School of Mathematics and Statistics
Math1131-Algebra

Lec07: Cross-product

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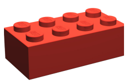
Learning outcomes for this lecture

At the the end of this lecture on the **cross product**,

- ☐ you should know that the cross product of two vectors in $\mathbb{R}^3$ (It only exists in dimension 3) is a **vector** which is **orthogonal** to both the initial vectors;
- ☐ you should be able to **calculate the coordinates** of the cross product $\overrightarrow{a} \times \overrightarrow{b}$ knowing the coordinates of the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$;
- ☐ you should know how to **perform calculations** involving the cross product. Beware, it is **neither commutative nor associative**; Also note that the cross product of parallel vectors is the zero vector;
- ☐ you should be able to use the **right hand rule** to predict the direction of the cross product;
- ☐ you should know that the length/magnitude of the cross product is the **area of a parallelogram**;
- ☐ you should be even more convinced that **drawing is really helpful**!

You can use this list as a check list to get ready for our next class: After studying the lecture notes, come back to this list, and for each item, check that you have indeed mastered it. Then tick the corresponding box ... or go back to the notes.

Determinants



2 by 2 determinants

A 2×2 *determinant* is a **number** calculated from a 2×2 matrix:

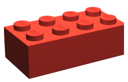
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example 1.

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 2 \times 3 = -2$$

$$\begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix} =$$

Cross product



Cross product of two vectors in $\mathbb{R}^3$

Let $\overrightarrow{\boldsymbol{a}} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\overrightarrow{\boldsymbol{b}} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ be vectors in $\mathbb{R}^3$.

The cross product of $\overrightarrow{\boldsymbol{a}}$ and $\overrightarrow{\boldsymbol{b}}$ is the vector

$$\overrightarrow{\boldsymbol{a}} \times \overrightarrow{\boldsymbol{b}} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

It is also called the vector product of $\overrightarrow{\boldsymbol{a}}$ and $\overrightarrow{\boldsymbol{b}}$.

NB: We can obtain the second and then the third component from the first component by permuting the subscripts:

$$1 \to 2 \to 3 \to 1.$$

NB: $\overrightarrow{\boldsymbol{a}} \cdot \overrightarrow{\boldsymbol{b}}$ is defined in all dimensions, but $\overrightarrow{\boldsymbol{a}} \times \overrightarrow{\boldsymbol{b}}$ is only defined in $\mathbb{R}^3$.

A trick to calculate the cross product

We often write the cross product as a 3 by 3 determinant

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & a_1 & b_1 \\ \vec{j} & a_2 & b_2 \\ \vec{k} & a_3 & b_3 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ -(a_1 b_3 - a_3 b_1) \\ a_1 b_2 - a_2 b_1 \end{pmatrix}.$$

Some people (your tutor?) write the vectors horizontally instead of vertically and develop along the first row instead of the first column, but this yields the same answer in the end, so no worries!

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Calculating cross product

Exercise 2. Let $\overrightarrow{\boldsymbol{u}} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\overrightarrow{\boldsymbol{v}} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$

a) Find $\overrightarrow{\boldsymbol{u}} \times \overrightarrow{\boldsymbol{v}}$

b) Check that $\overrightarrow{\boldsymbol{u}} \times \overrightarrow{\boldsymbol{v}}$ is perpendicular to both $\overrightarrow{\boldsymbol{u}}$ and $\overrightarrow{\boldsymbol{v}}$.

Checking our answers with Maple

```
> with(LinearAlgebra):
> u := <1,0,2>:
  v := <3,4,5>:

> # We calculate their cross-product
> w := CrossProduct(u,v);
# Does NOT work unless you first load the linear algebra package.
# The commands are case-sensitive.
```

$$w := \begin{bmatrix} -8 \\ 1 \\ 4 \end{bmatrix}$$

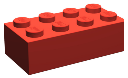
```
> # We now check orthogonality:
  w_dot_u := w.u;
  w_dot_v := w.v;
```

$$w\_dot\_u := 0$$

$$w\_dot\_v := 0$$

```
> # use SHIFT + ENTER to get a new line
```

$\overrightarrow{a} \times \overrightarrow{b}$ is orthogonal to both $\overrightarrow{a}$ and $\overrightarrow{b}$



Direction of the cross product

For any two vectors and in $\mathbb{R}^3$, $\overrightarrow{\boldsymbol{a}} \times \overrightarrow{\boldsymbol{b}}$ is orthogonal to $\overrightarrow{\boldsymbol{a}}$ and to $\overrightarrow{\boldsymbol{b}}$

Exercise 3. Show

$$\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

and

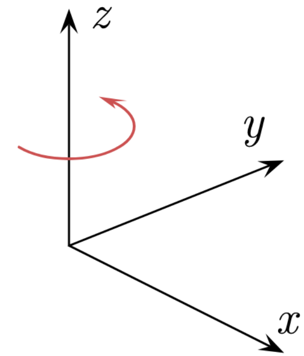
$$\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 0.$$

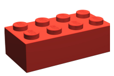
The right hand rule

Exercise 4. Find

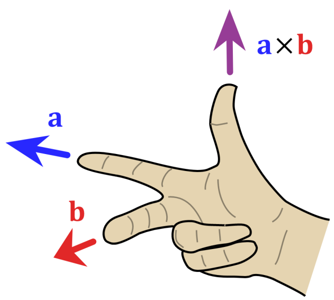
- $\hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}} =$
- $\hat{\boldsymbol{j}} \times \hat{\boldsymbol{k}} =$
- $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{i}} =$

Do you notice something?





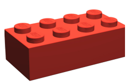
The **direction** of the cross product is given by the **right hand rule**.



Version 1: To use the right hand rule, hold out your right hand, point your index finger in the direction of the first vector, turn your middle finger in towards the direction of the second vector, and hold your thumb up. Your thumb should point in the direction of the of the cross product.

Version 2: If you push $\overrightarrow{\boldsymbol{a}}$ onto $\overrightarrow{\boldsymbol{b}}$ with the palm of your right hand, your thumb points in the direction of the cross product $\overrightarrow{\boldsymbol{a}} \times \overrightarrow{\boldsymbol{b}}$.

Properties of the cross product



Properties of the cross product

For all vectors $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{b}$ in $\mathbb{R}^3$ and all real numbers λ,

- $\overrightarrow{a} \times \overrightarrow{a} = \overrightarrow{0}$ (parallel $\Rightarrow$ cross product is the zero vector)
- $\overrightarrow{0} \times \overrightarrow{a} = \overrightarrow{a} \times \overrightarrow{0} = \overrightarrow{0}$
- $\overrightarrow{a} \times \overrightarrow{b} = -\overrightarrow{b} \times \overrightarrow{a}$ (anti-commutative law)
- $\overrightarrow{a} \times (\overrightarrow{b} + \overrightarrow{c}) = \overrightarrow{a} \times \overrightarrow{b} + \overrightarrow{a} \times \overrightarrow{c}$ (distributive law)
- $\overrightarrow{a} \times (\lambda \overrightarrow{b}) = (\lambda \overrightarrow{a}) \times \overrightarrow{b} = \lambda(\overrightarrow{a} \times \overrightarrow{b})$

Note that there is no associative law listed above!

$$\overrightarrow{i} \times (\overrightarrow{i} \times \overrightarrow{j}) =$$

$$(\overrightarrow{i} \times \overrightarrow{i}) \times \overrightarrow{j} =$$

Length of cross product

Exercise 5. Show that for all $\overrightarrow{\boldsymbol{a}}, \overrightarrow{\boldsymbol{b}} \in \mathbb{R}^3$, $|\overrightarrow{\boldsymbol{a}} \cdot \overrightarrow{\boldsymbol{b}}|^2 + |\overrightarrow{\boldsymbol{a}} \times \overrightarrow{\boldsymbol{b}}|^2 = |\overrightarrow{\boldsymbol{a}}|^2|\overrightarrow{\boldsymbol{b}}|^2$.

So if $\overrightarrow{\boldsymbol{a}}, \overrightarrow{\boldsymbol{b}} \in \mathbb{R}^3$ and θ is the angle between $\overrightarrow{\boldsymbol{a}}$ and $\overrightarrow{\boldsymbol{b}}$, then

$$\begin{aligned}
|\overrightarrow{\boldsymbol{a}} \times \overrightarrow{\boldsymbol{b}}|^2 &= |\overrightarrow{\boldsymbol{a}}|^2|\overrightarrow{\boldsymbol{b}}|^2 - |\overrightarrow{\boldsymbol{a}} \cdot \overrightarrow{\boldsymbol{b}}|^2 \\
&= |\overrightarrow{\boldsymbol{a}}|^2|\overrightarrow{\boldsymbol{b}}|^2 - |\overrightarrow{\boldsymbol{a}}|^2|\overrightarrow{\boldsymbol{b}}|^2 \cos^2\theta \\
&= |\overrightarrow{\boldsymbol{a}}|^2|\overrightarrow{\boldsymbol{b}}|^2(1 - \cos^2\theta) \\
&= |\overrightarrow{\boldsymbol{a}}|^2|\overrightarrow{\boldsymbol{b}}|^2 \sin^2\theta
\end{aligned}$$

Length of cross product

Hence

Length of the cross product and angle between two vectors

The length of the cross product of two vectors $\overrightarrow{\boldsymbol{a}}, \overrightarrow{\boldsymbol{b}} \in \mathbb{R}^3$ is

$$|\overrightarrow{\boldsymbol{a}} \times \overrightarrow{\boldsymbol{b}}| = |\overrightarrow{\boldsymbol{a}}||\overrightarrow{\boldsymbol{b}}|\sin\theta.$$

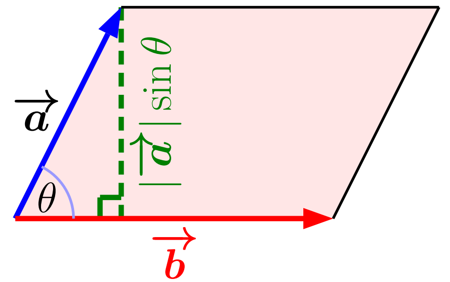
where θ is the angle between $\overrightarrow{\boldsymbol{a}}$ and $\overrightarrow{\boldsymbol{b}}$, with θ in the interval $[0, \pi]$.

Exercise 8.

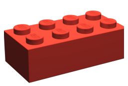
a) The length is maximum when ...

b) The length is minimum when ...

Area of a parallelogram



$$\text{Area} = \text{base} \times \text{altitude} = |\overrightarrow{b}||\overrightarrow{a}|\sin\theta = |\overrightarrow{a} \times \overrightarrow{b}|$$

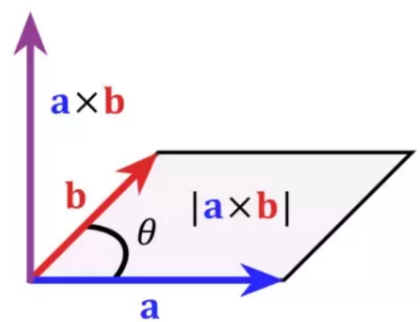


The length of the cross product is the area of a parallelogram

The length of the cross product of two vectors $\overrightarrow{a}, \overrightarrow{b} \in \mathbb{R}^3$ is

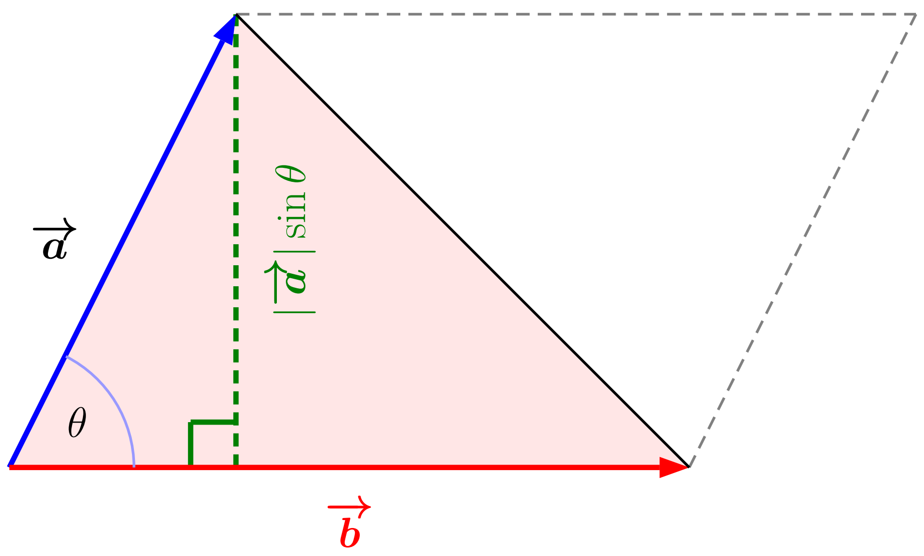
$$|\overrightarrow{a} \times \overrightarrow{b}| = |\overrightarrow{a}||\overrightarrow{b}|\sin\theta. \text{ (with } \theta \in [0,\pi]).$$

Therefore, it is equal to the area of the parallelogram with sides $\overrightarrow{a}$ and $\overrightarrow{b}$



https://www.geogebra.org/m/PhuZm5QQ

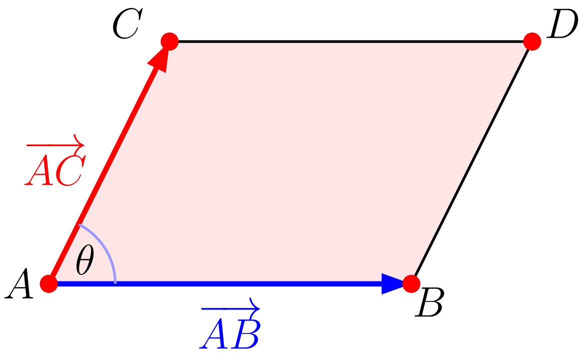
Area of a triangle



$$\text{Area} = \frac{1}{2}\text{base} \times \text{altitude} = \frac{1}{2}|\vec{b}||\vec{a}|\sin\theta = \frac{1}{2}|\vec{a} \times \vec{b}|$$

Area of a parallelogram

Example 6. Find the area of the parallelogram $ABDC$ formed by the points $A(1,0,1)$, $B(-2,1,3)$, $C(3,1,4)$ and D.



$$\text{Area of ABDC} = \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \left|\begin{pmatrix}-3\\1\\2\end{pmatrix} \times \begin{pmatrix}2\\1\\3\end{pmatrix}\right| = \left\|\begin{matrix}\overrightarrow{\boldsymbol{i}} & -3 & 2\\ \overrightarrow{\boldsymbol{j}} & 1 & 1\\ \overrightarrow{\boldsymbol{k}} & 2 & 3\end{matrix}\right\|$$

$$= \left|\begin{pmatrix}1\times 3 - 2\times 1\\ -((-3)\times 3 - 2\times 2)\\ -3\times 1 - 1\times 2\end{pmatrix}\right| = \left|\begin{pmatrix}1\\13\\-5\end{pmatrix}\right| = \sqrt{195}$$

Checking our answers with Maple

```
> with(LinearAlgebra):
> # use : instead of ; if you do not want the echo

  a := <1,0,1>:
  b := <-2,1,3>:
  c := <3,1,4>:

> # Vector AB x vector AC.

  n := CrossProduct(b-a,c-a);
```

$$n := \begin{bmatrix} 1 \\ 13 \\ -5 \end{bmatrix}$$

```
> # The length of vector n is the area of the parallelogram

  Area_parallelogram := sqrt(n.n);
```

$$Area\_parallelogram := \sqrt{195}$$