School of Mathematics and Statistics

Math1131 Mathematics 1A

CALCULUS LECTURE 5

THE PINCHING THEOREMS

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Math1131 CALCULUS
Continuity and the Pinching Theorems

Suppose that $f(x) \leq g(x) \leq h(x) \quad \forall x \in (b, \infty)$ and $\lim\limits_{x \longrightarrow \infty} f(x) = \lim\limits_{x \longrightarrow \infty} h(x) = L$. Then $\lim\limits_{x \longrightarrow \infty} g(x) = L$.

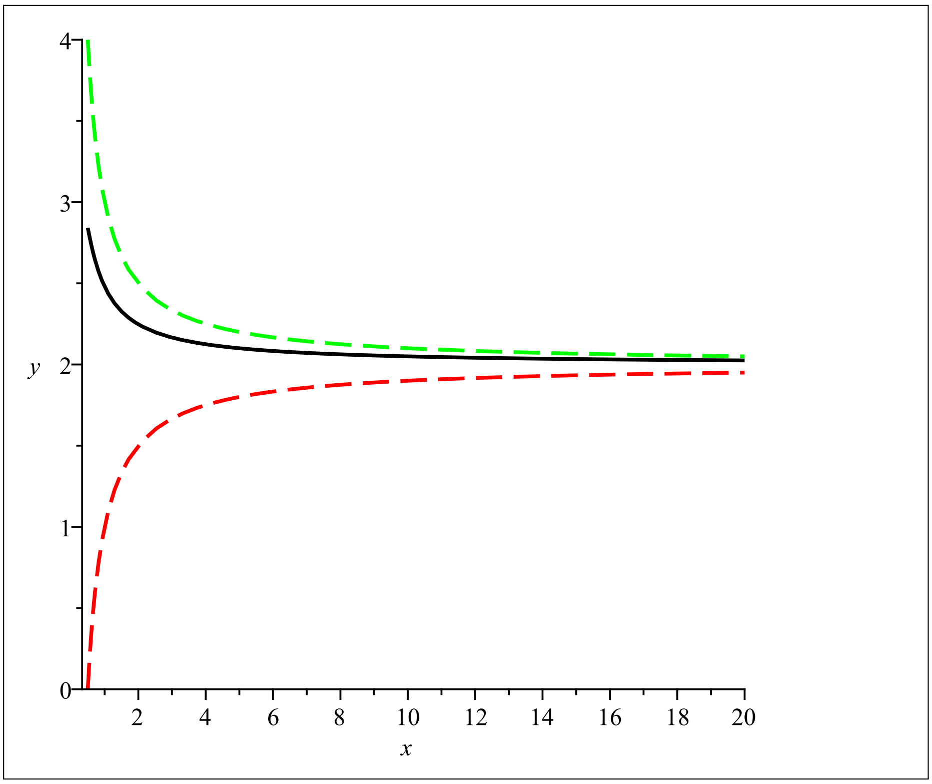
Let I be an open interval containing a point a and suppose that $f(x) \leq g(x) \leq h(x) \quad \forall x \in I$ (except perhaps at a). Assume also that $\lim\limits_{x \longrightarrow a} f(x) = \lim\limits_{x \longrightarrow a} h(x) = L$. Then $\lim\limits_{x \longrightarrow a} g(x) = L$.

A function f is said to be continuous at $x = a$ if $\lim\limits_{x \longrightarrow a} f(x) = f(a)$. Else it is discontinuous.

We now turn to two formal results called "Pinching Theorems" which will help us to pin down certain limits. Note that these theorems are also called Sandwich Theorems or Squeeze theorems. But first some new notation:

We have two special symbols which we use as abbreviations. Firstly $\forall$ is to be read as "for all" and also $\exists$ mean there exists. These two just make it a little easier to write down the mathematics. Now the Pinching theorem:

Pinching Theorem: Suppose that $f(x) \leq g(x) \leq h(x) \quad \forall x \in (b, \infty)$ and $\lim\limits_{x \longrightarrow \infty} f(x) = \lim\limits_{x \longrightarrow \infty} h(x) = L$. Then $\lim\limits_{x \longrightarrow \infty} g(x) = L$.



This should make sense! In the above h is green (top) and f is red (bottom). Both

h and f approach 2 as $x \to \infty$. Poor old g (black) is pinched between h and f and as a consequence has no choice but to also approach 2 as $x \to \infty$.

We usually warn you if we expect a formal use of the pinching theorem.

Example 1: Use the pinching theorem to find $\lim\limits_{x \longrightarrow \infty} e^{-x} \cos(10\sqrt{x})$.

We begin with the simple observation that $-1 \leq \cos(10\sqrt{x}) \leq 1$. Then

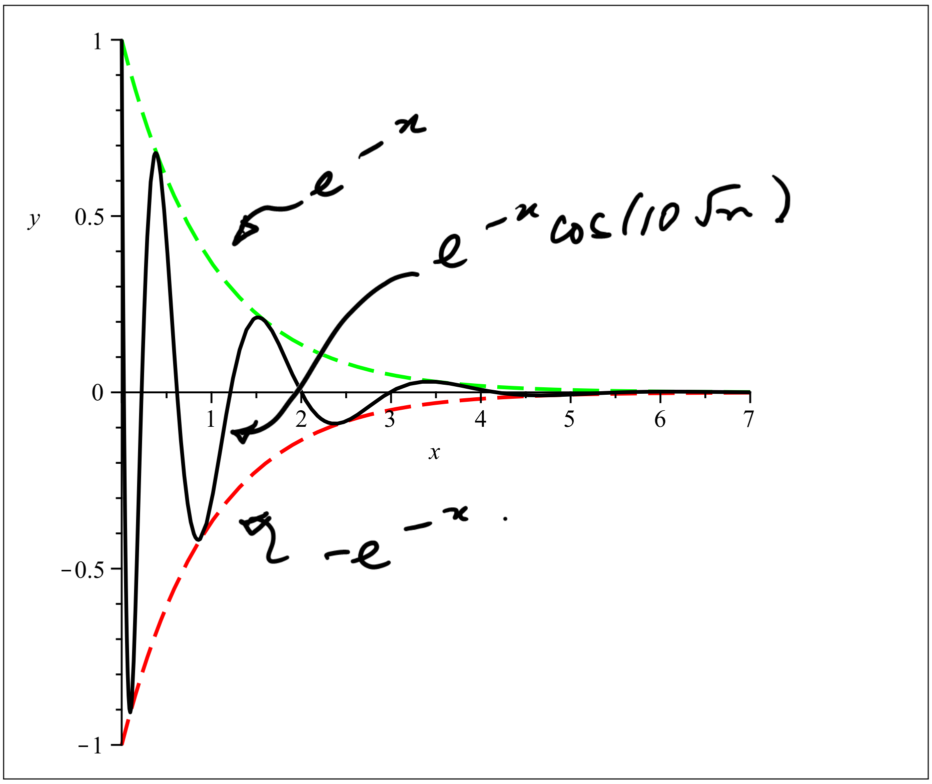
$$-1 \leq \cos(10\sqrt{x}) \leq 1$$

$$-e^{-x} \leq e^{-x} \cos(10\sqrt{x}) \leq e^{-x}.$$

Now $\lim\limits_{x \to \infty} e^{-x} = 0$ and $\lim\limits_{x \to \infty} (-e^{-x}) = 0$

$\therefore$ By Pinching theorem $\lim\limits_{x \to \infty} e^{-x} \cos(10\sqrt{x})$ is also be 0.

Therefore by the pinching theorem $\lim\limits_{x \longrightarrow \infty} e^{-x} \cos(10\sqrt{x}) = 0$



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There is also a finite version of the pinching theorem:

Pinching Theorem: Let I be an open interval containing a point a and suppose that $f(x) \le g(x) \le h(x) \quad \forall x \in I$ (except perhaps at a). Assume also that $\lim\limits_{x \longrightarrow a} f(x) = \lim\limits_{x \longrightarrow a} h(x) = L$. Then $\lim\limits_{x \longrightarrow a} g(x) = L$.

Example 2: Use the pinching theorem to show $\lim\limits_{x \longrightarrow 0} x^2 \sin(1/x) = 0$.

Note $x^2 \geq 0$

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

$$\lim_{x \to 0} x^2 = \lim_{x \to 0} -(x^2) = 0$$

$\therefore$ By Sandwich th^{m} $\lim\limits_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$ is also equal to 0

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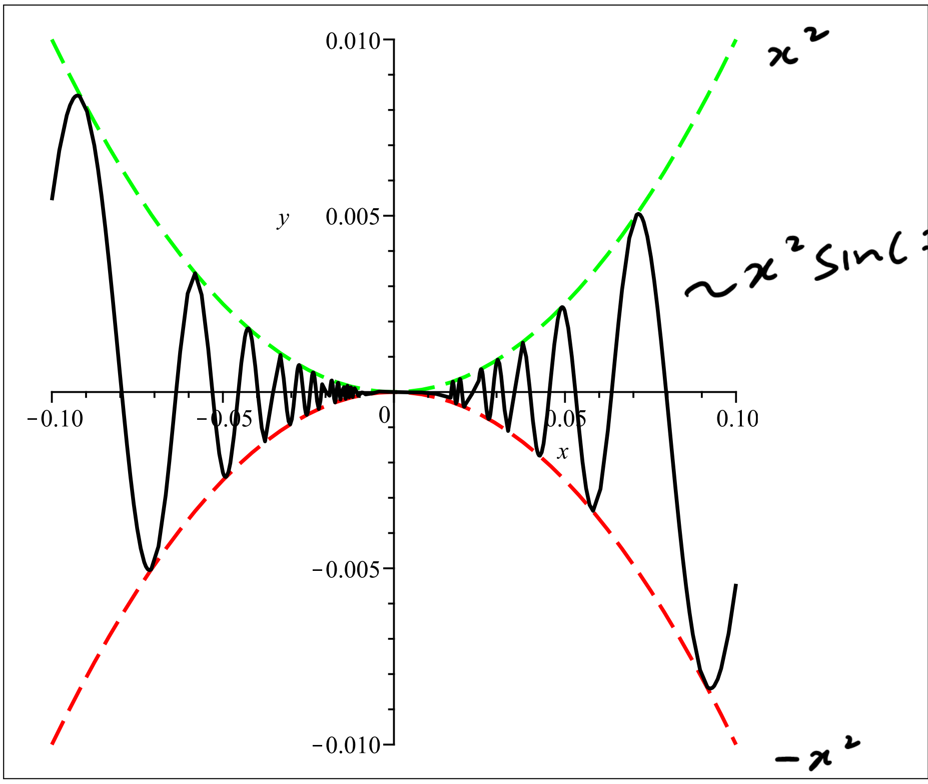


x^2

$\sim x^2 \sin\left(\frac{1}{x}\right)$

$-x^2$

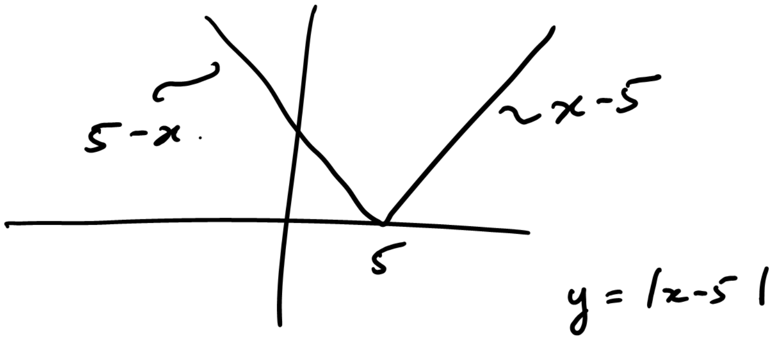
Before examining the concept of continuity lets do a little revision on the existence of limits:

Example 3:

a) Find $\lim\limits_{x \longrightarrow 5^+} \dfrac{|x-5|}{x-5}$.

b) Find $\lim\limits_{x \longrightarrow 5^-} \dfrac{|x-5|}{x-5}$.

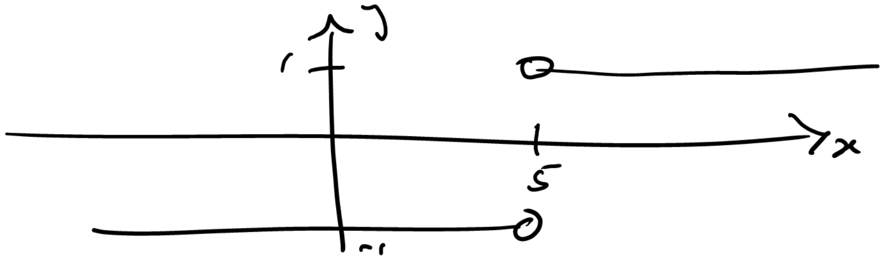
c) Does $\lim\limits_{x \longrightarrow 5} \dfrac{|x-5|}{x-5}$ exist ?



The best way to do these absolute value questions is to express the function in piece-meal form and draw a simple graph of the situation:

$$\frac{|x-5|}{x-5} = \begin{cases} \dfrac{x-5}{x-5} & x > 5 \\ \dfrac{5-x}{x-5} & x < 5 \end{cases}$$

$$= \begin{cases} 1 & x > 5 \\ -1 & x < 5 \end{cases}$$



$$\lim_{x \to 5^+} \frac{|x-5|}{x-5} = 1$$

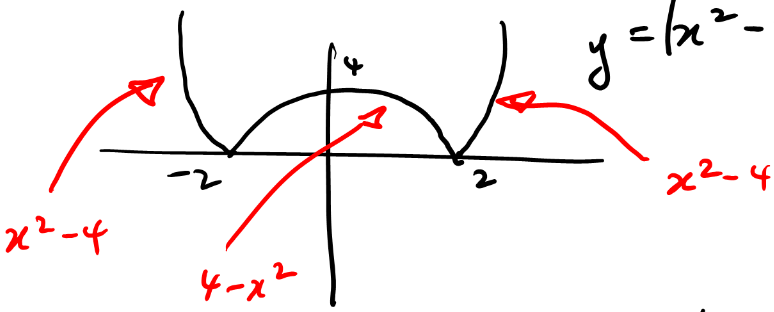
$$\lim_{x \to 5^-} \frac{|x-5|}{x-5} = -1 \qquad 1 \neq -1$$

Since $\lim\limits_{x \to 5^+} \neq \lim\limits_{x \to 5^-}$ we say that

$\lim\limits_{x \to 5} \dfrac{|x-5|}{x-5}$ does not exist ★

(D.N.E.)

Example 4: Does $\lim\limits_{x \longrightarrow 2} \dfrac{|x^2-4|}{x-2}$ exist ?

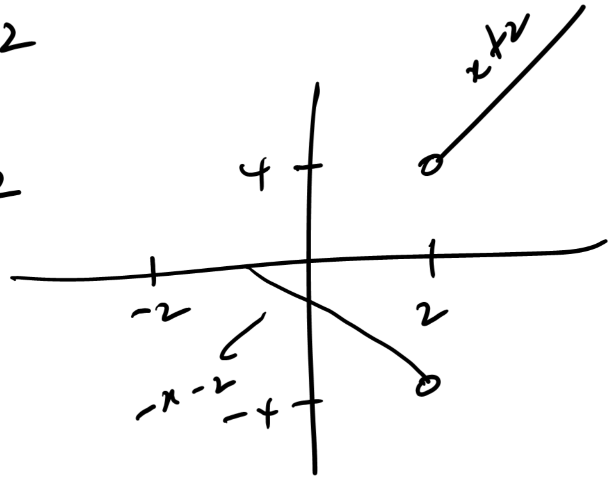


$y = |x^2-4|$

$$\frac{|x^2-4|}{x-2} = \begin{cases} \dfrac{x^2-4}{x-2} & x > 2 \\ \dfrac{4-x^2}{x-2} & -2 < x < 2 \end{cases}$$

$$= \begin{cases} \dfrac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} & x > 2 \\ \dfrac{-\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} & -2 < x < 2 \end{cases}$$

$$= \begin{cases} x+2 & x > 2 \\ -(x+2) & -2 < x < 2 \end{cases}$$



$$\lim_{x \to 2^+} \frac{|x^2-4|}{x-2} = 4$$

$$\lim_{x \to 2^-} \frac{|x^2-4|}{x-2} = -4$$

One sided limits are not equal.

$\therefore \lim\limits_{x \to 2} \dfrac{|x^2-4|}{x-2}$ D.N.E.

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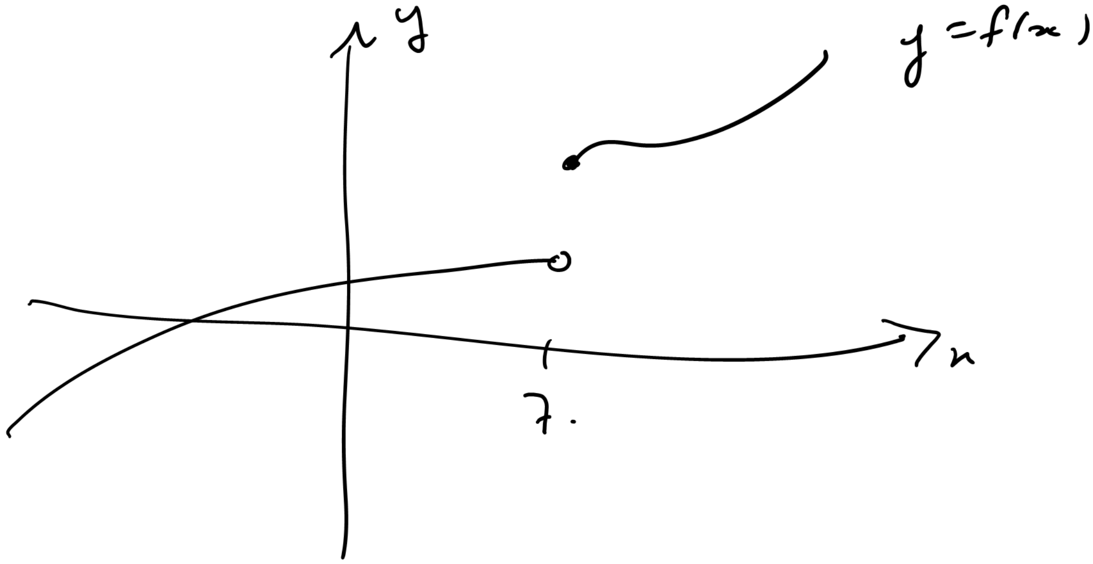
Continuity

We prefer our functions to be continuous. Abrupt jumps in graphs are a worry as the function then varies over large ranges in infinitesimal time intervals. For example one of the greatest concerns when trading in the stock-market are shares which close at one price on Friday afternoon and open at a drastically different price at the resumption of trading on Monday morning, leaving the trader with no chance to respond.

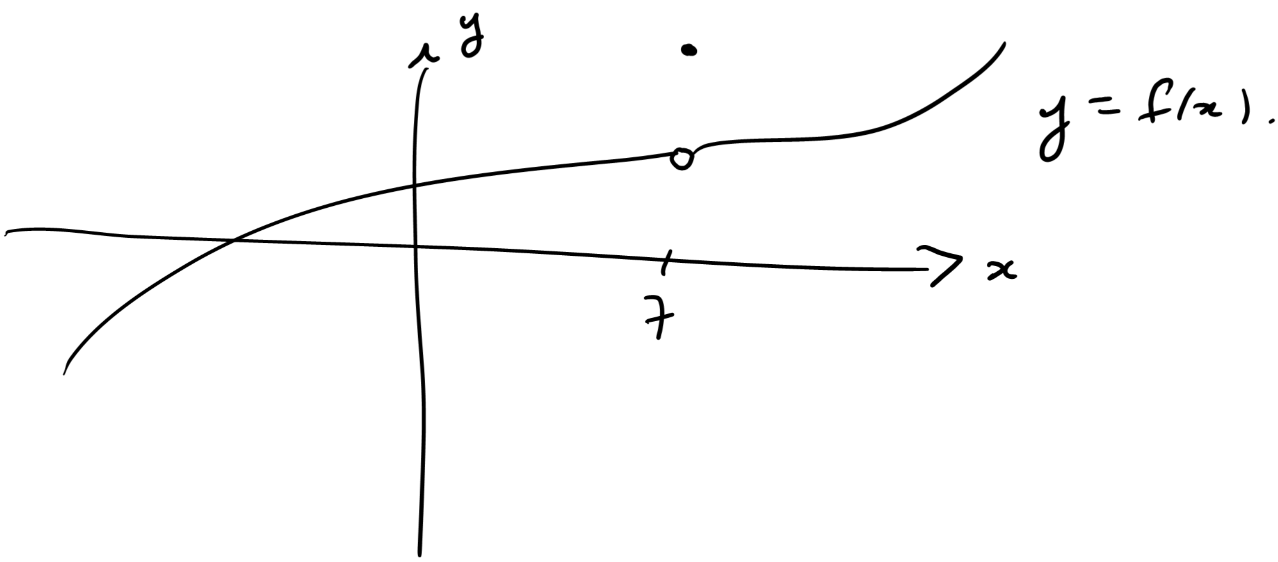
We will present a formal definition of continuity soon however for the moment you may think of a continuous function as one whose graph may be sketched without taking your pen off the paper.

Two different types of discontinuities are:

(I) A jump discontinuity:



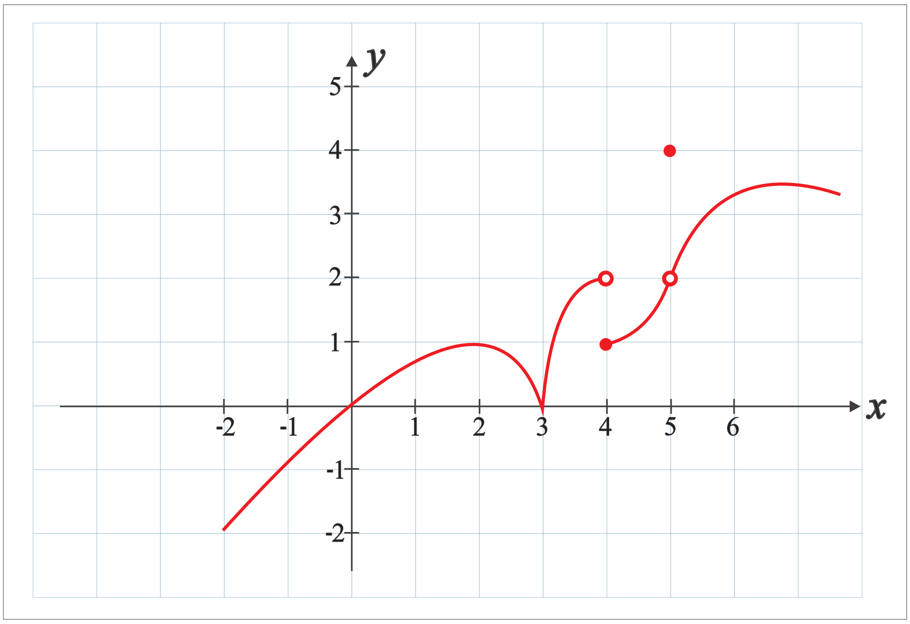
(II) A removable discontinuity:



Definition: A function f is said to be continuous at a point $x = a$ in its domain if $\lim_{x \longrightarrow a} f(x) = f(a)$. Else it is discontinuous at $x = a$.

What this means is that a function is continuous at $x = a$ if its behaviour **near** a perfectly matches its behaviour **at** a. Note that for continuity we need two things! The limit must exist and also be equal to the function value. You also have the intuitive idea of the pen lifting off the paper to help you out.

Example 5: Consider the graph of $y = f(x)$ presented below in red:



By considering appropriate limits determine whether the function is continuous at $x = 4, 5$ and 3. Classify the discontinuities.

$x = 4$: $\lim_{x\to 4^-} f(x) = 2$, $\lim_{x\to 4^+} f(x) = 1$

Left hand limit $\neq$ Right hand limit.

$\therefore \lim_{x\to 4} f(x)$ DNE $\therefore$ f is discontinuous at $x = 4$ (Jump).

$x = 5$: $\lim_{x\to 5^-} f(x) = 2$, $\lim_{x\to 5^+} f(x) = 2$

$\therefore \lim_{x\to 5} f(x) = 2$. But $f(5) = 4 \neq 2$

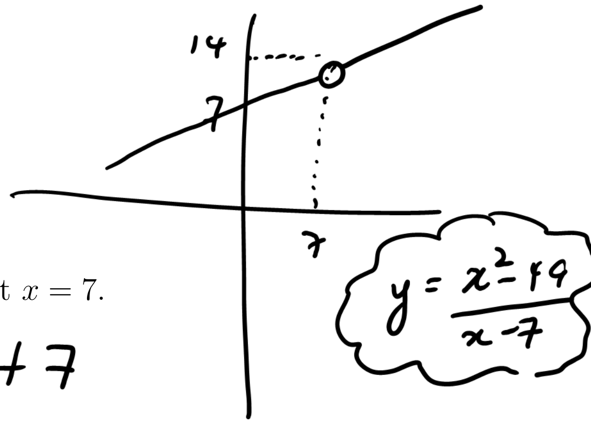
$\therefore$ f has a removable discontinuity at $x = 5$

$x = 3$: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = 0$ and $\underline{f(3) = 0}$

$\therefore \lim_{x\to 3} f(x)$ $\underline{\text{exists}}$ and equals 0 ★

$\therefore$ f is continuous at $\underline{x = 3}$

Example 6: Let $g(x) = \begin{cases} \dfrac{x^2-49}{x-7}, & x \neq 7; \\ \alpha, & x = 7 \end{cases}$



Sketch the graph of g and find α so that g is continuous at $x = 7$.

$$\frac{x^2-49}{x-7} = \frac{\cancel{(x-7)}(x+7)}{\cancel{(x-7)}} = x+7$$

Need $\alpha = 7+7 = 14$ for continuity

$$\lim_{x\to 7} \frac{x^2-49}{x-7} = \text{"}\frac{0}{0}\text{"}$$

$$= \lim_{x\to 7} \frac{\cancel{(x-7)}(x+7)}{\cancel{(x-7)}} = \lim_{x\to 7}(x+7) = \alpha$$

$$\therefore \alpha = 14$$

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Example 7: Let $f(x) = \dfrac{x^2-49}{x-7}$. Write down dom$(f)$ and find the values of x for which f discontinuous.

$$\text{Dom}(f) = \{x \in \mathbb{R} \mid x \neq 7\}$$

This function is continuous over its domain but not continuous over $\mathbb{R}$. We would **not** classify $x = 7$ as a point of discontinuity since $x = 7$ is not in the domain of the function f.

★ dom(f)=all real x except $x = 7$. ★

Example 8: Let $f(x) = \begin{cases} \dfrac{|\,x^2-4\,|}{x-2}, & x \neq 2; \\ \\ \alpha, & x = 2 \end{cases}$

Is there an α for which f is continuous at $x = 2$?

Recall that we showed in Example 4 that $\lim\limits_{x \longrightarrow 2} f(x)$ does not exist since the left and the right hand limits do not agree. That is the end of continuity at $x = 2$! No α will fix the situation.

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