



School of Mathematics and Statistics
Math1131-Algebra

Lec13: Fundamental Theorem of Algebra

Laure Helme-Guizon (Dr H)

Laure@unsw.edu.au

Jonathan Kress

j.kress@unsw.edu.au

Red-Centre, Rooms 3090 and 3073

2020 Term 1

Complex polynomials



Complex polynomials = Polynomials with coefficients in \mathbb{C}

A *complex polynomial* $p(z)$ is a complex valued function of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

where $a_0, \dots, a_n \in \mathbb{C}$ with $a_n \neq 0$ are the *coefficients* of $p(z)$.

- If the coefficients a_0, \dots, a_n are real numbers, then $p(z)$ is a *real polynomial*.
- $a_n z^n$ is the *leading term*; a_n is the *leading coefficient*. If $a_n = 1$ then $p(z)$ is *monic*.
- The *degree* of $p(z)$ is n .
- The degree of the zero polynomial $p(z) = 0$ is undefined.



Exercise 1. Which of the following functions are polynomials? “✓” or “✗”

a) $f(z) = z^3 + z + 1$

d) $f(z) = \sin z$

g) $f(z) = 2 + 3i$

b) $f(z) = z^5 + 6z^{-2} + 3z$

e) $f(z) = i$

h) $f(z) = 2z^3 - iz^2 + 4$

c) $f(z) = i\sqrt{3}z$

f) $f(z) = e^z$

i) $f(z) = \frac{z+1}{z-1}$

Roots and factors



Roots and factors

- If $p(\alpha) = 0$ then α is called a *root* of $p(z)$.
- If $p(z) = q(z)g(z)$ then $q(z)$ and $g(z)$ are *factors* of $p(z)$.
- A factor of degree 1 is called a *linear* factor and a factor of degree 2 is called a *quadratic* factor.

Example 2. $p(z) = z^2 + 1$ has roots $\pm i$ because

$$\begin{aligned}p(i) &= i^2 + 1 = -1 + 1 = 0 \\p(-i) &= (-i)^2 + 1 = -1 + 1 = 0.\end{aligned}$$

Example 3. $p(z) = z^2 - 1 = (z + 1)(z - 1)$ has linear factors $z + 1$ and $z - 1$.



Exercise 4. $p(z) = z^2 + 3 = \dots\dots\dots$

Therefore, its linear factors are $\dots\dots\dots$ and $\dots\dots\dots$

Long Division and the Remainder Theorem



Long Division for polynomials.

Given polynomials $a(x)$, $b(x)$, we can find polynomials $q(x)$ and $r(x)$ such that

$$a(x) = q(x)b(x) + r(x)$$

where the degree of the remainder $r(x)$ is less than the degree of the divisor $b(x)$.
(We are dividing a by b , so b is the *divisor*, q is the *quotient* and r is the *remainder*).



The remainder theorem

If the polynomial $p(z)$ is divided by the linear factor $z - \alpha$, then the remainder is simply the number $r = p(\alpha)$.

PROOF

Write $p(z) = q(z)(z - \alpha) + r(z)$.

The degree of $r(z)$ must be smaller than the degree of $z - \alpha$ which is 1, and so $r(z)$ must be constant.

Therefore, $r(z) = r(\alpha) = p(\alpha) - q(\alpha)(\alpha - \alpha) = p(\alpha)$.

The Remainder Theorem



Dividing $p(z)$ by $z - \alpha$ gives the remainder $r = p(\alpha)$.

Exercise 5. Find the remainder when $z^3 - 6z^2 + 11z - 7$ is divided by $z - 4$.

Do this by both long division and using the Remainder Theorem.



Checking some of our answers with Maple

```
> # Long division for polynomials  
a := z -> z^3 - 6*z^2 + 11*z - 7;  
b := z -> z - 4;  
a := z ↦ z3 - 6z2 + 11z - 7  
b := z ↦ z - 4  
# quotient when we divide a by b, the variable being z  
quo(a(z), b(z), z);  
z2 - 2z + 3  
# remainder when we divide a by b, the variable being z  
rem(a(z), b(z), z);  
5  
# Using the remainder theorem  
a(4);  
5
```

The Factor Theorem

Example 6. We have seen that $p(z) = z^2 + 1 = (z - i)(z + i)$ has roots i and $-i$, and its factors are $z - i$ and $z + i$.



The Factor Theorem

$z - \alpha$ is a factor of $p(z)$ if and only if $P(\alpha) = 0$.

PROOF

Let r be the remainder of $p(z)$ when divided by $z - \alpha$. Then

α is a root of $p(z)$



$$p(\alpha) = 0$$

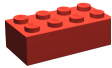


$$r = r(\alpha) = p(\alpha) = 0$$



$z - \alpha$ is a factor of $p(z)$

Factorisation



Fundamental Theorem of Algebra.

Each **complex** polynomial of degree $n \geq 1$ has at least one **complex** root.

This leads to...



The Factorisation Theorem

Each **complex** polynomial $p(z)$ of degree $n \geq 1$ has a factorisation

$$p(z) = a_n(z - \alpha_1)(z - \alpha_2) \dots (z - \alpha_n),$$

where the n complex numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ are **roots** of $p(z)$.

The roots are not necessarily distinct.

Exercise 7. Factorise the following polynomials into complex linear factors (= degree one factors).

a) $p_1(z) = z^2 - 4$

b) $p_2(z) = 2z^3 + 2z^2 - 4z$

c) $p_3(z) = z^3 - 8i$

Exercise 7, continued.

Factorise the following polynomials into linear factors (= degree one factors).

a) $p_1(z) = z^2 - 4$

b) $p_2(z) = 2z^3 + 2z^2 - 4z$

c) $p_3(z) = z^3 - 8i$

Factorising polynomials in \mathbb{C} with Maple

```
> # Example 7, part b
> p := z -> 2*z^3 + 2*z^2 - 4*z;
                                      $p := z \mapsto 2z^3 + 2z^2 - 4z$ 
> factor(p(z));
                                      $2z(z+2)(z-1)$ 
> # Example 7, part c
> p(z) := z^3 - 8*I;
                                      $p(z) := z^3 - 8I$ 
> # Naive approach
factor(p(z));
                                      $-(2Iz - z^2 + 4)(z + 2I)$ 
> # The problem is that if the coefficients are all integers then 'factor'
    computes all irreducible factors with integer coefficients. Thus factor does
    not necessarily factor into linear factors.
> # All roots in a+ib form
solve(p(z) = 0);
                                      $I + \sqrt{3}, I - \sqrt{3}, -2I$ 
> # In order to get linear factors, we force a factorisation using i and sqrt(3)
factor(p(z), {I, sqrt(3)});
                                      $(-z + I + \sqrt{3})(-z + I - \sqrt{3})(z + 2I)$ 
```

Real polynomials and conjugate roots



Real polynomials and conjugate roots

If α is a root of a **real** polynomial^a $p(z)$, then its conjugate $\bar{\alpha}$ is also a root of this polynomial.

^awhich means all the coefficients are real numbers rather than complex numbers

PROOF

Suppose that α is a root of a **real** polynomial

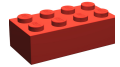
$$p(z) = a_n z^n + \dots + a_1 z + a_0,$$

that is, a_0, a_1, \dots, a_n are real and $p(\alpha) = 0$. Then

$$\begin{aligned} p(\bar{\alpha}) &= a_n \bar{\alpha}^n + \dots + a_1 \bar{\alpha} + a_0 \\ &= \overline{a_n \alpha^n + \dots + a_1 \alpha + a_0} \\ &= \overline{p(\alpha)} \\ &= \overline{0} \\ &= 0. \end{aligned}$$

Hence $\bar{\alpha}$ is also a root of $p(z)$.

Real polynomials and conjugate roots



Suppose that the real polynomial $p(z)$ has non-real root α . Then $\bar{\alpha}$ is also a root and $z - \alpha$ and $z - \bar{\alpha}$ are factors of $p(z)$. Hence $p(z)$ has the quadratic factor

$$\begin{aligned}(z - \alpha)(z - \bar{\alpha}) &= z^2 - (\alpha + \bar{\alpha})z + \alpha\bar{\alpha} \\ &= z^2 - 2\operatorname{Re}(\alpha)z + |\alpha|^2.\end{aligned}$$

Since $\operatorname{Re}(\alpha)$ and $|\alpha|^2$ are real, this quadratic factor is real.

As a consequence, every **real** polynomial can be factored into **real** linear and quadratic factors.

Exercise 8.

- a) Express $z^6 - 1$ as a product of linear factors.
- b) Express $z^6 - 1$ as a product of real linear and quadratic factors.



Assessed

Exercise 8, continued.

- a) Express $z^6 - 1$ as a product of linear factors.
- b) Express $z^6 - 1$ as a product of real linear and quadratic factors.



Complex polynomials and Maple: Ex 8

```

> p := z -> z^6-1;
> evalc([solve(z^6 = 1)]);
# Straight brackets to store the answers as a list so we can apply
# 'map' next

$$\left[ 1, -1, \frac{1}{2} - \frac{i\sqrt{3}}{2}, -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \frac{1}{2} + \frac{i\sqrt{3}}{2}, -\frac{1}{2} - \frac{i\sqrt{3}}{2} \right]$$

> map(polar, %);
# '%' means 'previous result'
# 'map' is used to apply 'polar' to each term of the previous list

$$\left[ \text{polar}(1, 0), \text{polar}(1, \pi), \text{polar}\left(1, -\frac{\pi}{3}\right), \text{polar}\left(1, \frac{2\pi}{3}\right), \text{polar}\left(1, \frac{\pi}{3}\right), \text{polar}\left(1, -\frac{2\pi}{3}\right) \right]$$

> factor(p(z));

$$(z-1)(z+1)(z^2+z+1)(z^2-z+1)$$

> # If the coefficients are all integers then 'factor' computes all
# irreducible factors with integer coefficients. Thus 'factor' does not
# necessarily factor into linear factors.
> # To get linear factors, we force a factorisation using i and sqrt(3)
factor(p(z), {I, sqrt(3)});

$$\frac{(z-1)(i\sqrt{3}-2z+1)(i\sqrt{3}+2z-1)(z+1)(i\sqrt{3}+2z+1)(i\sqrt{3}-2z-1)}{16}$$


```