School of Mathematics and Statistics
Math1131-Algebra

Lec12: Powers and roots of complex numbers

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Powers

Exercise 1. Calculate $(1+i\sqrt{3})^{10}$.

```
> z := (1 + sqrt(3)*I)^10;
```

$$z := \left(1 + \mathrm{I}\sqrt{3}\right)^{10}$$

```
> evalc(z);
  # to get the Cartesian = rectangular form
```

$$-512 - 512\,\mathrm{I}\sqrt{3}$$

Any non-zero complex number has n n^{th} roots

n^{th} root of a complex number

A complex number α is an n^{th} root of z if

$$\alpha^n = z.$$

TIP!

To find the n^{th} roots, write all the complex numbers in polar form.

Exercise 2. Find the 5^{th} roots of 1, that is, find all the complex numbers α such that

$$\alpha^5 = 1.$$

Checking our answers with Maple

```
> # Naive approach

  solve(z^5 = 1);
```

$$1, \frac{\sqrt{5}}{4}-\frac{1}{4}+\frac{I\sqrt{2}\sqrt{5+\sqrt{5}}}{4}, -\frac{\sqrt{5}}{4}-\frac{1}{4}+\frac{I\sqrt{2}\sqrt{5-\sqrt{5}}}{4}, -\frac{\sqrt{5}}{4}-\frac{1}{4}-\frac{I\sqrt{2}\sqrt{5-\sqrt{5}}}{4},$$
$$\frac{\sqrt{5}}{4}-\frac{1}{4}-\frac{I\sqrt{2}\sqrt{5+\sqrt{5}}}{4}$$

```
> # Let's convert those to polar coordinates

  SolsPolar := map(polar, [solve(z^5 = 1)]);
  #  we use straight brackets to store the answers as a list
```

$$SolsPolar := \left[\mathrm{polar}(1,0), \mathrm{polar}\left(\frac{\sqrt{(\sqrt{5}-1)^2+10+2\sqrt{5}}}{4}, \arctan\left(\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4\left(\frac{\sqrt{5}}{4}-\frac{1}{4}\right)}\right)\right),\right.$$
$$\mathrm{polar}\left(\frac{\sqrt{(-\sqrt{5}-1)^2+10-2\sqrt{5}}}{4}, \arctan\left(\frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4\left(-\frac{\sqrt{5}}{4}-\frac{1}{4}\right)}\right)+\pi\right),$$
$$\mathrm{polar}\left(\frac{\sqrt{(-\sqrt{5}-1)^2+10-2\sqrt{5}}}{4}, -\arctan\left(\frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4\left(-\frac{\sqrt{5}}{4}-\frac{1}{4}\right)}\right)-\pi\right),$$
$$\left.\mathrm{polar}\left(\frac{\sqrt{(\sqrt{5}-1)^2+10+2\sqrt{5}}}{4}, -\arctan\left(\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4\left(\frac{\sqrt{5}}{4}-\frac{1}{4}\right)}\right)\right)\right]$$

```
> map(simplify, SolsPolar);
  # 'Map' is used to apply 'simplify' to each term of the list 'SolsPolar'
```

$$\left[1, \mathrm{polar}\left(1, \frac{2\pi}{5}\right), \mathrm{polar}\left(1, \frac{4\pi}{5}\right), \mathrm{polar}\left(1, -\frac{4\pi}{5}\right), \mathrm{polar}\left(1, -\frac{2\pi}{5}\right)\right]$$

Roots

Exercise 3. Find the 6^{th} roots of $z = 1 + \sqrt{3}i$.

Some facts about the n^{th} roots of a complex number

Some facts about the n^{th} roots of a complex number

- There are n of them.
- They all have the same modulus, $|z|^{1/n}$, and so lie on a circle centred at 0.
- They are evenly spaced around that circle.

Exercise 4. Find the 3rd roots of -1.

Checking our answers with Maple

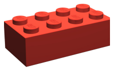
```
> evalc([solve(z^3 = -1)]);
  # Straight brackets to store the answers as a list so we can apply 'map'
  next
```

$$\left[-1, \frac{1}{2} - \frac{I\sqrt{3}}{2}, \frac{1}{2} + \frac{I\sqrt{3}}{2}\right]$$

```
> map(polar, %);
  #  '%' means 'previous result'
  # 'map' is used to apply 'polar' to each term of the previous list
```

$$\left[\mathrm{polar}(1, \pi), \mathrm{polar}\left(1, -\frac{\pi}{3}\right), \mathrm{polar}\left(1, \frac{\pi}{3}\right)\right]$$

The Binomial theorem also works for complex numbers



The Binomial Theorem

For small powers, we can calculate powers using the *Binomial Theorem*.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \qquad \text{where} \qquad \binom{n}{k} = {}^nC_k = \frac{n!}{k!(n-k)!}.$$

- Pascal's triangle can help you find the binomial coefficients.
- ${}^nC_k$ is read "*n choose k*".

Exercise 5. For example, calculate a) $(a+b)^3$ b) $(2-i)^5$

Checking some of our answers with Maple

Exercise 5, continued. Calculate a) $(a+b)^3$ b) $(2-i)^5$

```
> expand((a + b)^3 );
```

$$a^3 + 3\,a^2\,b + 3\,a\,b^2 + b^3$$

```
> expand((2 - I)^5 );
```

$$-38 - 41\,\mathrm{I}$$

$\cos(n\theta)$ and $\sin(n\theta)$ in terms of $\cos^k\theta$ and $\sin^k\theta$

From sines and cosines of multiples of θ to powers of $\sin\theta$ and $\cos\theta$

TIP! For this, use De Moivre's Theorem together with the Binomial theorem.

Exercise 6.

a) Find formulas for $\cos(4\theta)$ and $\sin(4\theta)$ in terms of powers of $\sin\theta$ and $\cos\theta$.

b) Hence or otherwise prove that $\cos 4\theta + 8\cos^2\theta \geqslant 1$ for all $\theta \in \mathbb{R}$.

$\cos(n\theta)$ and $\sin(n\theta)$ in terms of $\cos^k\theta$ and $\sin^k\theta$

From sines and cosines of multiples of θ to powers of $\sin\theta$ and $\cos\theta$

Exercise 6, continued.

a) Find formulas for $\cos(4\theta)$ and $\sin(4\theta)$ in terms of powers of $\sin\theta$ and $\cos\theta$.

b) Hence or otherwise prove that $\cos 4\theta + 8\cos^2\theta \geqslant 1$ for all $\theta \in \mathbb{R}$.

$\cos^n \theta$ and $\sin^n \theta$ in terms of $\cos(k\theta)$ and $\sin(k\theta)$

From *powers* of $\sin\theta$ and $\cos\theta$ to sines and cosines of *multiples* of θ

TIP!

For this, ◇ start from $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and/or $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$,

◇ use the binomial theorem,

◇ and then group the terms in pairs to get some sines and cosines back using $\cos k\theta = \frac{1}{2}(e^{ik\theta} + e^{-ik\theta})$ and $\sin k\theta = \frac{1}{2i}(e^{ik\theta} - e^{-ik\theta})$

Exercise 7. Write $\sin^3\theta$ in terms of sines of multiples of θ and hence evaluate

$$I = \int_0^{\pi/2} \sin^3\theta \, d\theta.$$

$\cos^n \theta$ and $\sin^n \theta$ in terms of $\cos(k\theta)$ and $\sin(k\theta)$

Exercise 7, continued. Write $\sin^3 \theta$ in terms of sines of multiples of θ and hence evaluate

$$I = \int_0^{\pi/2} \sin^3 \theta \, d\theta.$$

Checking some of our answers with Maple

```
> combine((sin(x))^3 , trig);

  # Powers to multiples
```

$$-\frac{\sin(3\,x)}{4}+\frac{3\,\sin(x)}{4}$$

```
> int((sin(x))^3, x = 0 .. Pi/2);

  # which makes it easy to integrate
```

$$\frac{2}{3}$$