MATH1131 Mathematics 1A – Algebra

Lecture 19: Matrix Transpose and Inverses

Lecturer: Sean Gardiner – sean.gardiner@unsw.edu.au

Based on slides by Jonathan Kress

Transpose

Definition

For any $m \times n$ matrix A, its transpose A^T is the $n \times m$ matrix whose rows are the columns of A. That is,

$$[A^T]_{ij} = [A]_{ji}$$

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if $A = \begin{pmatrix} 3 & 1 & 2 \\ 4 & 7 & 8 \end{pmatrix}$ then $A^T = \begin{pmatrix} 3 & 4 \\ 1 & 7 \\ 2 & 8 \end{pmatrix}$

and if $B = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$ then $B^T = \begin{pmatrix} 3 & 6 & 9 \end{pmatrix}$

Properties of transposes

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For all matrices $A, B \in M_{mn}$ and $C \in M_{nq}$, and scalars λ, μ,

- $(A^T)^T = A$.
- $(\lambda A + \mu B)^T = \lambda A^T + \mu B^T$.
- $(AC)^T = C^T A^T$.

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Proof

For $1 \le i \le m$ and $1 \le j \le n$,

$$[(A^T)^T]_{ij} = [A^T]_{ji} = [A]_{ij}.$$

Hence $(A^T)^T = A$. $\square$

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Proof

Both $(\lambda A + \mu B)^T$ and $\lambda A^T + \mu B^T$ are $n \times m$ matrices, and for all $1 \leq i \leq n$ and $1 \leq j \leq m$,

$$\begin{aligned} [(\lambda A + \mu B)^T]_{ij} \;=\; [(\lambda A + \mu B)]_{ji} \;&=\; \lambda [A]_{ji} + \mu [B]_{ji} \\ &=\; \lambda [A^T]_{ij} + \mu [B^T]_{ij} \;=\; [\lambda A^T + \mu B^T]_{ij}. \end{aligned}$$

Hence $(\lambda A + \mu B)^T = \lambda A^T + \mu B^T$. $\square$

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- $(A^T)^T = A$.
- $(\lambda A + \mu B)^T = \lambda A^T + \mu B^T$.
- $(AC)^T = C^T A^T$.

Proof

Both $(AC)^T$ and $C^T A^T$ are $q \times m$ matrices, and for $1 \leq i \leq q$ and $1 \leq j \leq m$,

$$[C^T A^T]_{ij} = \sum_{k=1}^{n} [C^T]_{in} [A^T]_{nj} = \sum_{k=1}^{n} [C]_{ni} [A]_{jn} = [AC]_{ji} = [(AC)^T]_{ji}$$

This means $C^T A^T = (AC)^T$. $\square$

Symmetric matrices

Definition

An $n \times n$ matrix A is said to be symmetric if $A = A^T$.

For example,

$$B = \begin{pmatrix} 1 & 6 & 8 \\ 6 & 5 & 7 \\ 8 & 7 & 2 \end{pmatrix}$$

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If A is an $m \times n$ matrix, then $C = A^T A$ is an $n \times n$ matrix and

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$$C^T = (A^T A)^T = A^T (A^T)^T$$

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If A is an $m \times n$ matrix, then $C = A^T A$ is an $n \times n$ matrix and

$$C^T = (A^T A)^T = A^T (A^T)^T = A^T A = C.$$

Transpose and dot product

If $\mathbf{u}$ and $\mathbf{v}$ are column vectors in $\mathbb{R}^n$, we can consider them as $n \times 1$ matrices. We now have a new way of writing their dot product:

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For example, if $\quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad$ and $\quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, then

$$\mathbf{u} \cdot \mathbf{v} \quad = \quad \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

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We can define the inverse of a matrix A similarly. Since the identity matrix behaves like the scalar identity 1, we can expect that the inverse A^{-1} has the property that $AA^{-1} = A^{-1}A = I$.

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Finding A^{-1} or even determining when it exists is more difficult than for scalars.

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If A has an inverse, then A is said to be invertible or non-singular.

If A does not have an inverse, we say that it is not invertible or singular.

Inverses – Examples

If $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, then $B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ is a right inverse of A:

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However, A has no left inverse, and thus no inverse. So A is not invertible.

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Thus, A is invertible.

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Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -1 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$, find AB and BA.

Hence solve the matrix equation

$$\begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix} X = \begin{pmatrix} 5 & 7 & -1 \\ 19 & 21 & -5 \end{pmatrix}.$$

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Since the left-hand side of the matrix equation is AX, multiplying both sides on the left by A^{-1} will yield:

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Properties

Theorem

1) The inverse of an invertible matrix is unique. The inverse of A is denoted by A^{-1}.
2) All invertible matrices are square. (However, not all square matrices are invertible.)
3) When A is a square matrix, if $AX = I$ or $XA = I$ then $X = A^{-1}$.

(Note that a matrix is called square if the number of its rows is equal to the number of its columns, that is, if it is an $n \times n$ matrix.)

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Proof of (1)

Suppose A is a matrix with inverses X and Y. Then

$$X = XI = X(AY) = (XA)Y = IY = Y.$$

Properties

Proof of (2) - not examinable

Suppose A is an invertible $m \times n$ matrix. Then its inverse A^{-1} is an $n \times m$ matrix.

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Suppose A is an invertible $m \times n$ matrix. Then its inverse A^{-1} is an $n \times m$ matrix.

Consider the system $A\mathbf{x} = \mathbf{0}$. If $\mathbf{x}$ is a solution, then

$$\mathbf{x} = I\mathbf{x} = (A^{-1}A)\mathbf{x} = A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{0} = \mathbf{0},$$

that is, the system $A\mathbf{x} = \mathbf{0}$ has a unique solution.

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that is, the system $A\mathbf{x} = \mathbf{0}$ has a unique solution. This is possible only if $n \leq m$.

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To see that not every square matrix is invertible, take, for example,
$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ or $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $A = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$ or ...

Properties

Proof of (3) - not examinable

Suppose A is a square matrix and X is a left inverse of A, that is, X is a matrix such that $XA = I$.

Since A and I are square matrices, it follows that X is also square, and they all have the same size, say $n \times n$.

Suppose $\mathbf{x}$ is a solution of the system $A\mathbf{x} = \mathbf{0}$. Then

$$\mathbf{x} = I\mathbf{x} = (XA)\mathbf{x} = X(A\mathbf{x}) = X\mathbf{0} = \mathbf{0},$$

that is, the system $A\mathbf{x} = \mathbf{0}$ has a unique solution, $\mathbf{x} = \mathbf{0}$.

Consider the augmented matrix $(A|\mathbf{0})$ of the system $A\mathbf{x} = \mathbf{0}$. Since the system has a unique solution, the reduced row echelon form must be $(I|\mathbf{0})$. But then for any $\mathbf{b} \in \mathbb{R}^n$, the reduced row echelon form of the system $A\mathbf{x} = \mathbf{b}$ must be $(I|\mathbf{c})$ for some $\mathbf{c} \in \mathbb{R}^n$. This means the system $A\mathbf{x} = \mathbf{b}$ has a unique solution, $\mathbf{x} = \mathbf{c}$.

$$\vdots$$

Properties

Proof of (3) continued - not examinable

Suppose $\mathbf{y}$ is a solution of the system $X\mathbf{y} = \mathbf{0}$, and let $\mathbf{x}$ be the unique solution of the system $A\mathbf{x} = \mathbf{y}$. Then

$$\mathbf{0} = X\mathbf{y} = X(A\mathbf{x}) = (XA)\mathbf{x}) = I\mathbf{x} = \mathbf{x}.$$

So $\mathbf{x} = \mathbf{0}$ and therefore $\mathbf{y} = A\mathbf{x} = A\mathbf{0} = \mathbf{0}$. Hence the system $X\mathbf{y} = \mathbf{0}$ has a unique solution, $\mathbf{y} = \mathbf{0}$.

Now let $\mathbf{b}_i$, $1 \leq i \leq n$, be the columns of the matrix $AX - I$. Since

$$X(AX - I) = X(AX) - XI = (XA)X - X = IX - X = X - X = 0,$$

we have $X\mathbf{b}_i = \mathbf{0}$ for $1 \leq i \leq n$. But this means $\mathbf{b}_i = \mathbf{0}$ for $1 \leq i \leq n$. Hence $AX - I = 0$ and therefore

$$AX = I.$$

So X is also a right inverse of A and hence the inverse of A.

$$\vdots$$

Proof of (3) continued - not examinable

Finally, if A is a square matrix and X a right inverse of A, that is, $AX = I$, then A is a left inverse of X. The argument above yields that A is a right inverse of A, that is, $XA = I$. But this means X is a left inverse of A and therefore the inverse of A.

Properties

Theorem

Let A, B be invertible matrices. Then

- A^{-1} is invertible and $(A^{-1})^{-1} = A$.
- AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
- A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

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Proof

Since A satisfies

$$AA^{-1} = I \qquad \text{and} \qquad A^{-1}A = I,$$

A is the inverse of A^{-1}. Hence A^{-1} is invertible and $(A^{-1})^{-1} = A$.

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Proof

Check:

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

and

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

So $B^{-1}A^{-1}$ is the inverse of AB.

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Proof

Check:

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I,$$

and

$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I.$$

So $(A^{-1})^T$ is the inverse of A^T.

Hence A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Examples

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Assuming all of the relevant inverses exist, simplify

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So rearranging, we find that

$$A^{-1} = \frac{1}{5}A - \frac{2}{5}I.$$