MATH1131 Mathematics 1A – Algebra

Lecture 21: Properties of Determinants

Lecturer: Sean Gardiner – sean.gardiner@unsw.edu.au

Based on slides by Jonathan Kress

Properties of determinants

Properties

- $\det(A^T) = \det(A)$
- $\det(AB) = \det(A)\det(B)$
- $R_i \leftrightarrow R_j$ for $i \neq j$ changes the sign of the determinant.
- $R_i \to \alpha R_i$ scales the determinant by α.
- $R_i \to R_i + \alpha R_j$ for $i \neq j$ does not change the determinant.

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Proof

Suppose A is size $n \times n$, and that the statement holds for matrices of size smaller than $n \times n$. (Certainly it holds for 1×1 matrices.)
The determinant of A^T expanding along the top row is:

$$\begin{aligned}\det(A^T) &= [A^T]_{11}|A^T_{11}| - [A^T]_{12}|A^T_{12}| + \cdots + (-1)^{n+1}[A^T]_{1n}|A^T_{1n}| \\ &= [A]_{11}|A_{11}| - [A]_{21}|A_{21}| + \cdots + (-1)^{n+1}[A]_{n1}|A_{n1}| \\ &= \det(A),\end{aligned}$$

expanding along the first column of A.

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Proof

The proof of this statement is omitted as it uses techniques beyond the scope of this course.

This property is an extremely useful and important one, especially because it means $\det(AB) = \det(BA)$.

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- $R_i \to \alpha R_i$ scales the determinant by α.
- $R_i \to R_i + \alpha R_j$ for $i \neq j$ does not change the determinant.

Proof

Consider first the matrix E_i obtained from swapping the adjacent rows R_i and R_{i+1} in the identity matrix I_n. Expanding along the ith row,

$$|E_i| = (-1)^{i+(i+1)} 1 |(E_i)_{i,i+1}| = (-1)^{2i+1}|I_{n-1}| = (-1) \times 1 = -1.$$

Now consider the matrix E obtained from swapping any two rows R_i and R_j of I_n. This is the same as swapping adjacent rows many times:

$$|E| = |(E_i E_{i+1} \cdots E_{j-2})(E_{j-1} \cdots E_{i+1} E_i) I_n| = (-1)^{2(j-i)-1} = -1.$$

So for any general matrix A, $\det(EA) = \det(E)\det(A) = -\det(A)$.

Properties of determinants

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- $\det(A^T) = \det(A)$
- $\det(AB) = \det(A)\det(B)$
- $R_i \leftrightarrow R_j$ for $i \neq j$ changes the sign of the determinant.
- $R_i \to \alpha R_i$ scales the determinant by α.
- $R_i \to R_i + \alpha R_j$ for $i \neq j$ does not change the determinant.

Proof

Consider first the matrix E obtained from multiplying the ith row of I_n by α.
Then since E is upper triangular, its determinant is the product of its diagonal entries:

$$\det(E) = 1 \times \cdots \times 1 \times \alpha \times 1 \times \cdots \times 1 = \alpha.$$

So for any general matrix A, $\det(EA) = \det(E)\det(A) = \alpha \det(A)$.

Properties of determinants

Properties

- $\det(A^T) = \det(A)$
- $\det(AB) = \det(A)\det(B)$
- $R_i \leftrightarrow R_j$ for $i \neq j$ changes the sign of the determinant.
- $R_i \to \alpha R_i$ scales the determinant by α.
- $R_i \to R_i + \alpha R_j$ for $i \neq j$ does not change the determinant.

Proof

Consider first the matrix E obtained from adding α times the jth row to the ith row of I_n.
Either E is upper triangular or its transpose is, so its determinant is the product of its diagonal entries:

$$\det(E) = 1 \times \cdots \times 1 = 1.$$

So for any general matrix A, $\det(EA) = \det(E)\det(A) = \det(A)$.

Properties of determinants

Some important consequences

- $\det(\alpha A) = \alpha^n \det(A)$, where A is an $n \times n$ matrix.
- $\det(A^{-1}) = \frac{1}{\det(A)}$.
- Swapping columns changes the sign of the determinant.
- If A has a zero row or column then $\det(A) = 0$.
- If one row/column of A is a multiple of another, then $\det(A) = 0$.
- A is invertible if and only if $\det(A) \neq 0$.

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αA is the matrix A with every row scaled by a factor of α.

So its determinant has been scaled by α for each of its n rows, i.e. it has been scaled by α^n.

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- If A has a zero row or column then $\det(A) = 0$.
- If one row/column of A is a multiple of another, then $\det(A) = 0$.
- A is invertible if and only if $\det(A) \neq 0$.

Since $\det(AB) = \det(A)\det(B)$, we have

$$\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1.$$

So $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

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- If A has a zero row or column then $\det(A) = 0$.
- If one row/column of A is a multiple of another, then $\det(A) = 0$.
- A is invertible if and only if $\det(A) \neq 0$.

Since $\det(A^T) = \det(A)$, swapping rows has the same effect as swapping columns.

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- If A has a zero row or column then $\det(A) = 0$.
- If one row/column of A is a multiple of another, then $\det(A) = 0$.
- A is invertible if and only if $\det(A) \neq 0$.

A matrix with a zero row can be thought of as a matrix whose row has been multiplied through by $\alpha = 0$, so the determinant must be 0.

Since $\det(A^T) = \det(A)$, the same applies for a zero column.

This is also obvious by the definition of the determinant, if we expand along the zero row/column.

Properties of determinants

Some important consequences

- $\det(\alpha A) = \alpha^n \det(A)$, where A is an $n \times n$ matrix.
- $\det(A^{-1}) = \frac{1}{\det(A)}$.
- Swapping columns changes the sign of the determinant.
- If A has a zero row or column then $\det(A) = 0$.
- If one row/column of A is a multiple of another, then $\det(A) = 0$.
- A is invertible if and only if $\det(A) \neq 0$.

In any matrix, if row i is α times row j, then performing the row operation $R_i \to R_i - \alpha R_j$ makes the ith row a zero row and does not change the determinant.

So the determinant of the original matrix must be 0, because the reduced matrix contains a zero row.

Since $\det(A^T) = \det(A)$, the same applies for columns.

Properties of determinants

Some important consequences

- $\det(\alpha A) = \alpha^n \det(A)$, where A is an $n \times n$ matrix.
- $\det(A^{-1}) = \frac{1}{\det(A)}$.
- Swapping columns changes the sign of the determinant.
- If A has a zero row or column then $\det(A) = 0$.
- If one row/column of A is a multiple of another, then $\det(A) = 0$.
- A is invertible if and only if $\det(A) \neq 0$.

Suppose the matrix A is reduced to its RREF matrix U using elementary row operations.
Notice that performing any row operation cannot change the determinant to 0, since swapping rows corresponds to multiplying by -1, and scaling rows corresponds to multiplying by some non-zero α.
So $\det(U) = 0$ if and only if $\det(A) = 0$.
So A is invertible $\iff U = I \iff \det(U) \neq 0 \iff \det(A) \neq 0$.

Determinants and inverses

Theorem

For a square matrix A, the following are equivalent:

- $\det(A) \neq 0$
- A is invertible
- $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$ for all $\mathbf{b} \in \mathbb{R}^n$
- $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$.

(That is, if any one of the above properties is true for a given matrix A, then all four are true for A.)

Determinants – Examples

Example

Suppose that A is an invertible 3×3 matrix and $\det(A) = 5$. Find:

- $\det(A^T)$,
- $\det(2A)$, and
- $\det(A^{-1})$.

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- $\det(A^T) = \det(A) = 5$.

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Suppose that A is an invertible 3×3 matrix and $\det(A) = 5$. Find:

- $\det(A^T)$,
- $\det(2A)$, and
- $\det(A^{-1})$.

- $\det(A^T) = \det(A) = 5$.

- Since A is size 3×3, $\det(2A) = 2^3 \det(A) = 40$.

Determinants – Examples

Example

Suppose that A is an invertible 3×3 matrix and $\det(A) = 5$. Find:

- $\det(A^T)$,
- $\det(2A)$, and
- $\det(A^{-1})$.

- $\det(A^T) = \det(A) = 5$.
- Since A is size 3×3, $\det(2A) = 2^3 \det(A) = 40$.
- $\det(A^{-1}) = \dfrac{1}{\det(A)} = \dfrac{1}{5}$.

Example

Find

$$\begin{vmatrix} 1 & -1 & 0 & 3 \\ 2 & -2 & 6 & -1 \\ 4 & -2 & 1 & 7 \\ 3 & 5 & -7 & 0 \end{vmatrix}$$

Determinants – Examples

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Determinants – Examples

$$\begin{vmatrix} 1 & -1 & 0 & 3 \\ 2 & -2 & 6 & -1 \\ 4 & -2 & 1 & 7 \\ 3 & 5 & -7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & 6 & -7 \\ 0 & 2 & 1 & -5 \\ 0 & 8 & -7 & -9 \end{vmatrix}$$

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$$= -1 \begin{vmatrix} 1 & -1 & 0 & 3 \\ 0 & 2 & 1 & -5 \\ 0 & 0 & 6 & -7 \\ 0 & 0 & -11 & 11 \end{vmatrix}$$

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The reduced matrix is in REF and therefore is upper triangular, so the determinant will be the product of its diagonal entries.

Determinants – Examples

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The reduced matrix is in REF and therefore is upper triangular, so the determinant will be the product of its diagonal entries.

That is, the determinant is $11 \times 1 \times 2 \times (-1) \times (-1) = 22$.

Determinants – Examples

Example

Find

$$\begin{vmatrix} 1 & 5 & 4 & 3 \\ 3 & -2 & 12 & -2 \\ 1 & -3 & 4 & 3 \\ 2 & -11 & 8 & 1 \end{vmatrix}$$

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Notice that the third column is 4 times the first column.

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Notice that the third column is 4 times the first column.

This means the determinant will be 0.

Determinants – Examples

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Find

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Notice that the third column is 4 times the first column.

This means the determinant will be 0.

Performing the operation $C_3 \to C_3 - 4C_1$ (where C_i represents the ith column) makes this especially clear:

$$\begin{vmatrix} 1 & 5 & 4 & 3 \\ 3 & -2 & 12 & -2 \\ 1 & -3 & 4 & 3 \\ 2 & -11 & 8 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 0 & 3 \\ 3 & -2 & 0 & -2 \\ 1 & -3 & 0 & 3 \\ 2 & -11 & 0 & 1 \end{vmatrix} = 0.$$

Example

For any $a, b, c \in \mathbb{R}$, find

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Determinants – Examples

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Determinants – Examples

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$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix} = (b-a)(c-a)(c-b).$$

(Notice that the step that extracts factors of $(b-a)$ and $(c-a)$ from rows 2 and 3 holds even if either factor is 0.)

Example

For what values of α is

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ \alpha & 0 & 1 \end{pmatrix}$$

invertible?

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A is invertible if and only if $\det(A) \neq 0$.

Determinants – Examples

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A is invertible if and only if $\det(A) \neq 0$.

Expanding along the bottom row,

$$\det(A) = \alpha \det \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} - 0 + 1 \det \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$$

Determinants – Examples

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A is invertible if and only if $\det(A) \neq 0$.

Expanding along the bottom row,

$$\det(A) = \alpha \det \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} - 0 + 1 \det \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \alpha - 5.$$

Determinants – Examples

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$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ \alpha & 0 & 1 \end{pmatrix}$$

invertible?

A is invertible if and only if $\det(A) \neq 0$.

Expanding along the bottom row,

$$\det(A) = \alpha \det \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} - 0 + 1 \det \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \alpha - 5.$$

So A is invertible whenever $\alpha \neq 5$.