School of Mathematics and Statistics

Math1131 Mathematics 1A

CALCULUS LECTURE 2

FUNCTIONS

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MATH1131 CALCULUS
FUNCTIONS

A function $f : A \to B$ is a rule which assigns each $x \in A$ to exactly one element $y \in B$.

$$(f \circ g)(x) = f(g(x))$$

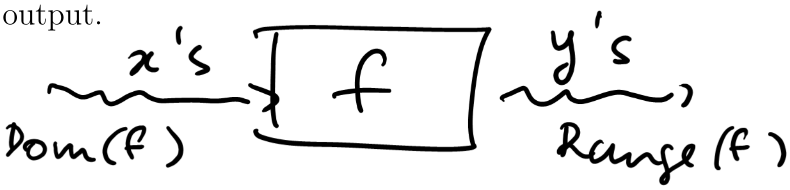
The abstract concept of a function is absolutely crucial for our later analysis of calculus. Indeed both integration and differentiations are process which are almost always applied to functions.

Definition: A function $f : A \to B$ is a rule which assigns each $x \in A$ to exactly one element $y \in B$.

The set A is referred to as the domain of f and denoted by $\text{Dom}(f)$, the set B is the co-domain of f and denoted by $\text{Codom}(f)$ and finally the set of all y values produced by the function is call the range of f and denoted by $\text{Range}(f)$.

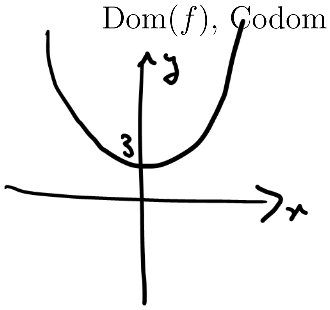
The co-domain is an announcement of everything that the function **might** produce. The range is what actually **is** produced.

The mental image to have of a function is a little machine which takes x's as input and spits out y's as output.



Remember also that every function has a graph and the graph often reveals its essential features.

Example 1: Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 + 3$. Sketch a graph of f and find $\text{Dom}(f)$, $\text{Codom}(f)$ and $\text{Range}(f)$. Evaluate $f(7)$ and find $f(w - 5)$.



$y = f(x) = x^2 + 3$

$x = 7 \to y = 7^2 + 3 = 49 + 3 = 52$

$f(7) = 52$

$\text{Dom}(f) = \mathbb{R}$, $\text{Codom}(f) = \mathbb{R}$.

$\text{Range}(f) : \{y \in \mathbb{R} : y \geq 3\} = [3, \infty)$

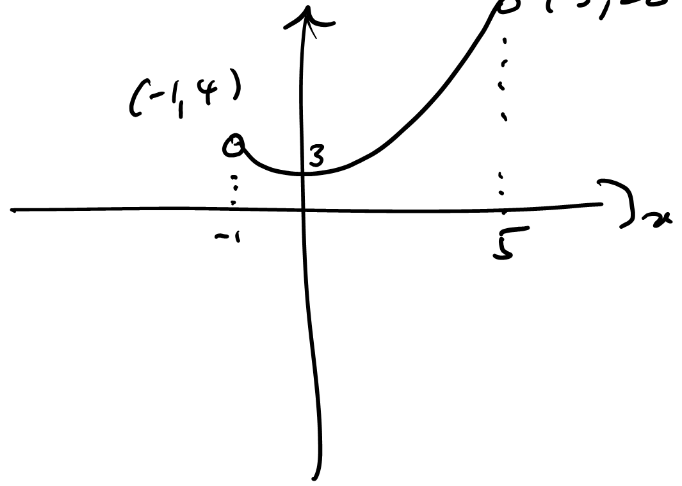
$\bigstar \quad \mathbb{R}, \quad \mathbb{R}, \quad [3, \infty), \quad 52, \quad w^2 - 10w + 28 \quad \bigstar$

$f(w-5) = (w-5)^2 + 3 = w^2 - 10w + 25 + 3$
$= w^2 - 10w + 28$

We can also artificially restrict the domain of a function if we need to:

Example 2: $f : (-1, 5) \to \mathbb{R}$ be given by $f(x) = x^2 + 3$. Sketch a graph of f and find Dom(f), Codom(f) and Range(f).

$(-1, 5)$ $\qquad$ $\mathbb{R}$



Range(f): $3 \leq y < 28$

$= [3, 28)$

$f(7) =$ undefined!!

★ $(-1, 5)$, $\quad \mathbb{R}$, $\quad [3, 28)$ ★

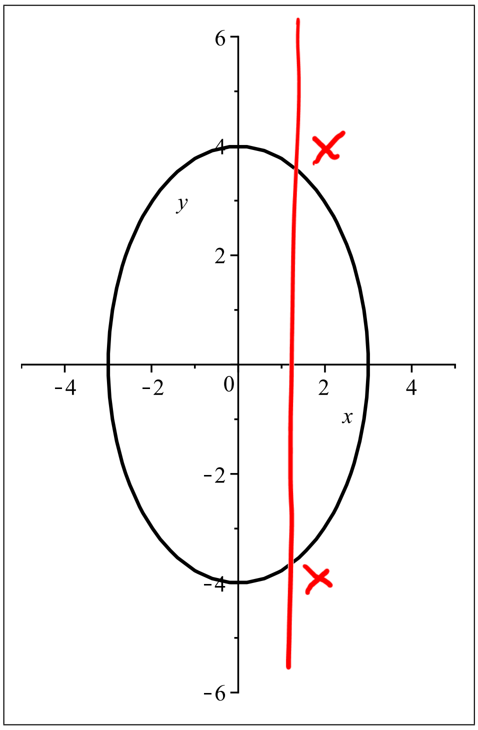
Question: What is the value of $f(7)$ in Example 2 above?

A small technicality before we continue. If a rule produces multiple y values for a single x value we call it a relation rather than a function. This is not the end of the world, however we do prefer the rigid mapping that springs from a definition of a function. It is always a worry to be picking over multiple possible y-values for relations which are not functions however we can cope with it if necessary.

Vertical Line Test: A relation is a function if a vertical line crosses its graph at most once.

Example 3: Identify which of the following are graphs of functions. In each case write down the domain and range.

a)

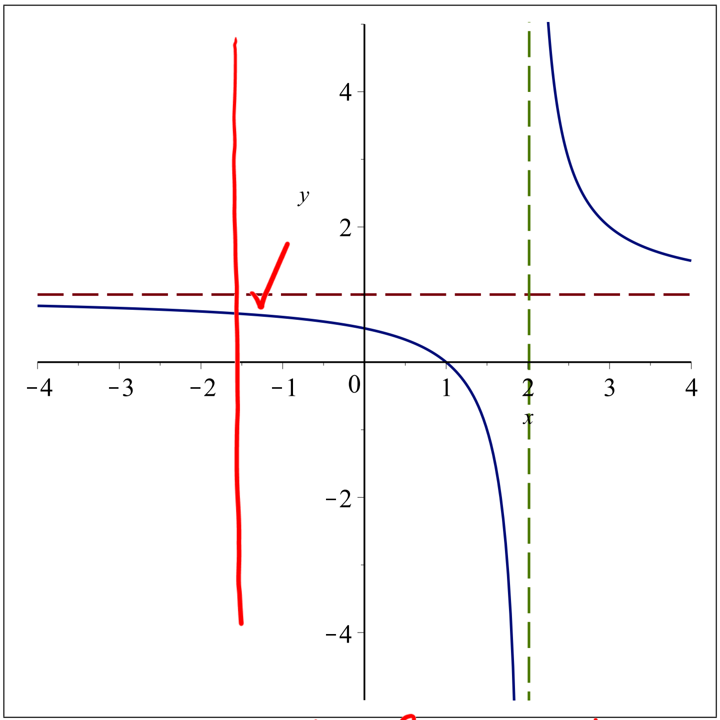


fails vertical line test. $\therefore$ NOT a function

Dom (g) : $[-3, 3]$

Range(g) : $[-4, 4]$

b)



passes vertical line test. $\therefore$ IS a function.

Dom(f) : $\{x \in \mathbb{R} \,/\, x \neq 2\}$

Range(f) : $\{y \in \mathbb{R} \mid y \neq 1\}$

★

When trying to find the domain of a function two issues to consider are that you may not take the square root of a negative number or divide by zero. Ranges are often determined intuitively or from a sketch.

Example 4: Find the domain of each of the following functions:

a) $f(x) = \dfrac{x-2}{x^2-9}$.

b) $f(x) = \sqrt{(x-5)^4(1-x)^3}$.

c) $f(x) = \dfrac{x-5}{x-5}$. What is the range of this function?

a) $y = \dfrac{x-2}{x^2-9}$

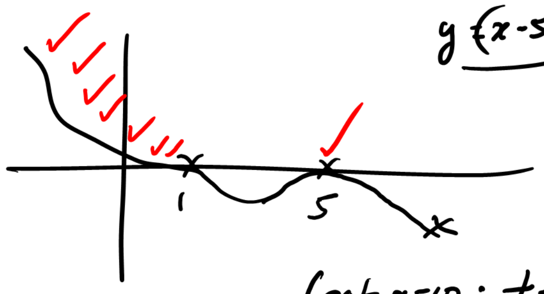
clearly $x \neq \pm 3$

$\therefore Dom(f) = \{x \in \mathbb{R} \,/\, x \neq \pm 3\}$

$= (-\infty,-3) \cup (-3,3) \cup (3,\infty)$

b) $y = \sqrt{(x-5)^4(1-x)^3}$

$y = \underline{(x-5)^4}\,\underline{(1-x)^3}$

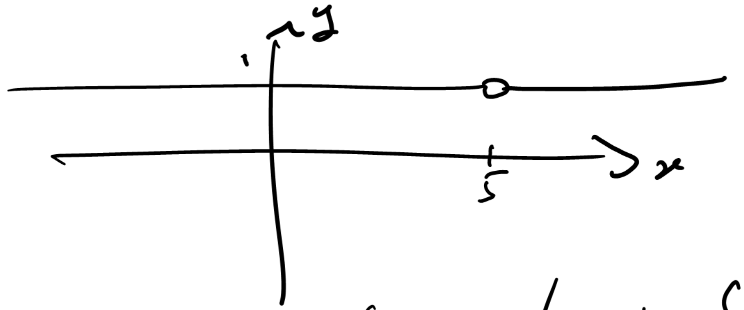


test $x=10$: $+ - = -$

$Dom(f) = (-\infty, 1] \cup \{5\}$

c) $f(x) = \dfrac{x-5}{x-5}$

$= 1 \, , \; x \neq 5$



$Dom(f) = \{x \in \mathbb{R} \,/\, x \neq 5\}$

$Range(f) = \{1\}$

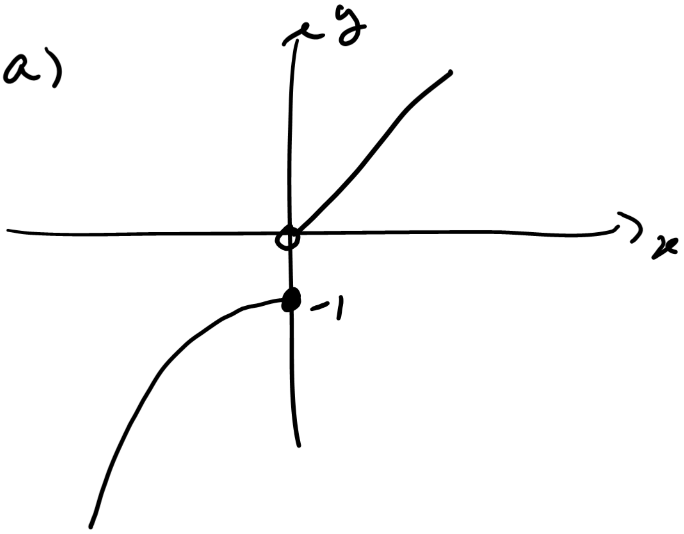
★

Example 5: Consider the function f defined to be

$$f(x) = \begin{cases} -x^2 - 1 \, , & x \leq 0; \\ 3x \, , & x > 0. \end{cases}$$

a) Sketch f.
b) Find $f(5)$.
c) Write down the domain and range of f.

a)



b) $f(5) = 15$

c) $Dom(f) = \mathbb{R}$

$Range(f) : \{y \in \mathbb{R} \,/\, y > 0 \text{ or } y \leq -1\}$

$= (0,\infty) \cup (-\infty,-1]$

★

B

Functions may be added, subtracted, multiplied and divided in the usual way. A new operation however is that of **composition** $\circ$. It is defined as

Definition: $(f \circ g)(x) = f(g(x))$.

Example 6: Let $f(x) = x^2$ and $g(x) = \dfrac{1}{x-5}$. Find $f \circ g$ and $g \circ f$.

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x-5}\right) = \frac{1}{(x-5)^2}$$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = \frac{1}{x^2-5}$$

$$\bigstar \quad (f \circ g)(x) = \left(\frac{1}{x-5}\right)^2, \quad (g \circ f)(x) = \frac{1}{x^2-5} \quad \bigstar$$

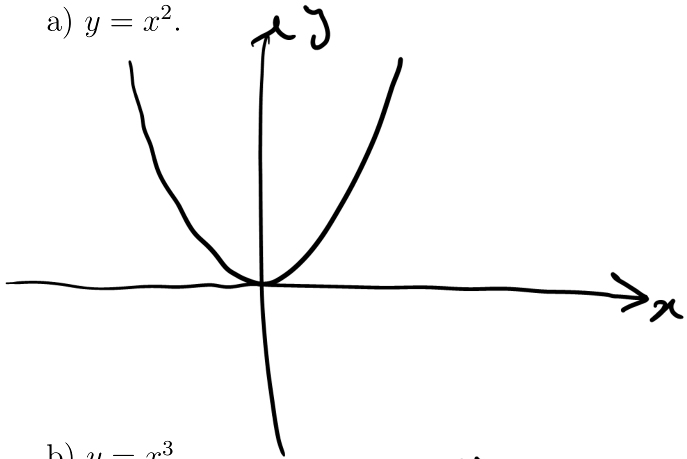
Observe from the above that in general $f \circ g \neq g \circ f$.

You must be able to sketch all the standard functions and be aware of their domains and ranges. See your printed notes for a comprehensive list of sketches:

Example 7: **HOMEWORK**

Sketch each of the following stating the domain and range in each case :

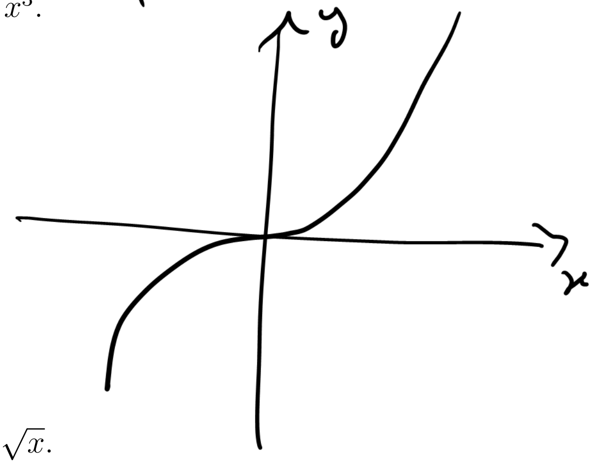
a) $y = x^2$.



Dom: $\mathbb{R}$

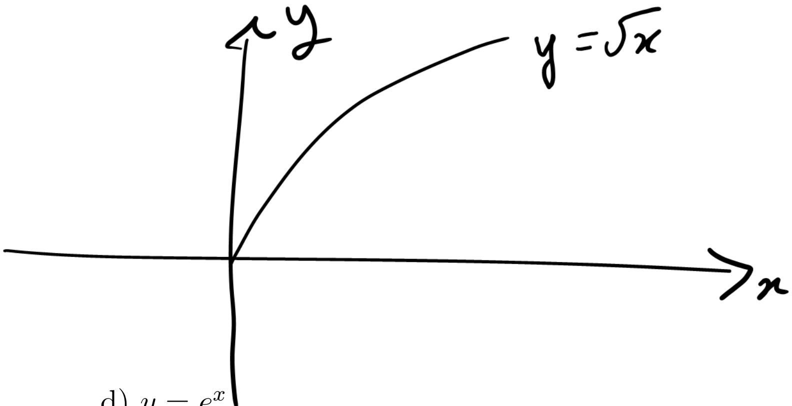
Range: $[0, \infty)$

b) $y = x^3$.



Dom = Range = $\mathbb{R}$

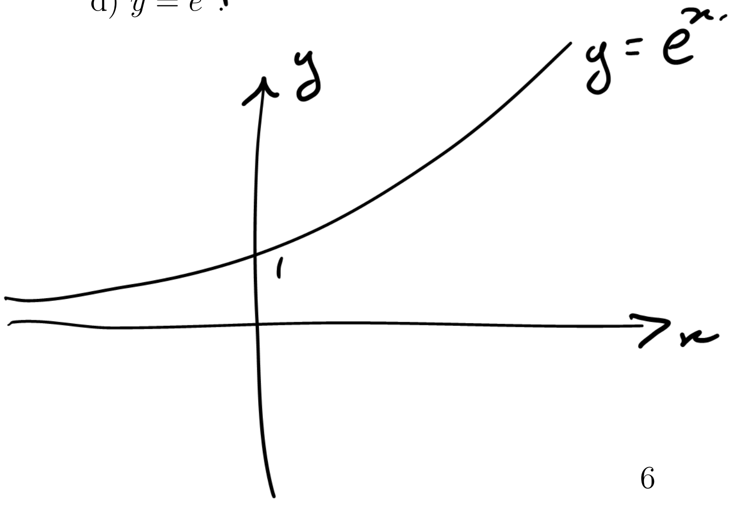
c) $y = \sqrt{x}$.



Dom: $[0, \infty)$

Range: $[0, \infty)$

d) $y = e^x$.

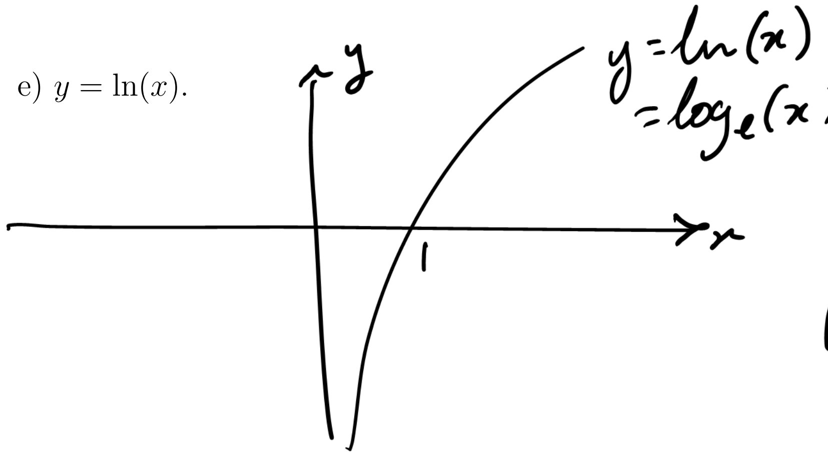


Dom: $\mathbb{R}$.

Range: $y > 0$

$= (0, \infty)$

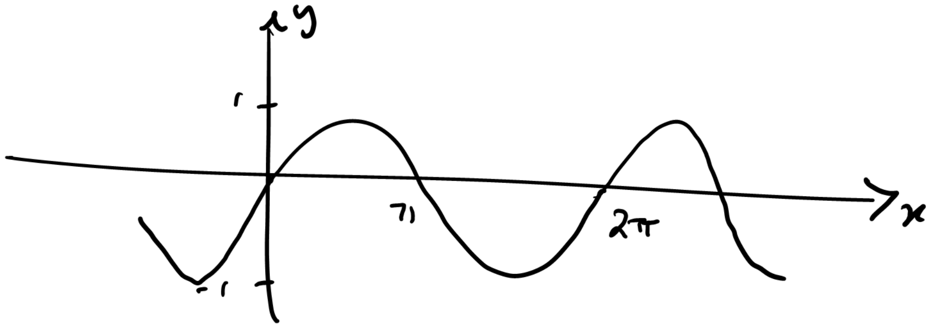
e) $y = \ln(x)$.



Dom: $(0, \infty)$

Range: $\mathbb{R}$.

f) $y = \sin(x)$.

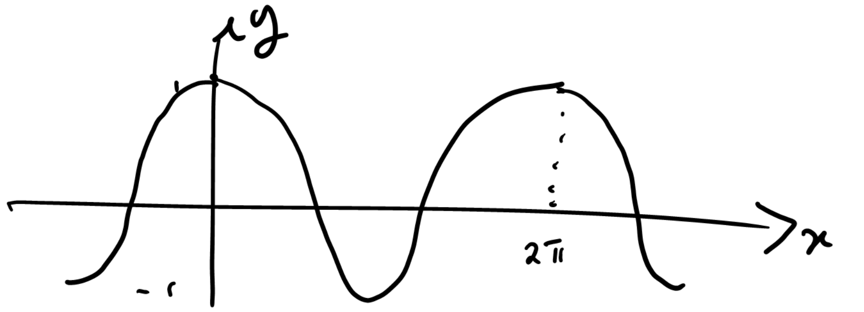


Dom: $\mathbb{R}$

Range: $[-1, 1]$

period $= 2\pi$

g) $y = \cos(x)$.

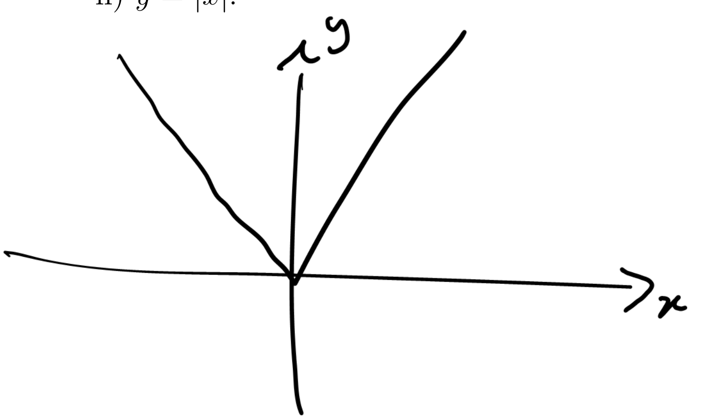


Dom: $\mathbb{R}$.

Range: $[-1, 1]$

period $= 2\pi$

h) $y = |x|$.



Dom: $\mathbb{R}$.

Range: $y \geq 0$.

★

Later on the immense machinery of calculus will be used to sketch graphs. For the moment however we consider intercepts, vertical asymptotes (division by zero) and horizontal asymptotes (behaviour for large x). Sometimes we simply manipulate standard graphs.

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