School of Mathematics and Statistics
Math1131-Algebra

Lec05: Scalar (dot) Product

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Learning outcomes for this lecture

At the the end of this lecture on the dot product = scalar product,

- ☐ you should know how to **calculate the dot product** of two vectors given their components; you should know **the scalar product of two vectors is NOT a vector**, it is a **scalar**! Hence the name *scalar* product
- ☐ you should be able to calculate the **length** of a vector and know how it is related to the scalar product (**the square of the length** is equal to **the dot product of the vector with itself**)
- ☐ you should know the formula that gives the dot product of two vectors in term of their lengths and the **cosine of the angle between them** and know how to use it to **calculate the angle** between two vectors
- ☐ you should be able to **perform calculations involving the dot product** (The rules are what you expect for a product –commutative, distributive– hence the the name)
- ☐ you should know what the **triangle inequality** says and what it means geometrically

You can use this list as a check list to get ready for our next class: After studying the lecture notes, come back to this list, and for each item, check that you have indeed mastered it. Then tick the corresponding box ... or go back to the notes.

Length in n dimensions

The length of a vector $\overrightarrow{\boldsymbol{a}} \in \mathbb{R}^n$ with

$$\overrightarrow{\boldsymbol{a}} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1\overrightarrow{\boldsymbol{e_1}} + a_2\overrightarrow{\boldsymbol{e_2}} + \cdots + a_n\overrightarrow{\boldsymbol{e_n}}$$

is defined to be

$$|\overrightarrow{\boldsymbol{a}}| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.$$

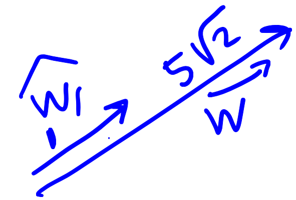
If $|\overrightarrow{\boldsymbol{a}}| = 1$ we say that $\overrightarrow{\boldsymbol{a}}$ is a unit vector and we sometimes write $\hat{\boldsymbol{a}}$ instead of $\overrightarrow{\boldsymbol{a}}$.

Properties of the length/magnitude/norm of a vector

- $|\overrightarrow{\boldsymbol{a}}|$ is a real number. (scalar)
- $|\overrightarrow{\boldsymbol{a}}| \geqslant 0$.
- $|\overrightarrow{\boldsymbol{a}}| = 0 \iff \overrightarrow{\boldsymbol{a}} = \overrightarrow{\boldsymbol{0}}$ ("$\iff$" is read "*if and only if*")
- $|\lambda\overrightarrow{\boldsymbol{a}}| = |\lambda||\overrightarrow{\boldsymbol{a}}|$ for $\lambda \in \mathbb{R}$.

$\|\lambda\vec{a}\| = |\lambda|\,\|\vec{a}\|$

Length in n dimensions



Example 1. Find two vectors of length 5 parallel to $\vec{w} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ -6 \end{pmatrix}$.

- $|\vec{w}| = \sqrt{1^2 + 2^2 + 3^2 + (-6)^2}$
  $= \sqrt{1 + 4 + 9 + 36} = \sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}$

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- $\hat{w}_1 = \frac{1}{5\sqrt{2}}\vec{w}$ and $\hat{w}_2 = -\hat{w}_1 = -\frac{1}{5\sqrt{2}}\vec{w}$ are two unit vectors parallel to $\vec{w}$

- $\vec{v}_1 = \frac{1}{\sqrt{2}}\vec{w}$ and $\vec{v}_2 = -\frac{1}{\sqrt{2}}\vec{w}$ are vectors of length 5 parallel to $\vec{w}$

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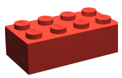
Scalar/dot product of two vectors



Scalar (dot) product of two vectors

The dot product or scalar product of two vectors $\vec{a}, \vec{b} \in \mathbb{R}^n$ with

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

is $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{k=1}^{n} a_k b_k.$ is a $\underline{\text{scalar}}$

Example 2. Calculate the following scalar products:

a) $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 1 \times 3 + 2 \times 4 = 11.$

b) $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \ldots 1 \times 1 + 2 \times 2 = 1^2 + 2^2 = 5$

c) $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \ldots -2 + 2 = 0$

d) $\begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \ldots 3 + 0 - 4 = -1$

Cosine rule for triangles

We will need it to establish a second formula for the scalar product, see next slide.

Cosine rule : Generalisation of Pythagoras' theorem to ALL triangles

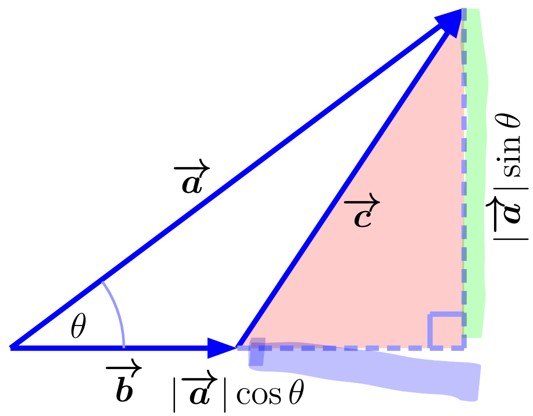
For any triangle, $|\vec{c}\,|^2 = |\vec{a}\,|^2 + |\vec{b}\,|^2 - 2|\vec{a}\,||\vec{b}\,|\cos\theta$
where $\theta \in [0, \pi]$ is the smallest angle between $\vec{a}$ and $\vec{b}$.

PROOF

Consider a triangle in $\mathbb{R}^n$ with sides $\vec{a}$, $\vec{b}$ and $\vec{c} = \vec{a} - \vec{b}$.

Let θ be the smallest angle between $\vec{a}$ and $\vec{b}$.

Apply Pythagoras' Theorem to the shaded triangle.



$$
\begin{aligned}
|\vec{c}\,|^2 &= \left(|\vec{a}\,|\cos\theta - |\vec{b}\,|\right)^2 + \left(|\vec{a}\,|\sin\theta\right)^2 \\
&= |\vec{a}\,|^2\cos^2\theta + |\vec{b}\,|^2 - 2|\vec{a}\,||\vec{b}\,|\cos\theta + |\vec{a}\,|^2\sin^2\theta \\
&= |\vec{a}\,|^2 + |\vec{b}\,|^2 - 2|\vec{a}\,||\vec{b}\,|\cos\theta
\end{aligned}
$$

A new formula for the scalar product in terms of the angle between the two vectors

From the cosine rule for triangles (rearranged),

$$\vec{c} = \vec{a} - \vec{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ \vdots \\ a_n - b_n \end{pmatrix}$$

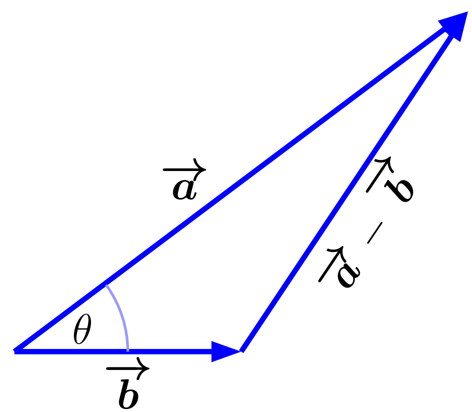
$$\begin{aligned} 2|\vec{\boldsymbol{a}}||\vec{\boldsymbol{b}}|\cos\theta &= |\vec{\boldsymbol{a}}|^2 + |\vec{\boldsymbol{b}}|^2 - |\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}}|^2 \\ &= |\vec{\boldsymbol{a}}|^2 + |\vec{\boldsymbol{b}}|^2 - \Big((a_1 - b_1)^2 + \cdots + (a_n - b_n)^2\Big) \\ &= |\vec{\boldsymbol{a}}|^2 + |\vec{\boldsymbol{b}}|^2 - \Big((a_1^2 + b_1^2 - 2a_1b_1) + \cdots + (a_n^2 + b_n^2 - 2a_nb_n)\Big) \\ &= \cancel{|\vec{\boldsymbol{a}}|^2} + \cancel{|\vec{\boldsymbol{b}}|^2} - \Big(\cancel{|\vec{\boldsymbol{a}}|^2} + \cancel{|\vec{\boldsymbol{b}}|^2} - 2\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}}\Big) \end{aligned}$$

So

$$\boxed{\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} = |\vec{\boldsymbol{a}}||\vec{\boldsymbol{b}}|\cos\theta}$$

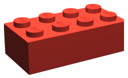
and

$$\cos\theta = \frac{\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}}}{|\vec{\boldsymbol{a}}||\vec{\boldsymbol{b}}|}.$$



Scalar/dot product and angle between two vectors

The previous slide proves that:

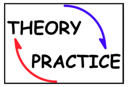


Scalar product and angle between two vectors

The dot product or scalar product of two vectors $\vec{a}, \vec{b} \in \mathbb{R}^n$ is

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta.$$

where θ is the smallest angle between $\vec{a}$ and $\vec{b}$, with θ in the interval $[0, \pi]$.



This formula is fundamental and should be committed to memory.
It can be used to calculate the angle between two vectors in $\mathbb{R}^n$.

Example 3. Find the angle θ between $\vec{a} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, with $\theta \in [0, \pi]$.

$\vec{a} \cdot \vec{b} = 2$

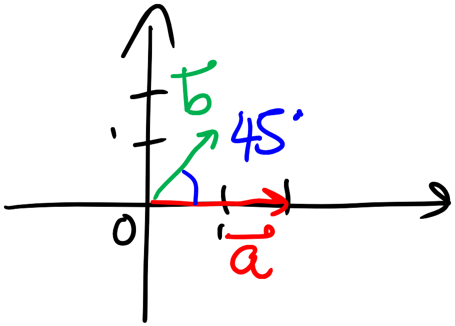
$|\vec{a}| = \sqrt{2^2 + 0^2} = 2$

$|\vec{b}| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$2 = 2\sqrt{2}\cos\theta$

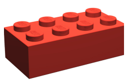
$\cos\theta = \frac{1}{\sqrt{2}}$

$\theta = 45° = \frac{\pi}{4}$



Scalar/dot product and Angle between two vectors

By rearranging the previous result, we get:



Angle between two vectors in $\mathbb{R}^n$

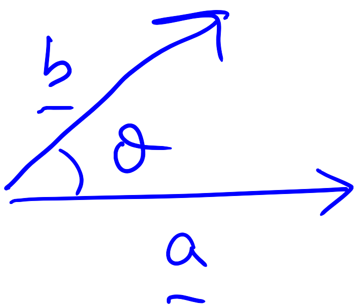
If $\overrightarrow{\boldsymbol{a}}$ and $\overrightarrow{\boldsymbol{b}}$ are two non-zero vectors in $\mathbb{R}^n$, then the angle between $\overrightarrow{\boldsymbol{a}}$ and $\overrightarrow{\boldsymbol{b}}$ is given by:

$$\cos\theta = \frac{\overrightarrow{\boldsymbol{a}} \cdot \overrightarrow{\boldsymbol{b}}}{|\overrightarrow{\boldsymbol{a}}||\overrightarrow{\boldsymbol{b}}|} \quad \text{where } \theta \in [0, \pi].$$

Example 4. Find the angle θ between $\overrightarrow{\boldsymbol{a}} = \begin{pmatrix} 2 \\ 0 \\ 3 \\ -1 \end{pmatrix}$ and $\overrightarrow{\boldsymbol{b}} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$.

$$|\underset{\sim}{a}| = \sqrt{2^2+0^2+3^2+(-1)^2} = \sqrt{14}$$

$$|\underset{\sim}{b}| = \sqrt{1^2+1^2+0^2+2^2} = \sqrt{6}$$

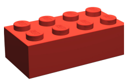


$$\underset{\sim}{a} \cdot \underset{\sim}{b} = 2\times 1 + 0\times 1 + 3\times 0 + (-1)\times 2$$

$$= 0$$

$$\cos\theta = \frac{0}{\sqrt{14}\sqrt{6}} = 0 \Rightarrow \theta = \frac{\pi}{2}$$

Algebraic properties of the scalar/dot product



The scalar/dot product behaves as expected for a product:

For all vectors $\overrightarrow{\boldsymbol{a}}, \overrightarrow{\boldsymbol{b}}, \overrightarrow{\boldsymbol{c}} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$,

- $\overrightarrow{\boldsymbol{a}} \cdot \overrightarrow{\boldsymbol{a}} = |\overrightarrow{\boldsymbol{a}}|^2$, so $|\overrightarrow{\boldsymbol{a}}| = \sqrt{\overrightarrow{\boldsymbol{a}} \cdot \overrightarrow{\boldsymbol{a}}}$ [Very useful!]
- $\overrightarrow{\boldsymbol{a}} \cdot \overrightarrow{\boldsymbol{b}} \in \mathbb{R}$ i.e, it is a scalar, NOT a vector!
- $\overrightarrow{\boldsymbol{a}} \cdot \overrightarrow{\boldsymbol{b}} = \overrightarrow{\boldsymbol{b}} \cdot \overrightarrow{\boldsymbol{a}}$ (commutative law)
- $\overrightarrow{\boldsymbol{a}} \cdot (\lambda \overrightarrow{\boldsymbol{b}}) = \lambda(\overrightarrow{\boldsymbol{b}} \cdot \overrightarrow{\boldsymbol{a}})$ (associative law of scalar multiplication)
- $\overrightarrow{\boldsymbol{a}} \cdot (\overrightarrow{\boldsymbol{b}} + \overrightarrow{\boldsymbol{c}}) = \overrightarrow{\boldsymbol{a}} \cdot \overrightarrow{\boldsymbol{b}} + \overrightarrow{\boldsymbol{a}} \cdot \overrightarrow{\boldsymbol{c}}$ (distributive law)

Exercise 5. For added fun, prove these laws.

Algebraic properties of the scalar/dot product

Exercise 6. Using the given laws, prove the following familiar looking formulas.

a) Expanding parentheses

$$(\overrightarrow{a} + \overrightarrow{b}) \cdot (\overrightarrow{c} + \overrightarrow{d}) = \overrightarrow{a} \cdot \overrightarrow{c} + \overrightarrow{a} \cdot \overrightarrow{d} + \overrightarrow{b} \cdot \overrightarrow{c} + \overrightarrow{b} \cdot \overrightarrow{d}$$

c) Difference of squares

$$|\overrightarrow{a}|^2 - |\overrightarrow{b}|^2 = (\overrightarrow{a} - \overrightarrow{b}) \cdot (\overrightarrow{a} + \overrightarrow{b})$$

b) Perfect squares

$$|\overrightarrow{a} + \overrightarrow{b}|^2 = |\overrightarrow{a}|^2 + 2\overrightarrow{a} \cdot \overrightarrow{b} + |\overrightarrow{b}|^2$$

$$|\overrightarrow{a} - \overrightarrow{b}|^2 = |\overrightarrow{a}|^2 - 2\overrightarrow{a} \cdot \overrightarrow{b} + |\overrightarrow{b}|^2$$

Algebraic properties of the scalar/dot product

Exercise 7.  In what sense does the dot product fail to be a product? In other words, what property does the product of real numbers have that the dot product does not have?

- no multiplicative inverse
- result of a different nature
- $(\vec{a}\cdot\vec{b})\cdot\vec{c} \overset{?}{=} \vec{a}\cdot(\vec{b}\cdot\vec{c})$ NOT HERE

Proofs in Geometry using the scalar product

FACT Two non-zero vectors make a right angle if and only if their scalar product is zero.

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

Exercise 8.

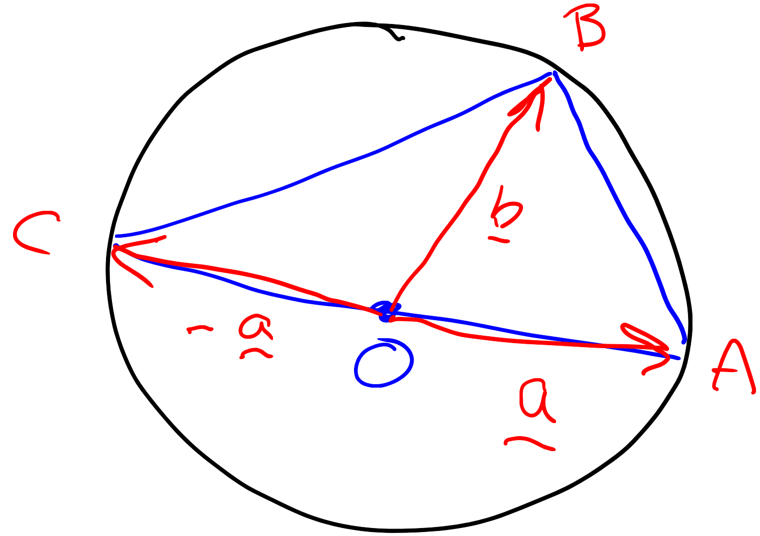
a) Prove the above fact.

" $\Longrightarrow$ "

" $\Longleftarrow$ "

Proofs in Geometry using the scalar product

b) (Thales Theorem). Show that the angle inside a semicircle is a right angle.



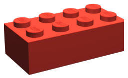
$\overrightarrow{AB} = b - a$

$\overrightarrow{BC} = -a - b$

$$\overrightarrow{AB} \cdot \overrightarrow{BC} = (b - a)\cdot(-a - b)$$
$$= -\cancel{b\cdot a} - b\cdot b + a\cdot a + \cancel{a\cdot b}$$
$$= |a|^2 - |b|^2 = 0$$

So $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{BC}$

Triangle inequality

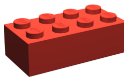


Triangle inequality

For all vectors $\overrightarrow{\boldsymbol{a}}$ and $\overrightarrow{\boldsymbol{b}}$ in $\mathbb{R}^n$, $\quad |\overrightarrow{\boldsymbol{a}} + \overrightarrow{\boldsymbol{b}}| \leqslant |\overrightarrow{\boldsymbol{a}}| + |\overrightarrow{\boldsymbol{b}}|$

Exercise 9. Draw a diagram to illustrate the triangle inequality
(See the Algebra Notes or later this lecture for a proof.)

Cauchy-Schwarz inequality



Cauchy-Schwarz inequality

For any vectors $\overrightarrow{\boldsymbol{a}}$ and $\overrightarrow{\boldsymbol{b}}$ in $\mathbb{R}^n$, $\quad |\overrightarrow{\boldsymbol{a}} \cdot \overrightarrow{\boldsymbol{b}}| \leqslant |\overrightarrow{\boldsymbol{a}}||\overrightarrow{\boldsymbol{b}}|$

Exercise 10. 

a) Prove Cauchy-Schwarz inequality.

b) Hence or otherwise, prove the Triangle inequality.

(The solution is in the Algebra Notes.)