, using vectors, that the quadrilateral formed by joining the midpoints of AB, BC, CD, and DA is a parallelogram.

6. **[R]** Use geometric vectors to solve the following problems. In each case, draw a careful picture and then use trigonometry to find the answer. If your picture is accurate, you may wish to use a ruler and protractor to confirm your result.

   a) An ant crawls 10 cm due east in a straight line and then crawls 5 cm northeast in a straight line. What is the ant's final displacement from its starting point?

   b) An ant is standing at the western edge of a moving walkway which is moving at 12 cm per sec in the direction due South. The ant starts to walk at 5 cm/sec across the walkway in the direction perpendicular to its edge. If the walkway is 40 cm wide, find the displacement of the ant from its starting point just as it steps off the walkway.

   c) An observer on a wharf sees a yacht sailing at 15 km per hour southeast. A sailor on the yacht is watching a container ship and sees it sailing at 25 km per hour due north. What is the velocity of the container ship as seen by the observer on the wharf?

   d) A rower is rowing across a river. His rowing speed is 2 km per hour and there is a current flowing in the river at 1 km per hour. Find the direction that the rower must row to go directly across the river. If the river is 300 metres wide, how long will it take him to cross the river?

7. **[X]** Town B is 18 km N 18° W from Town A. Town C is 25 km N 36° E from B. Town D is 20 km S 72° E from C. Town F is 15 km S 25° W from D. What are the distance and bearing of Town F from Town A?

8. **[X]** Let O, A, B, C be points in a plane. Suppose that X is the midpoint of BC, Y is the point on AC with $AY : YC = 3 : 1$, and Z is the point on AB with $AZ : ZB = 3 : 1$. Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, and $\overrightarrow{OC} = \mathbf{c}$.

   a) Write down the vectors $\overrightarrow{OX}$, $\overrightarrow{OY}$ and $\overrightarrow{OZ}$ in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

   b) Let T be the point on AX with $AT : TX = 6 : 1$. Write $\overrightarrow{OT}$ in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

   c) Show that T lies on both CZ and YB.

Problems 1.2 : Vector quantities and $\mathbb{R}^n$

9. **[R][V]** Find $\mathbf{u} + 2\mathbf{v} - 3\mathbf{w}$ (if possible) given that

   a) $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$;

   b) $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 7 \\ 6 \\ -1 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$;

   c) $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1 \end{pmatrix}$;

d) $\mathbf{u} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 10 \\ 2 \\ -3 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$;

e) $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{w} = -\mathbf{i} + \mathbf{j} - \mathbf{k}$.

10. [**R**] A car travels 3km due North then 5km Northeast. Use coordinate vectors to find the distance and direction from the starting point.

11. [**R**] Solve Problems 6 (a) and (b) using coordinate vectors.

12. [**R**] Suppose that $\mathbf{v} = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$ are vectors in $\mathbb{R}^3$; λ and μ are real numbers. Prove the scalar distributive law $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$ and the vector distributive law $\lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w}$.

13. [**X**] Prove the associative law of vector addition in $\mathbb{R}^n$. (Proposition 2 on page 12).

14. [**X**] Prove Proposition 3 on page 14.

Problems 1.3 : $\mathbb{R}^n$ and analytic geometry

15. [**R**][**V**] Let $\mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Draw coordinate axes and mark in the points whose coordinate vectors are $\mathbf{v}$, $-\mathbf{v}$, $\mathbf{w}$, $\mathbf{v} + \mathbf{w}$, $2\mathbf{v}$ and $\mathbf{v} - \mathbf{w}$.

16. [**R**] Given the following points A, B, C and D, are the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ parallel?

a) $A = (1, 2, 3)$, $B = (-2, 3, 4)$, $C = (-3, -4, 7)$, $D = (4, -6, -9)$;

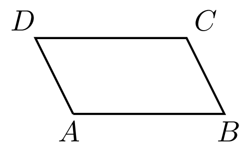
b) $A = (3, 2, 5)$, $B = (5, -3, -6)$, $C = (-2, 3, 7)$, $D = (0, -2, -4)$;

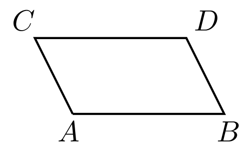
c) $A = (12, -4, 6)$, $B = (2, 6, -4)$, $C = (5, -2, 9)$, $D = (0, 3, 4)$.

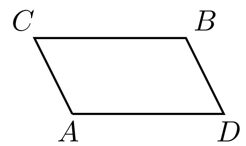
Do any of these sets of 4 points form a parallelogram?

17. [**R**] a) Prove that $A(1, 2, 1)$, $B(4, 7, 8)$, $C(6, 4, 12)$ and $D(3, -1, 5)$ are the vertices of a parallelogram.

b) Which of the following rough sketches correctly represents the parallelogram from part (a)?

i) 

ii) 

iii) 

18. [**R**] Show that the points $A(1, 2, 3)$, $B(3, 8, 1)$, $C(7, 20, -3)$ are collinear.

19. **[R][V]** Show that the points $A(-1,2,1)$, $B(4,6,3)$, $C(-1,2,-1)$ are not collinear.

20. **[R]** Show that the points A, B, C in $\mathbb{R}^3$ with coordinate vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 6 \\ -5 \\ -2 \end{pmatrix}$$

are collinear.

21. **[H]** If $A(-1,3,4)$, $B(4,6,3)$, $C(-1,2,1)$ and D are the vertices of a parallelogram, find all the possible coordinates for the point D.

22. **[H]** Consider three non-collinear points D, E, F in $\mathbb{R}^3$ with coordinate vectors $\mathbf{d}$, $\mathbf{e}$ and $\mathbf{f}$. There are exactly 3 points in $\mathbb{R}^3$ which, taken one at a time with D, E and F, form a parallelogram. Calculate vector expressions for the three points.

23. **[R][V]** Let $A = (2,3,-1)$ and $B = (4,-5,7)$. Find the midpoint of A and B. Find the point Q on the line through A and B such that B lies between A and Q and BQ is three times as long as AB.

24. **[R]** The coordinate vectors, relative to the origin O, of the points A and B are respectively $\mathbf{a}$ and $\mathbf{b}$. State, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vector of the point T which lies on AB and is such that $\overrightarrow{AT} = 2\overrightarrow{TB}$.

25. **[R]** List the standard basis vectors for $\mathbb{R}^5$.

26. **[R]** For each of the following vectors, find its length and find a vector of length one ("unit" vectors) parallel to it.

$$\mathbf{a} = \begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} 4 \\ 0 \\ 1 \\ -2 \\ 0 \end{pmatrix}.$$

27. **[R][V]** Find the distances between each of the following pairs of points with coordinate vectors:

$$\text{a)} \begin{pmatrix} 8 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix}; \qquad \text{b)} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ -7 \\ -7 \end{pmatrix}; \qquad \text{c)} \begin{pmatrix} 3 \\ 0 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 6 \\ 1 \\ 3 \end{pmatrix}.$$

28. **[R]** A triangle has vertices A, B and C which have coordinate vectors $\begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix}, \begin{pmatrix} 7 \\ -4 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 2 \\ 8 \end{pmatrix}$ respectively. Find the lengths of the sides of the triangle and deduce that the triangle is right-angled.

29. **[H]** Construct a cube in $\mathbb{R}^3$ with the length of each edge 1. Show that the face diagonal has length $\sqrt{2}$ and the long diagonal $\sqrt{3}$. Try to generalise this idea to $\mathbb{R}^4$ and show that there are now diagonals of length $\sqrt{2}, \sqrt{3}$ and 2. How many vertices does a 4-cube have?

30. **[X]** Find 10 vectors in $\mathbb{R}^{10}$, each pair of which is $5\sqrt{2}$ apart. Can you now find an 11th such vector?

Problems 1.4 : Lines

31. **[R][V]** Find the coordinate vector for the displacement vector $\overrightarrow{AB}$ and parametric vector forms for the lines through the points A and B with coordinates

a) $A\ (1,2),\ B\ (2,7)$; b) $A\ (1,2,-1),\ B\ (-1,-1,5)$;
c) $A\ (1,2,1),\ B\ (7,2,3)$; d) $A\ (1,2,-1,3),\ B\ (-1,3,1,1)$.

32. **[R]** Does the point $(3,5,7)$ lie on the line $\mathbf{x} = \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$?

33. **[R]** Find parametric vector forms for the following lines in $\mathbb{R}^2$:

a) $y = 3x + 4$; b) $3x + 2y = 6$; c) $y = -7x$;
d) $y = 4$; e) $x = -2$.

In each case indicate the direction of the line and a point through which the line passes.

34. **[R]** Find a parametric vector form and a Cartesian form for each of the following lines

a) through the points $(-4,1,3)$ and $(2,2,3)$;

b) through $(1,2,-3)$ parallel to the vector $\begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix}$;

c) through $(1,-1,1)$ parallel to the line joining the points $(2,2,1)$ and $(7,1,3)$;

d) through $(1,0,0)$ parallel to the line joining the points $(3,2,-1)$ and $(3,5,2)$.

35. **[R]** Let A, B, P be points in $\mathbb{R}^3$ with position vectors

$$\mathbf{a} = \begin{pmatrix} 7 \\ -2 \\ 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -5 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{p} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Let Q be the point on AB such that $AQ = \dfrac{2}{3} AB$.

a) Find $\mathbf{q}$, the position vector of Q.

b) Find the parametric vector equation of the line that passes through P and Q.

36. [**R**] Decide whether each of the following statements is true or false.

a) The lines $y = 3x - 4$ and $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 12 \end{pmatrix}$ are parallel.

b) The lines $\mathbf{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ 4 \end{pmatrix}$ and $2x + 3y = 8$ are parallel.

c) The lines $\mathbf{x} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 10 \\ 2 \\ 8 \end{pmatrix}$ and $\dfrac{x+10}{5} = y - 7 = \dfrac{z+3}{4}$ are parallel.

d) The line $\mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 10 \\ 0 \\ -4 \end{pmatrix}$ and the line

$$\frac{x+10}{5} = \frac{z+3}{-2} \quad \text{and} \quad y = -5$$

are parallel.

37. [**X**] Suppose A and B are points with coordinate vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. Write down a parametric vector form for

a) the line segment AB.

b) the ray from B through A.

c) all points P which lie on the line through A and B such that A is between P and B.

d) all points Q which lie on the line through A and B and are closer to B than A.

38. [**X**] Give a geometric interpretation of the following sets. In each set, $\lambda \in \mathbb{R}$.

a) $S = \left\{ \mathbf{x} : \mathbf{x} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 1 \\ 7 \end{pmatrix} \quad \text{for} \ \ 0 \leqslant \lambda \leqslant 1 \right\}.$

b) $S = \left\{ \mathbf{x} : \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 0 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 5 \\ 9 \\ 3 \\ 6 \end{pmatrix} \quad \text{for} \ \ -1 \leqslant \lambda \leqslant 5 \right\}.$

c) $S = \left\{ \mathbf{x} : \mathbf{x} = \lambda \begin{pmatrix} 6 \\ -2 \\ 7 \\ 2 \\ -1 \\ 5 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 0 \\ 4 \\ 8 \\ 3 \\ -5 \\ 4 \end{pmatrix} \quad \text{for} \ \ 0 \leqslant \lambda \leqslant 1 \right\}.$

d) $S = \left\{ \mathbf{x} : \mathbf{x} = \begin{pmatrix} 1 \\ 4 \\ -6 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ -1 \\ 5 \end{pmatrix} \quad \text{for } \lambda \geqslant 0 \right\}.$

e) $S = \left\{ \mathbf{x} : \mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -2 \\ 7 \end{pmatrix} \text{ for } |\lambda| \geqslant 2 \right\}$.

Problems 1.5 : Planes

39. [**R**] Find a parametric vector form for the planes passing through the points

a) $(0,0,0),\ (3,-1,2),\ (1,4,-6)$; b) $(1,4,-2),\ (2,6,4),\ (1,-10,3)$.

40. [**R**] For each of the following sets of vectors, decide if the set is a line or a plane, give a point on the line or plane, and give vectors parallel to the line or plane, i.e., geometrically describe the sets.

a) $S = \left\{ \mathbf{x} : \mathbf{x} = \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix} \text{ for } \lambda_1, \lambda_2 \in \mathbb{R} \right\}$.

b) $S = \left\{ \mathbf{x} : \mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 4 \end{pmatrix} + \lambda_1 \begin{pmatrix} -2 \\ 1 \\ 3 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 4 \\ -2 \\ -6 \\ -4 \end{pmatrix} \text{ for } \lambda_1, \lambda_2 \in \mathbb{R} \right\}$.

c) $\operatorname{span}\left(\begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -9 \\ -6 \\ -3 \\ -6 \end{pmatrix} \right)$.

d) $S = \left\{ \mathbf{x} : \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mathbf{y} \text{ for } \mathbf{y} \in \operatorname{span}\left(\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 8 \\ 2 \\ 4 \end{pmatrix} \right) \right\}$.

41. [**R**][**V**] Find parametric vector forms for the planes

a) through the point $(1,2,3)$ parallel to $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$;

b) through the points $(3,1,4),\ (-1,2,4),\ (6,7,-2)$;

c) through the points $(-2,4,1,6),\ (3,2,6,-1),\ (1,4,0,0)$;

d) $4x_1 - 3x_2 + 6x_3 = 12$, where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$;

e) $5x_2 - 6x_3 = 5$, where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$;

f) through the point $(1,2,3,4)$ parallel to the lines $\mathbf{x} = \begin{pmatrix} -3 \\ 1 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 0 \\ -4 \\ 5 \end{pmatrix}$ and

$$\frac{x_1 - 5}{7} = \frac{x_2 + 6}{2} = \frac{x_3 - 2}{-3} = \frac{x_4 + 1}{-5}.$$

42. **[R]** Find parametric vector forms to describe the following planes in $\mathbb{R}^3$.

 a) $x_1 + x_2 + x_3 = 0$. b) $3x_1 - x_2 + 4x_3 = 12$.
 c) $x_2 + 6x_3 = -1$. d) $x_3 = 2$.

43. **[H]** Show that the line $\mathbf{x} = t \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

 a) lies on the plane $4x - 5y - z = 0$, and b) is parallel to the plane $3x - 3y - z = 2$.

44. **[H]** a) Find the intersection of the line $\mathbf{x} = \begin{pmatrix} 2+t \\ 3-t \\ 4t \end{pmatrix}$ and the plane $2x + 3y + z = 16$.

 b) Find the intersection of the line $\mathbf{x} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$ and the plane $9x + 4y - z = 0$.

45. **[H]** a) Write the plane $\mathbf{x} = \begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$ in Cartesian form.

 b) Write the plane $\mathbf{x} = \begin{pmatrix} 6 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 6 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ in Cartesian form.

46. **[H]** Consider the line $\dfrac{x-3}{-2} = \dfrac{y+2}{3} = z - 1$ and the plane $2x + y + 3z = 23$ in $\mathbb{R}^3$.

 a) Find a parametric vector form for the line.
 b) Hence find where the line meets the plane.

47. **[H]** Let ℓ be the line $\dfrac{x-6}{5} = \dfrac{y-4}{2} = \dfrac{z-1}{-2}$ in $\mathbb{R}^3$.

 a) Express the line ℓ in parametric vector form.
 b) Find the coordinates of the point where ℓ meets the plane $2x + y - z = 1$.

48. **[X]** The following sets of points represent simple geometric figures in a plane. λ_1 and λ_2 are real numbers. For each problem draw a sketch in the (λ_1, λ_2) plane and a second sketch in $\mathbb{R}^2, \mathbb{R}^3$ or $\mathbb{R}^4$ (!!) as appropriate. For each problem identify the geometric shape.

a) $S = \left\{ \mathbf{x} : \mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{for } 0 \leqslant \lambda_1 \leqslant 1, \ \ 0 \leqslant \lambda_2 \leqslant 1 \right\}.$

b) $S = \left\{ \mathbf{x} : \mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{for } 0 \leqslant \lambda_1 \leqslant 1, \ \ 0 \leqslant \lambda_2 \leqslant \lambda_1 \right\}.$

c) $S = \left\{ \mathbf{x} \ : \mathbf{x} = \lambda_1 \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} \quad \text{for } 0 \leqslant \lambda_1 \leqslant 6, \ \ 0 \leqslant \lambda_2 \leqslant 8 \right\}.$

d) $S = \left\{ \mathbf{x} \ : \mathbf{x} = \lambda_1 \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} \quad \text{for } 0 \leqslant \lambda_1 \leqslant 6, \ \ 0 \leqslant \lambda_2 \leqslant \lambda_1 \right\}.$

e) $S = \left\{ \mathbf{x} \ : \mathbf{x} = \lambda_1 \begin{pmatrix} 2 \\ 1 \\ -2 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 4 \\ -2 \\ 3 \\ -1 \end{pmatrix} \quad \text{for } 0 \leqslant \lambda_1 \leqslant 6, \ \ 0 \leqslant \lambda_1 \leqslant \lambda_2 \right\}.$

49. [**X**] Write down the sets of points corresponding to the following:

a) A "parallelogram" with the three vertices $A(1, 3, 4, 2)$, $B(-2, 1, 0, 5)$ and $C(-4, 0, 6, 8)$. Hint: Look at Question 48 a), and assume B and C are adjacent to A.

b) The triangle with the three vertices given in part a) of this question. Hint: Look at Question 48 b).

c) All three parallelograms which have the three vertices given in part a).

50. [**X**] Given two planes in $\mathbb{R}^n$, $n \geqslant 3$:

$$\mathbf{x} = \mathbf{a} + s_1\mathbf{u}_1 + s_2\mathbf{u}_2 \quad \text{and} \quad \mathbf{x} = \mathbf{b} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2,$$

for $s_1, s_2, t_1, t_2 \in \mathbb{R}$. These two planes are said to be parallel if $\operatorname{span}(\mathbf{u}_1, \mathbf{u}_2) = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$.

Consider the pair of planes in $\mathbb{R}^4$ with equations

$$\mathbf{x} = s_1\mathbf{e}_1 + s_2\mathbf{e}_2 \quad \text{and} \quad \mathbf{x} = \mathbf{e}_4 + t_1\mathbf{e}_2 + t_2\mathbf{e}_3$$

for $s_1, s_2, t_1, t_2 \in \mathbb{R}$, where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_4\}$ is the standard basis of $\mathbb{R}^4$.

Show that these form a pair of skew planes; that is, they are non-parallel and non-intersecting.

Chapter 2

VECTOR GEOMETRY

"Why," said the Gryphon, "you first form into a line along the sea-shore—"
"Two lines!' cried the Mock Turtle.
Lewis Carroll, Alice in Wonderland.

In Chapters 1 and 4 we have shown how vectors can be used to solve geometric problems involving points, lines and planes.

Our aim in this chapter is to show how vectors can be used to solve geometric problems involving lengths, distances, areas, angles and volumes. For simplicity, and because of the fundamental importance of two and three dimensions in the physical sciences and engineering, we will concentrate on problems in two and three dimensions. However, we shall see that many of the two and three dimensional results can be easily generalised to $\mathbb{R}^n$. The key idea is to use **theorems** in $\mathbb{R}^2$ and $\mathbb{R}^3$ to motivate **definitions** in $\mathbb{R}^n$ for $n > 3$.

2.1 Lengths

We have defined lengths of vectors in $\mathbb{R}^n$ and distance between two points on page 18. We expect that lengths and distances in $\mathbb{R}^n$ should have the same essential properties as those in two and three dimensional spaces. For example, we expect them to be real non-negative numbers. Some of these essential properties for $\mathbb{R}^n$ are proved in the following proposition.

Proposition 1. For all $\mathbf{a} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$,

1. $|\mathbf{a}|$ is a real number,
2. $|\mathbf{a}| \geqslant 0$,
3. $|\mathbf{a}| = 0$ if and only if $\mathbf{a} = \mathbf{0}$,
4. $|\lambda\mathbf{a}| = |\lambda|\,|\mathbf{a}|$.

Proof. From Definition 3 on page 18

$$|\mathbf{a}| = \sqrt{a_1^2 + \cdots + a_n^2}.$$

As $\mathbf{a} \in \mathbb{R}^n$, all a_k^2 are real and non-negative, and hence $a_1^2 + \cdots + a_n^2 \geqslant 0$. Thus, properties 1 and 2 hold.

Property 3 holds since a sum of non-negative numbers is zero if and only if every term in the sum is zero.

The proof of Property 4 is as follows. From the definitions of $\lambda\mathbf{a}$ and the length of a vector, we have

$$|\lambda\mathbf{a}| = \sqrt{(\lambda a_1)^2 + \cdots + (\lambda a_n)^2} = |\lambda|\sqrt{a_1^2 + \cdots + a_n^2} = |\lambda|\ |\mathbf{a}|.$$

$\square$

Example 1. $|-6\mathbf{a}| = 6|\mathbf{a}|$. $\lozenge$

2.2 The dot product

Most introductory courses in trigonometry include a statement of the cosine rule for triangles. This rule can be stated in vector form as follows.

Proposition 1. Cosine Rule for Triangles. If the sides of a triangle in $\mathbb{R}^2$ or $\mathbb{R}^3$ are given by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, then

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}|\ |\mathbf{b}|\cos\theta,$$

where θ is the interior angle between $\mathbf{a}$ and $\mathbf{b}$.

Proof. From Figure 1, since $\triangle\, BPA$ is a right-angled triangle, we have

$$|\mathbf{c}|^2 = \left|\overrightarrow{BA}\right|^2 = \left|\overrightarrow{BP}\right|^2 + \left|\overrightarrow{PA}\right|^2.$$

But, $\left|\overrightarrow{BP}\right| = \left||\overrightarrow{OP}| - |\overrightarrow{OB}|\right| = \big||\mathbf{a}|\cos\theta - |\mathbf{b}|\big|$

and $\left|\overrightarrow{PA}\right| = |\mathbf{a}|\sin\theta,$

and hence

$$\begin{aligned}|\mathbf{c}|^2 &= \big||\mathbf{a}|\cos\theta - |\mathbf{b}|\big|^2 + |\mathbf{a}|^2\sin^2\theta \\ &= (|\mathbf{a}|\cos\theta - |\mathbf{b}|)^2 + |\mathbf{a}|^2\sin^2\theta \\ &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}|\ |\mathbf{b}|\cos\theta.\end{aligned}$$

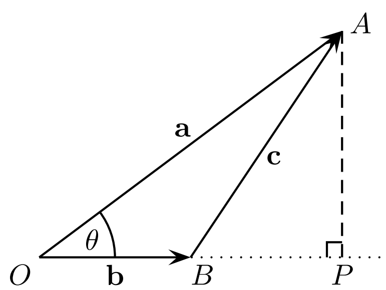


Figure 1: The Cosine Rule.

$\square$

If we write $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and use the formula for the length of $\mathbf{c} = \mathbf{a} - \mathbf{b}$, we have

$$\begin{aligned}|\mathbf{c}|^2 &= (a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 \\ &= \left(a_1^2 + a_2^2 + a_3^2\right) + \left(b_1^2 + b_2^2 + b_3^2\right) - 2\,(a_1b_1 + a_2b_2 + a_3b_3) \\ &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\,(a_1b_1 + a_2b_2 + a_3b_3)\,.\end{aligned}$$

On comparing this expression with the cosine rule, we find

$$a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{a}|\ |\mathbf{b}|\cos\theta.$$

This expression also applies immediately to vectors in the plane, where $a_3 = b_3 = 0$.

Let us now make the following definition.

Definition 1. *The* ***dot product*** *of two vectors* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ *is*

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + \cdots + a_n b_n = \sum_{k=1}^{n} a_k b_k.$$

Remark 1. You may have already encountered this product as the **scalar product** of two vectors. The name "scalar product" is helpful for reminding us that this operation gives us a scalar quantity (see also: Example 4 of Section 5.2).

For the special case of $\mathbb{R}^2$ or $\mathbb{R}^3$, we then have

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}| \cos\theta.$$

Students of physics and engineering should note that it is this geometric result which is usually taken as the definition of the dot product in physics and engineering courses.

Example 1. Find the dot product of $\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$, and hence find the cosine of the angle between the vectors.

Solution. The dot product is

$$\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = -2 + 3 + 8 = 9,$$

and the lengths are $\sqrt{21}$ and $\sqrt{14}$, and hence

$$\cos\theta = \frac{9}{\sqrt{(21)(14)}} = \frac{9}{7\sqrt{6}}.$$

$\Diamond$

Notice that the value of $\cos\theta$ does not uniquely define the value of the angle θ. It is conventional to define the angle between two vectors as an angle θ in the interval $[0, \pi]$, so that the value of $\cos\theta$ does uniquely define the value of θ.

Example 2. On choosing the angle in the interval $[0, \pi]$, the angle between the vectors $\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ of Example 1 is

$$\theta = \cos^{-1} \frac{9}{7\sqrt{6}} = 1.018\ldots \text{ radians}.$$

$\Diamond$

2.2.1 Arithmetic properties of the dot product

Some of the main properties of the dot product are summarised in the following proposition.

Proposition 2. For all vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and scalars $\lambda \in \mathbb{R}$,

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$, and hence $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$;
2. $\mathbf{a} \cdot \mathbf{b}$ is a scalar, i.e., is a real number;
3. **Commutative Law:** $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$;
4. $\mathbf{a} \cdot (\lambda \mathbf{b}) = \lambda(\mathbf{a} \cdot \mathbf{b})$;
5. **Distributive Law:** $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$;

Proof. Properties 1 and 2 follow immediately from the definitions of length and dot product.

The proof of the commutative law is as follows.

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= a_1 b_1 + \cdots + a_n b_n, \quad \text{and} \\ \mathbf{b} \cdot \mathbf{a} &= b_1 a_1 + \cdots + b_n a_n. \end{aligned}$$

But, $a_k b_k = b_k a_k$ (all a_k and b_k are real numbers), and hence the two expressions are equal.

The proof of Property 4 follows immediately on expanding the dot products on each side and using the properties of real numbers.

Finally, to prove the distributive law we note that

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= a_1(b_1 + c_1) + \cdots + a_n(b_n + c_n) \\ &= (a_1 b_1 + \cdots + a_n b_n) + (a_1 c_1 + \cdots + a_n c_n) \\ &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \end{aligned}$$

$\square$

2.2.2 Geometric interpretation of the dot product in $\mathbb{R}^n$

We have seen above that the dot product between two vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ has a geometric interpretation in terms of lengths and angles as

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\mathbf{a}|\ |\mathbf{b}| \cos\theta,$$

where $\theta \in [0, \pi]$ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Now, in $\mathbb{R}^n$, we have defined lengths of vectors (Definition 3 on page 18) and the dot product (Definition 1 of Section 2.2), but the idea of an angle has not been defined. It is reasonable to try to **define** an angle in $\mathbb{R}^n$ so that the geometric interpretation of the dot product in $\mathbb{R}^n$ is the same as the geometric interpretation of it in $\mathbb{R}^2$ or $\mathbb{R}^3$. We therefore define an angle in $\mathbb{R}^n$ as follows.

Definition 2. *If* $\mathbf{a}$, $\mathbf{b}$ *are non-zero vectors in* $\mathbb{R}^n$*, then the* ***angle*** θ *between* $\mathbf{a}$ *and* $\mathbf{b}$ *is given by*

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\ |\mathbf{b}|}, \qquad \text{where } \theta \in [0, \pi].$$

Example 3. Find the angle between $\mathbf{a} = \begin{pmatrix} -1 \\ 2 \\ -3 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -4 \\ 0 \\ -1 \end{pmatrix}$.

Solution. Using the definition of length, dot product and angle in $\mathbb{R}^n$, we have

$$|\mathbf{a}| = \sqrt{15}, \qquad |\mathbf{b}| = \sqrt{21}, \qquad \mathbf{a}\cdot\mathbf{b} = -2-8+0+1 = -9,$$

$$\cos\theta = -\frac{9}{\sqrt{(15)(21)}} = -\frac{3}{\sqrt{35}},$$

and hence, on choosing the angle between 0 and π, we have

$$\theta = \cos^{-1}\frac{-3}{\sqrt{35}} = 2.103\ldots \text{ radians.}$$

$\diamond$

Now, in Definition 2 of an angle in $\mathbb{R}^n$, we have assumed that the definition makes sense, i.e., that the equation

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\ |\mathbf{b}|}$$

can always be solved to obtain a real number as the value for the angle θ. But, since $-1 \leqslant \cos\theta \leqslant 1$ for real numbers, a real solution for θ is possible if and only if

$$-1 \leqslant \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\ |\mathbf{b}|} \leqslant 1.$$

A proof that this inequality is true for all non-zero vectors in $\mathbb{R}^n$ is given in the following important theorem.

Theorem 3 (The Cauchy-Schwarz Inequality)**.** *If* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, *then* $-|\mathbf{a}|\ |\mathbf{b}| \leqslant \mathbf{a}\cdot\mathbf{b} \leqslant |\mathbf{a}|\ |\mathbf{b}|$.

[**X**] *Proof.* Note first that the inequality is clearly true if either $\mathbf{a}$ or $\mathbf{b}$ is a zero vector.
For $\mathbf{b} \neq \mathbf{0}$, consider

$$q(\lambda) = |\mathbf{a} - \lambda\mathbf{b}|^2 \text{ for } \lambda \in \mathbb{R}.$$

Then, from the properties of lengths and dot products in $\mathbb{R}^n$, we have that $q \geqslant 0$ for all $\lambda \in \mathbb{R}$ and hence that

$$0 \leqslant q = (\mathbf{a} - \lambda\mathbf{b})\cdot(\mathbf{a} - \lambda\mathbf{b}) = |\mathbf{a}|^2 - 2\lambda\mathbf{a}\cdot\mathbf{b} + \lambda^2|\mathbf{b}|^2.$$

This $q(\lambda)$ is a quadratic function of λ which has a minimum at

$$\lambda = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}.$$

The minimum value is

$$q_0 = q\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right) = |\mathbf{a}|^2 - \frac{(\mathbf{a}\cdot\mathbf{b})^2}{|\mathbf{b}|^2}.$$

Now, as $q(\lambda) \geqslant 0$ for all λ, we have

$$0 \leqslant q_0 = |\mathbf{a}|^2 - \frac{(\mathbf{a}\cdot\mathbf{b})^2}{|\mathbf{b}|^2}.$$

Thus,

$$(\mathbf{a}\cdot\mathbf{b})^2 \leqslant |\mathbf{a}|^2|\mathbf{b}|^2,$$

and therefore

$$-|\mathbf{a}|\ |\mathbf{b}| \leqslant \mathbf{a}\cdot\mathbf{b} \leqslant |\mathbf{a}|\ |\mathbf{b}|,$$

and the proof is complete. ☐

Another useful inequality which follows immediately from the Cauchy-Schwarz inequality is the following inequality for lengths of vectors.

Theorem 4 (Minkowski's Inequality (or the Triangle Inequality))**.** *For* $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$*,*

$$|\mathbf{a}+\mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|.$$

Proof.

$$|\mathbf{a}+\mathbf{b}|^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = |\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2.$$

But, from the Cauchy-Schwarz inequality, $\mathbf{a}\cdot\mathbf{b} \leqslant |\mathbf{a}|\ |\mathbf{b}|$, and hence

$$|\mathbf{a}+\mathbf{b}|^2 \leqslant |\mathbf{a}|^2 + 2|\mathbf{a}|\ |\mathbf{b}| + |\mathbf{b}|^2 = \big(|\mathbf{a}|+|\mathbf{b}|\big)^2.$$

On taking positive square roots of both sides of this inequality, we then obtain the result to be proved. ☐

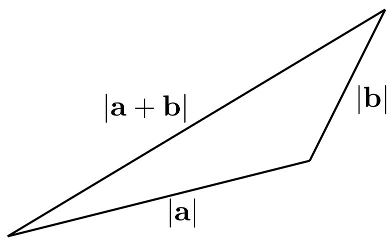


Figure 2: The Triangle Inequality.

As illustrated in Figure 2, the geometric interpretation of the triangle inequality is that the sum of two sides of a triangle is greater than or equal to the third side.

2.3 Applications: orthogonality and projection

The dot product has many applications. Two important applications are for testing if two vectors are at right angles and for projecting one vector on another.

2.3.1 Orthogonality of vectors

The dot product provides a simple test for vectors being at right angles to each other. We begin with the following definition.

Definition 1. *Two vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *in* $\mathbb{R}^n$ *are said to be* ***orthogonal*** *if* $\mathbf{a}\cdot\mathbf{b}=0$.

Note that the zero vector is orthogonal to every vector, including itself.

On using the formula for the dot product in terms of angles, we see that two vectors are orthogonal if

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\cos\theta = 0,$$

i.e., if either $|\mathbf{a}|$ or $|\mathbf{b}|$ or $\cos\theta$ is zero. As a vector is zero if its length is zero, and $\cos\theta = 0$ only when θ is a right angle, we have the result that vectors **a** and **b** are orthogonal if either vector is the zero vector or if the vectors are at right angles to each other.

NOTE. Two non-zero vectors at right angles to each other are also said to be **perpendicular** to each other or to be **normal** to each other.

By the definitions of length and orthogonality, the set of standard basis vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a set of vectors of unit length at right angles to each other. Sets of vectors with this property are of great practical importance, and they have been given a special name.

Definition 2. *An **orthonormal set of vectors** in $\mathbb{R}^n$ is a set of vectors which are of unit length and mutually orthogonal.*

The connection between the dot product and lengths and angles provides a simple test for a set of vectors to be an orthonormal set.

Example 1. The three standard basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ of $\mathbb{R}^3$ form an orthonormal set.

Proof. The three vectors are of unit length since

$$|\mathbf{e}_1|^2 = \mathbf{e}_1 \cdot \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1,$$

and similarly, $|\mathbf{e}_2|^2 = 1$ and $|\mathbf{e}_3|^2 = 1$.

The vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal since

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0.$$

Similarly, $\mathbf{e}_1$ and $\mathbf{e}_3$ are orthogonal since $\mathbf{e}_1 \cdot \mathbf{e}_3 = 0$. Finally, $\mathbf{e}_2$ and $\mathbf{e}_3$ are orthogonal since $\mathbf{e}_2 \cdot \mathbf{e}_3 = 0$.

Thus, the three vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are each of unit length and they are mutually orthogonal, and hence they form an orthonormal set. $\square$

Note that a compact form of writing the conditions for an orthonormal set in $\mathbb{R}^n$ are that, for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$,

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 0 & \text{for } i \neq j; \\ 1 & \text{for } i = j. \end{cases}$$

Example 2. Show that the three vectors $\mathbf{u}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$ form an orthonormal set. Find scalars $\lambda_1, \lambda_2, \lambda_3$ such that $\mathbf{e}_1 = \lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 + \lambda_3\mathbf{u}_3$.

SOLUTION. The three vectors are of unit length, since

$$\mathbf{u}_1 \cdot \mathbf{u}_1 = \left(\frac{1}{\sqrt{2}}\right)^2 + 0^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = 1,$$

and similarly $\mathbf{u}_2 \cdot \mathbf{u}_2 = 1$, $\mathbf{u}_3 \cdot \mathbf{u}_3 = 1$. They are mutually orthogonal since

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{3}}\right) + 0 \times \frac{1}{\sqrt{3}} + \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{3}}\right) = 0,$$

and similarly $\mathbf{u}_1 \cdot \mathbf{u}_3 = 0$, $\mathbf{u}_2 \cdot \mathbf{u}_3 = 0$. The three vectors therefore form an orthonormal set.

For writing $\mathbf{e}_1$ as a linear combination of the three vectors, again we use the dot product.

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{e}_1 &= \mathbf{u}_1 \cdot (\lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \lambda_3 \mathbf{u}_3) \\ \frac{1}{\sqrt{2}} &= \lambda_1(\mathbf{u}_1 \cdot \mathbf{u}_1) + \lambda_2(\mathbf{u}_1 \cdot \mathbf{u}_2) + \lambda_3(\mathbf{u}_1 \cdot \mathbf{u}_3), \end{aligned}$$

we have $\lambda_1 = \frac{1}{\sqrt{2}}$. Similarly,

$$\lambda_2 = \mathbf{u}_2 \cdot \mathbf{e}_1 = -\frac{1}{\sqrt{3}} \quad \text{and} \quad \lambda_3 = \mathbf{u}_3 \cdot \mathbf{e}_1 = \frac{1}{\sqrt{6}}.$$

In other words, we can write $\mathbf{e}_1$ as a linear combination of the three vectors:

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}}\mathbf{u}_1 - \frac{1}{\sqrt{3}}\mathbf{u}_2 + \frac{1}{\sqrt{6}}\mathbf{u}_3.$$

$\Diamond$

The method used in the above example can generally be used to write any vector in the span of an orthonormal set as a linear combination of vectors in this set.

Here is an application of orthogonality to the geometry of triangles.

Example 3. Show that the three altitudes of a triangle are concurrent, i.e., they intersect at a point.

SOLUTION. As shown in Figure 3, let the three vertices be A, B and C (coordinate vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$), and let D and E be the points at which the altitudes from A and B intersect the opposite sides of the triangle. Finally, let P (coordinate vector $\mathbf{p}$) be the point of intersection of the altitudes AD and BE.

Then, we can prove the result by showing that P is also on the altitude from C to AB.

Now, as P is on the altitude from A to the opposite side BC, we have that $\overrightarrow{AP}$ is perpendicular to $\overrightarrow{BC}$, and hence

$$(\mathbf{p}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{b})=0,$$

$$\text{i.e.,}\quad \mathbf{p}\cdot\mathbf{c}-\mathbf{p}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{c}+\mathbf{a}\cdot\mathbf{b}=0. \qquad (1)$$

Similarly, as P is on the altitude from B to the opposite side CA, we have

$$(\mathbf{p}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{c})=0,$$

$$\text{i.e.,}\quad \mathbf{p}\cdot\mathbf{a}-\mathbf{p}\cdot\mathbf{c}-\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{c}=0. \qquad (2)$$

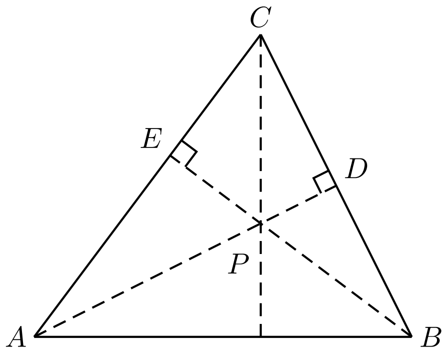


Figure 3: Altitudes of a Triangle.

Then, on adding (1) and (2) and grouping terms, we obtain

$$(\mathbf{p}-\mathbf{c})\cdot(\mathbf{a}-\mathbf{b})=0,$$

i.e., $\overrightarrow{CP}\cdot\overrightarrow{BA}=0$. Thus, $\overrightarrow{CP}$ is perpendicular to $\overrightarrow{BA}$ and hence P is on the altitude from C to the opposite side BA. The result is proved. $\Diamond$

2.3.2 Projections

The geometric idea of a projection of a vector $\mathbf{a}$ on a non-zero vector $\mathbf{b}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is shown in Figure 4, where $\overrightarrow{OP}$ is the projection.

Note that $\overrightarrow{OP}$ is parallel to $\mathbf{b}$ and $\overrightarrow{PA}=\mathbf{a}-\overrightarrow{OP}$ is perpendicular to $\mathbf{b}$. Note also that

$$|\overrightarrow{OP}|=|\mathbf{a}||\cos\theta|=\frac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{b}|},$$

because $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}|\ |\mathbf{b}|\cos\theta$. Since $\dfrac{\mathbf{b}}{|\mathbf{b}|}$ is a unit vector,

$$\begin{aligned}\overrightarrow{OP} &= |\overrightarrow{OP}|\frac{\mathbf{b}}{|\mathbf{b}|} \quad \text{when} \quad \theta \text{ is acute;}\\ \overrightarrow{OP} &= -|\overrightarrow{OP}|\frac{\mathbf{b}}{|\mathbf{b}|} \quad \text{when} \quad \theta \text{ is obtuse.}\end{aligned}$$

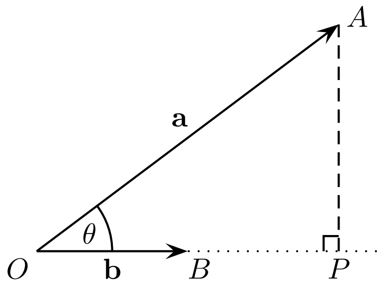


Figure 4: Projection of $\mathbf{a}$ on $\mathbf{b}$.

In both cases, $\overrightarrow{OP}=\left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}$. Thus we can formally define a projection in terms of the dot product as follows.

Definition 3. *For* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ *and* $\mathbf{b} \neq \mathbf{0}$, *the* ***projection of*** $\mathbf{a}$ ***on*** $\mathbf{b}$ *is*

$$proj_{\mathbf{b}}\mathbf{a} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}.$$

The geometric properties in $\mathbb{R}^2$ used to motivate this definition can be proved to be true for projections in $\mathbb{R}^n$ also. We have

Proposition 1. $\text{proj}_{\mathbf{b}}\mathbf{a}$ is the unique vector $\lambda\mathbf{b}$ parallel to the non-zero vector $\mathbf{b}$ such that

$$(\mathbf{a} - \lambda\mathbf{b}) \cdot \mathbf{b} = 0. \qquad (\#)$$

Proof. From (#) we have

$$0 = (\mathbf{a} - \lambda\mathbf{b}) \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} - \lambda|\mathbf{b}|^2.$$

For $\mathbf{b} \neq \mathbf{0}$, this equation can be solved for λ to obtain the unique solution

$$\lambda = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2},$$

and hence the unique solution of (#) is

$$\lambda\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b} = \text{proj}_{\mathbf{b}}\,\mathbf{a}.$$

The proof is complete. □

Alternative forms of writing the formula for a projection are sometimes useful. These are

$$\text{proj}_{\mathbf{b}}\mathbf{a} = (\mathbf{a} \cdot \widehat{\mathbf{b}})\widehat{\mathbf{b}} = |\mathbf{a}|\cos\theta\,\widehat{\mathbf{b}},$$

where $\widehat{\mathbf{b}} = \dfrac{\mathbf{b}}{|\mathbf{b}|}$ is the unit vector in the direction of $\mathbf{b}$ and where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

Example 4. Find the projections of a vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ on the three standard basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.

Solution. $\mathbf{a} \cdot \mathbf{e}_1 = a_1$, $|\mathbf{e}_1|^2 = \mathbf{e}_1 \cdot \mathbf{e}_1 = 1$, and hence

$$\text{proj}_{\mathbf{e}_1}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{e}_1}{|\mathbf{e}_1|^2}\,\mathbf{e}_1 = a_1\mathbf{e}_1.$$

Similarly, $a_2\mathbf{e}_2$ is the projection of $\mathbf{a}$ on $\mathbf{e}_2$ and $a_3\mathbf{e}_3$ is the projection of $\mathbf{a}$ on $\mathbf{e}_3$. ◊

Example 5. Find the projection of $\mathbf{a} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ on $\mathbf{b} = \begin{pmatrix} -4 \\ 1 \\ 5 \end{pmatrix}$.

SOLUTION. $\mathbf{a}\cdot\mathbf{b} = 3$, $|\mathbf{b}|^2 = 42$, and hence $\text{proj}_{\mathbf{b}}\mathbf{a} = \dfrac{1}{14}\begin{pmatrix}-4\\1\\5\end{pmatrix}$. ◊

Note that a simple formula for the length of the projection of $\mathbf{a}$ on $\mathbf{b}$ is

$$|\text{proj}_{\mathbf{b}}\mathbf{a}| = |\mathbf{a}\cdot\widehat{\mathbf{b}}| = \frac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{b}|}.$$

Example 6. Find the length of the projection of $\begin{pmatrix}3\\5\\1\end{pmatrix}$ on $\begin{pmatrix}-3\\1\\-4\end{pmatrix}$.

SOLUTION. As $\left|\begin{pmatrix}3\\5\\1\end{pmatrix}\cdot\begin{pmatrix}-3\\1\\-4\end{pmatrix}\right| = 8$, and as $\left|\begin{pmatrix}-3\\1\\-4\end{pmatrix}\right| = \sqrt{26}$, the length of the projection is $\dfrac{8}{\sqrt{26}}$.
◊

2.3.3 Distance between a point and a line in $\mathbb{R}^3$

In $\mathbb{R}^2$ and $\mathbb{R}^3$, the distance between a point B and a line $\mathbf{x} = \mathbf{a} + \lambda\mathbf{d}$ is the shortest distance between the point and the line. In the diagram, the distance is $|\overrightarrow{PB}|$, where P is the point on the line such that $\angle APB$ is a right angle (for a general proof of this result, see Question 13 in Problems for Chapter 2). We can easily find $|\overrightarrow{AB}|$ from the coordinates of A and B, and we can find $|\overrightarrow{AP}|$ as it is the length of the projection of $\overrightarrow{AB}$ on the direction $\mathbf{d}$. Thus,

$$\left|\overrightarrow{AB}\right| = |\mathbf{b}-\mathbf{a}|, \qquad \left|\overrightarrow{AP}\right| = \frac{\left|\overrightarrow{AB}\cdot\mathbf{d}\right|}{|\mathbf{d}|},$$

$$\text{and} \qquad \left|\overrightarrow{PB}\right| = \sqrt{|\overrightarrow{AB}|^2 - |\overrightarrow{AP}|^2}.$$

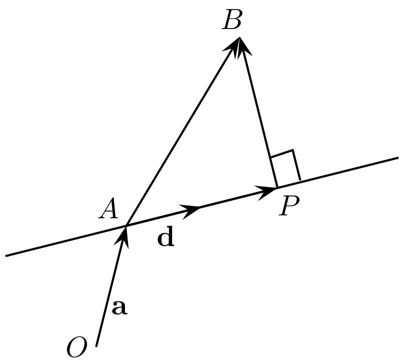


Figure 5: Shortest Distance between Point and Line.

Note that $\overrightarrow{AB}$ is the line segment joining some point A on the line to the given point B, while $\overrightarrow{AP}$ is the projection of this line segment on the direction of the line.

An alternative method of solving this problem is to use the fact that

$$\overrightarrow{PB} = \overrightarrow{AB} - \overrightarrow{AP} = \mathbf{b} - \mathbf{a} - \text{proj}_{\mathbf{d}}(\mathbf{b}-\mathbf{a}),$$

and then the shortest distance is $|\overrightarrow{PB}|$.

Example 7. Find the distance from the point $(2,-1,3)$ to the line through the points $(0,1,4)$ and $(4,2,9)$.

Solution. Let A, B and C be the points $(0,1,4)$, $(2,-1,3)$ and $(4,2,9)$, respectively. Suppose that P is the foot of the perpendicular from B to the line AC. Referring to Figure 5, the length of the projection of $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$ on the direction $\mathbf{d} = \overrightarrow{AC} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$ is

$$\left|\overrightarrow{AP}\right| = \frac{|\overrightarrow{AB} \cdot \mathbf{d}|}{|\mathbf{d}|} = \frac{|8-2-5|}{\sqrt{16+1+25}} = \frac{1}{\sqrt{42}}.$$

Together with $|\overrightarrow{AB}| = 3$, the distance from the point to the line is

$$\left|\overrightarrow{BP}\right| = \sqrt{3^2 - \frac{1}{42}} = \frac{\sqrt{377}}{\sqrt{42}}.$$

◊

2.4 The cross product

We now define the cross product of two vectors in three dimensions. One motivation for a cross product is to find a formula which gives a vector which is perpendicular to two other vectors.

Now, a vector $\mathbf{x}$ will be perpendicular to two vectors $\mathbf{a}$ and $\mathbf{b}$ if and only if the dot products $\mathbf{a} \cdot \mathbf{x}$ and $\mathbf{b} \cdot \mathbf{x}$ are both zero. Using coordinates, we can write

$$\begin{array}{ccccccc} a_1x_1 & + & a_2x_2 & + & a_3x_3 & = & 0 \\ b_1x_1 & + & b_2x_2 & + & b_3x_3 & = & 0 \end{array}$$

This pair of equations can be easily solved in the usual way to obtain a solution which can be written in the form

$$\mathbf{x} = \lambda \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix},$$

where λ is a real parameter. This expression for $\mathbf{x}$ looks like some kind of "product" of $\mathbf{a}$ and $\mathbf{b}$.

Definition 1. *The **cross product** of two vectors* $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ *and* $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ *in* $\mathbb{R}^3$ *is*

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

Note that the cross product of two vectors is a **vector**. For this reason the cross product is often called the **vector product** of two vectors, in contrast to the dot product which is a scalar and is often called the **scalar product** of two vectors. Note also that the cross product $\mathbf{a} \times \mathbf{b}$ has the important property that it is perpendicular to the two vectors **a** and **b**. As an exercise you might like to check directly that $\mathbf{a} \times \mathbf{b}$ is orthogonal to **a** by checking that $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.

There are several tricks available for remembering the formula for a cross product. The most common trick is to use determinant notation. The more general theory of determinants will be covered in a later chapter. We define here a 2×2 determinant by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

To find the cross product of two vectors $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ we write these as rows in a 3×3 determinant:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

To calculate this, we use the following procedure. Firstly, take the vector $\mathbf{e}_1$ and multiply it by the 2×2 determinant obtained by deleting the row and column in which $\mathbf{e}_1$ is contained. That is, we write

$$\mathbf{e}_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}.$$

Then take $-\mathbf{e}_2$ and repeat the process, followed by $\mathbf{e}_3$. Each of the 2×2 determinants can be found using the definition above.

We write

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \mathbf{e}_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{e}_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{e}_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\
&= \mathbf{e}_1(a_2b_3 - a_3b_2) - \mathbf{e}_2(a_1b_3 - a_3b_1) + \mathbf{e}_3(a_1b_2 - a_2b_1) \\
&= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}(a_2b_3 - a_3b_2) - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}(a_1b_3 - a_3b_1) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}(a_1b_2 - a_2b_1) \\
&= \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.
\end{aligned}$$

The determinant is expanded along the first row and as usual $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the standard basis vectors of $\mathbb{R}^3$. While this may appear complicated at first, with practice it is much easier than using the formula for the cross product.

Example 1. Find the cross product of $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix}$.

SOLUTION. Using the determinant formula, we have

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & -2 & 3 \\ 4 & -5 & 6 \end{vmatrix} = \mathbf{e}_1(-12+15) - \mathbf{e}_2(6-12) + \mathbf{e}_3(-5+8) = \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}.$$

◊

2.4.1 Arithmetic properties of the cross product

Some of the basic properties of the cross product are summarised in the following proposition.

Proposition 1. For all $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$,

1. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$, i.e., the cross product of a vector with itself is the zero vector.
2. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$. The cross product is **not commutative**. If the order of vectors in the cross product is reversed, then the sign of the product is also reversed.
3. $\mathbf{a} \times (\lambda \mathbf{b}) = \lambda(\mathbf{a} \times \mathbf{b})$ and $(\lambda \mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b})$.
4. $\mathbf{a} \times (\lambda \mathbf{a}) = \mathbf{0}$, i.e., the cross product of parallel vectors is zero.
5. **Distributive Laws**. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ and $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$.

Proof. Each of the properties listed in Proposition 1 can be proved by expanding each side of the properties using the definition of cross product given in Definition 1. For example, for Property 2, we have on expansion that

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} = -\begin{pmatrix} b_2a_3 - b_3a_2 \\ b_3a_1 - b_1a_3 \\ b_1a_2 - b_2a_1 \end{pmatrix} = -\mathbf{b} \times \mathbf{a}.$$

□

There are several useful relations for the cross products of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in $\mathbb{R}^3$.

Proposition 2. The three standard basis vectors in $\mathbb{R}^3$ satisfy the relations

1. $\mathbf{e}_1 \times \mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_3 = \mathbf{0}$,
2. $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1, \quad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$.

Proof. The proof of these relations follows immediately from Property 1 of Proposition 1 and from Definition 1. For example,

$$\mathbf{e}_1 \times \mathbf{e}_2 = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{e}_3.$$

□

NOTE. The cross product is **not** associative. In fact,

$$\begin{aligned} & (\mathbf{e}_1 \times \mathbf{e}_2) \times \mathbf{e}_2 = -\mathbf{e}_1, \\ \text{but} \qquad & \mathbf{e}_1 \times (\mathbf{e}_2 \times \mathbf{e}_2) = \mathbf{0}. \end{aligned}$$

Proposition 3. Suppose A, B are points in $\mathbb{R}^3$ that have coordinate vectors $\mathbf{a}$ and $\mathbf{b}$, and $\angle AOB = \theta$ then $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\ |\mathbf{b}| \sin\theta$.

Proof. If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ then both sides are 0 although θ is not defined.
Also, since $0 \leqslant \theta \leqslant \pi,\ \sin\theta \geqslant 0$ we need only prove

$$|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2\ |\mathbf{b}|^2 \sin^2\theta$$

$$\text{or} \quad |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2\ |\mathbf{b}|^2\ (1 - \cos^2\theta) = |\mathbf{a}|^2\ |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \quad \text{by Section 2.2.}$$

To prove this identity we expand both sides

$$\begin{aligned} \text{L.H.S.} &= \left| \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \right|^2 \\ &= a_2^2b_3^2 + a_3^2b_2^2 - 2a_2a_3b_2b_3 + a_3^2b_1^2 + a_1^2b_3^2 - 2a_3a_1b_3b_1 + a_1^2b_2^2 + a_2^2b_1^2 - 2a_1a_2b_1b_2 \end{aligned}$$

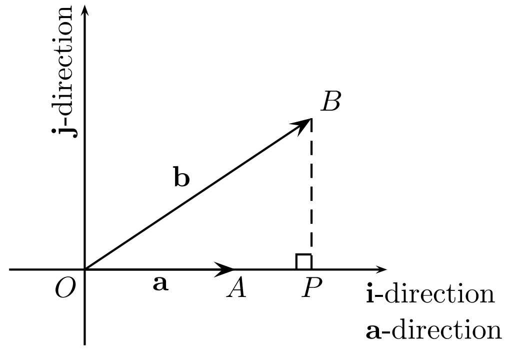
$$\begin{aligned} \text{R.H.S.} &= (a_1^2 + a_2^2 + a_3^2)\,(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\ &= a_1^2b_1^2 + a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_2^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 + a_3^2b_3^2 \\ &\quad - a_1^2b_1^2 - a_2^2b_2^2 - a_3^2b_3^2 - 2a_1a_2b_1b_2 - 2a_1a_3b_1b_3 - 2a_2a_3b_2b_3 \\ &= a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 - 2a_2a_3b_2b_3 - 2a_3a_1b_3b_1 - 2a_1a_2b_1b_2 \end{aligned}$$

as required. $\square$

2.4.2 A geometric interpretation of the cross product

We have constructed a cross product to be perpendicular to two given vectors. In this subsection we explore the geometric properties further.

Choose an orthonormal set of basis vectors so that $\mathbf{i}$ is in the direction of $\mathbf{a}$, and so that $\mathbf{i}$ and $\mathbf{j}$ are in the plane of $\mathbf{a}$ and $\mathbf{b}$, with $\mathbf{b} = b_1\mathbf{i} + b_2\,\mathbf{j},\ b_1 > 0$. Finally, $\mathbf{k}$ is taken at right angles to the plane of $\mathbf{a}$ and $\mathbf{b}$ with its direction determined by a "right-hand rule": using your right hand, point to $\mathbf{a}$ with your index finger and to $\mathbf{b}$ with your middle finger, then your thumb can be extended in the direction of $\mathbf{k}$. If we take the direction of $\mathbf{k}$ as normal to the page and pointing outwards, then the picture for the $\mathbf{i}, \mathbf{j}$ plane is as shown in Figure 6.



$\mathbf{a} = \overrightarrow{OA} = a_1\mathbf{i}$
$\mathbf{b} = \overrightarrow{OP} + \overrightarrow{PB} = b_1\mathbf{i} + b_2\mathbf{j}$
$\mathbf{a} \times \mathbf{b} = a_1 b_2 \mathbf{k}$

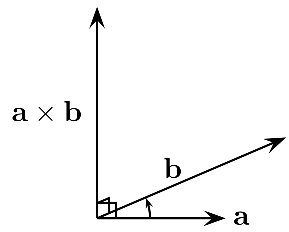


Figure 6: Geometry of $\mathbf{a} \times \mathbf{b}$.

Hence, we have

$$\mathbf{a} = \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ a_1 b_2 \end{pmatrix},$$

and we also have that the coordinates a_1 and b_2 are positive.

Now, since a_1 and b_2 are positive, the cross product is in the direction of $\mathbf{k}$, i.e., it is in a direction perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ as given by **the right-hand rule**.

Let θ denote the angle between $\mathbf{a}$ and $\mathbf{b}$. Note first that the angle θ is always chosen to be in the interval $[0, \pi]$ so that $\sin\theta \geqslant 0$. Also, the length of the vector $\mathbf{a} \times \mathbf{b}$ is

$$|\mathbf{a} \times \mathbf{b}| = a_1 b_2.$$

Now $a_1 = |\mathbf{a}|$, and from trigonometry, $b_2 = |\mathbf{b}| \sin\theta$, and hence $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\ |\mathbf{b}| \sin\theta$, in agreement with Proposition 3.

We have therefore shown that:

$\mathbf{a} \times \mathbf{b}$ *is a vector of length* $|\mathbf{a}|\ |\mathbf{b}| \sin\theta$ *in the direction perpendicular to both* $\mathbf{a}$ and $\mathbf{b}$ *as given by the right-hand rule.*

This statement is usually taken as the definition of the cross product in physics and engineering courses.

NOTE. The above proof is valid since if $\mathbf{a}, \mathbf{b}$ are arbitrary in $\mathbb{R}^3$ then we can apply a rotation to move $\mathbf{a}$ to $\alpha\mathbf{i}$. We then rotate about the x-axis to move $\mathbf{b}$ to $\beta\mathbf{i} + \gamma\mathbf{j}$ with $\gamma > 0$. Since rotations preserve lengths and angle sizes the results

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\ |\mathbf{b}| \sin\theta$$

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0,\ (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0 \quad \text{and}$$

$$(\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}) \quad \text{form a right-hand triple}$$

holds for arbitrary vectors $\mathbf{a}, \mathbf{b}$ in $\mathbb{R}^3$.

2.4.3 Areas

The length of the cross product $\mathbf{a} \times \mathbf{b}$ also has an alternative geometric interpretation in terms of the area of a parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$. Consider the picture of Figure 7.

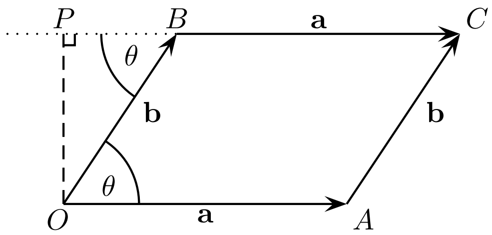


Figure 7: The Cross Product and the Area of a Parallelogram.

The area of the parallelogram $OACB$ in Figure 7 is "base times perpendicular height".

$$\text{Area} = |\overrightarrow{OA}||\overrightarrow{OP}| = |\mathbf{a}|\ |\mathbf{b}| \sin\theta = |\mathbf{a} \times \mathbf{b}|.$$

The vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, so it is normal to the plane of the parallelogram.

Example 2. Find the area of a parallelogram with vertices at points A $(1, 0, 1)$, B $(-2, 1, 3)$, and C $(3, 1, 4)$.

SOLUTION. One parallelogram can be formed with sides $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$. The cross product is $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 1 \\ 13 \\ -5 \end{pmatrix}$, and hence the area is $\left|\begin{pmatrix} 1 \\ 13 \\ -5 \end{pmatrix}\right| = \sqrt{195}$. $\diamond$

NOTE. There are three parallelograms which can be formed from three vertices. The areas of all three are equal. As an exercise you might like to find the other two parallelograms with the same three vertices and check that they have the same area.

[X] 2.4.4 Shortest distance between lines in $\mathbb{R}^3$

It is necessary to consider the case of parallel lines and the case of skew (non-parallel) lines separately. Be careful, non-parallel lines in $\mathbb{R}^3$ may not intersect.

Parallel Lines. The distance between two parallel lines is the same as the distance between a point on one of the lines and the other line. We only need to use the method in 2.3.3 to find the distance between a point on one line and the other line.

Skew Lines. The distance, i.e. the shortest distance, between two skew lines is obtained by drawing a perpendicular to both lines. The direction of the perpendicular is in the direction of the cross product of the directions of the lines. The shortest distance is the length of the projection on this perpendicular of a line segment joining any point on one line to any point on the other.

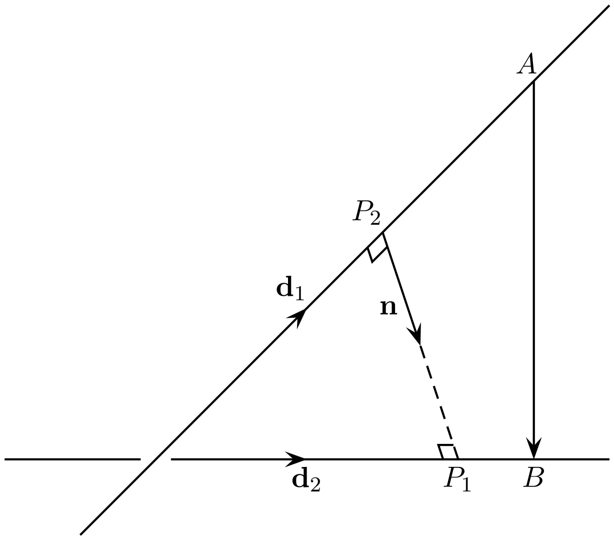


Figure 8: Shortest Distance between Skew Lines.

Example 3. Find the shortest distance between the lines

$$\mathbf{x} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{x} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}.$$

Solution. The cross product of the direction vectors of the lines is $\mathbf{n} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 \\ 1 \\ 10 \end{pmatrix}$, and hence $\mathbf{n}$ is perpendicular to both lines. A line segment from a point on one line to a point on the other is $\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

The shortest distance is the length of the projection of the line segment $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ on the perpendicular $\begin{pmatrix} -7 \\ 1 \\ 10 \end{pmatrix}$ and is given by, $\dfrac{\left|\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -7 \\ 1 \\ 10 \end{pmatrix}\right|}{\left|\begin{pmatrix} -7 \\ 1 \\ 10 \end{pmatrix}\right|} = \dfrac{4}{\sqrt{150}}$. ◊

2.5 Scalar triple product and volume

There are two products that can be constructed from three vectors in three dimensions. They are

Definition 1. *For* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$, *the* ***scalar triple product*** *of* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *is* $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

Definition 2. *For* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$, *the* ***vector triple product*** *of* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *is* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

In this section we shall briefly examine the properties of the scalar triple product and give it a geometric interpretation as a volume. Although the vector triple product is useful in physics and engineering, we shall not consider it any further in this mathematics course.

It is important to notice that in evaluating the scalar triple product, the **cross product must be calculated before the dot product**. The expression $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$ has no meaning, because $\mathbf{a} \cdot \mathbf{b}$ is a scalar and the cross product of a scalar and a vector has no meaning.

Some properties of the scalar triple product are listed in the following proposition.

Proposition 1. For $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$,

1. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, that is, the dot and cross can be interchanged.

2. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$, that is, the sign is reversed if the order of two vectors is reversed.

3. $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{a}) \cdot \mathbf{b} = 0$, that is, the scalar triple product is zero if any two vectors are the same.

4. The scalar triple product can be written using the determinant notation.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

This means that we replace $\mathbf{i}$ by a_1, $\mathbf{j}$ by a_2 and $\mathbf{k}$ by a_3 in the determinant form of the cross product.

Proof. The proof of Property 1 follows immediately on using the definitions of the dot and cross product to expand the two expressions. We have

$$\begin{aligned}\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - c_1b_2 \end{pmatrix} \\ &= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3c_1b_2.\end{aligned}$$

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} &= \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\ &= a_2b_3c_1 - a_3b_2c_1 + a_3b_1c_2 - a_1b_3c_2 + a_1b_2c_3 - a_2b_1c_3.\end{aligned}$$

The two fully expanded expressions are equal, and hence the result is proved.

Property 2 is an immediate consequence of the fact that $\mathbf{b} \times \mathbf{c} = -\mathbf{c} \times \mathbf{b}$.

Property 3 follows immediately from Property 1 and from the fact that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for all $\mathbf{a}$.

Property 4 can be proved by expanding both the scalar triple product and the determinant and noting that the expansions of the two are equal. □

NOTE. Property 3 gives one more proof that the cross product $\mathbf{a} \times \mathbf{b}$ is perpendicular to $\mathbf{a}$. Clearly, $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$, and hence $\mathbf{b}$ is also orthogonal to $\mathbf{a} \times \mathbf{b}$.

2.5.1 Volumes of parallelepipeds

A parallelepiped is a three-dimensional analogue of a parallelogram. Given three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ we can form a parallelepiped as shown in Figure 9.

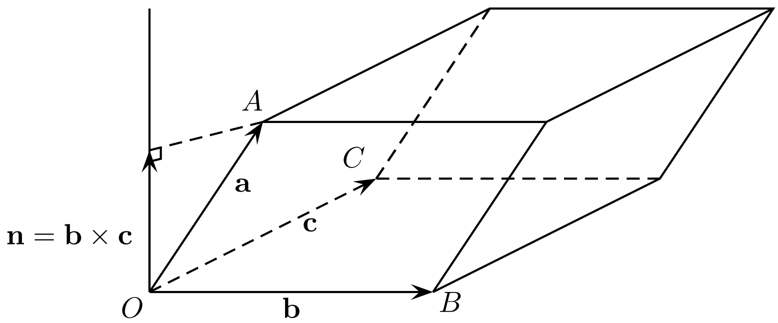


Figure 9: The Scalar Triple Product and the Volume of a Parallelepiped.

The parallelepiped formed from the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is called the **parallelepiped spanned by a, b, and c**.

From geometry, the volume of a parallelepiped is "area of base times perpendicular height". The base is the parallelogram whose sides are the vectors $\mathbf{b}$ and $\mathbf{c}$, and hence from the results of Section 2.4.2, the area of the base is $|\mathbf{b} \times \mathbf{c}|$ and the direction of the perpendicular to the base is the direction of $\mathbf{n} = \mathbf{b} \times \mathbf{c}$. From the results of Section 2.3.2, the length of the projection of the vector $\mathbf{a}$ on the perpendicular $\mathbf{n}$ is the perpendicular height, and hence

$$\text{Perpendicular height} = \frac{|\mathbf{a} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{b} \times \mathbf{c}|} = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{\text{Area of base}}.$$

Hence, the volume of the parallelepiped is given by the formula

$$\text{Volume} = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.$$

Example 1. Find the volume of the parallelepiped spanned by $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix}$, and $\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$.

SOLUTION. Using the determinant formula for the scalar triple product, we have

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \left(\begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} \right) = \begin{vmatrix} 1 & 2 & 3 \\ -2 & 4 & -1 \\ 3 & 5 & 1 \end{vmatrix} = 1(4+5) - 2(-2+3) + 3(-10-12) = -59,$$

and hence the volume is $|-59| = 59$. $\Diamond$

Notice that if the volume of the parallelepiped spanned by three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is zero, i.e., if $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0$, then the three vectors are in the same plane (that is, they are coplanar).

2.6 Planes in $\mathbb{R}^3$

Another useful application of vectors and dot and cross products is to the geometry of planes in three dimensions. There are three common forms for the equation of a plane in $\mathbb{R}^3$. These are "parametric vector form", "Cartesian form", and "point-normal form". Each of these equations has a direct geometric interpretation. In this section we shall discuss the geometric interpretation of each of these forms and we shall show how to convert one form to another. We shall also show how to find the distance between a point and a plane in $\mathbb{R}^3$.

2.6.1 Equations of planes in $\mathbb{R}^3$

Parametric Vector Form. In Section 1.5, we showed that the equation of a plane through a given point with position vector $\mathbf{c}$ and parallel to two given non-parallel vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ could be written in a parametric vector form

$$\mathbf{x} = \mathbf{c} + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 \text{ for } \lambda_1, \lambda_2 \in \mathbb{R}.$$

Cartesian Form. In Section 1.5, we have shown that the linear equation in three unknowns,

$$a_1x_1 + a_2x_2 + a_3x_3 = b,$$

represents a plane in $\mathbb{R}^3$. This linear equation is also often called the **Cartesian form** of the equation of a plane in $\mathbb{R}^3$. We have already shown that this Cartesian form can be converted to a parametric vector form by solving the linear equation.

It is important to note that a single linear equation $a_1x_1+\cdots+a_nx_n = b$ is the equation of a plane only if $n = 3$. For $n = 2$, the equation is $a_1x_1+a_2x_2 = b$ which represents a line. In general, solving a single linear equation with n unknowns yields a parametric vector form of solution containing $n-1$ parameters, whereas a parametric vector equation of a plane must contain 2 parameters.

The Cartesian form can be given a geometric interpretation in terms of intercepts on the three coordinate axes.

We first divide the Cartesian form by the right hand side and rearrange it as

$$\frac{x_1}{d_1} + \frac{x_2}{d_2} + \frac{x_3}{d_3} = 1,$$

where $\quad d_1 = \dfrac{b}{a_1}, \quad d_2 = \dfrac{b}{a_2}, \quad d_3 = \dfrac{b}{a_3}.$

Then, to obtain the intercept on the x_1 axis, we set $x_2 = x_3 = 0$ in the equation, and find $x_1 = d_1$. By a similar argument, the intercept on the x_2 axis is d_2 and the intercept on the x_3 axis is d_3.

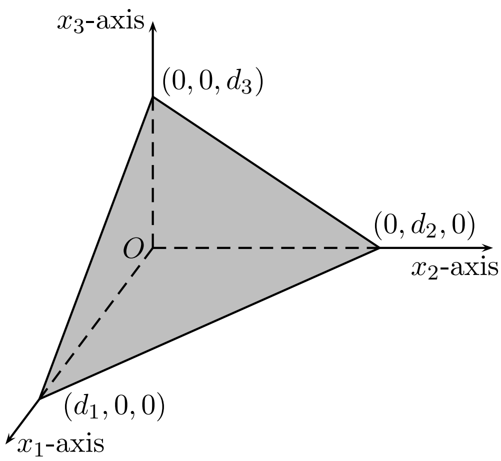


Figure 10: $a_1x_1 + a_2x_2 + a_3x_3 = b$.

The rule for obtaining intercepts is therefore to rewrite the equation with 1 on the right, and then the intercepts are the reciprocals of the coefficients of the variables. Notice that if the coefficient of x_1 is zero, then the plane is parallel to the x_1 axis. A similar result applies if the coefficient of x_2 or x_3 is zero.

Example 1. Write down the equation of a plane which has the intercepts $4, 7, -2$ on the three coordinate axes.

SOLUTION. The coefficients of the variables are the reciprocals of the intercepts, and hence the equation is

$$\frac{x_1}{4} + \frac{x_2}{7} + \frac{x_3}{-2} = 1, \qquad \text{or} \qquad 14x_1 + 8x_2 - 28x_3 = 56.$$

$\Diamond$

Point-Normal Form. A point-normal form is an equation of the form $\mathbf{n} \cdot (\mathbf{x} - \mathbf{c}) = 0$, where $\mathbf{n}$ and $\mathbf{c}$ are fixed vectors and $\mathbf{x}$ is the position vector of a point in space. If we rewrite this equation as $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{c}$ and then expand in terms of coordinates, we obtain

$$n_1x_1 + n_2x_2 + n_3x_3 = n_1c_1 + n_2c_2 + n_3c_3 = b,$$

which is a Cartesian equation of a plane. Thus, the equation $\mathbf{n} \cdot (\mathbf{x} - \mathbf{c}) = 0$ is the equation of a plane. As for a Cartesian form, a point-normal equation is an equation of a plane for three-dimensional vectors only.

The name "point-normal form" for the equation $\mathbf{n} \cdot (\mathbf{x} - \mathbf{c}) = 0$ is based on the following geometric interpretation (see Figure 11). Clearly, $\mathbf{x} = \mathbf{c}$ is a solution of the equation, and hence $\mathbf{c}$ is the position vector of a point on the plane. If $\mathbf{x}$ is any other point in the plane, then the line segment $\mathbf{x} - \mathbf{c}$ lies in the plane. The equation then says that the vector $\mathbf{n}$ is normal to all line segments lying in the plane. The vector $\mathbf{n}$ is called a normal to the plane. A point-normal form is therefore the most convenient form of equation to use when a point on the plane and a normal to the plane are known.

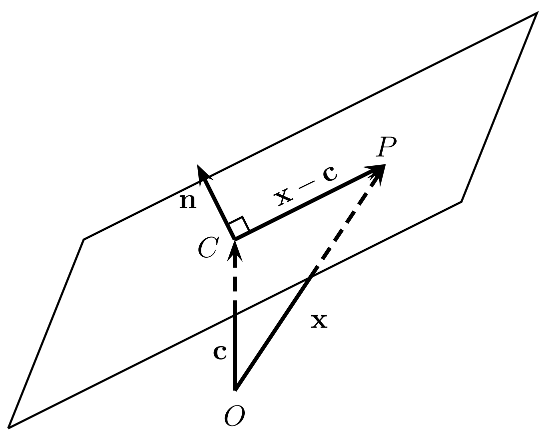


Figure 11: Point-Normal Form $\mathbf{n} \cdot (\mathbf{x} - \mathbf{c}) = 0$.

Example 2. Find a point-normal form for the plane which passes through the point $(1, -2, 3)$ and has $\begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$ as a normal vector.

SOLUTION. $$\begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix} \cdot \left(\mathbf{x} - \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \right) = 0. \qquad \Diamond$$

We shall now show how to convert from one form of equation to another by giving examples of the methods.

Example 3 (Conversion from point-normal to Cartesian form)**.** Find an equation, in Cartesian form, of the plane which passes through the point $(2, 0, 4)$ and whose normal is $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$.

SOLUTION. A point-normal form is $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} \cdot \left(\mathbf{x} - \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \right) = 0$, and hence a corresponding Cartesian form is $2x_1 - 3x_2 + 5x_3 = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} = 24.$ $\Diamond$

Example 4 (Conversion from Cartesian to point-normal form)**.** Find a point-normal form of for the plane described by the Cartesian equation

$$3x_1 - 7x_2 + 5x_3 = 21.$$

SOLUTION. A comparison of Cartesian and point-normal forms shows that the coefficients of x_1, x_2, x_3 are just the coordinates of a normal to the plane. Thus, a normal to the plane is $\mathbf{n} = \begin{pmatrix} 3 \\ -7 \\ 5 \end{pmatrix}$. To find some point on the plane, we let $x_2 = x_3 = 0$, and then from the equation we find $x_1 = \dfrac{21}{3} = 7$. Hence $(7, 0, 0)$ is a point on the plane.

A point-normal form is therefore $\begin{pmatrix} 3 \\ -7 \\ 5 \end{pmatrix} \cdot \left(\mathbf{x} - \begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix} \right) = 0.$ $\Diamond$

Example 5 (Conversion from parametric vector to point-normal form)**.** Find a point-normal form for the place described by the parametric vector equation

$$\mathbf{x} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 6 \\ 1 \end{pmatrix}.$$

Solution. The plane is parallel to $\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 6 \\ 1 \end{pmatrix}$. The cross product of these two vectors is therefore normal to the plane. Thus, a normal to the plane is $\mathbf{n} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} \times \begin{pmatrix} -2 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} -21 \\ -8 \\ 6 \end{pmatrix}$. As $(3, -5, 1)$ is a point on the plane, a point-normal form is $\begin{pmatrix} -21 \\ -8 \\ 6 \end{pmatrix} \cdot \left(\mathbf{x} - \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} \right) = 0.$ ♢

Example 6 (Conversion from parametric vector to Cartesian form)**.** Find an equation, in Cartesian form, for the plane through the point $(2, -1, 3)$ parallel to the vectors $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$.

Solution. The simplest way to carry out this conversion is to first find a point-normal form as in Example 5, and then to find a Cartesian form as in Example 3.

A normal to the plane is $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -9 \\ -7 \\ 5 \end{pmatrix}$.

A point-normal form is $\begin{pmatrix} -9 \\ -7 \\ 5 \end{pmatrix} \cdot \left(\mathbf{x} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right) = 0.$

Hence a Cartesian form is $-9x_1 - 7x_2 + 5x_3 = 4$. ♢

Example 7 (Conversion from point-normal to parametric vector form)**.** Find a parametric vector form for the plane which is normal to $\begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}$ and passes through the point $(0, 2, 1)$.

Solution. One way to carry out this conversion is to first convert the point-normal we were given into a Cartesian form as in Example 3, and then convert the Cartesian form we found into a parametric vector form as in section 1.5.3.

A point-normal form is $\begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix} \cdot \left(\mathbf{x} - \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right) = 0$. Hence, a Cartesian form is

$$-2x_1 + 4x_3 = \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = 4.$$

The variable x_2 does not appear in the equation, but don't forget it. As in section 1.5.3, we set $x_2 = \lambda_1$ to be a real parameter. Also, x_3 can have any value in the equation, and hence we set $x_3 = \lambda_2$ to be a second real parameter. Then, we obtain $x_1 = -2 + 2x_3 = -2 + 2\lambda_2$. Then, on rewriting the solution in vector form, we obtain a parametric vector form of the equation of the plane to be

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 + 2\lambda_2 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

◊

Example 8. Find a point-normal form for the plane that passes through the three points $A\,(3,1,2)$, $B\,(0,-2,1)$, and $C\,(1,2,3)$.

SOLUTION. The plane is parallel to the two line segments

$$\overrightarrow{AB} = \begin{pmatrix} -3 \\ -3 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \overrightarrow{AC} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

Hence a normal to the plane is

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -3 \\ -3 \\ -1 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ -9 \end{pmatrix}.$$

The point $A\,(3,1,2)$ is a point on the plane, and hence a point-normal form is

$$\begin{pmatrix} -2 \\ 5 \\ -9 \end{pmatrix} \cdot \left(\mathbf{x} - \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \right) = 0.$$

◊

NOTE. Although we have not stated it explicitly, most of the solutions given to the examples in this section are not unique. For example, any multiple of a normal to a plane is still a normal, any multiple of a vector parallel to a plane is still a vector parallel to the plane, there are an infinite number of points on a plane etc. However, the intercepts of a plane on the coordinate axes are unique.

2.6.2 Distance between a point and a plane in $\mathbb{R}^3$

The distance from the point B to the plane is $\left|\overrightarrow{PB}\right|$, where $\overrightarrow{PB}$ is normal to the plane. If A is any point on the plane, $\overrightarrow{PB}$ is the projection of $\overrightarrow{AB}$ on any vector normal to the plane. The shortest distance is the "perpendicular distance" to the plane, and it is the length of the projection on a normal to the plane of a line segment from any point on the plane to the given point.

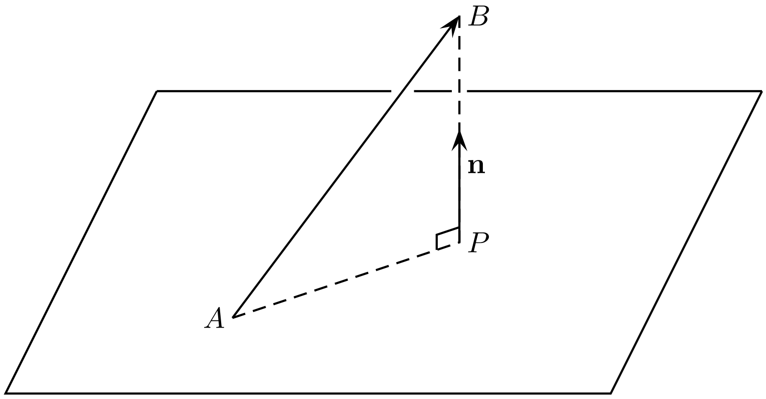


Figure 12: Shortest Distance between Point and Plane.

Example 9. Find the distance between the point $(2,-1,3)$ and the plane

$$\mathbf{x}=\begin{pmatrix}0\\4\\2\end{pmatrix}+\lambda_1\begin{pmatrix}4\\1\\2\end{pmatrix}+\lambda_2\begin{pmatrix}-3\\1\\5\end{pmatrix}.$$

SOLUTION. $(0,4,2)$ is a point on the plane and $\mathbf{n}=\begin{pmatrix}4\\1\\2\end{pmatrix}\times\begin{pmatrix}-3\\1\\5\end{pmatrix}=\begin{pmatrix}3\\-26\\7\end{pmatrix}$ is a vector normal to the plane. A vector from the point $(0,4,2)$ on the plane to the given point $(2,-1,3)$ is $\begin{pmatrix}2\\-1\\3\end{pmatrix}-\begin{pmatrix}0\\4\\2\end{pmatrix}=\begin{pmatrix}2\\-5\\1\end{pmatrix}$.

The shortest distance is the length of the projection of $\begin{pmatrix}2\\-5\\1\end{pmatrix}$ on $\mathbf{n}$, which is

$$\frac{\left|\begin{pmatrix}2\\-5\\1\end{pmatrix}\cdot\mathbf{n}\right|}{|\mathbf{n}|}=\frac{143}{\sqrt{734}}.$$

◊

2.7 Geometry and Maple

Rather than list all the commands that you might need in this chapter, we suggest that you have a look through the commands available in the `LinearAlgebra` package using the on-line manual. The command

```
?LinearAlgebra;
```

opens a help page with a list of the commands options available in the linear algebra package. If you think that `CrossProduct` looks promising, you should click on its hyperlink, or type in:

```
?CrossProduct
```

in the Maple worksheet. The on-line help usually gives a few examples. Try these first then you can experiment with a few problems of your own.

The package `geom3d` contains many useful procedures for solving problems in three-dimensional geometry. In fact, every one of the problems for Section 5.7 can be solved with procedures in `geom3d`. The following is an outline of some of the things which you can do with `geom3d`.

The method of assigning a name to a point, line or plane is not quite what you might expect. To say that A is the point $(1, 2, 3)$, you do NOT enter `A:=[1,2,3];`, you enter `point(A,[1,2,3]);`. The command `line` is used to assign a name to a line. The line may be specified by giving two points on it or a point on it and a direction parallel to it. The command `plane` is used to assign a name to a plane. The plane may be specified by giving a Cartesian equation for it or three (non-collinear) points on it or a point on it and a normal direction or a point on it and two lines parallel to it. The command `sphere` is used to assign a name to a sphere. A sphere may be specified by giving its Cartesian equation or four points on it or the end-points of a diameter or its center and its radius. To display the specifications of one of these things you need to use `detail` (see the example below). For a plane, the detail includes a Cartesian equation for the plane. (If you want to find a normal to a plane p use

`NormalVector(p);`)

Be warned that if you specify a plane or sphere by means of an equation then Maple will want you to specify the names of the variables which are associated with the three axes. You can do this by listing them as a third argument to the `plane` or `sphere` command, as in

```
plane(P,x+y+z=1,[x,y,z]);
```

If you leave out the `[x,y,z]` then Maple will, rather strangely, prompt you to `enter the name of the x-axis`, to which you reply `x;`, and similarly for the other two axes.

When you have set up objects of these types you can, for example, use the command `distance` to find the distance between two of them or the command `intersection` to find the intersection of two of them (except the intersection of a line and a sphere) or the command `FindAngle` to find the angle between two of them. You can use the Maple `help` to find out more about any of these commands and to find out about the many other commands available in `geom3d`.

In the following example, we first label the points $A(0, 1, 2)$ and $B(2, 3, 1)$ and the line AB through A and B. Applying `detail` to AB shows that the direction of the line is $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ and the line can be expressed in parametric vector form as

$$\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Then we assign the label P to the plane through $C(4, 5, 6)$ with normal $(1, 1, 1)$ and use `detail` to find that P can be described by the Cartesian equation

$$x + y + z = 15.$$

Then we assign the label X to the point of intersection of the line AB and the plane P and find that the coordinates of X are $(8, 9, -2)$. Finally, we find that the plane ABC through the three points A, B, C can be described by the Cartesian equation

$$12x - 12y + 12 = 0.$$

Notice that we use a colon : to suppress the output of most of the commands because the output would just be an echo of assigned names.

```
with(geom3d):
point(A,[0,1,2]),point(B,[2,3,1]):
line(AB,[A,B]):
detail(AB);
   Warning, assume that the name of the parameter in the parametric equations is _t
   Warning, assuming that the names of the axes are _x, _y, and _z
   name of the object:  AB
   form of the object:  line3d
   equation of the line:  [_x = 2*_t, _y = 1+2*_t, _z = 2-_t]
point(C,[4,5,6]):
plane(P,[C,[1,1,1]]):
detail(P);
   Warning, assuming that the names of the axes are _x, _y and _z
   name of the object:  P
   form of the object:  plane3d
   equation of the plane:  -15+_x+_y+_z = 0
intersection(X,AB,P):
detail(X);
   name of the object:  X
   form of the object:  point3d
   coordinates of the point:  [8, 9, -2]
plane(ABC,[A,B,C]):
detail(ABC);
   Warning, assuming that the names of the axes are _x, _y and _z
   name of the object:  ABC
   form of the object:  plane3d
   equation of the plane:  12+12*_x-12*_y = 0
```

Problems for Chapter 2

Questions marked with [**R**] are routine, [**H**] harder, [**M**] Maple and [**X**] are for MATH1141 only. You should make sure that you can do the easier questions before you tackle the more difficult questions. Questions marks with a [**V**] have video solutions available on Moodle.

Problems 2.2: The dot product

1. [**R**][**V**] Find the angles between the following pairs of vectors:

$$\text{a)}\ \begin{pmatrix}-2\\2\\0\end{pmatrix}, \begin{pmatrix}0\\3\\0\end{pmatrix};\quad \text{b)}\ \begin{pmatrix}1\\0\\3\end{pmatrix}, \begin{pmatrix}-2\\5\\1\end{pmatrix};\quad \text{c)}\ \begin{pmatrix}7\\1\\-2\end{pmatrix}, \begin{pmatrix}3\\-11\\5\end{pmatrix};\quad \text{d)}\ \begin{pmatrix}3\\0\\1\\4\end{pmatrix}, \begin{pmatrix}-2\\6\\1\\3\end{pmatrix}.$$

2. [**H**] Find the cosines of the internal angles of the triangles whose vertices have the following coordinate vectors:

$$\text{a)}\ A\begin{pmatrix}4\\0\\2\end{pmatrix}, B\begin{pmatrix}6\\2\\1\end{pmatrix} \text{ and } C\begin{pmatrix}5\\1\\6\end{pmatrix};\qquad \text{b)}\ A\begin{pmatrix}0\\2\\1\end{pmatrix}, B\begin{pmatrix}-1\\3\\0\end{pmatrix} \text{ and } C\begin{pmatrix}3\\1\\2\end{pmatrix};$$

$$\text{c)}\ A\begin{pmatrix}1\\-2\\0\\3\end{pmatrix}, B\begin{pmatrix}0\\4\\-2\\5\end{pmatrix} \text{ and } C\begin{pmatrix}-2\\1\\0\\3\end{pmatrix}.$$

3. [**R**] A cube has vertices at the 8 points $O\,(0,0,0)$, $A\,(1,0,0)$, $B\,(1,1,0)$, $C\,(0,1,0)$, $D\,(0,0,1)$, $E\,(1,0,1)$, $F\,(1,1,1)$, $G\,(0,1,1)$. Sketch the cube, and then find the angle between the diagonals $\overrightarrow{OF}$ and $\overrightarrow{AG}$.

4. [**H**][**V**] Prove the following properties of dot products for vectors $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{R}^3$. :

   a) $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$, b) $\mathbf{a}\cdot(\lambda\mathbf{b})=\lambda(\mathbf{a}\cdot\mathbf{b})$, c) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$.

5. [**X**] Prove that $\big|\,|\mathbf{a}|-|\mathbf{b}|\,\big|\leqslant|\mathbf{a}-\mathbf{b}|$ for all $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$.

   HINT. See the proof of Minkowski's inequality in Section 2.2.2.

6. [**H**] Use the dot product to prove that the diagonals of a square intersect at right angles.

Problems 2.3 : Applications: orthogonality and projection

7. [**R**] Let $\mathbf{u}_1=\begin{pmatrix}\frac{1}{\sqrt2}\\0\\-\frac{1}{\sqrt2}\end{pmatrix}$, $\mathbf{u}_2=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\mathbf{u}_3=\begin{pmatrix}\frac{1}{\sqrt2}\\0\\\frac{1}{\sqrt2}\end{pmatrix}$ and $\mathbf{a}=\begin{pmatrix}2\\-3\\1\end{pmatrix}$. Show that the set of vectors $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ is an orthonormal set. Find scalars $\lambda_1,\lambda_2,\lambda_3$ such that $\mathbf{a}=\lambda_1\mathbf{u}_1+$

$\lambda_2\mathbf{u}_2 + \lambda_3\mathbf{u}_3$.

HINT. See Examples 2 of Section 2.3.

8. **[H]** Consider the triangle ABC in $\mathbb{R}^3$ formed by the points $A(3,2,1)$, $B(4,4,2)$ and $C(6,1,0)$.

   a) Find the coordinates of the midpoint M of the side BC.
   b) Find the angle BAC.
   c) Find the area of the triangle ABC.
   d) Find the coordinates of the point D on BC such that AD is perpendicular to BC.

9. **[R][V]** Find the following projections:

   a) the projection of $\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$ on $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$,

   b) the projection of $\begin{pmatrix} 2 \\ -1 \\ 2 \\ 4 \end{pmatrix}$ on $\begin{pmatrix} -1 \\ 3 \\ 0 \\ 2 \end{pmatrix}$,

   c) the projection of $\begin{pmatrix} -2 \\ 2 \\ 7 \end{pmatrix}$ on the direction of the line $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$.

10. **[R]** Find the shortest distances between

    a) the point $(-2,1,5)$ and the line $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ 3 \\ -4 \end{pmatrix}$;

    b) the point $(0,3,8)$ and the line $\dfrac{x_1 - 1}{1} = \dfrac{x_2 - 2}{-1} = \dfrac{x_3 - 3}{4}$;

    c) **[X]** the point $(11,2,-1)$ and the line of intersection of the planes

$$\mathbf{x} \cdot \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = 0 \qquad \text{and} \qquad \mathbf{x} = \lambda_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix}.$$

11. **[X]** A point P in $\mathbb{R}^n$ has coordinate vector $\mathbf{p}$. Find the coordinate vector of the point Q which is the reflection of P in the line ℓ which passes through the point $\mathbf{a}$ parallel to the direction $\mathbf{d}$.

    NOTE. Define Q to be the point which lies in the same plane as P and ℓ with ℓ bisecting the interval PQ.

12. **[X]** Fix $\mathbf{a},\ \mathbf{b} \in \mathbb{R}^n$ with $\mathbf{b} \neq \mathbf{0}$. Let $q(\lambda) = |\mathbf{a} - \lambda\mathbf{b}|^2$.

    a) Show $q(\lambda)$ is a minimum when $\lambda = \lambda_0 = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}$.

b) Determine $q(\lambda_0)$ and hence show that $-|\mathbf{a}|\ |\mathbf{b}| \leqslant \mathbf{a}\cdot\mathbf{b} \leqslant |\mathbf{a}|\ |\mathbf{b}|$.

13. [**X**] Let B be a point in $\mathbb{R}^n$ with coordinate vector $\mathbf{b}$. Let $\mathbf{x} = \mathbf{a} + \lambda\mathbf{d},\quad \lambda \in \mathbb{R}$ be the equation of a line. Do the following:

a) Show that the square of the distance from B to an arbitrary point $\mathbf{x}$ on the line is given by

$$q(\lambda) = |\mathbf{b}-\mathbf{a}|^2 - 2\lambda(\mathbf{b}-\mathbf{a})\cdot\mathbf{d} + \lambda^2|\mathbf{d}|^2.$$

b) Find the shortest distance between the point B and the line by minimising $q(\lambda)$.

c) If P is the point on the line closest to B, show that

$$\overrightarrow{PB} = \mathbf{b} - \mathbf{a} - \operatorname{proj}_{\mathbf{d}}(\mathbf{b}-\mathbf{a}),$$

and show that $\overrightarrow{PB}$ is orthogonal to the direction $\mathbf{d}$ of the line.
NOTE. This problem proves that the shortest distance between a point and a line is obtained by "dropping a perpendicular from the point to the line".

Problems 2.4 : The cross product

14. [**R**] Find the cross product $\mathbf{a}\times\mathbf{b}$ of the following pairs of vectors:

a) $\mathbf{a} = \begin{pmatrix} 0 \\ 2 \\ -4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, b) $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 6 \\ 1 \end{pmatrix}$,

c) $\mathbf{a} = \begin{pmatrix} 1 \\ 9 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix}$.

15. [**R**][**V**] Find a vector which is perpendicular to $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}$.

16. [**H**] Prove the following properties of cross products for vectors $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbb{R}^3$:

a) $\mathbf{a}\times\mathbf{a} = \mathbf{0}$; b) $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$;
c) $\mathbf{a}\times(\lambda\mathbf{b}) = \lambda(\mathbf{a}\times\mathbf{b})$; d) $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$.

17. [**R**] Find the areas of, and the normals to the planes of, the following parallelograms:

a) the parallelogram spanned by $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$;

b) a parallelogram which has vertices at the three points $A\,(0,2,1)$, $B\,(-1,3,0)$ and $C\,(3,1,2)$ and sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

18. **[R][V]** Find the areas of the triangles with the following vertices:

    a) $A\,(0,2,1)$, $B\,(-1,3,0)$ and $C\,(3,1,2)$;

    b) $A\,(2,2,0)$, $B\,(-1,0,2)$ and $C\,(0,4,3)$.

19. **[R]** Let D, E, F be the points with coordinate vectors

$$\mathbf{d} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}, \ \mathbf{e} = \begin{pmatrix} 6 \\ 7 \\ 8 \end{pmatrix}, \ \mathbf{f} = \begin{pmatrix} 7 \\ 8 \\ 10 \end{pmatrix}$$

    a) Calculate $\cos(\angle DEF)$ as a surd.

    b) Calculate the area of ΔDEF as a surd.

20. **[X]** Find the shortest distances between

    a) the line through $(1,2,3)$ parallel to $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ and the line through $(0,2,5)$ parallel to $\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$;

    b) the line through the points $(1,3,1)$ and $(1,5,-1)$ and the line through the points $(0,2,1)$ and $(1,2,-3)$;

    c) the lines $\mathbf{x} = \begin{pmatrix} 2 \\ 7 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ -5 \end{pmatrix}$ and $\dfrac{x_1 - 1}{-10} = \dfrac{x_2 - 2}{1} = \dfrac{x_3 - 3}{4}$.

21. **[X]** Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors in $\mathbb{R}^3$ which satisfy the relations $\mathbf{b} = \mathbf{c} \times \mathbf{a}$ and $\mathbf{c} = \mathbf{a} \times \mathbf{b}$.

    a) Show that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are a set of mutually orthogonal vectors.

    b) Show that $\mathbf{b}$ and $\mathbf{c}$ are of equal length and that if $\mathbf{b} \neq \mathbf{0}$, then $\mathbf{a}$ is a unit vector (i.e. a vector of length 1).

22. **[X]** A tetrahedron has vertices A, B, C and D with coordinate vectors for the points being $\mathbf{a} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$. Find parametric vector equations for the two altitudes of the tetrahedron which pass through the vertices A and B, and determine whether the two altitudes intersect or not.

    NOTE. An altitude of a tetrahedron through a vertex is a line through the vertex and perpendicular to the opposite face.

23. **[X]** Points A, B, C and D have coordinate vectors $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$, respectively.

a) Find a parametric vector equations of the line through A and B and the line through C and D.

b) Find the shortest distance between the lines AB and CD.

c) Find the point P on AB and point Q on CD such that PQ is the shortest distance between the lines AB and CD.

Problems 2.5 : Scalar triple product and volume

24. **[R][V]** Show that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ can be written in the form

$$\left| \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \right|,$$

where the vectors $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ are replaced by the scalars a_1, a_2, a_3.

25. **[R]** Find the volumes of the following parallelepipeds:

a) the parallelepiped spanned by $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$;

b) a parallelepiped which has vertices at the four points $A\,(2, 1, 3)$, $B\,(-2, 1, 4)$, $C\,(0, 4, 1)$ and $D\,(3, -1, 0)$, with sides $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$.

26. **[R]** Show that the four points A, B, C, O with coordinate vectors $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ are coplanar.

Problems 2.6 : Planes in $\mathbb{R}^3$

27. **[R][V]** Find parametric vector, point-normal, and Cartesian forms for the following planes:

a) the plane through $(1, 2, -2)$ perpendicular to $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$;

b) the plane through $(1, 2, -2)$ parallel to $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$;

c) the plane through the three points $(1, 2, -2)$, $(-1, 1, 2)$ and $(2, 3, 1)$;

d) the plane with intercepts -1, 2 and -4 on the x_1, x_2 and x_3 axes;

e) [**X**] the plane through $(1,2,-2)$ which is parallel to $\begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$ and the line of intersection of the planes

$$\mathbf{x} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0 \qquad \text{and} \qquad \mathbf{x} = \lambda_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

28. [**R**] Consider four points O, A, B, C in $\mathbb{R}^3$ with coordinate vectors

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \ \mathbf{c} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}.$$

Let Π be the plane through A and parallel to the lines OB and OC.

a) Find a parametric vector form for Π.
b) Find a vector $\mathbf{n}$ normal to Π.
c) Use the point normal form to find a Cartesian equation for Π.

29. [**R**] Find the following projections:

a) the projection of $\begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix}$ on the normal to the plane $2x_1 + 2x_2 + x_3 = 4$;

b) [**X**] the projection of $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ on the line of intersection of the planes

$$\mathbf{x} \cdot \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = 0 \qquad \text{and} \qquad \mathbf{x} = \lambda_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix}.$$

30. [**R**][**V**] Find the shortest distances between

a) the point $(2,6,-5)$ and the plane $\left(\mathbf{x} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right) \cdot \begin{pmatrix} -2 \\ 4 \\ 4 \end{pmatrix} = 0$;
b) the point $(1,4,1)$ and the plane $2x_1 - x_2 + x_3 = 5$;
c) the point $(1,2,1)$ and the plane with intercepts at $3, -1, 2$ on the three axes;
d) the origin and the plane through the three points $(2,1,3)$, $(5,3,1)$ and $(5,1,2)$.

31. [**R**] Let P be the plane in $\mathbb{R}^3$ through the points $A = (1,2,0)$, $B = (0,1,2)$, and $C = (-1,3,1)$.

a) Find a parametric vector form for the plane P.
b) Find a vector $\mathbf{n}$ normal to the plane P.

c) Find a point normal form for the plane P.

d) Find the shortest distance from the point $Q = (2, 4, 5)$ to the plane P.

32. [**X**] a) Let $\mathbf{a}$ and $\mathbf{v}$ be two non-zero vectors in $\mathbb{R}^3$. Show how to write $\mathbf{v}$ as $\mathbf{c} + \mathbf{d}$ where $\mathbf{c}$ is parallel to $\mathbf{a}$ and $\mathbf{d}$ is perpendicular to $\mathbf{a}$.

b) Consider the plane

$$\Pi = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \ \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$

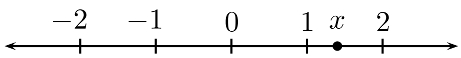
and the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. By using a) (or otherwise) express $\mathbf{v}$ as $\mathbf{c} + \mathbf{d}$ where $\mathbf{d}$ is parallel to Π and $\mathbf{c}$ is perpendicular to Π. (We call $\mathbf{d}$ the projection of $\mathbf{v}$ onto Π).

Chapter 3

COMPLEX NUMBERS

"Ignorance of Axioms", the Lecturer continued,
"is a great drawback in life. It wastes so much time
to have to say them over and over again."
Lewis Carroll, Sylvie and Bruno Concluded.

The main purpose of this chapter is to introduce the system of complex numbers. In the calculus part of this subject we concentrate on the set $\mathbb{R}$ of real numbers which we think of as corresponding to the points on the number line.



x is real. $x \in \mathbb{R}$.

If one wants to solve all quadratic equations then the real numbers do not suffice. In particular, if $b^2 - 4ac < 0$ then it is not possible to find real solutions to $ax^2 + bx + c = 0$. In this chapter, we will construct a larger set of numbers in which such an equation can be solved. Indeed, in this larger set, every polynomial equation has at least one solution. Much of mathematics becomes simpler (not more complicated) when complex numbers are used and applications to areas such as physics, chemistry, electrical and mechanical engineering, oceanography, economics and the theory of dynamical systems are made simpler by using complex numbers.

3.1 A review of number systems

Before going into the details of the complex number system, it is worth while reviewing some of the basic properties of the number systems which you have already met in primary and high school mathematics. This review is not intended as a rigorous development of the theory of number systems – such a rigorous development is quite difficult and is sometimes given in more advanced courses.

The first system of numbers that you will have met consists of the set of **natural numbers** (or counting numbers)

$$\mathbb{N} = \{0, 1, 2, \ldots\},$$

together with rules of addition and multiplication. This set of numbers has the property that addition or multiplication of natural numbers always produces another natural number, whereas subtraction or division may not. Thus $3-5$ and $\frac{3}{5}$ are not natural numbers. We say that the set of natural numbers is **closed** under the operations of addition and multiplication, whereas the set is not closed under the operations of subtraction or division.

A very limited class of equations have solutions in $\mathbb{N}$. For example, in $\mathbb{N}$, $x+3=7$ and $5x+2=17$ can be solved, but $x+7=3$ and $5x+2=18$ can not! Thus, to solve all linear equations, a larger set of numbers is required.

A set of numbers that is closed under subtraction (i.e., for which subtraction is always possible) can be obtained by **extending** the natural number system by introducing a new number (-1). Then, after using the usual rules for addition and multiplication, the set of **integers**

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

is obtained.

However, the set $\mathbb{Z}$ is not closed under division as, for example, $\frac{3}{5}$ is not an integer. At this stage $x+7=3$ has a unique solution but $5x+2=18$ still has no solution. To solve such an equation, we need fractions. Thus, we extend the system of integers to the set of **rational numbers**, $\mathbb{Q}$, defined by

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z} \ \text{ for } \ q \neq 0 \right\}.$$

This system is then closed under the four standard operations of arithmetic of addition, subtraction, multiplication and division (division by zero excluded).

Now that the rationals are in the set that we are focusing on, all equations in one variable with rational coefficients, such as $5x+2=18$ or $\frac{3}{2}x+\frac{1}{4}=1$ can be solved. All solutions will be rational although sometimes a solution may happen to be an integer. Indeed, if we now consider the general equation in one variable $ax+b=c$ with $a, b, c \in \mathbb{Q}$ then this has a unique solution $x=(c-b)/a$ unless $a=0$.

The rationals are the first and primary example of a mathematical concept called a **field**. (See definition 1.) A field is a set (of numbers) which satisfy "twelve number laws". These laws, or axioms as they are called, form a minimal list of properties that one needs in order to be able to add, subtract, multiply and divide (by non-zero numbers).

Much elementary mathematics can be carried out using rational numbers, as is done by your calculator. The set of real numbers, $\mathbb{R}$, contains all the rationals, along with numbers such as $\sqrt{2}, \sqrt{3}, \pi, e$, etc. which are **not** rational. The proofs that $\sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots$ are irrational are very straightforward. The proofs that π and e are irrational are harder. You will see a proof of the irrationality of e later in the year. The last question in the HSC Extension 2 2003 paper outlines a proof of the irrationality of π.

Using real numbers, we can now solve some quadratic and higher degree equations. For example, $x^2-3=0$ has 2 real solutions $x=\pm\sqrt{3}$ and $x^3-x^2-3x+3=0$ has 3 real solutions $x=1, \ \pm\sqrt{3}$. The set of real numbers also satisfies all twelve number laws and thus forms a field. Hence all equations $ax+b=c$ with $a, b, c \in \mathbb{R}, \ a \neq 0$, have a unique solution.

Note that the general motivation behind the development of number systems sketched above is that some set of numbers is extended to a new set by introducing new "numbers" so that some operation (e.g., subtraction, division, or finding lengths of sides of squares) is always possible in the extended set.

We conclude this section by introducing the definition of a field.

Definition 1. *Let $\mathbb{F}$ be a non-empty set of elements for which a rule of addition $(+)$ and a rule of multiplication are defined. Then the system is a* **field** *if the following twelve axioms (or fundamental number laws) are satisfied.*

1. **Closure under Addition**. If $x, y \in \mathbb{F}$ then $x + y \in \mathbb{F}$.
2. **Associative Law of Addition**. $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{F}$.
3. **Commutative Law of Addition**. $x + y = y + x$ for all $x, y \in \mathbb{F}$.
4. **Existence of a Zero**. There exists an element of $\mathbb{F}$ (usually written as 0) such that $0 + x = x + 0 = x$ for all $x \in \mathbb{F}$.
5. **Existence of a Negative**. For each $x \in \mathbb{F}$, there exists an element $w \in \mathbb{F}$ (usually written as $-x$) such that $x + w = w + x = 0$.
6. **Closure under Multiplication**. If $x, y \in \mathbb{F}$ then $xy \in \mathbb{F}$.
7. **Associative Law of Multiplication**. $x(yz) = (xy)z$ for all $x, y, z \in \mathbb{F}$.
8. **Commutative Law of Multiplication**. $xy = yx$ for all $x, y \in \mathbb{F}$.
9. **Existence of a One**. There exists a non-zero element of $\mathbb{F}$ (usually written as 1) such that $x1 = 1x = x$ for all $x \in \mathbb{F}$.
10. **Existence of an Inverse for Multiplication**. For each non-zero $x \in \mathbb{F}$, there exists an element w of $\mathbb{F}$ (usually written as $1/x$ or x^{-1}) such that $xw = wx = 1$.
11. **Distributive Law**. $x(y + z) = xy + xz$ for all $x, y, z \in \mathbb{F}$.
12. **Distributive Law**. $(x + y)z = xz + yz$, for all $x, y, z \in \mathbb{F}$.

3.2 Introduction to complex numbers

We cannot find a real number x such that $x^2 + 1 = 0$, since the square of real number is never negative. In the sixteenth century, mathematicians wrote down formal solutions to quadratic equations such as $x^2 + 1 = 0$ in the form $x = \pm\sqrt{-1}$. If this object $\sqrt{-1}$ were treated just like an ordinary number, but with the property that $\sqrt{-1}\sqrt{-1} = -1$, then indeed these values of x satisfy the above equation. The problem was that no-one really had any idea what writing down $\sqrt{-1}$ meant! No real-life problems that were being treated at the time ever had meaningful solutions that involved these "imaginary" numbers.

In the eighteenth and nineteenth centuries however, these imaginary numbers became more and more useful. Leonhard Euler introduced the now standard notation i for $\sqrt{-1}$ and showed that the use of these numbers allows one to express deep relationships between the trigonometric functions and the exponential function.

Let us begin with the general quadratic equation $ax^2 + bx + c = 0,\ a \neq 0$, then by multiplying both sides of the equation by $4a$ and completing the square, we have

$$\begin{aligned} 4a^2x^2 + 4abx + 4ac &= 0 \\ (2ax+b)^2 &= b^2 - 4ac \end{aligned}$$

Hence, if $\Delta = b^2 - 4ac \geqslant 0$

$$\begin{aligned} 2ax + b &= \pm\sqrt{\Delta} \\ \text{or} \quad x &= \frac{-b \pm \sqrt{\Delta}}{2a} \end{aligned}$$

and the equation has been solved.

If $\Delta = b^2 - 4ac < 0$, we can express the solutions using the complex number i. For example, if $x^2 = -4$, then we can write

$$x^2 = \pm\sqrt{-4} = \pm 2\sqrt{-1} = \pm 2i.$$

So, returning to our quadratic equation $ax^2 + bx + c = 0$ with $\Delta = b^2 - 4ac < 0$, we write $\Delta = -(4ac - b^2)$

$$\text{thus} \quad \sqrt{\Delta} = \pm i\sqrt{4ac - b^2} \quad \text{and}$$

$$x = \frac{-b \pm i\sqrt{4ac - b^2}}{2a} \quad \text{are our 2 solutions.}$$

We now define the set $\mathbb{C}$ of **complex numbers** by:

$$\mathbb{C} = \{a + bi \,|\, a, b \in \mathbb{R},\ i^2 = -1\}.$$

A complex number written in the form $a + bi$, where $a,\ b \in \mathbb{R}$, is said to be in **Cartesian form**. The real number a is called the *real part* of $a + bi$, and b is called the *imaginary part*. The set $\mathbb{C}$ contains all the real numbers (when $b = 0$). Numbers of the form bi, with b real ($b \neq 0$), are called **purely imaginary numbers**. The set of complex numbers also satisfies the twelve number laws, and so it also forms a field.

Example 1. Some examples of complex numbers in Cartesian form are

$$3 + 4i, \quad 2 - i = 2 + (-1)i, \quad -5i = 0 + (-5)i, \quad 6 = 6 + 0i \quad \text{and} \quad \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}.$$

$\Diamond$

NOTE. The symbol $\Diamond$ indicates the end of an example.

3.3 The rules of arithmetic for complex numbers

Let $z = a + bi$ and $w = c + di$, where $a, b, c, d \in \mathbb{R}$.

Addition and subtraction. We define the sum, $z + w$, by

$$z + w = (a + c) + (b + d)i.$$

and the difference, $z - w$, by

$$z - w = (a - c) + (b - d)i.$$

That is, we add or subtract the real parts and the imaginary parts separately.

Example 1. If $z = 6 + 5i$ and $w = -3 + 4i$ then $z + w = 3 + 9i$ and $z - w = 9 + i$. $\Diamond$

Multiplication. Expanding out, we have

$$(a + bi)(c + di) = ac + bci + adi + (bi)(di) = ac + (bc + ad)i + (bd)i^2.$$

Since $i^2 = -1$, this can be simplified to $(ac - bd) + (bc + ad)i$. Hence we define the product, zw, by

$$zw = (ac - bd) + (bc + ad)i.$$

Example 2. If $z = 6 + 5i$ and $w = -3 + 4i$ then

$$zw = (6 + 5i)(-3 + 4i) = -18 - 20 - 15i + 24i = -38 + 9i.$$

$\Diamond$

(It is wise to multiply out the terms producing real numbers first and then purely imaginary numbers second.)

Division. To divide two complex numbers we use a similar process to that used in "rationalising the denominator". For example,

$$\frac{1}{2 + \sqrt{3}} = \frac{(2 - \sqrt{3})}{(2 + \sqrt{3})(2 - \sqrt{3})} = \frac{2 - \sqrt{3}}{4 - 3} = 2 - \sqrt{3},$$

rationalises the denominator by multiplying both numerator and denominator by the denominator with the sign of $\sqrt{3}$ changed.

A similar process can be applied when dividing complex numbers:

$$\frac{z}{w} = \frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac + bd + (bc - ad)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\, i.$$

Thus we define the quotient $\frac{z}{w}$, $(w \neq 0)$ by

$$\frac{z}{w} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\, i.$$

Note that the quotient of two complex numbers is also a complex number.

Example 3. If $z = 3 - 4i$ and $w = 1 + 2i$ then

$$\frac{z}{w} = \frac{3 - 4i}{1 + 2i} = \frac{(3 - 4i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{-5 - 10i}{5} = -1 + 2i.$$

$\Diamond$

NOTE. The formula for division by $w = c + id$ given above fails if and only if $c^2 + d^2 = 0$. But, since c and d are real, $c^2 + d^2 = 0$ if and only if $c = 0$ and $d = 0$, that is, if and only if $w = 0$. Thus the formula for complex division fails if and only if the denominator is 0.

Before concluding this section on complex number arithmetic, we should point out that we now have three examples of fields; namely, the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$.

Proposition 1. [**X**] *The following properties hold for addition of complex numbers:*

1. **Uniqueness of Zero.** There is one and only one zero in $\mathbb{C}$.
2. **Cancellation Property.** If $z, v, w \in \mathbb{C}$ satisfy $z + v = z + w$, then $v = w$.

Proposition 2. [**X**] *The following properties hold for multiplication of complex numbers:*

1. $0z = 0$ for all complex numbers z.
2. $(-1)z = -z$ for all complex numbers z.
3. **Cancellation Property**. If $z, v, w \in \mathbb{C}$ satisfy $zv = zw$ and $z \neq 0$, then $v = w$.
4. If $z, w \in \mathbb{C}$ satisfy $zw = 0$, then either $z = 0$ or $w = 0$ or both.

One final point should be made. So far we have stressed the similarities between real number arithmetic and complex number arithmetic. However, there are also many important differences. One important difference is that while it makes sense to say that a real number is positive or that one real number is greater than (or less than) another, it does **not** make sense to say that a complex number is positive or that one complex number is greater than (or less than) another. That is, complex numbers cannot be *ordered*.

3.4 Real parts, imaginary parts and complex conjugates

Definition 1. *The **real part** of $z = a + bi$ (written $\mathrm{Re}(z)$), where $a, b \in \mathbb{R}$, is given by*

$$\mathrm{Re}(z) = a.$$

Definition 2. *The **imaginary part** of $z = a + bi$ (written $\mathrm{Im}(z)$), where $a, b \in \mathbb{R}$, is given by*

$$\mathrm{Im}(z) = b.$$

Example 1. If $z = 6 - 5i$ then $\mathrm{Re}(z) = 6$ and $\mathrm{Im}(z) = -5$. ◊

NOTE.

1. The imaginary part of a complex number is a real number.
2. Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. That is
$$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d,$$
where $a, b, c, d \in \mathbb{R}$.

Example 2. Find real numbers a, b such that $(2a + b) + (3a - 2b)i = 4 - i$.

Solution. By comparing the real parts and imaginary parts, we have
$$2a + b = 4 \quad \text{and} \quad 3a - 2b = -1.$$
Hence $a = 1, b = 2$. ♢

The **complex conjugate** of a complex number is defined as follows.

Definition 3. *If $z = a + bi$, where $a, b \in \mathbb{R}$, then the* ***complex conjugate*** *of z is $\overline{z} = a - bi$.*

Example 3. $\overline{3 - 4i} = 3 + 4i, \qquad \overline{6} = 6, \qquad \overline{-10i} = 10i.$ ♢

Example 4. Let $z = 2 + i$, $w = 1 + 2i$. We have
$$\overline{z + w} = \overline{3 + 3i} = 3 - 3i.$$
Note that $\overline{z} + \overline{w} = (2 - i) + (1 - 2i) = 3 - 3i$. So $\overline{z + w} = \overline{z} + \overline{w}$. ♢

Properties of the Complex Conjugate.

1. $\overline{\overline{z}} = z$.
2. $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{z - w} = \overline{z} - \overline{w}$.
3. $\overline{zw} = \overline{z}\,\overline{w}$ and $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$.
4. $\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$ and $\mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z})$.
5. If $z = a + bi$, then $z\overline{z} = a^2 + b^2$, so $z\overline{z} \in \mathbb{R}$ and $z\overline{z} \geqslant 0$.

The proofs of most of these properties are straightforward. For example, the proof of property 4 is as follows.

If $z = a + bi$ then $\overline{z} = a - bi$ and
$$\begin{aligned} z + \overline{z} &= (a + bi) + (a - bi) = 2a = 2\mathrm{Re}(z) \\ z - \overline{z} &= (a + bi) - (a - bi) = 2bi = 2i\mathrm{Im}(z). \end{aligned}$$
The proofs of the remaining properties are left as exercises.

So far, we handle complex numbers by writing them in $a + bi$ form. With the above properties, we can write the real and imaginary parts of a complex number in terms of the complex number and its conjugate. We can also prove some general results in a simpler way.

Example 5. Let $z, w \in \mathbb{C}$ such that $z\overline{z} = w\overline{w}$. Prove that $\dfrac{z+w}{z-w}$ is purely imaginary.

Proof. To show that $\dfrac{z+w}{z-w}$ is purely imaginary, we only need to show that $\mathrm{Re}\left(\dfrac{z+w}{z-w}\right)$ is 0. Using $\alpha + \overline{\alpha} = 2\mathrm{Re}(\alpha)$,

$$
\begin{aligned}
\mathrm{Re}\left(\frac{z+w}{z-w}\right) &= \frac{1}{2}\left(\frac{z+w}{z-w} + \overline{\left(\frac{z+w}{z-w}\right)}\right) \\
&= \frac{1}{2}\left(\frac{z+w}{z-w} + \frac{\overline{z+w}}{\overline{z-w}}\right) \\
&= \frac{(z+w)(\overline{z}-\overline{w}) + (z-w)(\overline{z}+\overline{w})}{2(z-w)\overline{(z-w)}} \\
&= \frac{z\overline{z} - z\overline{w} + w\overline{z} - w\overline{w} + z\overline{z} + z\overline{w} - w\overline{z} - w\overline{w}}{2(z-w)\overline{(z-w)}} \\
&= \frac{2(z\overline{z} - w\overline{w})}{2(z-w)\overline{(z-w)}}
\end{aligned}
$$

which is 0 since $z\overline{z} = w\overline{w}$. The result follows. ☐

NOTE. The symbol ☐ indicates the end of a proof.

3.5 The Argand diagram

In the rest of this chapter, unless otherwise stated, we shall assume $a, b, x, y \in \mathbb{R}$ when we say $z = a + bi$ or $z = x + yi$ is a complex number.

There is a simple and extremely useful geometric picture of complex numbers which is obtained by identifying a complex number $z = a + bi$ with the point in the xy-plane whose coordinates are (a, b). For example, $4 + 3i$ is represented by $(4, 3)$, 4 by $(4, 0)$ and $3i$ by $(0, 3)$. The coordinate plane with complex numbers plotted in this way is called an **Argand diagram**, as shown in Figure 1.

Note that a number $z = a + bi$ is plotted with $\mathrm{Re}(z) = a$ in the x-direction and $\mathrm{Im}(z) = b$ in the y-direction. For this reason, the x-axis is usually called the **real axis** and the y-axis is usually called the **imaginary axis**.

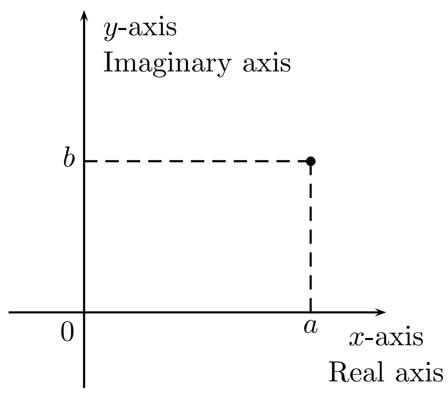


Figure 1: The Argand Diagram.

Example 1. Plot the numbers 4, -4, i, $-2i$, $-2+3i$, $3-4i$, $-3-4i$ on an Argand diagram. The solution is shown in Figure 2. $\diamondsuit$

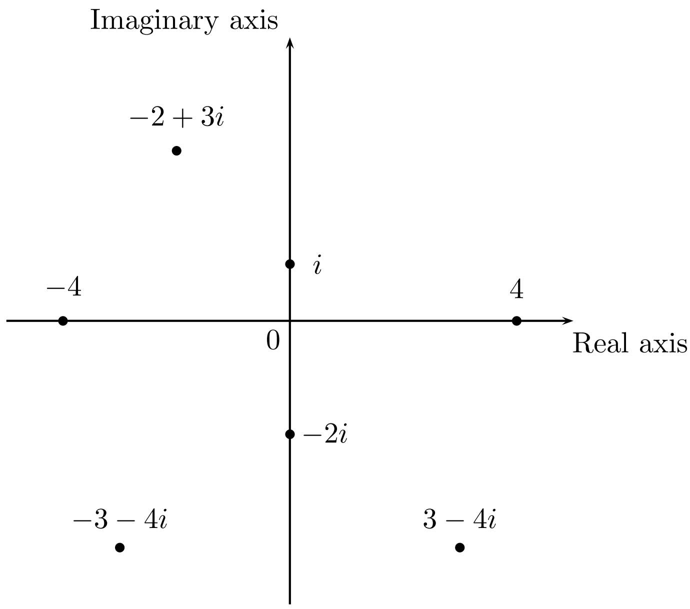


Figure 2: Plots of 4, -4, i, $-2i$, $-2+3i$, $3-4i$, $-3-4i$, on the Argand Diagram.

Note that we can write either i and $-2i$ on the y-axis above, or simply mark 1 and -2, remembering that this is the imaginary axis.

Plotting real and imaginary parts leads to a simple geometric picture of addition and subtraction of complex numbers. Addition and subtraction is done by adding or subtracting x-coordinates and y-coordinates separately, as shown in Figure 3.

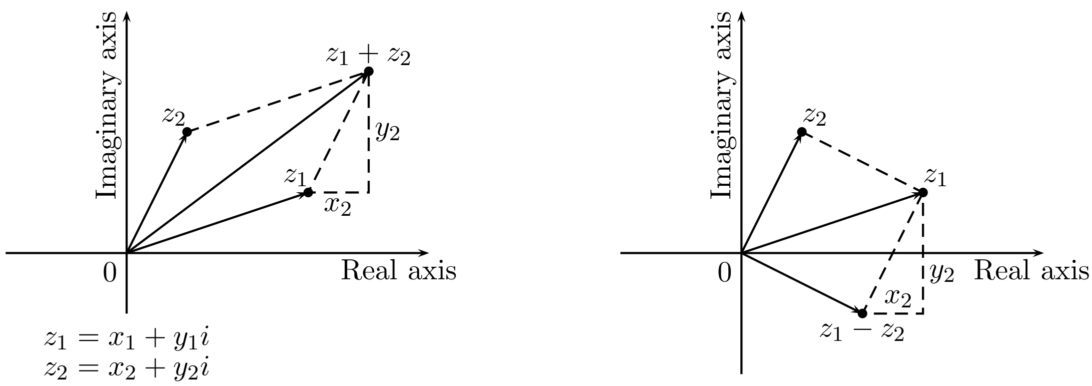


Figure 3: Addition and Subtraction of Complex Numbers.

3.6 Polar form, modulus and argument

An alternative representation for complex numbers, which proves to be very useful, is obtained by using plane polar coordinates r and θ instead of the Cartesian coordinates x and y. The coordinate r is the distance of a point from the origin, and θ is an angle measured from the positive x-axis, as shown in Figure 4.

Take a complex number $z \neq 0$, then from Figure 4, Pythagoras' theorem gives

$$r = \sqrt{x^2 + y^2} \quad \text{with} \quad r > 0.$$

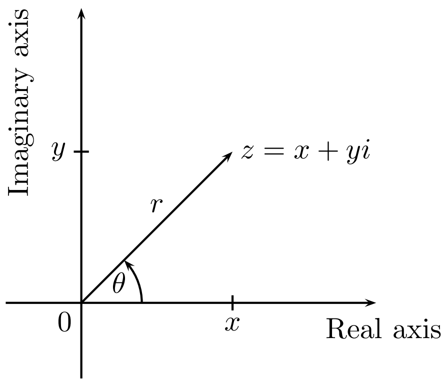


Figure 4: Polar Coordinates of a Complex Number.

By trigonometry,

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} \qquad \text{and} \qquad \sin\theta = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}.$$

Thus the relations between the real and imaginary parts of $z = x + yi$ and the polar coordinates r and θ are

$$\mathrm{Re}(z) = x = r\cos\theta \qquad \text{and} \qquad \mathrm{Im}(z) = y = r\sin\theta,$$

and hence a complex number $z \neq 0$ can be written using the polar coordinates r and θ as:

$$z = r(\cos\theta + i\sin\theta).$$

It is important to note here that the angle θ for a given complex number $z = x + yi$ is not uniquely defined; since adding or subtracting 2π produces exactly the same values for x and y and hence the same complex number z. This result is summarised in the following proposition.

Proposition 1 (Equality of Complex Numbers)**.** *Two complex numbers*

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2), \quad z_1,\ z_2 \neq 0$$

are equal if and only if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2k\pi$, where k is any integer.

[**X**] *Proof.* Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

If $r_1 = r_2$ and $\theta_1 = \theta_2 + 2k\pi$, then $\cos\theta_1 = \cos\theta_2,\ \sin\theta_1 = \sin\theta_2$,

$$\begin{aligned} x_1 &= r_1\cos\theta_1 = r_2\cos\theta_2 = x_2 \\ y_1 &= r_1\sin\theta_1 = r_2\sin\theta_2 = y_2 \end{aligned}$$

and $z_1 = z_2$.

Conversely, if $z_1 = z_2$, then we have $x_1 = x_2$ and $y_1 = y_2$. Hence,

$$r_1 = \sqrt{x_1^2 + y_1^2} = \sqrt{x_2^2 + y_2^2} = r_2,$$

and since $r_1,\ r_2 \neq 0$,

$$\cos\theta_1 = \frac{x_1}{r_1} = \frac{x_2}{r_2} = \cos\theta_2 \qquad \text{and} \qquad \sin\theta_1 = \frac{y_1}{r_1} = \frac{y_2}{r_2} = \sin\theta_2,$$

and hence, $\cos(\theta_1 - \theta_2) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 = \cos^2\theta_1 + \sin^2\theta_1 = 1$
so $\theta_1 - \theta_2 = 2k\pi,\ \theta_1 = \theta_2 + 2k\pi$ for $k \in \mathbb{Z}$. □

The polar coordinate r that we have associated with a complex number is often called the **modulus** of the complex number. The formal definition is:

Definition 1. *For $z = x + yi$, where $x, y \in \mathbb{R}$, we define the* ***modulus*** *of z to be*
$$|z| = \sqrt{x^2 + y^2}.$$

The quantity $|z|$ is also called the **magnitude** of z or the **absolute value** of z. Note that it has a geometric interpretation as the distance $r = |z|$ of the point z from the origin in an Argand diagram. Note also that $z\overline{z} = |z|^2$.

Example 1. $|-4| = 4$, $|2i| = 2$, and $|3 - 4i| = \sqrt{3^2 + (-4)^2} = 5$. ◊

Argument and Principal Argument.

The polar coordinate θ that we have associated with a complex number is often called an **argument** of the complex number and is written as $\arg(z)$. As mentioned above, such an angle can be increased or decreased by 2π without changing the corresponding complex number. It is desirable to define a particular argument that is unique for a given complex number. We do this by choosing a value θ of the argument so that $-\pi < \theta \leqslant \pi$. This is called the **principal argument** of z and is written as $\operatorname{Arg}(z)$.

The following diagrams illustrate the possible positions of a complex number z in each of the four quadrants.

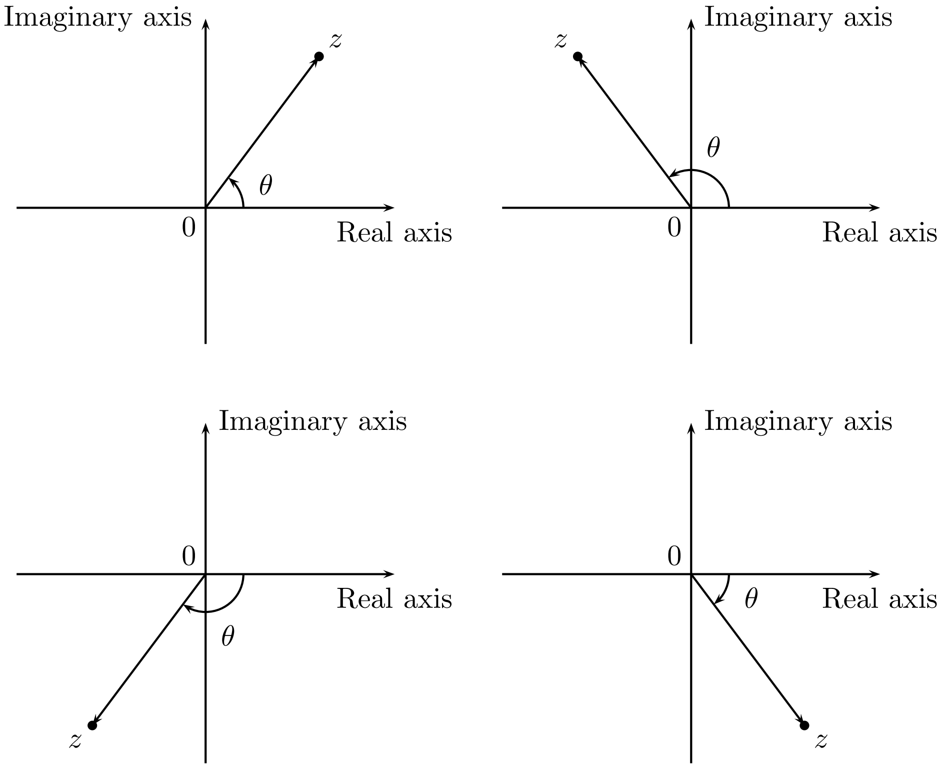


Figure 5: Principal Argument.

We can see from the diagrams, that if z lies in the first or second quadrant, we measure the principal argument θ anticlockwise. For z in the 3rd or the 4th quadrant, we measure θ, as a negative angle, clockwise. Note that we leave $\operatorname{Arg}(0)$ undefined.

One useful strategy is always to draw a diagram, then use the tangent ratio to find the *acute* angle α formed by the corresponding triangle and use this to find the principal argument.

Example 2. Find the arguments of $1+i$, $-1+\sqrt{3}\,i$, $-\sqrt{3}-i$ and $1-\sqrt{3}\,i$.

SOLUTION.

We first plot $z = 1+i$ on an Argand diagram as in Figure 6. The complex number lies in the 1st quadrant, and so $0 < \operatorname{Arg}(z) < \pi/2$. As shown in the figure,

$$\tan\theta = 1, \quad \text{and}$$

$$\operatorname{Arg}(1+i) = \theta = \frac{\pi}{4}.$$

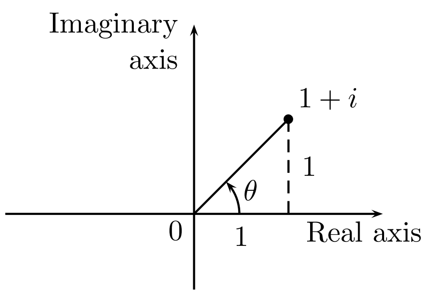


Figure 6.

From Figure 7, the complex number $-1+\sqrt{3}\,i$ lies in the 2nd quadrant. Also,

$$\tan\alpha = \sqrt{3}, \quad \text{so} \quad \alpha = \frac{\pi}{3}.$$

Hence

$$\operatorname{Arg}\left(-1+\sqrt{3}\,i\right) = \theta = \pi - \alpha = \frac{2\pi}{3}.$$

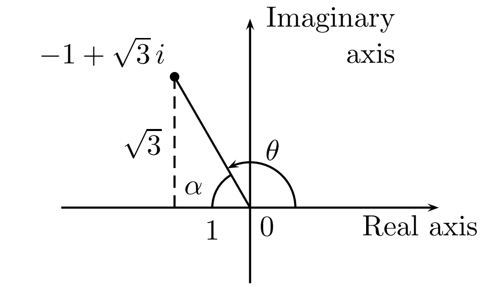


Figure 7.

From Figure 8, the complex number $-\sqrt{3}-i$ lies in the 3rd quadrant. Also,

$$\tan\alpha = \frac{1}{\sqrt{3}}, \quad \text{so} \quad \alpha = \frac{\pi}{6}.$$

Hence

$$\operatorname{Arg}\left(-\sqrt{3}-i\right) = -\pi + \alpha = -\frac{5\pi}{6}.$$

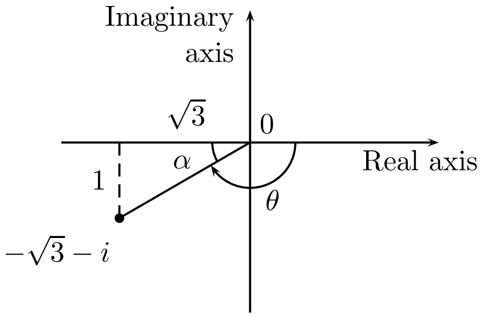


Figure 8.

From Figure 9, the complex number $1-\sqrt{3}\,i$ lies in the 4th quadrant. Also

$$\tan\alpha = \sqrt{3}, \quad \text{so} \quad \alpha = \frac{\pi}{3}.$$

Hence,

$$\operatorname{Arg}\left(1-\sqrt{3}\,i\right) = -\alpha = -\frac{\pi}{3}.$$

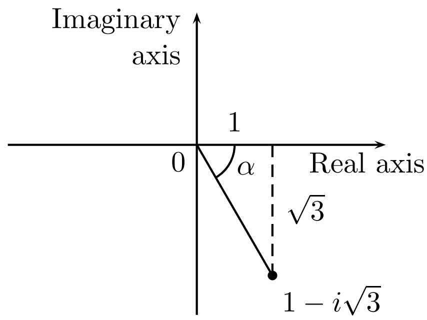


Figure 9.

$\Diamond$

In the special case that a given complex number is real or purely imaginary, we can easily read the principal argument from an Argand diagram.

Example 3. Find the principal argument for each of the numbers 4, -4, i and $-2i$.

Solution. From Figure 2 in Section 3.5, we can easily read the arguments as

$$\operatorname{Arg}(4) = 0, \quad \operatorname{Arg}(-4) = \pi, \quad \operatorname{Arg}(i) = \frac{\pi}{2}, \quad \operatorname{Arg}(-2i) = -\frac{\pi}{2}.$$

$\Diamond$

Example 4. Find the "$a + ib$" form of the complex number with modulus 4 and argument $-\dfrac{\pi}{6}$.

Solution.

$$z = 4\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = 4\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 2\sqrt{3} - 2i.$$

◊

3.7 Properties and applications of the polar form

We begin by proving a useful lemma.

Lemma 1. *For any real numbers θ_1 and θ_2*

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2).$$

Proof. Expanding the left hand side, we obtain

$$(\cos\theta_1\ \cos\theta_2 - \sin\theta_1\ \sin\theta_2) + i(\cos\theta_1\ \sin\theta_2 + \sin\theta_1\ \cos\theta_2).$$

Then, using the standard trigonometric formulae

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\ \cos\theta_2 - \sin\theta_1\ \sin\theta_2,$$

and

$$\sin(\theta_1 + \theta_2) = \cos\theta_1\ \sin\theta_2 + \sin\theta_1\ \cos\theta_2,$$

the result follows. □

Lemma 1 can be used to derive a very important and useful theorem for integer powers of complex numbers in polar forms. This theorem is called De Moivre's Theorem.

Theorem 2 (De Moivre's Theorem)**.** *For any real number θ and integer n*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \qquad (\#)$$

Proof. We shall prove this theorem by proving that the condition (#) holds for the four separate cases of $n > 0$, $n = 0$, $n = -1$ and $n < -1$, and hence that it holds for all $n \in \mathbb{Z}$.

Case 1. $n > 0$. The proof is by induction (see Section 3.11).

We first note that (#) is obviously true for $n = 1$.

We now show that, if (#) is true for some value of $n > 0$, then it is also true for $n+1$. Assuming (#) is true for some value of $n > 0$, we have

$$\begin{aligned}(\cos\theta + i\sin\theta)^{n+1} &= (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^n \\ &= (\cos\theta + i\sin\theta)(\cos n\theta + i\sin n\theta) \\ &= \cos(n+1)\theta + i\sin(n+1)\theta \qquad \text{(from lemma 1)},\end{aligned}$$

and hence (#) is true for $n+1$.

Now, we have already seen that (#) is true for $n=1$, and hence, from the first principle of induction, it is also true for all integers $n \geqslant 1$.

Case 2. $n=0$. The condition (#) is true for this case, provided we use the convention that $z^0 = 1$ for any complex number z.

Case 3. $n=-1$. By definition, $z^{-1} = \dfrac{1}{z}$. Then, applying the division rule for complex numbers to $z = \cos\theta + i\sin\theta$, we have

$$\begin{aligned}
(\cos\theta + i\sin\theta)^{-1} &= \frac{1}{\cos\theta + i\sin\theta} \\
&= \frac{1}{\cos\theta + i\sin\theta} \times \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta} \\
&= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} \\
&= \cos(-\theta) + i\sin\,(-\theta),
\end{aligned}$$

where we have used the trigonometric identities

$$\cos\theta = \cos(-\theta), \qquad \sin\theta = -\sin(-\theta) \qquad \text{and} \qquad \cos^2\theta + \sin^2\theta = 1.$$

The condition (#) is therefore true for $n=-1$ also.

Case 4. $n < -1$. Let $m = -n$ so that $m > 1$. Note that

$$\begin{aligned}
(\cos\theta + i\sin\theta)^n &= (\cos\theta + i\sin\theta)^{-m} \\
&= \left((\cos\theta + i\sin\theta)^{-1}\right)^m \\
&= (\cos\,(-\theta) + i\sin(-\theta))^m && \text{(from case 3)} \\
&= \cos(-m\theta) + i\sin(-m\theta) && \text{(from case 1)} \\
&= \cos(n\theta) + i\sin(n\theta).
\end{aligned}$$

□

De Moivre's Theorem provides a simple formula for integer powers of complex numbers. However, it can also be used to suggest a meaning for complex powers of complex numbers. To make this extension to complex powers it is actually sufficient just to give a meaning to the exponential function for imaginary exponents. We first make the following definition.

Definition 1. (Euler's Formula). *For real θ, we define*

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This definition may appear somewhat arbitrary at first, but there are several reasons why it is reasonable. First recall that for real constants a, θ and ϕ, we have for integer n,

$$(e^{a\theta})^n = e^{an\theta}, \tag{1}$$

$$e^{a\theta}e^{a\phi} = e^{a(\theta+\phi)}, \tag{2}$$

$$e^0 = 1, \tag{3}$$

$$\frac{d}{d\theta}\left(e^{a\theta}\right) = ae^{a\theta}. \tag{4}$$

In fact, properties (3) and (4) are often taken as a definition of the exponential function for real numbers.

The next point to notice is that if a is replaced by i and if $e^{i\theta}$ is replaced by $(\cos\theta + i\sin\theta)$ in these four formulae, then all four formulae are still satisfied. In fact, equation (1) would be De Moivre's Theorem, equation (2) would be lemma 1, equation (3) would obviously be true, and equation (4) would be

$$\frac{d}{d\theta}(\cos\theta + i\sin\theta) = i(\cos\theta + i\sin\theta) = -\sin\theta + i\cos\theta,$$

which is also true (provided we assume that differentiation of expressions containing the symbol i can be carried out in the same way as if i were a real constant). Thus, $\cos\theta + i\sin\theta$ has exactly the same properties that we would like $e^{i\theta}$ to have, and so the definition

$$e^{i\theta} = \cos\theta + i\,\sin\theta$$

is consistent with our experience with other exponential functions.

Note also that, since cosine is an even function and sine is an odd function, so

$$e^{-i\theta} = \cos(-\theta) + i\,\sin(-\theta) = \cos\theta - i\,\sin\theta,$$

which is the conjugate of $e^{i\theta}$.

We should also say that in more sophisticated treatments of complex functions, $e^{i\theta}$ will be defined by other means (usually a "power series") and Euler's formula then becomes a theorem.

De Moivre's Theorem and Euler's formula have a wide variety of uses, ranging from calculation of powers of complex numbers to calculation of roots of complex numbers to derivation of trigonometric formulae etc. We shall examine some of these applications in the remainder of this chapter.

3.7.1 The arithmetic of polar forms

Using Euler's formula, we can rewrite the complex number $z = r(\cos\theta + i\sin\theta)$ in an alternative and more usual form. We will call this form, the **polar form** of the complex number.

Definition 2. *The* ***polar form*** *for a non-zero complex number* z *is*

$$z = re^{i\theta}$$

where $r = |z|$ *and* $\theta = \text{Arg}(z)$.

Four important special cases are:

$$1 = e^0, \qquad i = e^{i\pi/2}, \qquad -1 = e^{i\pi}, \qquad -i = e^{-i\pi/2}.$$

Some further examples of polar forms are as follows.

Example 1. $-4 = 4e^{i\pi}, \quad -4i = 4e^{-i\pi/2}, \quad 1 - i\sqrt{3} = 2e^{-i\pi/3}, \quad -1 - i\sqrt{3} = 2e^{-2i\pi/3}.$ $\diamondsuit$

(Note: At High School you may have taken $r(\cos\theta + i\sin\theta)$ as the polar form and abbreviated it to $r\,\text{cis}\,(\theta)$. This notation is not used in this course.)

Since $e^{i\theta} = e^{i(\theta+2k\pi)}$ for all k an integer, it is sometimes convenient to express the polar form using a general argument, that is,

$$z = e^{i\theta}, \quad \text{where } \theta = \text{Arg}(z) + 2k\pi,\ k \in \mathbb{Z}.$$

Using Euler's formula, we can rewrite the equality proposition for polar forms (Proposition 1 of Section 3.6) as

$$z_1 = r_1 e^{i\theta_1} = r_2 e^{i\theta_2} = z_2$$

if and only if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2k\pi$ for $k \in \mathbb{Z}$.

The polar form is very useful for multiplication and division of complex numbers. The formulae for multiplication and division of polar forms are:

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)},$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$$

It is sometimes useful to express these results in terms of the modulus and argument. In this case, we have for multiplication,

$$|z_1 z_2| = r_1 r_2 = |z_1|\,|z_2| \qquad \text{and} \qquad \text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) + 2k\pi,$$

where k is an integer, chosen so that $-\pi < \text{Arg}(z_1 z_2) \leqslant \pi$. That is, the rule is to *multiply* the moduli and *add* the arguments. For division, we have

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} \qquad \text{and} \qquad \text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2) + 2k\pi,$$

where k is an integer chosen so that $-\pi < \text{Arg}(z_1/z_2) \leqslant \pi$. That is, the rule is to *divide* the moduli and *subtract* the arguments.

Example 2. Use polar forms to find the modulus and argument of $(-1 - i)(1 - i\sqrt{3})$ and $(-1 - i)/(1 - i\sqrt{3})$.

SOLUTION. The modulus and argument of $-1-i$ and $1-i\sqrt{3}$ are

$$\begin{aligned} |-1-i| &= \sqrt{2} \quad &\text{and} \quad \operatorname{Arg}(-1-i) &= -\frac{3\pi}{4}, \\ |1-i\sqrt{3}| &= 2 \quad &\text{and} \quad \operatorname{Arg}\left(1-i\sqrt{3}\right) &= -\frac{\pi}{3}. \end{aligned}$$

Hence, for multiplication, we have

$$\left|(-1-i)\left(1-i\sqrt{3}\right)\right| = |-1-i|\,\left|1-i\sqrt{3}\right| = 2\sqrt{2},$$

and, for some integer k,

$$\operatorname{Arg}\left((-1-i)\left(1-i\sqrt{3}\right)\right) = -\frac{3\pi}{4} + \left(-\frac{\pi}{3}\right) + 2k\pi = -\frac{13\pi}{12} + 2k\pi.$$

Then, on choosing $k=1$, to obtain a principal argument in the interval $(-\pi,\pi]$, we have

$$\operatorname{Arg}\left((-1-i)\left(1-i\sqrt{3}\right)\right) = -\frac{13\pi}{12} + 2\pi = \frac{11\pi}{12}.$$

For division, we have

$$\left|\frac{-1-i}{1-i\sqrt{3}}\right| = \frac{|-1-i|}{\left|1-i\sqrt{3}\right|} = \frac{\sqrt{2}}{2}$$

and, for some integer k,

$$\operatorname{Arg}\left(\frac{-1-i}{1-i\sqrt{3}}\right) = -\frac{3\pi}{4} - \left(-\frac{\pi}{3}\right) + 2k\pi = -\frac{5\pi}{12},$$

where $k=0$ has been chosen to obtain a principal argument in the interval $(-\pi,\ \pi]$. ◊

The next example shows a reasonably simple method for finding the square roots of a complex number in Cartesian form. It rests on the observation that if $z \in \mathbb{C}$ and $z = a+bi$ with $a,\, b \in \mathbb{R}$, then

$$|z^2| = |z \cdot z| = |z|\;|z| = |z|^2 \qquad \text{or} \qquad |(a+bi)^2| = a^2+b^2,$$

and also $\operatorname{Re}(z^2) = a^2-b^2$, $\operatorname{Im}(z^2) = 2ab$.

Example 3. Find the square roots of $-5-12i$.

SOLUTION. We want to find all complex solutions to $z^2 = -5-12i$. Writing $z = a+bi$, this is equivalent to finding all real solutions (a,b) to the equation $(a+bi)^2 = -5-12i$. Expanding the left-hand side and equating real and imaginary parts, we have

$$\begin{aligned} a^2-b^2 &= -5 \\ 2ab &= -12. \end{aligned}$$

Also,

$$a^2+b^2 = |(a+ib)^2| = |-5-12i| = \sqrt{(-5)^2+(-12)^2} = 13.$$

So we want to solve the system of equations

$$\begin{aligned} a^2 + b^2 &= 13 \\ a^2 - b^2 &= -5 \\ 2ab &= -12. \end{aligned}$$

Solving the first pair in this system, we get $a^2 = 4$ and $b^2 = 9$. This means that $a = \pm 2$ and $b = \pm 3$. The third equation is satisfied precisely when we choose opposite signs for a and b. Therefore the two square roots of $-5 - 12i$ are

$$z = 2 - 3i \quad \text{and} \quad -2 + 3i,$$

or more compactly,

$$z = \pm(2 - 3i).$$

$\Diamond$

Example 4. Use the quadratic formula to solve the equation

$$z^2 - (4 + i)z + (5 + 5i) = 0.$$

SOLUTION. We solve this using the quadratic formula. The proof of the quadratic formula uses only field axioms and the fact of the existence of square roots and thus carries over directly to quadratic polynomials with complex coefficients. The details are left as an exercise.

Proceeding, the solutions to $z^2 - (4 + i)z + (5 + 5i) = 0$ are

$$z = \frac{4 + i \pm r}{2}$$

where r is a square root of $(4 + i)^2 - 4(5 + 5i) = -5 - 12i$.

From the previous example, the square roots of $-5 - 12i$ are $\pm(2 - 3i)$. Thereby the roots to the quadratic are

$$z = \frac{4 + i + (2 - 3i)}{2} = 3 - i \qquad \text{and} \qquad z = \frac{4 + i - (2 - 3i)}{2} = 1 + 2i$$

We can easily check that $z = 3 - i$ and $z = 1 + 2i$ are roots to the equation by substitution. $\Diamond$

Example 5. Show that the set of numbers of unit modulus, that is, the set

$$S = \{z \in \mathbb{C} : |z| = 1\},$$

is closed under multiplication and division.

SOLUTION. For closure under multiplication we must prove that the product $z_1z_2 \in S$ for all z_1, $z_2 \in S$. Now, if $z_1, z_2 \in S$, then $|z_1| = 1$ and $|z_2| = 1$. Then

$$|z_1z_2| = |z_1||z_2| = 1$$

also. Thus, $z_1z_2 \in S$, and hence S is closed under multiplication.

For closure under division we must prove that $z_1/z_2 \in S$ for all z_1, $z_2 \in S$. For z_1/z_2, we have

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = 1,$$

as $|z_1| = 1$ and $|z_2| = 1$. Thus $z_1/z_2 \in S$, and hence S is closed under division. $\Diamond$

Example 6. [**X**] Suppose a and b are real numbers (not both zero) and $w = (az+b)/(bz+a)$. Show that, if $|z| = 1$, then $|w| = 1$.

Solution. We have

$$|w| = \left|\frac{az+b}{bz+a}\right| = \frac{|az+b|}{|bz+a|}.$$

Now, $|z| = 1$, so $z = \cos\theta + i\sin\theta,\ \theta$ real.

$$\begin{aligned} |az+b| = |a(\cos\theta + i\sin\theta) + b| &= \sqrt{(a\cos\theta+b)^2 + (a\sin\theta)^2} \\ &= \sqrt{a^2 + 2ab\cos\theta + b^2} \\ \text{and} \qquad |bz+a| = |b(\cos\theta + i\sin\theta) + a| &= \sqrt{a^2 + 2ab\cos\theta + b^2}. \end{aligned}$$

So $|w| = 1$. $\Diamond$

3.7.2 Powers of complex numbers

With the exception of very simple cases such as squares and cubes, the simplest method of calculating powers of a complex number z is to use the polar form. Thus, if

$$z = re^{i\theta},$$

then the properties of exponentials give

$$z^n = r^n e^{in\theta}.$$

Example 7. Calculate $\left(1+i\sqrt{3}\right)^{10}$.

Solution. For $z = 1 + i\sqrt{3}$, we have $|z| = 2$, $\operatorname{Arg}(z) = \frac{\pi}{3}$ and so $z = 2e^{i\pi/3}$. Hence,

$$\left(1+i\sqrt{3}\right)^{10} = \left(2e^{i\pi/3}\right)^{10} = 2^{10}e^{10i\pi/3} = 2^{10}e^{-2i\pi/3} = 2^{10}\left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right) = 2^9\left(-1-i\sqrt{3}\right).$$

$\Diamond$

3.7.3 Roots of complex numbers

The polar form can also be used to find roots of complex numbers, where a root of a complex number is defined as follows.

Definition 3. *A complex number z is an* ***nth root*** *of a number z_0 if z_0 is the nth power of z, that is, z is the nth root of z_0 if $z^n = z_0$.*

If $z_0 \neq 0$ the nth roots of z_0 can be found by equating the polar forms of z^n and z_0. Thus, if $z = re^{i\theta}$ and $z_0 = r_0 e^{i\theta_0}$, we have

$$z^n = r^n e^{in\theta} = r_0 e^{i\theta_0},$$

and hence, on equating the moduli and arguments, we have

$$r^n = r_0 \qquad \text{and} \qquad n\theta = \theta_0 + 2k\pi \quad \text{for} \quad k \in \mathbb{Z}.$$

Thus

$$r = r_0^{1/n} \quad \text{and} \quad \theta = \frac{\theta_0 + 2k\pi}{n} = \frac{\theta_0}{n} + \frac{2k\pi}{n} \quad \text{for} \quad k \in Z,$$

and

$$z = re^{i\theta} = r_0^{\frac{1}{n}}\, e^{i\left(\frac{\theta_0 + 2k\pi}{n}\right)}. \tag{*}$$

As k ranges over the integers, z takes precisely n different values. These n values can be found by letting k take any n consecutive values. Thus z_0 has precisely n distinct nth roots.

Example 8. Find all fifth roots of unity.

Solution. The modulus and argument of $z_0 = 1$ are $r_0 = |1| = 1$ and $\theta_0 = \text{Arg}(1) = 0$.

Hence, the fifth roots of unity are complex numbers, z, given by

$$z^5 = e^{2k\pi i} \quad \text{for } k \in \mathbb{Z},$$

By (*) above, these roots are

$$z = e^{i\left(\frac{0+2k\pi}{5}\right)},$$

where the five consecutive integers for k are chosen to be $-2, -1, 0, 1, 2$ so that each argument $\theta = \frac{2k\pi}{5}$ lies in $(-\pi, \pi]$.

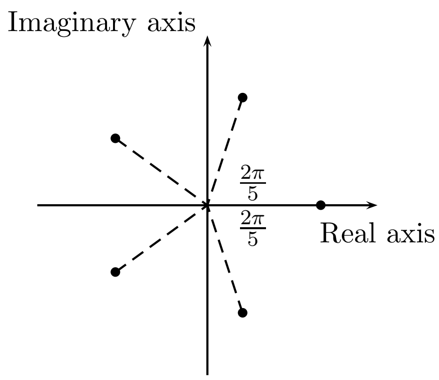


Figure 10: The Five Fifth Roots of Unity.

Polar forms for the five fifth roots of unity are therefore

$$e^{-4\pi i/5},\ e^{-2\pi i/5},\ 1,\ e^{2\pi i/5},\ e^{4\pi i/5}.$$

$\Diamond$

Note that these solutions occur in conjugate pairs, eg. $e^{2\pi i/5}$ and $e^{-2\pi i/5}$ and so on.
As shown in Figure 10, these fifth roots are equally spaced out on a circle of radius 1 with angles $\dfrac{2\pi}{5}$ between them.

Example 9. Find all sixth roots of -2.

Solution. The modulus and argument of -2 are $|-2| = 2$ and $\text{Arg}(-2) = \pi$.

Hence, the sixth roots of -2 are complex numbers z, such that

$$z^6 = 2e^{(\pi+2k\pi)i} \quad \text{for } k \in \mathbb{Z}.$$

So, the sixth roots are

$$z = 2^{\frac{1}{6}}\, e^{i\frac{(2k+1)\pi}{6}}, \text{ for } k = -3, -2, -1, 0, 1, 2.$$

Writing $\alpha = 2^{\frac{1}{6}}$, the six roots are: $\alpha e^{-i\frac{5\pi}{6}}$, $\alpha e^{-i\frac{\pi}{2}}$, $\alpha e^{-i\frac{\pi}{6}}$, $\alpha e^{i\frac{\pi}{6}}$, $\alpha e^{i\frac{\pi}{2}}$, and $\alpha e^{i\frac{5\pi}{6}}$.

In "$a + ib$" form, these sixth roots are

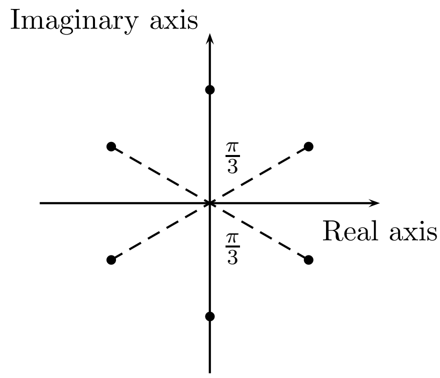


Figure 11: The Six Sixth Roots of -2.

$$-2^{\frac{1}{6}}\left(\frac{\sqrt{3}+i}{2}\right),\ -2^{\frac{1}{6}}i,\ 2^{\frac{1}{6}}\left(\frac{\sqrt{3}-i}{2}\right),\ 2^{\frac{1}{6}}\left(\frac{\sqrt{3}+i}{2}\right),\ 2^{\frac{1}{6}}i,\ -2^{\frac{1}{6}}\left(\frac{\sqrt{3}-i}{2}\right)$$

$\diamondsuit$

There is again a simple geometric picture of this result. From figure 11, the 6 roots lie on a circle of radius $2^{\frac{1}{6}}$ with angles $\dfrac{2\pi}{6} = \dfrac{\pi}{3}$ between them.

In summary, there are always exactly n nth roots of a non-zero complex number. On the Argand diagram, these nth roots of z_0 lie on a circle of radius $|z_0|^{\frac{1}{n}}$ with angles $\dfrac{2\pi}{n}$ between them.

3.8 Trigonometric applications of complex numbers

A large variety of trigonometric formulae can be obtained by using complex numbers and the Binomial Theorem. This theorem is stated below, and a proof is given in Section 3.12.

Theorem 1 (Binomial Theorem)**.** *If $a, b \in \mathbb{C}$ and $n \in \mathbb{N}$, then*

$$\begin{aligned}(a+b)^n &= a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + nab^{n-1} + b^n \\ &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k,\end{aligned}$$

where the numbers $\displaystyle\binom{n}{k} = \frac{n!}{k!(n-k)!}$ are the binomial coefficients.

For small values of n the binomial coefficients may be easily calculated using **Pascal's triangle**.

n	BINOMIAL COEFFICIENTS
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

Figure 12: Pascal's Triangle.

Note that each coefficient (except the 1's at the end) is obtained by adding the two coefficients immediately above it.

Example 1. For $n = 4$, the coefficients are 1, 4, 6, 4, 1, and hence

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

$\Diamond$

Sine and cosine of multiples of θ.

It is now easy to express $\cos n\theta$ or $\sin n\theta$ in terms of powers of $\cos\theta$ or $\sin\theta$ by using De Moivre's Theorem.

Example 2. Find a formula for $\cos 4\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$.

Solution. Using De Moivre's Theorem and the Binomial Theorem gives

$$\begin{aligned}\cos 4\theta + i\sin 4\theta &= (\cos\theta + i\sin\theta)^4 \\ &= \cos^4\theta + 4\cos^3\theta\,(i\sin\theta) + 6\cos^2\theta\,(i\sin\theta)^2 + 4\cos\theta\,(i\sin\theta)^3 + (i\sin\theta)^4 .\end{aligned}$$

Then, on using $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and separately equating real and imaginary parts on the left and right hand sides gives

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

and

$$\sin 4\theta = 4\left(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta\right).$$

Note that the formula for $\cos 4\theta$ can be rewritten in term of powers of $\cos\theta$ only by using $\sin^2\theta = 1 - \cos^2\theta$ to replace the sine terms by cosines. $\Diamond$

Powers of sine and cosine.

Euler's formula shows that the trigonometric functions, sine and cosine, are closely related to the exponential function with an imaginary exponent.

Using Euler's formula, we note that

$$\begin{aligned} e^{in\theta} &= \cos n\theta + i\sin n\theta, \\ e^{-in\theta} &= \cos(-n\theta) + i\sin\,(-n\theta) = \cos n\theta - i\sin n\theta. \end{aligned}$$

On first adding and then subtracting these formulae, we obtain the important formulae

$$\cos n\theta = \frac{1}{2}\left(e^{in\theta} + e^{-in\theta}\right), \qquad \sin n\theta = \frac{1}{2i}\left(e^{in\theta} - e^{-in\theta}\right).$$

In particular, we have

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right), \qquad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right).$$

We can apply the above formulae to derive trigonometric formulae which relate powers of $\sin\theta$ or $\cos\theta$ to sines or cosines of multiples of θ.

Example 3. Find a formula for $\sin^3\theta$ in terms of sines of multiples of θ.

SOLUTION. Using the formula $\sin n\theta = \dfrac{1}{2i}\left(e^{in\theta} - e^{-in\theta}\right)$, for $n = 1$ and $n = 3$, and the Binomial Theorem gives

$$\begin{aligned}
\sin^3\theta &= \left(\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right)^3 = -\frac{1}{8i}\left(e^{i\theta} - e^{-i\theta}\right)^3 \\
&= -\frac{1}{8i}\left(e^{i3\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-i3\theta}\right) \\
&= -\frac{1}{4}\left(\frac{1}{2i}\left(e^{i3\theta} - e^{-i3\theta}\right) - 3\cdot\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right) \\
&= \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta.
\end{aligned}$$

♢

Example 4. Find a formula for $\cos^5\theta$ in terms of cosines of multiples of θ.

SOLUTION. Using the formula $\cos n\theta = \frac{1}{2}\left(e^{in\theta} + e^{-in\theta}\right)$, for $n = 1, 3, 5$, and the Binomial Theorem gives

$$\begin{aligned}
\cos^5\theta &= \left(\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\right)^5 = \frac{1}{32}\left(e^{i\theta} + e^{-i\theta}\right)^5 \\
&= \frac{1}{32}\left(e^{i5\theta} + 5e^{i3\theta} + 10e^{i\theta} + 10e^{-i\theta} + 5e^{-i3\theta} + e^{-i5\theta}\right) \\
&= \frac{1}{16}\left(\frac{1}{2}\left(e^{i5\theta} + e^{-i5\theta}\right) + \frac{5}{2}\left(e^{i3\theta} + e^{-i3\theta}\right) + 5\left(e^{i\theta} + e^{-i\theta}\right)\right) \\
&= \frac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right).
\end{aligned}$$

♢

Formulae of this type are very useful in integration. For example, using the above identity,

$$\int \cos^5\theta\, d\theta = \frac{1}{16}\left(\frac{1}{5}\sin 5\theta + \frac{5}{3}\sin 3\theta + 10\sin\theta\right) + C.$$

[X] **Example 5.** Find the sum of

$$\cos\theta + \cos(2\theta) + \cdots + \cos(n\theta).$$

SOLUTION. We use the fact that $\cos\theta$ is the real part of $e^{i\theta}$. The required sum is then the real part of the sum

$$S_n = e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta}.$$

Since $e^{ki\theta} = \left(e^{i\theta}\right)^k$, the sum S_n is a geometric progression with ratio of successive terms given by $R = e^{i\theta}$. Then, on using the formula for the sum of a geometric progression (see Section 3.11), we have

$$S_n = R + R^2 + \cdots + R^n = R\left(1 + R + \cdots + R^{n-1}\right) = R\frac{1-R^n}{1-R} = e^{i\theta}\frac{1-e^{in\theta}}{1-e^{i\theta}}.$$

We require the real part of S_n. The simplest way of finding the real part is to use the following trick. Note that

$$1 - e^{i\theta} = e^{i\theta/2}\left(e^{-i\theta/2} - e^{i\theta/2}\right) = -2ie^{i\theta/2}\sin\left(\frac{\theta}{2}\right),$$

and hence that

$$S_n = e^{i\theta}\frac{e^{in\theta/2}\sin\left(\frac{n\theta}{2}\right)}{e^{i\theta/2}\sin\left(\frac{\theta}{2}\right)} = e^{i(n+1)\theta/2}\,\frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}.$$

The required sum of the cosine terms is therefore

$$\sum_{k=1}^{n}\cos(k\theta) = \mathrm{Re}(S_n) = \cos\left(\frac{(n+1)\theta}{2}\right)\frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}.$$

◊

Sums of this type are used in X-ray diffraction, solid-state physics, chemistry, signal processing in electrical engineering, and tomography, as well as in many other areas where periodic functions and waves must be analysed.

3.9 Geometric applications of complex numbers

We have seen that the Argand diagram can be used to represent every complex number as a point in a plane. This plane is frequently called the **complex plane.** Relations between complex numbers can therefore be given a geometric interpretation in this complex plane, and, conversely, the geometry of a plane can be represented by algebraic relations between complex numbers. This connection between geometry and complex numbers has proved to be a very useful and powerful tool in mathematics and in areas such as physics, electrical engineering, fluid dynamics, oceanography, aerodynamics and mechanical engineering, among others.

In this section, we shall look at some simple geometrical examples. From Figure 3 in Section 3.5, we can see that the complex numbers 0, z_1, z_2 and $z_1 + z_2$ form a parallelogram. For the geometric interpretation of the difference of two complex numbers, we shall use a different approach.

Example 1. Give a geometric interpretation of $|z - w|$ and $\mathrm{Arg}\,(z - w)$.

Solution. Let $z = x + iy$, $w = a + ib$. Note that $z - w = (x - a) + i(y - b)$. Now

$$|z - w| = \sqrt{(x - a)^2 + (y - b)^2}$$

which is the distance between the points representing z and w.

To understand the geometric interpretation of $\text{Arg}\,(z - w)$, we represent the complex number $z - w$ by the directed line segment (the vector) from w to z. We plot z and w on the Argand diagram for each of the four cases as shown in Figure 13. The angle α satisfies

$$\sin \alpha = \frac{y - b}{|z - w|} \qquad \text{and} \qquad \cos \alpha = \frac{x - a}{|z - w|},$$

and hence $\alpha = \text{Arg}(z - w)$.

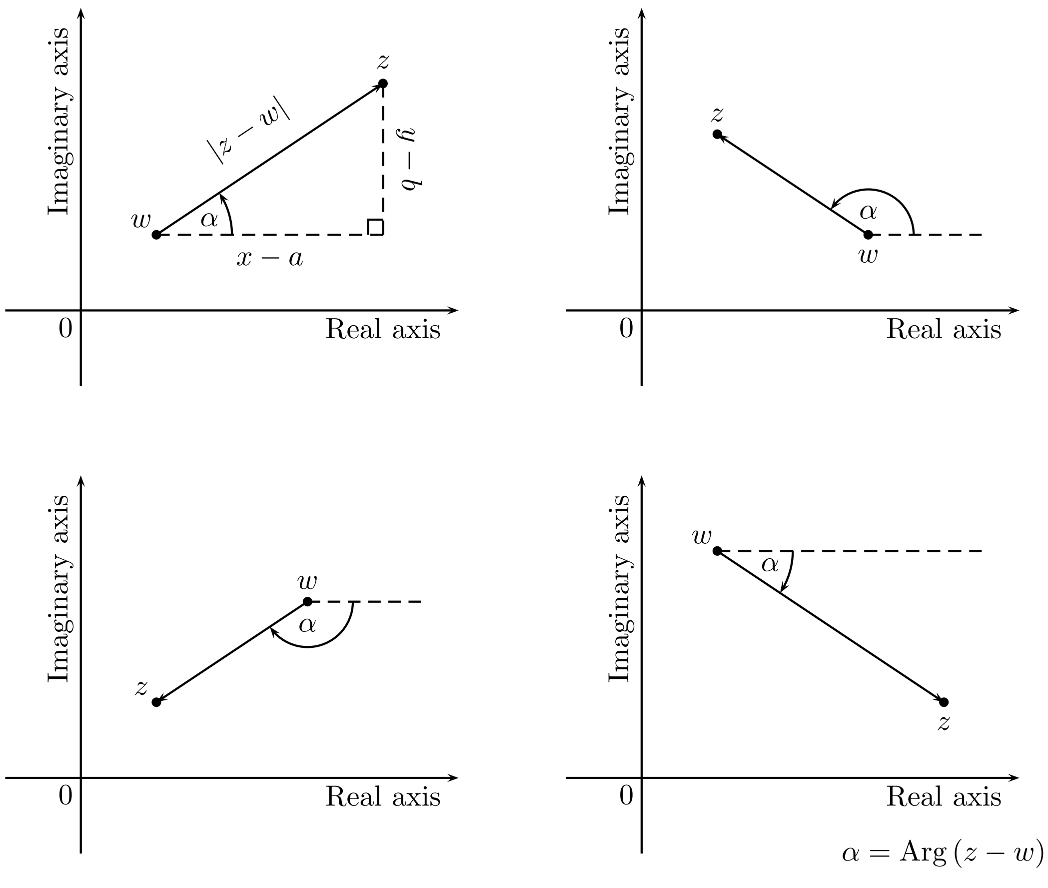


Figure 13: Geometric Interpretation of $\text{Arg}\,(z - w)$.

The fact that $|z-w|$ is the distance between the points z and w and that $\text{Arg}\,(z - w)$ is the angle between the arrow from w to z and a line in the direction of the positive real axis are important and should be remembered. ◊

Example 2. Give a geometric interpretation of each of the following sets of points

a) $|z - 4 - i| = 3$, and b) $|z - 4 - i| \leqslant 3$.

Solution. $|z-4-i|$ is the distance of a point z from the point $4+i$. Therefore $|z-4-i|=3$ says z is at a distance of 3 from the point $4+i$. Thus, the equation describes a circle of radius 3 with centre at the point $4+i$. Similarly, $|z-4-i| \leqslant 3$ is the region in which all points are at a distance at most 3 from the point $4+i$, and hence the inequality describes the disc of radius 3 with centre at $4+i$ (with the boundary circle included). The plots are shown in Figures 14 and 15.

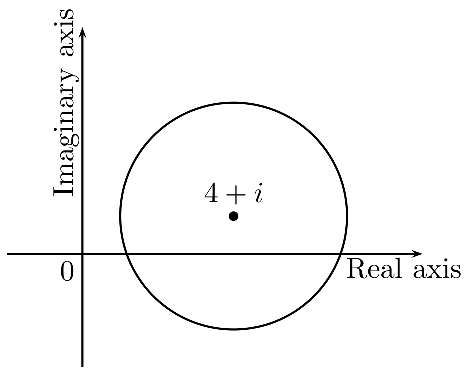


Figure 14: $\{z \in \mathbb{C} : |z-4-i| = 3\}$.

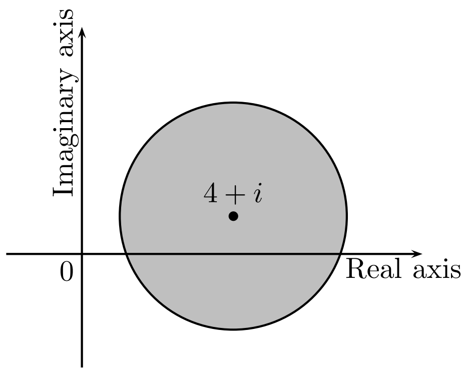


Figure 15: $\{z \in \mathbb{C} : |z-4-i| \leqslant 3\}$.

An alternative way of arriving at a geometric interpretation of a set is to use the "$x+yi$" (or Cartesian) form of a complex number. For example, using $z = x+yi$, we can square the equation $|z-4-i| = 3$ and rewrite it to obtain

$$|z-4-i|^2 = |(x-4)+(y-1)i|^2 = (x-4)^2 + (y-1)^2 = 9,$$

which corresponds to the equation of a circle with centre at $x=4$ and $y=1$ and of radius 3. $\Diamond$

Example 3. Sketch the set $\left\{z \in \mathbb{C} : -\dfrac{\pi}{3} \leqslant \operatorname{Arg}(z-4-i) \leqslant \dfrac{\pi}{4}\right\}$.

Solution. The above set is the set of all complex numbers z such that

$$-\frac{\pi}{3} \leqslant \operatorname{Arg}(z-4-i) \leqslant \frac{\pi}{4}.$$

Now, as shown in Example 1, $\operatorname{Arg}(z-4-i)$ represents the angle between the line segment from $4+i$ to z and a line parallel to the real axis, and hence the required sketch is as shown in Figure 16. Notice that the point $z = 4+i$ is omitted from the set. This is because the function $\operatorname{Arg}(z-4-i)$ is undefined there. The boundary line segments are

$$\left\{z \in \mathbb{C} : \operatorname{Arg}(z-4-i) = -\frac{\pi}{3}\right\},$$

$$\left\{z \in \mathbb{C} : \operatorname{Arg}(z-4-i) = \frac{\pi}{4}\right\}.$$

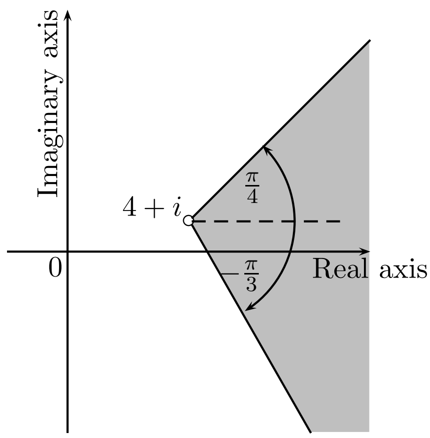
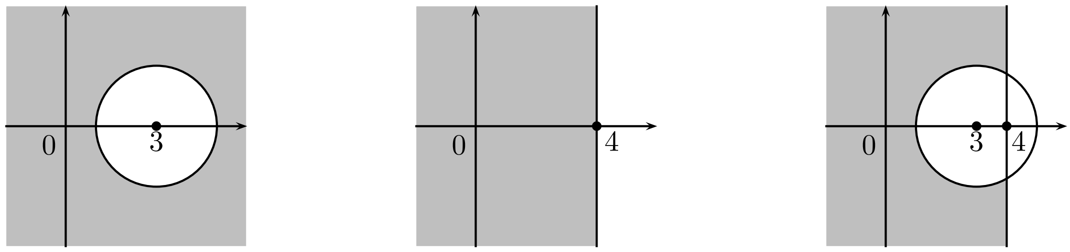


Figure 16: $\left\{z : -\dfrac{\pi}{3} \leqslant \operatorname{Arg}(z-4-i) \leqslant \dfrac{\pi}{4}\right\}$.

$\Diamond$

Example 4. Sketch the set $\{z \in \mathbb{C} : |z-3| \geqslant 2 \text{ and } \mathrm{Re}(z) \leqslant 4\}$.

SOLUTION. The given set is the set of all complex numbers z which satisfy both $|z-3| \geqslant 2$ and $\mathrm{Re}(z) \leqslant 4$. The inequality $|z-3| \geqslant 2$ describes the region on and outside the circle of radius 2 centred at 3, while $\mathrm{Re}(z) = x \leqslant 4$ corresponds to the half-plane on and to the left of the line $x = 4$. The required set is the intersection of these two regions as both inequalities must be satisfied. The plots are shown in Figure 17. In practise, only the last diagram need to be shown.



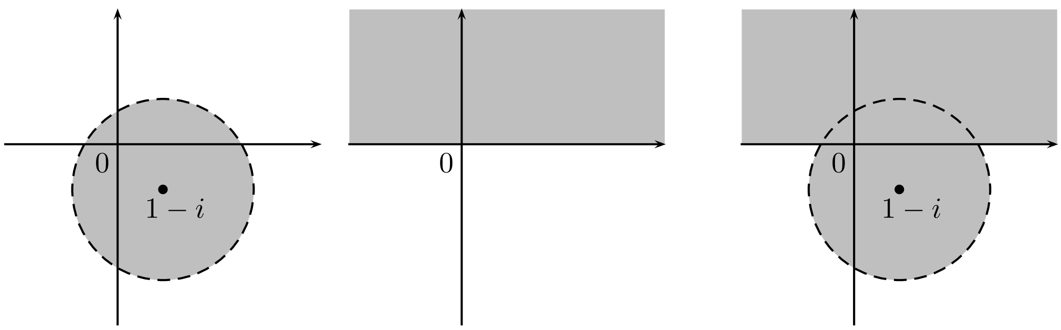
$\{z \in \mathbb{C} : |z-3| \geqslant 2\}$ $\{z \in \mathbb{C} : \mathrm{Re}(z) \leqslant 4\}$ $\{z \in \mathbb{C} : |z-3| \geqslant 2 \text{ and } \mathrm{Re}(z) \leqslant 4\}$

Figure 17: Argand Diagrams for Example 4.

$\Diamond$

Example 5. Sketch the set $\{z \in \mathbb{C} : |z-1+i| < 2 \text{ or } \mathrm{Im}(z) \geqslant 0\}$.

SOLUTION. The inequality $|z-1+i| < 2$ represents the open disc of radius 2 with centre at $1-i$.



$\{z \in \mathbb{C} : |z-1+i| < 2\}$ $\{z \in \mathbb{C} : \mathrm{Im} z \geqslant 0\}$ $\{z \in \mathbb{C} : |z-1+i| < 2 \text{ or } \mathrm{Im}(z) \geqslant 0\}$

Figure 18: Argand Diagrams for Example 5.

To indicate that the circle $|z-1+i| = 2$ is not included in the set, we draw it with dashes. Also, $\mathrm{Im}(z) = y \geqslant 0$ describes the half-plane on and above the real axis. The required set is the union of these two regions (as **either** inequality being satisfied puts the point in the set). $\Diamond$

3.10 Complex polynomials

We saw at the beginning of this chapter that there are polynomials which have no roots in the real numbers. Instead we defined the number i to be the solution for $x^2 + 1 = 0$. We shall now investigate in more detail the properties of polynomials and their roots when we work over the complex numbers.

Definition 1. *Suppose n is a natural number and $a_0, a_1, \ldots, a_n$ are complex numbers with $a_n \neq 0$. Then the function $p : \mathbb{C} \longrightarrow \mathbb{C}$ such that*

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

is called a polynomial of degree n.
The zero polynomial is defined to be the function $p(z) = 0$ and we do not define its degree.

Note. If $a_0, a_1, \ldots, a_n$ are real and z takes only real values, then we say that the polynomial is defined over $\mathbb{R}$.

Example 1. $p_1(z) = 1 + 3z^2$, $p_2(z) = 1 + i - 4iz^3$ are examples of polynomials.

Note that $p_1(z)$ has real coefficients. If it takes complex values, we say it is a polynomial defined over $\mathbb{C}$. However, if it only takes real values, we say it is a polynomial defined over $\mathbb{R}$.

3.10.1 Roots and factors of polynomials

An important mathematical and practical problem concerning polynomials is that of factorising them into simpler polynomials. This factorisation problem is closely related to the problem of finding the roots (or zeroes) of polynomials.

Definition 2. *A number α is a* **root** *(or* **zero***) of a polynomial p if $p(\alpha) = 0$.*

Definition 3. *Let p be a polynomial. Then, if there exist polynomials p_1 and p_2 such that $p(z) = p_1(z)p_2(z)$ for all complex z, then p_1 and p_2 are called* **factors** *of p.*

Theorem 1 (Remainder Theorem). *The remainder r which results when $p(z)$ is divided by $z - \alpha$ is given by $r = p(\alpha)$.*

Theorem 2 (Factor Theorem). *A number α is a root of p if and only if $z - \alpha$ is a factor of $p(z)$.*

The major difference between polynomials over the complex numbers and polynomials over the real numbers is contained in the following theorem.

Theorem 3 (The Fundamental Theorem of Algebra). *A polynomial of degree $n \geqslant 1$ has at least one root in the complex numbers.*

We shall not try to prove the fundamental theorem here. The proof is usually given in courses on functions of a complex variable.

There are several important points to note about the fundamental theorem.

The theorem is not true in general for polynomials over $\mathbb{R}$. For example, the real quadratic q defined by $q(z) = 1 + z^2$ does not have any real roots. In contrast, it has the two complex roots $\pm i$.

We can combine the Fundamental Theorem of Algebra and the Factor Theorem to prove the following extremely important theorem.

Theorem 4 (Factorisation Theorem). *Every polynomial of degree $n \geqslant 1$ has a factorisation into n linear factors of the form*

$$p(z) = a(z - \alpha_1)(z - \alpha_2) \dots (z - \alpha_n), \tag{\#}$$

where the n complex numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ are roots of p and where a is the coefficient of z^n.

[**X**] *Proof.* To prove the theorem we use induction on the degree of p.
Condition (#) is obviously true for $\deg(p) = 1$. We now prove that, for $n \geqslant 1$, (#) is true for polynomials of degree $n + 1$ whenever it is true for polynomials of degree n.

Let p be a polynomial of degree $n + 1$, where $n \geqslant 1$. Then, by the Fundamental Theorem there exists a root $\alpha_1 \in \mathbb{C}$ of p, and by the Factor Theorem $z - \alpha_1$ is a factor of p. Hence $p(z) = (z - \alpha_1)p_1(z)$, where p_1 is a polynomial of degree n. Now, if (#) is true for polynomials of degree n, then there are n complex numbers $\alpha_2, \dots, \alpha_{n+1}$ and a complex number a such that

$$p_1(z) = a(z - \alpha_2) \dots (z - \alpha_{n+1}).$$

Thus,

$$p(z) = (z - \alpha_1)p_1(z) = a(z - \alpha_1)(z - \alpha_2) \dots (z - \alpha_{n+1}).$$

Hence, (#) is true for polynomials of degree $n + 1$ whenever it is true for polynomials of degree n, and as (#) is true for $n = 1$, it is true for all $n \geqslant 1$, by induction.

Finally, note that in the factorisation shown in (#) a is the coefficient of the z^n term in p. $\square$

Example 2. Find the roots and factors of the quadratics

$$p_1(z) = z^2 + 2z + 1, \qquad p_2(z) = 2z^2 - 9z + 4 \qquad \text{and} \qquad p_3(z) = z^2 + z + 1.$$

Solution. It is straightforward to factorise $p_1(z)$:

$$p_1(z) = (z + 1)^2.$$

So the polynomial has a repeated root -1, with multiplicity 2.

Likewise, we have

$$p_2(z) = (2z - 1)\,(z - 4).$$

The roots of this polynomial are $\frac{1}{2}$ and 4.

The polynomial $p_3(z)$ does not easily factor. Using the quadratic formula, the roots are $z = \dfrac{-1 \pm i\sqrt{3}}{2}$, so

$$p_3(z) = \left(z - \frac{1}{2}\left(-1 + i\sqrt{3}\right)\right)\left(z - \frac{1}{2}\left(-1 - i\sqrt{3}\right)\right).$$

$\Diamond$

The Factorisation Theorem guarantees that a polynomial of degree n always has n roots, but it does not tell us how to actually find these roots. For polynomials of degree $n \leqslant 4$, exact formulae for the roots in terms of the coefficients have been found. For quadratics, the exact formula for the roots is well known and very easy to use. For cubic polynomials (degree 3) the exact formula is called Cardano's formula and it is also reasonably easy to use. For quartic polynomials (degree 4) there is also an exact formula for the roots but the formula is very complicated. However, for polynomials of degree $n > 4$, there are no formulae for the roots in terms of square roots, cube roots, etc. A proof of this fact is given in courses on Galois Theory.

In general, it is either difficult or impossible to find exact roots, and hence an exact factorisation, for higher degree polynomials. It is usually necessary to resort to approximate numerical methods to find the roots of a polynomial. At the present time the best general-purpose numerical method is based on finding the "eigenvalues" of a "companion matrix" for the polynomial. These numerical methods are discussed in advanced courses on numerical matrix algebra.

There are, however, some very simple types of higher degree polynomials for which exact roots, and hence an exact factorisation, can be found. One such case is for polynomials of the form $p(z) = z^n - a$, since in this case the solutions of

$$z^n - a = 0 \qquad \text{or equivalently of} \qquad z^n = a,$$

are the nth roots of the number a. This problem has been discussed in Section 3.7.3.

Example 3. Factorise $z^6 + 1$.

SOLUTION. We first solve $z^6 + 1 = 0$ or $z^6 = -1$ by finding the sixth roots of -1. Since the modulus and argument of -1 are $|-1| = 1$ and $\text{Arg}(-1) = \pi$, we have

$$z^6 = e^{(\pi + 2k\pi)i} \quad \text{for } k \in \mathbb{Z}.$$

So, the sixth roots in polar forms are

$$z = e^{i\frac{(2k+1)\pi}{6}}, \text{ for } k = -3, -2, -1, 0, 1, 2.$$

That is, $e^{-i\frac{5\pi}{6}}$, $e^{-i\frac{\pi}{2}}$, $e^{-i\frac{\pi}{6}}$, $e^{i\frac{\pi}{6}}$, $e^{i\frac{\pi}{2}}$, and $e^{i\frac{5\pi}{6}}$.

By the Factor Theorem, we have

$$z^6 + 1 = \left(z - e^{i\pi/2}\right)\left(z - e^{-i\pi/2}\right)\left(z - e^{i\pi/6}\right)\left(z - e^{-i\pi/6}\right)\left(z - e^{5i\pi/6}\right)\left(z - e^{-5i\pi/6}\right).$$

An alternative form can be obtained by putting the roots into the "$a + ib$" form. You should do this as an exercise, and you should obtain the result

$$z^6 + 1 = (z-i)(z+i)\left(z - \frac{1}{2}\left(\sqrt{3} + i\right)\right)\left(z - \frac{1}{2}\left(\sqrt{3} - i\right)\right)\left(z - \frac{1}{2}\left(-\sqrt{3} + i\right)\right)\left(z - \frac{1}{2}\left(-\sqrt{3} - i\right)\right).$$

$\Diamond$

3.10.2 Factorisation of polynomials with real coefficients

This is an important special case of the general theory. In the examples of factorisation we have given above in Examples 2 and 3, all of the polynomials have real coefficients. For the quadratics in Example 2, we showed one case of two equal real roots, one case of two distinct real roots, and one case of two complex roots. In Example 3, we obtained six complex roots of z^6+1. However, if you examine the complex roots in Examples 2 and 3, you will see that the roots occur in conjugate pairs. Thus, in the quadratic example the roots $(-1+i\sqrt{3})/2$ and $(-1-i\sqrt{3})/2$ are a conjugate pair, while in the sixth-root example there are three conjugate pairs $\left\{e^{-i\pi/2}, e^{i\pi/2}\right\}$, $\left\{e^{-i\pi/6}, e^{i\pi/6}\right\}$, and $\left\{e^{-5i\pi/6}, e^{5i\pi/6}\right\}$. These results are examples of a useful proposition which is:

Proposition 5. If α is a root of a polynomial p with **real** coefficients, then the complex conjugate $\overline{\alpha}$ is also a root of p.

Proof. Let p be a polynomial with real coefficients given by

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \quad \text{for all} \quad z \in \mathbb{C},$$

where $a_0, a_1, \ldots, a_n$ are real numbers. Then, since α is a root of p,

$$p(\alpha) = a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n = 0.$$

On taking the complex conjugate of this equation, and using the facts that if a_k is real and b is any complex number, then

$$\overline{a_k b} = \overline{a_k}\ \overline{b} = a_k \overline{b}$$

and that the complex conjugate of α^k is $(\overline{\alpha})^k$, we have

$$0 = \overline{p(\alpha)} = a_0 + a_1\overline{\alpha} + a_2\overline{\alpha}^2 + \cdots + a_n\overline{\alpha}^n = p(\overline{\alpha}).$$

Hence, $\overline{\alpha}$ is also a root of p. □

An immediate consequence of this proposition is that the roots of a complex polynomial with real coefficients are either real or occur in conjugate pairs. This fact can be used to obtain a factorisation of a polynomial with real coefficients into linear or quadratic factors with real coefficients.

Proposition 6. If p is a polynomial with real coefficients, then p can be factored into linear and quadratic factors all of which have real coefficients.

Proof. For a polynomial with real coefficients, if α is a root which is not real then so is $\overline{\alpha}$, and hence both $z-\alpha$ and $z-\overline{\alpha}$ are factors of p. On multiplying these factors, we obtain

$$(z-\alpha)(z-\overline{\alpha}) = z^2 - (\alpha+\overline{\alpha})z + \alpha\overline{\alpha}.$$

Then, from section 3.3, we have that

$$\alpha + \overline{\alpha} = 2\mathrm{Re}(\alpha), \qquad \text{and} \qquad \alpha\overline{\alpha} = |\alpha|^2,$$

where $\mathrm{Re}(\alpha)$ and $|\alpha|^2$ are both real numbers. Hence,

$$q(z) = (z-\alpha)(z-\overline{\alpha}) = z^2 - 2\mathrm{Re}(\alpha)\,z + |\alpha|^2$$

is a quadratic with real coefficients. Thus, all complex factors can be replaced in pairs by quadratic factors with real coefficients, and the proof is complete. □

Example 4. Factorise $z^5 + 32$ into linear and quadratic factors with real coefficients.

SOLUTION. We first solve $z^5 = -32$ to find the fifth roots of -32. Using the usual procedure, we obtain the five solutions

$$2e^{i\pi/5}, \qquad 2e^{i3\pi/5}, \qquad 2e^{-i\pi/5}, \qquad 2e^{i\pi} = -2, \qquad 2e^{-3i\pi/5},$$

which consist of one real root and two conjugate pairs.

The factors corresponding to a conjugate pair can be replaced by a quadratic factor with real coefficients. For example,

$$\begin{aligned}\left(z - 2e^{i\pi/5}\right)\left(z - 2e^{-i\pi/5}\right) &= z^2 - 2\left(e^{i\pi/5} + e^{-i\pi/5}\right) z + 4 \\ &= z^2 - 4z \cos\frac{\pi}{5} + 4.\end{aligned}$$

The required factorisation is therefore

$$\begin{aligned}z^5 + 2 &= (z+2)\left(z - 2\,e^{i\pi/5}\right)\left(z - 2\,e^{-i\pi/5}\right)\left(z - 2\,e^{3i\pi/5}\right)\left(z - 2\,e^{-3i\pi/5}\right) \\ &= (z+2)\left(z^2 - 4z\cos\frac{\pi}{5} + 4\right)\left(z^2 - 4z\cos\frac{3\pi}{5} + 4\right).\end{aligned}$$

Note that this factorisation is certainly not obvious. If you multiply out the right hand side and compare coefficients you will find some very surprising relations between $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$. ◊

3.11 Appendix: A note on proof by induction

In mathematics and logic a proposition is a meaningful statement which can only be either true or false. Proof by induction is a method of proof which is often used to prove that some proposition $P(n)$ is true for all integers $n \geqslant n_0$, where n_0 is a fixed integer.

Example 1. For each integer $n \geqslant 1$, let $P(n)$ be the proposition

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1).$$

In this example, $P(n)$ gives the formula for the sum of the first n positive integers. ◊

Example 2. For each integer $n \geqslant 0$, let $P(n)$ be the proposition

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r},$$

where $r \neq 1$ is a fixed number. In this example, $P(n)$ gives the formula for the sum of $n+1$ terms of a geometric progression with ratio r. ◊

Example 3. For each integer $n \geqslant 2$, let $P(n)$ be the proposition that "n can be completely factored into prime numbers". ◊

As we shall see, all the propositions $P(n)$ in examples 1, 2 and 3 are true.

There are several versions of "proof by induction", but the two most commonly used are based on what are sometimes called the first and second principles of induction.

First Principle of Induction. Let $n_0 \in \mathbb{Z}$, and let $P(n)$, for $n \geqslant n_0$, be propositions. Then, if

1. $P(n_0)$ is true, and
2. for each $n \geqslant n_0$, $P(n+1)$ is true whenever $P(n)$ is true,

then $P(n)$ is true for all $n \geqslant n_0$.

Let us return to the previous examples.

Example 1. We prove the formula for the sum of the first n integers, i.e., that:
For $n \geqslant 1$,

$$1+2+3+\cdots+n=\frac{1}{2}n(n+1).$$

Proof. We first note that the proposition $P(1)$ states that $1=\frac{1}{2}\times 1\times 2$, which is clearly true.

Now, if $P(n)$ is true for some fixed $n \geqslant 1$, then the sum of $n+1$ integers is

$$\begin{aligned}1+2+\cdots+n+(n+1)&=\frac{1}{2}n(n+1)+(n+1)\\&=\frac{1}{2}(n+1)(n+2)=\frac{1}{2}(n+1)\big((n+1)+1\big),\end{aligned}$$

and hence $P(n+1)$ is also true. Thus, for all $n \geqslant 1$, $P(n+1)$ is true whenever $P(n)$is true.

Therefore, from the first principle of induction, $P(n)$ is true for all integers $n \geqslant 1$. ☐

Example 2. We prove the formula for the sum of a geometric progression, i.e., that:
For $n \in \mathbb{N}$ and $r \neq 1$,

$$1+r+\cdots+r^n=\frac{1-r^{n+1}}{1-r}.$$

Proof. We first note that the proposition $P(0)$ states that

$$1=\frac{1-r}{1-r},$$

which is clearly true for $r \neq 1$.

Now, if $P(n)$ is true for some fixed integer $n \geqslant 0$, then the sum of $n+2$ terms is

$$1+r+\cdots+r^n+r^{n+1}=\frac{1-r^{n+1}}{1-r}+r^{n+1}=\frac{1-r^{n+2}}{1-r}=\frac{1-r^{(n+1)+1}}{1-r},$$

and hence $P(n+1)$ is also true. Thus, for any integer $n \geqslant 0$, $P(n+1)$ is true whenever $P(n)$ is true.

Therefore, from the first principle of induction, $P(n)$ is true for all $n \geqslant 0$. ☐

We now briefly describe the second principle of induction.

Second Principle of Induction. Let $n_0 \in \mathbb{Z}$, and let $P(n)$, for $n \geqslant n_0$, be propositions. Then, if

1. $P(n_0)$ is true, and

2. for each $n \geqslant n_0$, $P(n+1)$ is true whenever all propositions $P(m)$ are true for $m = n_0, \ \ldots, \ n$ (i.e. $n_0 \leqslant m \leqslant n$),

then $P(n)$ is true for all $n \geqslant n_0$.

Example 3. Every integer $n \geqslant 2$ can be factored into primes.

Proof. The proposition $P(n)$ is that n can be factored into primes. $P(2)$ then asserts that 2 can be factored into primes. This result is clearly true, since 2 is itself a prime.

We now show that, if $P(m)$ is true for all integers m with $2 \leqslant m \leqslant n$, then $P(n+1)$ is also true.

Now, the integer $n+1$ must either be prime or not prime.
Case 1. $n+1$ is prime. Then $n+1$ is already factored and hence $P(n+1)$ is true in this case.
Case 2. $n+1$ is not prime. Then, there are two integers m_1 and m_2 less than $n+1$ such that $n+1 = m_1 m_2$. But, as $P(m)$ is true for $2 \leqslant m \leqslant n$, m_1 and m_2 can both be factored into primes, and hence $n+1$ can also be factored into primes. Thus, $P(n+1)$ is true in this case also.

We have shown that $P(2)$ is true, and that $P(n+1)$ is true whenever $P(m)$ is true for all m with $2 \leqslant m \leqslant n$. Hence, from the second principle of induction, $P(n)$ is true for all integers $n \geqslant 2$. $\square$

In fact the two apparently different types of induction are the same, as we can easily show. Simply let the proposition $Q(n)$ be "$P(m)$ is true for $m = n_0, \ \ldots, \ n$". Then use the first type of proof.

3.12 Appendix: The Binomial Theorem

Theorem 1. *If $a, b \in \mathbb{C}$ and $n \in \mathbb{N}$, then*

$$\begin{aligned}(a+b)^n &= a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + nab^{n-1} + b^n \\ &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\, b^k,\end{aligned}$$

where the numbers

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

are the binomial coefficients.

Note. We are using the convention that $a^0 = 1$ and $0! = 1$.

Proof. The proof of the theorem is based on the first principle of induction. In this case, the proposition $P(n)$ to be proved true is that the formula given above for $(a+b)^n$ is correct.
For $n = 1$, $P(1)$ asserts that $(a+b)^1 = a^1 + b^1$, which is clearly true.

Now, if the formula is correct for some integer $n \geqslant 1$, then we have

$$\begin{aligned}(a+b)^{n+1} &= (a+b)(a+b)^n \\ &= (a+b)\sum_{k=0}^{n}\binom{n}{k}a^{n-k}\,b^k.\end{aligned}$$

On multiplying out, we find that the coefficient of a^{n+1} and b^{n+1} are 1 and that the coefficient of $a^{n+1-k}b^k$ is

$$\binom{n}{k}+\binom{n}{k-1}.$$

Now,

$$\begin{aligned}\binom{n}{k}+\binom{n}{k-1} &= \frac{n!}{k!(n-k)!}+\frac{n!}{(k-1)!(n-k+1)!} \\ &= \frac{n!}{k!(n-k+1)!}(n-k+1+k) \\ &= \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k}.\end{aligned}$$

Hence,

$$\begin{aligned}(a+b)^{n+1} &= a^{n+1}+\sum_{k=1}^{n}\binom{n+1}{k}a^{n+1-k}b^k+b^{n+1} \\ &= \sum_{k=0}^{n+1}\binom{n+1}{k}a^{n+1-k}b^k.\end{aligned}$$

and hence $P(n+1)$ is true.

We have therefore shown that $P(1)$ is true, and that, for all $n \geqslant 1$, $P(n+1)$ is true whenever $P(n)$ is true. Thus, $P(n)$ is true for all $n \geqslant 1$ by induction. $\square$

NOTE. The relation

$$\binom{n}{k-1}+\binom{n}{k}=\binom{n+1}{k}$$

shows why Pascal's triangle works.

3.13 Complex numbers and Maple

Maple is a Symbolic Computing Package which enables computers to do algebra and calculus. Information about computing and Maple is available from the School of Mathematics web site, my eLearning Vista, the MATH1131/1141 information booklet and the Computing Notes for the First Year Mathematics Courses.

This section gives a few hints on how it can be used for complex numbers. The symbol `I` is used for the complex number i. Multiplication is denoted by $*$ and exponentiation by the 'circumflex' or 'uparrow' ^. Complex number arithmetic is normally done automatically. However, in some situations the command `evalc` is required to get a complex expression in $a + ib$ from, where a, b are real. Some examples are:

```
(3+5*I)^3;
```

gives the correct value of $(3 + 5i)^3 = -198 + 10i$ on the screen.

```
Re(%);
```

gives the real part of this last expression. If you now try:

```
conjugate(%%);
```

the complex conjugate of the expression before last will appear. The command

```
evalc(polar(2,Pi));
```

can be used to enter complex numbers (here $2e^{\pi i}$) in polar form and this command gives the correct value -2.

Instructions on how to convert complex numbers into polar form are available 'on line' by means of

```
?convert[polar]
```

Alternatively, you can use Maple Help menu. Various other conversion operations are also available.

Problems for Chapter 3

"Why," said the Dodo, "the best way to explain it is to do it."
- Lewis Carroll, Alice in Wonderland.

Questions marked with [**R**] are routine, [**H**] harder, [**M**] Maple and [**X**] are for MATH1141 only. You should make sure that you can do the easier questions before you tackle the more difficult questions. Questions marks with a [**V**] have video solutions available on Moodle.

Problems 3.1 : A review of number systems

1. [**R**] Solve (if possible) the following equations for $x \in \mathbb{N}$, $x \in \mathbb{Z}$, $x \in \mathbb{Q}$ and $x \in \mathbb{R}$.

   a) $x + 25 = 0$, $3x - 9 = 0$, $3x + 9 = 0$, $3x + 10 = 0$.
   b) $x^2 + 4x - 5 = 0$, $2x^2 - 13x + 15 = 0$, $x^2 - x - 1 = 0$, $x^2 + 3x + 4 = 0$.
   c) $\sin(\pi x/3) = 0$, $\sin(x/3) = 0$.

2. [**R**] Is the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ closed under addition? Prove your answer.

3. [**H**] Can any finite set of integers be closed under addition? Prove your answer.

4. [**R**] Is the set $\{-1, 1\}$ closed under multiplication and division?

Problems 3.3 : The rules of arithmetic for complex numbers

5. [**R**][**V**] Let $z = 2 + 3i$, $w = -1 + 2i$. Calculate $3z$, z^2, $z + 2w$, $z(w + 3)$, $\dfrac{z}{w}$, $\dfrac{w}{z}$.

6. [**R**] Write the following expressions in $a + ib$ or "Cartesian" form:

   a) $\dfrac{1+i}{1+2i}$, b) $\dfrac{2-i}{3+i} - \dfrac{3-i}{2+i}$.

7. [**R**] If $z = a + ib$, express the following in "Cartesian" form:

   a) z^2, b) $\dfrac{1}{z}$ c) $\dfrac{z+1}{z-1}$

8. [**R**][**V**] Use the quadratic formula to find all complex roots of the following polynomials.
   a) $z^2 + z + 1$, b) $z^2 + 2z + 3$, c) $z^2 - 6z + 10$,
   d) $-2z^2 + 6z - 3$, e) $z^4 + 5z^2 + 4$.

9. [**H**] Show that $\left[\left(\sqrt{3}+1\right) + \left(\sqrt{3}-1\right)i\right]^3 = 16(1+i)$.

10. [**R**] Simplify $(r+s)^2$ where r is the square root of $3 + 4i$ with non negative real part and s is the square root of $3 - 4i$ with non negative real part.

11. **[H][V]** Simplify $\left(\dfrac{a+bi}{a-bi}\right)^2 - \left(\dfrac{a-bi}{a+bi}\right)^2$ where a and b are real numbers not both zero.

Problems 3.4 : Real parts, imaginary parts and complex conjugates

12. **[R]** Find $\text{Re}(z)$, $\text{Im}(z)$ and $\overline{z}$ for $z = -1+i,\ 2+3i,\ 2-3i,\ \dfrac{2-i}{1+i},\ \dfrac{1}{(1+i)^2}$.

13. **[R]** Let $z = 1+2i$ and $w = 3-4i$. Calculate z^2 and $\dfrac{\overline{z}}{w}$, expressing the answers in Cartesian form.

14. **[R][V]** Given that $2z+3w = 1+12i$ and $\overline{z}-\overline{w} = 3-i$, find z and w.

15. **[R]** By evaluating each side of the equations, check that $\overline{zw} = \overline{z}\,\overline{w}$, and $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$ are satisfied by the complex numbers $z = 2+3i$, $w = -1+2i$.

16. **[R]** Prove that for any two complex numbers z and w

a) $\text{Im}(z) = \dfrac{1}{2i}(z-\overline{z})$ b) $2\text{Re}(z) = z+\overline{z}$

c) $\overline{(z-w)} = \overline{z}-\overline{w}$ d) $\overline{\left(\dfrac{1}{z}\right)} = \dfrac{1}{\overline{z}}$,

e) $\overline{zw} = \overline{z}\,\overline{w}$ f) $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$.

17. **[H]**
   a) Use the properties of the complex conjugate to show that if the complex number α is a root of a quadratic equation $ax^2+bx+c=0$ with a,b,c being **real** coefficients, then so is $\overline{\alpha}$.
   b) Write down the monic quadratic polynomial with real coefficients which has $3-2i$ as one of its roots.
   c) Does the result of a) generalise to higher degree polynomials?

Problems 3.5 : The Argand diagram and 3.6 : Polar form, modulus and argument

18. **[R][V]** Find the modulus, principal argument and polar form of each of the following numbers and plot them on an Argand diagram:

a) $6+6i$, b) -4, c) $\sqrt{3}-i$, d) $\dfrac{-1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}}$, e) $-7+3i$.

19. **[R]** If $z = 4+3i$ and $w = 2+i$ find $|3z-3iw|$, $\text{Im}\big((1-i)z - 3|w|\big)$.

20. **[H]** If $z = 1+i$, calculate the powers z^j for $j = 1,2,\ldots,10$ and plot them on an Argand diagram. Is there a pattern? What is the smallest positive integer n such that z^n is a real number?

21. [**R**] Find the "$a+ib$" form of the complex numbers whose moduli and principal arguments are

a) $|z| = 3,\ \mathrm{Arg}(z) = \dfrac{\pi}{3}$;   b) $|z| = 3,\ \mathrm{Arg}(z) = \dfrac{5\pi}{6}$;

c) $|z| = 3,\ \mathrm{Arg}(z) = -\dfrac{2\pi}{3}$;   d) $|z| = 3,\ \mathrm{Arg}(z) = -\dfrac{\pi}{6}$;

e) [**H**] $|z| = 3,\ \mathrm{Arg}(z) = \dfrac{\pi}{8}$.

22. [**R**][**V**] a) Show that $z\,\overline{z} = |z|^2$. Hence, or otherwise, show that if $|z| = 1$, then $\overline{z} = z^{-1}$.

b) Show that $|z| = |\overline{z}|$ for all $z \in \mathbb{C}$.

c) If $z = r(\cos\theta + i\sin\theta)$, show that a polar form for the complex conjugate is $\overline{z} = r\,(\cos(-\theta) + i\sin(-\theta))$.

23. [**H**] Show that $\mathrm{Re}\left(\dfrac{1-z}{1+z}\right) = 0$ for any complex z with $|z| = 1$.

24. [**H**] Use $z\overline{z} = |z|^2$ to prove the identity $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$.

25. [**H**] Use $z\overline{z} = |z|^2$ to show that

$$|1 - \overline{z}w|^2 - |z - w|^2 = (1 - |z|^2)\,(1 - |w|^2)$$

and deduce that $|1 - \overline{z}w|^2 = |z - w|^2$ if either z or w lies on the unit circle.

26. [**R**] Plot the following complex numbers on an Argand diagram:

a) $2e^{\frac{i\pi}{4}}$,   b) $3e^{\frac{5i\pi}{6}}$,   c) $e^{-\frac{2i\pi}{3}}$,   d) $2e^{-\frac{i\pi}{2}}$,   e) $4e^{i\pi}$.

27. [**R**] Let $z = (1 - i)$ and $w = 2e^{i\pi/3}$. Calculate w^6, $z - w$ and $\dfrac{w}{\overline{z}}$ and express your answers in Cartesian form.

28. [**R**] For $z = 3e^{-5\pi i/6}$ and $w = 1 + i$, find $\mathrm{Re}\left(iw + \overline{z}^2\right)$.

29. [**R**] Solve $|e^{i\theta} - 1| = 2$ for $-\pi < \theta \leqslant \pi$.

30. [**R**] Find $\mathrm{Arg}(-1+i)$ and $\mathrm{Arg}(-\sqrt{3}+i)$ and hence find the principal arguments of the complex numbers $(-1+i)(-\sqrt{3}+i)$ and $\dfrac{-1+i}{-\sqrt{3}+i}$.

31. [**H**][**V**] Let $z = (1 + \sqrt{3}i)$ and $w = (1 + i)$. Find $\mathrm{Arg}\,z$ and $\mathrm{Arg}\,w$ and hence $\mathrm{Arg}(zw)$. Evaluate zw and hence show that $\cos\dfrac{7\pi}{12} = \dfrac{1-\sqrt{3}}{2\sqrt{2}}$. Find a similar expression for $\sin\dfrac{7\pi}{12}$.

32. [**R**] Find polar forms for $z = 1 + i\sqrt{3}$ and $w = 1 - i$, and hence find first the polar forms and then the "$a + ib$" forms of zw, z^9, and $\left(\dfrac{z}{\overline{w}}\right)^{12}$.

Problems 3.7 : Properties and applications of the polar form

33. **[R]** Find the polar, and hence also the Cartesian form for:

    a) $\left(\sqrt{3}+i\right)^5$, b) $\left(\dfrac{-1+i}{\sqrt{2}}\right)^{1002}$, c) $\left(\dfrac{1+\sqrt{3}i}{2}\right)^{-8}$.

34. **[H]** Find the square roots (in Cartesian Form) of

    a) $21-20i$, b) $-16+30i$, c) $24+70i$.

35. **[H]** a) Explain why multiplying a complex number z by $e^{i\theta}$ rotates the point represented by z anticlockwise about the origin, through an angle θ.

    b) The point represented by the complex number $1+i$ is rotated anticlockwise about the origin through an angle of $\dfrac{\pi}{6}$. Find its image in polar and Cartesian form.

    c) Find the complex number (in Cartesian form) obtained by rotating $6-7i$ anticlockwise about the origin through an angle $\dfrac{3\pi}{4}$.

36. **[H]** If $z = re^{i\theta}, 0 \leqslant \theta \leqslant \dfrac{\pi}{2}$, show that

    a) $\left|(1-i)z^2\right| = \sqrt{2}r^2$ b) $\operatorname{Arg}\left((1-i)z^2\right) = 2\theta - \dfrac{\pi}{4}$,

    c) $\left|\dfrac{1+i\sqrt{3}}{z}\right| = \dfrac{2}{r}$, d) $\operatorname{Arg}\left(\dfrac{1+i\sqrt{3}}{z}\right) = \dfrac{\pi}{3} - \theta$.

37. **[H][V]** Find the roots (in Cartesian Form) of

    a) $z^2 - 3z + (3-i) = 0$, b) $z^2 - (7-i)z + (14-5i) = 0$,

    c) $z^2 + (4-i)z + (1+13i) = 0$.

38. **[R]** Find the seventh roots of -1 and plot the roots on an Argand diagram.

39. **[R]** Find the sixth roots of i and plot the roots on an Argand diagram.

40. **[H]** Find the fifth roots of $16 - 16i\sqrt{3}$ and plot the roots on an Argand diagram.

41. **[H]** Find all $z \in \mathbb{C}$ satisfying $(z-6+i)^3 = -27$.

42. **[H][V]** Show that if ω is an nth root of unity ($\omega \neq 1$ and $n > 1$) then

    $$\omega + \omega^2 + \cdots + \omega^n = 0.$$

    Hint: Sum the geometric progression.

43. **[X]** Show that the set $\{z \in \mathbb{C} : |z| \leqslant 1\}$ is closed under multiplication. Is the set closed under division (zero excluded)? Is the set closed under addition or subtraction?

44. [**X**] Use the properties of complex conjugates to show that if $a, b \in \mathbb{R}$ and $|z| = 1$, then $|a + bz| = |az + b|$. Hint: You might find the results of Question 22 useful.

45. [**X**] Suppose a and b are real numbers (not both zero) and $w = \dfrac{az + bz^{-1}}{bz + az^{-1}}$. Show that if $|z| = 1$, then $|w| = 1$.
Hint: You might find the results of Question 22 useful.

46. [**X**] Let z, w be complex numbers.

   a) Using polar forms, show that $|\mathrm{Re}(z\overline{w})| \leqslant |z|\,|w|$.

   b) Use the result in (a) to show that $|z + w| \leqslant |z| + |w|$, and interpret the result geometrically. Hint: Write $|z + w|^2 = (z + w)(\overline{z} + \overline{w})$ and expand.

47. [**X**] Let

$$z = \frac{i(1 + is)}{1 - is} \qquad \text{where} \quad s \in \mathbb{R}.$$

   a) Show that

$$\mathrm{Arg}(z) = \begin{cases} \dfrac{\pi}{2} + 2\tan^{-1} s & \text{for} \quad s \leqslant 1 \\[2ex] -\dfrac{3\pi}{2} + 2\tan^{-1} s & \text{for} \quad s > 1. \end{cases}$$

   b) Describe geometrically what happens to z as s increases from $-\infty$ to ∞.

48. [**H**] Suppose $\theta, \phi \neq \dfrac{\pi}{2}(2k + 1)$ where k is an integer. Use the fact that, for $z \neq -1$,

$$z = \frac{1 + z}{1 + z^{-1}}$$

   a) to find the real and imaginary parts of

$$\frac{1 + \cos 2\theta + i \sin 2\theta}{1 + \cos 2\theta - i \sin 2\theta};$$

   b) to show that if n is a positive integer then

$$\left(\frac{1 + \sin\phi + i\cos\phi}{1 + \sin\phi - i\cos\phi}\right)^n = \cos n\left(\frac{\pi}{2} - \phi\right) + i \sin n\left(\frac{\pi}{2} - \phi\right).$$

49. [**X**] For $n > 1$, let $\omega_1, \omega_2, ..., \omega_n$ be the n distinct nth roots of 1 and let A_k be the point on the Argand diagram which represents ω_k. Let P represent any point z on the unit circle, and let PA_k denote the distance from P to A_k.

   a) Prove that $(PA_k)^2 = (z - \omega_k)(\overline{z} - \overline{\omega_k})$.

b) Deduce that $\displaystyle\sum_{k=1}^{n}(PA_k)^2 = 2n$.

c) Now let P represent the point x on the real axis, $-1 < x < 1$, prove that

$$\prod_{k=1}^{n} PA_k = 1 - x^n.$$

Problems 3.8 : Trigonometric applications of complex numbers

50. **[R][V]** Using De Moivre's theorem and the binomial theorem, prove the identity

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

51. **[R]** a) Use De Moivre's Theorem to express $\cos 6\theta$ and $\sin 6\theta$ in terms $\cos\theta$ and $\sin\theta$.
    b) Write $\cos 6\theta$ in terms of $\cos\theta$ only

52. **[H]** Express $\cos 7\theta$ and $\sin 7\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$.

53. **[R]** a) Derive a formula for $\cos\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$.
    b) Deduce a formula for $\cos^6\theta$ in terms of $\cos k\theta$, $1 \leqslant k \leqslant 6$.
    c) Show that $\displaystyle\int_0^{\frac{\pi}{2}} \cos^6\theta \; d\theta = \frac{5\pi}{32}$.

54. **[R][V]** Express $\sin^5\theta$ and $\cos^4\theta$ in terms of sines or cosines of multiples of θ, and hence find their integrals.

55. **[X]** a) Use De Moivre's Theorem to express $\cos 5\theta$ as a polynomial $p(x)$ in $x = \cos\theta$.
    b) Put $\theta = 36^\circ = \frac{\pi}{5}$ and show that $x = \cos\frac{\pi}{5}$ is a root of $P(x) = 16x^5 - 20x^3 + 5x + 1$.
    c) Check that $P(x) = (x+1)(4x^2 - 2x - 1)^2$.
    d) What are the 5 roots of $P(x)$? Give full reasons for your answer.
    e) Deduce that $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} + \cos\frac{7\pi}{5} + \cos\frac{9\pi}{5} = 1$ and $\cos\frac{\pi}{5}\cos\frac{3\pi}{5}\cos\frac{7\pi}{5}\cos\frac{9\pi}{5} = \frac{1}{16}$.

56. **[X]** Let $\omega_1, \omega_2, \ldots, \omega_n$ be the n distinct nth roots of unity ($n \geqslant 1$). Show that if k is an integer then

$$\omega_1^k + \omega_2^k + \cdots + \omega_n^k$$

equals 0 or n. Find the values of k for which the sum is n.
Hint: Write the roots in polar form and sum the resulting geometric progression. See Example 5 of Section 3.8.

57. **[X]** Show that if θ is not a multiple of 2π, then the imaginary part of

$$\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} \quad \text{is} \quad \frac{\sin\left(\frac{1}{2}(n+1)\theta\right)\sin(\frac{1}{2}n\theta)}{\sin\frac{1}{2}\theta}.$$

Hint. See Example 5 of Section 3.8.

58. [**X**] Find the sum of

$$\sin\theta + \sin 2\theta + \cdots + \sin n\theta.$$

Hint. See previous exercise.

59. [**X**] a) Calculate the sum of the series

$$S = e^{i\theta} - \frac{e^{3i\theta}}{3^2} + \frac{e^{5i\theta}}{3^4} - \frac{e^{7i\theta}}{3^6} + \cdots$$

b) Hence show that

$$\sin\theta - \frac{\sin 3\theta}{3^2} + \frac{\sin 5\theta}{3^4} - \frac{\sin 7\theta}{3^6} + \cdots = \frac{72\sin\theta}{82 + 18\cos 2\theta}.$$

Problems 3.9 : Geometric applications of complex numbers

60. [**R**][**V**] Sketch the set of points on the complex plane corresponding to each of the following:

a) $|z-i| \leqslant 2$,

b) $|z-i| \leqslant 2$ or $-\dfrac{\pi}{3} \leqslant \operatorname{Arg}(z-i) \leqslant \dfrac{2\pi}{3}$,

c) $|z| \geqslant 2$ and $|\operatorname{Im}(z)| \leqslant 3$,

d) $\operatorname{Re}(z) \geqslant \operatorname{Im}(z)$,

e) $|z-i| = |z+i|$,

f) $|z-1-i| < 1$ and $-\dfrac{\pi}{4} < \operatorname{Arg}(z-1-i) \leqslant \dfrac{\pi}{2}$,

g) $|z-i| = 2|z+i|$,

h) [**X**] $|z-i| + |z+i| = 6$.

61. [**R**] Sketch the following on two carefully labelled Argand diagrams.

a) $S_1 = \{z \ : \ \operatorname{Re}(z) \geqslant 3\operatorname{Im}(z) \quad \text{and} \quad |z-(3+i)| > 2\}$,

b) $S_2 = \left\{z \ : \ |z-i| < |z+i| \quad \text{and} \quad -\dfrac{\pi}{6} \leqslant \operatorname{Arg}(z-i) \leqslant \dfrac{\pi}{6}\right\}$.

62. [**R**] Let $S = \{z \in \mathbb{C} \ : \ \operatorname{Im}(z) > -4 \quad \text{and} \quad |z-1-i| \geqslant 3\}$.

a) Sketch S on a carefully labelled Argand diagram.

b) Does $2+4i$ belong to S?

63. [**R**] Let z be a complex number. Prove that $|z - \operatorname{Re}(z)| \leqslant |z-x|$ for all real numbers x. Draw a sketch to illustrate the result.

64. [**H**] Let z, w be complex numbers.

a) Sketch the subset of the complex plane defined by $w = e^{i\alpha}$ for $-\pi < \alpha \leqslant \pi$.

b) Given that $\operatorname{Arg}(z) = \theta$, prove that $|z - e^{i\theta}| \leqslant |z - e^{i\alpha}|$ for all $\alpha \in \mathbb{R}$.

c) Give a geometric interpretation of the result in part b).

Problems 3.10 : Complex polynomials

65. **[R]** Use the remainder theorem to find the following remainders when.

    a) $2 + 3z - z^2 + 6z^3$ is divided by $z - 5$,
    b) $1 - 6z + 5z^2 - 8z^3 + 2z^4$ is divided by $z + 2$,
    c) $3z + 2z^2 + z^3$ is divided by $z - 1 - i$.

66. **[R][V]** Use the remainder theorem and the factor theorem to show that $z - 2$ is a factor of $p(z) = 30 - 17z - 3z^2 + 2z^3$. Then divide p by $z - 2$ and hence find all linear factors of p.

67. **[R]** Use the method of the previous question to show that $z - 1$ and $z + 2$ are factors of $p(z) = -8 - 6z + 7z^2 + 6z^3 + z^4$. Then find all linear factors of p.

68. **[R]** Find all linear factors of

    a) $z^5 + i$, b) $z^6 + 8$.

69. **[H]** a) Factorise $x^8 - 1$ into real linear and real quadratic factors.
    b) Repeat for $x^6 + 8$.

70. **[R][V]** Factorise $z^4 + 4$ over the rational numbers.

71. **[R]** Factorise the polynomial $z^4 + i$ into complex linear factors.

72. **[R][V]** a) Solve the equation $z^6 = -1$ where $z \in \mathbb{C}$.
    b) Plot your solutions from part a) as points in the Argand diagram.
    c) Write $z^6 + 1$ as a product of complex linear factors.
    d) Write $z^6 + 1$ as a product of real quadratic factors.

73. **[H]** Let $p(z) = z^6 + z^4 + z^2 + 1$.

    a) By using the identity, $(z^2 - 1)p(z) = z^8 - 1$, find all 6 complex roots of $p(z)$ in polar form.
    b) Hence factorise $p(z)$ into complex linear factors.
    c) Factorise $p(z)$ into a product of 3 real irreducible quadratic polynomials.

74. **[H]** Let $p(z) = 1 + z + z^2 + z^3 + z^4$.

    a) Solve $z^5 - 1 = 0$ and hence factorise $p(z)$ into linear factors.
    b) Find all linear and quadratic factors with real coefficients for $p(z)$.
    c) Divide the equation $p(z) = 0$ by z^2. Let $x = z + \frac{1}{z}$ and deduce that $x^2 + x - 1 = 0$.
    d) Deduce that
    $$\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4} \quad \text{and} \quad \cos\frac{4\pi}{5} = \frac{-1-\sqrt{5}}{4}$$

75. **[H]** Consider $f(t) = t^6 + t^5 - t^4 - 5t^3 - 6t^2 - 6t - 4$. Given that $-1+i$ is a root of f and that f also has two real integer roots,

a) factorise f into complex linear factors,

b) factorise f into linear and quadratic factors with real coefficients.

76. **[H][V]** Let $f(z) = z^5 - 2z^4 + 2z^3 - 5z^2 + 10z - 10$. Given that $1+i$ is a root, find all solutions to $f(z) = 0$.

77. **[X]** a) By considering $z^9 - 1$ as a difference of two cubes, write

$$1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8$$

as a product of two real factors one of which is a quadratic.

b) Solve $z^9 - 1 = 0$ and hence write down the six solutions of $z^6 + z^3 + 1 = 0$.

c) By letting $y = z + \frac{1}{z}$ and dividing $z^6 + z^3 + 1 = 0$ by z^3, deduce that

$$\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} = 0.$$

78. **[X]** Let $p(z) = 3z - z^3 + 5z^4 + z^6$. You are told that the six roots of p, say $\alpha_1, \dots, \alpha_6$, are distinct.

a) Prove that at least two of these roots are real.

b) Show that

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 = 0.$$

c) Hence or otherwise, show that there is at least one root with positive real part, and at least one root with negative real part.

d) Show that if $|z| > 3$, then

$$|3z - z^3 + 5z^4| < |z^6|.$$

e) Hence or otherwise show that for $j = 1, \dots, 6$, $\quad |\alpha_j| \leqslant 3$.

79. **[X]** Cardan (approximately 1545) gave a formula for the roots of the cubic equation

$$x^3 + ax + b = 0$$

in the form $x = u - v$, where

$$u = \left\{-\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + \frac{a^3}{27}}\right\}^{\frac{1}{3}} \qquad v = \left\{\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + \frac{a^3}{27}}\right\}^{\frac{1}{3}}$$

and where the cube roots u and v must be selected to satisfy $uv = \frac{a}{3}$. It can be shown that there are three pairs of values of u and v which satisfy the above conditions.

If $a \neq 0$ a simpler way of writing Cardan's formula is that the three roots of the cubic are of the form

$$x = u - \frac{a}{3u},$$

where u satisfies

$$u^3 = -\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + \frac{a^3}{27}}.$$

a) Use the simpler version of Cardan's formula to find all three roots of $x^3 - 6x + 4 = 0$. (Note that complex numbers are used in the calculation even though all three roots are real).

b) Use the fact that one root of the cubic $x^3 - 6x + 4$ is 2 to factor the cubic as $(x-2)q(x)$ where $q(x)$ is a quadratic. Hence find all roots of the cubic. Hence deduce that

$$\cos\frac{5\pi}{12} = \frac{-1+\sqrt{3}}{2\sqrt{2}} \qquad \text{and} \qquad \cos\frac{11\pi}{12} = -\frac{1+\sqrt{3}}{2\sqrt{2}}.$$

80. **[X]** a) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

b) Make the substitution $x = k\cos\theta$ in the equation $x^3 - ax - b = 0$ and show that the left hand side will become $\cos 3\theta$ if we choose k such that $k^2 = \frac{4}{3}a$ (assume $a > 0$!).

c) With the above choice of k show that the cubic equation can then be written as

$$\cos 3\theta = \frac{4b}{k^3} \quad \text{provided} \; -1 \leqslant \frac{4b}{k^3} \leqslant 1.$$

d) Use the method outlined above to find the three real roots of $x^3 - 6x - 4 = 0$.

81. **[X]** Consider the polynomial

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where the all coefficients $a_0, a_1, \ldots, a_n$ are integers. Show that, if r/s is a rational root of p for which the integers r and s have no common factors, then r is a divisor of a_0 and s is a divisor of a_n. Hence find all rational roots of

a) $-5 + 3x - x^2 + 3x^3$, b) $5 - 22x + x^2 + 28x^3$, c) $4 - x - 100x^2 + 25x^3$.

82. **[X]** Let M, N be positive integers. If $x^M(1-x)^N$ is divided by $(1+x^2)$, and the remainder is $ax+b$, show that $a = (\sqrt{2})^N \sin\dfrac{(2M-N)\pi}{4}$ and $b = (\sqrt{2})^N \cos\dfrac{(2M-N)\pi}{4}$.

Problems 3.11 : Appendix: A note on proof by induction

83. **[R][V]** Prove by mathematical induction that for all positive integers n,

$$1.2 + 2.3 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2).$$

84. **[R]** Prove that, for all integers $n \geqslant 1$,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}\,n(n+1)(2n+1).$$

85. **[R]** Prove that, for all integers $n \geqslant 1$,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}\,n^2(n+1)^2.$$

86. **[H]** Prove that, for all integers $n \geqslant 1$,

$$1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{1}{30}\,n(n+1)(6n^3 + 9n^2 + n - 1).$$

87. **[X]** Prove, by induction, that the sum to k terms of

$$1^2 - 3^2 + 5^2 - 7^2 + 9^2 - 11^2 + \cdots$$

is $-8n^2$ if $k = 2n$ and $8n^2 + 8n + 1$ if $k = 2n + 1$.

88. **[X]** A sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ is defined recursively by

$$a_1 = 1, \quad a_{n+1} = \sum_{j=1}^{n} \frac{a_j}{2^j} \quad \text{for } n \geqslant 1.$$

Use the second Principle of Induction to prove that $a_n \leqslant 1$ for all $n \geqslant 1$.

89. **[X]** Suppose we draw n lines in the plane with no three lines concurrent and no two lines parallel. Let s_n denote the number of regions into which these lines divide the plane. For example, $s_1 = 2, s_2 = 4, s_3 = 7, \ldots$. Prove that $s_{n+1} = s_n + (n+1)$. Deduce by induction that $s_n = \frac{1}{2}n(n+1) + 1$.

Problems 3.13 : Complex numbers and Maple

90. **[M]** Write a Maple command (or commands) to evaluate the complex number $(\sqrt{2} + 7i)^{13}$ in Cartesian or "$a + ib$" form.

91. **[M]** Use Maple to evaluate $(5 + i)^4\,(239 - i)$. Then use de Moivre's Theorem to show that $\dfrac{\pi}{4} = 4\,\cot^{-1} 5 - \cot^{-1} 239$.

"I can do Addition," she said, "if you give me time
— but I can't do Subtraction under ANY circumstances!"
Lewis Carroll, Through the Looking Glass.

Chapter 4

LINEAR EQUATIONS AND MATRICES

One glass lemonade (Why ca'n't you drink water, like me?)
three sandwiches (They never put in half mustard enough.
I told the young woman so, to her face ...)
and seven biscuits. Total one-and-twopence.
Lewis Carroll, A Tangled Tale.

Linear equations and matrices are very important because they are used as mathematical models in virtually all areas in which mathematics is applied in the modern world and because they also appear at the heart of computational algorithms for solving a vast array of quite diverse mathematical problems.

The processes described in this chapter are appropriate for the general theory of handling systems of linear equations. They work well for most small-scale examples.

An understanding of linear equations and matrices is essential for much of the later work in algebra in this book so you must master the material in this chapter.

4.1 Introduction to linear equations

You are already familiar with solving linear equations in one variable such as $2x = 6$, and system of two simultaneous linear equations in two variables such as $\left\{ \begin{array}{rcl} 2x + y & = & 3 \\ 3x - y & = & 7 \end{array} \right.$.

A linear equation in two variables such as $x - 2y = 1$ has infinitely many solutions, since once y is specified, then x is determined and conversely. It is convenient to write the solutions to such an equation parametrically, by introducing a parameter λ.

Example 1. Express solutions of $x - 2y = 1$ parametrically.

SOLUTION. We put $y = \lambda$ then $x = 2\lambda + 1$, so we can write

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2\lambda + 1 \\ \lambda \end{pmatrix}, \qquad \lambda \in \mathbb{R},$$

as the parametric solution to $x - 2y = 1$. Notice that this is not unique, since we could also have put $x = \lambda$ and solved for y. $\Diamond$

Two linear simultaneous equations in two unknowns may have:

(1) No solution, for example $\left\{ \begin{array}{rcl} x+y & = & 3 \\ x+y & = & 5 \end{array} \right.$.

(2) Unique solution, for example $\left\{ \begin{array}{rcl} 2x+y & = & 3 \\ 3x-y & = & 7 \end{array} \right.$.

(3) Infinitely many solutions, for example $\left\{ \begin{array}{rcl} x-2y & = & 1 \\ 2x-4y & = & 2 \end{array} \right.$.

Since these equations correspond to straight lines in the plane, case (1) represents two distinct parallel lines; case (2), two non-parallel lines and in case (3), the two lines are the same.

Now consider some examples of two equations in *three* unknowns. Assume that none of the equations is of the form $0x_1 + 0x_2 + 0x_3 = b$. The solutions of such simultaneous equations can be interpreted as the points of intersection (if any) of two planes in $\mathbb{R}^3$. From geometry, we expect that two planes either

1. intersect along a line, or
2. are parallel and do not intersect, or
3. are parallel with all points in common, (that is, they are the same plane).

We convert each system to another *equivalent system*, i.e. a system which has the same solution set, until we get one which gives us a simple form of the solution set.

Example 2. Find the solution of the system of equations

$$x_1 + x_2 + x_3 = 5 \qquad (1)$$
$$3x_1 + 4x_2 + 7x_3 = 20 \qquad (2)$$

Solution. To solve this pair, we first eliminate x_1 from equation (2) by subtracting 3 times equation (1) from equation (2). The new equation (2) is then

$$x_2 + 4x_3 = 5 \qquad (2')$$

Therefore the original system is equivalent to

$$x_1 + x_2 + x_3 = 5 \qquad (1)$$
$$x_2 + 4x_3 = 5 \qquad (2')$$

Equation $(2')$ has infinitely many solutions which can be represented using a parameter. Put $x_3 = \lambda$, then

$$x_2 = 5 - 4\lambda.$$

Substituting this back to equation (1), we have

$$x_1 + (5 - 4\lambda) + \lambda = 5,$$

so
$$x_1 = 3\lambda.$$

We write the solutions as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3\lambda \\ 5-4\lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}, \qquad \lambda \in \mathbb{R}.$$

The geometric interpretation of this example is that equations (1) and (2) each represents a plane, and the solutions of system represent the points of intersection of the two planes. In this particular example, the intersection of the two planes is a line. This line passes through the point $(0, 5, 0)$ and is parallel to the vector $\begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$. ◊

Example 3. Find the solution of the system

$$\begin{aligned} 2x_1 - 3x_2 + x_3 &= 20 && (1) \\ 4x_1 - 6x_2 + 2x_3 &= 34. && (2) \end{aligned}$$

SOLUTION. As usual, we eliminate x_1 from equation (2). On subtracting 2 × equation (1) from equation (2), we obtain the new equation (2) as

$$0x_1 + 0x_2 + 0x_3 = -6, \qquad (2')$$

which clearly has no solution. Thus the system has no solution. The geometric interpretation is that the planes represented by equations (1) and (2) are parallel and do not intersect. ◊

Example 4. Find the solutions of the system

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 20 && (1) \\ 2x_1 - 6x_2 + 2x_3 &= 40. && (2) \end{aligned}$$

SOLUTION. Eliminating x_1 from equation (2) gives the new equation (2) as

$$0x_1 + 0x_2 + 0x_3 = 0. \qquad (2')$$

The given system is thus equivalent to

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 20 && (1) \\ 0x_1 + 0x_2 + 0x_3 &= 0 && (2') \end{aligned}$$

Thus the solutions to (1) and (2) are just the solutions to (1) and conversely. There are infinitely many solutions, but here two of the unknowns need to be specified in order to obtain the third. Thus **two** parameters λ_1, λ_2 need to be introduced. Put $x_2 = \lambda_1$ and $x_3 = \lambda_2$, then

$$x_1 = 20 + 3\lambda_1 - \lambda_2,$$

and we write the solutions as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 20 + 3\lambda_1 - \lambda_2 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 20 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \qquad \text{for } \lambda_1,\ \lambda_2 \in \mathbb{R}.$$

Hence the system has an infinite number of solutions. The geometric interpretation of this example is that equations (1) and (2) represent the same plane and the 'solution' is just a parametric vector form of expression for that plane. $\diamondsuit$

Now, what about a system of more than two equations in two or three variables? Let us exclude the cases that one or more of the equations in the system is of the form $0x_1 + 0x_2 = b$ or $0x_1+0x_2+0x_3 = b$. In such cases, the system is either has no solutions or is equivalent to a system of two or less equations.

For a system of three equations in two variables, we can interpret the three equations as three lines in a plane. Geometrically, we have three cases.

1. The three lines are concurrent, so the system has a unique solution.
2. The three lines do not have a point in common, so the system has no solution. (See Figure 1 for cases in which the three lines are distinct.)
3. The three equations represent the same line. The system has infinitely many solutions.

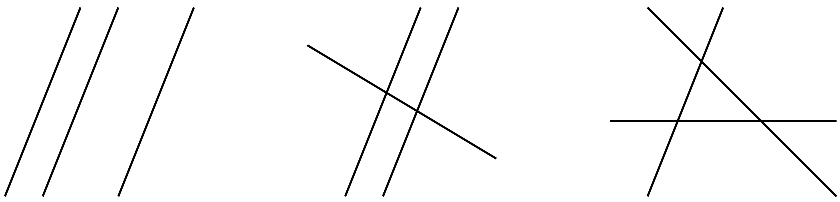


Figure 1: Three Lines with no Point in Common.

For a system of three equations in three variables, the three equations represent three planes in three dimensional space. Geometrically, we have four cases.

1. The three planes intersect at one point, so the system has a unique solution.
2. The three planes do not have a point in common, so the system has no solution. (See Figure 2 for cases in which the three planes are distinct.)
3. The three planes intersect in a line, so there are infinitely many solutions.
4. The three equations represent the same plane. Again, the system has infinitely many solutions.

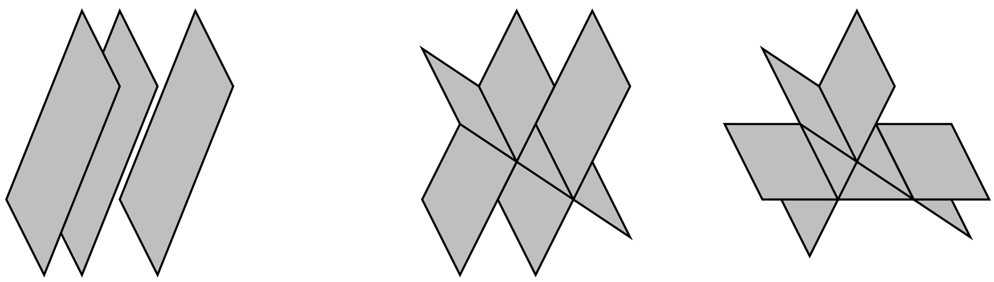


Figure 2: Three Planes with no Point in Common.

4.2 Systems of linear equations and matrix notation

We will now consider the general system of simultaneous linear equations and develop a systematic method of solution.

Definition 1. *A* ***system*** *of m* ***linear equations*** *in n variables is a set of m linear equations in n variables which must be simultaneously satisfied. Such a system is of the form*

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}\right. \qquad (*)$$

Note carefully the position of the subscripts in (*): a_{ij} is the coefficient of the variable x_j in the ith equation, that is, the **equation index is first** and the **variable index is second**.

A **solution** to a system of equations is the set of values of the variables which simultaneously satisfy all the equations. We normally write a solution in form of a column vector in $\mathbb{R}^n$. For instance, if $x_1 = \alpha_1, \dots, x_n = \alpha_n$ satisfy the equations simultaneously, the vector $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ is a solution. A system of equations is said to be **consistent** if it has at least one solution. Otherwise, the system is said to be **inconsistent**.

In particular, the system is called homogeneous when all the b_i's are zero.

Definition 2. *The system (*) is said to be* ***homogeneous*** *if $b_1 = 0, \ \dots, \ b_m = 0$.*

Example 1.

$$\left\{\begin{array}{ccccccccc} 2x_1 & + & 3x_2 & - & x_3 & + & x_4 & = & 6 \\ x_1 & - & 4x_2 & + & 5x_3 & & & = & 2 \\ x_1 & & & - & 6x_3 & & & = & 5 \\ x_1 & + & 2x_2 & & & - & x_4 & = & 7 \end{array}\right.$$

is a system of 4 linear equations in 4 unknowns. ◊

The variables $x_1, x_2, \dots, x_n$ in the system (*) are really just place marker for the coefficients. Hence it is convenient to adopt a shorthand notation for (*) which records only the coefficients. This is called the **augmented matrix** for the system (*):

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right).$$

Example 2. The system in Example 1 has augmented matrix:

$$\left(\begin{array}{cccc|c} 2 & 3 & -1 & 1 & 6 \\ 1 & -4 & 5 & 0 & 2 \\ 1 & 0 & -6 & 0 & 5 \\ 1 & 2 & 0 & -1 & 7 \end{array}\right).$$

NOTE.

1. A rectangular array of numbers is called a **matrix**. So the above matrix form of the system of linear equations is called the corresponding augmented matrix. The vertical line is not part of the matrix. It is used to separate the right hand side of the system from the coefficients on the left hand side.

2. We call $\left(\begin{array}{cccc|c} a_{i1} & a_{i2} & \cdots & a_{in} & b_i \end{array}\right)$ the ith *row* of the augmented matrix which is denoted by R_i. The entries in R_i of the matrix correspond to the coefficients and the right hand side of the i-th equation.

3. We call $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$ the jth *column* which is denoted by C_j. The entries in C_j correspond to the coefficients of x_j when $1 \leqslant j \leqslant n$. The last column C_{n+1} is the right hand side of the system.

We denote the matrix of coefficients by A, the column of the right hand side as the vector $\mathbf{b}$ and the column vector of the variables as $\mathbf{x}$. That is

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}. \qquad \text{and} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Using such notation, the augmented matrix can be abbreviated by $(A|\mathbf{b})$[1].

The drawback of using augmented matrix is the omission of the variables. We introduce the notion of the **matrix equation** form. The system of equations is represented by the matrix equation $A\mathbf{x} = \mathbf{b}$. The matrix A is called the **coefficient matrix** and $\mathbf{x}$ is called the **unknown vector**.

When we write $A\mathbf{x}$, we mean the vector

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \ \dots \ + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ \dots \ + a_{mn}x_n \end{pmatrix}.$$

[1]Formally speaking, $(A|\mathbf{b})$ should be $\left(\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \middle| \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}\right)$, but for simplicity we denote the augmented matrix $\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right)$ by $(A|\mathbf{b})$. .

This is our first meeting with "matrix multiplication", which we shall investigate in more detail in Chapter 5.

Finally we can also write the system of linear equations in **vector equation** or **vector form**.

$$x_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

This vector equation can be written more concisely as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b},$$

where

$$\mathbf{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} \quad \text{for} \quad 1 \leqslant j \leqslant n \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Notice that the vector $\mathbf{a}_j$ contains the coefficients of the variable x_j in the system of linear equations. The vector $\mathbf{b}$ is called the right-hand-side vector.

This vector form of the system of equations shows that solving the system corresponds to expressing the right-hand-side vector $\mathbf{b}$ as a linear combination of the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$.

Example 3. The system of linear equations

$$\left\{ \begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 1 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & -1 \\ 7x_1 & - & 5x_2 & - & 9x_3 & = & 0 \end{array} \right.$$

may be written in vector form as

$$x_1\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + x_2\begin{pmatrix} 2 \\ 5 \\ -5 \end{pmatrix} + x_3\begin{pmatrix} 3 \\ 6 \\ -9 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix};$$

or represented by the matrix equation $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -5 & -9 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix};$$

or represented by the augmented matrix

$$(A|\mathbf{b}) = \left(\begin{array}{rrr|r} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & -1 \\ 7 & -5 & -9 & 0 \end{array}\right).$$

$\Diamond$

Example 4. Write the system of linear equations corresponding to the augmented matrix

$$\left(\begin{array}{rrrr|r} 1 & 3 & 6 & 7 & -2 \\ -2 & 0 & 5 & -4 & 3 \\ 7 & 0 & 0 & -5 & -10 \end{array}\right).$$

SOLUTION. Because there are 4 columns before the "|", the system has 4 variables, which we will call x_1, x_2, x_3 and x_4 (although other names are acceptable). Because $(A|\mathbf{b})$ has 3 rows, the system has 3 equations. Reading off the coefficients of row 1, row 2, and row 3 in turn, we obtain the system

$$\left\{\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & - & 6x_3 & + & 7x_4 & = & -2 \\ -2x_1 & & & + & 5x_3 & - & 4x_4 & = & 3 \\ 7x_1 & & & & & - & 5x_4 & = & -10 \end{array}\right.$$

Although it is conventional to omit variables with zero coefficients when writing a system of linear equations, it is essential in matrix notation that every row contains exactly the same number of entries and that every column also contains exactly the same number of entries so 0s must *not* be omitted from matrices or vectors. ♢

Example 5. Write the system of linear equations corresponding to the matrix equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 3 & -2 & 6 & 7 & -8 \\ 5 & 3 & -2 & -7 & 4 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 7 \\ -3 \end{pmatrix}.$$

SOLUTION. The matrix A has 5 columns, so there are 5 variables, which we call x_1, x_2, x_3, x_4, x_5. Also, A has 2 rows, so there are 2 equations.

Reading off the coefficients from the two rows, we obtain the system

$$\left\{\begin{array}{rcrcrcrcrcr} 3x_1 & - & 2x_2 & + & 6x_3 & + & 7x_4 & - & 8x_5 & = & 7 \\ 5x_1 & + & 3x_2 & - & 2x_3 & - & 7x_4 & + & 4x_5 & = & -3 \end{array}\right.$$

♢

Example 6. Write down the $A\mathbf{x} = \mathbf{b}$ form of the system of linear equations corresponding to the vector equation

$$x_1 \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} -4 \\ 8 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 6 \\ 7 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \\ 2 \end{pmatrix}.$$

SOLUTION. The vector associated with x_j is the jth column of the coefficient matrix so as a matrix equation:

$$\begin{pmatrix} 2 & 5 & -4 & 6 \\ -3 & 0 & 8 & 7 \\ 4 & -1 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \\ 2 \end{pmatrix}.$$

♢

4.3 Elementary row operations

When you solve a pair a simultaneous linear equations, you add (or subtract) a multiple of one equation from the other equation. Each equation in a system of linear equations is represented by one row of the corresponding augmented matrix. This means that the operation of adding a multiple of one equation from another equation can be carried out on the augmented matrix by adding a multiple of one row from another row. This is an example of what we call a **row operation**.

In this section we introduce the three **elementary row operations**. These operations are applied to the augmented matrix in order to find the solution to a system of m linear equations in n variables. Each of these operations gives us a new augmented matrix which is **equivalent** to the old augmented matrix in the following sense: the system represented by the new matrix has exactly the same solution set as the system represented by the old matrix.

We also show how to record row operations to make it easier to check your calculations.

4.3.1 Interchange of equations

If two equations are interchanged, then the set of solutions of the new system is clearly the same as that of the original system. The corresponding operation for an augmented matrix is to interchange two complete rows.

Example 1. Consider the system

$$\begin{array}{lrcrcrcl} (1) & & & 2x_2 & + & 3x_3 & = & 5 \\ (2) & -x_1 & + & 3x_2 & + & x_3 & = & 6 \\ (3) & 2x_1 & + & 4x_2 & + & 7x_3 & = & 8 \end{array} \quad \text{or} \quad \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{rrr|r} 0 & 2 & 3 & 5 \\ -1 & 3 & 1 & 6 \\ 2 & 4 & 7 & 8 \end{array} \right).$$

Interchanging equations (1) and (2), or rows R_1 and R_2, yields

$$\begin{array}{lrcrcrcl} (1') = (2) & -x_1 & + & 3x_2 & + & x_3 & = & 6 \\ (2') = (1) & & & 2x_2 & + & 3x_3 & = & 5 \\ (3) & 2x_1 & + & 4x_2 & + & 7x_3 & = & 8 \end{array} \quad \text{or} \quad \begin{array}{c} \text{New } R_1 = \text{old } R_2 \\ \text{New } R_2 = \text{old } R_1 \\ R_3 \end{array} \left(\begin{array}{rrr|r} -1 & 3 & 1 & 6 \\ 0 & 2 & 3 & 5 \\ 2 & 4 & 7 & 8 \end{array} \right).$$

This should be recorded as

$$\left(\begin{array}{rrr|r} 0 & 2 & 3 & 5 \\ -1 & 3 & 1 & 6 \\ 2 & 4 & 7 & 8 \end{array} \right) \xrightarrow{R_2 \leftrightarrow R_1} \left(\begin{array}{rrr|r} -1 & 3 & 1 & 6 \\ 0 & 2 & 3 & 5 \\ 2 & 4 & 7 & 8 \end{array} \right).$$

$\Diamond$

4.3.2 Adding a multiple of one equation to another

The operation of adding a multiple of one equation to another does not change the solution set of the system of equations. For example, consider the system of equations

$$\left\{\begin{array}{ll} a_1x_1+\cdots+a_nx_n & =b \\ \alpha_1x_1+\cdots+\alpha_nx_n & =\beta \end{array}\right. \tag{1}$$

If we add λ times equation (1) to equation (2), we obtain the system

$$\left\{\begin{array}{rl} a_1x_1+\cdots+a_nx_n & =b \\ (\alpha_1+\lambda a_1)x_1+\cdots+(\alpha_n+\lambda a_n)x_n & =\beta+\lambda b \end{array}\right. \tag{2}$$

Proposition 1. The solution of the system (1) is the same as the solution of the system (2).

Proof. Let $\mathbf{x}=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ be a solution of system (1). Then we have

$$\begin{aligned} a_1x_1+\ \dots\ +a_nx_n &= b \\ \text{and} \qquad (\alpha_1+\lambda a_1)x_1+\cdots+(\alpha_n+\lambda a_n)x_n &= (\alpha_1x_1+\cdots+\alpha_nx_n)+\lambda(a_1x_1+\cdots+a_nx_n) \\ &= \beta+\lambda b, \end{aligned}$$

and so system (2) is satisfied.

Conversely, let $\mathbf{x}=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ be a solution of system (2). Then, we have

$$\begin{aligned} a_1x_1+\ \dots\ +a_nx_n &= b \\ \text{and} \qquad \alpha_1x_1+\cdots+\alpha_nx_n &= (\alpha_1+\lambda a_1)x_1+\cdots+(\alpha_n+\lambda a_n)x_n-\lambda(a_1x_1+\cdots+a_nx_n) \\ &= (\beta+\lambda b)-\lambda b=\beta, \end{aligned}$$

and so system (1) is satisfied. $\square$

In the augmented-matrix notation, the addition of a scalar multiple of one equation to another is equivalent to addition of a scalar multiple of one complete row to another row.

Example 2. Consider the system

$$\begin{array}{lrcrcrcr} (1) & x_1 & + & 2x_2 & + & 3x_3 & = & 5 \\ (2) & -x_1 & - & 4x_2 & + & x_3 & = & 6 \\ (3) & 2x_1 & + & 10x_2 & + & 7x_3 & = & 8 \end{array} \qquad \text{or} \qquad \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{rrr|r} 1 & 2 & 3 & 5 \\ -1 & -4 & 1 & 6 \\ 2 & 10 & 7 & 8 \end{array}\right)$$

On adding equation (1) to equation (2) in the above system, we obtain

$$\begin{array}{lrcrcrcr} (1) & x_1 & + & 2x_2 & + & 3x_3 & = & 5 \\ (2') & & - & 2x_2 & + & 4x_3 & = & 11 \\ (3) & 2x_1 & + & 10x_2 & + & 7x_3 & = & 8 \end{array} \qquad \text{or} \qquad \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{rrr|r} 1 & 2 & 3 & 5 \\ 0 & -2 & 4 & 11 \\ 2 & 10 & 7 & 8 \end{array}\right),$$

where the new $R_2 =$ old $R_2 + R_1$.

This should be recorded as

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 5 \\ -1 & -4 & 1 & 6 \\ 2 & 10 & 7 & 8 \end{array}\right) \xrightarrow{R_2 = R_2 + R_1} \left(\begin{array}{rrr|r} 1 & 2 & 3 & 5 \\ 0 & -2 & 4 & 11 \\ 2 & 10 & 7 & 8 \end{array}\right).$$

$\diamondsuit$

4.3.3 Multiplying an equation by a non-zero number

We can multiply one equation by a non-zero constant without changing the solutions of the system. For example the systems of equations corresponding to the two matrices

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 8 \\ 0 & 0 & 5 & 6 & 9 \\ 0 & 0 & 0 & 7 & 3 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 8 \\ 0 & 0 & 1 & \frac{6}{5} & \frac{9}{5} \\ 0 & 0 & 0 & 1 & \frac{3}{7} \end{array}\right)$$

are equivalent.

This should be recorded as

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 8 \\ 0 & 0 & 5 & 6 & 9 \\ 0 & 0 & 0 & 7 & 3 \end{array}\right) \xrightarrow[R_3 = \frac{1}{7}R_3]{R_2 = \frac{1}{5}R_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 8 \\ 0 & 0 & 1 & \frac{6}{5} & \frac{9}{5} \\ 0 & 0 & 0 & 1 & \frac{3}{7} \end{array}\right).$$

Row operations should be done one after the other. Although in the above we have done two row operations at once, there is no difference from performing the operations one after the other. However, we should not make the following **mistake**.

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 5 \\ -1 & -4 & 1 & 6 \\ 2 & 10 & 7 & 8 \end{array}\right) \xrightarrow[R_1 = R_1 + R_2]{R_2 = R_2 + R_1} \left(\begin{array}{rrr|r} 0 & -2 & 4 & 11 \\ 0 & -2 & 4 & 11 \\ 2 & 10 & 7 & 8 \end{array}\right)$$

The two systems are not equivalent. If we do want to perform the row operations $R_2 = R_2 + R_1$ then $R_1 = R_1 + R_2$, we should have

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 5 \\ -1 & -4 & 1 & 6 \\ 2 & 10 & 7 & 8 \end{array}\right) \xrightarrow{R_2=R_2+R_1} \left(\begin{array}{rrr|r} 1 & 2 & 3 & 5 \\ 0 & -2 & 4 & 11 \\ 2 & 10 & 7 & 8 \end{array}\right) \xrightarrow{R_1=R_1+R_2} \left(\begin{array}{rrr|r} 1 & 0 & 7 & 16 \\ 0 & -2 & 4 & 11 \\ 2 & 10 & 7 & 8 \end{array}\right).$$

4.4 Solving systems of equations

This section describes a process for solving systems of linear equations.

The process consists of two distinct stages.

1. In the first stage, known as **Gaussian elimination**, we use two types of row operation (interchange of rows and subtracting a multiple of one row from another row) to produce an equivalent system in a simpler form which is known as **row-echelon form**. From the row-echelon form we can tell many things about solutions of the system. In particular, we can tell whether the system has no solution, a unique solution or infinitely many solutions.

2. If the system does have solutions, the second stage is to find them. It can be carried out by either of two methods.

   (a) We can use further row operations to obtain an even simpler form which is called **reduced row-echelon form**. From this form we can read off the solution(s).

   (b) From the row echelon form, we can read off the value (possibly in terms of parameters) for at least one of the variables. We substitute this value into one of the other equations and get the value for another variable, and so on. This process, which is called **back-substitution**, will be described fully later.

4.4.1 Row-echelon form and reduced row-echelon form

We begin by defining some special forms of matrix.

Definition 1. *In any matrix*

1. *a **leading row** is one which is not all zeros,*
2. *the **leading entry** in a leading row is the first (i.e. leftmost) non-zero entry,*
3. *a **leading column** is a column which contains the leading entry for some row.*

For example, in the following matrix the 1st row is a leading row with leading entry 5 and the 2nd row is a non-leading row. The 2nd column is a leading column but the 1st and 3rd columns are non-leading columns.

$$\begin{pmatrix} 0 & 5 & 7 \\ 0 & 0 & 0 \end{pmatrix}$$

Definition 2. *A matrix is said to be in **row-echelon form** if*

1. *all leading rows are above all non-leading rows (so any all-zero rows are at the bottom of the matrix), and*
2. *in every leading row, the leading entry is further to the right than the leading entry in any row higher up in the matrix.*

The following are examples of augmented matrices which are in row-echelon form. The circled entries are leading entries. We shall see later that the position of leading entries in a row-echelon form gives important information about solubility of the system.

(1) $\left(\begin{array}{ccc|c} 2 & 3 & 4 & 11 \\ 0 & -3 & 2 & 7 \\ 0 & 0 & 4 & 8 \end{array}\right)$, (2) $\left(\begin{array}{cccc|c} -5 & 0 & 4 & 1 & 0 \\ 0 & 2 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right)$,

(3) $\left(\begin{array}{ccc|c} 6 & 2 & 3 & 4 \\ 0 & 0 & 7 & 7 \end{array}\right)$, (4) $\left(\begin{array}{ccccc|c} -5 & 0 & 2 & 4 & 1 & 0 \\ 0 & 0 & 2 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$,

(5) $\left(\begin{array}{ccc|c} 2 & 3 & 4 & 11 \\ 0 & -3 & 2 & 7 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right)$, (6) $\left(\begin{array}{ccccc|} -5 & 0 & 4 & 1 & 0 \\ 0 & 2 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$.

The following are examples of augmented matrices which are NOT in row-echelon form.

(7) $\left(\begin{array}{ccc|c} 0 & 6 & 3 & 4 \\ 5 & 7 & 0 & 0 \end{array}\right)$, (8) $\left(\begin{array}{ccc|c} -5 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 \end{array}\right)$,

(9) $\left(\begin{array}{cccc|c} 5 & 3 & 5 & -4 & 6 \\ 0 & 0 & 6 & -6 & 7 \\ 0 & 0 & 7 & 0 & -8. \end{array}\right)$

Example (8) does not satisfy Condition 1 of Definition 2. Examples (7) and (9) do not satisfy Condition 2.

Warning. The definition of row-echelon form varies from one book to another and the terminology in Definition 1 is not universally adopted.

Definition 3. *A matrix is said to be in* ***reduced row-echelon form*** *if it is in row-echelon form and in addition*

3. *every leading entry is 1, and*
4. *every leading entry is the only non-zero entry in its column.*

The following are examples of augmented matrices which are in *reduced* row-echelon form.

$$(10)\quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right), \qquad (11)\quad \left(\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

$$(12)\quad \left(\begin{array}{ccccc|c} 1 & -1 & 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{array}\right).$$

Later in this section, we shall see that we can easily read the solutions of a system with its augmented matrix in reduced row-echelon form. The solution set of a system with its augmented matrix in row-echelon from can be found by a process called *back-substitution*.

4.4.2 Gaussian elimination

In this subsection we shall see how the operations of interchanging two rows and of adding a multiple of a row from another row can be used to take a system of equations $A\mathbf{x} = \mathbf{b}$ and transform it into an equivalent system $U\mathbf{x} = \mathbf{y}$ such that the augmented matrix $(U|\mathbf{y})$ is in row-echelon form. The method we shall describe is called the Gaussian-elimination algorithm.

We now describe the steps in the algorithm and illustrate each step by applying it to the following augmented matrices.

$$\text{a)}\quad (A|\mathbf{b}) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 1 & 3 \\ 1 & 3 & 3 & 2 \end{array}\right), \qquad \text{b)}\quad (A|\mathbf{b}) = \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 2 & 3 & -1 \\ 0 & 3 & -3 & 3 & 3 & -6 \\ 0 & 0 & 0 & 1 & 3 & -2 \\ 0 & 6 & -6 & 6 & 3 & -9 \end{array}\right).$$

Step 1. Select a pivot element.

We shall use the following rule to choose what is called a **pivot element**: start at the left of the matrix and move to the right until you reach a column which is not all zeros, then go down this column and choose the first non-zero entry as the pivot entry. The column containing the pivot entry is called the **pivot column** and the row containing the pivot entry is called the **pivot row**.

NOTE. When solving linear equations on a computer, a more complicated pivot selection rule is generally used in order to minimise "rounding errors".

In examples (a) and (b), our pivot selection rule selects the circled entries below as pivot entries for the first step of Gaussian elimination.

$$\text{a)}\quad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 1 & 3 \\ 1 & 3 & 3 & 2 \end{array}\right), \qquad \text{b)}\quad \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 2 & 3 & -1 \\ 0 & 3 & -3 & 3 & 3 & -6 \\ 0 & 0 & 0 & 1 & 3 & -2 \\ 0 & 6 & -6 & 6 & 3 & -9 \end{array}\right).$$

In example (a), row 1 is the pivot row and column 1 is the pivot column. In example (b), row 2 is the pivot row and column 2 is the pivot column.

Step 2. By a row interchange, swap the pivot row and the top row if necessary.

The rule is that the first pivot row of the augmented matrix must finish as the first row of $(U|\mathbf{y})$. You can achieve this by interchanging row 1 and the pivot row.

In example (a), the pivot row is already row 1, so no row interchange is needed. In example (b), the pivot row is row 2, so rows 1 and 2 of the augmented matrix must be interchanged. This is shown as

$$\text{b)}\quad \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 2 & 3 & -1 \\ 0 & 3 & -3 & 3 & 3 & -6 \\ 0 & 0 & 0 & 1 & 3 & -2 \\ 0 & 6 & -6 & 6 & 3 & -9 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccccc|c} 0 & 3 & -3 & 3 & 3 & -6 \\ 0 & 0 & 0 & 2 & 3 & -1 \\ 0 & 0 & 0 & 1 & 3 & -2 \\ 0 & 6 & -6 & 6 & 3 & -9 \end{array}\right).$$

Step 3. Eliminate (i.e., reduce to 0) all entries in the pivot column below the pivot element.

We can do this by adding suitable multiples of the pivot row to the *lower* rows. After this process the pivot column is in the correct form for the final row-echelon matrix $(U|\mathbf{y})$.

In example (a), we can use the row operations $R_2 = R_2 + (-2)\,R_1$ and $R_3 = R_3 + (-1)\,R_1$ to reduce the pivot column to the required form. In recording the row operations, we can simply write $R_2 = R_2 - 2R_1$ and $R_3 = R_3 - R_1$.

$$\text{a)}\quad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 1 & 3 \\ 1 & 3 & 3 & 2 \end{array}\right) \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 2 & -1 \end{array}\right).$$

In example (b), the row operation $R_4 = R_4 - 2\,R_1$ reduces the pivot column to the required form. This gives

$$\text{b)}\quad \left(\begin{array}{ccccc|c} 0 & 3 & -3 & 3 & 3 & -6 \\ 0 & 0 & 0 & 2 & 3 & -1 \\ 0 & 0 & 0 & 1 & 3 & -2 \\ 0 & 6 & -6 & 6 & 3 & -9 \end{array}\right) \xrightarrow{R_4 = R_4 - 2R_1} \left(\begin{array}{ccccc|c} 0 & 3 & -3 & 3 & 3 & -6 \\ 0 & 0 & 0 & 2 & 3 & -1 \\ 0 & 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & -3 & 3 \end{array}\right).$$

Step 4. Repeat steps 1 to 3 on the submatrix of rows and columns strictly to the right of and below the pivot element and stop when the augmented matrix is in row-echelon form.

Note that the top row in step 2 here should mean the top row of the submatrix. In example (a), the required operation is

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 2 & -1 \end{array}\right) \xrightarrow{R_3 = R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 1 & -4 \end{array}\right).$$

Then we have reduced the matrix to row-echelon from

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 1 & -4 \end{array}\right).$$

In example (b), the required operations are

$$\left(\begin{array}{cccccc|c} 0 & 3 & -3 & 3 & 3 & -6 \\ 0 & 0 & 0 & 2 & 3 & -1 \\ 0 & 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & -3 & -3 \end{array}\right) \xrightarrow{R_3 = R_3 - \frac{1}{2}R_2} \left(\begin{array}{ccccc|c} 0 & 3 & -3 & 3 & 3 & -6 \\ 0 & 0 & 0 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 & -3 & -3 \end{array}\right)$$

$$\xrightarrow{R_4 = R_4 + 2R_3} \left(\begin{array}{ccccc|c} 0 & 3 & -3 & 3 & 3 & -6 \\ 0 & 0 & 0 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) = (U|\mathbf{y}).$$

In both examples we have now reached a matrix which is in row-echelon form, so we have completed the process of Gaussian elimination for these examples.

NOTE.

1. If you are doing Gaussian elimination without the aid of a computer, you don't have to stick rigidly to the pivot selection rule which we stated above. For example, the value 1 is very convenient as a pivot entry, so when the leftmost non-zero column has a 1 in it you may find it best to use this 1 as your pivot entry in preference to some other non-zero entry which is above the 1.

   For example, if you have the augmented matrix

$$\left(\begin{array}{cc|c} 13 & 27 & 174 \\ 1 & 2 & 13 \end{array}\right)$$

   you should do

$$\left(\begin{array}{cc|c} 13 & 27 & 174 \\ 1 & 2 & 13 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 2 & 13 \\ 13 & 27 & 174 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 - 13R_1} \left(\begin{array}{cc|c} 1 & 2 & 13 \\ 0 & 1 & 5 \end{array}\right) \xrightarrow{R_1 = R_1 - 2R_2} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 5 \end{array}\right)$$

   rather than applying the rule stated above, which would give

$$\left(\begin{array}{cc|c} 13 & 27 & 174 \\ 1 & 2 & 13 \end{array}\right) \xrightarrow{R_2 = R_2 - \frac{1}{13}R_1} \left(\begin{array}{cc|c} 13 & 27 & 174 \\ 0 & -\frac{1}{13} & -\frac{5}{13} \end{array}\right) \text{ etc.}$$

2. The legal row operations are just

$$\begin{aligned} &R_i \leftrightarrow R_j, \\ &R_i = \lambda R_i \quad \text{where } \lambda \neq 0, \\ &R_i = R_i + \lambda R_j, \end{aligned}$$

and they are supposed to be done one after the other. Be careful to avoid doing simultaneously two operations like

$$R_1 = R_1 - 2R_2 \quad \text{and} \quad R_2 = R_1 - 2R_2.$$

See page 139. If all the entries in R_i are integers and have a common factor λ, we should use $R_i = \frac{1}{\lambda}R_i$ to produce integers with smaller magnitude.

4.4.3 Transformation to reduced row-echelon form

Given a matrix in row-echelon form, we can transform it into *reduced* row-echelon form by using two types of elementary row operations — multiplication of a row by a constant and adding a multiple of one row to another row.

The procedure is as follows.

Start with the lowest row which is not all zeros. Multiply it by a suitable constant to make its leading entry 1. Then add multiples of this row to higher rows to get all zeros in the column above the leading entry of this row. Repeat this procedure with the second lowest non-zero row, and so on.

We shall now see how this procedure applies to examples 1 to 6 of Section 4.4.1 (all of which are already in row-echelon form) and how the solution (if any) of each system relates to the row-echelon form.

Example 1. The transformation to reduced row-echelon form goes like this.

$$\left(\begin{array}{ccc|c} 2 & 3 & 4 & 11 \\ 0 & -3 & 2 & 7 \\ 0 & 0 & 4 & 8 \end{array}\right) \xrightarrow{R_3 = \frac{1}{4}R_3} \left(\begin{array}{ccc|c} 2 & 3 & 4 & 11 \\ 0 & -3 & 2 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow[R_2 = R_2 - 2R_3]{R_1 = R_1 - 4R_3} \left(\begin{array}{ccc|c} 2 & 3 & 0 & 3 \\ 0 & -3 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{R_2 = -\frac{1}{3}R_2} \left(\begin{array}{ccc|c} 2 & 3 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{R_1 = R_1 - 3R_2} \left(\begin{array}{ccc|c} 2 & 0 & 0 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{R_1 = \frac{1}{2}R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

The final augmented matrix represents the system

$$\begin{array}{rcrcrcr} x_1 & + & 0x_2 & + & 0x_3 & = & 3 \\ 0x_1 & + & x_2 & + & 0x_3 & = & -1 \\ 0x_1 & + & 0x_2 & + & x_3 & = & 2, \end{array}$$

so the solution is $x_1 = 3,\ x_2 = -1,\ x_3 = 2.$ ♢

Notice that in the original row-echelon form, **every column of the coefficient matrix** (the part of the augmented matrix to the left of the vertical bar) **is a leading column** and that the system of equations has a **unique solution**.

Example 2. In this case we can get to reduced row-echelon form just by multiplying each row by a suitable constant: $R_1 = -\frac{1}{5}R_1$, $R_2 = \frac{1}{2}R_2$ and $R_3 = \frac{1}{2}R_3$. The reduced row-echelon form is then

$$\left(\begin{array}{cccc|c} 1 & 0 & -\frac{4}{5} & -\frac{1}{5} & 0 \\ 0 & 1 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Since the third row of the reduced row-echelon form represents the equation

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 1,$$

the system has no solution. $\Diamond$

In general, if an augmented matrix has a row of the form

$$(0 \;\; 0 \;\; 0 \cdots \;\; 0 \mid \alpha), \text{ where } \alpha \neq 0,$$

then the corresponding system of equations has no solution. Note that if the **right-hand-side column** is a **leading** column, then the system of equations has **no solution**, i.e. **inconsistent**.

Example 3. The transformation to reduced row-echelon form goes like this.

$$\left(\begin{array}{ccc|c} 6 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow{R_1 = R_1 - 3R_2} \left(\begin{array}{ccc|c} 6 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{R_1 = \frac{1}{6}R_1} \left(\begin{array}{ccc|c} 1 & \frac{1}{3} & 0 & \frac{1}{6} \\ 0 & 0 & 1 & 1 \end{array}\right).$$

The final augmented matrix represents the system

$$x_1 + \tfrac{1}{3}x_2 = \tfrac{1}{6}, \qquad x_3 = 1.$$

The second equations tells us that x_3 must equal 1. For the first equation, we need to set x_2 as a parameter, say λ. This means that the system must have infinitely many solutions. In fact $\mathbf{x}$ will be a solution if and only if

$$x_1 = \tfrac{1}{6} - \tfrac{1}{3}\lambda, \qquad x_2 = \lambda, \qquad x_3 = 1.$$

This can be rewritten in vector form as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} - \frac{1}{3}\lambda \\ \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{pmatrix} \quad \text{for } \lambda \in \mathbb{R}.$$

This is a parametric vector form of a line in $\mathbb{R}^3$ through the point $(\frac{1}{6},\ 0,\ 1)$ parallel to the vector $\begin{pmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{pmatrix}$. $\Diamond$

A variable x_i is a *leading variable* if the ith column of the row-echelon matrix is a leading column. It is a *non-leading variable* if the ith column of the matrix is a non-leading column. Notice that the leading variables in the echelon form are x_1 and x_3 while x_2 is non-leading, so this example illustrates a general rule which says that **non-leading variables can be chosen arbitrarily** by setting parameter, and then the leading variables can be written in terms of the parameters.

Example 4. In this example, the third row corresponds to the equation

$$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 0$$

which is satisfied by every $\mathbf{x}$, and so is superfluous. However, such rows should not be deleted as doing so may cause confusion in some applications that you will encounter in later courses.

The transformation to reduced echelon form gives

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 1 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Columns 2, 4 and 5 are non-leading columns, so the variables x_2, x_4 and x_5 are non-leading variables and in terms of parameters we have $x_2 = \lambda_1$, $x_4 = \lambda_2$ and $x_5 = \lambda_3$. We can then read off the values of the leading variables as

$$x_1 = \tfrac{1}{5} + \tfrac{1}{5}\lambda_2 - \tfrac{1}{5}\lambda_3, \quad x_3 = \tfrac{1}{2} - \tfrac{3}{2}\lambda_2 - \lambda_3.$$

This set of solutions can be expressed in vector form as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} \frac{1}{5} \\ 0 \\ -\frac{3}{2} \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} -\frac{1}{5} \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{for } \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}.$$

◊

Examples 3 and 4 illustrate that **the parameters in the solution are the values of the non-leading variables**, so **the number of parameters in the solution equals the number of non-leading columns in the row-echelon form of the coefficient matrix**.

In general, if a system is consistent and there are non-leading columns in a row-echelon form of the system, then the system has *infinitely many solutions*.

Example 5. In this example, row 4 is all zeros and so is superfluous. The remaining rows are the same as in example 1, so the solution is identical to the solution in example 1. ◊

Example 6. Transformation to reduced row-echelon form gives

$$\left(\begin{array}{cccc|c} 1 & 0 & -\frac{4}{5} & -\frac{1}{5} & 0 \\ 0 & 1 & \frac{3}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We let the values of the non-leading variables x_3 and x_4 be arbitrary parameters, say λ_1 and λ_2. We can then read off the values for the leading variables x_1 and x_2 (in terms of λ_1 and λ_2) and express the set of solutions in vector form as

$$\mathbf{x} = \lambda_1 \begin{pmatrix} \frac{4}{5} \\ -\frac{3}{2} \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} \frac{1}{5} \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{for } \lambda_1, \lambda_2 \in \mathbb{R}.$$

This is a parametric vector form of a plane in $\mathbb{R}^4$ which contains the origin and is parallel to the vectors $\begin{pmatrix} \frac{4}{5} \\ -\frac{3}{2} \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{5} \\ -1 \\ 0 \\ 1 \end{pmatrix}$. $\Diamond$

We end this subsection by summarising the rule for getting solutions from a reduced echelon form of the system:

Assign an arbitrary parameter as the value for each non-leading variable. Then read off expressions for the leading variables in terms of the arbitrary parameters which you have introduced.

4.4.4 Back-substitution

When solving small systems by hand, you may prefer to use this procedure as an alternative to the transformation of a row-echelon form into reduced row-echelon form. Instead of doing further row operations on the augmented matrix, we go to the equations represented by the row-echelon form and proceed as follows.

Assign an arbitrary parameter value to each non-leading variable. Then read off from the last non-trivial equation an expression for the last leading variable in terms of your arbitrary parameters. Substitute this expression back into the second last equation to get an expression for the second last leading variable, and so on.

Example 7. We will redo example 4 of the last subsection, using back-substitution instead of transformation to reduced echelon form. The row-echelon form is

$$\left(\begin{array}{ccccc|c} -5 & 0 & 2 & 4 & 1 & 0 \\ 0 & 0 & 2 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Solution. The non-leading variables are $x_2,\ x_4$ and x_5 so we let $x_2 = \lambda_1,\ x_4 = \lambda_2$ and $x_5 = \lambda_3$. Then the second row corresponds to the equation

$$2x_3 + 3\lambda_2 + 2\lambda_3 = 1$$

which gives

$$x_3 = \frac{1}{2} - \frac{3}{2}\lambda_2 - \lambda_3.$$

Substituting this back into the equation represented by the first row gives

$$-5x_1 + 2\left(\frac{1}{2} - \frac{3}{2}\lambda_2 - \lambda_3\right) + 4\lambda_2 + \lambda_3 = 0.$$

We solve this for x_1 and get

$$x_1 = \frac{1}{5} + \frac{1}{5}\lambda_2 - \frac{1}{5}\lambda_3.$$

Note that this is the same solution as we got by transforming to reduced row-echelon form. $\Diamond$

4.5 Deducing solubility from row-echelon form

We can determine the number of solutions to a system by examining the **position of leading entries** in a row-echelon form for the augmented matrix of the system. The rules for doing this are summarised in the following proposition.

Proposition 1. If the augmented matrix for a system of linear equations can be transformed into an equivalent row-echelon form $(U|\mathbf{y})$ then:

1. The system has **no solution** if and only if the right hand column $\mathbf{y}$ is a **leading column**.
2. If the right hand column $\mathbf{y}$ is a NOT a leading column then the system has:
    a) A **unique** solution if and only if **every variable is a leading variable**.
    b) **Infinitely many** solutions if and only if there is **at least one non-leading variable**. In this case, each non-leading variable becomes a parameter in the general expression for all solutions and the number of parameters in the solution equals the number of non-leading variables.

It often happens in practice that you want to consider the system $A\mathbf{x} = \mathbf{b}$ for one fixed coefficient matrix A but many different vectors $\mathbf{b}$. For this reason it is useful to know the answer to the following question: if we know a row-echelon form U for the *coefficient matrix* A (not the augmented matrix $(A|\mathbf{b})$), what can we say about solubility of the system $A\mathbf{x} = \mathbf{b}$ for an arbitrary right hand side $\mathbf{b}$?

If U has an all-zero row then it will obviously be possible to find at least one $\mathbf{b} \in \mathbb{R}^n$ such that the row-echelon form $(U|\mathbf{y})$ for $(A|\mathbf{b})$ has a row of the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & | & \alpha \end{pmatrix}, \ \alpha \neq 0$$

and this represents an equation which has no solution. On the other hand, if U has no non-leading row (see page 140) then no $\mathbf{b}$ can give rise to an impossible row of the above type and we can always get a solution by back-substitution. In this case the number of solutions will be infinite if and only if there are non-leading columns in U.

The following proposition sums up what we can say about solubility for arbitrary $\mathbf{b}$.

Proposition 2. If A is an $m \times n$ matrix which can be transformed by elementary row operations into a row-echelon form U then the system $A\mathbf{x} = \mathbf{b}$ has

a) **at least one** solution for each $\mathbf{b}$ in $\mathbb{R}^m$ if and only if U has **no non-leading rows**,

b) **at most one** solution for each $\mathbf{b}$ in $\mathbb{R}^m$ if and only if U has **no non-leading columns**,

c) **exactly one** solution for each $\mathbf{b}$ in $\mathbb{R}^m$ if and only if U has **no non-leading rows** and **no non-leading columns**.

When U has neither non-leading row nor non-leading column, $A\mathbf{x} = \mathbf{b}$ always has a unique solution. Otherwise, whether the matrix equation has solution depends on the vector $\mathbf{b}$. We shall study examples on finding a condition on $\mathbf{b}$ for a solution to exist and we shall discuss how to find a *specific formula* for a solution $\mathbf{x}$ expressed in terms of an arbitrary right hand side $\mathbf{b}$ in the next section.

4.6 Solving $A\mathbf{x} = \mathbf{b}$ for indeterminate $\mathbf{b}$

To solve the equation $A\mathbf{x} = \mathbf{b}$, we apply row operations to reduce the corresponding augmented matrix $(A|\mathbf{b})$.

Example 1. Solve the system of equations

$$\begin{cases} 2x_1 + 3x_2 = b_1 \\ 3x_1 + 4x_2 = b_2 \end{cases}$$

SOLUTION. We reduce the augmented matrix as follows.

$$\left(\begin{array}{cc|c} 2 & 3 & b_1 \\ 3 & 4 & b_2 \end{array}\right) \xrightarrow{R_2 = R_2 - \frac{3}{2}R_1} \left(\begin{array}{cc|c} 2 & 3 & b_1 \\ 0 & -\frac{1}{2} & -\frac{3}{2}b_1 + b_2 \end{array}\right).$$

From the second row of the row-echelon form, we have

$$-\frac{1}{2}x_2 = -\frac{3}{2}b_1 + b_2 \qquad \text{or} \qquad x_2 = 3b_1 - 2b_2.$$

Substitute it into the equation represented by the first row,

$$2x_1 + 3(3b_1 - 2b_2) = b_1 \qquad \text{or} \qquad x_1 = -4b_1 + 3b_2.$$

◊

NOTE. We could write the system of equations as

$$\begin{cases} 2x_1 + 3x_2 = 1b_1 + 0b_2 \\ 3x_1 + 4x_2 = 0b_1 + 1b_2 \end{cases}$$

Provided the interpretation of the columns is clearly understood the row reduction can be rewritten as

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \xrightarrow{R_2 = R_2 - \frac{3}{2}R_1} \left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right).$$

Then we can proceed to reduce the matrix to reduced row-echelon from or apply back-substitution as before.

In the above example, the system always has a unique solution for any $\mathbf{b}$. The system of equations in the next example does not always have solution. A condition on $\mathbf{b}$ has to be satisfied for a solution to exist.

Example 2. Find a condition for $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ so that the system of equations

$$\begin{cases} 2x_1 + 3x_2 = b_1 \\ 4x_1 + 6x_2 = b_2 \end{cases}$$

has solutions. Find the solution set if the condition is satisfied.

SOLUTION. The reduction process yields

$$\left(\begin{array}{cc|c} 2 & 3 & b_1 \\ 4 & 6 & b_2 \end{array}\right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{cc|c} 2 & 3 & b_1 \\ 0 & 0 & b_2 - 2b_1 \end{array}\right).$$

The second row represents the equation $0x_1 + 0x_2 = -2b_1 + b_2$. Thus for solutions to exist, the condition $-2b_1 + b_2 = 0$ has to be satisfied. If solutions exist, we have to assign the non-leading variable x_2 a parameter λ. Then the equation corresponding to the first row becomes

$$2x_1 + 3\lambda = b_1, \quad \text{that is} \quad x_1 = -\frac{3}{2}\lambda + \frac{1}{2}b_1.$$

The solution set is then

$$\left\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \begin{pmatrix} \frac{1}{2}b_1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix},\ \lambda \in \mathbb{R}\right\}.$$

$\diamondsuit$

Sometimes there is more than one condition for solutions to exist.

Example 3. Find conditions on $b_1,\ b_2,\ b_3,\ b_4$ for the following system of equations to be consistent. Find the solutions when these conditions are satisfied.

$$\begin{cases} x_1 + 2x_2 = b_1 \\ 3x_1 + 5x_2 = b_2 \\ 5x_1 + 7x_2 = b_3 \\ 7x_1 + 9x_2 = b_4 \end{cases}$$

SOLUTION.

$$\left(\begin{array}{cc|c} 1 & 2 & b_1 \\ 3 & 5 & b_2 \\ 5 & 7 & b_3 \\ 7 & 9 & b_4 \end{array}\right) \xrightarrow[\substack{R_3 = R_3 - 5R_1 \\ R_4 = R_4 - 7R_1}]{R_2 = R_2 - 3R_1} \left(\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -1 & b_2 - 3b_1 \\ 0 & -3 & b_3 - 5b_1 \\ 0 & -5 & b_4 - 7b_1 \end{array}\right)$$

$$\xrightarrow[R_4 = R_4 - 5R_2]{R_3 = R_3 - 3R_2} \left(\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -1 & -3b_1 + b_2 \\ 0 & 0 & 4b_1 - 3b_2 + b_3 \\ 0 & 0 & 8b_1 - 5b_2 + b_4 \end{array}\right)$$

From rows 3 and 4, the original system of equations has a solution if and only if $b_3 = -4b_1 + 3b_2$ and $b_4 = -8b_1 + 5b_2$. If the condition is satisfied, we get $x_2 = 3b_1 - b_2$ from row 2. Then from row 1, we have $x_1 = b_1 - 2(3b_1 - b_2) = -5b_1 + 2b_2$. $\diamondsuit$

4.7 General properties of the solution of $A\mathbf{x} = \mathbf{b}$

We have seen that every system of linear equations can be written as $A\mathbf{x} = \mathbf{b}$ and every such system has either no solution, a unique solution or an infinite number of solutions. For all systems of linear equations that we have solved in this chapter which have infinitely many solutions, the solutions can be written in parametric vector form as

$$\mathbf{x} = \mathbf{x}_p + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_k\mathbf{v}_k,$$

where $\mathbf{v}_1, \cdots, \mathbf{v}_k$ are non-zero vectors and where the parameters $\lambda_1, \ldots, \lambda_k$ are scalars. It is not difficult to see that this is generally true for all systems of linear equations with infinitely many solutions.

Furthermore, we shall prove that the vector $\mathbf{x}_p$ is a solution of the system $A\mathbf{x} = \mathbf{b}$, and the vectors $\mathbf{v}_i$ which go with the parameters in the solution are solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. It does not depend on how we reduce the matrix to an echelon form.

We begin with some important propositions about solutions of $A\mathbf{x} = \mathbf{0}$.

Proposition 1. $A\mathbf{x} = \mathbf{0}$ has $\mathbf{x} = \mathbf{0}$ as a solution.

Proof. Since $A\mathbf{x} = \mathbf{0}$ represents the system of equations

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{cases},$$

so $x_1 = x_2 = \cdots = x_n = 0$ satisfies all the equations. $\square$

Proposition 2. If $\mathbf{v}$ and $\mathbf{w}$ are solutions of $A\mathbf{x} = \mathbf{0}$ then so are $\mathbf{v} + \mathbf{w}$ and $\lambda\mathbf{v}$ for any scalar λ.

Proof. Since $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$ are solutions, we have

$$\begin{cases} a_{11}v_1 & + & a_{12}v_2 & + & \cdots & + & a_{1n}v_n & = & 0 \\ a_{21}v_1 & + & a_{22}v_2 & + & \cdots & + & a_{2n}v_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}v_1 & + & a_{m2}v_2 & + & \cdots & + & a_{mn}v_n & = & 0 \end{cases}, \tag{1}$$

$$\begin{cases} a_{11}w_1 & + & a_{12}w_2 & + & \cdots & + & a_{1n}w_n & = & 0 \\ a_{21}w_1 & + & a_{22}w_2 & + & \cdots & + & a_{2n}w_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}w_1 & + & a_{m2}w_2 & + & \cdots & + & a_{mn}w_n & = & 0 \end{cases}. \tag{2}$$

If we add the ith equations of (1) and (2) for $1 \leqslant i \leqslant m$, we shall get

$$\begin{cases} a_{11}(v_1+w_1) & + & a_{12}(v_2+w_2) & + & \cdots & + & a_{1n}(v_n+w_n) & = & 0 \\ a_{21}(v_1+w_1) & + & a_{22}(v_2+w_2) & + & \cdots & + & a_{2n}(v_n+w_n) & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}(v_1+w_1) & + & a_{m2}(v_2+w_2) & + & \cdots & + & a_{mn}(v_n+w_n) & = & 0 \end{cases}.$$

Hence $\mathbf{v} + \mathbf{w}$ is a solution of $A\mathbf{x} = \mathbf{0}$.

If we multiply both sides of every equation of (1) by λ, then

$$\begin{cases} a_{11}(\lambda v_1) & + & a_{12}(\lambda v_2) & + & \cdots & + & a_{1n}(\lambda v_n) & = & 0 \\ a_{21}(\lambda v_1) & + & a_{22}(\lambda v_2) & + & \cdots & + & a_{2n}(\lambda v_n) & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}(\lambda v_1) & + & a_{m2}(\lambda v_2) & + & \cdots & + & a_{mn}(\lambda v_n) & = & 0 \end{cases}.$$

Hence $\lambda\mathbf{v}$ is also a solution. $\square$

A further important proposition about homogeneous systems can be obtained by induction from the last proposition.

Proposition 3. Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be solutions of $A\mathbf{x} = \mathbf{0}$ for $1 \leqslant j \leqslant k$. Then $\lambda_1\mathbf{v}_1 + \cdots + \lambda_k\mathbf{v}_k$ is also a solution of $A\mathbf{x} = \mathbf{0}$ for all $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$.

We now turn to the properties of the solutions of $A\mathbf{x} = \mathbf{b}$.

Proposition 4. If $\mathbf{v}$ and $\mathbf{w}$ are solutions of $A\mathbf{x} = \mathbf{b}$ then $\mathbf{v} - \mathbf{w}$ is a solution of $A\mathbf{x} = \mathbf{0}$.

Proof. Since $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$ are solutions of $A\mathbf{x} = \mathbf{b}$, we have

$$\begin{cases} a_{11}v_1 & + & a_{12}v_2 & + & \cdots & + & a_{1n}v_n & = & b_1 \\ a_{21}v_1 & + & a_{22}v_2 & + & \cdots & + & a_{2n}v_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}v_1 & + & a_{m2}v_2 & + & \cdots & + & a_{mn}v_n & = & b_n \end{cases}, \tag{1}$$

$$\begin{cases} a_{11}w_1 & + & a_{12}w_2 & + & \cdots & + & a_{1n}w_n & = & b_1 \\ a_{21}w_1 & + & a_{22}w_2 & + & \cdots & + & a_{2n}w_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}w_1 & + & a_{m2}w_2 & + & \cdots & + & a_{mn}w_n & = & b_n \end{cases}. \tag{2}$$

If we subtract the ith equation of (2) from the ith equation of (1) for $1 \leqslant i \leqslant m$, we shall get

$$\begin{cases} a_{11}(v_1 - w_1) & + & a_{12}(v_2 - w_2) & + & \cdots & + & a_{1n}(v_n - w_n) & = & 0 \\ a_{21}(v_1 - w_1) & + & a_{22}(v_2 - w_2) & + & \cdots & + & a_{2n}(v_n - w_n) & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}(v_1 - w_1) & + & a_{m2}(v_2 - w_2) & + & \cdots & + & a_{mn}(v_n - w_n) & = & 0 \end{cases}.$$

Hence $\mathbf{v} - \mathbf{w}$ is a solution of $A\mathbf{x} = \mathbf{0}$. ☐

We can now prove the following result.

Proposition 5. Let $\mathbf{x}_p$ be a solution of $A\mathbf{x} = \mathbf{b}$.

1. If $\mathbf{x} = \mathbf{0}$ is the only solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$ then $\mathbf{x}_p$ is the unique solution of $A\mathbf{x} = \mathbf{b}$.

2. If the homogeneous equation has non-zero solutions $\mathbf{v}_1, \ldots, \mathbf{v}_k$ then

$$\mathbf{x} = \mathbf{x}_p + \lambda_1\mathbf{v}_1 + \cdots + \lambda_k\mathbf{v}_k,$$

   is also a solution of $A\mathbf{x} = \mathbf{b}$ for all $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$.

Proof. For part 1, let $\mathbf{v}$ be another solution of $A\mathbf{x} = \mathbf{b}$. By Proposition 4, $\mathbf{v} - \mathbf{x}_p$ is a solution of $A\mathbf{x} = \mathbf{0}$. Since $\mathbf{0}$ is the only solution of $A\mathbf{x} = \mathbf{0}$, we have $\mathbf{v} - \mathbf{x}_p = \mathbf{0}$ and that is $\mathbf{v} = \mathbf{x}_p$. Hence $\mathbf{x}_p$ is the unique solution of $A\mathbf{x} = \mathbf{b}$.

For part 2, let $\mathbf{v} = \lambda_1\mathbf{v}_1 + \cdots + \lambda_k\mathbf{v}_k$. By Proposition 3, we know $\mathbf{v}$ is also a solution of $A\mathbf{x} = \mathbf{0}$. With a similar method used in proving Proposition 2, we can show that $\mathbf{x}_p + \mathbf{v}$ is a solution of $A\mathbf{x} = \mathbf{b}$. ☐

NOTE. The form of solutions in part 2 of Proposition 5 raises the question of what is the minimum number of vectors $\mathbf{v}_1, \cdots, \mathbf{v}_k$ required to yield all the solutions of $A\mathbf{x} = \mathbf{b}$. We will return to this question in Chapter 6 when we discuss the ideas of "spanning sets" and "linear independence". For the present, it is sufficient to note that the solution method we have used in this chapter does find **all** solutions of $\mathbf{Ax} = \mathbf{b}$.

Example 1. In Section 4.4.2 we found a row-echelon form for the augmented matrix of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 0 & 0 & 0 & 2 & 3 \\ 0 & 3 & -3 & 3 & 3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 6 & -6 & 6 & 3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} -1 \\ -6 \\ -2 \\ -9 \end{pmatrix}$$

as

$$\left(\begin{array}{ccccc|c} 0 & 3 & -3 & 3 & 3 & -6 \\ 0 & 0 & 0 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

By back-substitution, $x_5 = -1$, $x_4 = 1$. By setting $x_1 = \lambda_1$ and $x_3 = \lambda_2$, we also have $x_2 = \lambda_2 - 2$. Hence the solutions are

$$\mathbf{x} = \begin{pmatrix} \lambda_1 \\ \lambda_2 - 2 \\ \lambda_2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \\ -1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{for} \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

The first vector is a particular solution (corresponding to $\lambda_1 = \lambda_2 = 0$) of $A\mathbf{x} = \mathbf{b}$ and the vectors which go with the parameters are solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

As an exercise, can you write down the two systems of 4 equations in 5 variables corresponding to $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ and verify that the first vector is a solution to the non-homogeneous system and the two other vectors are solutions to the homogeneous system? $\diamondsuit$

4.8 Applications

Linear equations appear in many applications. In this section we first present some geometric applications and then some applications in some other fields.

4.8.1 Geometry

In Section 4.1, we have shown some simple geometric applications of linear equations to lines in $\mathbb{R}^2$ and planes in $\mathbb{R}^3$. In this section we show how some common problems involving lines and planes in $\mathbb{R}^n$ can be solved using linear equations.

Example 1. Does $\begin{pmatrix} 3 \\ 0 \\ 5 \\ 6 \end{pmatrix}$ belong to span $\left(\begin{pmatrix} 1 \\ -2 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 1 \\ 2 \end{pmatrix} \right)$?

SOLUTION. The vector $\begin{pmatrix} 3 \\ 0 \\ 5 \\ 6 \end{pmatrix} \in \operatorname{span}\left(\begin{pmatrix} 1 \\ -2 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 1 \\ 2 \end{pmatrix} \right)$ if and only if

$$\begin{pmatrix} 3 \\ 0 \\ 5 \\ 6 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ -2 \\ 3 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 4 \\ 1 \\ 2 \end{pmatrix} \quad \text{for some } \lambda_1, \lambda_2 \in \mathbb{R}.$$

On equating corresponding coordinates on both sides of this vector equation, we have

$$\begin{cases} \begin{array}{rcrcr} \lambda_1 & & & = & 3 \\ -2\lambda_1 & + & 4\lambda_2 & = & 0 \\ 3\lambda_1 & + & \lambda_2 & = & 5 \\ 2\lambda_1 & + & 2\lambda_2 & = & 6 \end{array} \end{cases}$$

The augmented matrix is

$$(A|\mathbf{b}) = \left(\begin{array}{rr|r} 1 & 0 & 3 \\ -2 & 4 & 0 \\ 3 & 1 & 5 \\ 2 & 2 & 6 \end{array} \right)$$

This is equivalent to

$$(U|\mathbf{y}) = \left(\begin{array}{rr|r} 1 & 0 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & -\frac{11}{2} \\ 0 & 0 & 0 \end{array} \right).$$

As the system has no solution, $\begin{pmatrix} 3 \\ 0 \\ 5 \\ 6 \end{pmatrix}$ is not in $\operatorname{span}\left(\begin{pmatrix} 1 \\ 2 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 1 \\ 2 \end{pmatrix} \right)$. $\Diamond$

Example 2. Is $\mathbf{v} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ parallel to the plane

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}?$$

SOLUTION. The vector $\mathbf{v} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ is parallel to the plane if and only if it is a linear combination of the two vectors $\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$. Hence, $\mathbf{v}$ is parallel to the plane if and only if there are real numbers λ_1 and λ_2 such that

$$\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = \lambda_1 \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}.$$

On equating corresponding components of the vectors on both sides, we obtain

$$\begin{cases} -\lambda_1 & + & 3\lambda_2 & = & 2 \\ 3\lambda_1 & + & \lambda_2 & = & 4 \\ 2\lambda_1 & + & 4\lambda_2 & = & 6 \end{cases}$$

The augmented matrix is

$$(A|\mathbf{b}) = \left(\begin{array}{cc|c} -1 & 3 & 2 \\ 3 & 1 & 4 \\ 2 & 4 & 6 \end{array}\right),$$

which becomes, by Gaussian elimination,

$$(U|\mathbf{y}) = \left(\begin{array}{cc|c} -1 & 3 & 2 \\ 0 & 10 & 10 \\ 0 & 0 & 0 \end{array}\right).$$

Since the right hand column is non-leading, this system has a solution for λ_1 and λ_2, and hence $\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ **is** parallel to the plane.

[Note that this question can also be solved by geometric considerations using dot and cross products.] ◊

Example 3. Find the intersection of the line through $(3, 2, 1, 4)$ parallel to $\begin{pmatrix} 1 \\ 2 \\ -3 \\ 6 \end{pmatrix}$ and the plane through $(3, 1, -4, 7)$ parallel to $\begin{pmatrix} 2 \\ 1 \\ 4 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \\ 0 \\ 6 \end{pmatrix}$.

Solution. The equation of the line is

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -3 \\ 6 \end{pmatrix} \quad \text{for } \lambda \in \mathbb{R},$$

and the equation of the plane is

$$\mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ -4 \\ 7 \end{pmatrix} + \mu_1 \begin{pmatrix} 2 \\ 1 \\ 4 \\ 5 \end{pmatrix} + \mu_2 \begin{pmatrix} -1 \\ 3 \\ 0 \\ 6 \end{pmatrix} \quad \text{for } \mu_1, \mu_2 \in \mathbb{R}.$$

An intersection occurs for values of λ, μ_1, μ_2 for which

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -3 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -4 \\ 7 \end{pmatrix} + \mu_1 \begin{pmatrix} 2 \\ 1 \\ 4 \\ 5 \end{pmatrix} + \mu_2 \begin{pmatrix} -1 \\ 3 \\ 0 \\ 6 \end{pmatrix}.$$

On rearranging, we have

$$\lambda \begin{pmatrix} 1 \\ 2 \\ -3 \\ 6 \end{pmatrix} - \mu_1 \begin{pmatrix} 2 \\ 1 \\ 4 \\ 5 \end{pmatrix} - \mu_2 \begin{pmatrix} -1 \\ 3 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -5 \\ 3 \end{pmatrix}.$$

On equating coordinates on both sides, we obtain a set of 4 equations in 3 unknowns

$$\begin{cases} \lambda - 2\mu_1 + \mu_2 = 0 \\ 2\lambda - \mu_1 - 3\mu_2 = -1 \\ -3\lambda - 4\mu_1 = -5 \\ 6\lambda - 5\mu_1 - 6\mu_2 = 3 \end{cases}$$

After forming an augmented matrix, and using Gaussian elimination, we obtain

$$(U|\mathbf{y}) = \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 3 & -5 & -1 \\ 0 & 0 & -\frac{41}{3} & -\frac{25}{3} \\ 0 & 0 & 0 & \frac{681}{123} \end{array}\right).$$

The system of equations has no solution, so the given line and plane in $\mathbb{R}^4$ do not intersect. $\Diamond$

4.8.2 Chemical engineering

A greatly simplified example which illustrates an application of systems of linear equations to oil refining is as follows.

Example 4. An oil company has three refineries at Sydney, Melbourne and Brisbane. Each refinery makes four products: super petrol, unleaded petrol, diesel fuel and aviation fuel. The amount of each product that each refinery can make per hour is given in the following table.

	Sydney	Melbourne	Brisbane
Super (litres/hour)	10000	20000	5000
Unleaded (litres/hour)	2000	5000	1000
Diesel (litres/hour)	500	800	200
Aviation fuel (litres/hour)	100	200	100

The oil company has to decide how many hours per day to run each refinery in order to produce a required amount of product each day.

Do the following:

(a) Set up a linear equation model for the company.

(b) On Monday, the oil company requires **exactly** 610,000 litres of super, 137,000 litres of unleaded, 26,400 litres of diesel and 7,200 litres of aviation fuel. Find the number of hours each refinery should be run or show that the desired production levels cannot be met.

(c) On Tuesday, the amount of super, unleaded and diesel required is the same, but the amount of aviation fuel required is increased to 7,800 litres. Again find the number of hours each refinery should be run or show that the desired production levels cannot be met.

SOLUTION. Let

$$\begin{aligned} x_1 &= \text{number of hours for which Sydney refinery runs,} \\ x_2 &= \text{number of hours for which Melbourne refinery runs,} \\ x_3 &= \text{number of hours for which Brisbane refinery runs,} \end{aligned}$$

and let b_1, b_2, b_3 and b_4 be the amounts of super, unleaded, diesel and aviation fuel respectively required each day. Then the equations are:

$$\begin{array}{lrcrcrcl} \text{Super} & 10000\,x_1 & + & 20000\,x_2 & + & 5000\,x_3 & = & b_1 \\ \text{Unleaded} & 2000\,x_1 & + & 5000\,x_2 & + & 1000\,x_3 & = & b_2 \\ \text{Diesel} & 500\,x_1 & + & 800\,x_2 & + & 200\,x_3 & = & b_3 \\ \text{Aviation} & 100\,x_1 & + & 200\,x_2 & + & 100\,x_3 & = & b_4 \end{array}$$

The coefficient matrix, unknowns vector and right-hand side are therefore

$$A = \begin{pmatrix} 10000 & 20000 & 5000 \\ 2000 & 5000 & 1000 \\ 500 & 800 & 200 \\ 100 & 200 & 100 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

On Monday

$$(A|\mathbf{b}) = \left(\begin{array}{rrr|r} 10000 & 20000 & 5000 & 610000 \\ 2000 & 5000 & 1000 & 137000 \\ 500 & 800 & 200 & 26400 \\ 100 & 200 & 100 & 7200 \end{array}\right)$$

is equivalent to

$$(U|\mathbf{y}) = \left(\begin{array}{rrr|r} 10000 & 20000 & 5000 & 610000 \\ 0 & 1000 & 0 & 15000 \\ 0 & 0 & -50 & -1100 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

which has the solution $x_3 = 22,\ x_2 = 15,\ x_1 = 20$.

On Tuesday

$$(A|\mathbf{b}) = \left(\begin{array}{rrr|r} 10000 & 20000 & 5000 & 410000 \\ 2000 & 5000 & 1000 & 137000 \\ 500 & 800 & 200 & 26400 \\ 100 & 200 & 100 & 7800 \end{array}\right)$$

is equivalent to

$$(U|\mathbf{y}) = \left(\begin{array}{ccc|c} 10000 & 2000 & 5000 & 610000 \\ 0 & 1000 & 0 & 15000 \\ 0 & 0 & -50 & -1100 \\ 0 & 0 & 0 & 600 \end{array}\right)$$

which has no solution, so production levels can not be met. $\diamondsuit$

The model used here is too simple to reflect accurately what is done in practice. In practice the problem is generalised in two ways. First, the equations are replaced by inequalities; i.e., the problem is changed from one of having to produce **exactly** the required amount of product to one of having to produce **at least** the required amount of product. Secondly, the problem is changed to one of having to **minimise the cost of production** subject to the constraints of having to produce the required amount of product. This generalised problem is called a **linear programming problem**. Linear programming is very important in economic, financial and manufacturing applications and so on where problems involving thousands of variables and inequalities are routinely solved, and result in savings of millions of dollars.

4.8.3 Economics

A simplified example of an application of linear equations to economics is as follows.

Example 5. The island of Wotsit-Matta has a simple economy in which the only goods produced are wheat, iron and pigs. The Wotsit-Mattas have worked out that the following amounts of wheat, iron and pigs are required on 31 December of each year for each unit of wheat, iron and pigs produced in the following year:

Requirements	Wheat (per tonne)	Iron (per tonne)	Pigs (per hundred)
Wheat(tonnes)	.5333	4.286	2.0000
Iron(tonnes)	.02557	.2857	.0500
Pigs (hundreds)	.0400	.5714	.5000

That is, to produce 1 tonne of wheat in a particular year we need to have available on 31 December of the previous year 0.5333 tonnes of wheat, 0.02557 tonnes of iron and 0.0400 hundred pigs, and similarly for iron and pigs.

Assuming that **all** of the wheat, iron and pigs available at 31 December of one year are used to produce wheat, iron and pigs in the following year, find the amount of wheat, iron and pigs at 31 December, 1988 and then at 31 December, 1989 if

(a) the Wotsit-Mattas have 180 tonnes of wheat, 8.401 tonnes of iron and 2400 pigs available at 31 December, 1987, and

(b) the Wotsit-Mattas have 180 tonnes of wheat, 8 tonnes of iron and 2400 pigs available at 31 December, 1987.

Solution. We first set up a linear equations model for the Wotsit-Matta economy. Note that the unknowns are the amounts of wheat, iron and pigs produced in a year, and the right-hand sides are the amounts available from the previous year.

We let

$$\begin{aligned} x_1 &= \text{number of tonnes of wheat produced in a year} \\ x_2 &= \text{number of tonnes of iron produced in a year} \\ x_3 &= \text{number of hundreds of pigs produced in a year} \end{aligned}$$

and let

$$\begin{aligned} b_1 &= \text{number of tonnes of wheat available from the previous year} \\ b_2 &= \text{number of tonnes of iron available from the previous year} \\ b_3 &= \text{number of hundreds of pigs available from the previous year} \end{aligned}$$

Then the linear-equation model for the economy is

$$\begin{array}{lrcrcrcl} \text{Wheat used} & .5333x_1 & + & 4.286x_2 & + & 2x_3 & = & b_1 \\ \text{Iron used} & .02667x_1 & + & .2857x_2 & + & .05x_3 & = & b_2 \\ \text{Pigs used} & .04x_1 & + & .5714x_2 & + & .5x_3 & = & b_3 \end{array}$$

We can now solve for the production in years 1988 and 1989.

Case 1.(a). On 31 December, 1987, $\mathbf{b} = \begin{pmatrix} 180 \\ 8.401 \\ 24 \end{pmatrix}$. On solving for the production in 1988 we obtain $\mathbf{x} = \begin{pmatrix} 180 \\ 8.403 \\ 24 \end{pmatrix}$, that is, 180 tonnes wheat, 8.403 tonnes iron and 2400 pigs.

This 1988 production is then used as the right-hand side $\mathbf{b}$ in the equations for the 1989 production. On solving $A\mathbf{x} = \begin{pmatrix} 180 \\ 8.403 \\ 24 \end{pmatrix}$ as before, we obtain the solution $\mathbf{x} = \begin{pmatrix} 179.9 \\ 8.425 \\ 23.98 \end{pmatrix}$, and so the 1989 production is 179.9 tonnes wheat, 8.425 tonnes iron and 2398 pigs.

Case 2.(b). On 31 December, 1987, $\mathbf{b} = \begin{pmatrix} 180 \\ 8 \\ 24 \end{pmatrix}$. We find that the solution for 1988 production is 200.1 tonnes wheat, 4.658 tonnes iron and 2667 pigs.

On using this 1988 production as the right-hand side for the 1989 production, we obtain the 1989 production as 432.2 tonnes wheat, −34.37 tonnes iron and 5787 pigs.

◊

Note. The negative result for iron in case (b) means that the Wotsit-Matta economy is in deep trouble. Possible actions that the Wotsit-Mattas might take include importing iron from Australia (if they can find the foreign exchange) or alternatively, having a great party and eating sufficient wheat and pigs to stop the overheating in these sections of their economy. A complete mathematical explanation of the marked difference in behaviour between cases (a) and (b) requires a knowledge of eigenvalues and eigenvectors (which will be studied in MATH1231).

4.9 Matrix reduction and Maple

This section deals with the reduction of partitioned matrices to row echelon and reduced row echelon forms. It should be noted that Maple can handle symbolic entries, so we can solve systems that involve general right-hand-side vectors. The first instruction needed is:

```
with(LinearAlgebra):
```

which loads the linear algebra package. If you wish to experiment with a matrix with, say, 3 rows and 4 columns and with small integer entries, just try:

```
RandomMatrix(3,4);
```

though if you want to vary your random matrix you should try first:

```
randomize(301092);
```

Integers other than 301092 will give rise to other random matrices. To assign the name A to this, you use:

```
A:=RandomMatrix(3,4);
```

The 3×3 identity matrix may be obtained with the command:

```
Id:=IdentityMatrix(3);
```

An easy way to enter the values of a matrix is to command:

```
C:=< <1,4> | <2,5> | <3,6> >;
```

Note that we are inputting the entries of the matrix *columnwise* to give a 2×3 matrix called C. You may recover the second column with the command:

```
Column(%,2);
```

If you want to augment two matrices A_1, A_2 with the same number (r) of rows to obtain what is denoted in the main text by the partitioned matrix $(A_1|A_2)$, the command

```
< A1 | A2 >;
```

will do the trick. However Maple is blind to the partitioning. The following example, using a given 3×4 matrix A, is worth trying:

```
b:=< b1 , b2 , b3 >;
Ab:=< A | b >;
```

Here `Ab` is the name we have given to the resulting 3×5 matrix.

The reduction to echelon form of the matrix A is achieved by:

```
GaussianElimination(A);
```

The reduced row-echelon form is obtained with the command:

```
ReducedRowEchelonForm(%);
```

either after using `GaussianElimination` or directly. Here the `%` refers to the previous expression. `GaussianElimination` does not always do what one would like. Try `GaussianElimination(A)` with

```
A:=< <1,2,3> | <1,2,3> | <1,2,3> | <a,b,c> >;
```

There is a facility for finding the general solution of a set of linear equations. For example, the pair $1x_1 + 2x_2 + 3x_3 + 7x_4 = b_1$, $2x_1 + 4x_2 + 5x_3 + 9x_4 = b_2$ can be handled by the commands:

```
A:=< <1,2> | <2,4> | <3,5> | <7,9> >;
b:=< b1,b2 >;
Ab:=< A | b >;
GaussianElimination(%);
BackwardSubstitute(%);
```

The same answer is obtained from:

```
GaussianElimination(Ab);
```

```
BackwardSubstitute(%);
```

Experimenting with this may help motivate the reduced row-echelon form. You should also experiment with:

```
BackwardSubstitute(Ab);
```

to see what can go wrong if you leave out the Gaussian elimination step.

Rows 1 and 3 of the matrix C may be swapped with the command:

```
RowOperation(C,[1,3]);
```

You may replace row 4 by row 4 plus $\frac{7}{2}$ times row 3 in the matrix A to row 4 with

```
RowOperation(A,[4,3],7/2);
```

You may multiply row 4 of the matrix A by $\frac{7}{2}$ by the command:

```
RowOperation(A,4,7/2);
```

You should now practise reducing a 'random' matrix A by hand and comparing your answer with that obtained from `GaussianElimination` and `ReducedRowEchelonForm`. Other useful commands may be found by typing in:

```
?Pivot
?LinearSolve
```

Problems for Chapter 4

Questions marked with [**R**] are routine, [**H**] harder, [**M**] Maple and [**X**] are for MATH1141 only. You should make sure that you can do the easier questions before you tackle the more difficult questions. Questions marks with a [**V**] have video solutions available on Moodle.

Problems 4.1 : Introduction to linear equations

1. [**R**] Find the solution set of each of the following linear equation.

   a) $2x_1 - 5 = 0$ as an equation of one variable, then as an equation in two variables, and then three variables.

   b) $x_1 + 2x_2 = 4$ as an equation of two variables, then three variables.

   c) $2x_1 - 3x_2 + x_3 = 2$ as an equation of three variables.

2. [**R**][**V**] Determine algebraically whether the following systems of equations have a unique solution, no solution, or an infinite number of solutions. Draw graphs to illustrate your answers.

   a) $$\begin{array}{rcrcr} 3x_1 & + & 2x_2 & = & 6 \\ 9x_1 & + & 6x_2 & = & 36 \end{array}$$

   b) $$\begin{array}{rcrcr} 3x_1 & + & 2x_2 & = & 6 \\ 9x_1 & + & 4x_2 & = & 36 \end{array}$$

   c) $$\begin{array}{rcrcr} x_1 & - & 5x_2 & = & 5 \\ 6x_1 & - & 30x_2 & = & 30 \end{array}$$

3. [**H**] Find conditions on the coefficients a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2 so that the system of equations

   $$\begin{array}{rcrcl} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{array}$$

   has a) a unique solution, b) no solution, and c) an infinite number of solutions. For simplicity, assume $a_{11} \neq 0$.

4. [**X**] Repeat the previous question with no simplifying assumptions. That is, find general conditions which apply for all possible values of the coefficients.

5. [**R**] Find and geometrically describe the solutions for the following systems of linear equations.

   a) $$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 5 \\ 2x_1 & + & 5x_2 & + & 8x_3 & = & 12 \end{array}$$

   b) $$\begin{array}{rcrcrcr} 4x_1 & + & 5x_2 & - & 2x_3 & = & 16 \\ 8x_1 & + & 10x_2 & - & 4x_3 & = & 20 \end{array}$$

   c) $$\begin{array}{rcrcrcr} 4x_1 & + & 5x_2 & - & 2x_3 & = & 16 \\ 8x_1 & + & 10x_2 & - & 4x_3 & = & 32 \end{array}$$

6. [**X**] Prove algebraically that two distinct planes in $\mathbb{R}^3$ either intersect in a line or are parallel with no points in common. Use a linear equation in three unknowns to represent a plane in $\mathbb{R}^3$.

7. [**R**] Show that $x_1 = 2 - 2\lambda,\ x_2 = \lambda,\ x_3 = 3 + 2\lambda$, where λ is any real number, satisfy the system of equations

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & - & x_3 & = & -1 \\ 2x_1 & + & 4x_2 & & & = & 4 \\ & & 6x_2 & - & 3x_3 & = & -9 \end{array}$$

Problems 4.2 : Systems of linear equations and matrix notation

8. [**R**][**V**] Write each of the following systems of equations in vector form, as a matrix equation $A\mathbf{x} = \mathbf{b}$, and in the augmented matrix $(A|\mathbf{b})$ form.

a) $$\begin{array}{rcrcrcl} 3x_1 & - & 3x_2 & + & 4x_3 & = & 6 \\ 5x_1 & + & 2x_2 & - & 3x_3 & = & 7 \\ -x_1 & - & x_2 & + & 6x_3 & = & 8 \end{array}$$

b) $$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 7x_3 & + & 8x_4 & = & -2 \\ 3x_1 & + & 2x_2 & - & 5x_3 & - & x_4 & = & 7 \\ & & 3x_2 & + & 6x_3 & - & 6x_4 & = & 5 \end{array}$$

9. [**R**] Write the system of equations, the matrix equation and the augmented matrix form corresponding to the vector equation

$$x_1 \begin{pmatrix} 1 \\ 0 \\ -6 \\ 7 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 6 \\ -1 \\ 9 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 6 \\ -4 \\ 11 \end{pmatrix} = \begin{pmatrix} 10 \\ -2 \\ 0 \\ 5 \end{pmatrix}.$$

Problems 4.3 : Elementary row operations

10. [**R**] For each of the following matrices, find the appropriate elementary row operations to describe the transformation from one matrix to the next. Also continue the row reduction until the matrix is in row echelon form.

a) $$\left(\begin{array}{rrr|r} 1 & 4 & 2 & 3 \\ 2 & 6 & 3 & 0 \\ 4 & -2 & 4 & 4 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 4 & 2 & 3 \\ 0 & -2 & -1 & -6 \\ 0 & -18 & -4 & -8 \end{array}\right),$$

b) $$\left(\begin{array}{rrr|r} 3 & 4 & 1 & 3 \\ 2 & 8 & 0 & 2 \\ 0 & 8 & 3 & 0 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & -4 & 1 & 1 \\ 1 & 4 & 0 & 1 \\ 0 & 8 & 3 & 0 \end{array}\right).$$

11. [**M**] Write down the output when the Maple command `RowOperation(A,[2,1],3);` is applied to the matrix

$$A = \left(\begin{array}{rrr|r} 2 & 4 & 1 & 2 \\ 3 & 2 & 4 & 1 \\ 1 & 3 & 1 & 3 \end{array}\right).$$

Problems 4.4 : Solving systems of equations

12. **[R]** For each of the following augmented matrices do the following. Determine whether the matrix is in row-echelon form as defined in Section 4.4.1. If the matrix is in row-echelon form, identify the leading elements, leading rows, leading columns, and non-leading columns.

a) $\left(\begin{array}{ccc|c} 3 & 2 & 1 & 10 \\ 0 & 4 & 2 & 8 \\ 0 & 0 & -7 & 14 \end{array}\right)$, b) $\left(\begin{array}{ccc|c} 3 & 2 & 1 & 10 \end{array}\right)$, c) $\left(\begin{array}{ccc|c} 3 & 2 & 1 & 10 \\ 4 & 0 & 2 & 8 \\ 0 & 0 & -7 & 14 \end{array}\right)$,

d) $\left(\begin{array}{ccc|c} 3 & 2 & 1 & 10 \\ 0 & 4 & 2 & 8 \end{array}\right)$, e) $\left(\begin{array}{ccc|c} 0 & 3 & 1 & 6 \\ 0 & 0 & 1 & 5 \end{array}\right)$, f) $\left(\begin{array}{ccc|c} 3 & 2 & 1 & 10 \\ 0 & 4 & 2 & 8 \\ 0 & 0 & -7 & 14 \\ 0 & 0 & 0 & 0 \end{array}\right)$,

g) $\left(\begin{array}{ccc|c} 3 & 2 & 1 & 10 \\ 0 & 4 & 2 & 8 \\ 0 & 0 & -7 & 14 \\ 0 & 0 & 0 & 6 \end{array}\right)$, h) $\left(\begin{array}{ccc|c} 3 & 2 & 1 & 10 \\ 0 & 4 & 2 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \end{array}\right)$.

13. **[R]** Find the solutions to the following systems of equations. If possible give a geometric interpretation of the solution.

a)
$$\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & + & x_3 & = & 10 \\ & & 4x_2 & + & 2x_3 & = & 8 \\ & & & - & 7x_3 & = & 14 \end{array}$$

b)
$$\begin{array}{rcrcrcrcr} 3x_1 & + & 2x_2 & + & x_3 & + & x_4 & = & 10 \\ & & 4x_2 & + & 2x_3 & - & 4x_4 & = & 8 \\ & & & - & 7x_3 & + & 14x_4 & = & 14 \end{array}$$

14. **[R][V]** For each of the following systems of equations, do the following:

i) Write down the corresponding augmented matrix.

ii) Use Gaussian elimination to transform the augmented matrix into row-echelon form.

iii) Solve each system of equations writing your answer in vector form.

a) $$\begin{array}{rcrcl} x_1 & - & 2x_2 & = & 5 \\ 3x_1 & + & x_2 & = & 8 \end{array}$$

b) $$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 3x_3 & = & 3 \\ 2x_1 & + & 4x_2 & + & 10x_3 & = & 14 \end{array}$$

c) $$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & = & 11 \\ 2x_1 & - & x_2 & + & 3x_3 & = & 10 \\ 4x_1 & + & x_2 & - & x_3 & = & 4 \end{array}$$

d) $$\begin{array}{rcrcrcr} 2x_1 & - & 2x_2 & + & 4x_3 & = & -3 \\ 3x_1 & - & 3x_2 & + & 6x_3 & = & -4 \\ & & 5x_2 & + & 2x_3 & = & 9 \end{array}$$

e) $$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 4x_3 & = & 10 \\ -3x_1 & + & 3x_2 & + & 15x_3 & = & 15 \\ -2x_1 & - & x_2 & + & x_3 & = & -5 \end{array}$$

f) $$\begin{array}{rcrcrcr} x_1 & - & 4x_2 & - & 5x_3 & = & -6 \\ 2x_1 & - & x_2 & - & x_3 & = & 2 \\ 3x_1 & + & 9x_2 & + & 12x_3 & = & 30 \end{array}$$

g) $$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 4 \\ & & x_2 & - & x_3 & + & x_4 & = & 1 \\ 3x_1 & + & 2x_2 & & & - & 2x_4 & = & 3 \\ 5x_1 & + & 3x_2 & & & - & x_4 & = & 9 \end{array}$$

h) $$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 4 \\ & & x_2 & - & x_3 & + & x_4 & = & 1 \\ 3x_1 & + & 2x_2 & & & - & 2x_4 & = & 3 \end{array}$$

15. [**R**] For each of the following augmented matrices, find a *reduced* row-echelon form. Then write down all solutions of the corresponding system of equations and try to give a geometric interpretation of the solutions.

a) $\left(\begin{array}{rrr|r} 2 & 4 & 1 & 4 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{array}\right)$, b) $\left(\begin{array}{rrrr|r} 1 & 2 & 3 & 4 & 1 \\ 0 & -1 & 5 & 6 & 2 \\ 0 & 0 & 1 & 7 & 3 \end{array}\right)$.

Problems 4.5 : Deducing solubility from row-echelon form

16. [**R**][**V**] For each of the following augmented matrices, **without** solving, decide whether or not the corresponding system of linear equations has a unique solution, no solution or infinitely many solutions.

a) $\left(\begin{array}{rrr|r} 3 & 2 & 1 & 10 \\ 0 & 4 & 2 & 8 \\ 0 & 0 & -7 & 14 \end{array}\right)$, b) $\left(\begin{array}{rrr|r} 3 & 2 & 1 & 10 \\ 0 & 4 & 2 & 8 \\ 0 & 0 & -7 & 14 \\ 0 & 0 & 0 & 6 \end{array}\right)$, c) $\left(\begin{array}{rrr|r} 3 & 2 & 1 & 10 \end{array}\right)$,

d) $\left(\begin{array}{rrr|r} 3 & 2 & 1 & 10 \\ 0 & 4 & 2 & 8 \end{array}\right)$, e) $\left(\begin{array}{rrr|r} 3 & 2 & 1 & 10 \\ 0 & 4 & 2 & 8 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$.

17. [**H**][**V**] Determine which values of k, if any, will give a) a unique solution b) no solution

c) infinitely many solutions to the system of equations

$$\begin{array}{rcrcrcl} x & + & y & + & kz & = & 2 \\ 3x & + & 4y & + & 2z & = & k \\ 2x & + & 3y & - & z & = & 1. \end{array}$$

18. **[H]** For which values of λ do the equations

$$\begin{array}{rcrcrcr} x & + & 2y & + & \lambda z & = & 1 \\ -x & + & \lambda y & - & z & = & 0 \\ \lambda x & - & 4y & + & \lambda z & = & -1 \end{array}$$

have a) no solutions, b) infinitely many solutions, c) a unique solution?

19. **[H]** Consider the equation

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 2 & -1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ a \\ a+2b \end{pmatrix}.$$

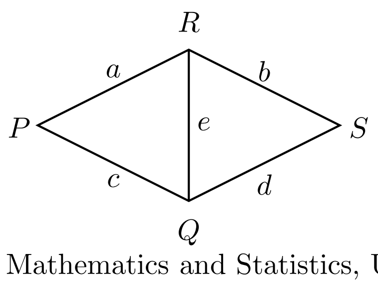
For what values of a and b does the equation have

a) a unique solution,
b) no solution,
c) infinitely many solutions?
d) In the case of (c), determine all solutions.

20. **[H]** You are an auditor for a company whose four executives make regular business trips on four routes and you suspect that at least one of the executives has been overstating her expenses. You don't know how much it costs to travel each route, but you know that it is the same for all the executives. You know the number of trips each executive made on each route in a certain period and you know the total expenses claimed by each executive for this period. If the numbers of trips are as shown in the table below, do you have sufficient information to be sure that someone is cheating? State your reasoning clearly.

	Route			
	1	2	3	4
Executive A	0	1	1	2
Executive B	1	2	0	1
Executive C	3	4	0	1
Executive D	2	1	3	3

21. **[H][V]** P, Q, R and S are four cities connected by highways which are labelled as shown in the diagram.



A hire car operator in P makes a note of the number of kilometres travelled by five customers who made trips starting and ending at P. He knows that the routes travelled by the five customers were as follows: abdc abdea cddc cdbec aedbec

Can he determine the length of each of the five highways? State your reasoning clearly.

Problems 4.6 : Solving $A\mathbf{x} = \mathbf{b}$ for indeterminate $\mathbf{b}$

22. [**R**] For each of the following systems of linear equations, find $x_1,\ x_2$ and x_3 in terms of $b_1,\ b_2$ and b_3.

a) $$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & 3x_3 & = & b_1 \\ & & x_2 & - & 3x_3 & = & b_2 \\ -2x_1 & + & 3x_2 & - & 2x_3 & = & b_3 \end{array}$$

b) $$\begin{array}{rcrcrcl} 2x_1 & & & - & 4x_3 & = & b_1 \\ 3x_1 & + & x_2 & - & 2x_3 & = & b_2 \\ -2x_1 & - & x_2 & - & x_3 & = & b_3 \end{array}$$

23. [**R**] Show that the system of equations $x + y + 2z = a$, $x + z = b$ and $2x + y + 3z = c$ are consistent if and only if $c = a + b$.

24. [**R**] For the following systems, find conditions on the right-hand-side vector $\mathbf{b}$ which ensure that the system has a solution.

a) $$\begin{array}{rcrcrcl} 2x_1 & & & - & 4x_3 & = & b_1 \\ 3x_1 & + & x_2 & - & 2x_3 & = & b_2 \\ -2x_1 & - & x_2 & & & = & b_3 \end{array}$$

b) $$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & + & 3x_3 & - & x_4 & = & b_1 \\ 2x_1 & - & x_2 & & & + & 2x_4 & = & b_2 \\ x_1 & - & 2x_2 & - & 3x_3 & + & 3x_4 & = & b_3 \\ & & 3x_2 & + & 6x_3 & - & 4x_4 & = & b_4 \end{array}$$

Problems 4.7 : General properties of the solution of $A\mathbf{x} = \mathbf{b}$

25. [**R**] Show that $\mathbf{x} = \begin{pmatrix} 7 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$ are the solutions of

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 2x_3 & = & 3 \\ 2x_1 & - & 6x_2 & + & 4x_3 & = & 2 \\ -2x_1 & + & 4x_2 & - & 4x_3 & = & -6 \end{array}$$

and that $\mathbf{x} = \lambda \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$ are the solutions of the corresponding homogeneous system

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 2x_3 & = & 0 \\ 2x_1 & - & 6x_2 & + & 4x_3 & = & 0 \\ -2x_1 & + & 4x_2 & - & 4x_3 & = & 0 \end{array}$$

Problems 4.8 : Applications

Some of these problems can be solved using more elementary geometric techniques. But try your best to also find solutions that involve solving systems of linear equations.

26. [**R**] Does the point $(-3, 3, 6)$ lie on the plane

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}?$$

27. [**R**] Is the vector $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ in span $\left(\begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \right)$?

28. [**R**] Is the vector $\begin{pmatrix} 1 \\ 1 \\ 4 \\ 12 \end{pmatrix}$ in span $\left(\begin{pmatrix} 3 \\ -1 \\ 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 4 \\ 3 \end{pmatrix} \right)$?

29. [**R**] Can $\begin{pmatrix} 3 \\ 1 \\ -2 \\ 4 \end{pmatrix}$ be expressed as a linear combination of $\begin{pmatrix} 1 \\ 0 \\ -3 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 5 \\ 6 \end{pmatrix}$?

30. [**R**] Do the lines $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 12 \\ 15 \\ 7 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ intersect?

31. [**R**] Is the vector $\begin{pmatrix} 5 \\ 7 \\ -1 \end{pmatrix}$ parallel to the plane $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$ for $\lambda_1, \lambda_2 \in \mathbb{R}$?

32. [**H**] Show that the line $\dfrac{x-1}{2} = \dfrac{y}{3} = \dfrac{z+1}{-1}$ is parallel to the plane

$$\mathbf{x} = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \lambda_1,\ \lambda_2 \in \mathbb{R}.$$

33. [**H**] Find the intersection (if any) of the line $\mathbf{x} = \begin{pmatrix} 0 \\ 18 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ for $\mu \in \mathbb{R}$ and the plane

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \text{ for } \lambda_1, \lambda_2 \in \mathbb{R}.$$

34. [**R**] Find the intersection (if any) of the planes $8x_1 + 8x_2 + x_3 = 35$ and

$$\mathbf{x} = \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad \text{for } \lambda_1, \lambda_2 \in \mathbb{R}.$$

35. [**H**] Are the planes

$$\mathbf{x} = \begin{pmatrix} 1 \\ -4 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 1 \\ -2 \\ 7 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ 1 \\ 5 \\ 2 \end{pmatrix} \quad \text{for } \lambda_1, \lambda_2 \in \mathbb{R}$$

and

$$\mathbf{x} = \begin{pmatrix} 2 \\ -4 \\ 1 \\ 3 \end{pmatrix} + \mu_1 \begin{pmatrix} 3 \\ -1 \\ 2 \\ 4 \end{pmatrix} + \mu_2 \begin{pmatrix} -1 \\ 4 \\ 2 \\ 6 \end{pmatrix} \quad \text{for } \mu_1, \mu_2 \in \mathbb{R}$$

parallel?

36. [**R**] Show that the 3 planes with Cartesian equations

$$\begin{aligned} x + 3y + 2z &= 5 \\ 2x + y - z &= 2 \\ 7x + 11y + 4z &= 13 \end{aligned}$$

do not intersect at one point.

37. [**H**] Consider the following system of equations

$$\begin{aligned} x + y - z &= 1 \\ 2x - 4y + 2z &= 2 \\ 3x - 3y + z &= 3 \end{aligned}$$

a) Use Gaussian elimination and back–substitution to find the solution(s), if any, of the above equations.

b) Use your result in part a) to decide whether the three planes represented by the equations are parallel, intersecting in a straight line, intersecting at a point or have some other configuration.

38. [**R**] Find a polynomial $p(x)$ of degree 2 satisfying $p(1) = 5,\ p(2) = 7,\ p(3) = 13$.

39. [**R**] The total of the ages of my brother, my sister and myself is 140 years. I am seven times the difference between their ages (my sister is older than my brother) and in seven years I will be half their combined ages now. How old are we?

40. [**R**] In a trip to Asia a traveller spent \$90 a day for hotels in Bangkok, \$60 a day in Singapore and \$60 a day in Kuala Lumpur. For food the traveller spent \$60 a day in Bangkok, \$90 a day in Singapore, and \$60 a day in Kuala Lumpur. In addition the traveller spent \$30 a day in other expenses in each city. The traveller's diary shows that the total hotel bill was \$1020, total food bill was \$960, and total other expenses were \$420. Find the number of days the traveller spent in each city, or show that the diary must be wrong.

41. [**R**] A dietician is planning a meal consisting of three foods. A serving of the first food contains 5 units of protein, 2 units of carbohydrates and 3 units of iron. A serving of the second food contains 10 units of protein, 3 units of carbohydrates and 6 units of iron. A serving of the third food contains 15 units of protein, 2 units of carbohydrates and 1 unit of iron. How many servings of each food should be used to create a meal containing 55 units of protein, 13 units of carbohydrates and 17 units of iron?

42. [**X**] A farmer owns a 12-hectare farm on which he grows wheat, oats and barley. Each hectare of cereal crop planted has certain requirements for labour, fertiliser and irrigation water as shown in the following table.

Crop	Labour (hours per week)	Fertiliser (kilograms)	Irrigation Water kilolitres)
Wheat (per hectare)	6	150	72
Oats (per hectare)	6	100	48
Barley (per hectare)	2	70	36
Amount available	48	700	612

Answer the following questions

a) Set up a linear equation model for the system.

b) Find the solution (if any) of the model.

c) Replace the equations by inequalities assuming that not all of the available land, labour time, fertiliser and irrigation water have to be used. Then introduce 4 new "slack" variables which represent the amounts of unused land, labour, fertiliser and irrigation water respectively.

d) Can you find any reasonable solutions for the systems of linear equations in (c), i.e. solutions in which the variables are non-negative?

43. [**X**] In this problem we shall calculate the area of a spherical triangle. Consider the surface of a sphere of unit radius of area 4π. A great circle on a sphere is the intersection of that sphere with a plane through the centre. If two great circles meet at antipodal points P, P' let the angle θ between them be the angle $0 < \theta < \pi$ between the tangents to the two circles at P. ($\pi - \theta$ is also the angle between the two great circles). Finally define a spherical triangle to be the region bounded by 3 great circles meeting A, B, C with angles α, β, γ.

a) The areas bounded by 2 great circles are called lunes. Show their areas are 2θ, 2θ, $2(\pi - \theta)$, $2(\pi - \theta)$.

b) Show the surface of the sphere is divided by a spherical triangle into 8 regions equal in area in pairs.

c) Use parts a) and b) to set up a simple system of 4 linear equations in the 4 areas.

d) Hence show area $ABC = \alpha + \beta + \gamma - \pi$.

Problems 4.9 : Matrix reduction and Maple

44. **[M]** The Maple session below (for which the package `LinearAlgebra` has been loaded) calculates the intersection of 3 planes Π_1, Π_2 and Π_3.

a) Write down the cartesian equations for Π_1, Π_2 and Π_3.

b) Give a full geometric description of the intersection of Π_1, Π_2 and Π_3.

c) Express the intersection of Π_1, Π_2 and Π_3 in cartesian form.

```
> A:=<<1,3,2>|<2,6,4>|<-1,-1,-1>>;
```

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -1 \\ 2 & 4 & -1 \end{bmatrix}$$

```
> b:=<2,12,7>;
```

$$b := \begin{bmatrix} 2 \\ 12 \\ 7 \end{bmatrix}$$

```
> LinearSolve(A,b);
```

$$\begin{bmatrix} 5 - 2\_t_2 \\ \_t_2 \\ 3 \end{bmatrix}$$

45. **[M]**

```
> with(LinearAlgebra):
```

```
> A:=<<1,3,4,7>|<-2,6,2,-8>|<1,8,7,6>|<a,b,c,d>>;
```

$$A := \begin{bmatrix} 1 & -2 & 1 & a \\ 3 & 6 & 8 & b \\ 4 & 2 & 7 & c \\ 7 & -8 & 6 & d \end{bmatrix}$$

```
> GaussianElimination(A);
```

$$\begin{bmatrix} 1 & -2 & 1 & a \\ 0 & 12 & 5 & b - 3a \\ 0 & 0 & -\frac{7}{6} & c - \frac{3a}{2} - \frac{5b}{6} \\ 0 & 0 & 0 & d - a + 2b - 3c \end{bmatrix}$$

```
>
```

a) The above is a Maple session designed to calculate where 4 planes in $\mathbb{R}^3$ meet. What are the equations of the planes?

b) What are the condition(s) on a, b, c, d for the planes to meet at a point?

c) If $a = 1,\ b = 2,\ c = 3$ and the planes meet, where do they meet?

Chapter 5

MATRICES

"It seems very pretty," she said when she had finished it,
"but it's RATHER hard to understand!...
...Somehow it seems to fill my head with ideas
— only I don't exactly know what they are!"
Lewis Carroll, Through the Looking Glass.

In Chapter 4, matrices are treated as devices to help in writing systems of linear equations. They are also important in many other areas of mathematics and their applications.

In this chapter, we shall define the addition of two matrices, the multiplication of two matrices and the multiplication of a matrix by a scalar. Under these operations of addition and scalar multiplication matrices behave much like vectors and indeed like real numbers. For instance, both matrices and real numbers obey associative laws and distributive laws. Yet, there are differences between matrices and real numbers. Multiplication of real numbers is commutative, but, as we shall see, matrix multiplication is not. Every non-zero real number has an multiplicative inverse which is the reciprocal of that number. However, not all matrices have inverses. We shall study the matrix arithmetic and algebra and an important function on the set of square matrices—the determinant function.

5.1 Matrix arithmetic and algebra

In this section we describe the arithmetic of matrices, including equality and addition of matrices, multiplication of a matrix by a scalar, and multiplication of matrices. In general, **division is not defined for matrices**, although an inverse can be defined for some matrices (see Section 5.3).

Definition 1. *An $m \times n$ (read "m by n") **matrix** is an array of m rows and n columns of numbers of the form*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

*The number a_{ij} in the matrix A lies in row i and column j. It is called the ijth **entry** or ijth **element** of the matrix.*

NOTE.

1. An $m \times 1$ matrix is called a column vector, while a $1 \times n$ matrix is called a row vector.

2. We use M_{mn} to stand for the set of all $m \times n$ matrices, i.e., the set of all matrices with m rows and n columns.

   In general, we will assume that the entries in a matrix can be complex numbers. However, it is sometimes necessary to distinguish between real matrices, in which all entries a_{ij} are real numbers, and complex matrices, in which all a_{ij} are complex numbers. In this case, we use $M_{mn}(\mathbb{R})$ for the set of all real $m \times n$ matrices and $M_{mn}(\mathbb{C})$ for the set of all complex $m \times n$ matrices. Likewise, $M_{mn}(\mathbb{Q})$ is used for the set of all rational $m \times n$ matrices.

3. We say an $m \times n$ matrix is of **size** $m \times n$.

4. When we say "let $A = (a_{ij})$", we are specifying a matrix of fixed size, in which, for each given i, j, the ijth entry is a_{ij}. On the other hand, for a given matrix A, we denote the entry in the ith row and jth column by $[A]_{ij}$.

Example 1. The matrix $A = \begin{pmatrix} 1 & 2 & -1 & 2 \\ 3 & -4 & 2 & 5 \\ -6 & -3 & 1 & 4 \end{pmatrix}$ is of size 3×4 and $[A]_{24} = 5$. $\diamondsuit$

When the number of rows or columns of a matrix is larger than 9, we need to use a comma to separate the row number and the column number. If B is a matrix of size 5×12, we shall use $[B]_{2,4}$ and $[B]_{3,11}$ to denote the entry in the second row and fourth column and the entry in the third row and the eleventh column, respectively.

5.1.1 Equality, addition and multiplication by a scalar

The rules for equality of matrices, addition of matrices and multiplication of a matrix by a scalar are straightforward generalisations of the corresponding rules for vectors in $\mathbb{R}^n$.

Definition 2. *Two matrices A and B are defined to be* **equal** *if*

1. *the number of rows of A equals the number of rows of B,*
2. *the number of columns of A equals the number of columns of B,*
3. *$[A]_{ij} = [B]_{ij}$ for all i and j.*

To summarise, equal matrices have the same size and their corresponding entries are equal.

Addition of matrices is defined as follows.

Definition 3. *If A and B are $m \times n$ matrices, then the* **sum** *$C = A + B$ is the $m \times n$ matrix whose entries are*

$$[C]_{ij} = [A]_{ij} + [B]_{ij} \qquad \text{for all} \quad i, j.$$

To understand the proofs of the properties of matrices, we need to know the notation $[A]_{ij}$ well. The row i column j entry of the matrix $A + B$ is denoted by $[A + B]_{ij}$. The definition simply says that this entry is the same as $[A]_{ij} + [B]_{ij}$, that is the sum of the corresponding entries of the matrices A and B.

Note that addition of matrices of different sizes is **not defined**.

Example 2.

$$\begin{pmatrix} 2 & 3 & 5 \\ 4 & -3 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 4 \\ 2 & 4 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 9 \\ 6 & 1 & 1 \end{pmatrix}$$

but

$$\begin{pmatrix} -1 & 1 & 4 \\ 2 & 4 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

is not defined as the sizes are different. ◊

Proposition 1. Let A, B and C be $m \times n$ matrices.

1. $A + B = B + A$. (Commutative Law of Addition)
2. $(A + B) + C = A + (B + C)$. (Associative Law of Addition)

Proof. We only prove the Associative law.

$$\begin{aligned}
[(A+B)+C]_{ij} &= [A+B]_{ij} + [C]_{ij} && \text{(by definition, the sum of } (A+B) \text{ and } C) \\
&= ([A]_{ij} + [B]_{ij}) + [C]_{ij} && \text{(by definition, the sum of } A \text{ and } B) \\
&= [A]_{ij} + ([B]_{ij} + [C]_{ij}) && \text{(associative law of numbers)} \\
&= [A]_{ij} + [B+C]_{ij} && \text{(by definition, the sum of } B \text{ and } C) \\
&= [A+(B+C)]_{ij} && \text{(by definition, the sum of } A \text{ and } (B+C))
\end{aligned}$$

Hence $(A + B) + C = A + (B + C)$. □

We now define what is meant by a zero matrix.

Definition 4. *A* ***zero*** *matrix (written* $\mathbf{0}$*) is a matrix in which every entry is zero.*

Example 3.

$$\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{is the zero matrix in } M_{22}.$$

and

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{is the zero matrix in } M_{23}.$$

$\diamondsuit$

Note that when writing the zero matrix by hand, you can write $\underline{\underline{0}}$ to distinquish it from the zero vector and the number 0.

Proposition 2. Let A be a matrix and $\mathbf{0}$ be the zero matrix, both in M_{mn}. Then

$$A + \mathbf{0} = \mathbf{0} + A = A.$$

Proof. Similar to the proof of the associative law of addition but skipping the detail explanation, we have

$$[A + \mathbf{0}]_{ij} = [A]_{ij} + 0 = [A]_{ij}.$$

Hence $A + \mathbf{0} = \mathbf{A}$. We can also prove that $\mathbf{0} + A = A$ in a similar way. $\square$

Definition 5. *For any matrix* $A \in M_{mn}$*, the* ***negative*** *of* A *is the* $m \times n$ *matrix* $-A$ *with entries*

$$[-A]_{ij} = -[A]_{ij} \quad \textit{for all } 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n.$$

Using the above definition of the negative of a matrix, we can define subtraction by

$$A - B = A + (-B) \quad \text{for all } A, B \in M_{mn}.$$

As with addition, we cannot subtract one matrix from another one of different size.

Example 4. Let $A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & -1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -4 & 5 \\ 3 & 0 & -2 \end{pmatrix}$. We have

$$-B = \begin{pmatrix} -2 & 4 & -5 \\ -3 & 0 & 2 \end{pmatrix} \quad \text{and} \quad A - B = \begin{pmatrix} -1 & 7 & -7 \\ -1 & -1 & 5 \end{pmatrix}.$$

$\diamondsuit$

Proposition 3. If A is an $m \times n$ matrix, then

$$A + (-A) = (-A) + A = \mathbf{0}.$$

We now define multiplication of a matrix by a scalar, that is, by a number (real or complex).

Definition 6. *If A is an $m \times n$ matrix and λ is a scalar, then the* **scalar multiple** *$B = \lambda A$, of A is the $m \times n$ matrix whose entries are*

$$[B]_{ij} = \lambda [A]_{ij} \text{ for all } 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n.$$

that is, each entry of the matrix is multiplied by the scalar λ.

Example 5.

$$3 \begin{pmatrix} 2 & 3 & 5 \\ 4 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 9 & 15 \\ 12 & -9 & 6 \end{pmatrix}.$$

$\Diamond$

Proposition 4. Let λ, μ be scalars and A, B be matrices in M_{mn}.

1. $\lambda(\mu A) = (\lambda\mu)A$ (Associative Law of Scalar Multiplication)
2. $(\lambda + \mu)A = \lambda A + \mu A$ (Scalar Multiplication is distributive over Scalar Addition)
3. $\lambda(A + B) = \lambda A + \lambda B$ (Scalar Multiplication is distributive over Matrix Addition)

The above rules for equality, matrix addition and multiplication by a scalar are essentially the same as the corresponding rules for vectors in $\mathbb{R}^n$. The matrix operations also obey the basic laws of arithmetic as do addition and multiplication by a scalar in $\mathbb{R}^n$. For instance, in both M_{mn} and $\mathbb{R}^n$, the commutative laws, associative laws and distributive laws hold.

5.1.2 Matrix multiplication

In Section 4.2, we represented the system of linear equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

by the matrix equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

It makes sense to use the above relation between the matrix equation $A\mathbf{x} = \mathbf{b}$ and the system of linear equations as the basis for the definition of the product of two matrices. The column vector $\mathbf{x} \in \mathbb{R}^n$ may be regarded as an $n \times 1$ matrix and the column vector $\mathbf{b} = A\mathbf{x} \in \mathbb{R}^m$ may be regarded as an $m \times 1$ matrix.

Definition 7. *If $A = (a_{ij})$ is an $m \times n$ matrix and $\mathbf{x}$ is an $n \times 1$ matrix with entries x_i, then the* **product** $\mathbf{b} = A\mathbf{x}$ *is the $m \times 1$ matrix whose entries are given by*

$$b_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = \sum_{k=1}^{n} a_{ik}x_k \quad \text{for} \quad 1 \leqslant i \leqslant m.$$

Example 6. If A is a 3×2 matrix and $\mathbf{x}$ is a 2×1 matrix given by

$$A = \begin{pmatrix} 2 & -3 \\ 5 & -1 \\ -7 & 4 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

then the product $A\mathbf{x}$ is the 3×1 matrix

$$A\mathbf{x} = \begin{pmatrix} 2(6) + (-3)(9) \\ 5(6) + (-1)(9) \\ (-7)(6) + 4(9) \end{pmatrix} = \begin{pmatrix} -15 \\ 21 \\ -6 \end{pmatrix}.$$

$\Diamond$

To obtain a suitable general definition of multiplication of two matrices, we replace the matrix $\mathbf{x}$ with 1 column by a matrix X with p columns. We can then think of the matrix A multiplying each of the columns of X using the above rule.

Definition 8. *Let A be an $m \times n$ matrix and X be an $n \times p$ matrix and let $\mathbf{x}_j$ be the jth column of X. Then the* **product** $B = AX$ *is the $m \times p$ matrix whose jth column $\mathbf{b}_j$ is given by*

$$\mathbf{b}_j = A\mathbf{x}_j \qquad \text{for} \quad 1 \leqslant j \leqslant p.$$

The matrix X is the **augmented matrix** of the p column vectors, i.e. $X = (\mathbf{x}_1|\cdots|\mathbf{x}_p)$. So the definition can be written as

$$A(\mathbf{x}_1|\cdots|\mathbf{x}_p) = (A\mathbf{x}_1|\cdots|A\mathbf{x}_p).$$

By combining Definitions 6 and 7, we can give an equivalent definition of a matrix product in terms of the entries of the matrices as:

Definition 9. *(Alternative) If A is an $m \times n$ matrix and X is an $n \times p$ matrix, then the* **product** AX *is the $m \times p$ matrix whose entries are given by the formula*

$$[AX]_{ij} = [A]_{i1}[X]_{1j} + \cdots + [A]_{in}[X]_{nj} = \sum_{k=1}^{n} [A]_{ik}[X]_{kj} \quad \text{for } 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant p.$$

NOTE.

1. Let A be an $m \times n$ matrix and B be an $r \times s$ matrix. The product AB is defined **only** when the number of columns of A is the same as the number of rows of B, i.e. $n = r$. If $n = r$, the size of AB will be $m \times s$.

$$\begin{array}{ccc} A & \qquad\qquad & B \\ m \times n & & r \times s \end{array}$$

if equal

size of AB

2. Matrix multiplication is a row-times-column process: we get the (row i, column j) entry of AX by going across the ith row of A and down the jth column of X multiplying and adding as we go.

Example 7. For

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{pmatrix} \qquad \text{and} \qquad X = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 3 \\ 3 & -3 & 1 \end{pmatrix}$$

the product $AX = (b_{ij})$ can be easily obtained.

$$A\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 14 \\ -2 \end{pmatrix}, \qquad A\begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -14 \\ 2 \end{pmatrix}, \qquad A\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 11 \\ 1 \end{pmatrix}.$$

$$\text{Hence} \qquad AX = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 3 \\ 3 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 14 & -14 & 11 \\ -2 & 2 & 1 \end{pmatrix}.$$

$\diamondsuit$

Warning. For some A and X, $AX \neq XA$, i.e., matrix multiplication does **not** satisfy the commutative law. This can happen when one product is defined and the other is not. It can happen even when both are defined. Thus, for the A and X just given, the product XA is not defined as the number of columns of X does not equal the number of rows of A.

$$\begin{array}{ccc} X & \qquad\qquad & A \\ 3 \times 3 & & 2 \times 3 \end{array}$$

not equal

XA is not defined

Here is an example in which both AX and XA are defined but are different:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad \text{and} \qquad X = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix},$$

in which case

$$AX = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 4 & 8 \end{pmatrix},$$

whereas

$$XA = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 2 & 8 \end{pmatrix}.$$

A matrix is said to be **square** if it has the same number of rows as columns. For example, of the four matrices displayed below, the second and fourth are square:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

The **diagonal** of a square matrix consists of the positions on the line from the top left to the bottom right. More precisely, the diagonal entries of an $n \times n$ square matrix (a_{ij}) are $a_{11},\ a_{22}, \ldots,\ a_{nn}$. The matrix is said to be an **identity matrix** if its diagonal entries are all 1 and all other entries are 0. The following are identity matrices

$$(1) \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For each integer n there is one and only one $n \times n$ identity matrix. It is denoted by I_n, or just by I if there is no risk of ambiguity.

Definition 10. *An **identity** matrix (written I) is a square matrix with 1's on the diagonal and 0's off the diagonal.*

Example 8.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{is the identity matrix in } M_{22}.$$

and

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{is the identity matrix in } M_{33}.$$

◊

Although matrix multiplication does not satisfy a commutative law of multiplication, the operation does satisfy some laws.

Proposition 5 (Properties of Matrix Multiplication)**.** Let A, B, C be matrices and λ be a scalar.

1. If the product AB exists, then $A(\lambda B) = \lambda(AB) = (\lambda A)B$.
2. **Associative Law of Matrix Multiplication**. If the products AB and BC exist, then $A(BC) = (AB)C$.
3. $AI = A$ and $IA = A$, where I represents identity matrices of the appropriate (possibly different) sizes.
4. **Right Distributive Law**. If $A + B$ and AC exist, then $(A + B)C = AC + BC$.
5. **Left Distributive Law**. If $B + C$ and AB exist, then $A(B + C) = AB + AC$.

Proof. We will prove the right distributive law, and leave the proof of the remainder as a problem at the end of the chapter. As $A + B$ exists, A and B must be the same size. Also, as AC exists, the number of columns of A must be equal to the number of rows of C. Therefore we let A, B be $m \times n$ matrices and C be an $n \times p$ matrix. Then BC also exists and the row i, column j entry of $(A + B)C$ is

$$\begin{aligned}
[(A+B)C]_{ij} &= \sum_{k=1}^{n} [A+B]_{ik}[C]_{kj} \\
&= \sum_{k=1}^{n} ([A]_{ik} + [B]_{ik})[C]_{kj} \\
&= \sum_{k=1}^{n} [A]_{ik}[C]_{kj} + \sum_{k=1}^{n} [B]_{ik}[C]_{kj} \\
&= [AC]_{ij} + [BC]_{ij} \\
&= [AC + BC]_{ij}.
\end{aligned}$$

Hence $(A + B)C = AC + BC$ as claimed. $\square$

Example 9. If $A \in M_{23}$ is given by

$$A = \begin{pmatrix} 1 & 2 & 4 \\ -3 & 5 & 7 \end{pmatrix},$$

then by direct multiplication

$$AI = \begin{pmatrix} 1 & 2 & 4 \\ -3 & 5 & 7 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A,$$

and

$$IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ -3 & 5 & 7 \end{pmatrix} = A.$$

Note that in this case the right and left I's must be of different sizes for the matrix products to exist. $\diamondsuit$

Example 10. Here is an example to verify the associative law of multiplication. For

$$A = \begin{pmatrix} 2 & 1 & -2 \\ 3 & 4 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 5 \\ 0 & -3 \\ 2 & 6 \end{pmatrix}, \qquad C = \begin{pmatrix} -5 & 2 \\ -3 & -6 \end{pmatrix},$$

we have

$$A(BC) = \begin{pmatrix} 2 & 1 & -2 \\ 3 & 4 & 2 \end{pmatrix} \begin{pmatrix} -10 & -32 \\ 9 & 18 \\ -28 & -32 \end{pmatrix} = \begin{pmatrix} 45 & 18 \\ -50 & -88 \end{pmatrix}$$

and

$$(AB)C = \begin{pmatrix} -6 & -5 \\ 1 & 15 \end{pmatrix} \begin{pmatrix} -5 & 2 \\ -3 & -6 \end{pmatrix} = \begin{pmatrix} 45 & 18 \\ -50 & -88 \end{pmatrix}.$$

$\Diamond$

Using the properties of matrix operations, we can simplify expressions in unknown matrices in almost the same way as simplifying expressions in algebra.

Example 11. By distributive laws and associative laws, we can write

$$A(A + 2B)A = A^3 + 2ABA.$$

Here A^3 obviously means AAA. Thanks to the associative law of matrix multiplication, we do not have to specify ABA to be $(AB)A$ or $A(BA)$ because $(AB)A = A(BA)$. Note that we cannot write the second term as $2A^2B$ because matrix multiplication is **not** commutative.

Example 12. Expand the expression $(A + I)^2$.

SOLUTION.

$$\begin{aligned} (A+I)^2 &= (A+I)(A+I) \\ &= (A+I)A + (A+I)I \\ &= (A^2 + IA) + (AI + I) \\ &= A^2 + A + A + I \\ &= A^2 + 2A + I \end{aligned}$$

Can you identify all the rules used in each step? $\Diamond$

Note that, in general, $(A + B)^2 \neq A^2 + 2AB + B^2$.

5.1.3 Matrix arithmetic and systems of linear equations

In Section 4.2 we used the matrix notation $A\mathbf{x} = \mathbf{b}$ as a shorthand notation for a system of linear equations. Then we used this as the motivation to define multiplication in Definition 7 in 5.1.2. That is, when we multiply the coefficient matrix A with the unknown column vector $\mathbf{x}$, we obtain a column vector $A\mathbf{x}$. If we equate the components of the column vector $A\mathbf{x}$ with those of $\mathbf{b}$, we shall recover the system of equations. Thus $A\mathbf{x} = \mathbf{b}$ is not simply a shorthand notation, it *is* a matrix equation. In other words, we have the following proposition.

Proposition 6. Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$.

The vector $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ is a solution to the matrix equation $A\mathbf{x} = \mathbf{b}$, i.e. $A\mathbf{v} = \mathbf{b}$ if and only if $x_1 = v_1, \ldots, x_n = v_n$ is a solution to the system of equations

$$\begin{cases} a_{11}x_1 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & = & b_m \end{cases}.$$

Using this we can rewrite the proof of the propositions in Section 4.7 in a much shorter way. As an example, let us rewrite the proof of the following.

Proposition 2 (Section 4.7). If $\mathbf{v}$ and $\mathbf{w}$ are solutions of $A\mathbf{x} = \mathbf{0}$ then so are $\mathbf{v} + \mathbf{w}$ and $\lambda\mathbf{v}$ for any scalar λ.

Proof. Since $\mathbf{v}$ and $\mathbf{w}$ are solutions of $A\mathbf{x} = \mathbf{0}$, we have

$$A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Hence $\mathbf{v} + \mathbf{w}$ is also a solution.

On the other hand, we have

$$A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda\mathbf{0} = \mathbf{0}.$$

Thus $\lambda\mathbf{v}$ is also a solution. $\square$

5.2 The transpose of a matrix

Definition 1. *The* ***transpose*** *of an $m \times n$ matrix A is the $n \times m$ matrix A^T (read 'A transpose') with entries given by*

$$[A^T]_{ij} = [A]_{ji}.$$

Example 1. The transpose of

$$A = \begin{pmatrix} 3 & 2 \\ -4 & 0 \\ 0 & 1 \\ 1 & -5 \end{pmatrix} \quad \text{is} \quad A^T = \begin{pmatrix} 3 & -4 & 0 & 1 \\ 2 & 0 & 1 & -5 \end{pmatrix}.$$

$\lozenge$

Example 2. The transpose of

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \in M_{23} \qquad \text{is} \qquad A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix} \in M_{32}.$$

◊

NOTE. The columns of A^T are the rows of A and the rows of A^T are the columns of A.

5.2.1 Some uses of transposes

Vectors. As noted in Chapter 3, column vectors are often written as the transpose of 'row vectors'.

Example 3. The transpose of the column vector

$$\mathbf{x} = \begin{pmatrix} 2 \\ 5 \\ -2 \\ 1 \end{pmatrix}$$

is the row vector $\mathbf{x}^T = \begin{pmatrix} 2 & 5 & -2 & 1 \end{pmatrix} = \mathbf{y}$; conversely, $\mathbf{x} = \mathbf{y}^T$. ◊

Products of Vectors. So far we have not defined the product of two vectors. However, the product of a row vector and a column vector makes sense as a special case of matrix multiplication.

Example 4. Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors in $\mathbb{R}^n$. Then, using the usual rule for matrix multiplication, we have

$$\mathbf{a}^T\mathbf{b} = \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$

This product is a 1×1 matrix[1], which we sometimes regard simply as a scalar, and so $\mathbf{a}^T\mathbf{b}$ is often called the **scalar product** of the two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$. Note also that $\mathbf{b}^T\mathbf{a} = \mathbf{a}^T\mathbf{b}$. ◊

Example 5. If

$$\mathbf{a} = \begin{pmatrix} 1 \\ 4 \\ -2 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 2 \\ -3 \\ 0 \\ 6 \end{pmatrix}$$

then

$$\mathbf{a}^T\mathbf{b} = \begin{pmatrix} 1 & 4 & -2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 0 \\ 6 \end{pmatrix} = 2 - 12 + 0 + 18 = 8 = \begin{pmatrix} 2 & -3 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ -2 \\ 3 \end{pmatrix} = \mathbf{b}^T\mathbf{a}.$$

[1] Technically, we should write the matrix product $\mathbf{a}^T\mathbf{b}$ as a matrix with brackets. In practice we write the result as a single number without brackets.Thus if $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, then we write $\mathbf{a}^T\mathbf{b} = 8$, not $\mathbf{a}^T\mathbf{b} = (8)$.

The order of multiplication is very important. If $\mathbf{a}$ and $\mathbf{b}$ are (column) vectors, then the product $\mathbf{a}\mathbf{b}^T$ is not a scalar but a matrix. ◊

Example 6. For $\mathbf{a} \in \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^n$

$$\mathbf{a}\mathbf{b}^T = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix} = \begin{pmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_mb_1 & a_mb_2 & \cdots & a_mb_n \end{pmatrix}.$$

◊

Example 7. If

$$\mathbf{a} = \begin{pmatrix} 1 \\ 4 \\ -2 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 2 \\ -3 \\ 0 \\ 6 \end{pmatrix}$$

then

$$\mathbf{a}\mathbf{b}^T = \begin{pmatrix} 2 & -3 & 0 & 6 \\ 8 & -12 & 0 & 24 \\ -4 & 6 & 0 & -12 \\ 6 & -9 & 0 & 18 \end{pmatrix}.$$

◊

Note that the expressions $\mathbf{a}\mathbf{b}$, $\mathbf{a}^T\mathbf{b}^T$ have no meaning. The expression $\mathbf{b}\mathbf{a}^T$ has a meaning as a matrix. However, $\mathbf{b}\mathbf{a}^T \neq \mathbf{a}\mathbf{b}^T$ in general.

Example 8. For the vectors $\mathbf{a}$ and $\mathbf{b}$ of Example 7

$$\mathbf{b}\mathbf{a}^T = \begin{pmatrix} 2 & 8 & -4 & 6 \\ -3 & -12 & 6 & -9 \\ 0 & 0 & 0 & 0 \\ 6 & 24 & -12 & 18 \end{pmatrix}.$$

Note that the matrix in Example 8 is the transpose of the matrix in Example 7. ◊

5.2.2 Some properties of transposes

Proposition 1. The transpose of a transpose is the original matrix, i.e., $(A^T)^T = A$.

Proof. A transpose is obtained by changing the order of the subscripts on each entry, and hence

$$[(A^T)^T]_{ij} = [A^T]_{ji} = [A]_{ij}.$$

Thus all matrix entries of $(A^T)^T$ are equal to corresponding entries of A, and hence the matrices are equal. □

Example 9.

$$\left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T\right)^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

$\diamondsuit$

Proposition 2. If $A, B \in M_{mn}$ and $\lambda, \mu \in \mathbb{R}$, then $(\lambda A + \mu B)^T = \lambda A^T + \mu B^T$.

Proof. For $A, B \in M_{mn}$ and $\lambda, \mu \in \mathbb{R}$, the matrix $(\lambda A + \mu B)^T$ has matrix entries

$$\begin{aligned} [(\lambda A + \mu B)^T]_{ij} &= [\lambda A + \mu B]_{ji} \\ &= [\lambda A]_{ji} + [\mu B]_{ji} \\ &= \lambda a_{ji} + \mu b_{ji} \\ &= \lambda [A^T]_{ij} + \mu [B^T]_{ij} \\ &= [\lambda A^T + \mu B^T]_{ij}. \end{aligned}$$

Thus, as corresponding matrix entries are equal, we have $(\lambda A + \mu B)^T = \lambda A^T + \mu B^T$, and the result is proved. $\square$

Proposition 3. If AB exists, then $(AB)^T = B^T A^T$.

That is, if a product exists, then the transpose of a product is the product of the transposes, but with the order of multiplication **reversed**.

Proof. Since AB exists, the number of columns of A must equal the number of rows of B. Suppose $A \in M_{mn}$ and $B \in M_{np}$. Then the matrix entries of $(AB)^T$ are

$$[(AB)^T]_{ij} = [AB]_{ji} = \sum_{k=1}^{n} [A]_{jk}[B]_{ki}.$$

Now, by definition of transpose, $B^T \in M_{pn}$ and $A^T \in M_{nm}$. Thus the product $B^T A^T$ exists. The matrix entries of this product are

$$[B^T A^T]_{ij} = \sum_{k=1}^{n} [B^T]_{ik}[A^T]_{kj} = \sum_{k=1}^{n} [B]_{ki}[A]_{jk} = \sum_{k=1}^{n} [A]_{jk}[B]_{ki} = [(AB)^T]_{ij}.$$

The result is proved. $\square$

Example 10. Let

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -3 & 6 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -2 & 5 & -1 \\ 4 & -1 & 5 \\ 1 & 3 & 2 \end{pmatrix}.$$

Then

$$(AB)^T = \begin{pmatrix} 5 & 0 & 7 \\ -12 & 36 & -6 \end{pmatrix}^T = \begin{pmatrix} 5 & -12 \\ 0 & 36 \\ 7 & -6 \end{pmatrix}$$

and

$$B^TA^T = \begin{pmatrix} -2 & 4 & 1 \\ 5 & -1 & 3 \\ -1 & 5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & -3 \\ -1 & 6 \end{pmatrix} = \begin{pmatrix} 5 & -12 \\ 0 & 36 \\ 7 & -6 \end{pmatrix}.$$

$\Diamond$

Example 11. For column vectors, $\mathbf{a}, \mathbf{b}$, we have from Propositions 1 and 3 that

$$(\mathbf{a}\mathbf{b}^T)^T = (\mathbf{b}^T)^T\mathbf{a}^T = \mathbf{b}\mathbf{a}^T,$$

as noted for the special case of the vectors in Examples 7 and 8. $\Diamond$

Symmetric matrices are especially important in certain applications of matrices to geometry and physics,

> **Definition 2.** *A matrix is said to be a* ***symmetric*** *matrix if* $A = A^T$.

Note that the entries of a symmetric matrix A satisfy $[A]_{ij} = [A^T]_{ji} = [A]_{ji}$ for all i, j, and hence a symmetric matrix must be square.

Example 12. The matrix

$$A = \begin{pmatrix} 1 & 3 & -4 & 6 \\ 3 & 7 & 10 & -11 \\ -4 & 10 & -2 & 5 \\ 6 & -11 & 5 & 4 \end{pmatrix}$$

is a symmetric matrix. $\Diamond$

Example 13. Let A and B be symmetric matrices. Prove that $A + B$ is symmetric but AB may not.

Proof. Since A and B are symmetric, we have $A^T = A$ and $B^T = B$. By Proposition 2,

$$(A + B)^T = A^T + B^T = A + B.$$

Hence $A + B$ is symmetric.

Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$. Both A and B are symmetric. However,

$$AB = \begin{pmatrix} 4 & 4 \\ 1 & 2 \end{pmatrix}$$

is not symmetric. $\square$

5.3 The inverse of a matrix

In Section 5.1, we defined the operations of addition, subtraction and multiplication for matrices. We also emphasised that **division is not defined for matrices**. The closest we can come to defining 'division' for matrices is to define the **inverse** A^{-1} of a matrix A. Not all matrices, however, have inverses.

Warning. If x and y are numbers, then $x^{-1} = \dfrac{1}{x}$ means 1 divided by x, and $\dfrac{x}{y}$ means x divided by y. However, for a matrix A, we **cannot** use $\dfrac{I}{A}$ to represent A^{-1}. For matrices A and B, writing $\dfrac{A}{B}$ would be ambiguous. The reason is that matrix multiplication is not commutative, and so it is not clear whether $\dfrac{A}{B}$ is supposed to mean AB^{-1} or $B^{-1}A$.

Definition 1. *A matrix X is said to be an* ***inverse*** *of a matrix A if both*

$$AX = I \qquad \text{and} \qquad XA = I,$$

where I is an identity (or unit) matrix of the appropriate size.

If a matrix A has an inverse, then A is said to be an **invertible** matrix. An invertible matrix is also called a **non-singular** matrix. If a matrix A is not an invertible matrix, then it is called a **singular** matrix.

Proposition 1. All invertible matrices are square, that is, have the same number of rows as columns.

[**X**] *Proof.* Suppose that the matrix A has more columns than rows. Then the equation $A\mathbf{x} = \mathbf{0}$ has some non-zero solution $\mathbf{x}$. Left multiplication by the hypothetical inverse X yields $\mathbf{x} = I\mathbf{x} = (XA)\mathbf{x} = X\mathbf{0} = \mathbf{0}$, which is a contradiction. So no matrix with more columns than rows has an inverse. If the matrix A has more rows than columns and has an inverse X then X has more columns than rows and has an inverse A. As before this leads to a contradiction. $\square$

Example 1. The matrix

$$X = \begin{pmatrix} 1 & -\frac{1}{3} & -\frac{2}{3} \\ 1 & \frac{2}{3} & -\frac{2}{3} \\ 2 & \frac{2}{3} & -\frac{5}{3} \end{pmatrix}$$

is an inverse of the matrix

$$A = \begin{pmatrix} 2 & 3 & -2 \\ -1 & 1 & 0 \\ 2 & 4 & -3 \end{pmatrix},$$

since (check calculations)

$$AX = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = XA.$$

$\Diamond$

Definition 2. *A matrix X is said to be a* **right inverse** *of A if A is $r \times c$, X is $c \times r$ and $AX = I_r$. A matrix Y is said to be a* **left inverse** *of A if A is $r \times c$, Y is $c \times r$ and $YA = I_c$.*

NOTE.

1. The conditions in the definition of an invertible matrix are very restrictive, so many square matrices are not invertible.

2. However, invertible matrices are very important as we shall begin to see below.

3. If A has left inverse then $c \leqslant r$. If A has a right inverse then $r \leqslant c$. (See proof of Proposition 1).

5.3.1 Some useful properties of inverses

Theorem 2. *If the matrix A has both a left inverse Y and a right inverse X, then $Y = X$. In particular, if both Y and X are inverses of A, then $Y = X$.*

That is, if A has a left and a right inverse, then A is both invertible and square.

Proof. $Y = YI_r = Y(AX) = (YA)X = I_cX = X$.

Thus, $AX = I = XA$ and so A is invertible. Hence A is a square matrix by Proposition 1. ☐

Notation: Because the inverse of a matrix A (if one exists) is unique we can denote it by a special symbol: A^{-1} (read 'A inverse').

Proposition 3. If A is an invertible matrix, then A^{-1} is also an invertible matrix and the inverse of A^{-1} is A. That is, $(A^{-1})^{-1} = A$.

Proof. If $X = A^{-1}$ exists, then $AX = I$ and $XA = I$. But these two conditions also imply that A is the inverse of X, and hence X is invertible and $X^{-1} = A$. Replacing X by A^{-1} then gives the result $(A^{-1})^{-1} = A$. ☐

Proposition 4. If A and B are invertible matrices and the product AB exists, then AB is also an invertible matrix and $(AB)^{-1} = B^{-1}A^{-1}$.

Note the reversal of the order in the product of the inverses.

Proof. On multiplying AB from the right by $B^{-1}A^{-1}$, we have

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

Thus the matrix $X = B^{-1}A^{-1}$ satisfies $(AB)X = I$.

Similarly, on multiplying AB from the left by $B^{-1}A^{-1}$, we have

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

Thus $X = B^{-1}A^{-1}$ also satisfies $X(AB) = I$. Thus, this X is the inverse of AB, and we have $B^{-1}A^{-1} = \mathrm{X} = (AB)^{-1}$. ☐

This proposition can easily be extended to products with 3 or more factors to obtain, for example,

$$(ABC)^{-1} = C^{-1}(AB)^{-1} = C^{-1}B^{-1}A^{-1}.$$

These results are sometimes useful in simplifying complicated expressions.

Example 2. Assuming F, G, and H are invertible and all products exist, simplify

$$A = HG(FHG)^{-1}FG.$$

$\diamondsuit$

SOLUTION. Replacing $(FHG)^{-1}$ by $G^{-1}H^{-1}F^{-1}$, we have

$$A = HGG^{-1}H^{-1}F^{-1}FG.$$

Then using $GG^{-1} = I$ and $F^{-1}F = I$, we have

$$A = HIH^{-1}IG = HH^{-1}G = IG = G.$$

$\diamondsuit$

5.3.2 Calculating the inverse of a matrix

Suppose A is a 2×2 invertible matrix. Write the columns of A^{-1} as $\mathbf{x}_1$, $\mathbf{x}_2$ then $AA^{-1} = I$ can be written as

$$AA^{-1} = A(\mathbf{x}_1|\mathbf{x}_2) = (\mathbf{e}_1|\mathbf{e}_2),$$

where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the standard basis for $\mathbb{R}^2$. Then $\mathbf{x}_1$ is the solution of $A\mathbf{x} = \mathbf{e}_1$ and $\mathbf{x}_2$ is the solution of $A\mathbf{x} = \mathbf{e}_2$. We could find $\mathbf{x}_1$ by reducing $(A|\mathbf{e}_1)$ to the reduced row-echelon form and similarly for $\mathbf{x}_2$. This can be done simultaneously as shown in the following example.

Example 3. Find the inverse of $A = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$.

SOLUTION.

$$(A|\,\mathbf{e}_1\;\mathbf{e}_2) = \left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \xrightarrow{R_2 = R_2 - \frac{3}{2}R_1} \left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right).$$

If we further reduce the matrix to reduced row-echelon form by

$$\begin{aligned} \left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right) &\xrightarrow{R_2 = -2R_2} \left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 1 & 3 & -2 \end{array}\right) \\ &\xrightarrow{R_1 = R_1 - 3R_2} \left(\begin{array}{cc|cc} 2 & 0 & -8 & 6 \\ 0 & 1 & 3 & -2 \end{array}\right) \\ &\xrightarrow{R_1 = \frac{1}{2}R_1} \left(\begin{array}{cc|cc} 1 & 0 & -4 & 3 \\ 0 & 1 & 3 & -2 \end{array}\right). \end{aligned}$$

Hence $\mathbf{x}_1 = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $A^{-1} = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix}$. $\Diamond$

In general, if A is an invertible $n \times n$ matrix and the columns of the unique inverse, A^{-1}, are $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, then $\mathbf{x}_i$ is the unique solution of $A\mathbf{x} = \mathbf{e}_i$, where $1 \leqslant i \leqslant n$. Here $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is the standard basis for $\mathbb{R}^n$. Conversely if any equation $A\mathbf{x} = \mathbf{e}_i$ does not have a unique solution, then A is not invertible.

This suggests a method of finding the inverse of a (invertible) matrix A:

1. Form the augmented matrix $(A \mid I)$ with n rows and $2n$ columns.

2. Use Gaussian-elimination to convert $(A \mid I)$ to row-echelon form $(U \mid C)$. Then, if all entries in the bottom row of U are zero, stop. In such circumstances A has no inverse.

3. Otherwise, use further row operations to reduce $(U|C)$ to reduced row-echelon form $(I \mid B)$. The right hand half of this reduced row-echelon form is the inverse.

The next proposition sums up the implications of these observations.

Proposition 5. A matrix A is invertible if and only if it can be reduced by elementary row operations to an identity matrix I and if $(A \mid I)$ can be reduced to $(I \mid B)$ then $B = A^{-1}$.

Example 4. Determine if the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

is invertible and, if it is invertible, find its inverse.

SOLUTION. We solve $AX = I$ by forming an augmented matrix with the 3 columns of I on the right, as in

$$(A \mid I) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right).$$

As usual in solving equations, we first use Gaussian elimination to reduce the augmented matrix to row-echelon form. We obtain

$$(U \mid Y) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -4 & -2 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 \end{array}\right).$$

Because the bottom row of U contains a non-zero entry, A is invertible.

To obtain the inverse $X = A^{-1}$, we use further row operations to get U into reduced row-echelon form and find

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{5}{6} & -\frac{1}{6} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right).$$

We now have an identity matrix on the left, so the solution X of the equations $AX = I$ can be read off immediately as the matrix on the right of the $|$. The solution is

$$A^{-1} = X = \begin{pmatrix} -\frac{1}{2} & \frac{5}{6} & -\frac{1}{6} \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

As a check, multiplication of A and the A^{-1} obtained should give I. ◊

Example 5. Determine if

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

is invertible and, if it is invertible, find its inverse.

Solution. The augmented matrix $[A \mid I]$ is

$$(A \mid I) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right).$$

On reducing $[A \mid I]$ to row-echelon form by Gaussian elimination, we have

$$(U \mid Y) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right).$$

Here, all entries in the bottom row of U are zero, so A is not invertible. ◊

5.3.3 Inverse of a 2×2 matrix

There is a simple formula of the inverse of an invertible 2×2 matrix.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \qquad \text{provided } ad - bc \neq 0.$$

We can easily prove this by checking that the product $\dfrac{1}{ad - bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is I. This formula is useful and should be committed to memory.

Example 6. Find the inverse of $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$ by the above formula.

Solution. $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}^{-1} = \dfrac{1}{2 \times 4 - 3 \times 3} \begin{pmatrix} 4 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix}.$ ◊

[X] 5.3.4 Elementary row operations and matrix multiplication

In this subsection we shall see that the three elementary row operations on a matrix A in Chapter 4 can be interpreted as multiplication of A by "elementary matrices". As applications, we can use this to prove that the algorithm to calculate the inverse of a matrix works and prove that the left inverse of a **square** matrix is two-sided in the next subsection.

Example 7. 1. If $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then $E_1A = \begin{pmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{pmatrix}$ (check this), that is, the second row has been multiplied by 3.

2. If $E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$ then $E_2C = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ m & n & o & p \\ i & j & k & l \end{pmatrix}$, that is, the 3rd and 4th rows have been swapped.

3. If $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix}$ then $E_3A = \begin{pmatrix} a & b & c \\ d & e & f \\ g-5a & h-5b & i-5c \end{pmatrix}$, that is, row three is replaced by row three minus five times row one. $\diamondsuit$

Proposition 6. The three elementary row operations can be effected by left multiplication by matrices. These matrices are all invertible.

Proof. Let
$$E_1 = \begin{array}{c} \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \\ \begin{matrix} \uparrow \\ i \end{matrix} \end{array} \qquad E_2 = \begin{array}{c} \begin{pmatrix} 1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & 1 & & & & & & & \\ & & & 0 & & & & 1 & & \\ & & & & 1 & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & 1 & & & \\ & & & 1 & & & & 0 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 1 \end{pmatrix} \\ \begin{matrix} \uparrow & \qquad\qquad & \uparrow \\ i & & j \end{matrix} \end{array}$$

$$E_3 = \begin{array}{c} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & \lambda & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \\ \begin{matrix} \uparrow & \qquad & \uparrow \\ i & & j \end{matrix} \end{array}.$$ The entries in the blank space are zeroes.

As demonstrated by Example 7 it is easy to see that, when multiplied on the left of a matrix,

- E_1 multiplies row i by λ
- E_2 interchanges row i and row j
- E_3 adds λ times row i to row j

If is also clear that

$$E_1^{-1} = \begin{pmatrix} 1 &&&& \\ & \ddots &&& \\ && \lambda^{-1} && \\ &&& \ddots & \\ &&&& 1 \end{pmatrix}, \ E_2^{-1} = E_2, \ E_3^{-1} = \begin{array}{c} \begin{pmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 &&&& \\ &&& \ddots &&& \\ && -\lambda && 1 && \\ &&&&& \ddots & \\ &&&&&& 1 \end{pmatrix} \\ \begin{matrix} \uparrow & \qquad & \uparrow \\ i & & j \end{matrix} \end{array}.$$

Indeed E_1^{-1}, E_2^{-1} and E_3^{-1} "undo" the operations done by E_1, E_2 and E_3. □

Theorem 7. *If an augmented matrix $(A \mid I)$ is reduced to a matrix of the form $(I \mid X)$ then X is a left inverse of A.*

Proof. On the reduction of $(A \mid I) \to (I \mid X)$, each step is an elementary row operation. By Proposition 6 each row operation is also left multiplication by a matrix of type E_1, E_2 or E_3. So there are a sequence of matrices $S_1, \cdots, S_k$, each of which is of type E_1, E_2 or E_3 such that

$$S_k\, S_{k-1} \cdots S_1\, A = I.$$

However the same operations are performed on I yielding X, so we have

$$S_k\, S_{k-1} \cdots S_1\, I = X$$

hence $XA = I$, that is X is a left inverse of A. □

Theorem 8. *The left inverse of A in Theorem 7 is also a right inverse of A.*

Proof. Continuing from the proof of Theorem 7, $X = S_k\, S_{k-1} \cdots S_1$ is a left inverse of A. By Proposition 6 $S_1, \dots, S_k$ are invertible with inverses $S^{-1}, \dots, S_k^{-1}$. Let $B = S_1^{-1}, S_2^{-1}, \dots, S_k^{-1}$. Now

$$XB = S_k\, S_{k-1} \dots S_2\, S_1\, S_1^{-1}\, S_2^{-1} \dots S_k^{-1} = I$$

and $BX = S_1^{-1}\, S_2^{-1} \dots S_k^{-1}\, S_k\, S_{k-1} \dots S_2\, S_1 = I$. But $XA = I$ also, hence

$$B = BI = B(XA) = (BX)A = IA = A$$

thus B is both the left and right inverse of X as is A! So, X actually is A^{-1}. □

Corollary 9. *The algorithm described in 5.3.2 does work. That is, the matrix calculated by the algorithm is the inverse of the given matrix.*

5.3.5 Inverses and solution of $A\mathbf{x} = \mathbf{b}$

Proposition 10. If A is a square matrix, then A is invertible if and only if $A\mathbf{x} = 0$ implies $\mathbf{x} = 0$.

Proof. Suppose A is invertible and $A\mathbf{x} = 0$ then $A^{-1}A\mathbf{x} = 0,\ I\mathbf{x} = 0,\ \mathbf{x} = 0.$

Conversely suppose $A\mathbf{x} = 0$ implies $\mathbf{x} = 0$ then the reduced echelon form of A is the identity matrix and, as discussed in 5.3.4, A^{-1} exists. ☐

Proposition 11. Let A be an $n \times n$ square matrix. Then A is invertible if and only if the matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for all vectors $\mathbf{b} \in \mathbb{R}^n$. In this case, the unique solution is $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof. Suppose A is invertible and $A\mathbf{x} = \mathbf{b}$ then $A^{-1}A\mathbf{x} = A^{-1}\mathbf{b},\ I\mathbf{x} = A^{-1}\mathbf{b},\ \mathbf{x} = A^{-1}\mathbf{b}.$

Conversely suppose $A\mathbf{x} = \mathbf{b}$ has a unique solution for all vectors $\mathbf{b}$. Then, in particular, $A\mathbf{x} = 0$ has a unique solution, and this is clearly $\mathbf{x} = 0$. Hence, by Proposition 10, A is invertible. ☐

NOTE. In principle, this proposition can be used to obtain a solution of $A\mathbf{x} = \mathbf{b}$ by finding an inverse A^{-1} and then forming $\mathbf{x} = A^{-1}\mathbf{b}$. However, there are several serious practical problems which occur if the proposition is used in this way. Some of these problems are as follows.

1. To obtain the inverse of an $n \times n$ matrix A it is necessary to solve n equations of the form $A\mathbf{x}_j = \mathbf{e}_j$. Even with the use of the most efficient available method (the LU-factorisation method), the calculation of an inverse takes longer than the solution of the original equation $A\mathbf{x} = \mathbf{b}$.

2. It is very easy to forget that many matrices do not have inverses. It is then very easy to make the mistake of saying that the 'solution' of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b}$ when, in fact, the equations might actually have no solution or might have an infinite number of solutions.

3. The Gaussian elimination and back-substitution method of solving equations always works, whereas the $\mathbf{x} = A^{-1}\mathbf{b}$ formula only works in the special case that $A\mathbf{x} = \mathbf{b}$ has a unique solution.

4. In large-scale numerical calculations, use of the inverse may produce more 'rounding error'.

However, the propositions can be used to prove some important results.

Corollary 12. *Let A be a square matrix.*

1. *If X is a left inverse of A, then X is the (two-sided) inverse of A. That is,*

$$\text{if } XA = I, \text{ then } X = A^{-1}.$$

2. *If X is a right inverse of A, then X is the (two-sided) inverse of A.*

Proof. For part 1, assume $A\mathbf{x} = \mathbf{0}$. Since X is a left inverse of A,

$$\mathbf{x} = I\mathbf{x} = (XA)\mathbf{x} = X(A\mathbf{x}) = X\mathbf{0} = \mathbf{0}.$$

Then by Proposition 10, A is invertible. Hence

$$XA = I \quad \Rightarrow \quad XAA^{-1} = IA^{-1} \quad \Rightarrow \quad X = A^{-1}.$$

We can use similar arguments to prove part 2. ☐

5.4 Determinants

In this section we shall define the determinant function. If A is a square matrix, then the determinant of A, written $\det(A)$, is a number. Determinants arise in many areas of mathematics. As we shall see in Chapter 5, they can be used to find the volume of a parallelepiped. Determinants are very important in the theory of eigenvalues and eigenvectors, covered in Chapter 8. In later years you will need to calculate determinants to perform changes of variables in multivariable calculus.

From a theoretical point of view, the determinant is also important in determining whether a system of n linear equations in n unknowns has a solution for every right-hand-side vector, or equivalently, whether an $n \times n$ matrix A is invertible. As we shall see, A is invertible if and only if $\det(A) \neq 0$. Unfortunately, calculating $\det(A)$, even by the most efficient methods, takes the same length of time as finding the row-echelon form for A, so determinants are not used for numerical calculations of this kind. On the other hand, many general statements can be proved by using the properties of the determinant function.

Determinants are defined only for square matrices. We shall give definitions for arbitrary-size $n \times n$ matrices, but most of our examples will be restricted to determinants of 2×2 and 3×3 matrices. The case of 1×1 matrices is included for completeness. An alternative notation for $\det(A)$ is $|A|$.

5.4.1 The definition of a determinant

The determinant of a 1×1 matrix A is defined to be its sole entry. As the notation $|A|$ tends to cause confusion with the absolute value, it should be used with caution for 1×1 matrices. Thus $|(-1)| = -1$ whilst $|-1| = 1$.

There are several ways of defining determinants, each of which has advantages and disadvantages. We use a recursive definition which begins with the definition of a 2×2 determinant. Here 'recursive' indicates that the definition of a determinant for square matrices of a given size invokes that for matrices of a smaller size.

Definition 1. *The **determinant** of a* 2×2 *matrix*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \textit{is} \qquad \det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

Example 1. If $A = \begin{pmatrix} 2 & -4 \\ -3 & 4 \end{pmatrix}$ then $|A| = \det(A) = 2(4) - (-4)(-3) = -4$. $\diamondsuit$

The definition of the determinant of a general $n \times n$ matrix can be built up recursively from the definition of the determinant of a 2×2 matrix.

We first define the concept of a minor of an entry of a matrix.

Definition 2. *For a matrix A, the (row i, column j) **minor** is the determinant of the matrix obtained from A by deleting row i and column j from A.*

Notation. We shall use the symbol $|A_{ij}|$ to represent the (row i, column j) minor in a matrix A.

Example 2. The (row 2, column 3) minor of the (row 2, column 3) entry in $A = \begin{pmatrix} 2 & -3 & 4 \\ 8 & -1 & -4 \\ 6 & 5 & -7 \end{pmatrix}$ is, on deleting row 2 and column 3, $|A_{23}| = \begin{vmatrix} 2 & -3 \\ 6 & 5 \end{vmatrix} = 28.$ ♢

Definition 3. *The* ***determinant*** *of an $n \times n$ matrix A is*

$$\begin{aligned} |A| &= a_{11}|A_{11}| - a_{12}|A_{12}| + a_{13}|A_{13}| - a_{14}|A_{14}| + \cdots + (-1)^{1+n}a_{1n}|A_{1n}| \\ &= \sum_{k=1}^{n}(-1)^{1+k}a_{1k}|A_{1k}|. \end{aligned}$$

Note that each term in the definition of the determinant $|A|$ is a product of an entry in the first row of A with its corresponding minor. The signs of the terms alternate $+, -, +, -, \ldots$ starting with a $+$ on the (row 1, column 1) entry. The formula is called "expanding along the first row of the determinant."

Example 3. Evaluate the determinant

$$\det(A) = \begin{vmatrix} 5 & 1 & 7 \\ -2 & 3 & -4 \\ 6 & -1 & 2 \end{vmatrix}.$$

Solution.

$$\det(A) = 5\begin{vmatrix} 3 & -4 \\ -1 & 2 \end{vmatrix} - 1\begin{vmatrix} -2 & -4 \\ 6 & 2 \end{vmatrix} + 7\begin{vmatrix} -2 & 3 \\ 6 & -1 \end{vmatrix} = 5(6-4) - (-4+24) + 7(2-18) = -122.$$

♢

5.4.2 Properties of determinants

In this section we present some of the basic properties of determinants. Where it is possible to do so in an efficient manner by elementary methods, we give proofs of the properties for the general $n \times n$ case. For properties where the general proof requires more advanced methods, we give proofs for the 2×2 or 3×3 cases instead.

Proposition 1. $\det(A^T) = \det(A)$

Proof. For the 3×3 case. If

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \text{then} \qquad A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

By direct evaluation of $\det(A)$ and $\det(A^T)$ from Definition 3, we obtain

$$\det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

and

$$\det(A^T) = a_{11}\begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{32} \\ a_{13} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{vmatrix}$$
$$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13}$$

These two expressions are equal, and hence $\det(A) = \det(A^T)$. □

One immediate consequence of Proposition 1 is that a determinant can also be evaluated by 'expansion down the first column', that is

$$|A| = a_{11}|A_{11}| - a_{21}|A_{21}| + a_{31}|A_{31}| - a_{41}|A_{41}| + \cdots + (-1)^{n+1}a_{n1}|A_{n1}| = \sum_{k=1}^{n}(-1)^{k+1}a_{k1}|A_{k1}|.$$

Example 4. Use expansion along the first column to evaluate the determinant of Example 3.

SOLUTION. Expanding along the first column, we have

$$\det(A) = 5\begin{vmatrix} 3 & -4 \\ -1 & 2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 7 \\ -1 & 2 \end{vmatrix} + 6\begin{vmatrix} 1 & 7 \\ 3 & -4 \end{vmatrix} = 5(6-4) + 2(2+7) + 6(-4-21) = -122.$$

◊

Proposition 2. If any two rows (or any two columns) of A are interchanged, then the sign of the determinant is reversed. More precisely if the matrix B is obtained from the matrix A by interchanging two rows (or columns), then $\det B = -\det A$.

Proof. For 2×2 case. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then, on interchanging columns 1 and 2 of A,

$$\det\begin{pmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{pmatrix} = a_{12}a_{21} - a_{11}a_{22} = -\det(A).$$

□

One important application of this proposition is that a determinant can be evaluated by expanding along any row or any column, with the signs chosen from the following array.

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If we evaluate the determinant by expanding the ith row or the jth column, where $1 \leqslant i,\ j \leqslant n$, we have

$$|A| = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} |A_{ik}| = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} |A_{kj}|.$$

Example 5. Use expansion along the second row to evaluate $\det(A)$ of the previous two examples.

SOLUTION. To evaluate $\det(A)$ by expanding along the second row, we choose the signs from the second row of $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$. So

$$\det(A) = -(-2) \begin{vmatrix} 1 & 7 \\ -1 & 2 \end{vmatrix} + 3 \begin{vmatrix} 5 & 7 \\ 6 & 2 \end{vmatrix} - (-4) \begin{vmatrix} 5 & 1 \\ 6 & -1 \end{vmatrix} = -122.$$

◊

Obviously, we should evaluate a determinant by expanding along that row or column containing the greatest number of $0's$. We will soon see an even better method.

A second application of Proposition 2 is as follows.

Proposition 3. If a matrix contains a zero row or column then its determinant is zero.

Proof. Clearly, if we evaluate the determinant by expansion along the zero row or column, the value obtained will be zero. □

Most of the remaining important properties of determinants centre around the question of what happens to the value of the determinant when one column (or row) of a matrix is changed, with all other columns (or rows) remaining unchanged.

Proposition 4. If a row (or column) of A is multiplied by a scalar, then the value of $\det A$ is multiplied by the same scalar. That is, if the matrix B is obtained from the matrix A by multiplying a row (or column) of A by the scalar λ, then $\det B = \lambda \det A$.

Proof. On multiplying the first row of

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

by a scalar λ, we obtain

$$B = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Then, on expanding along the first row, we have

$$\det(B) = \sum_{k=1}^{n} (-1)^{1+k} \lambda a_{1k} \, |A_{1k}| = \lambda \sum_{k=1}^{n} (-1)^{1+k} a_{1k} \, |A_{1k}| = \lambda \det(A),$$

and hence the result is proved for a multiple of the first row. Then, from Propositions 1 and 2, the result is also true for a scalar multiple of any row or column. □

An immediate consequence of Propositions 2 and 4 is the following useful result.

Proposition 5. If any column of a matrix is a multiple of another column of the matrix (or any row is a multiple of another row), then the value of $\det(A)$ is zero.

Proof. The result is clearly true if the scalar multiple is zero.

Now, assume that the ith column of a matrix A is λ times the jth column. Then, if we multiply the ith column by λ^{-1} we obtain a matrix B, whose determinant, from Proposition 4, has the value $\det(B) = \lambda^{-1}\det(A)$.

This matrix B has its ith and jth columns equal. On interchanging columns i and j, we have, from Proposition 2, that the determinant of the new matrix has the value $-\det(B)$. But, interchanging equal columns does not change the matrix, so the new matrix is still B with determinant $\det(B)$. Hence, $\det(B) = -\det(B)$, and thus $\det(B) = 0$ and $\det(A) = 0$.

The proof of the proposition for rows follows immediately from Proposition 1. $\square$

Example 6.

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & -5 & 7 \\ -3 & -6 & -9 \end{vmatrix} = 0$$

as row 3 is a scalar multiple of row 1. This result can easily be checked directly by using expansion along the second row to evaluate the determinant. $\Diamond$

Propositions 2 and 4 show the effect of the two elementary row operations of interchanging two rows and of multiplying a row by a scalar. The effect of the third elementary row operation of adding a multiple of one row to another row is given in the following proposition.

Proposition 6. If a multiple of one row (or column) is added to another row (or column), then the value of the determinant is not changed.

[**X**] *Proof.* On adding λ times row i to row 1 of

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \qquad \text{we obtain} \qquad B = \begin{pmatrix} a_{11} + \lambda\, a_{i1} & a_{12} + \lambda\, a_{i2} & \cdots & a_{1n} + \lambda\, a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Then, on expanding along the first row, we have

$$\begin{aligned} \det(B) &= \sum_{k=1}^{n} (-1)^{1+k}\,(a_{1k} + \lambda\, a_{ik})\,|A_{1k}| \\ &= \sum_{k=1}^{n} (-1)^{1+k} a_{1k}\,|A_{1k}| + \lambda \sum_{k=1}^{n} (-1)^{1+k} a_{ik}\,|A_{1k}| \\ &= \det(A) + \lambda \sum_{k=1}^{n} (-1)^{1+k} a_{ik}\,|A_{1k}|\,. \end{aligned}$$

The last sum is the determinant of the matrix which is obtained from A by replacing the first row of A by the ith row of A. Thus, the first and ith rows of this matrix are the same, and hence, from Proposition 5, its determinant is zero. Thus the second sum in the above equation is zero, and hence $\det(B) = \det(A)$. We have therefore proved the result for subtraction of scalar multiples of any row from the first row. Then, from Propositions 1 and 2, the result is true for any two rows or any two columns. $\square$

The final useful proposition that we shall give here is as follows.

Proposition 7. If A and B are square matrices such that the product AB exists, then

$$\det(AB) = \det(A)\det(B).$$

All known proofs of this result are non–trivial, so we shall not give a proof here.

Example 7. For

$$A = \begin{pmatrix} 2 & -4 \\ -3 & 5 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} -1 & -8 \\ 2 & 9 \end{pmatrix},$$

we have

$$|A| = -2, \qquad |B| = 7, \qquad |AB| = \begin{vmatrix} -10 & -52 \\ 13 & 69 \end{vmatrix} = -14 = |A||B|.$$

$\diamondsuit$

5.4.3 The efficient numerical evaluation of determinants

With the exception of 2×2 and 3×3 matrices, the direct evaluation of a determinant from Definition 3 is a lengthy procedure. For example, the value of an $n \times n$ determinant is given in terms of n minors of size $(n-1) \times (n-1)$, each of which is given in terms of $n-1$ minors of size $(n-2) \times (n-2)$, etc. The evaluation ending with the evaluation of 2×2 minors. In fact, the total number of terms to be evaluated is $n!$. Even for a small number such as $n = 6$, this is already $6! = 720$ terms.

With the exception of the 2×2 and 3×3 cases, by far the most efficient method of evaluating determinants of matrices with numerical entries is to use Gaussian elimination to reduce the matrix to row-echelon form. The basic results required for this efficient method of evaluation are given in the following two propositions.

Proposition 8. If U is a square row-echelon matrix, then $\det(U)$ is equal to the product of the diagonal entries of U.

Proof. Let

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix}$$

be an $n \times n$ matrix in row-echelon form. Then the proposition states that

$$\det(U) = u_{11}u_{22}\cdots u_{nn}.$$

We shall prove this result by induction. We first introduce some notation.

Let $U(j)$ be the submatrix of U which is defined by

$$U(j) = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1j} \\ 0 & u_{22} & \cdots & u_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{jj} \end{pmatrix}.$$

Note that $U(j)$ is a square row-echelon form matrix for $1 \leqslant j \leqslant n$, with $U(1) = [u_{11}]$, $\det(U(1)) = u_{11}$ and $U(n) = U$.

We shall now prove that if the proposition is true for $\det\big(U(j)\big)$, then it is also true for $\det\big(U(j+1)\big)$. Now,

$$\det\big(U(j+1)\big) = \begin{vmatrix} u_{11} & u_{12} & \cdots & u_{1,j+1} \\ 0 & u_{22} & \cdots & u_{2,j+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{j+1,j+1} \end{vmatrix}.$$

We can evaluate $\det\big(U(j+1)\big)$ by expanding along the last row, and we obtain

$$\det\big(U(j+1)\big) = (-1)^{2j} u_{j+1,j+1} \det\big(U(j)\big) = u_{j+1,j+1} \det\big(U(j)\big).$$

If we now assume that the proposition is true for $\det\big(U(j)\big)$, we can replace $\det\big(U(j)\big)$ by the product of its diagonal entries to obtain

$$\det\big(U(j+1)\big) = u_{11} \cdots u_{jj} u_{j+1,j+1}.$$

Hence, if the result is true for $\det\big(U(j)\big)$ it is also true for $\det\big(U(j+1)\big)$.

However, the result is true for $j = 1$, since in this case we have $\det\big(U(1)\big) = u_{11}$. It is therefore true for $j = 2, 3, \ldots$, and the proposition is proven for all $n \geqslant 1$ by induction. $\square$

Example 8.

$$\begin{vmatrix} 3 & 2 & 9 & 3 & 12 \\ 0 & -4 & -5 & 6 & 2 \\ 0 & 0 & 6 & -7 & -3 \\ 0 & 0 & 0 & 5 & 20 \\ 0 & 0 & 0 & 0 & 8 \end{vmatrix} = (3)(-4)(6)(5)(8) = -2880.$$

$\Diamond$

Proposition 9. If A is a square matrix and U is an equivalent row-echelon form obtained from A by Gaussian elimination using row interchanges and adding a multiple of one row to another, then $\det(A) = \epsilon \det(U)$, where $\epsilon = +1$ if an even number of row interchanges have been made, and $\epsilon = -1$ if an odd number of row interchanges have been made.

Proof. From Proposition 2, each row interchange reverses the sign of the determinant. Thus the sign is unchanged if $0, 2, 4, \ldots$ row interchanges have been made, whereas the sign is reversed if $1, 3, 5 \ldots$ row interchanges have been made.

From Proposition 6, the value of the determinant is unchanged if a multiple of one row is added to another row. Thus row addition operations do not change the value of the determinant. The proof is complete. $\square$

Propositions 8 and 9 show that the value of a determinant can be found by the highly efficient Gaussian elimination method.

Example 9. Evaluate the determinant of

$$A = \begin{pmatrix} 1 & -1 & 0 & 3 \\ 2 & -2 & 6 & -1 \\ 4 & -2 & 1 & 7 \\ 3 & 5 & -7 & 2 \end{pmatrix}.$$

SOLUTION. The first leading element for Gaussian elimination is already in the first row, so no row interchange is required. Hence, at this stage, $\epsilon = 1$. Adding multiples of the first row to subsequent rows does not change the value of the determinant. As is usual in Gaussian elimination, the multiple are chosen to make the entries below the leading element all zero. After partly reducing the original matrix we have:

$$|A| = \begin{vmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & 6 & -7 \\ 0 & 2 & 1 & -5 \\ 0 & 8 & -7 & -7 \end{vmatrix}.$$

The second leading element is in row 3. We therefore interchange rows 2 and 3 to bring the second pivot element to its required row 2 position. This interchange changes the sign of the determinant, and hence we now have $\epsilon = -1$. After interchanging rows 2 and 3 and then adding suitable multiples of the new row 2 from the new row 3 and row 4, we have

$$|A| = (-1)\begin{vmatrix} 1 & -1 & 0 & 3 \\ 0 & 2 & 1 & -5 \\ 0 & 0 & 6 & -7 \\ 0 & 0 & -11 & 13 \end{vmatrix}.$$

The third pivot element is in its correct row 3 position, so no row interchange is required. Hence ϵ remains at its previous value of -1. After further reducing the matrix we have

$$|A| = (-1)\begin{vmatrix} 1 & -1 & 0 & 3 \\ 0 & 2 & 1 & -5 \\ 0 & 0 & 6 & -7 \\ 0 & 0 & 0 & \frac{1}{6} \end{vmatrix} = (-1)(1 \times 2 \times 6 \times \frac{1}{6}) = -2.$$

The experienced evaluator of determinants will discard entries in these matrices as the calculation progresses. This leads to the shortened calculation:

$$\begin{vmatrix} 1 & -1 & 0 & 3 \\ 2 & -2 & 6 & -1 \\ 4 & -2 & 1 & 7 \\ 3 & 5 & -7 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & 6 & -7 \\ 0 & 2 & 1 & -5 \\ 0 & 8 & -7 & -7 \end{vmatrix} = (-1)\begin{vmatrix} 2 & 1 & -5 \\ 0 & 6 & -7 \\ 0 & -11 & 13 \end{vmatrix} = -2\begin{vmatrix} 6 & -7 \\ -11 & 13 \end{vmatrix} = -2.$$

$\diamondsuit$

Although we can reduce any matrix to a row echelon form by the two operations—row exchange and row addition, using the row operation—multiply a row by a scalar helps, especially when we do the calculations by hand. This row operation exploits the result of Proposition 4. We illustrate the technique by the following example.

[**X**] **Example 10.** Factorise $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$.

SOLUTION.

$$\begin{array}{cll}
 & \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} & \\
= & \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2-a^2 & b^3-a^3 \\ 0 & c^2-a^2 & c^3-a^3 \end{vmatrix} & \begin{array}{l} R_2 = R_2 - R_1 \\ R_3 = R_3 - R_1 \end{array} \\
= & (b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & a+b & a^2+ab+b^2 \\ 0 & a+c & a^2+ac+c^2 \end{vmatrix} & \begin{array}{l} \text{Proposition 4} \\ \text{factorise } (b-a) \text{ from } R_2 \\ \text{factorise } (c-a) \text{ from } R_3 \end{array} \\
= & (b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & a+b & a^2+ab+b^2 \\ 0 & c-b & a(c-b)+c^2-b^2 \end{vmatrix} & R_3 = R_3 - R_2 \\
= & (b-a)(c-a)(c-b)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & a+b & a^2+ab+b^2 \\ 0 & 1 & a+b+c \end{vmatrix} & \begin{array}{l} \text{Proposition 4} \\ \text{factorise } (c-b) \text{ from } R_3 \end{array} \\
= & (b-a)(c-a)(c-b)[(a+b)(a+b+c)-(a^2+ab+b^2)] & \text{expand along the first column} \\
= & (a-b)(b-c)(c-a)(ab+bc+ca) &
\end{array}$$

$\diamondsuit$

5.4.4 Determinants and solutions of $A\mathbf{x} = \mathbf{b}$

The propositions of Section 5.4.3 can be used to establish an important relation between the value of $\det(A)$ and the solution of $A\mathbf{x} = \mathbf{b}$ for a **square** matrix A. The basic result is as follows.

Proposition 10. Let A be an $n \times n$ matrix.

1. *If* $\det(A) \neq 0$, *the equation* $A\mathbf{x} = \mathbf{b}$ *has a solution and the solution is unique for all* $\mathbf{b} \in \mathbb{R}^n$.

2. *If* $\det(A) = 0$, *the equation* $A\mathbf{x} = \mathbf{b}$ *either has no solution or an infinite number of solutions for a given* $\mathbf{b}$.

Proof.
CASE 1. $\det(A) \neq 0$. From Propositions 8 and 9 , $\det(A) \neq 0$ implies that all diagonal entries of an equivalent row-echelon matrix U for A are non-zero. Then, since A, and hence U, is a square

matrix, U has no zero rows and no non-leading columns. As there are no zero rows, the equations have a solution for all $\mathbf{b}$, and, as there are no non-leading columns, each solution is unique.

CASE 2. $\det(A) = 0$. In this case, U contains at least one zero diagonal entry, so at least one zero row and at least one non-leading column. Thus there is a non-zero solution to $A\mathbf{x} = \mathbf{0}$. The result follows from Proposition 5 of Section 4.7. $\square$

NOTE. In the case when $\det(A) \neq 0$ there is an explicit formula called *Cramer's rule*, available for the solution of $A\mathbf{x} = \mathbf{b}$. Each entry x_i of $\mathbf{x}$ is specified as the quotient of two determinants.

Example 11. Describe the type of solution for each of the following systems.

$$\left\{\begin{array}{rcrcl} 2x_1 & + & 3x_2 & = & b_1 \\ 4x_1 & + & 6x_2 & = & b_2 \end{array}\right. \qquad \text{and} \qquad \left\{\begin{array}{rcrcrcl} x_1 & - & x_2 & + & 3x_3 & = & b_1 \\ 2x_1 & + & 3x_2 & + & x_3 & = & b_2 \\ 3x_1 & + & x_2 & + & 4x_3 & = & b_3 \end{array}\right. .$$

SOLUTION. The determinants of the coefficient matrices of the equations are

$$|A| = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0 \qquad \text{and} \qquad |B| = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 4 \end{vmatrix} = -5 \neq 0.$$

Hence, depending on the values of b_1 and b_2, the first system of equations either has no solution or an infinite number of solutions, whereas the second system of equations has a unique solution for all $\mathbf{b}$. $\diamondsuit$

NOTE. With the possible exception of 2×2 and 3×3 matrices, Proposition 10 does not provide a practical method of determining the number of solutions of $A\mathbf{x} = \mathbf{b}$, because the most efficient method for evaluating $\det(A)$ is first to use Gaussian elimination to reduce A to row-echelon form U and then to multiply diagonal entries of U. However, as we have seen before, the number of solutions that an equation $A\mathbf{x} = \mathbf{b}$ has can be seen directly from the row-echelon form itself, without the need for any further calculation.

Proposition 10 leads immediately to two results which are sometimes useful.

Proposition 11. For a square matrix A, the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$ has a non-zero solution if and only if $\det(A) = 0$.

Proof. $A\mathbf{x} = \mathbf{0}$ always has $\mathbf{x} = \mathbf{0}$ as a solution. From Proposition 10, if $\det(A) \neq 0$, then $\mathbf{x} = \mathbf{0}$ is the unique solution, whereas, if $\det(A) = 0$, then there are an infinite number of non-zero solutions. $\square$

Proposition 12. A square matrix A is invertible if and only if $\det(A) \neq 0$.

Proof. The result follows immediately on combining the results of Proposition 11 and of Proposition 10 of Section 5.3. $\square$

Example 12. Use determinants to check if the matrices

$$A = \begin{pmatrix} 2 & 5 \\ -4 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 3 & 4 \\ -2 & 4 & -8 \\ -3 & 1 & -12 \end{pmatrix}$$

are invertible.

Solution. As $|A| = 22 \neq 0$, A is invertible. As column 3 of B is a multiple of column 1 of B, $\det(B) = 0$. Thus B has no inverse. $\Diamond$

There is a very simple relationship between the determinant of a matrix and the determinant of the inverse of the matrix.

Proposition 13. If A is an invertible matrix, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Proof. If A^{-1} exists, then

$$\det(AA^{-1}) = \det(I) = 1.$$

But, from Proposition 7, the determinant of a product is the product of the determinants, and hence

$$1 = \det(AA^{-1}) = \det(A)\det(A^{-1}).$$

Finally, as A is invertible, $\det(A) \neq 0$, and hence $\det(A^{-1}) = \dfrac{1}{\det(A)}$. $\square$

5.5 Matrices and Maple

The Linear Algebra package becomes available on Maple only if it is 'loaded' with the command:

```
with(LinearAlgebra):
```

Maple treats matrices as individual entities, which can be added and subtracted (use `+` and `-`) and multiplied by scalars (use `*`). Matrix multiplication is done with the dot.

The matrix $A = \begin{pmatrix} 11 & 12 \\ 21 & 22 \end{pmatrix}$ is entered by the command:

```
A := < < 11,21 > | < 12,22 > >;
```

Maple ignores irrelevant blanks! To display the matrix A you enter:

```
A;
```

In Chapter 3 we showed how to enter identity matrices. You can now try:

```
B := 3*A-2*IdentityMatrix(2);
C := B.A;
```

You can transpose a given matrix A and set B as this transpose with:

```
B := Transpose(A);
```

The determinant of a *square* matrix A can be evaluated by

```
Determinant(A);
```

The inverse of a *square* matrix A can be calculated by:

```
A^(-1);
```

though you should read Subsection 5.3.2 and then try reducing an augmented matrix.

Problems for Chapter 5

Questions marked with [**R**] are routine, [**H**] harder, [**M**] Maple and [**X**] are for MATH1141 only. You should make sure that you can do the easier questions before you tackle the more difficult questions. Questions marks with a [**V**] have video solutions available on Moodle.

Problems 5.1 : Matrix arithmetic and algebra

1. [**R**] Given the matrices

$$A = \begin{pmatrix} 2 & -3 & 4 \\ 3 & 2 & -2 \\ 1 & -1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 1 \\ 3 & 4 \\ -1 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} -3 & 2 \\ 1 & -4 \\ 6 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 3 & 1 \\ 1 & -2 & -3 \end{pmatrix}.$$

Find the following matrices if they exist, or explain why they don't exist. (I stands for an identity matrix of the appropriate size).

a) $3A$, b) $-2B$, c) $A+B$, d) $B+C$, e) $A+3I$,
f) $B+3I$, g) AB, h) BA, i) BC, j) CD,
k) A^2, l) B^2, m) $(BD)^2$.

2. [**H**] Suppose A and B are matrices such that both AB and BA are defined.

a) Show that AB and BA are both square matrices.

b) If $AB = BA$, show that A and B are both square and of the same size.

c) If A and B are square matrices such that $AB = BA$, show that $(A-B)(A+B) = A^2 - B^2$.

d) Find two 2×2 matrices A, B for which $(A-B)(A+B) \neq A^2 - B^2$.

e) Prove that $(A+B)^2 = A^2 + B^2 + 2AB$ if and only if $AB = BA$.

3. [**H**] Let A and B be matrices of the same size. By considering the general entries $[A]_{ij}$, $[B]_{ij}$, $[A+B]_{ij}$ and $[B+A]_{ij}$, prove the commutative laws of addition, i.e. $A+B = B+A$.

4. [**H**] Suppose λ is a scalar and $A, B \in M_{mn}$. Prove that $\lambda(A+B) = \lambda A + \lambda B$.

5. [**H**] Let A and B be two matrices such that AB is defined. By considering the general entry in both sides of the equation, show that $A(\lambda B) = \lambda AB$ where λ is any real number.

6. [**R**] Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 \\ 2 & -2 \\ -1 & 4 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & 2 \\ 3 & -2 \\ -2 & 4 \end{pmatrix}.$$

Show that $AB = AC$ and deduce that matrices cannot in general be cancelled from products.

7. **[R][V]** Let
$$A=\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}.$$
Show that $A^2 = A + 5I$ and hence find A^6 as a linear combination of A and I.

8. **[R]** Let
$$N=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$
Find N^2 and N^3. Show that $(I+N)(I-N+N^2) = I$.

9. **[H][V]** Let A and B be $n \times n$ real matrices such that $A^2 = I,\ B^2 = I$ and $(AB)^2 = I$. Prove that $AB = BA$.

10. **[H]** Let A be a 2×2 real matrix such that $AX = XA$ for all 2×2 real matrices X. Show that $A = \alpha I$ for some $\alpha \in \mathbb{R}$.

11. **[H]** Suppose
$$A=\begin{pmatrix} 1 & -2 & 3 \\ 4 & 0 & 1 \\ 3 & 2 & -1 \end{pmatrix}, \qquad B=\begin{pmatrix} 7 & 0 & 3 \\ 2 & -1 & 6 \\ -1 & 0 & 5 \end{pmatrix}.$$
   a) Write down a column vector $\mathbf{v}$ such that $A\mathbf{v}$ is the second column of A.
   b) Write down a row vector $\mathbf{v}$ such that $\mathbf{v}B$ is the third row of B.
   c) Write down a column vector $\mathbf{v}$ such that $A\mathbf{v}$ is the second column of AB.
   d) Write down a row vector $\mathbf{v}$ such that $\mathbf{v}B$ is the first row of AB.

12. **[X]**
   a) Prove the associative law of matrix multiplication.
   b) Suppose that $A \in M_{mn}$ and I is an $m \times m$ matrix. Prove that $IA = A$.

Problems 5.2 : The transpose of a matrix

13. **[R]** Find the transposes of the following matrices:
$$A=\begin{pmatrix} 1 & -2 \\ -3 & 0 \\ 4 & 5 \end{pmatrix}, \qquad B=\begin{pmatrix} 2 & -5 & 4 & 3 \\ -4 & 6 & 5 & 5 \\ 5 & 0 & 8 & 6 \end{pmatrix}, \qquad C=\begin{pmatrix} 1 & 4 & 2 \\ 4 & -3 & 6 \\ 2 & 6 & 7 \end{pmatrix}.$$

14. **[R]** Let $\mathbf{a} = (1, 3, -2)^T$ and $\mathbf{b} = (0, 4, 2)^T$. Evaluate all of the following expressions that make sense and find those which are equal:
$$\mathbf{ab}, \quad \mathbf{a}^T\mathbf{b}, \quad \mathbf{ab}^T, \quad \mathbf{a}^T\mathbf{b}^T, \quad \mathbf{b}^T\mathbf{a}, \quad \mathbf{ba}^T.$$

15. **[R][V]** Suppose that A is a square matrix.

a) Show that the matrix $B = (A + A^T)$ is symmetric.

b) Show that the matrix $C = AA^T$ is symmetric.

c) A matrix M with the property that $M^T = -M$ is called a *skew symmetric* matrix. Show that $D = (A - A^T)$ is a skew symmetric matrix.

d) **[H]** Can you show how to write any square matrix as the sum of a symmetric and a skew symmetric matrix?

16. **[H]** Show by constructing an example that, in general, $A^TA \neq AA^T$, even if A is square.

17. **[H]** Suppose there exists a real matrix G such that $GG^T = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ where $\lambda, \mu \in \mathbb{R}$. Prove that λ and μ are non-negative. If $\lambda = 45$ and $\mu = 20$ find an example of such a matrix G with integer entries.

18. **[X]** Show that a matrix $A \in M_{mn}$ is a symmetric matrix if and only if:

i) A is square (i.e., $m = n$), *and*

ii) $\mathbf{x}^TA\mathbf{y} = (A\mathbf{x})^T\mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Problems 5.3 : The inverse of a matrix

19. **[R][V]** Find the inverses of those of the following 2×2 matrices that have inverses.

$$\text{a)}\ \begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix}, \quad \text{b)}\ \begin{pmatrix} -4 & 7 \\ 3 & -5 \end{pmatrix}, \quad \text{c)}\ \begin{pmatrix} 6 & 12 \\ 3 & 6 \end{pmatrix}, \quad \text{d)}\ \begin{pmatrix} 8 & 9 \\ 3 & 4 \end{pmatrix}, \quad \text{e)}\ \begin{pmatrix} 0 & 1 \\ 1 & 7 \end{pmatrix}.$$

20. **[R]** Use the matrix inversion algorithm of Section 5.3 to decide if the following matrices are invertible, and find the inverses for those which are invertible.

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 2 & 3 \\ -1 & 4 & -2 \end{pmatrix}, \ C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{pmatrix}, \ D = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 3 & 1 \\ 1 & -7 & -2 \end{pmatrix}.$$

21. **[H]** Write down the inverse of each of the following matrices

$$\text{a)}\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix} \qquad \text{b)}\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 \\ -2 & 0 & 0 \end{pmatrix}.$$

22. **[R]** Decide if the following matrices are invertible, and find the inverses for those that are invertible.

$$A = \begin{pmatrix} 1 & 2 & -1 & -1 \\ 1 & 2 & -2 & -2 \\ 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 3 & 3 & 1 & 5 \\ 1 & 0 & 2 & 4 \\ 0 & -4 & 2 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & -2 & -7 \\ 2 & 4 & 3 & 14 \\ -1 & -2 & 3 & 11 \\ 3 & 5 & 2 & 12 \end{pmatrix}.$$

23. [**R**] Given that A, B and C are invertible $n \times n$ matrices simplify

a) $A(CB^2A)^{-1}C$, b) $(ABA^{-1})^6$, c) $A(A^{-1}+A)^2A^{-1}$,

d) $A(I+(I-A)+\cdots+(I-A)^m)$.
HINT: Write the first A as $I-(I-A)$.

24. [**R**] a) Simplify $(B^{-1}A)^{-1}$.

b) Find $(B^{-1}A)^{-1}$ if $A^{-1} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$.

25. [**H**] a) Prove that $(A^T)^{-1} = (A^{-1})^T$ for any invertible matrix A.

b) If A, B, C are invertible matrices of the same size simplify

i) $A^{-1}(BA^T)^TB$, ii) $A^T(CA^T)^{-1}C^T$.

26. [**R**][**V**] Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 3 & -2 & 2 \end{pmatrix}$.

a) Calculate A^{-1}. b) Solve $A\mathbf{x} = \mathbf{c}$ for $\mathbf{x}$, where $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$.

27. [**H**] A square matrix Q is said to be an orthogonal matrix if it has the property that $Q^TQ = I$. That is, $Q^T = Q^{-1}$. Show that the matrix

$$Q = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

is orthogonal. Hence write down the solution of $Q\mathbf{x} = \mathbf{b}$ for $\mathbf{b} \in \mathbb{R}^3$.

28. [**H**] Show $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is orthogonal. Show that $\mathbf{x} \in \mathbb{R}^2$ and $Q\mathbf{x}$ are equidistant from the origin. [**X**] Show that Q acts as a rotation on $\mathbb{R}^2$.

29. [**X**] A complex generalisation of Question 27 is the following. A square matrix Q is said to be a unitary matrix if it has the property that $\overline{Q}^T Q = I$, where $\overline{Q}$ is the matrix obtained from Q by taking complex conjugates of each entry of Q. Give an example of a 2×2 unitary matrix with non-real entries.

30. [**X**] Let Q be a square $n \times n$ orthogonal matrix, i.e., a square matrix for which $Q^TQ = I$, where I is an identity matrix. Show that $(Q\mathbf{x}) \cdot (Q\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

HINT. $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T\mathbf{y}$.

31. **[X]** Let Q be a square $n \times n$ orthogonal matrix. Show that the columns of Q are a set of n orthonormal vectors in $\mathbb{R}^n$. Show that the rows of Q also form a set of n orthonormal vectors in $\mathbb{R}^n$.

32. **[X]** Let Q be a square $n \times n$ orthogonal matrix. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the n standard basis column vectors of $\mathbb{R}^n$. Show that the set of vectors $\{Q\mathbf{e}_1, Q\mathbf{e}_2, \ldots, Q\mathbf{e}_n\}$ also form a set of orthonormal vectors.

33. **[X]** Show that the matrix

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}}i & -\frac{1}{\sqrt{2}}i & 0 \\ \frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}}i & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

is unitary. Then use the result of Question 29 to write down the solution of $Q\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = (b_1\ b_2\ b_3)^T$ with b_1, b_2, b_3 complex.

34. **[X]** a) Suppose $ab \neq 0$. Write down the inverse of $\begin{pmatrix} a & 0 \\ c & b \end{pmatrix}$.

b) Let A, B, C be 2×2 matrices where A and B are invertible and let O be the 2×2 zero matrix. Find the inverse of the 4×4 matrix $\begin{pmatrix} A & O \\ C & B \end{pmatrix}$.

Problems 5.4 : Determinants

35. **[R]** Evaluate the determinants of the following 2×2 matrices and hence determine whether or not they are invertible.

a) $\begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix}$, b) $\begin{pmatrix} -4 & 7 \\ 3 & -5 \end{pmatrix}$, c) $\begin{pmatrix} 5 & 2 \\ 10 & 4 \end{pmatrix}$, d) $\begin{pmatrix} 8 & 9 \\ 3 & 4 \end{pmatrix}$, e) $\begin{pmatrix} 11 & 13 \\ 12 & 14 \end{pmatrix}$.

36. **[R]** Evaluate the determinants for the following matrices by reducing to row echelon form.

a) $\begin{pmatrix} -1 & 1 & 2 \\ 2 & 4 & -1 \\ 0 & -1 & 1 \end{pmatrix}$, b) $\begin{pmatrix} 1 & -2 & 4 \\ 3 & 1 & -2 \\ 1 & 5 & -10 \end{pmatrix}$, c) $\begin{pmatrix} 1 & 0 & 4 \\ 3 & 1 & -2 \\ 1 & 5 & -10 \end{pmatrix}$.

37. **[R]** Find the determinant of the matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 7 & 1 \\ 1 & 8 & 3 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix}$.

38. **[H]** Suppose $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ has determinant 5. Find

a) $\det \begin{pmatrix} 3a & 3b & 3c \\ 2d & 2e & 2f \\ -g & -h & -i \end{pmatrix}$, b) $\det \begin{pmatrix} a+2d & b+2e & c+2f \\ d-g & e-h & f-i \\ g & h & i \end{pmatrix}$,

c) $\det \begin{pmatrix} d & e & f \\ g & h & i \\ a & b & c \end{pmatrix}$, d) $\det(7A)$.

39. [**R**] Given that A is a 3×3 matrix with $\det A = -2$. Calculate:

a) $\det A^T$, b) $\det A^{-1}$, c) $\det A^5$.

40. [**R**] Evaluate $\det(A)$, $\det(B)$ and hence $\det(AB)$, where

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & 5 \\ 3 & 4 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & -1 & 0 \\ -3 & 2 & 4 \\ 2 & 5 & 0 \end{pmatrix}.$$

41. [**R**] For what values of a is the matrix $\begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & a \end{pmatrix}$ invertible?

42. [**H**] Long long ago, a mathematician wrote C and C^{-1} on a piece of paper. Unfortunately insects have damaged the paper and all that is left is

$$C = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 2 & -1 \\ * & * & * \end{bmatrix} \quad \text{and} \quad C^{-1} = \begin{bmatrix} * & 0 & -1 \\ 2 & * & -1 \\ 5 & 1 & * \end{bmatrix}$$

a) Find C^{-1}. b) Find C. c) Find $\det C$.

43. [**H**] Show that

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1+c \end{pmatrix} = abc.$$

44. [**H**] Let U_1 and U_2 be two $n \times n$ row-echelon matrices. Prove that $\det(U_1)\det(U_2) = \det(U_1U_2)$.

45. [**H**] Let $A = \begin{pmatrix} \alpha & 1 & -1 \\ \alpha & 2\alpha+2 & \alpha \\ \alpha-3 & \alpha-3 & \alpha-3 \end{pmatrix}$.

a) Factorise $\det(A)$.

b) Hence, find the values of α will there be a nonzero solution of $A\mathbf{x} = \mathbf{0}$.

46. [**R**] Show by constructing an example that in general $\det(A+B) \neq \det(A) + \det(B)$.

47. [**R**] Show by constructing an example that in general $\det(\lambda A) \neq \lambda \det(A)$.

48. [**H**] Use the product rule for determinants to show that a square orthogonal matrix Q (see Question 27) has a value for $\det(Q)$ of $+1$ or -1.

49. [**X**] Use the product rule for determinants to show that a square unitary matrix Q (see Question 29) has $\det(Q) = e^{i\theta}$ for some angle θ.

50. [**X**] Let A and B be two matrices which differ only in the first column, i.e., let

$$A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix},$$

where $\mathbf{a}_1$ and $\mathbf{b}_1$ are the first columns of A and B and where $\mathbf{a}_i,\ i = 2, 3, \ldots, n$, are the remaining columns of both A and B. Let C be the matrix

$$C = \begin{pmatrix} \mathbf{a}_1 + \mathbf{b}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}$$

obtained by replacing the first column of A (or B) by the sum of the first columns of A and B.

Show that

$$\det(C) = \det(A) + \det(B).$$

Explain why the result of this question also holds for adding two matrices which differ only in one column (not necessarily the first) or which differ only in one row.

51. [**X**] Prove the following relationships between "volumes" and determinants:

a) In two dimensions, let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, and consider the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$. Show that a parametric vector form for the parallelogram is

$$\mathbf{x} = \lambda_1 \mathbf{a} + \lambda_2 \mathbf{b} \ \text{ for } \ 0 \leqslant \lambda_1 \leqslant 1, \ \ 0 \leqslant \lambda_2 \leqslant 1,$$

and then show that the area of the parallelogram is equal to $|\det(A)|$, where A is the matrix with rows $\mathbf{a}$ and $\mathbf{b}$.

b) In three dimensions, let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$, and consider the parallelepiped spanned by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. A parametric vector form for the equation of the parallelepiped is

$$\mathbf{x} = \lambda_1 \mathbf{a} + \lambda_2 \mathbf{b} + \lambda_3 \mathbf{c} \ \text{ for } \ 0 \leqslant \lambda_1 \leqslant 1, \ 0 \leqslant \lambda_2 \leqslant 1, \ 0 \leqslant \lambda_3 \leqslant 1.$$

Show that the volume of the parallelepiped is equal to $|\det(A)|$, where A is the matrix with rows $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

c) What is the one-dimensional version of these results?

52. **[X]** Show that

$$\det\begin{pmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{pmatrix} = (a-b)(b-c)(c-a).$$

53. **[X]** Show that

$$\det\begin{pmatrix}1+x & 2 & 3 & 4\\ 1 & 2+x & 3 & 4\\ 1 & 2 & 3+x & 4\\ 1 & 2 & 3 & 4+x\end{pmatrix} = x^3(x+10).$$

54. **[X]** Factorise the determinant

$$\det\begin{pmatrix}y+z+2x & y & z\\ x & z+x+2y & z\\ x & y & x+y+2z\end{pmatrix}.$$

55. **[X]** Factorise the determinant

$$\det\begin{pmatrix}z & 1 & 2\\ 1 & z & 3\\ 1 & 1 & z+1\end{pmatrix}$$

and hence solve the simultaneous equations

$$zx+y=2, \qquad x+zy=3, \qquad x+y=z+1.$$

56. **[X]** Suppose α, β and γ are the roots of the cubic equation $x^3+px+q=0$ and $s_k=\alpha^k+\beta^k+\gamma^k$. Find s_1, s_2, s_3 in terms of p and q and show that

$$\det\begin{pmatrix}s_1 & s_2 & s_3\\ s_2 & s_3 & s_1\\ s_3 & s_1 & s_2\end{pmatrix} = 8p^3+27q^3.$$

57. **[X]** Let $A(x_1,y_1), B(x_2,y_2), C(x_3,y_3)$ be three points in the plane.

a) Suppose A, B and C are collinear. Show that

$$\det\begin{pmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{pmatrix} = 0.$$

b) Now suppose that A, B, C are not collinear. By considering the areas of some trapezia (or otherwise), show that the area of the triangle with vertices A, B, C is given by $|D|$ where

$$2D = \det\begin{pmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{pmatrix}.$$

ANSWERS TO SELECTED PROBLEMS

Chapter 1

1. a) $\mathbf{a}+\mathbf{h}$, b) $\mathbf{a}-\mathbf{h}$, c) $\mathbf{a}+\frac{1}{2}\mathbf{h}$, d) $\frac{3}{4}\,\mathbf{a}$, e) $\frac{3}{4}\,\mathbf{a}-\frac{1}{2}\,\mathbf{h}$.

2. a) $\mathbf{0}$, b) $2\overrightarrow{CA}$.

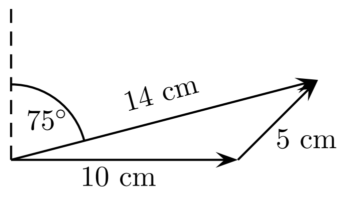
3. a) $-4\mathbf{a}+5\mathbf{b}$, b) $(2p+3r)\mathbf{a}+(2q-3s)\mathbf{b}$.

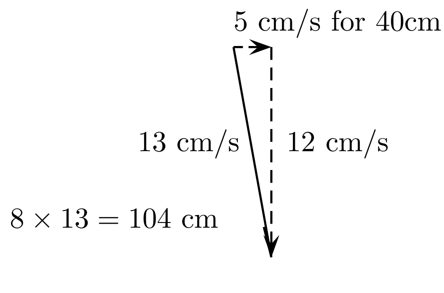
4. a) $\frac{1}{2}(\mathbf{b}+\mathbf{a})$, $\frac{1}{2}(\mathbf{b}+\mathbf{c})$

6. For part (a) and (b) we denote the vector of length 1 cm towards the east by $\mathbf{i}$ and the vector of length 1 cm towards the north by $\mathbf{j}$.

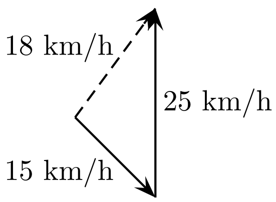
   a) ≈ 14 cm N 75° E.
   $\approx 13.5\mathbf{i}+3.5\mathbf{j}$

   b) ≈ 104 cm S 23° E,
   or exactly $40\mathbf{i}-96\mathbf{j}$

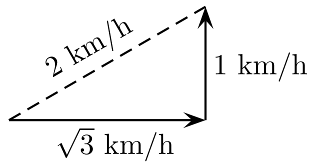




c) ≈ 18 km/h N $36°$ E.



d) The rower must row $30°$ upstream, ≈ 10.4 minutes to cross



7. Approximately 28.0 km N $51°\,9'$ E from A.

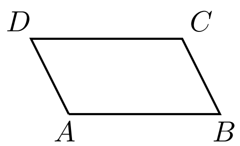
8. a) $\dfrac{1}{2}\,\mathbf{b} + \dfrac{1}{2}\,\mathbf{c}$, $\quad \dfrac{1}{4}\,\mathbf{a} + \dfrac{3}{4}\,\mathbf{c}$, $\quad \dfrac{1}{4}\,\mathbf{a} + \dfrac{3}{4}\,\mathbf{b}$. $\qquad$ b) $\dfrac{1}{7}\,\mathbf{a} + \dfrac{3}{7}\,\mathbf{b} + \dfrac{3}{7}\,\mathbf{c}$.

9. a) $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\quad$ b) $\begin{pmatrix} 16 \\ 15 \\ -5 \end{pmatrix}$, $\quad$ c) $\begin{pmatrix} -7 \\ 2 \\ -6 \\ -1 \end{pmatrix}$, $\quad$ d) Not possible, $\quad$ e) $7\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$.

10. 7.43, N $28°$ E.

16. a) not parallel, $\quad$ b) parallel, $\quad$ c) parallel.
Only in b) is $ABCD$ a parallelogram.

17. The correct sketch is:



21. $(4, 5, 0)$, $(-6, -1, 2)$, $(4, 7, 6)$

22. $\mathbf{d} + \mathbf{e} - \mathbf{f}$, $\quad \mathbf{d} + \mathbf{f} - \mathbf{e}$, $\quad \mathbf{e} + \mathbf{f} - \mathbf{d}$.

23. The midpoint is $(3, -1, 3)$. The point Q is $(10, -29, 31)$.

24. $\mathbf{t} = \dfrac{1}{3}\mathbf{a} + \dfrac{2}{3}\mathbf{b}$

25. $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$

26. $6, \frac{1}{6}\begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix}$; $\sqrt{14}, \frac{1}{\sqrt{14}}\begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix}$; $\sqrt{21}, \frac{1}{\sqrt{21}}\begin{pmatrix} 4 \\ 0 \\ 1 \\ -2 \\ 0 \end{pmatrix}$.

27. a) 15, b) 12, c) $\sqrt{62}$.

28. $\sqrt{35}, \sqrt{6}, \sqrt{41}$.

29. A 4–cube has 16 vertices, say, $V = \{(a,b,c,d) \mid a,b,c,d = 0,1\}$.

30. $(5,0^9)^T$, $(0,5,0^8)^T \dots, (0^9,5)^T$. Yes, $(\alpha,\alpha,\dots,\alpha)^T$ where $(5-\alpha)^2 + 9\alpha^2 = 50$.

31. a) $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 5 \end{pmatrix}, \lambda \in \mathbb{R}$; b) $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ -3 \\ 6 \end{pmatrix}, \lambda \in \mathbb{R}$;

c) $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$; d) $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 1 \\ 2 \\ -2 \end{pmatrix}, \lambda \in \mathbb{R}$.

32. Yes, it corresponds to $\lambda = 1$.

33. a) $\mathbf{x} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \lambda \in \mathbb{R}$; b) $\mathbf{x} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -3 \end{pmatrix}, \lambda \in \mathbb{R}$;

c) $\mathbf{x} = \lambda\begin{pmatrix} 1 \\ -7 \end{pmatrix}, \lambda \in \mathbb{R}$; d) $\mathbf{x} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \lambda \in \mathbb{R}$;

e) $\mathbf{x} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$.

34. a) $\mathbf{x} = \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix}$ or $\frac{x_1+4}{6} = x_2 - 1, \quad x_3 = 3$.

b) $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix}$ or $\frac{x_1-1}{4} = \frac{x_2-2}{-5} = \frac{x_3+3}{6}$.

c) $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$ or $\frac{x_1-1}{5} = \frac{x_2+1}{-1} = \frac{x_3-1}{2}$.

d) $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$ or $x_1 = 1, \ x_2 = x_3$.

35. $\begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$. a) $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R}$.

36. a) true, b) false, c) true, d) true.

37. a) $\mathbf{x} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}), \quad 0 \leqslant \lambda \leqslant 1$; b) $\mathbf{x} = \mathbf{b} + \lambda(\mathbf{a} - \mathbf{b}), \quad \lambda \geqslant 0$;
c) $\mathbf{x} = \mathbf{b} + \lambda(\mathbf{a} - \mathbf{b}), \quad \lambda \geqslant 1$; d) $\mathbf{x} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}), \quad \lambda \geqslant \frac{1}{2}$.

38. a) Line segment joining $(1, 3, 6)$ and $(-2, 4, 13)$.
b) Line segment joining $(3, -3, -5, -3, -13)$ and $(-9, 27, 49, 15, 23)$.
c) Line segment joining $(0, 4, 8, 3, -5, 4)$ and $(6, -2, 7, 2, -1, 5)$.
d) Ray from point $(1, 4, -6, 2)$ parallel to $\big(3, 0, -1, 5\big)$.
e) Line through $(3, 1, -4)$ parallel to $\begin{pmatrix} 6 \\ -2 \\ 7 \end{pmatrix}$ with segment from $(-9, 5, -18)$ to $(15, -3, 10)$ removed.

39. a) $\mathbf{x} = \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 4 \\ -6 \end{pmatrix}; \quad \lambda, \mu \in \mathbb{R}$.
b) $\mathbf{x} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -14 \\ 5 \end{pmatrix}; \quad \lambda, \mu \in \mathbb{R}$.

40. a) Plane through the origin parallel to $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$.
b) Line through $(3, 1, 2, 4)$ parallel to $\begin{pmatrix} -2 \\ 1 \\ 3 \\ 2 \end{pmatrix}$.
c) Line through origin parallel to $\begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \end{pmatrix}$.
d) Plane through $(1, 2, 3)$ parallel to $\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 2 \\ 4 \end{pmatrix}$.

41. a) $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$ for $\lambda_1, \lambda_2 \in \mathbb{R}$;

b) $\mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} + \lambda_1 \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 6 \\ -6 \end{pmatrix}$ for $\lambda_1, \lambda_2 \in \mathbb{R}$;

c) $\mathbf{x} = \begin{pmatrix} -2 \\ 4 \\ 1 \\ 6 \end{pmatrix} + \lambda_1 \begin{pmatrix} 5 \\ -2 \\ 5 \\ -7 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 0 \\ -1 \\ -6 \end{pmatrix}$ for $\lambda_1, \lambda_2 \in \mathbb{R}$;

d) $\mathbf{x} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ for $\lambda_1, \lambda_2 \in \mathbb{R}$;

e) $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ \frac{6}{5} \\ 1 \end{pmatrix}$ for $\lambda_1, \lambda_2 \in \mathbb{R}$;

f) $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 4 \\ 0 \\ -4 \\ 5 \end{pmatrix} + \lambda_2 \begin{pmatrix} 7 \\ 2 \\ -3 \\ -5 \end{pmatrix}$ for $\lambda_1, \lambda_2 \in \mathbb{R}$.

42. a) $\mathbf{x} = \lambda_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$; $\lambda_1, \lambda_2 \in \mathbb{R}$.

b) $\mathbf{x} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix}$; $\lambda_1, \lambda_2 \in \mathbb{R}$.

c) $\mathbf{x} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ -6 \\ 1 \end{pmatrix}$; $\lambda_1, \lambda_2 \in \mathbb{R}$.

d) $\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$; $\lambda_1, \lambda_2 \in \mathbb{R}$.

44. a) $(3, 2, 4)$, b) $(3, -4, 11)$.

45. a) $6x - 3y + 2z = -12$, b) $6x - 12y + 13z = 100$.

46. a) $\mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$ for $\lambda \in \mathbb{R}$. b) $(-13, 22, 9)$.

47. a) $\mathbf{x} = \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 2 \\ -2 \end{pmatrix}$ for $\lambda \in \mathbb{R}$. b) $(1, 2, 3)$.

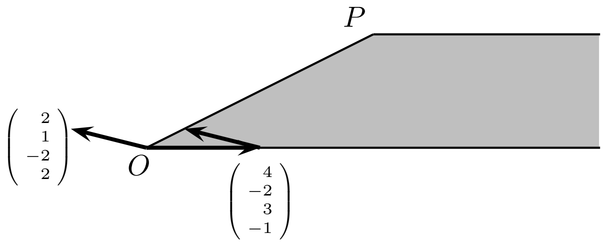
48. a) Parallelogram with vertices $(0, 1)$, $(1, 3)$, $(2, 4)$, $(3, 6)$.

b) Triangle with vertices $(0,1)$, $(1,3)$, $(3,6)$.

c) Parallelogram with vertices $(0,0,0)$, $(12,6,-12)$, $(32,-16,24)$, $(44,-10,12)$.

d) Triangle with vertices $(0,0,0)$, $(12,6,-12)$, $(36,-6,6)$.

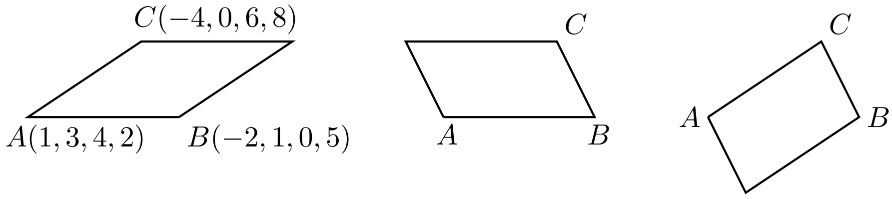
e) An unbounded region with vertices O and P and two of the three sides parallel to $\begin{pmatrix} 4 \\ -2 \\ 3 \\ -1 \end{pmatrix}$. At P, $\lambda_1 = \lambda_2 = 6$ and so $\overrightarrow{OP} = \begin{pmatrix} 36 \\ -6 \\ 6 \\ 6 \end{pmatrix}$.



49. a) See c).

b) $\mathbf{x} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} -3 \\ -2 \\ -4 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ -1 \\ 6 \\ 3 \end{pmatrix}$ for $0 \leqslant \lambda_1 \leqslant 1,\ 0 \leqslant \lambda_2 \leqslant \lambda_1$.

c) The three parallelograms are:



The algebraic definitions are:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} -3 \\ -2 \\ -4 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} -5 \\ -3 \\ 2 \\ 6 \end{pmatrix} \quad \text{for } 0 \leqslant \lambda_1 \leqslant 1,\ 0 \leqslant \lambda_2 \leqslant 1.$$

$$\mathbf{x} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} -3 \\ -2 \\ -4 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ -1 \\ 6 \\ 3 \end{pmatrix} \quad \text{for } 0 \leqslant \lambda_1 \leqslant 1,\ 0 \leqslant \lambda_2 \leqslant 1.$$

$$\mathbf{x} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 1 \\ -6 \\ -3 \end{pmatrix} + \lambda_2 \begin{pmatrix} -5 \\ -3 \\ 2 \\ 6 \end{pmatrix} \text{ for } 0 \leqslant \lambda_1 \leqslant 1,\ 0 \leqslant \lambda_2 \leqslant 1.$$

Chapter 2

1. a) $\frac{\pi}{4}$ b) $\cos^{-1}\left(\frac{1}{10\sqrt{3}}\right) \approx 86°41'$, c) $\frac{\pi}{2}$, d) $\cos^{-1}\left(\frac{7}{10\sqrt{13}}\right) \approx 78°48'$.

2. a) $0, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{3}}$; b) $\frac{4}{3\sqrt{2}}, -\frac{5}{\sqrt{33}}, \frac{8}{\sqrt{66}}$; c) $\frac{7}{3\sqrt{10}}, -\frac{1}{\sqrt{42}}, \frac{8}{\sqrt{105}}$.

3. $\cos^{-1}\left(\frac{1}{3}\right) \approx 70°32'$.

7. $\lambda_1 = \mathbf{a} \cdot \mathbf{u}_1 = \frac{1}{\sqrt{2}}$, $\lambda_2 = \mathbf{a} \cdot \mathbf{u}_2 = -3$, $\lambda_3 = \mathbf{a} \cdot \mathbf{u}_3 = \frac{3}{\sqrt{2}}$.

8. a) $\begin{pmatrix} 5 \\ \frac{5}{2} \\ 1 \end{pmatrix}$. b) $\frac{\pi}{2}$. c) $\frac{\sqrt{66}}{2}$. d) $\frac{1}{17}\begin{pmatrix} 80 \\ 50 \\ 22 \end{pmatrix}$.

9. a) $\frac{1}{3}\begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}$, b) $\frac{3}{14}\begin{pmatrix} -1 \\ 3 \\ 0 \\ 2 \end{pmatrix}$, c) $\begin{pmatrix} -3 \\ 3 \\ 6 \end{pmatrix}$.

10. a) 7, b) 3, c) $\sqrt{6}$.

11. $\mathbf{q} = -\mathbf{p} + 2\mathbf{a} + 2\,\mathrm{proj}_{\mathbf{d}}(\mathbf{p} - \mathbf{a})$.

12. b) $q(\lambda_0) = \mathbf{a} \cdot \mathbf{a} - \frac{(\mathbf{a} \cdot \mathbf{b})^2}{\mathbf{b} \cdot \mathbf{b}}$

14. a) $\begin{pmatrix} 16 \\ -4 \\ -2 \end{pmatrix}$, b) $\begin{pmatrix} -23 \\ -11 \\ 20 \end{pmatrix}$, c) $\begin{pmatrix} -45 \\ 9 \\ -18 \end{pmatrix}$.

15. $\begin{pmatrix} 12 \\ -8 \\ 6 \end{pmatrix}$.

17. a) $2\sqrt{21}$, $\begin{pmatrix} 8 \\ -4 \\ 2 \end{pmatrix}$; b) $2\sqrt{2}$, $\begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix}$.

18. a) $\sqrt{2}$; b) $\dfrac{15}{2}$.

19. a) $-\dfrac{4}{3\sqrt{2}}$. b) $\dfrac{1}{\sqrt{2}}$.

20. a) 2, b) $\dfrac{5\sqrt{2}}{6}$, c) 7.

23. a) Line through A and B is $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$, $\lambda_1 \in \mathbb{R}$.

Line through C and D is $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$, $\lambda_2 \in \mathbb{R}$.

b) Shortest distance is $\dfrac{3}{\sqrt{17}}$.

c) Point P is $\left(-\dfrac{21}{17}, \dfrac{38}{17}, \dfrac{53}{17}\right)$ and Q is $\left(-\dfrac{30}{17}, \dfrac{32}{17}, \dfrac{47}{17}\right)$.

25. a) 14, b) 53.

27. As usual, the answers for equations of planes are not unique.

a) $\mathbf{x} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$; $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \cdot \left(\mathbf{x} - \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}\right) = 0$;
$x_1 - x_2 - 2x_3 = 3$.

b) $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$; $\begin{pmatrix} -5 \\ 5 \\ -5 \end{pmatrix} \cdot \left(\mathbf{x} - \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}\right) = 0$;
$x_1 - x_2 + x_3 = -3$.

c) $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} + \lambda_1 \begin{pmatrix} -2 \\ -1 \\ 4 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -7 \\ 10 \\ -1 \end{pmatrix} \cdot \left(\mathbf{x} - \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}\right) = 0$;
$7x_1 - 10x_2 + x_3 = -15$.

d) $\mathbf{x} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1/4 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 2 \\ -1 \end{pmatrix} \cdot \left(\mathbf{x} - \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}\right) = 0$;
$4x_1 - 2x_2 + x_3 = -4$.

e) $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 14 \\ 2 \\ -6 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 17 \\ 15 \end{pmatrix} \cdot \left(\mathbf{x} - \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \right) = 0;$

$4x_1 + 17x_2 + 15x_3 = 8.$

28. a) $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$, for $\lambda,\ \mu \in \mathbb{R}$.

b) $\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$. c) $x_1 + x_2 + x_3 = 7.$

29. a) $\begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$, b) $\frac{1}{2}\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$.

30. a) 3, b) $\sqrt{6}$, c) $\frac{13}{7}$, d) $\frac{25}{7}$.

31. a) $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$, $\lambda,\ \mu \in \mathbb{R}$.

b) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. c) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \left(\mathbf{x} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right) = 0.$ d) $\frac{8}{\sqrt{3}}$.

32. a) $\mathbf{c} = \text{proj}_{\mathbf{a}}\mathbf{v},\ \mathbf{d} = \mathbf{v} - \mathbf{c}$. b) $\mathbf{c} = \begin{pmatrix} \frac{3}{11} \\ -\frac{1}{11} \\ -\frac{1}{11} \end{pmatrix}$, $\mathbf{d} = \begin{pmatrix} \frac{8}{11} \\ \frac{12}{11} \\ \frac{12}{11} \end{pmatrix}$.

Chapter 3

1.

	$x \in \mathbb{N}$	$x \in \mathbb{Z}$	$x \in \mathbb{Q}$	$x \in \mathbb{R}$
a)	-	-25	-25	-25
	3	3	3	3
	-	-3	-3	-3
	-	-	$-\frac{10}{3}$	$-\frac{10}{3}$
b)	1	$1,-5$	$1,-5$	$1,-5$
	5	5	$5,\frac{3}{2}$	$5,\frac{3}{2}$
	-	-	-	$\frac{1\pm\sqrt{5}}{2}$
	-	-	-	-
c)	$3j, j \in \mathbb{N}$	$3j, j \in \mathbb{Z}$	$3j, j \in \mathbb{Z}$	$3j, j \in \mathbb{Z}$
	0	0	0	$3k\pi, k \in \mathbb{Z}$

2. No.

3. Yes. The set $\{0\}$ and the empty set $\emptyset = \{\ \}$.

4. Yes.

5. $3z = 6 + 9i$, $\quad z^2 = -5 + 12i$, $\quad z + 2w = 7i$, $\quad z(w + 3) = -2 + 10i$, $\quad \dfrac{z}{w} = \dfrac{1}{5}(4 - 7i)$, $\dfrac{w}{z} = \dfrac{1}{13}(4 + 7i)$.

6. a) $\dfrac{1}{5}(3 - i)$, $\quad$ b) $-\dfrac{1}{2}(1 - i)$.

7. a) $a^2 - b^2 + 2abi$, $\quad$ b) $\dfrac{a}{a^2 + b^2} - i\dfrac{b}{a^2 + b^2}$, $\quad$ c) $\dfrac{1}{(a-1)^2 + b^2}\big((a^2 - 1 + b^2) - 2ib\big)$.

8. a) $\frac{1}{2}(-1 \pm \sqrt{3}\,i)$, $\quad$ b) $-1 \pm \sqrt{2}\,i$, $\quad$ c) $3 \pm i$, $\quad$ d) $\frac{1}{2}(3 \pm \sqrt{3})$, $\quad$ e) $\pm i$, $\pm 2i$.

10. 16

11. $\dfrac{8abi(a^2 - b^2)}{(a^2 + b^2)^2}$

12.

z	$\mathrm{Re}(z)$	$\mathrm{Im}(z)$	$\overline{z}$
$-1+i$	-1	1	$-1-i$
$2+3i$	2	3	$2-3i$
$2-3i$	2	-3	$2+3i$
$\frac{2-i}{1+i}$	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1+3i}{2}$
$\frac{1}{(1+i)^2}$	0	$-\frac{1}{2}$	$\frac{i}{2}$

13. $-3+4i, \quad \dfrac{11}{25}-\dfrac{2}{25}i.$

14. $z=2+3i, \quad w=-1+2i.$

17. b) $z^2-6z+13$

18.

z	$\lvert z\rvert$	$\mathrm{Arg}(z)$	Polar Form
$6+6i$	$6\sqrt{2}$	$\frac{\pi}{4}$	$6\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$
-4	4	π	$4(\cos\pi+i\sin\pi)$
$\sqrt{3}-i$	2	$\frac{-\pi}{6}$	$2\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)$
$\frac{-1}{\sqrt{2}}-\frac{i}{\sqrt{2}}$	1	$\frac{-3\pi}{4}$	$\cos\frac{3\pi}{4}-i\sin\frac{3\pi}{4}$
$-7+3i$	$\sqrt{58}$	α	$\sqrt{58}(\cos\alpha+i\sin\alpha)$

Here $\alpha=\pi-\tan^{-1}\frac{3}{7}$.

19. $\sqrt{234}, \quad -1.$

20. $n=4$

21. a) $\dfrac{3}{2}\left(1+\sqrt{3}i\right)$, b) $\dfrac{3}{2}\left(-\sqrt{3}+i\right)$, c) $-\dfrac{3}{2}\left(1+\sqrt{3}i\right)$, d) $\dfrac{3}{2}\left(\sqrt{3}-i\right)$,

e) $\dfrac{3}{2}\left(\sqrt{2+\sqrt{2}}+i\sqrt{2-\sqrt{2}}\right)$ (Double angle formula used).

27. $64, \quad -(1+\sqrt{3})i, \quad \dfrac{1+\sqrt{3}}{2}+\dfrac{-1+\sqrt{3}}{2}i.$

28. $\dfrac{7}{2}.$

29. $\pi.$

30. $\mathrm{Arg}\,(-1+i)=\dfrac{3\pi}{4}$; $\mathrm{Arg}\left(-\sqrt{3}+i\right)=\dfrac{5\pi}{6}$;

$\mathrm{Arg}\left((-1+i)\left(-\sqrt{3}+i\right)\right)=-\dfrac{5\pi}{12}$; $\mathrm{Arg}\left(\dfrac{-1+i}{-\sqrt{3}+i}\right)=-\dfrac{\pi}{12}.$

31. $\sin \dfrac{7\pi}{12} = \dfrac{1+\sqrt{3}}{2\sqrt{2}}$.

32. $zw = 2\sqrt{2}e^{i\pi/12} = 2\sqrt{2}\left[\cos\left(\dfrac{\pi}{12}\right) + i\sin\left(\dfrac{\pi}{12}\right)\right]$; $z^9 = -512$; $\left(\dfrac{z}{\overline{w}}\right)^{12} = 64e^{i\pi} = -64$.

33. a) $16\left(-\sqrt{3}+i\right)$, b) $-i$, c) $-\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$.

34. a) $\pm(5-2i)$, b) $\pm(3+5i)$, c) $\pm(7+5i)$.

35. b) $\sqrt{2}e^{\frac{5\pi i}{12}} = \frac{1}{2}\left(\left(\sqrt{3}-1\right) + i\left(\sqrt{3}+1\right)\right)$, c) $\frac{1}{\sqrt{2}}(1+13i)$.

37. a) $2+i,\ 1-i$; b) $4+i,\ 3-2i$; c) $1-2i,\ -5+3i$.

38. $e^{i\pi/7},\ e^{3i\pi/7},\ e^{5i\pi/7},\ e^{i\pi},\ e^{-i\pi/7},\ e^{-3i\pi/7},\ e^{-5i\pi/7}$.

39. $e^{in\pi/12}$ for $n = -11, -7, -3, 1, 5, 9$.

40. $2e^{in\pi/15}$ for $n = -13, -7, -1, 5, 11$.

41. $\dfrac{15}{2} + i\left(\dfrac{3\sqrt{3}}{2} - 1\right)$, $3-i$, $\dfrac{15}{2} - i\left(\dfrac{3\sqrt{3}}{2} + 1\right)$.

48. a) Real part $= \cos(2\theta)$. Imaginary part $= \sin(2\theta)$.

51. a) $\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$
$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$
b) $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$

52. $\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$
$\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$.

53. a) $\cos\theta = \dfrac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$.
b) $\cos^6\theta = \dfrac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$.

54.
$$\begin{aligned}
\sin^5\theta &= \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)\\
\int \sin^5\theta\, d\theta &= \frac{1}{16}\left(-\frac{1}{5}\cos 5\theta + \frac{5}{3}\cos 3\theta - 10\cos\theta\right) + C,\\
\cos^4\theta &= \frac{1}{8}[3 + 4\cos(2\theta) + \cos(4\theta)]\\
\int \cos^4\theta\, d\theta &= \frac{1}{8}\left[3\theta + 2\sin(2\theta) + \frac{1}{4}\sin(4\theta)\right] + C.
\end{aligned}$$

55. a) $\cos 5\theta = 16x^5 - 20x^3 + 5x$

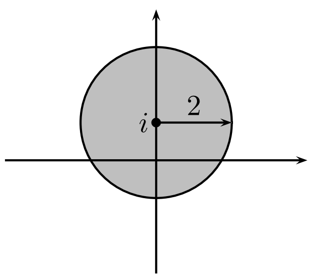
d) $-1,\ \cos\frac{\pi}{5},\ \cos\frac{3\pi}{5},\ \cos\frac{7\pi}{5},\ \cos\frac{9\pi}{5}$.

56. The sum is n when k is an integer multiple of n and 0 otherwise.

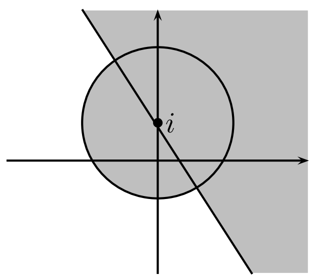
58. $\dfrac{\sin\left(\frac{1}{2}(n+1)\theta\right)\sin\left(\frac{1}{2}n\theta\right)}{\sin\frac{1}{2}\theta}$.

59. a) $\dfrac{9e^{i\theta}}{9+e^{i2\theta}}$.

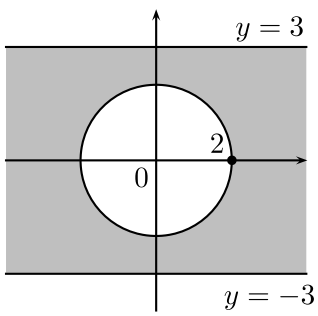
60. a) $|z-i| \leqslant 2$



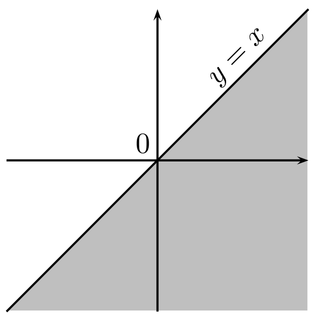
b) $|z-i| \leqslant 2$ or $-\dfrac{\pi}{3} \leqslant \operatorname{Arg}(z-i) \leqslant \dfrac{2\pi}{3}$



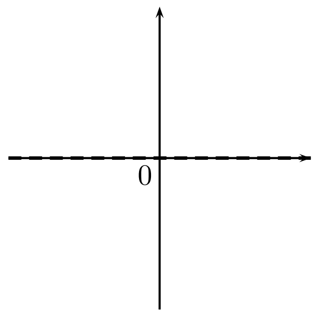
c) $|z| \geqslant 2$ and $|\mathrm{Im}(z)| \leqslant 3$



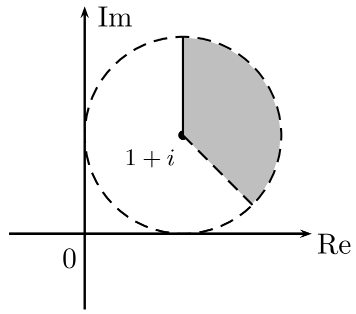
d) $y \leqslant x$



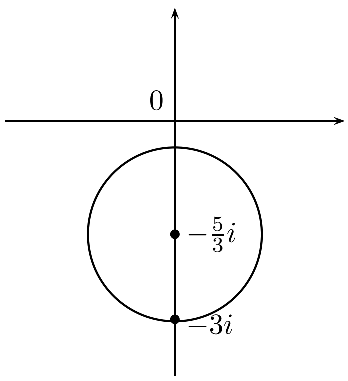
e) The real axis



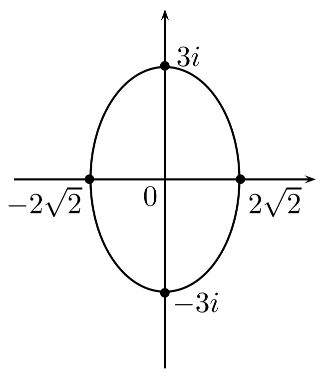
f) $|z-1-i|<1$ & $-\dfrac{\pi}{4}<\operatorname{Arg}(z-1-i)\leqslant\dfrac{\pi}{2}$



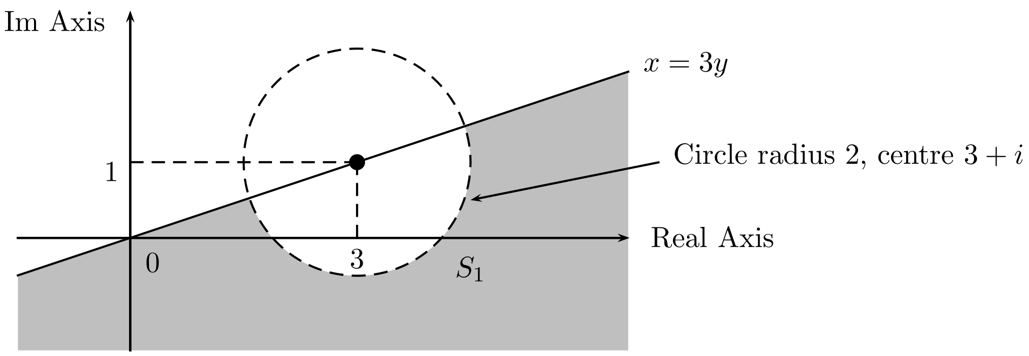
g) Circle: $x^2+\left(y+\dfrac{5}{3}\right)^2=\left(\dfrac{4}{3}\right)^2$



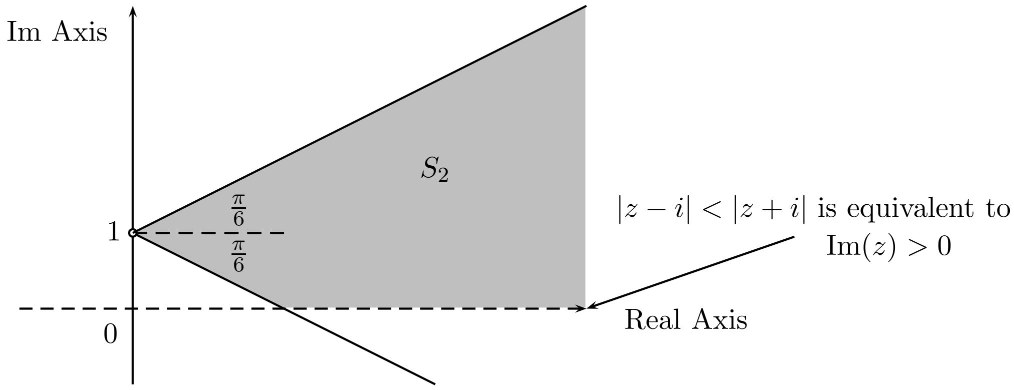
h) Ellipse: $\left(\dfrac{x}{2\sqrt{2}}\right)^2+\left(\dfrac{y}{3}\right)^2=1$



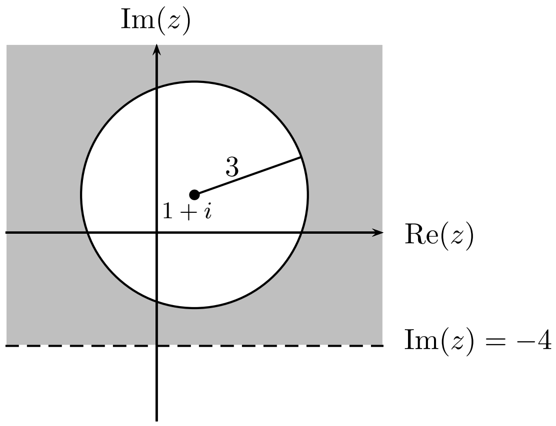
61. a) $\operatorname{Re}(z)\geqslant 3\operatorname{Im}(z)$ and $|z-(3+i)|>2$



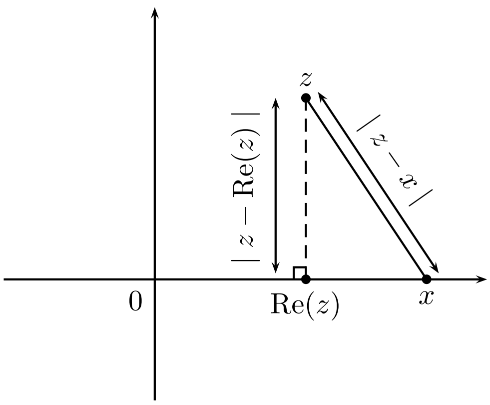
b) $|z-i|<|z+i|$ and $-\dfrac{\pi}{6} \leqslant \operatorname{Arg}(z-i) \leqslant \dfrac{\pi}{6}$



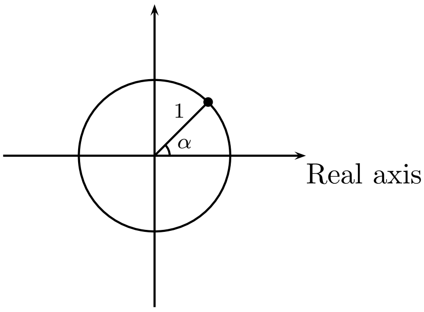
62. a) $\operatorname{Im}(z)>-4$ and $|z-1-i| \geqslant 3$ b) Yes.



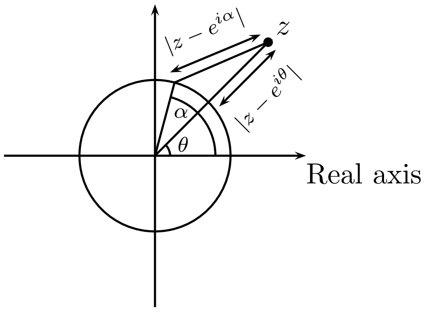
63. $|\, z-x \,| \geqslant |\, z-\operatorname{Re}(z) \,|$



64. a) $w = e^{i\alpha}$, $-\pi < \alpha \leqslant \pi$



c) $|\, z - e^{i\alpha} \,| \geqslant |\, z - e^{i\theta} \,|$, $\theta = \operatorname{Arg}(z)$



65. a) 742, b) 129, c) $1 + 9i$.

66. $p(z) = (z-2)(2z-5)(z+3)$.

67. $p(z) = (z-1)(z+1)(z+2)(z+4)$.

68. a) $\left(z - e^{-\frac{i\pi}{10}}\right)\left(z - e^{\frac{3\pi i}{10}}\right)\left(z - e^{\frac{7\pi i}{10}}\right)\left(z - e^{-\frac{i\pi}{2}}\right)\left(z - e^{-\frac{9\pi i}{10}}\right)$.

b) $\left(z - \sqrt{2}e^{\frac{i\pi}{6}}\right)\left(z - \sqrt{2}e^{\frac{i\pi}{2}}\right)\left(z - \sqrt{2}e^{\frac{5\pi i}{6}}\right)\left(z - \sqrt{2}e^{-\frac{i\pi}{6}}\right)\left(z - \sqrt{2}e^{-\frac{i\pi}{2}}\right)\left(z - \sqrt{2}e^{-\frac{5\pi i}{6}}\right)$.

69. a) $(x-1)(x+1)(x^2+1)(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$.

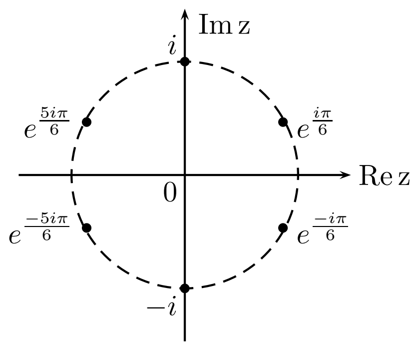
b) $(x^2+2)(x^2+\sqrt{6}x+2)(x^2-\sqrt{6}x+2)$.

70. $(z^2+2z+2)(z^2-2z+2)$

71. $(z - e^{-i\pi/8})\,(z - e^{i3\pi/8})\,(z - e^{i7\pi/8})\,(z - e^{-i5\pi/8})$

72. a) $e^{-\frac{5\pi i}{6}}$, $e^{-\frac{\pi i}{2}}$, $e^{-\frac{\pi i}{6}}$, $e^{\frac{\pi i}{6}}$, $e^{\frac{\pi i}{2}}$, $e^{\frac{5\pi i}{6}}$.

b) Note that the solutions are evenly spaced around the unit circle centred on 0.



c) $\left(z - e^{-\frac{5\pi i}{6}}\right)\left(z - e^{-\frac{\pi i}{2}}\right)\left(z - e^{-\frac{\pi i}{6}}\right)\left(z - e^{\frac{\pi i}{6}}\right)\left(z - e^{\frac{\pi i}{2}}\right)\left(z - e^{\frac{5\pi i}{6}}\right)$.

d) $\left(z^2+1\right)\left(z^2+\sqrt{3}\,z+1\right)\left(z^2-\sqrt{3}\,z+1\right)$.

73. a) $e^{i\pi/4}$, $e^{i\pi/2}$, $e^{i3\pi/4}$, $e^{-i\pi/4}$, $e^{-i\pi/2}$, $e^{-3\pi/4}$.

b) $\left(z-e^{i\pi/4}\right)\left(z-e^{-i\pi/4}\right)\left(z-e^{i3\pi/4}\right)\left(z-e^{-i3\pi/4}\right)\left(z-e^{i\pi/2}\right)\left(z-e^{-i\pi/2}\right)$

c) $\left(z^2-\sqrt{2}z+1\right)\left(z^2+\sqrt{2}z+1\right)\left(z^2+1\right)$.

74. a) $(z-e^{2i\pi/5})(z-e^{-2i\pi/5})(z-e^{4i\pi/5})(z-e^{-4i\pi/5})$

b) $\left(z^2-2z\cos\left(\frac{2\pi}{5}\right)+1\right)\left(z^2-2z\cos\left(\frac{4\pi}{5}\right)+1\right)$

75. a) $(t+1-i)\ (t+1+i)\ (t-2)\ (t+1)\ (t+i)\ (t-i)$,

b) $(t^2+2t+2)\ (t-2)\ (t+1)\ (t^2+1)$.

76. $1+i$, $1-i$, $\sqrt[3]{5}$, $\frac{\sqrt[3]{5}}{2}\left(-1+i\sqrt{3}\right)$, $\frac{\sqrt[3]{5}}{2}\left(-1-i\sqrt{3}\right)$.

77. a) $(z^2+z+1)(z^6+z^3+1)$. b) $e^{\pm 2i\pi/9}$, $e^{\pm 4i\pi/9}$, $e^{\pm 8i\pi/9}$.

79. a) One of the roots is $(-2+2i)^{1/3}+(-2-2i)^{1/3}$.

80. d) $-2,\ 2\sqrt{2}\cos\frac{\pi}{12},\ 2\sqrt{2}\cos\frac{7\pi}{12}$.

81. a) 1; b) $-1,\ \frac{5}{7},\ \frac{1}{4}$; c) $4,\ \pm\frac{1}{5}$.

90. `evalc((sqrt(2)+7*I)^13);`

Chapter 4

1. a) $\left\{\frac{5}{2}\right\}$, $\left\{\begin{pmatrix}\frac{5}{2}\\ \lambda\end{pmatrix} : \lambda\in\mathbb{R}\right\}$, $\left\{\begin{pmatrix}\frac{5}{2}\\ \lambda\\ \mu\end{pmatrix} : \lambda,\mu\in\mathbb{R}\right\}$

b) $\left\{\begin{pmatrix}4-2\lambda\\ \lambda\end{pmatrix} : \lambda\in\mathbb{R}\right\}$, $\left\{\begin{pmatrix}4-2\lambda\\ \lambda\\ \mu\end{pmatrix} : \lambda,\mu\in\mathbb{R}\right\}$

c) $\left\{\begin{pmatrix}\lambda\\ \mu\\ 2-2\lambda+3\mu\end{pmatrix} : \lambda,\mu\in\mathbb{R}\right\}$

2. a) No solution. b) Unique solution $\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}8\\ -9\end{pmatrix}$.

c) Infinite number of solutions on the line $\mathbf{x}=\begin{pmatrix}5\\ 0\end{pmatrix}+\lambda\begin{pmatrix}5\\ 1\end{pmatrix}$, $\lambda\in\mathbb{R}$.

3. For $a_{11} \neq 0$ the conditions are as follows.
    a) If $a_{11}a_{22} - a_{12}a_{21} \neq 0$, then solution is unique.
    b) If $a_{11}a_{22} - a_{12}a_{21} = 0$ and $a_{11}b_2 - a_{21}b_1 \neq 0$, then there is no solution.
    c) If $a_{11}a_{22} - a_{12}a_{21} = 0$ and $a_{11}b_2 - a_{21}b_1 = 0$, then there are an infinite number of solutions.

4. The general conditions are as follows.
    a) If $a_{11}a_{22} - a_{12}a_{21} \neq 0$, then solution is unique.
    b) There is no solution if $a_{11}a_{22} - a_{12}a_{21} = 0$ and either
        i) $a_{11}b_2 - a_{21}b_1 \neq 0$, or
        ii) $a_{12}b_2 - a_{22}b_1 \neq 0$, or
        iii) $a_{11} = a_{12} = a_{21} = a_{22} = 0$ and b_1, b_2 are not both zero.
    c) There are an infinite number of solutions otherwise.

5. a) Solution set $= \left\{ \begin{pmatrix} 1+\lambda \\ 2-2\lambda \\ \lambda \end{pmatrix} : \lambda \in \mathbb{R} \right\}$.

    Planes intersect in line $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \ \lambda \in \mathbb{R}$.

    b) No solution. Planes are parallel.

    c) Solution set $= \left\{ \begin{pmatrix} 4 - \frac{5}{4}\lambda + \frac{1}{2}\mu \\ \lambda \\ \mu \end{pmatrix} : \lambda,\ \mu \in \mathbb{R} \right\}$. Equations represent the same plane.

8. a) In vector form,

$$x_1 \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} 4 \\ -3 \\ 6 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 8 \end{pmatrix}.$$

    As a matrix equation and augmented matrix,

$$\begin{pmatrix} 3 & -3 & 4 \\ 5 & 2 & -3 \\ -1 & -1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 8 \end{pmatrix}; \qquad (A|\mathbf{b}) = \left(\begin{array}{rrr|r} 3 & -3 & 4 & 6 \\ 5 & 2 & -3 & 7 \\ -1 & -1 & 6 & 8 \end{array} \right).$$

    b) In vector form,

$$x_1 \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} + x_3 \begin{pmatrix} 7 \\ -5 \\ 6 \end{pmatrix} + x_4 \begin{pmatrix} 8 \\ -1 \\ -6 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ 5 \end{pmatrix}.$$

    As a matrix equation and augmented matrix,

$$\begin{pmatrix} 1 & 3 & 7 & 8 \\ 3 & 2 & -5 & -1 \\ 0 & 3 & 6 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ 5 \end{pmatrix}; \qquad (A|\mathbf{b}) = \left(\begin{array}{rrrr|r} 1 & 3 & 7 & 8 & -2 \\ 3 & 2 & -5 & -1 & 7 \\ 0 & 3 & 6 & -6 & 5 \end{array} \right).$$

9. The system of equation is

$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & & & = & 10 \\ & & 6x_2 & + & 6x_3 & = & -2 \\ -6x_1 & - & x_2 & - & 4x_3 & = & 0 \\ 7x_1 & + & 9x_2 & + & 11x_3 & = & 5 \end{array}$$

The augmented matrix form is

$$A = \left(\begin{array}{rrr|r} 1 & -3 & 0 & 10 \\ 0 & 6 & 6 & -2 \\ -6 & -1 & -4 & 0 \\ 7 & 9 & 11 & 5 \end{array}\right).$$

10. a) $R_2 = R_2 - 2R_1,\ R_3 = R_3 - 4R_1;$ b) $R_1 = R_1 - R_2,\ R_2 = \frac{1}{2}R_2.$

11. $\left(\begin{array}{rrr|r} 2 & 4 & 1 & 2 \\ 9 & 14 & 7 & 7 \\ 1 & 3 & 1 & 3 \end{array}\right).$

12. All but c) and h) are in row-echelon form.

13. a) $\mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$. Point of intersection of 3 planes.

b) $\mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}.$

A line in $\mathbb{R}^4$ through the point $(2, 3, -2, 0)$ and parallel to $\begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$.

14. a) $\mathbf{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. b) $\mathbf{x} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$ c) $\mathbf{x} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$.

d) No solution. e) $\mathbf{x} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$ f) No solution.

g) $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \end{pmatrix}$. h) $\mathbf{x} = \begin{pmatrix} -3 \\ 6 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ -1 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$

15. a) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right)$.

Solution: $\mathbf{x} = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$, which is the position vector of a point in $\mathbb{R}^3$.

b) $\left(\begin{array}{cccc|c} 1 & 0 & 0 & -75 & -34 \\ 0 & 1 & 0 & 29 & 13 \\ 0 & 0 & 1 & 7 & 3 \end{array}\right)$.

Solution: $\mathbf{x} = \begin{pmatrix} -34 \\ 13 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 75 \\ -29 \\ -7 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}$, which is a line in $\mathbb{R}^4$.

16. a) Unique solution, b) no solution, c) infinitely many solutions, d) infinitely many solutions, e) unique solution.

17. a) $k \neq 3$, b) no such value of k, c) $k = 3$.

18. a) $\lambda = \pm 2$, b) $\lambda = 1$, c) all other values of λ.

19. a) $a \neq 0$, b) $a = 0$, $b \neq 0$, c) $a = b = 0$, d) $\mathbf{x} = \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ \frac{5}{2} \\ -1 \\ 3 \end{pmatrix}$ $\lambda \in \mathbb{R}$.

20. Perhaps, if the costs are negative or very large then you can be sure that someone is cheating.

21. No.

22. a) $\begin{array}{rcrcrcr} x_1 & = & 7b_1 & + & 5b_2 & + & 3b_3 \\ x_2 & = & 6b_1 & + & 4b_2 & + & 3b_3 \\ x_3 & = & 2b_1 & + & b_2 & + & b_3 \end{array}$ b) $\begin{array}{rcrcrcr} x_1 & = & \frac{3}{2}b_1 & - & 2b_2 & - & 2b_3 \\ x_2 & = & -\frac{7}{2}b_1 & + & 5b_2 & + & 4b_3 \\ x_3 & = & \frac{1}{2}b_1 & - & b_2 & - & b_3 \end{array}$

24. a) $b_3 - \frac{1}{2}b_1 + b_2 = 0$. b) $b_1 - b_2 + b_3 = 0$ and $-2b_1 + b_2 + b_4 = 0$.

26. Yes.

27. No.

28. Yes, since $\begin{pmatrix} 1 \\ 1 \\ 4 \\ 12 \end{pmatrix} = 3 \begin{pmatrix} 3 \\ -1 \\ 4 \\ 6 \end{pmatrix} - 2 \begin{pmatrix} 4 \\ -2 \\ 4 \\ 3 \end{pmatrix}$.

29. No.

30. Yes, at $(6, 13, 11)$.

31. Yes, since $\begin{pmatrix} 5 \\ 7 \\ -1 \end{pmatrix} = 3 \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} - 4 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

33. Meet at $(6, 9, 4)$.

34. The planes intersect at the line $\mathbf{x} = \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 4 \\ 8 \end{pmatrix} \quad \lambda \in \mathbb{R}$.

35. Planes are not parallel as $\lambda_1 \begin{pmatrix} 2 \\ 1 \\ -2 \\ 7 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ 1 \\ 5 \\ 2 \end{pmatrix} = \mu_1 \begin{pmatrix} 3 \\ -1 \\ 2 \\ 4 \end{pmatrix} + \mu_2 \begin{pmatrix} -1 \\ 4 \\ 2 \\ 6 \end{pmatrix}$ only when $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 0$.

37. a) $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix}, \ \lambda \in \mathbb{R}$. b) The planes intersect in a line.

38. $p(x) = 2x^2 - 4x + 7$

39. I am 42, my brother is 46 and my sister is 52.

40. 6 days in Bangkok, 4 each in Singapore and Kuala Lumpur.

41. $3, 1, 2$.

42. a) Letting x_1 be the number of hectares of wheat, x_2 be the number of hectares of oats and x_3 be the number of hectares of barley gives the equations

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 12 \\ 6x_1 & + & 6x_2 & + & 2x_3 & = & 48 \\ 150x_1 & + & 100x_2 & + & 70x_3 & = & 700 \\ 72x_1 & + & 48x_2 & + & 36x_3 & = & 612 \end{array}$$

b) There is no solution.

c) The inequalities are

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & \leqslant & 12 \\ 6x_1 & + & 6x_2 & + & 2x_3 & \leqslant & 48 \\ 150x_1 & + & 100x_2 & + & 70x_3 & \leqslant & 700 \\ 72x_1 & + & 48x_2 & + & 36x_3 & \leqslant & 612 \end{array}$$

and with slack variables s_1, s_2, s_3, s_4 the equations are

$$\begin{array}{rcrcrcccccccccl} x_1 & + & x_2 & + & x_3 & + & s_1 & & & & & & & = & 12 \\ 6x_1 & + & 6x_2 & + & 2x_3 & & & + & s_2 & & & & & = & 48 \\ 150x_1 & + & 100x_2 & + & 70x_3 & & & & & + & s_3 & & & = & 700 \\ 72x_1 & + & 48x_2 & + & 36x_3 & & & & & & & + & s_4 & = & 612 \end{array}$$

d) Some sensible solutions are to either plant $4\frac{2}{3}$ hectares wheat and no oats and barley, or 7 hectares oats and no wheat and barley, or 10 hectares barley and no wheat and oats. There are also an infinite number of other reasonable solutions. In each case it is the fertiliser which is restricting the planting.

44. a) Π_1 is $x + 2y - z = 2$, $\quad \Pi_2$ is $3x + 6y - z = 12$, $\quad \Pi_3$ is $2x + 4y - z = 7$.

b) $\mathbf{x} = \begin{pmatrix} 5 - 2t_2 \\ t_2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix} + t_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \quad t_2 \in \mathbb{R}$

The intersection is a line through $(5, 0, 3)$ and parallel to $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$.

c) $x = -2y + 5$ and $z = 3$.

45. a) $\begin{cases} x - 2y + z = a \\ 3x + 6y + 8z = b \\ 4x + 2y + 7z = c \\ 7x - 8y + 6z = d \end{cases}$. b) $d - a + 2b - 3c = 0$. c) $\left(\frac{4}{7}, -\frac{1}{7}, \frac{1}{7} \right)$.

Chapter 5

1. a) $3A = \begin{pmatrix} 6 & -9 & 12 \\ 9 & 6 & -6 \\ 3 & -3 & 9 \end{pmatrix}$.

b) $-2B = \begin{pmatrix} 4 & -2 \\ -6 & -8 \\ 2 & -10 \end{pmatrix}$.

c) $A + B$ is not defined.

d) $B + C = \begin{pmatrix} -5 & 3 \\ 4 & 0 \\ 5 & 7 \end{pmatrix}$.

e) $A + 3I = \begin{pmatrix} 5 & -3 & 4 \\ 3 & 5 & -2 \\ 1 & -1 & 6 \end{pmatrix}$.

f) $B + 3I$ is not defined.

g) $AB = \begin{pmatrix} -17 & 10 \\ 2 & 1 \\ -8 & 12 \end{pmatrix}$.

h) BA is not defined.

i) BC is not defined.

j) $CD = \begin{pmatrix} -4 & -13 & -9 \\ -2 & 11 & 13 \\ 14 & 14 & 0 \end{pmatrix}$.

k) $A^2 = \begin{pmatrix} -1 & -16 & 26 \\ 10 & -3 & 2 \\ 2 & -8 & 15 \end{pmatrix}$.

l) B^2 is not defined.

m) $(BD)^2 = \begin{pmatrix} -86 & 81 & 167 \\ -47 & 38 & 85 \\ -187 & 171 & 358 \end{pmatrix}$.

7. $96A + 205I$.

8. $N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad N^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

11. a) $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, b) $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$, c) $\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$, d) $\begin{pmatrix} 1 & -2 & 3 \end{pmatrix}$.

13. $A^T = \begin{pmatrix} 1 & -3 & 4 \\ -2 & 0 & 5 \end{pmatrix}, \quad B^T = \begin{pmatrix} 2 & -4 & 5 \\ -5 & 6 & 0 \\ 4 & 5 & 8 \\ 3 & 5 & 6 \end{pmatrix}, \quad C^T = \begin{pmatrix} 1 & 4 & 2 \\ 4 & -3 & 6 \\ 2 & 6 & 7 \end{pmatrix} = C.$

14. $\mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{a} = 8$, $\mathbf{a}\mathbf{b}^T = \begin{pmatrix} 0 & 4 & 2 \\ 0 & 12 & 6 \\ 0 & -8 & -4 \end{pmatrix}$, $\mathbf{b}\mathbf{a}^T = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 12 & -8 \\ 2 & 6 & -4 \end{pmatrix}$, $\mathbf{ab}$ and $\mathbf{a}^T\mathbf{b}^T$ are not defined.

17. A possible $G = \begin{pmatrix} 3 & 6 \\ -4 & 2 \end{pmatrix}$.

19. a) $\begin{pmatrix} 4 & -7 \\ -1 & 2 \end{pmatrix}$, b) $\begin{pmatrix} 5 & 7 \\ 3 & 4 \end{pmatrix}$, c) no inverse, d) $\dfrac{1}{5}\begin{pmatrix} 4 & -9 \\ -3 & 8 \end{pmatrix}$, e) $\begin{pmatrix} -7 & 1 \\ 1 & 0 \end{pmatrix}$.

20. $A^{-1} = \begin{pmatrix} 1 & 3 & -4 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 8 & -2 & -3 \\ \frac{1}{2} & 0 & 0 \\ -3 & 1 & 1 \end{pmatrix}$, C is not invertible,

$D^{-1} = \dfrac{1}{4}\begin{pmatrix} 1 & 1 & 1 \\ 5 & -3 & 1 \\ -17 & 11 & -5 \end{pmatrix}$.

21. a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{6} \end{pmatrix}$ b) $\begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}$

22. $A^{-1} = \begin{pmatrix} 4 & -3 & -2 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & -2 & -2 & 1 \\ 0 & 1 & 2 & -1 \end{pmatrix}$; $B^{-1} = \begin{pmatrix} 6 & -2 & 1 & 0 \\ 9 & -4 & 3 & -1 \\ 25 & -11 & 8 & -2 \\ -14 & 6 & -4 & 1 \end{pmatrix}$;

C^{-1} does not exist.

23. a) $\left(B^{-1}\right)^2$, b) AB^6A^{-1}, c) $(A + A^{-1})^2$, d) $I - (I - A)^{m+1}$.

24. a) $A^{-1}B$. b) $\begin{pmatrix} 2 & 4 & 4 \\ 1 & -2 & 3 \\ 1 & 0 & 3 \end{pmatrix}$.

25. b) i) B^TB, ii) $C^{-1}C^T$.

26. a) $\begin{pmatrix} -2 & 0 & 1 \\ 2 & 1 & -1 \\ 5 & 1 & -2 \end{pmatrix}$. b) $\begin{pmatrix} -2c_1 + c_3 \\ 2c_1 + c_2 - c_3 \\ 5c_1 + c_2 - 2c_3 \end{pmatrix}$.

27. $\mathbf{x} = Q^T\mathbf{b}$.

29. e.g. $\begin{pmatrix} \frac{1}{\sqrt{2}}i & -\frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}}i \end{pmatrix}$.

33. From Question 29, Q is invertible, and hence $Q\mathbf{x} = \mathbf{b}$ has the solution $\mathbf{x} = Q^{-1}\mathbf{b} = \overline{Q}^T\mathbf{b}$.

34. a) $\dfrac{1}{ab}\begin{pmatrix} b & 0 \\ -c & a \end{pmatrix}$. b) $\begin{pmatrix} A^{-1} & 0 \\ -B^{-1}CA^{-1} & B^{-1} \end{pmatrix}$.

35. a) 1, b) -1, c) 0, d) 5, e) -2. All are invertible except $\begin{pmatrix} 5 & 2 \\ 10 & 4 \end{pmatrix}$.

36. a) -9, b) 0, c) 56.

37. -126.

38. a) -30, b) 5, c) 5, d) $5 \times 7^3 = 1715$.

39. a) -2, b) $-\dfrac{1}{2}$, c) -32.

40. -83, -108, 8964.

41. $a \neq 1$.

42. a) $\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 5 & 1 & -3 \end{pmatrix}$. b) $\begin{pmatrix} -2 & -1 & 1 \\ 1 & 2 & -1 \\ -3 & -1 & 1 \end{pmatrix}$. c) 1.

45. a) $(\alpha - 3)\ (\alpha + 1)\ (\alpha + 2)$. b) -1, -2, 3.

46. For example, $A = B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

47. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\lambda = -1$.

54. $2(x + y + z)^3$.

55. $(z - 1)(z^2 + 2z - 4)$, $x = -1, y = 1 \pm \sqrt{5}, z = -1 \pm \sqrt{5}$.

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