Chapter 3: Continuous Functions

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MATH1131

UNSW

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Revision: Continuity at a point

Definition

Let f be defined on some open interval containing the point a. We say that f is continuous at a if

$$\lim_{x \to a} f(x) = f(a);$$

otherwise we say that f is discontinuous at a.

The value $f(a)$ needs to be defined and the limit needs to exist!

If f is continuous at every point $a \in \mathbb{R}$, then f is called continuous everywhere.

Combining continuous functions

Theorem

Suppose that the functions f and g are continuous at a point a. Then

$$f+g, \quad f-g, \quad fg$$

are continuous at a.

If $g(a) \neq 0$ then

$$f/g$$

is also continuous at a.

Proof

Suppose that f and g are continuous at a. Then,

$$\lim_{x \to a} f(x) = f(a), \qquad \lim_{x \to a} g(x) = g(a)$$

by the definition of continuity at a point. Therefore,

$$\begin{aligned}
\lim_{x \to a} (f+g)(x) &= \lim_{x \to a} (f(x) + g(x)) && \text{(def. of } f+g\text{)} \\
&= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) && \text{(limit rule)} \\
&= f(a) + g(a) && (f, g \text{ cont.}) \\
&= (f+g)(a) && \text{(def. of } f+g\text{).}
\end{aligned}$$

Hence $f+g$ is continuous at a.

The proofs that the functions $f-g$, fg and f/g are continuous at a are similar.

Composition of continuous functions

Even larger classes of continuous functions may be obtained in the following manner:

Theorem

Suppose that f is continuous at a and that g is continuous at $f(a)$. Then $g \circ f$ is continuous at a.

Proof.

$$\begin{aligned}
\lim_{x \to a} (g \circ f)(x) &= \lim_{x \to a} (g(f(x)) && \text{(def. of } g \circ f\text{)} \\
&= g(\lim_{x \to a} f(x)) && \text{(cont. of } g\text{)} \\
&= g(f(a)) && \text{(cont. of } f\text{)} \\
&= (g \circ f)(a). && \text{(def. of } g \circ f\text{)}
\end{aligned}$$

Hence $g \circ f$ is continuous at a.

Examples

We said in the previous chapter that the elementary functions are continuous everywhere on their domain.

Let us start with the simplest functions:

- constant functions
- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$
- the sine function.

These functions are continuous everywhere on $\mathbb{R}$ (the proof uses the formal definition of the limit at a point, which you don't need to know yet).

But we can use these three functions, together with the operations above, to prove the continuity of new functions.

Example 1. Show that polynomials and rational functions are continuous at every point of their respective domains.

Solution. Any polynomial can be obtained from f and constant functions via addition and multiplication, e.g. $x^3 - 4x^2 + 5 = [(x \times x \times x)] + [(-4) \times x \times x] + 5$, and hence is continuous everywhere.

Similarly, any rational function is of the form $\dfrac{p(x)}{q(x)}$, where p and q are two (continuous) polynomials, and is therefore continuous at every point a for which $q(a) \neq 0$.

Example 2. Show that cosine function is continuous everywhere.

Solution. Recall that

$$\cos x = \sin(\pi/2 - x) \quad \forall x \in \mathbb{R}.$$

Thus, we can write $\cos(x) = g(h(x))$, where $g(x) = \sin x$ and $h(x) = \pi/2 - x$.

Now, since h is continuous everywhere (as a linear polynomial) and the sine function is also continuous everywhere, the cosine function is also continuous everywhere.

Example 3. Why is $f(x) = \sqrt{\cos^2(x) + 3}$ continuous everywhere?

Solution.

Short answer: It is a combination of continuous functions and hence is continuous.

Longer answer: Let $g_1(x) = \cos x$, $g_2(x) = x^2 + 3$ and $g_3(x) = \sqrt{x}$. Then

$$f(x) = g_3(g_2(g_1(x)))$$

Now g_1, g_2 and g_3 are continuous everywhere they are defined. Hence the composition f is also continuous everywhere.

Example 4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \cos(ax) & x \le \pi \\ bx & x > \pi. \end{cases}$$

For what values of a and b will f be continuous?

Example 4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \cos(ax) & x \leq \pi \\ bx & x > \pi. \end{cases}$$

For what values of a and b will f be continuous?

Solution. From the definition of f we see that f is continuous when $x \neq \pi$. Indeed, since $f(x) = \cos(ax)$ when $x \leq \pi$, and the cosine function is continuous, f is continuous. Similarly for $x > \pi$.
The only potential problem with continuity is at $x = \pi$. For continuity at π we must have (by definition) $\lim_{x \to \pi} f(x) = f(\pi) = \cos(a\pi)$. As the rule for f has a change in form at π we need to use left and right limits to evaluate $\lim_{x \to \pi} f(x)$.
We have

$$\lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^-} \cos(ax) = \cos(a\pi)$$

and

$$\lim_{x \to \pi^+} f(x) = \lim_{x \to \pi^+} bx = b\pi.$$

Thus for continuity of f we require $\cos(a\pi) = b\pi \implies b = \frac{\cos(a\pi)}{\pi}$.

Continuity on intervals

In the previous chapter we define continuity of a function f at a point. We now define what means continuity on an interval.

Continuity on (a, b)

Suppose that f is a real-valued function defined on an open interval (a, b). We say that f is continuous on (a, b) if f is continuous at every point of the interval (a, b).

Continuity on $[a, b]$

Suppose f is a real-valued function defined on a closed interval $[a, b]$. We say that

- f is continuous at the endpoint a if

$$\lim_{x \to a^+} f(x) = f(a),$$

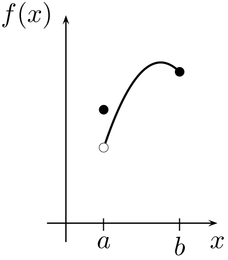
- f is continuous at the endpoint b if

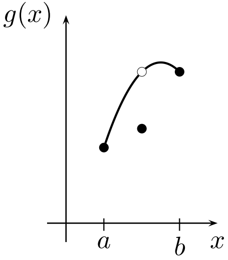
$$\lim_{x \to b^-} f(x) = f(b),$$

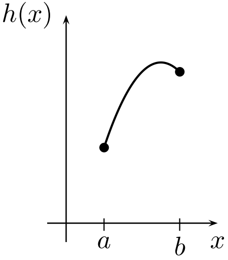
- f is continuous on $[a, b]$ if f is continuous on (a, b) and at each of the endpoints a and b.

Example

Example. Consider the functions f, g and h, whose graphs are shown below.







All three functions are defined on the interval $[a, b]$.

- f is continuous on the open interval (a, b) and at the endpoint b.
- g is continuous at the endpoints a and b but not continuous on the open interval (a, b).
- h is continuous on the closed interval $[a, b]$.

The intermediate value theorem

Look at the following two claims:

1. A plane takes off and after 12 minutes it is at 20,000 feet. At some point, it must have passed through an altitude of 10,000 feet.
2. Yesterday GreenEnergy shares were \$2.34 a share. Today they are trading at \$1.47 a share. At some point they must have been trading at \$2.00 a share.

The first of these is true, the second not. The difference lies in the properties of the two functions involved:

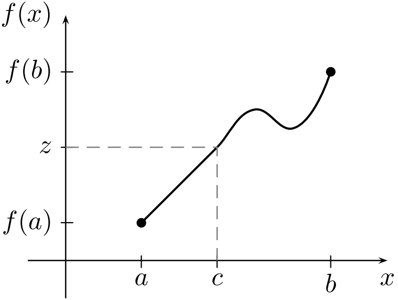
$A(t) =$ altitude at time t
$S(t) =$ share price at time t.

The first is a continuous function on a nice domain $[0, 12]$. The second is much more complicated (not continuous)!

The intermediate value theorem (IVT)

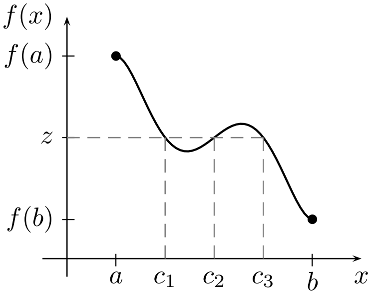
Theorem

Suppose that f is continuous on the closed interval $[a, b]$. If z lies between $f(a)$ and $f(b)$ then there exists at least one real number c in $[a, b]$ such that $f(c) = z$.



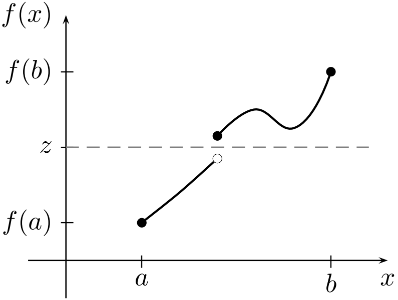
A few more remarks on IVT

- The number c in $[a, b]$ may not be unique.



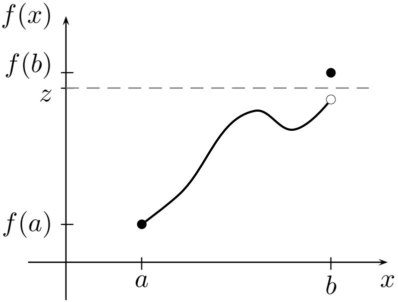
Here, there exists three numbers c_i with $f(c_i) = z$.

- Continuity of f is crucial.



Here, for $z \in [f(a), f(b)]$ $\nexists$ $c \in [a, b]$ such that $f(c) = z$.

- Continuity on a closed interval $[a, b]$ is crucial.



Here, for $z \in [f(a), f(b)]$ $\nexists$ $c \in [a, b]$ such that $f(c) = z$.

Applications of IVT

Often, the IVT is used to show that the equation

$$f(x) = 0$$

has a solution in $[a, b]$.

In this case you need to check:

- f is continuous on $[a, b]$, and
- $f(a)f(b) < 0$.

(i.e. $f(a)$ and $f(b)$ are the opposite sign of each other – meaning that to get from $f(a)$ to $f(b)$ you have to cross through zero.)

Example

Show that there exists a solution $c \in [1, 2]$ of the equation

$$\sqrt{c} = c^2 - 1$$

and approximate its value.

Solution. Consider the function $f(x) = \sqrt{x} - x^2 + 1$. Since f is continuous on [1,2], $f(1) = 1 > 0$ and $f(2) = \sqrt{2} - 3 < 0$, by IVT we have that there exists $c \in [1, 2]$ such that $f(c) = 0$. That is $\sqrt{c} - c^2 + 1 = 0$ or $\sqrt{c} = c^2 - 1$.

Let's find an approximate value of $c \rightarrow$ cut the interval in half!

$f(1.5) \sim -0.026 < 0 \Rightarrow c \in [1, 1.5]$
$f(1.25) \sim 0.55 > 0 \Rightarrow c \in [1.25, 1.5]$
$f(1.375) \sim 0.28 > 0 \Rightarrow c \in [1.375, 1.5]$
$f(1.4375) \sim 0.13 > 0 \Rightarrow c \in [1.4375, 1.5]$
$f(1.46875) \sim 0.05 > 0 \Rightarrow c \in [1.46875, 1.5]$
$f(1.484375) \sim 0.01 > 0 \Rightarrow c \in [1.484375, 1.5]$
$f(1.4921875) \sim -0.005 < 0 \Rightarrow c \in [1.484375, 1.4921875]$

Example. Show that the equation $\ln(x+1) = \cos x$ has at least one positive solution.

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We rearrange $\ln(x+1) = \cos x$ to $f(x) = \ln(x+1) - \cos x$. Note: $f(x)$ is the sum of 2 elementary (and continuous) functions, so $f(x)$ is continuous.
We then try to find a positive x value, $x = a$, such that $f(a) < 0$, and also another positive x value, $x = b$, such that $f(b) > 0$. Then the IVT says that there is a x value, $x = c$, such that $f(c) = 0$, (as $f(a) < 0 < f(b)$).
So now we need to find possible a and b values.
Now $\ln 1 = 0$, so choosing $x = 0$, $\quad f(0) = 0 - 1 < 0$.
$\cos\left(\frac{\pi}{2}\right) = 0$, so choosing $x = \frac{\pi}{2}$, $f\left(\frac{\pi}{2}\right) = \ln\left(\frac{\pi}{2}\right) - 0 > 0$.
(In fact, $f(\pi/2) \approx 0.9442$.)
Hence $0 \in (f(0), f(\pi/2))$. By the IVT there exists $c \in (0, \pi/2)$ such that $f(c) = 0$, i.e., $\ln(c+1) = \cos c$.
In other words, the equation $\ln(x+1) = \cos x$ has a positive solution c.

Example

Show that if f is continuous on $[0, 1]$ with $0 \leq f(x) \leq 1$, then there exists $c \in [0, 1]$ such that $f(c) = c$.

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Solution. There are two cases:

1. Case 1: f satisfies $f(0) = 0$ or $f(1) = 1$.
   Here the statement is trivially true with $c = 0$ or $c = 1$.
2. Case 2: f satisfies $f(0) \neq 0$ and $f(1) \neq 1$.

   Here,
   $$0 < f(0) \leq 1 \quad \text{and} \quad 0 \leq f(1) < 1.$$
   Define
   $$g : [0,1] \to \mathbb{R} \quad g(x) = f(x) - x.$$
   Then g is continuous on the closed interval $[0,1]$.

Moreover, $g(0) = f(0) - 0 = f(0) > 0$ and $g(1) = f(1) - 1 < 0$.
Since $0 \in [g(1), g(0)]$, by IVT there exists $c \in [0,1]$ such that $g(c) = 0$. Thus, there exists $c \in [0,1]$ such that $f(c) = c$.

The maximum-minimum theorem

Definition

Suppose that f is defined on a closed interval $[a, b]$.

- We say that a point c in $[a, b]$ is an absolute minimum point for f on $[a, b]$ if

  $$f(c) \leq f(x) \quad \text{for all} \quad x \in [a, b].$$

  The corresponding value $f(c)$ is called the absolute minimum value of f on $[a, b]$. If f has an absolute minimum point on $[a, b]$ then we say that f attains a minimum on $[a, b]$.
- We say that a point d in $[a, b]$ is an absolute maximum point for f on $[a, b]$ if

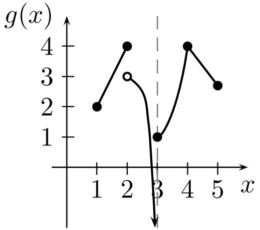
  $$f(x) \leq f(d) \quad \text{for all} \quad x \in [a, b].$$

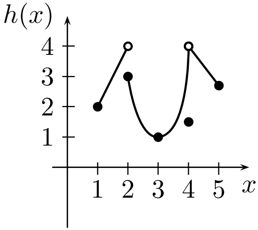
  The corresponding value $f(d)$ is called the absolute maximum value of f on $[a, b]$. If f has an absolute maximum point on $[a, b]$ then we say that f attains a maximum on $[a, b]$.

An absolute maximum point and an absolute minimum point are sometimes referred to as a global maximum point and a global minimum point.

Example

Consider the functions g and h, which are illustrated below.



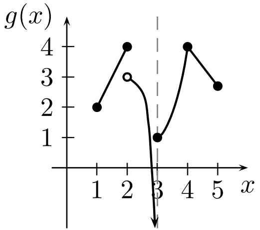


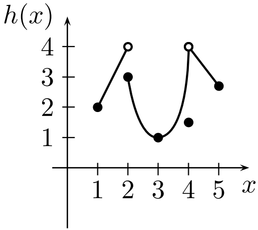
The absolute minimum and maximum points of g and h on $[1, 5]$ are recorded in the following table.

	g	h
Absolute maximum value Absolute maximum points		.
Absolute minimum value Absolute minimum points	.	

Example

Consider the functions g and h, which are illustrated below.





The absolute minimum and maximum points of g and h on $[1, 5]$ are recorded in the following table.

	g	h
Absolute maximum value	4	n.a.
Absolute maximum points	2, 4	none
Absolute minimum value	n.a.	1
Absolute minimum points	none	3

This example shows that a function $f : [a, b] \to \mathbb{R}$ need not have an absolute maximum point (or an absolute minimum point) on a closed interval $[a, b]$. But...

The maximum-minimum theorem

Theorem

If f is continuous on a closed interval $[a, b]$ then f attains an absolute minimum and absolute maximum on $[a, b]$. That is, there exist points c and d in $[a, b]$ such that

$$f(c) \leq f(x) \leq f(d)$$

for all x in $[a, b]$.

If you drop any of these conditions the theorem is false!

Remark: Locating the absolute max and min is not that straightforward! ... see Chapter 5!

Examples

- The function $f : [1, 2] \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ has both an absolute maximum point and an absolute minimum point on $[1, 2]$ since it is continuous on $[1, 2]$.

  But, the function $g : (1, 2) \to \mathbb{R}$ defined by $g(x) = \frac{1}{x}$ has neither an absolute maximum point nor an absolute minimum point on $(1, 2)$.

  Thus, we can not drop the assumption that the interval $[a, b]$ is closed in the Max-Min theorem.

- The function $h : [-1, 1] \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} \frac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

  has neither an absolute maximum point nor an absolute minimum point on $[-1, 1]$.

Bounded functions

Definition

Suppose that $f : A \to \mathbb{R}$. We say that f is bounded on A if there exists some positive number M such that

$$|f(x)| \leq M, \quad \text{for all } x \in A.$$

The domain A is a clearly vital part of this definition. The function $f(x) = x^2$ is bounded on the domain $[0, 100]$, but not on the domain $\mathbb{R}$.

The Max-Min Theorem implies:

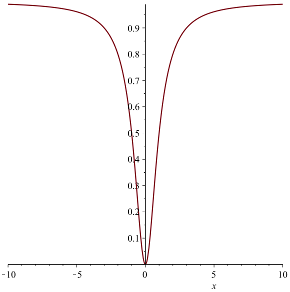
Theorem

If f is continuous on a closed interval $[a, b]$, then it is a bounded function on $[a, b]$.

- *Don't get bounded intervals and bounded functions confused!*
- Note that a function can be bounded without having an absolute maximum or minimum value...

Example. Is the function $f : [0, \infty) \to \mathbb{R}$, $f(x) = \frac{x^2}{1+x^2}$ bounded? Does it attain an absolute minimum or maximum?

$>$plots[interactive]$(x^2/(x^2+1))$



- f is bounded on $[0, \infty)$: $|f(x)| \leq 1$ for all $x \in [0, \infty)$
- f attains an absolute minimum (0) at $x = 0$
- but f does not attain an absolute maximum (horizontal asymptote!)

Note: This is another example that we can not drop the assumption that the interval $[a, b]$ is closed in the Max-Min theorem.

Summary: What did we learn in this chapter?

- Combination of continuous functions (p. 3, 5)
- Continuity on intervals (p. 10)
- Intermediate value theorem (IVT, p. 14)
- IVT to show f(x)=0 has a solution (p. 18)
- Absolute maximum / minimum (p. 22)
- Maximum-minimum theorem (p. 25)
- Bounded functions (p. 27)