THE UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1131 Calculus

Section 3: - Properties of Continuous Functions.

Having defined what we mean by continuity, we wish to see what the practical and theoretical implications of this definition are.

Functions can be defined globally on all of $\mathbb{R}$ or locally on some closed interval $[a, b]$ and in the latter case, we need to extend our concept of continuity.

Definition: If f is a function defined on a closed interval $[a, b]$, then we say that f is cts at $x = a$ if $\lim_{x\to a^+} f(x) = f(a)$ and similarly, f is cts at $x = b$ if $\lim_{x\to b^-} f(x) = f(b)$.

Ex: $f(x) = \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

$\lim_{x\to -1^+} f(x) = 0 = f(-1)$

$\lim_{x\to 1^-} f(x) = 0 = f(1)$

For $a \in (-1, 1)$

$\lim_{x\to a} f(x) = \sqrt{1-a^2} = f(a)$

$\therefore$ f is cts on $[-1, 1]$.

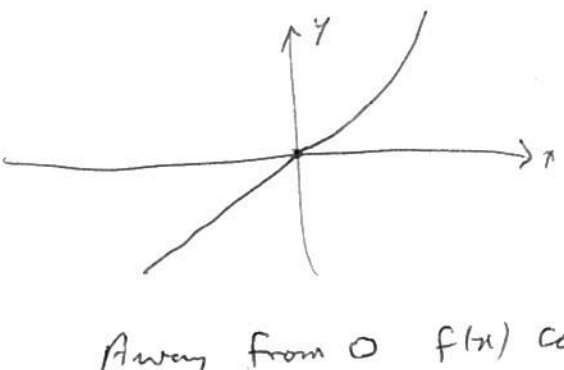
Split Functions:

A function may be defined *piece-wise* using a split definiton.

Ex:

$$f(x) = \begin{cases} x & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

Prove that f is continuous everywhere.



Away from 0 $f(x)$ consists of functions which are clearly continuous.

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0$

$\therefore$ f is cts also at 0.

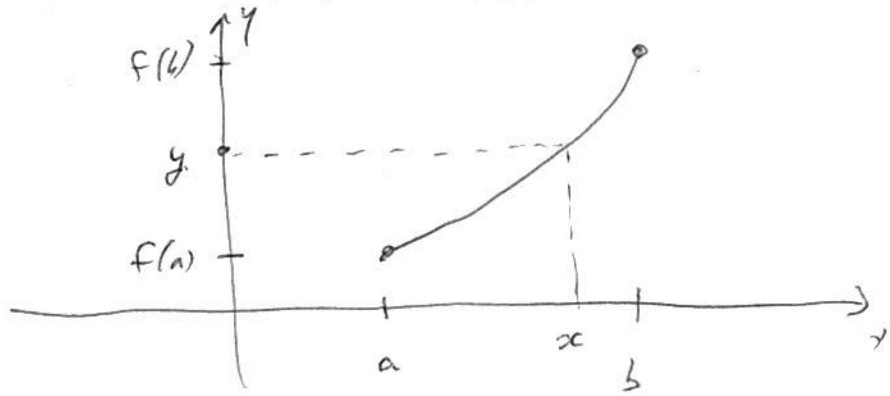
$\therefore$ f is cts everywhere.

The Intermediate Value Theorem:

The following theorem is very useful in locating zeros of functions. Although it appears intuitively obvious, it is somewhat hard to prove and the proof will not be given.

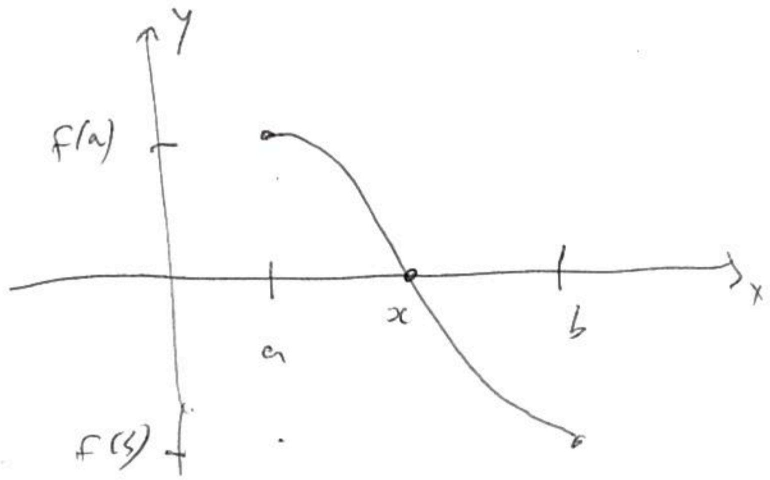
Theorem: (IVT)

Suppose that f is cts on a closed interval $[a, b]$ and y lies between $f(a)$ and $f(b)$, then there is at least one $x \in [a, b]$ such that $f(x) = y$.



The theorem is more often used in the following form:

Corollary: Suppose that f is cts on a closed interval $[a, b]$ and that $f(a)$ and $f(b)$ have opposite sign. Then there exists at least one $x \in [a, b]$ such that $f(x) = 0$.



Ex: Let $f(x) = x^3 - x - 1$ in the interval $[1, 2]$.

$f(1) = -1 < 0$

$f(2) = 8 - 2 = 6 > 0$

f is cts on $[1, 2]$

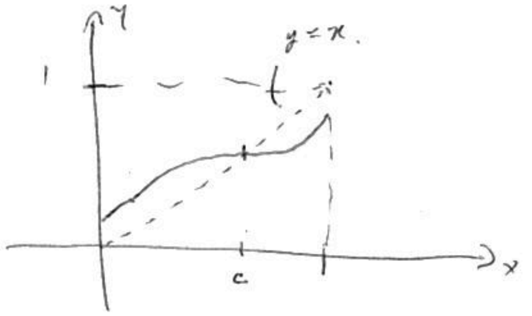
$\therefore$ By I.V.T. f has a zero somewhere in the interval $[1, 2]$.

[In fact at

$$x = \frac{(108+12\sqrt{69})^{1/3}}{6} + \frac{2}{(108+12\sqrt{69})^{1/3}} \simeq 1.3247$$

Note that a polynomial of even degree has an even (possibly zero) number of roots (counting multiplicity) and a polynomial of odd degree has an odd number of roots (counting multiplicity).

Ex: Suppose that f is a cts function defined on the interval $[0, 1]$ which has its range also in the interval $[0, 1]$. Prove that there is a real number $c \in [0, 1]$ such that $f(c) = c$.



If $f(0) = 0$ or $f(1) = 1$ then we have a soln. to $f(x) = x$.

So suppose $f(0) > 0$ and $f(1) < 1$.

Consider the function

$$g(x) = f(x) - x.$$

g is cts since $f(x)$ and x are continuous.

Now $g(0) = f(0) - 0 = f(0) > 0$

$g(1) = f(1) - 1 < 0$ since $f(1) < 1$.

$\therefore$ By I.V.T. there exists at c s.t. $g(c) = 0$

$\therefore f(c) - c = 0$

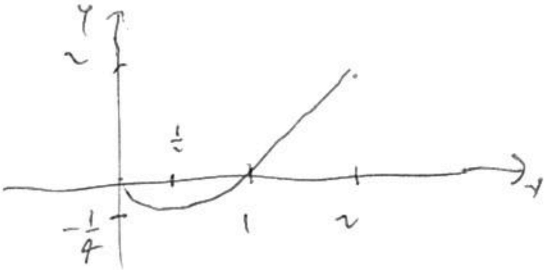
$\therefore f(c) = c$

Maxima and Minima:

Suppose f is a function defined on an interval $[a, b]$. We say that f has a **global maximum** at $x = c$ (also called *absolute maximum*), if $f(c) \geq f(x)$ for all $x \in [a, b]$.

We similarly define the term **global minimum**.

For example, the function $f(x) = x(x-1)$ defined on $[0, 2]$ has a maximum and a minimum since f is cts on the closed interval $[0,2]$



In fact the min. is $-\frac{1}{4}$ and the max. is 2.

Theorem: (Min-Max Theorem)

Suppose that f is cts on a closed interval $[a, b]$ then f has a global maximum and a global minimum on $[a, b]$.

In other words, there are real numbers c and d in the interval $[a, b]$ such that

$$f(c) \leq f(x) \leq f(d) \qquad \text{for all } x \in [a, b].$$

Notice that the result is no longer true if we use open intervals or if the function is not cts.

For example $f(x) = \frac{1}{x}$ on the interval $[-1, 1]$ or on the interval $(0, 1)$.

Again, while this result seems intuitively obvious, it is not trivial to prove and we shall delete the proof.

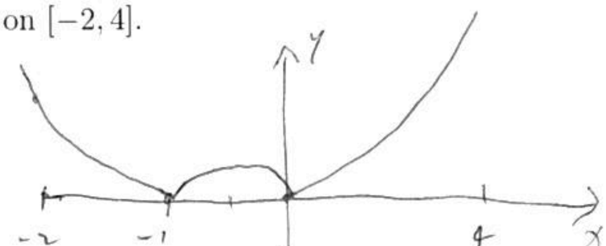
You should be able to state precisely, both of the above theorems.

Ex: Find the min and max of $f(x) = |x||x+1|$ on $[-2, 4]$.

For $x \leq -1$, $f(x) = -x \times -(x+1)$
$= x(x+1)$

For $-1 \leq x \leq 0$, $f(x) = -x(x+1)$

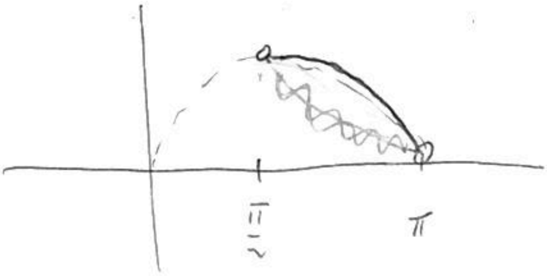
For $x \geq 0$, $f(x) = x(x+1)$



$f(-2) = 2$
$f(4) = 20$
$f(-\frac{1}{2}) = \frac{1}{4}$

f has a min. of 0 and a max. of 20

Ex: Repeat for $f(x) = \sin x$ on $(\pi/2, \pi)$.



f has no max and no min.

($(\frac{\pi}{2}, \pi)$ is an open interval.)

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Section 4: - Differentiable Functions.

The subject we call *calculus* began with the ancient Greeks, who considered finding the area bounded by curved figures. Indeed, as well as being able to find the volume of a sphere, Archimedes was able to find the area bounded by a parabola and a line. This was done in a purely geometric way. In late antiquity the problem of finding tangents to curves arose (notably the curves were conics) and some progress was made on this problem. The methods used were ingenious and for the most part *ad hoc*.

The fact that these two problems were related, was not discovered and proven until the 17th century by Newton and about the same time by Leibniz. Our modern version of calculus and the notation we use were essentially due to Leibniz.

In this section we shall study the theoretical ideas of differentiation.

Definition: Suppose f is defined on some open interval containing x. We say that f is **differentiable** at x if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists, and in this case, we denote the limit by $f'(x)$ of $\frac{df}{dx}$ or $\frac{d}{dx}(f(x))$.

It is called the derivative of f at x.

We say that f is **differentiable** on (a, b) is f is differentiable (write diffble) at each x in (a, b).

Ex: Find the derivatives of $f(x) = x^3$, and $f(x) = \sqrt{x}$, from first principles.

$f(x) = x^3$

$$\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3hx + h^2 = 3x^2.$$

$f(x) = \sqrt{x}$

$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{1}{2\sqrt{x}}.$$

At $x=0$, Ex: Show that $f(x) = |x|$ is not differentiable at $x = 0$

$$\lim_{h\to 0^+} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

$$\lim_{x\to 0^-} \frac{|0+h|-|0|}{h} = \lim_{h\to 0} \frac{-h}{h} = -1$$

$\therefore \lim_{h\to 0} f(x)$ does NOT exist

So $f(x)$ is not diff^{ble} at $x=0$

Ex: Check whether the following are differentiable at $x = 0$.

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and

$$g(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

For $f(x)$,

At $x=0$,

$$\lim_{h\to 0} \frac{h\sin\frac{1}{h} - 0}{h} = \lim_{h\to 0} \sin\left(\frac{1}{h}\right)$$

does NOT exist, so

f is not diff^{ble} at $x=0$

For $g(x)$ at $x=0$

$$\lim_{h\to 0} \frac{h^2\sin\frac{1}{h} - 0}{h} = \lim_{h\to 0} h\sin\frac{1}{h}$$

$= 0$ by Pinching Thm.

$\therefore$ g is diff^{ble} at $x=0$

Equivalent Definition: We could equally say that f is differentiable at $x = a$ if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists.

N.B. This says that for x near a
$f(x) \simeq (x-a)f'(a) + f(a)$
i.e. f is approximately a straight line!
& $\lim_{x \to a} f(x) = f(a)$
so f cts at $x =$

Proof: Replace $x - a$ by h.

Theorem: If f is differentiable at $x = a$ then f is cts at $x = a$.

Corollary: If f is NOT cts at $x = a$ then f is NOT diffble at $x = a$.

Ex:

$$f(x) = \begin{cases} e^x & x > 0 \\ x^2 & x \leq 0 \end{cases}$$

is not cts at $x = 0$ hence it cannot be diffble at $x = 0$.

Ex: Discuss whether or not $f(x) = x|x|$ is differentiable.

For $x > 0$ $f(x) = x^2$ which is diffble
For $x < 0$ $f(x) = -x^2$ " " " .
Look at what happens at $x = 0$

$$\lim_{h \to 0^+} \frac{h|h| - 0}{h} = \lim_{h \to 0^+} \frac{h^2}{h} = 0$$

$$\lim_{h \to 0^-} \frac{h|h| - 0}{h} = \lim_{h \to 0^-} \frac{-h^2}{h} = 0$$

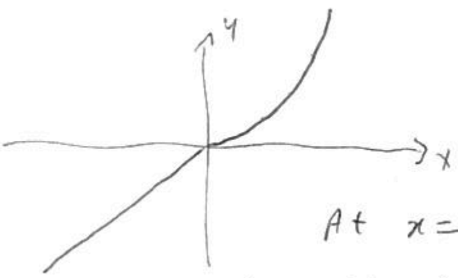
$\therefore f$ is diffble at $x = 0$
$\therefore f$ diffble everywhere.

Split Functions:

A function may be defined *piece-wise* using a split definiton.

Ex:

$$f(x) = \begin{cases} x & x < 0 \\ x^2 & x \geq 0 \end{cases}$$



Away from 0, f is diffble.

At $x = 0$

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0^+} \frac{h^2 - 0}{h} = 0$$

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0^-} \frac{h - 0}{h} = 1$$

$\therefore f$ NOT diffble at $x = 0$

Theorem: Suppose

$$f(x) = \begin{cases} p(x) & x \geq a \\ q(x) & x < a \end{cases}$$

with $p(x)$ and $q(x)$ differentiable in some interval containing a. Then f is differentiable at $x = a$ iff f is cts at $x = a$ and $p'(a) = q'(a)$.

Ex:

$$f(x) = \begin{cases} \sin x & x < \pi \\ ax + b & x \geq \pi \end{cases}$$

Given that f is diffble at π find a and b.

For f to be diffble, we first require f to be cts, so

$$\lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x) = f(\pi)$$

$$\therefore \sin \pi = a\pi + b$$

$$\therefore \boxed{a\pi + b = 0}$$

Also $$\frac{d}{dx}(\sin x)\Big|_{x=\pi} = \frac{d}{dx}(ax+b)\Big|_{x=\pi}$$

$$\cos \pi = a$$

$$\therefore \boxed{a = -1}$$

So then $\boxed{b = -\pi}$

Rules for Differentiation: Suppose f and g are both differentiable at $x = a$, then at $x = a$ we have

(i) $f \pm g$ is diffble and $(f \pm g)'(x) = f'(x) \pm g'(x)$.

(ii) $f(x) = C$ has derivative 0.

(iii) $cf(x)$ is differentiable and $(cf)'(x) = cf'(x)$ where c is a constant.

(iv) fg is differentiable and $(fg)'(x) = f(x)g'(x) + f'(x)g(x)$.

(v) $\frac{f}{g}$ is diffble, provided $g(a) \neq 0$ and $\left(\frac{f}{g}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$.

(vi) $f \circ g$ is differentiable and $(f \circ g)'(x) = f'(g(x)).g'(x)$.

Proof: See printed notes.

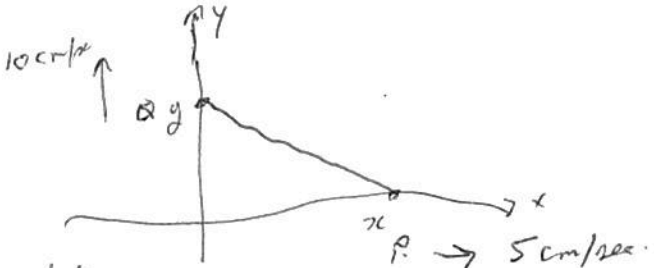
Rates of Change:

Suppose $y(t)$ is a quantity that can vary with time (t).
The **average** rate of change of y between times t and $t+h$ is clearly $\dfrac{y(t+h)-y(t)}{h}$.

If the limit of this rate exists as $h \to 0$, then that limit, $\frac{dy}{dt}$ will represent the *instantaneous rate of change* of y at time t.

You have already met this idea in relation to displacement, velocity, (which is the instantaneous rate of change of displacement) and acceleration (which is the instantaneous rate of change of velocity).

Ex: A point P is moving to the right along the x axis at a constant rate of 5 cm/sec and a point Q is moving up the y axis at a constant rate of 10 cm/sec. How fast is the distance between the two points changing when $OP = 30$ and $OQ = 40$?



Let

$D = \sqrt{x^2+y^2} = (x^2+y^2)^{\frac{1}{2}}$

By chain rule,

$$\frac{dD}{dt} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\left[2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right]$$

At $x = 30$, $y = 40$, $\frac{dx}{dt} = 5$, $\frac{dy}{dt} = 10$

$$\frac{dD}{dt} = \frac{1}{2\times 50}(300 + 800)$$

$= 11$ cm/sec

Ex: A sphere is increasing at a rate of 16 m³/sec. Find the rate of increase of the surface area when $r = 8$ m.

$V = \frac{4}{3}\pi r^3$, $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\therefore \frac{dr}{dt} = \frac{16}{4\pi r^2} = \frac{4}{\pi r^2}$

$S = 4\pi r^2$

$\frac{dS}{dt} = 8\pi r \frac{dr}{dt} = 8\pi r \cdot \frac{4}{\pi r^2}$

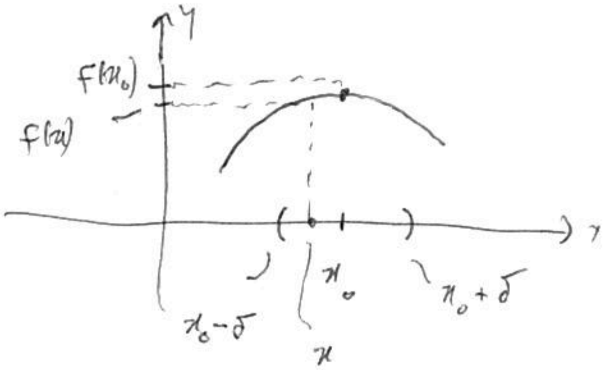
$= \frac{32}{r} = 4$ m²/sec at $r = 8$

Local maxima and minima and stationary points:

Local Maxima and Minima:

Definition: Suppose that f is a function and x_0 is a point in the domain of f. Consider the interval $(x_0 - \delta, x_0 + \delta)$, where we think of δ as being a small positive real number. If $f(x) \leq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta)$ then we say that x_0 is a **local maximum**.

Also, if $f(x) \geq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta)$ then we say that x_0 is a **local minimum**.



Theorem: Suppose that f is defined on the interval (a, b), and has a local maximum (or minimum) at $c \in (a, b)$.
If f is differentiable at c then $f'(c) = 0$.

Proof: Suppose that f has a minimum at c. Then $f(x) - f(c)$ must be positive (or 0) for all $x \in (c - \delta, c + \delta)$ for some $\delta > 0$. Hence $\lim\limits_{x \to c^+} \dfrac{f(x) - f(c)}{x - c} \geq 0$ since the denominator is also positive.

Further $\lim\limits_{x \to c^-} \dfrac{f(x) - f(c)}{x - c} \leq 0$ for all $x \in (c - \delta, c + \delta)$ since the denominator is always negative.
Now f is differentiable, so the above limits exists as $x \to c$ hence by the pinching theorem we have $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} = 0$, but this says $f'(c) = 0$.

Implicit Differentiation:

While most functions you have met were presented in an explicit form, that is, with y as the subject, a function may also be given implicitly.

For example $x^2 + y^2 = 9, y \geq 0$ gives an implicit form of the semi-circle radius 3 centre O. In order to find $\frac{dy}{dx}$, we could solve for y explicitly and then differentiate.

This is not always possible. For example, the function implicitly defined by $y^5 + xy = 3$

cannot be algebraically solved for y (although you could make x the subject and find $\frac{dx}{dy}$).

To differentiate such an expression, we use **implicit differentiation** which is really just an application of the chain rule.

Ex: Find $\frac{dy}{dx}$ for $x^4 + y^4 = 1$, and for $y^5 + xy = 3$.

$$x^4 + y^4 = 1$$

$$\frac{d}{dx}: \quad 4x^3 + 4y^3 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x^3}{y^3}.$$

$$y^5 = xy + 3$$

$$\frac{d}{dx}: \quad 5y^4 \frac{dy}{dx} = y + x\frac{dy}{dx}$$

$$\frac{dy}{dx}(5y^4 - x) = y$$

$$\frac{dy}{dx} = \frac{y}{5y^4 - x}.$$

Ex: Find the gradient at the point $(1,1)$ for the curve (in fact it is an hyperbola).
$x^2 - 3xy + 2y^2 + y = 1$

$$\frac{d}{dx}: \quad 2x - 3y - 3xy' + 4yy' + y' = 0$$

$$y'(4y + 1 - 3x) = 3y - 2x$$

$$y' = \frac{3y - 2x}{4y + 1 - 3x}.$$

$$\therefore \quad y' = \frac{1}{2} \text{ at } (1,1)$$

Ex: Prove that $\frac{d}{dx}x^\alpha = \alpha x^{\alpha-1}$ then α is rational.

Write $\alpha = \frac{p}{q}$, then $y^q = x^p$. Hence $qy^{q-1}y' = px^{p-1}$. Solving for y' and substituting we have

$$y' = \frac{p}{q}x^{p-1}y^{1-q} = \frac{p}{q}x^{p-1}x^{\frac{p}{q}(1-q)} = \frac{p}{q}x^{\frac{p}{q}-1} = \alpha x^{\alpha-1}.$$