MATH1131 Mathematics 1A – Algebra

Lecture 2: Algebraic Vectors

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Based on slides by Jonathan Kress

Geometric Vectors

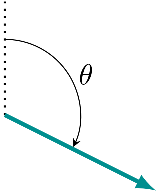
Description

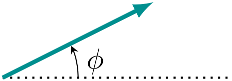
Geometric vectors are quantities that have a length and direction.

The length of a vector is denoted with vertical bars:

$$|\overrightarrow{AB}| = \text{the length of } \overrightarrow{AB}$$

In 2 dimensions, the direction of a vector can be described by the angle between the vector and a fixed direction such as North or the "x-axis":



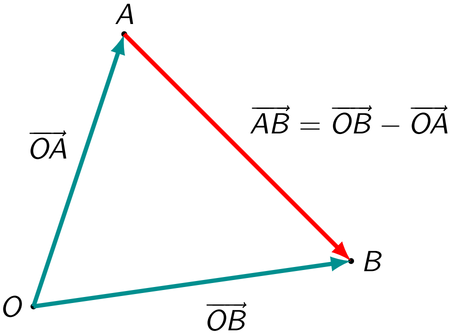


It gets much more difficult to describe the direction of a vector in higher dimensions using angles. In general we care less about a vector's particular direction than its direction in relation to other vectors.

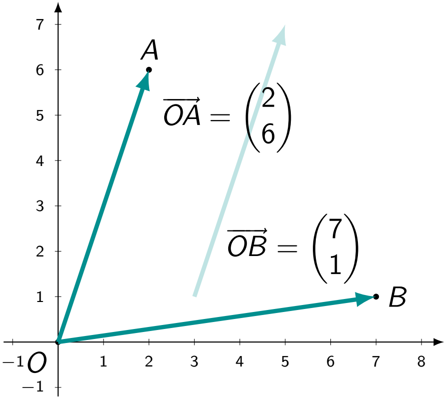
Often we have a special point in space called the origin, denoted O.

For given points A and B, the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are their position vectors.

$\overrightarrow{AB}$ is called the displacement vector from A to B. $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.



Another way of describing geometric vectors is by defining a coordinate system. We identify any position vector (i.e. shifted so that its tail is at the origin) with the coordinates of the point at its tip.



Objects like $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 1 \end{pmatrix}$ are called algebraic vectors.

Note: By convention, we always write these as columns.

The space described by all two-dimensional coordinates is called $\mathbb{R}^2$.

An algebraic vector $\mathbf{x} \in \mathbb{R}^2$ is an ordered pair of real numbers x_1 and x_2 (called the components of $\mathbf{x}$), written

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Suppose also $\mathbf{y} \in \mathbb{R}^2$ with $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $\lambda \in \mathbb{R}$ is a scalar. Then

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$$

and

$$\lambda\mathbf{x} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}.$$

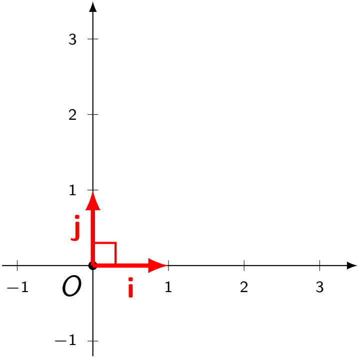
We add and scale component-by-component, and two vectors are equal if all their components are equal.

For all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ and scalars $\lambda, \mu \in \mathbb{R}$:

Associative Law of Addition	$(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$
Commutative Law of Addition	$\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$
Zero Exists	Some element $\mathbf{0} \in \mathbb{R}^2$ satisfies $\mathbf{u}+\mathbf{0} = \mathbf{u}$ for all $\mathbf{u}$
Negative Exists	Some element $(-\mathbf{u}) \in \mathbb{R}^2$ satisfies $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$
Associative Law of Scalar Multiplication	$\lambda(\mu\mathbf{u}) = (\lambda\mu)\mathbf{u}$
Multiplication by identity	$1\mathbf{u} = \mathbf{u}$
Scalar Distributive Law	$(\lambda+\mu)\mathbf{u} = \lambda\mathbf{u}+\mu\mathbf{u}$
Vector Distributive Law	$\lambda(\mathbf{u}+\mathbf{v}) = \lambda\mathbf{u}+\lambda\mathbf{v}$

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

if $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ then $-\mathbf{u} = \begin{pmatrix} -u_1 \\ -u_2 \end{pmatrix}$



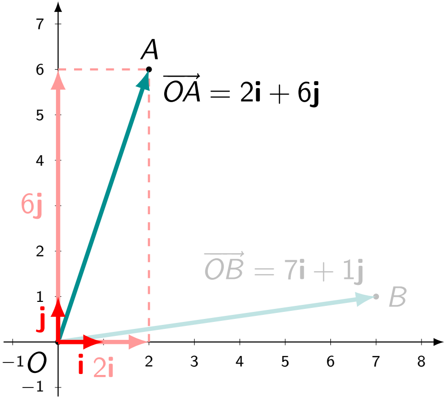
When we specify a coordinate system, what we are really doing is specifying two non-parallel directions.

Let $\mathbf{i}$ be the vector of length 1 unit pointing in the positive horizontal direction, and $\mathbf{j}$ be the vector of length 1 unit pointing in the positive vertical direction.

Then $\mathbf{i}$ and $\mathbf{j}$ have the same unit length and are at right angles (orthogonal) to each other. We say they are orthonormal.

Representations

Vector components



Any position vector in the plane can be expressed (uniquely!) as the sum of scalar multiples of $\mathbf{i}$ and $\mathbf{j}$.

In the diagram,

$$\overrightarrow{OA} = 2\mathbf{i} + 6\mathbf{j}.$$

Remember the coefficients 2 and 6 are called the components of $\overrightarrow{OA}$.

The component vectors above are $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$.

Are our geometric and algebraic definitions of vectors equivalent?

- We can map geometric vectors to algebraic vectors and back.
- The operations of vector addition and scalar multiplication are also consistent:

Geometrically

$$\begin{aligned}\overrightarrow{OA} + \overrightarrow{OB} &= (2\mathbf{i} + 6\mathbf{j}) + (7\mathbf{i} + 1\mathbf{j}) \\ &= (2+7)\mathbf{i} + (6+1)\mathbf{j} \\ &= 9\mathbf{i} + 7\mathbf{j}\end{aligned}$$

and

$$\begin{aligned}2\overrightarrow{OA} &= 2(2\mathbf{i} + 6\mathbf{j}) \\ &= (2\times 2)\mathbf{i} + (2\times 6)\mathbf{j} \\ &= 4\mathbf{i} + 12\mathbf{j}\end{aligned}$$

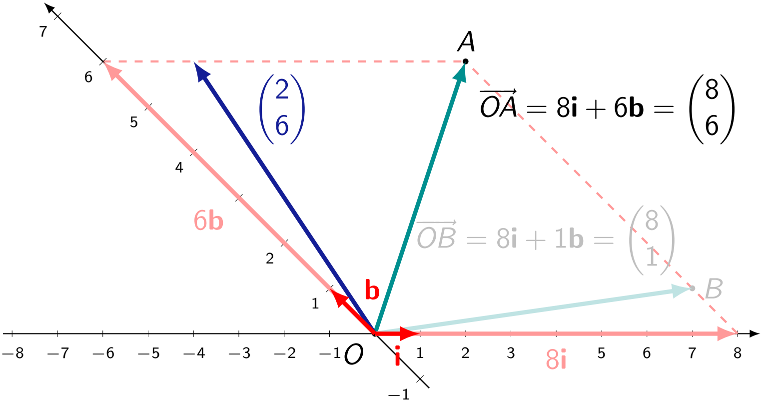
Algebraically

$$\begin{aligned}\overrightarrow{OA} + \overrightarrow{OB} &= \begin{pmatrix}2\\6\end{pmatrix} + \begin{pmatrix}7\\1\end{pmatrix} \\ &= \begin{pmatrix}2+7\\6+1\end{pmatrix} = \begin{pmatrix}9\\7\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}2\overrightarrow{OA} &= 2\begin{pmatrix}2\\6\end{pmatrix} \\ &= \begin{pmatrix}2\times 2\\2\times 6\end{pmatrix} = \begin{pmatrix}4\\12\end{pmatrix}\end{aligned}$$

What if we choose a different coordinate system?



We can still assign algebraic vectors in $\mathbb{R}^2$ here, but their components are different.

Algebraic vectors in two dimensions

From now on we will be working mostly with algebraic vectors, using geometric vectors for illustrative purposes only.

To represent vectors in $\mathbb{R}^2$, we usually choose standard basis vectors that have unit length and are mutually orthogonal, namely:

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

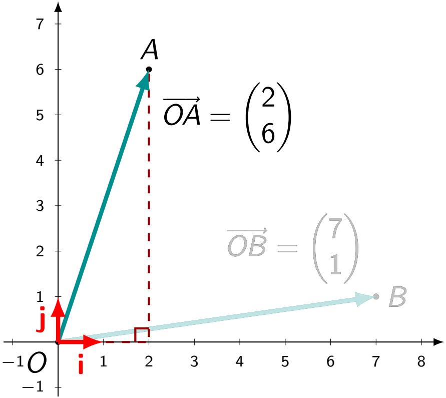
The algebraic vector $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is represented by $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$.

The length of $\mathbf{u}$ is given by the formula:

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2}.$$

Algebraic vectors in two dimensions

Lengths



By definition:

$$|\overrightarrow{OA}| = \sqrt{2^2 + 6^2} = \sqrt{40}$$

Using Pythagoras:

$$|\overrightarrow{OA}| = \sqrt{2^2 + 6^2} = \sqrt{40}$$

Consider $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{y} \in \mathbb{R}^3$ written in components

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \text{and} \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},$$

and consider a scalar $\lambda \in \mathbb{R}$. Then we have

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix},$$

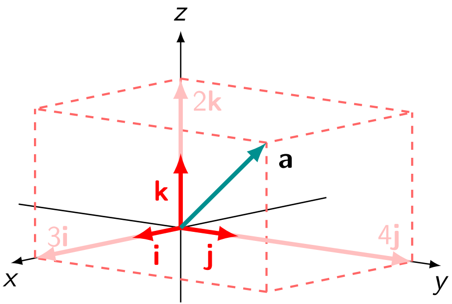
and

$$\lambda\mathbf{x} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{pmatrix}.$$

As before, we add and scale component-by-component, and two vectors are equal if all their components are equal.

To represent the vectors in $\mathbb{R}^3$, we again choose standard basis vectors that have unit length and are mutually orthogonal:

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$



Here

$$\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix},$$

and

$$|\mathbf{a}| = \sqrt{3^2 + 4^2 + 2^2} = \sqrt{29}.$$

Algebraic vectors in n dimensions

Consider $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$ written in components

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \text{and} \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

and consider a scalar $\lambda \in \mathbb{R}$. Then we have

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix},$$

and

$$\lambda \mathbf{x} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}.$$

As always, we add and scale component-by-component, and two vectors are equal if all their components are equal.

The standard basis vectors in $\mathbb{R}^n$ are

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \quad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

so we can write

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n.$$

For example, in three dimensions, $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$ and $\mathbf{e}_3 = \mathbf{k}$.

The length of $\mathbf{a} \in \mathbb{R}^n$, where

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n,$$

is defined to be

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.$$

If $|\mathbf{a}| = 1$, we say that $\mathbf{a}$ is a unit vector.

For any nonzero vector $\mathbf{a} \in \mathbb{R}^n$,

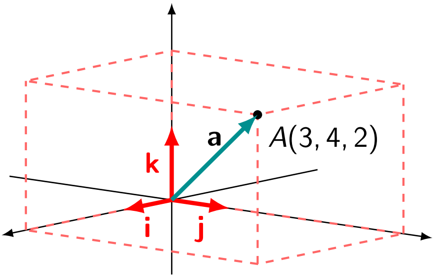
$$\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a}$$

is a unit vector in the same direction as $\mathbf{a}$.

Points and vectors: notation

Note: We write vectors as columns, and points as rows (with commas).

If A is the point $(3, 4, 2)$ in $\mathbb{R}^3$, the point is written as $A(3, 4, 2)$, and its position vector is written as $\mathbf{a} = \overrightarrow{OA} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$.



Algebraic vector examples

Example

Let $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \in \mathbb{R}^2$. Find $\mathbf{v} + \mathbf{w}$ and $3\mathbf{w}$.

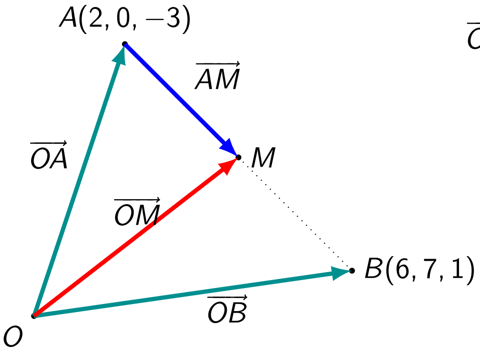
$$\mathbf{v} + \mathbf{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1+3 \\ 1+(-2) \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}.$$

$$3\mathbf{w} = 3\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \times 3 \\ 3 \times (-2) \end{pmatrix} = \begin{pmatrix} 9 \\ -6 \end{pmatrix}.$$

Algebraic vector examples

Example

Let $A(2,0,-3)$ and $B(6,7,1)$ be two points in $\mathbb{R}^3$ and let M be their midpoint. Find $\overrightarrow{OM}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ and also by just taking the average of their components.



$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OA} + \overrightarrow{AM} \\ &= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} \\ &= \overrightarrow{OA} + \frac{1}{2}(\overrightarrow{OB} - \overrightarrow{OA}) \\ &= \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}).\end{aligned}$$

Algebraic vector examples

Example

Let $A(2, 0, -3)$ and $B(6, 7, 1)$ be two points in $\mathbb{R}^3$ and let M be their midpoint. Find $\overrightarrow{OM}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ and also by just taking the average of their components.

$$\text{So}\quad \overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}) = \frac{1}{2}\left(\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \begin{pmatrix} 6 \\ 7 \\ 1 \end{pmatrix}\right)$$

$$= \begin{pmatrix} \frac{1}{2}(2+6) \\ \frac{1}{2}(0+7) \\ \frac{1}{2}(-3+1) \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 7/2 \\ -1 \end{pmatrix}.$$

Collinear points

Example

The points A, B and C are collinear if and only if $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$.

Are the points $A(1, 2, 3, 1)$, $B(1, -2, 3, 2)$, and $C(1, -10, 3, 4)$ collinear?

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ -2 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \\ 0 \\ 1 \end{pmatrix},$$

and

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 1 \\ -10 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -8 \\ 0 \\ 2 \end{pmatrix}.$$

Collinear points

Example

The points A, B and C are collinear if and only if $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$.

Are the points $A(1, 2, 3, 1)$, $B(1, -2, 3, 2)$, and $C(1, -10, 3, 4)$ collinear?

Here $$\overrightarrow{AB} = \begin{pmatrix} 0 \\ -4 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ -8 \\ 0 \\ 2 \end{pmatrix} = \tfrac{1}{2}\overrightarrow{BC},$$

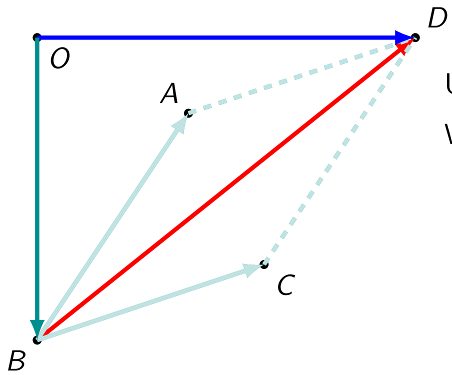
so $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$ since it is a scalar multiple of it.

Therefore A, B, and C are collinear.

Parallelograms

Example

Suppose that $A(2,3,-1,2)$, $B(2,4,-1,-2)$, and $C(-1,-2,1,0)$ are 3 points in $\mathbb{R}^4$. Find the coordinates of the point D such that $ABCD$ is a parallelogram.



Use the parallelogram law!

We know $\overrightarrow{BA} + \overrightarrow{BC} = \overrightarrow{BD}$.

$$\text{So} \quad \begin{aligned} \overrightarrow{OD} &= \overrightarrow{OB} + \overrightarrow{BD} \\ &= \overrightarrow{OB} + \overrightarrow{BA} + \overrightarrow{BC}. \end{aligned}$$

Parallelograms

Example

Suppose that $A(2,3,-1,2)$, $B(2,4,-1,-2)$, and $C(-1,-2,1,0)$ are 3 points in $\mathbb{R}^4$. Find the coordinates of the point D such that $ABCD$ is a parallelogram.

$$\begin{aligned}
\text{That is,} \qquad \overrightarrow{OD} &= \overrightarrow{OB} + \overrightarrow{BA} + \overrightarrow{BC} \\
&= \begin{pmatrix} 2 \\ 4 \\ -1 \\ -2 \end{pmatrix} + \begin{pmatrix} 2-2 \\ 3-4 \\ -1-(-1) \\ 2-(-2) \end{pmatrix} + \begin{pmatrix} -1-2 \\ -2-4 \\ 1-(-1) \\ 0-(-2) \end{pmatrix} \\
&= \begin{pmatrix} -1 \\ -3 \\ 1 \\ 4 \end{pmatrix}
\end{aligned}$$

So the coordinates of D are $(-1,-3,1,4)$.