School of Mathematics and Statistics
Math1131-Algebra

Lec18. Matrices: Transposes and Inverses

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Matrix multiplication to extract rows or columns

Exercise 1. Let

$$B = \begin{pmatrix} 3 & 1 & -5 & 4 \\ 8 & 7 & 2 & 2 \\ 9 & -1 & -2 & -7 \end{pmatrix}.$$

a) Find a (column) vector $\overrightarrow{\boldsymbol{u}}$ such that $B\overrightarrow{\boldsymbol{u}}$ is the third column of B.

b) Find a (row) vector $\overrightarrow{\boldsymbol{v}}$ such that $\overrightarrow{\boldsymbol{v}}B$ is the second row of B.

c) Find a vector $\overrightarrow{\boldsymbol{w}}$ such that $B\overrightarrow{\boldsymbol{w}}$ is 2 times the first column of B plus 5 times the third column of B.

Matrix multiplication to extract rows or columns

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c) $\begin{pmatrix} 3 & 1 & -5 & 4 \\ 8 & 7 & 2 & 2 \\ 9 & -1 & -2 & -7 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 5 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 8 \\ 9 \end{pmatrix} + 5\begin{pmatrix} -5 \\ 2 \\ -2 \end{pmatrix}$

$\begin{pmatrix} 2 \\ 0 \\ 5 \\ 0 \end{pmatrix} = \underset{\sim}{w}$

$= \begin{pmatrix} 2\times 3 + 5\times(-5) \\ \\ \end{pmatrix}$

$\underset{\sim}{w} = 2\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

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$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \underset{\sim}{u}$

b) $\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 & -5 & 4 \\ 8 & 7 & 2 & 2 \\ 9 & -1 & -2 & -7 \end{pmatrix}$

$= \begin{pmatrix} 8 & 7 & 2 & 2 \end{pmatrix}$

UNSW SYDNEY

Transpose of a matrix

Transpose of a matrix.

For any $m \times n$ matrix A, its transpose A^T is the $n \times m$ matrix whose columns are the rows of A.
That is,

$$[A^T]_{ij} = [A]_{ji}$$

Example. For example, if

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 4 & 7 & 8 \end{pmatrix}$$

then

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$$(A^T)_{13} = (A)_{31}$$

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then

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and

$$B = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \iff B^T = \begin{pmatrix} 3 & 6 & 9 \end{pmatrix}$$

Transpose of a matrix

Exercise 2. Let $A = \begin{pmatrix} 3 & -5 \\ -7 & 8 \\ 0 & -1 \end{pmatrix}$

a) A is a 3×2 matrix and its transpose is a 2×3 matrix.

b) The transpose of A is denoted A^T which we read "A *transpose*".

c) The transpose of A is:

$$\begin{pmatrix} 3 & -7 & 0 \\ -5 & 8 & -1 \end{pmatrix}$$

UNSW SYDNEY

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Transpose of a product of matrices.

- If A and B are matrices for which the *sum* $A + B$ is defined then

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- If A and B are matrices for which the *product* AB is defined then
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TIP!

Note that when we take the transpose of a product, the matrices are swapped. (Nothing special happens when we take the transpose of a sum).

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- If A and B are matrices for which the *product* AB is defined then
$$(AB)^T = B^T A^T. \quad (*)$$

TIP!

Note that when we take the transpose of a product, the matrices are swapped. (Nothing special happens when we take the transpose of a sum).

Exercise 3. Let A, B and C are matrices for which the product ABC is defined. Conjecture (= guess) what $(ABC)^T$ is, and then prove your conjecture.

- conjecture: $(ABC)^T = C^T B^T A^T$
- $(ABC)^T = ((AB)C)^T \overset{(*)}{=} C^T (AB)^T \overset{(*)}{=} C^T (B^T A^T)$
$= C^T B^T A^T$

(For fast students) Could you extend this result to a product of n matrices?

Symmetric matrices

The transpose of the transpose of a matrix is the original matrix.

For any matrix A, we have $(A^T)^T = A$.

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Taking the transpose of a matrix is an involution.
Complex conjugation is another example of an involution.

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Definition (Symmetric matrix).

An $n \times n$ matrix A is said to be *symmetric* *if and only if* $A = A^T$.

For example, $B = \begin{pmatrix} 1 & 6 & 8 \\ 6 & 5 & 7 \\ 8 & 7 & 2 \end{pmatrix}$ is symmetric.

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Note that a symmetric matrix must necessarily be square (necessary condition).
Symmetric matrices have remarkable properties which are studied in second year (MATH2501).

Symmetric matrices

Exercise 4. Let $A = \begin{pmatrix} 3 & -5 & \dots \\ \dots & 8 & \dots \\ 0 & \dots & \dots \end{pmatrix}$

Write values of the missing entries which make A symmetric.

These matrices are called symmetric because ....

Exercise 5. Show that $C = A^T A$ is symmetric for any matrix A.

Symmetric matrices

Exercise 4. Let $A = \begin{pmatrix} 3 & -5 & 0 \\ -5 & 8 & \dots \\ 0 & \dots & \dots \end{pmatrix}$

Write values of the missing entries which make A symmetric.

These matrices are called symmetric because .... they are unchange under reflection in the diagonal

Exercise 5. Show that $C = A^T A$ is symmetric for any matrix A.

We need to show $C^T = C$

$$C^T = (A^T A)^T$$
$$= A^T (A^T)^T$$
$$= A^T A$$
$$= C$$

So C is symmetric

suppose A is $m \times n$

$$\underbrace{\underset{n\times m}{A^T}\ \underset{m\times n}{A}}_{n\times n}$$

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Symmetric matrices

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These matrices are called symmetric because ....

Exercise 5. Show that $C = A^T A$ is symmetric for any matrix A.

$$C^T = (A^T A)^T = A^T (A^T)^T = A^T A = C.$$

Transpose and dot product

Dot product of two vectors written as a matrix product.

If $\vec{u}$ and $\vec{v}$ are column vectors, we can consider them as $n \times 1$ matrices.
The *dot product* of $\vec{u}$ and $\vec{v}$ is equal to the *matrix product* of the transpose of $\vec{u}$, which is a row vector, and the column vector $\vec{v}$:

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}.$$

Example.
If

$$\vec{u} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \vec{v} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

then

$$\vec{u} \cdot \vec{v} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = 3 \times 2 + 1 \times 0 + (-1) \times 1 = 5.$$

and

$$\vec{u}^T \vec{v} = \begin{pmatrix} 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = 3 \times 2 + 1 \times 0 + (-1) \times 1 = 5.$$

Inverses

Consider the equation

$$ax = b$$

for real numbers a, b, x with $a \neq 0$. How do we solve this for x?

Inverses

$$2x = 6$$

$$\iff \frac{1}{2} \cdot 2x = \frac{1}{2} \cdot 6$$

$$\iff 1 \cdot x = 3$$

$$\iff x = 3$$

Consider the equation

$$ax = b$$

for real numbers a, b, x with $a \neq 0$. How do we solve this for x?

We multiply both sides by a^{-1} to give

$$a^{-1}ax = a^{-1}b$$

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$$x = \frac{b}{a}.$$

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Could we do the same for matrices?

Inverses

The matrices that acts like the number 1 are the identity matrices I.

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Suppose A, B and X are matrices and we want to solve the matrix equation

$$AX = B.$$

Inverses

$a^{-1}a = 1$

The matrices that acts like the number 1 are the identity matrices I.

Suppose A, B and X are matrices and we want to solve the matrix equation

$$AX = B.$$

If we could find a matrix A^{-1} with the property that

$$A^{-1}A = I$$

then we could multiply both sides of the equation on the left by A^{-1} to give

$$\begin{aligned} A^{-1}AX &= A^{-1}B \\ \implies \qquad IX &= A^{-1}B \\ \implies \qquad X &= A^{-1}B. \end{aligned}$$

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But can we find a suitable A^{-1}?

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But can we find a suitable A^{-1}?

Sadly enough, not always. Some matrices have an inverse ... and others just do not! For instance if a matrix is not square (i.e. its number of rows is different from its number of columns), it cannot have an inverse.

Inverse of a matrix

Example 6. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -1 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$.

a) Find AB and BA.

b) Hence solve the matrix equation $\begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix} X = \begin{pmatrix} 5 & 7 & -1 \\ 19 & 21 & -5 \end{pmatrix}$.

C

$A \quad X = C$

a) $AB = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 4-3 & -1+1 \\ 12-12 & -3+4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$

$BA = I_2$

b) $AX = C$

Multiply both sides on the left by B

$BAX = BC$

$I_2 X = BC$

$\boxed{X = BC}$

$AX = C$

$AXB = CB$

?

stuck.

Inverse of a matrix

Example 6, continued. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -1 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$. $= A^{-1}$

a) Find AB and BA.

b) Hence solve the matrix equation $\begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix} X = \begin{pmatrix} 5 & 7 & -1 \\ 19 & 21 & -5 \end{pmatrix}$.

$$\boxed{AX = C}$$

$X = BC$

$$BC = \begin{pmatrix} 4 & -1 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 5 & 7 & -1 \\ 19 & 21 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 28-21 & -4+5 \\ -\frac{15}{2}+\frac{19}{2} & -\frac{21}{2}+\frac{21}{2} & \frac{3}{2}-\frac{5}{2} \end{pmatrix}$$

$$\boxed{X = BC = \begin{pmatrix} 1 & 7 & 1 \\ 2 & 0 & -1 \end{pmatrix}}$$

Checking our answers with Maple

```
> with(LinearAlgebra):
> # Enter the matrices column by column

  A := < <1,3>|<2,8> >;
  B := < <4,-3/2>|<-1,1/2> >;
```

$$A := \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$$

$$B := \begin{bmatrix} 4 & -1 \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

```
> # For matrix multiplication, use . not *

  AB := A.B;
  BA := B.A;
```

$$AB := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
> # We solve the equation

  C := < <5,19>|<7,21>|<-1,-5> >;
  X := B.C;
```

$$C := \begin{bmatrix} 5 & 7 & -1 \\ 19 & 21 & -5 \end{bmatrix}$$

$$X := \begin{bmatrix} 1 & 7 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

```
> # We check our answer
  A.X;
```

$$\begin{bmatrix} 5 & 7 & -1 \\ 19 & 21 & -5 \end{bmatrix}$$

$AX = C$

Inverse of a matrix

Definitions and properties (Inverse of a matrix).

- For a matrix A,
  - B is a *left inverse* of A means $BA = I$.
  - B is a *right inverse* of A means $AB = I$.
  - If B is both a left and right inverse of A then B is an *inverse* of A.
- If A has an inverse then that inverse is unique and is denoted A^{-1} (say "A inverse"). In that case, we say that A is *invertible* or *non-singular*.
- Conditions for matrices to have an inverse :
  - Only *square* matrices can have an inverse (necessary condition).
  - If a square matrix has a left *or* right inverse then this inverse works on both sides so the matrix is invertible (sufficient condition).
- Inverse of a product :
  - If A and B are invertible matrices for which the *product* AB is defined then

$$(AB)^{-1} = B^{-1}A^{-1}.$$

TIP! So just like with *transpose*, when we take the *inverse* of a *product*, the matrices are *swapped*

Note that no rule exists for $(A+B)^{-1}$.

Inverse of a matrix

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$$
$$= AIA^{-1}$$
$$= A\,A^{-1}$$
$$= I$$

$$(B^{-1}A^{-1})(AB)$$

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Note that no rule exists for $(A+B)^{-1}$.

Inverse of a 2×2 matrix

For 2×2 matrices, we have a formula for the inverse.

- The 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible *if and only if* $ad - bc \neq 0$.
- and in that case its inverse is : $A^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

  You should remember this!
- Note that if $ad - bc = 0$ then this matrix has no inverse and we say it is *not* invertible or singular.

Example 7. Use this to find $\begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}^{-1}$.

Exercise 8. PROOF Check that the formula given above works.

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$$\begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}^{-1} = \frac{1}{1 \times 8 - 2 \times 3} \begin{pmatrix} 8 & -2 \\ -3 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 8 & -2 \\ -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -1 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

Exercise 8. PROOF Check that the formula given above works.

$$A^{-1}A = \frac{1}{ad-bc} \begin{pmatrix} d & b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \frac{1}{ad-bc} \begin{pmatrix} ad - bc & \cancel{db - bd}\ 0 \\ \cancel{-ca + ac}\ 0 & -cb + ad \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Determinant and inverse of a 2×2 matrix

Determinant and inverse of a 2×2 matrix.

- For the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the *number* $ad - bc$ (given by the "gamma rule") is called the *determinant* of A and is denoted $\det(A)$ or $|A|$.
- A^{-1} exists *if and only if* $\det(A) \neq 0$.
- If $det(A) \neq 0$, $A^{-1} = \dfrac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \dfrac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Example 9. For each of the following, determine if they are invertible, and if so, find their inverse.

$$A = \begin{pmatrix} 3 & 9 \\ 2 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} -3 & -1 \\ 2 & 6 \end{pmatrix}$$

How can you check that your inverse is correct?

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How can you check that your inverse is correct?

$\det(A) = 3 \times 6 - 2 \times 9 = 0$

A^{-1} does not exist

A is not invertible

$\det(B) = -3 \times 6 - 2 \times (-1)$
$= -16$

$B^{-1} = \dfrac{1}{-16} \begin{pmatrix} 6 & 1 \\ -2 & -3 \end{pmatrix}$

Checking our answers with Maple

```
> with(LinearAlgebra):
> # Enter the matrices column by column

  B := < <-3,2>|<-1,6> >;
```

$$B := \begin{bmatrix} -3 & -1 \\ 2 & 6 \end{bmatrix}$$

```
> # We calculate the determinant of B

  Determinant(B);
```

$$-16$$

```
> # We find the inverse of B

  B^(-1);
  -16*B^(-1);
```

$$\begin{bmatrix} -\frac{3}{8} & -\frac{1}{16} \\ \frac{1}{8} & \frac{3}{16} \end{bmatrix}$$

$$\begin{bmatrix} 6 & 1 \\ -2 & -3 \end{bmatrix}$$

Inverses and transposes

Exercise 10 [Two important results]. When the given inverses exist, prove the following

1. $(AB)^{-1} = B^{-1}A^{-1}$ — *Note that like for transpose, the matrices are swapped!*
2. $(A^{-1})^T = (A^T)^{-1}$. — *You can take the inverse and the transpose in any order. This will be useful and is worth remembering!*

① done

② $(A^{-1})^T A^T = (A A^{-1})^T$

$= (I)^T = I$

$\underbrace{(A^{-1})^T}_{} A^T = I$

so $(A^{-1})^T = (A^T)^{-1}$

$(AB)^T = B^T A^T$

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Inverses and transposes

Example 11. Assuming all of the relevant inverses exist, simplify

$$M = B^2(AB)^{-1}A(B^{-1})^T(AB)^T$$

$$= B\,\boxed{B\,B^{-1}}\,\boxed{A^{-1}A}\,\boxed{(B^T)^{-1}\,B^T}\,A^T$$

$$= B\;I\;I\;I\;A^T$$

$$= BA^T$$

Inverses and transposes

Example 11. Assuming all of the relevant inverses exist, simplify

$$M = B^2(AB)^{-1}A(B^{-1})^T(AB)^T$$

Examples

$$\underset{m\times n}{A}\ \underset{m\times n}{A}$$

Exercise 12.

Given that $A^2 = 2A + 5I$, express A^4 and A^{-1} as linear combinations of A and I.

$$A^2 = 2A + 5I$$

$\times A$

$$\begin{aligned}
A^3 &= 2A^2 + 5A \\
&= 2(2A + 5I) + 5A \\
&= 4A + 10I + 5A \\
A^3 &= 9A + 10I
\end{aligned}$$

$\times A$

$$\begin{aligned}
A^4 &= 9A^2 + 10A \\
&= 9(2A + 5I) + 10A
\end{aligned}$$

$$\boxed{A^4 = 28A + 45I}$$

$$\begin{aligned}
A^2 &= 2A + 5I \\
A^2 - 2A &= 5I \\
A(A - 2I) &= 5I
\end{aligned}$$

$\times \frac{1}{5}$

$$\begin{aligned}
\frac{1}{5}A(A - 2I) &= I \\
A\left[\frac{1}{5}(A - 2I)\right] &= I
\end{aligned}$$

$$\boxed{A^{-1} = \frac{1}{5}(A - 2I)}$$

$$A \boxed{} = I$$

A^{-1}