THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 2 2018

MATH1131
MATHEMATICS 1A

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS IS APPENDED TO THE PAPER

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 1

1. i) Find the following limits giving brief reasons for each.

a) $\lim_{x\to 0^+} x^2 - x - 6$

b) $\lim_{x\to 3} \dfrac{x^2 - x - 6}{x - 3}$

c) $\lim_{x\to \infty} \dfrac{x^2 - x - 6}{\sqrt{3x^4 + 7}}$

d) $\lim_{x\to 0} \dfrac{x(x^2 - x - 6)}{\sin x}$

ii) Find all real numbers a and b (if any) such that the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} x^2 - x - 6 & x \geq 0 \\ ax + b & x < 0. \end{cases}$$

is differentiable everywhere.

iii) Does $(x^2 - x - 6)\sin(x)$ attain a minimum for $x \in [510, 521]$? Give reasons for your answer.

iv) Find the critical points of

$$h(x) = |x^2 - x - 6|$$

on the interval $I = [0, 4]$. Deduce the maximum and minimum values of the function h over I.

v) Using the Maple session below or otherwise, find the area between the curves $y = (x^2 - x - 6)e^x$ and $y = 0$ for x between 0 and 4.

```
> f := x^2 - x - 6;
```

$$f := x^2 - x - 6$$

```
> int(f*exp(x), x=0..3);
```

$$3 - 3e^3$$

```
> int(f*exp(x), x=3..4);
```

$$3e^3 + e^4$$

vi) Let $p : \mathbb{R} \to (0, \infty)$ and $q : \mathbb{R} \to (0, \infty)$ be differentiable and $y = \dfrac{p(x)}{q(x)}$. By differentiating $\ln y$, prove the quotient rule for differentiation.

vii) Given that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, $f(1) = 5$ and $\displaystyle\int_0^1 f(x)\,dx = 2$, find $\displaystyle\int_0^1 xf'(x)\,dx$.

Please see over ...

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 2

2. i) Let

$$y = \int_{x^2}^{1} \sqrt{1 + \cos t}\, dt.$$

Find $\dfrac{dy}{dx}$.

ii) a) State the Mean Value Theorem carefully, including all necessary hypotheses.

b) By considering the function $f(t) = \tan^{-1} t$ on an appropriate interval, use the Mean Value Theorem to show that

$$\frac{x}{1 + x^2} < \tan^{-1} x < x$$

for any positive number $x > 0$.

iii) Determine which, if any, of the following improper integrals converge. Give reasons for your answers.

a) $\displaystyle\int_1^{\infty} \frac{\sin(x^3) + x^2}{x^4 + x^2}\, dx,$

b) $\displaystyle\int_2^{\infty} \frac{1}{(\ln x)^3}\, dx.$

[You may use the fact, without proof, that $(\ln x)^3 < x$ for $x > 100$.]

iv) a) State the definition of $\sinh x$ in terms of the exponential function.

b) Use this to prove the identity

$$4 \sinh^3 x = \sinh 3x - 3 \sinh x.$$

v) Consider the function $f(x) = x^2 + \sin x$ defined on the interval $(1, \infty)$.

a) Show that f is an increasing function.

b) What is the domain of $g = f^{-1}$?

c) By considering $f(2\pi)$, find $g(4\pi^2)$ and $g'(4\pi^2)$.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 3

3. i) Suppose that $z = 1 + i$ and $w = \sqrt{3} + i$.

a) Find zw in Cartesian form.

b) Show that $\text{Arg}(zw) = \dfrac{5\pi}{12}$.

c) Hence show that

$$\cos\left(\frac{5\pi}{12}\right) = \frac{-1+\sqrt{3}}{2\sqrt{2}}.$$

ii) Consider the vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} \text{ and } \mathbf{c} = \begin{pmatrix} -1 \\ 0 \\ 6 \end{pmatrix}.$$

a) Calculate $\mathbf{a} \times \mathbf{b}$.

b) Hence or otherwise calculate $(4\mathbf{b}) \times (2\mathbf{a})$.

c) Show that $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.

d) Is $\mathbf{c}$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$? Give reasons for your answer.

iii) A plane Π passing through the point $P(1, 2, 3)$ is perpendicular to the line

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}.$$

a) Find a Cartesian equation of the plane Π.

b) Find a parametric vector form for the plane.

c) Find the intersection of the line and the plane.

d) Find the shortest distance from the point $(1, 0, 1)$ to the plane.

iv) Use the following Maple output to assist you in answering the questions below.

```
> with(LinearAlgebra):
> m:=<<1,-1,p>|<2,p,-4>|<p,-1,p>>;
```

$$m := \begin{bmatrix} 1 & 2 & p \\ -1 & p & -1 \\ p & -4 & p \end{bmatrix}$$

```
> t:=<1,0,-1>;
```

$$t := \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

```
> <m|t>;
```

$$\begin{bmatrix} 1 & 2 & p & 1 \\ -1 & p & -1 & 0 \\ p & -4 & p & -1 \end{bmatrix}$$

```
> G:=GaussianElimination(%);
```

$$G := \begin{bmatrix} 1 & 2 & p & 1 \\ 0 & p+2 & -1+p & 1 \\ 0 & 0 & 3\,p-p^2-2 & 1-p \end{bmatrix}$$

Consider the following system of linear equations:

$$\begin{array}{rcrcrcr} x & + & 2y & + & pz & = & 1 \\ -x & + & py & - & z & = & 0 \\ px & - & 4y & + & pz & = & -1 \end{array}$$

a) How many solutions does this system have when $p = 2$? Give reasons.

b) How many solutions does this system have when $p = 1$? Give reasons.

c) Discuss the case where $p = -2$.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 4

4. i) Let A and B be 2×2 matrices.

a) Use a counterexample to show that $\det(A+B)$ does not equal $\det(A)+\det(B)$ in general.

b) Use the fact that $\det(AB) = \det(A)\det(B)$ to prove that if A is an invertible matrix then $\det(A^{-1}) = \det(A)^{-1}$.

ii) Consider the following maple output.

```
> p := z^5-7*z^4+21*z^3-33*z^2+28*z-10;
```

$$p(z) = z^5 - 7z^4 + 21z^3 - 33z^2 + 28z - 10$$

```
> subs(z=1+I, p);
```

$$0$$

```
> subs(z=2-I, p);
```

$$0$$

a) Show that 1 is a root of the polynomial

$$p(z) = z^5 - 7z^4 + 21z^3 - 33z^2 + 28z - 10.$$

b) Use the maple output to factorise $p(z)$ into complex linear factors.

c) Express $p(z)$ as a product of real linear and quadratic factors.

iii) Consider the matrices

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & -4 \\ 0 & 10 \\ 1 & -6 \end{pmatrix}.$$

Here C is a 2×3 matrix and D is a 3×2 matrix.

a) Calculate CD.

b) Hence find the 2×3 matrix Y which solves the matrix equation

$$DY = \begin{pmatrix} -3 & 6 & -5 \\ 10 & -10 & 10 \\ -5 & 8 & -7 \end{pmatrix}.$$

iv) Consider the following system of equations:

$$\begin{array}{rrrl} x & +\,4y & -\,z & = -1 \\ 2x & +\,4y & & = \;\;4 \\ x & -\,4y & +\,3z & = \;\;11 \end{array}$$

a) Write the system in augmented matrix form and reduce it to row echelon form.

b) Solve the system, expressing your solution in vector form.

c) If x, y and z solve the above system and are all greater than or equal to zero, what is the largest possible value of z?

v) Consider the fixed points $A(1,0,0)$ and $B(-1,0,0)$ and variable point X with coordinates (x,y,z) in $\mathbb{R}^3$. Show that X lies on the unit sphere centred at the origin if and only if the triangle AXB is right angled at X.

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BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln|k|$$
$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$
$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$
$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$
$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$
$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$
$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$
$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$
$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$
$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$
$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$
$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$
$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$
$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$
$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$
$$\begin{aligned}\int \frac{dx}{a^2-x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, && |x| < a\\ &= \frac{1}{a}\coth^{-1}\frac{x}{a} + C, && |x| > a > 0\\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, && x^2 \neq a^2\end{aligned}$$
$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$
$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + C$$
$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$

END OF EXAMINATION