THE UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1131 Calculus

Section 7: - Curve Sketching.

Although one can now sketch curves on the computer, we still need to be able to give rough sketches of functions by hand.

In this section we will list a number of basic features one should look for when sketching the graph of a function.

Checklist:

A. Domain and Range

B. x and y intercepts

C. Symmetries (even, odd, periodic)

D. Horizontal and vertical asymptotes

E. Oblique asymptotes and asymptotic behaviour

F. Stationary points and inflections of various types using Calculus.

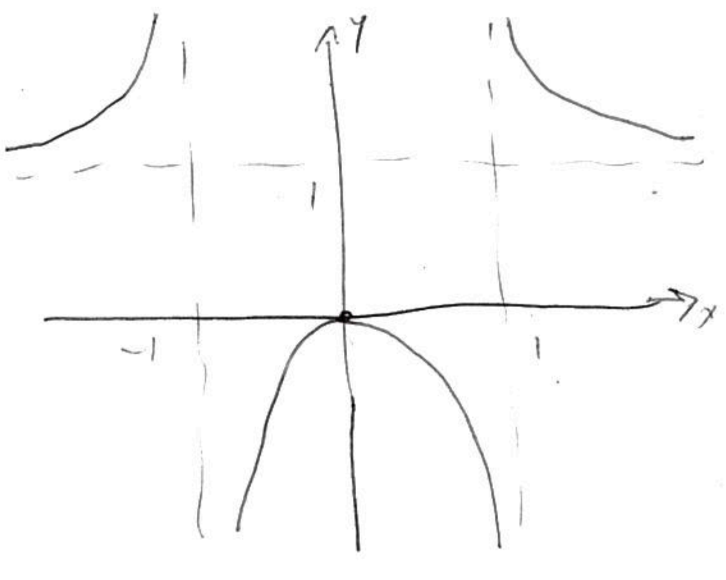
Notice that I have placed the use of Calculus last!

Ex: Sketch $y = \frac{x^2}{x^2-1}$. $= f(x)$

$\lim_{x \to \infty} y = 1$

f is even.

Vertical asymptotes at $x = \pm 1$



~~Ex: Sketch $y = x^2 e^x$.~~

Ex: Sketch $y = \frac{x^2+3}{x-1}$ (Notice that this has oblique asymptote $y = x + 1$.)

$$y = \frac{x^2 - 1 + 4}{x-1}$$

$$= (x+1) + \frac{4}{x+1}$$

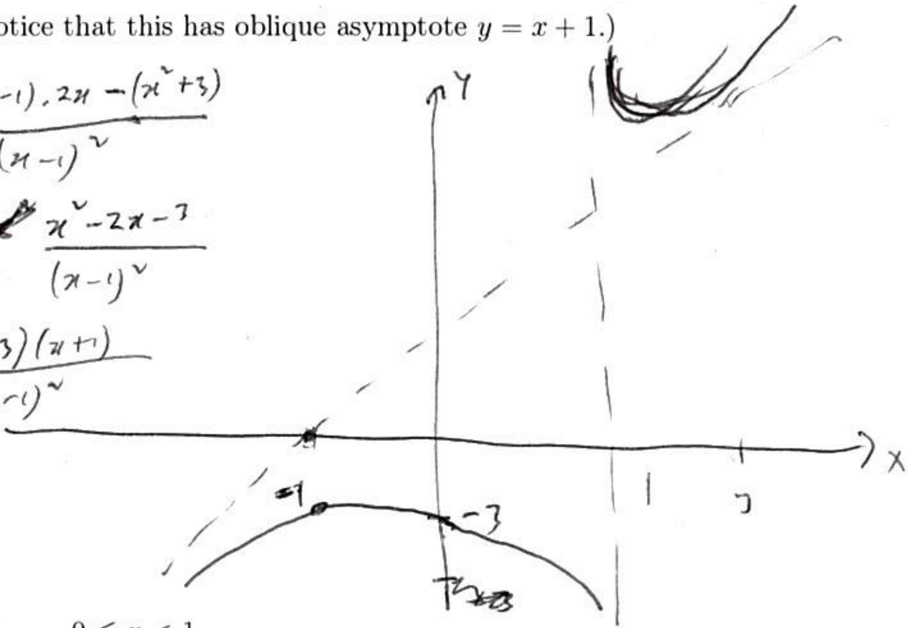
$\therefore$ $y \sim x+1$ for large x.

SP's at $x = 3, -1$.

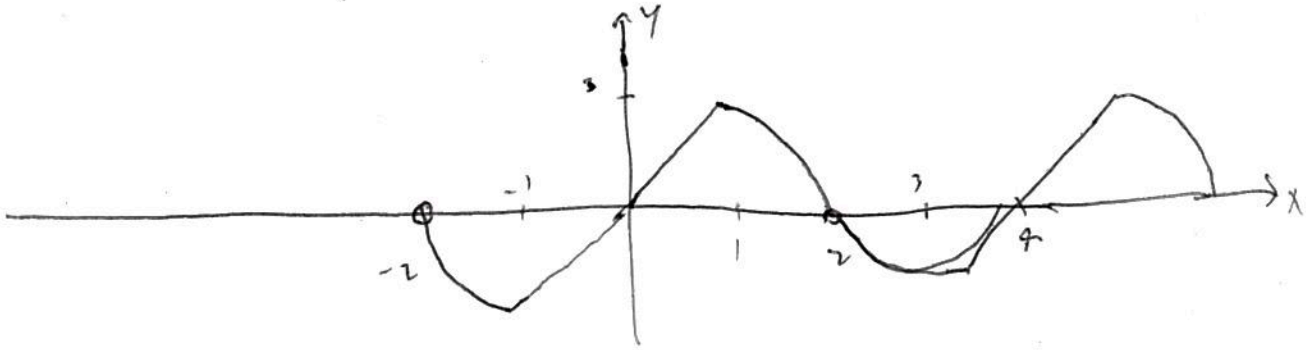
$$y' = \frac{(x-1).2x - (x^2+3)}{(x-1)^2}$$

$$= \frac{x^2 - 2x - 3}{(x-1)^2}$$

$$= \frac{(x-3)(x+1)}{(x-1)^2}$$



Ex: Sketch $f(x) = \begin{cases} 3x & 0 \le x < 1 \\ 4 - x^2 & 1 \le x < 2 \end{cases}$, with f odd and $f(x+4) = f(x)$.



Parametric and Polar-Co-ordinates.

Given a function $y = f(x)$, then a **parametrization** of this function is a way of splitting the variables x and y into two separate equations which are linked by a new variable called a **parameter**.

There are many reasons for wanting to do this. In the study of projectile motion, it is easier to look at the horizontal motion and vertical motion separately, despite the fact that they are clearly linked. The parameter in question is time t.

You have also seen the standard parametrization of the circle $x^2 + y^2 = r^2$, which is $x = r\cos\theta, y = r\sin\theta$, where the parameter θ measures the angle with the positive x axis made by a ray from the centre to the circumference of the circle.

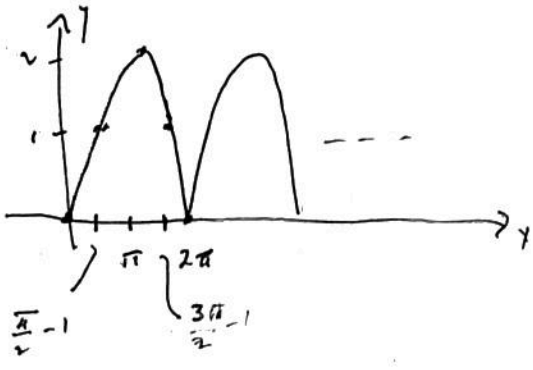
In addition, you will have studied the standard parametrization of the parabola $x^2 = 4ay$, which is $x = 2at, y = at^2$, where the parameter t gives the derivative $\frac{dy}{dx}$.

One can always trivially parametrize a curve of the form $y = f(x)$ by simply writing $x = t, y = f(t)$. Conversely, however, there are many parametrically defined curves which cannot easily be expressed explicitly in the form $y = f(x)$. For example, $x = at - a\sin t, y = a + at\cos t$.

Ex: Sketch $x = \theta - \sin\theta, y = 1 - \cos\theta$.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	0	$\frac{\pi}{2}-1$	π	$\frac{3\pi}{2}-1$	2π
y	0	1	2	1	0

Repeats



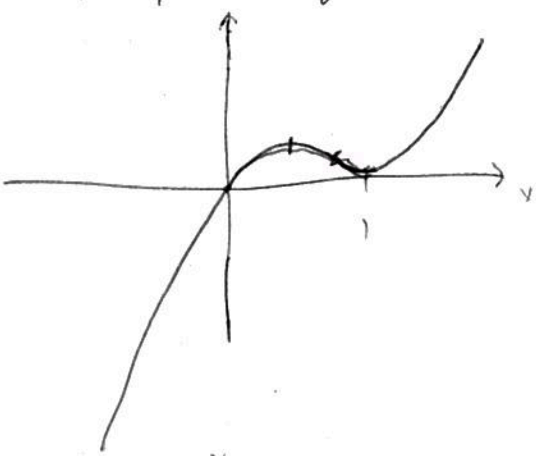
Ex: Sketch $\begin{cases} x = \cos 3t \\ y = \sin 2t \end{cases}$ using MAPLE.

Graphs

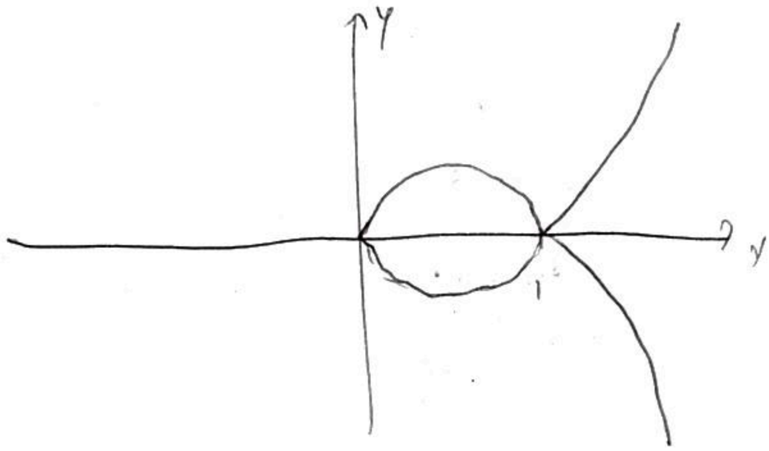
Implicitly Defined Curves:

Ex: Sketch $y^2 = x(x-1)^2$.

Firstly sketch $y = x(x-1)^2$



$y^2 = x(x-1)^2$



[What is happening at $x=0$ & $x=1$?

$2yy' = (x-1)^2 + 2x(x-1)$

$= (x-1)[(x-1)+2x]$

$= (x-1)(3x-1)$

$2\sqrt{x}(x-1)y' = (x-1)(3x-1)$

For x near 1^+

$y' \simeq 1$

y' not defined at $x=0$]

To find the derivative $\frac{dy}{dx}$ for such curves we use the chain rule:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Ex: Find $\frac{dy}{dx}$ for $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{-\sin t} = -\cot(t)$$

Ex: Find $\frac{dy}{dx}$ at the point $t = 1$ for the curve $x = \frac{3t}{1+t^3}, y = \frac{3t^2}{1+t^3}$. (This is the *Folium of Diocles/Descartes* and has cartesian eqn. $x^3 + y^3 = 3xy$.)

([illegible])

$$\frac{dy}{dt} = \frac{(1+t^3).6t - 3t^2.3t^2}{(1+t^3)^2}$$

$$= \frac{12-9}{4} \quad \text{at } t=1$$

$$= \frac{3}{4}$$

$$\frac{dx}{dt} = \frac{(1+t^3)3 - 3t.3t^2}{(1+t^3)^2}$$

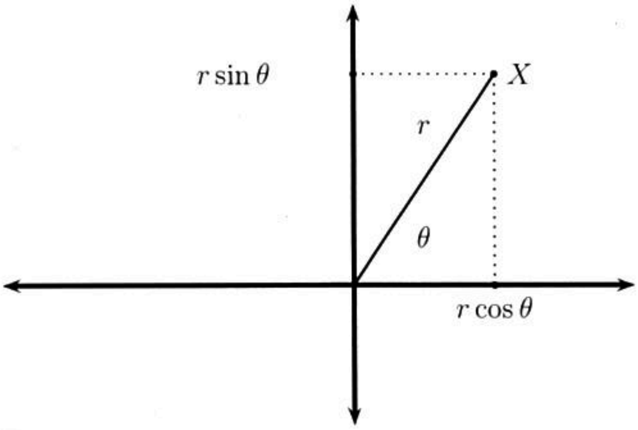
$$= \frac{6-9}{4} = \frac{-3}{4}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = -1$$

Polar Co-ordinates:

As mentioned above, there is a standard way to parametrize the circle, using polar co-ordinates. We can generalise this slightly and arrive at a new co-ordinate system for describing every point in the plane.

Given a point $X(x, y) \neq (0, 0)$, then we measure its distance r from the origin and the angle θ it makes with the positive x axis. Knowing these two quantities, the point X is uniquely determined. The ordered pair (r, θ) are called the **polar co-ordinates** of the point X.



Using simple trigonometry and Pythagoras' theorem, we can write down the equations relating the two co-ordinate systems.

$$x = r\cos\theta, \quad y = r\sin\theta,$$

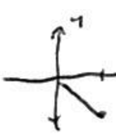
and

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}.$$

Note that the statement $\tan\theta = \frac{y}{x}$ needs to be used **carefully** and a diagram should be drawn!

NB. r is always positive, and in the case $r = 0$ then θ is not defined. The point $(0, 0)$ in Cartesian co-ordinates, is thus special. We take (by convention) $-\pi < \theta \leq \pi$.

Ex: a. Convert $(1, -1)$ to polar co-ordinates.
b. Convert $(3, \frac{5\pi}{6})$ to Cartesian co-ordinates.

a) 

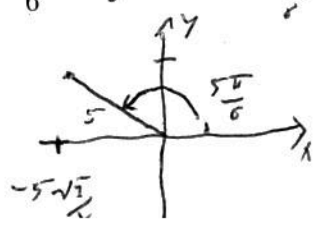
$r = \sqrt{2}, \quad \theta = -\frac{\pi}{4}$

$(1, -1)_{cart} = (\sqrt{2}, -\frac{\pi}{4})_{polar}$

b) ~~xs~~ $r = 5, \quad \theta = \frac{5\pi}{6}$

$x = 5\cos\frac{5\pi}{6} = -5\sqrt{3}/2$

$y = 5\sin\frac{5\pi}{6} = \frac{5\sqrt{3}}{2}$



$(3, \frac{5\pi}{6})_{polar} = (-\frac{5\sqrt{3}}{2}, \frac{5\sqrt{}}{2}$

Ca

Symmetries

$r = f(\theta)$ - polar curve

① If $f(-\theta) = f(\theta)$ then

curve is symmetical about the X-axis.

eg. $r = 2 + \cos\theta$

$r(-\theta) = 2 + \cos(-\theta)$

$= 2 + \cos\theta$

② If $f(\pi - \theta) = f(\theta)$

then curve is symmetical about the Y-axis.

eg. $r = \cos(2\theta)$

$r(\pi - \theta) = \cos(2\pi - 2\theta)$

$= \cos 2\pi \cos 2\theta + \sin 2\pi \sin 2\theta$

$= \cos 2\theta.$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4} n^2 (n+1)^2$$

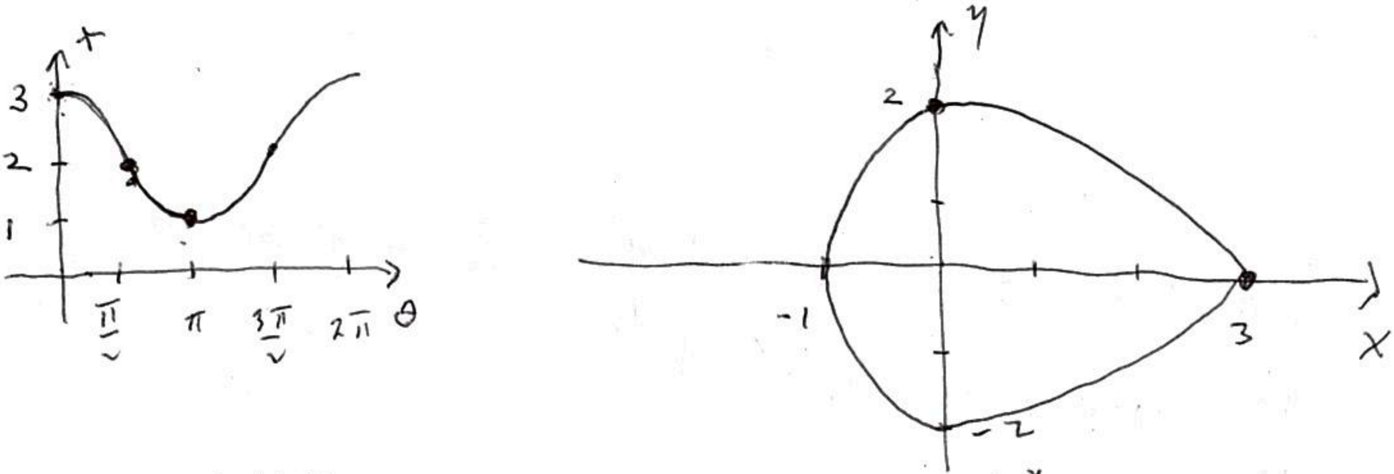
$$\frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right) = \frac{1}{n^3} \cdot \frac{1}{6} n(n+1)(2n+1)$$

$$= \frac{1}{6n^2}(n+1)(2n+1)$$

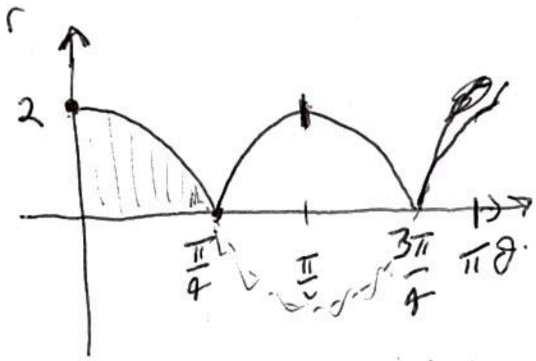
$$= \frac{1}{6n^2}(2n^2 + 3n + 1)$$

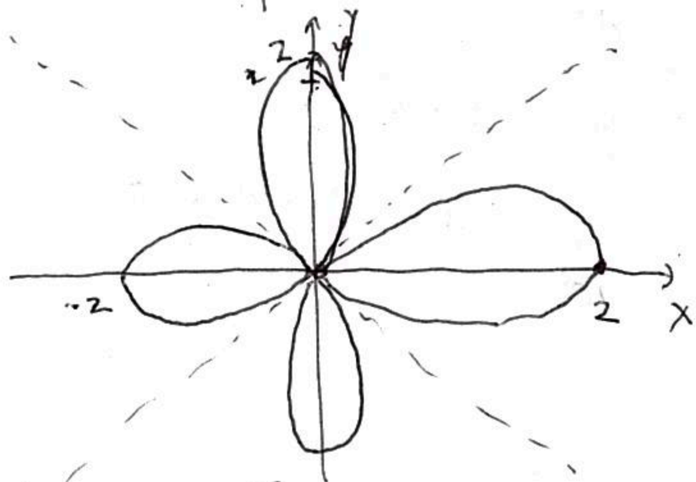
$$= \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

Ex: $r = 2 + \cos\theta$. (This is a limaçon, from the French for 'little snail').



Ex: $r = 2|\cos(2\theta)|$.



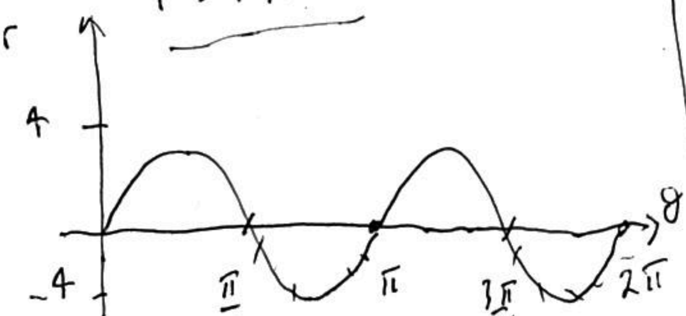


Note: If we put $\theta \longleftrightarrow -\theta$

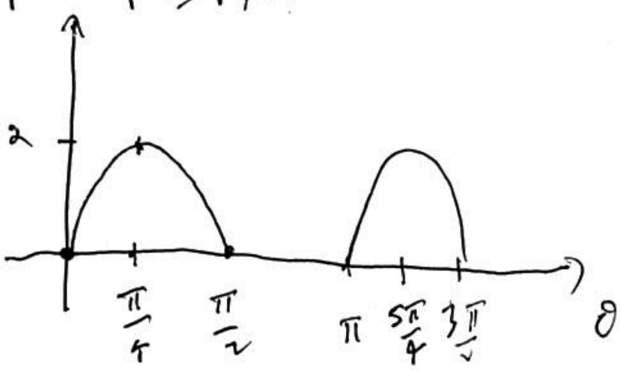
No change in r, symmetric w.r.t x axis

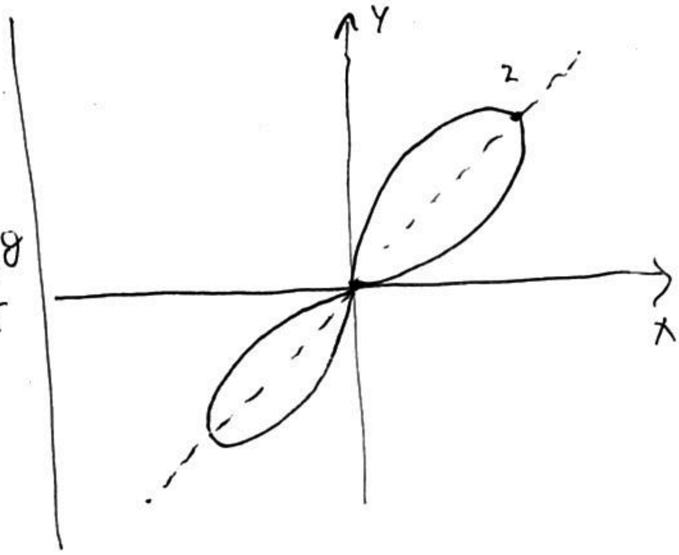
Ex: $r^2 = 4\sin(2\theta)$. (This is a lemniscate, from the Latin *lemniscatus* -ribbon shaped.)

$r = 4\sin 2\theta$



$r^2 = 4\sin 2\theta$





The Tangent to a Polar Curve:

Theorem: If $x = r(\theta)\cos\theta, y = r(\theta)\sin\theta$ are differentiable functions then

$$\frac{dy}{dx} = \frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta}.$$

$x = r(\theta)\cos\theta$

$\frac{dx}{d\theta} = -r\sin\theta + \cos\theta \cdot r'$

$y = r\sin\theta$

$\frac{dy}{d\theta} = r\cos\theta + \sin\theta\, r'$

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}$

Ex: Find where the tangent to the polar curve $r = 1 + \sin\theta$ is horizontal and where it is vertical. What is happening at $\theta = 3\pi/2$?

$r = 1 + \sin\theta$

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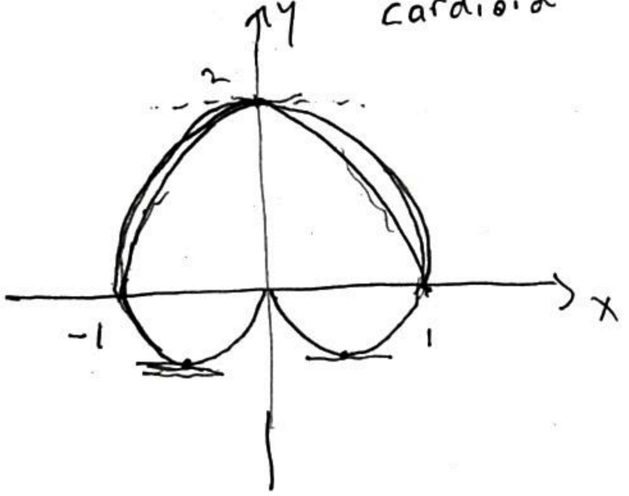

$x = r\cos\theta$

$= (1+\sin\theta)\cos\theta$

$$\frac{dx}{d\theta} = -\sin\theta + \cos^2\theta - \sin^2\theta$$

$$= -(2\sin^2\theta + \sin\theta - 1)$$

$y = r\sin\theta$

$= (1+\sin\theta)\sin\theta$

$$\frac{dy}{d\theta} = \cos\theta(1 + 2\sin\theta)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos\theta(1+2\sin\theta)}{-(2\sin^2\theta + \sin\theta - 1)}$$

$\frac{dy}{dx} = 0$ when NUM $= 0$, DEM $\neq 0$

NUM $= 0$, when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

or when $1 + 2\sin\theta = 0$

$\sin\theta = -\frac{1}{2} \Rightarrow \theta = \frac{-\pi}{6}, \frac{-5\pi}{6}$

At $\theta = \frac{3\pi}{2}$, DEM $= 0$ also so $\frac{dy}{dx}$ not defined.

Vertical tangent when DEM $= 0$

$2\sin^2\theta + \sin\theta - 1 = 0$

$(2\sin\theta - 1)(\sin\theta + 1) = 0$

$\theta = \frac{3\pi}{2}$ ✗

$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

Lect. Notes

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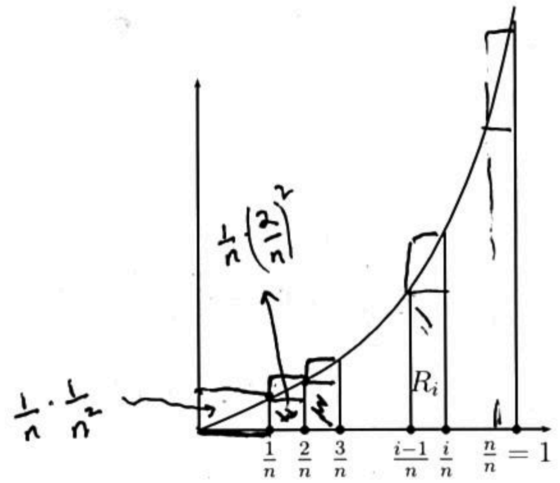
Section 8: - Integration.

We consider the problem of giving meaning to, and finding, the 'area under a curve'. This problem goes back to Archimedes.

We begin by an example:

Ex: $f(x) = x^2$ on $[0, 1]$.



Divide the interval $[0, 1]$ into n equal parts of width $\frac{1}{n}$.

This is called *a partition* of $[0, 1]$.

We now look at the region R bounded by the curve, the x-axis and the line $x = 1$.

Look at a general region R_i bounded by the curve, the x-axis and the lines $x = \frac{i-1}{n}, x = \frac{i}{n}$, where $0 \le i \le n - 1$ and construct two rectangles U_i and L_i so that U_i has height $\left(\frac{i}{n}\right)^2$ and L_i has height $\left(\frac{i-1}{n}\right)^2$.

Adding up all the U_i's we have

$$\frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \dots + \left(\frac{n}{n}\right)^2\right] = \frac{1}{n^3}\left(1^2 + 2^2 + \dots + n^2\right) = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}.$$

This is called the *upper Riemann Sum* for f on P and is denoted by $\overline{S_P}$.

Similarly, adding up all the L_i's we have

$$\frac{1}{n}\left[\left(\frac{0}{n}\right)^2+\left(\frac{1}{n}\right)^2+...+\left(\frac{n-1}{n}\right)^2\right]=\frac{1}{n^3}\left(0^2+1^2+...+(n-1)^2\right)=\frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2}.$$

This is called the *lower Riemann Sum* for f on P and is denoted by $\underline{S_P}$.

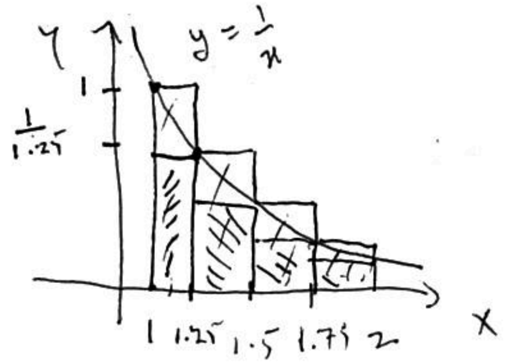
Clearly $\underline{S_P} \leq$ "area of R" $\leq \overline{S_P}$, that is, $\frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2} \leq$ "area of R" $\leq \frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}$.

So, as $n \to \infty$, we have that the "area of R" is $\frac{1}{3}$. $\int_0^1 x^2\,dx = \frac{1}{3}$.

We can take this as the **definition** of the area of R.

Approximating Areas using Partitions:

Ex: Use the partition $P = \{1, 1.25, 1.5, 1.75, 2\}$ to approximate the area under $y = \frac{1}{x}$ between $x = 1$ and $x = 2$ and hence find an upper and lower bound for $\log 2$.



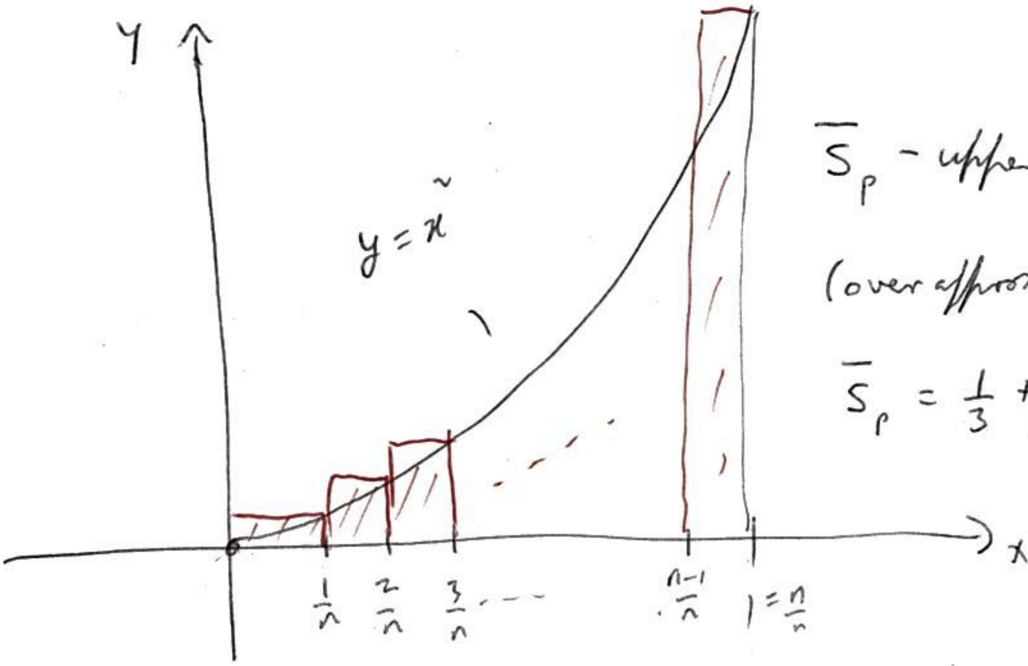
$$\overline{S_P} = 0.25\left[1+\frac{1}{1.25}+\frac{1}{1.5}+\frac{1}{1.75}\right] \simeq 0.7595$$

$$\underline{S_P} = 0.25\left[\frac{1}{1.25}+\frac{1}{1.5}+\frac{1}{1.75}+\frac{1}{2}\right] \simeq 0.6345$$

$$\int_1^2 \frac{1}{x}\,dx = \ln x\Big|_1^2 = \ln 2$$

$$\therefore\ 0.6345 \leq \ln 2 \leq 0.7595$$

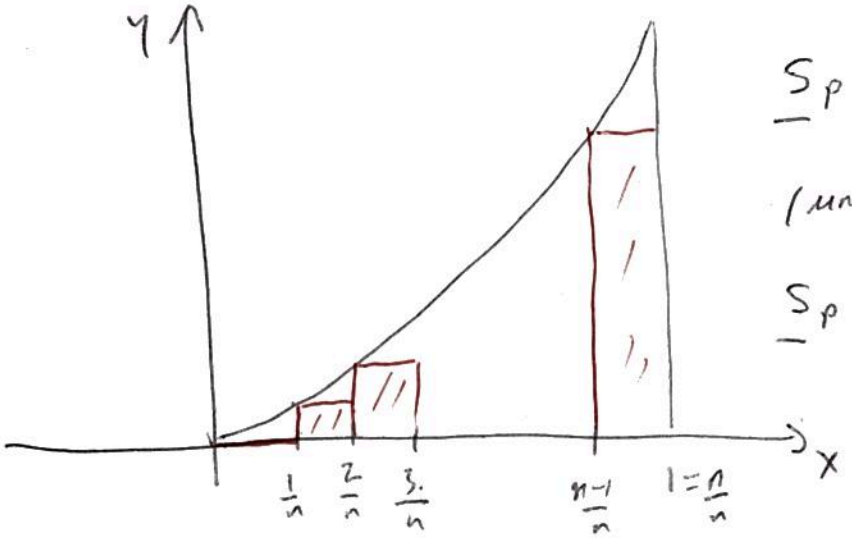
$$[\ln 2 \simeq 0.6931]$$



$\overline{S}_P$ - upper Riemann sum

(over approximation)

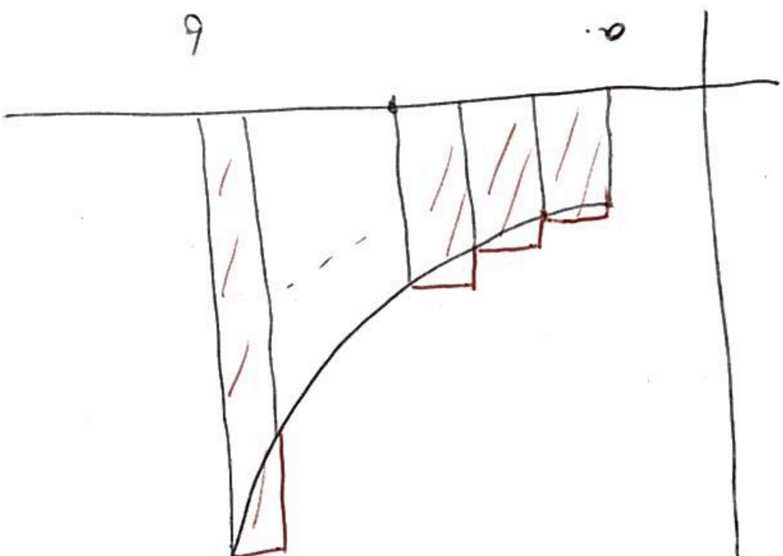
$\overline{S}_P = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$



$\underline{S}_P$ - lower Riemann sum

(under approximation)

$\underline{S}_P = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$



Now for the more general case:

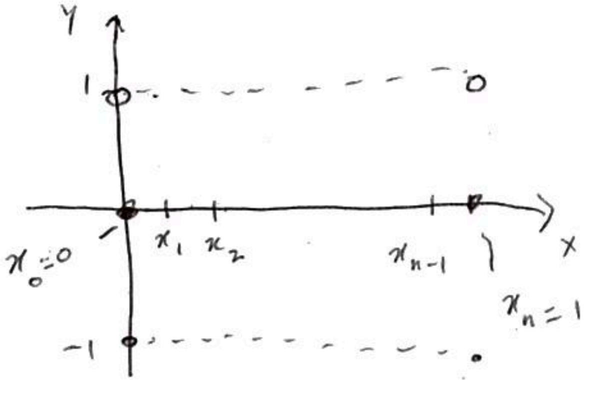
Suppose we have a function $f(x)$ which is defined on the interval $[a,b]$. We divide this interval into n parts (not necessarily equal), and form the **partition** $P_n = \{x_0 = a, x_1, x_2, ..., x_n = b\}$. In each interval $[x_{i-1}, x_i]$ draw a rectangle of height $f(c_i)$ for some $c_i \in [x_{i-1}, x_i]$.
Let Δx_i be the width of the rectange (i.e. $\Delta x_i = x_i - x_{i-1}$). Now add up the areas of all these rectangles, giving $A = \sum_{i=1}^{n} f(c_i)\Delta x_i$.

In each rectangle, we can choose the c_i so that the height of the rectangle is equal to the maximum value of $f(x)$ for $x \in [x_{i-1}, x_i]$. The corresponding sum will be called the **upper Riemann sum, $\overline{S_P}$**. Similarly, we can take c_i so that the height of the rectangle is equal to the minimum value $f(x)$ for $x \in [x_{i-1}, x_i]$. The corresponding sum will be called the **lower Riemann sum, $\underline{S_P}$**.

Finally, if, for all possible partitions, P, the limit of the lower Riemann sum and the upper Riemann sum both exist and are equal, we say that the function f is **integrable** and we write

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\,dx.$$

Ex: $f(x) = 1$ if x is irrational and -1 if x is rational.



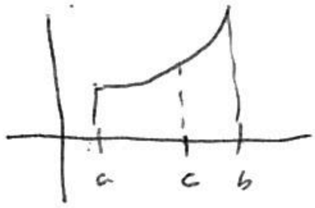
$\overline{S}_P = (x_1-x_0)\cdot 1 + (x_2-x_1) + (x_3-x_2) + \dots + (x_n-x_{n-1}) = x_n - x_0 = 1$

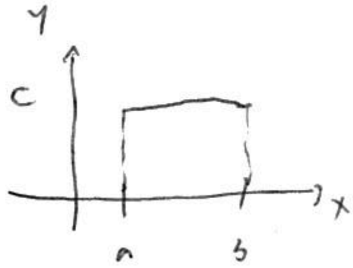
$\underline{S}_P = (x_1-x_0)\times -1 + (x_2-x_1)\cdot -1 + \dots + (x_n-x_{n-1}) = -(x_n-x_0) = -1.$

$\overline{S}_P \neq \underline{S}_P$, f is NOT Riemann integrable.

If $f(x)$ is positive for all $x \in [a,b]$ and integrable on $[a,b]$, then we define the area bounded by the curve $y = f(x)$, the x axis and the lines $x = a, x = b$ to be the integral $\int_a^b f(x)\,dx$.

Theorem: If there is a partition P such that the upper and lower Riemann sums for f can be made arbitrarily close, then f is Riemann integrable.





Properties of Integrals:

(i) If $f(x) = C$ for all $x \in [a, b]$ then $\int_a^b f(x)\,dx = C(b-a)$.

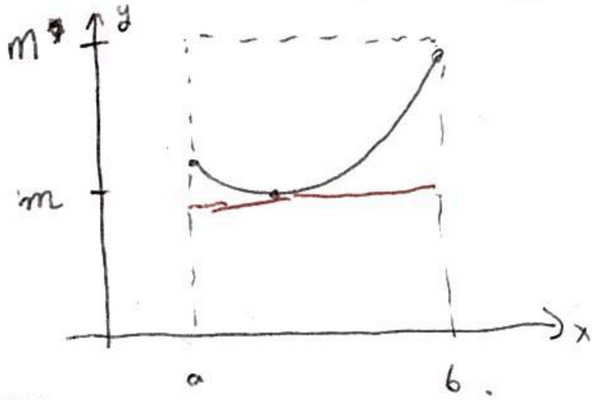
(ii) If $a < c < b$ and f is integrable on $[a, b]$, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

(iii) If f and g are both integrable on $[a, b]$ then $\int_a^b (\alpha f(x) + g(x))\,dx = \alpha \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

(iv) If $f \geq 0$ and f is integrable on $[a, b]$ then $\int_a^b f(x)\,dx \geq 0$.

(v) If f and g are both integrable on $[a, b]$ and $f(x) \geq g(x)$ on $[a, b]$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

(vi) (ML Theorem) If f is integrable on $[a, b]$ and m, M are real numbers such that $m \leq f(x) \leq M$ for all $x \in [a, b]$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.



Definitions:

Suppose $a < b$ and f is integrable on $[a, b]$ then we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx, \text{ and } \int_a^a f(x)\,dx = 0.$$

Primitives:

We now begin to relate the problem of integration to that of differentiation.

Definition:
Let f be a continuous function on some interval (a, b).
A function F with the properties that:

(i) F is differentiable on (a, b) and

(ii) $F'(x) = f(x)$ for all $x \in (a, b)$

is called a **primitive** (or antiderivative) of f.

$f(x) = 2x$
$F(x) = x^2$ or $x^2 + 3$ or $x^2 - \pi$ primitive

Notice that if F is a primitive of f then so is $F(x) + C$, where C is a constant. In fact, any two primitives for f can only differ by a constant. (For the proof see Calc.notes).

Ex: $f(x) = x^n$ has primitive $F(x) = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$.

Note that many functions do not have 'elementary primitives', i.e. we cannot write the primitives in terms of the standard functions. e.g. $f(x) = e^{x^2}$.

The Fundamental Theorem of Calculus:

The great discovery of Newton and Leibniz was to realise that differentiation and integration are in some sense opposite processes. This statement is made more precise by the so called Fundamental Theorem(s) of Calculus. There are two versions of this. The first says that the derivative of the integral gets us back to the function we started with, while the second relates primitives to areas. We cannot 'prove' either of these with modern rigour (but then neither could Leibniz or Newton), but we can give a reasonable argument to explain why these results are true. I am going to combine both into one and then state the results separately afterwards.

Suppose that f is continuous on the interval $[a, b]$.

Take a point $x \in [a, b]$, and consider the area under the curve between a and x, which we denote by

$$A(x) = \int_a^x f(t)\, dt.$$

The function $A(x)$ is continuous, however we will not include the proof of this.

We now make some further restrictions to simplify our analysis.

Suppose that f is integrable, positive and increasing on $[a, b]$ and let F be a primitive of f.

For the moment, consider the area under the curve between x and $x + h$ and observe that we can under and over approximate this area by:

$$f(x)h \leq A(x + h) - A(x) \leq f(x + h)h.$$

Dividing by h, we see that

$$f(x) \leq \frac{A(x+h) - A(x)}{h} \leq f(x+h).$$

If we now take a limit as $h \to 0$, then by the Pinching Theorem, we have

$$f(x) = \frac{dA}{dx}.$$

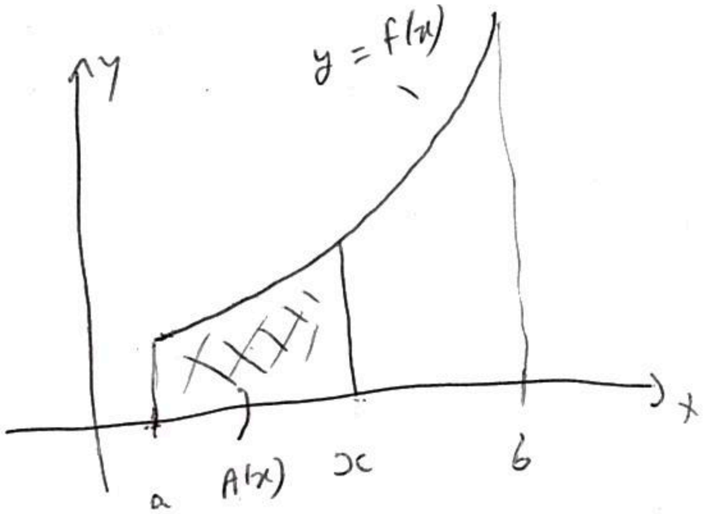
Thus $A(x)$ is a primitive of f. Now since primitives differ at most by a constant, we have $A(x) = F(x) + C$ for some constant C.

Now $A(a) = \displaystyle\int_a^a f(t)\,dt = 0$ so $C = -F(a)$. Also if we set $x = b$ we have

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

F.T.C.

Suppose f is cts on $[a,b]$ **and** f increasing on $[a,b]$

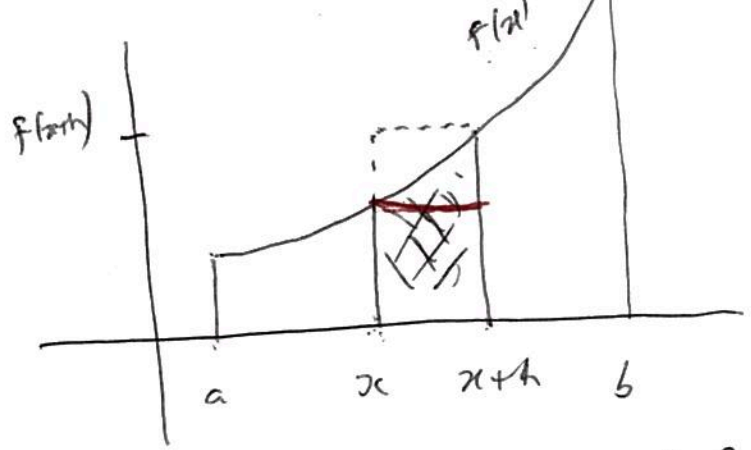


Define

$A(x)$ to be the area from a to x under the curve.

N.B. $A(a) = 0$

$$A(b) = \int_a^b f(x)\,dx.$$



Let F be a primitive of f, so $F'(x) = f(x)$

Shaded area is

$$A(x+h) - A(x)$$

Using rectangles

$$h\,f(x) \le A(x+h) - A(x) \le h\,f(x+h)$$

$$\therefore f(x) \le \frac{A(x+h) - A(x)}{h} \le f(x+h)$$

By pinching theorem, as $h \to 0$

$$\lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = f(x)$$

i.e. $\boxed{\frac{dA}{dx} = f(x)}$

$\therefore$ A is a primitive for f.

$\therefore$ $A(x) = F(x) + C$

$A(a) = 0 \Rightarrow C = -F(a)$

$$A(b) = \int_a^b f(x)\,dx$$

$$= F(b) - F(a)$$

$$\frac{du}{dt} = -1 \qquad -\int_a^0 f(-u)\,du = \int_a^0 f(u)\,du$$

f odd

$$= -\int_0^a f(u)\,du$$

Write the integral as

$$\int_{-a}^{a} f(t)\,dt = \int_{-a}^{0} f(t)\,dt + \int_{0}^{a} f(t)\,dt.$$

Now make the substitution $u = -t$ in the first integral giving,

$$\int_{-a}^{a} f(t)\,dt = \int_{a}^{0} f(u)\,du + \int_{0}^{a} f(t)\,dt = 0.$$

Ex: Find $\int_{-\pi}^{\pi} \underline{x^4 \sin(x^3)}\,dx.$ $= 0$

$f(x) = x^4 \sin(x^3)$

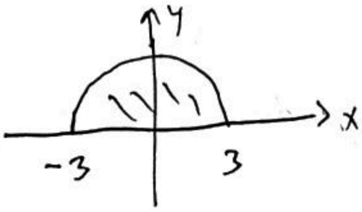
$f(-x) = (-x)^4 \sin\left((-x)^3\right)$

$= -x^4 \sin(x^3)$

f is odd

Ex: Find $\int_{-3}^{3} \sqrt{9 - x^2}\,dx.$ $= \frac{1}{2} \times \pi \cdot 3^2$

$y = \sqrt{9 - x^2}$ $\qquad = \frac{9\pi}{2}$.



Theorem: Suppose that f is integrable on $\mathbb{R}$.
If f is periodic with period T, then for any $a \in \mathbb{R}$ we have

$$\int_{a}^{a+T} f(t)\,dt = \int_{0}^{T} f(t)\,dt.$$

Proof: Again we split the integral as

$$\int_{a}^{a+T} f(t)\,dt = \int_{a}^{0} f(t)\,dt + \int_{0}^{a+T} f(t)\,dt$$

and put $t = u - T$ in the first integral giving

$$\int_{a}^{a+T} f(t)\,dt = \int_{a+T}^{T} f(u)\,du + \int_{0}^{a+T} f(t)\,dt = \int_{0}^{T} f(t)\,dt.$$

Ex: Evaluate $I = \int x e^{x^2}\, dx$.

Let $u = x^2$

$$\frac{du}{dx} = 2x$$

$$\left(\text{"} \frac{du}{2} = x\,dx \text{"} \right)$$

$$I = \frac{1}{2}\int e^{u}\, du$$

$$= \frac{1}{2} e^{u} + C$$

$$= \frac{1}{2} e^{x^2} + C$$

Ex: Evaluate $I = \int_0^2 \frac{1}{2+\sqrt{x}}\, dx$.

Put $x = u^2$

$$\boxed{\frac{dx}{du} = 2u}$$

$x = 2, \quad u = \sqrt{2}$

$x = 0, \quad u = 0$

$$I = \int_0^{\sqrt{2}} \frac{1}{2+u} \cdot 2u\, du$$

$$= 2\int_0^{\sqrt{2}} \frac{u}{u+2}\, du = 2\int_0^{\sqrt{2}} \frac{u+2-2}{u+2}\, d$$

$$= 2\int_0^{\sqrt{2}} 1 - \frac{2}{u+2}\, du = 2\left[u - 2\ln(u+2\right.$$

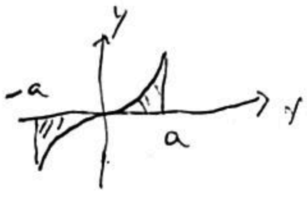
$$\cancel{= 2[2 - 2\ln 4 + 2\ln 2]}$$

$$= 2\left[\sqrt{2} - 2\ln(2+\sqrt{2}) + 2\ln 2\right.$$

Theorem: Suppose f is integrable on $[-a, a]$, where $a \in \mathbb{R}^+$.

(i) If f is odd then $\int_{-a}^{a} f(t)\, dt = 0$.

(i) If f is even then $\int_{-a}^{a} f(t)\, dt = 2\int_0^a f(t)\, dt$



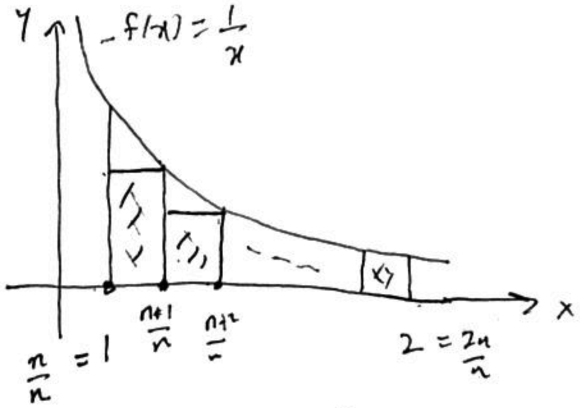
Proof: (of (i))

Recall that f odd means $f(-u) = -f(u)$.

Integrals and Series:

We can use integrals to find the value of certain series.

Ex: Use the function $f(x) = \frac{1}{x}$ in $[1,2]$ with partition $\{\frac{n}{n}, \frac{n+1}{n}, ..., \frac{2n}{n}\}$ to show that
$\lim_{n\to\infty} \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n} = \log 2.$



Since $\frac{1}{x}$ is cts on $[1,2]$,

$$\int_1^2 \frac{1}{x}\,dx = \lim_{n\to\infty} \underline{S}_P$$

$$\ln x \Big|_1^2 = \lim_{n\to\infty} \frac{1}{n}\left[\frac{n}{n+1} + \frac{n}{n+2} + \dots + \frac{n}{2n}\right]$$

$$\ln 2 = \lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}\right)$$

Integration Techniques:

While the rules for differentiation enable us to differentiate a very wide variety of functions, integrals are much mroe difficult and we only have a small number of tools available. Among these are:

Integration by Substitution:

Theorem: Suppose g is a differentiable function then
If $x = g(u)$, then

$$\int f(x)\,dx = \int f(g(u))\,\frac{dg}{du}\,du.$$

$$\frac{dx}{du} = g'(u)$$

The conditions f positive and increasing, can be removed and we have:

Theorem A: (First Fundamental Theorem of Calculus)
Suppose f is cts on $[a, b]$. Then the function F defined by

$$F(x) = \int_a^x f(t)\, dt$$

is continuous on $[a, b]$ and differentiable on (a, b), with $F'(x) = f(x)$ for all $x \in (a, b)$.

Theorem B: (Second Fundamental Theorem of Calculus)
If f is cts on $[a, b]$ and F is a primitive of f, then

$$\int_a^b f(t)\, dt = F(b) - F(a).$$

Ex: Find the area under $y = \frac{1}{\sqrt{1-x^2}}$ between $x = 0$ and $x = \frac{1}{2}$.

$$\text{Area} = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \sin^{-1} x \Big|_0^{\frac{1}{2}}$$

$$= \sin^{-1}\tfrac{1}{2} - \sin^{-1} 0$$

$$= \frac{\pi}{6} \text{ units}^2.$$

Ex: Let $G(x) = \int_0^x e^{t^2}\, dt$. Find $\frac{dG}{dx}$.

$$G'(x) = e^{x^2}$$

by 1^{st} F. Theorem

Let $F(t)$ be a prin. f e^{t^2}

const

$$\text{Then } G(x) = F(t)\Big|_0^x = F(x) - F(0)$$

$$\text{Then } G'(x) = F'(x) = e^{x^2}$$

Ex: Let $G(x) = \int_{x^2}^{x^4} \sin(t^2)\, dt$. Find $\frac{dG}{dx}$.

Let $F(t)$ be a prin. f $\sin(t^2)$

$$G(x) = F(t)\Big|_{x^2}^{x^4}$$

$$= F(x^4) - F(x^2)$$

$$G'(x) = F'(x^4) \cdot 4x^3 - F'(x^2) \cdot 2x$$

$$= 4x^3 \cdot \sin((x^4)^2) - 2x \sin((x^2)^2)$$

$$= 4x^3 \sin(x^8) - 2x \sin(x^4)$$

$$\int_1^\infty \frac{1}{x}\,dx \to \infty \quad \underline{\text{diverges}}$$

On the other hand, suppose we find the volume of the solid generated when we rotate this curve about the x axis between 1 and N. Then

$$V = \pi \int_1^N \frac{1}{x^2}\,dx = \pi(1 - \frac{1}{N})$$

and this clearly has the limit π as $N \to \infty$.
In this case we say that the improper integral $\int_1^\infty \frac{1}{x^2}\,dx$ converges to 1.

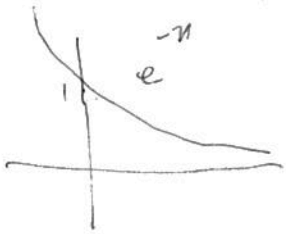
Definition: Suppose that

$$\int_a^N f(x)\,dx \to L, \quad \text{as} \quad N \to \infty$$

then we say that the improper integral $\int_a^\infty f(x)\,dx$ converges to L, and that f is *integrable* on $[a, \infty)$.

On the other hand if the integral $\int_a^N f(x)\,dx$ does NOT have a (finite) limit as $N \to \infty$ then we say that the improper integral diverges to infinity.

One can similarly define improper integrals at $-\infty$.



Ex: Look at $\int_0^\infty xe^{-x^2}\,dx$.

Look at $I = \int x e^{-x^2}\,dx$

Put $u = -x^2$

$$\frac{du}{dx} = -2x$$

$$I = \int -\frac{1}{2} e^{+u}\,du$$

$$= -\frac{1}{2} e^{u} + C$$

$$= -\frac{1}{2} e^{-x^2}$$

$$\int_0^\infty x e^{-x^2}\,dx$$

$$= \lim_{N\to\infty} \int_0^N x e^{-x^2}\,dx$$

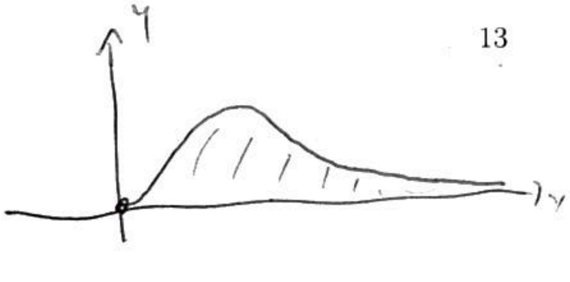
$$= \lim_{N\to\infty} -\frac{1}{2} e^{-x^2}\Big|_0^N$$

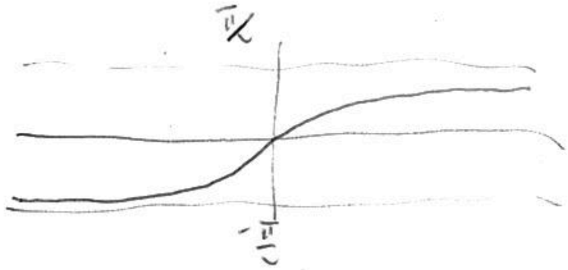
$$= \lim_{N\to\infty} -\frac{1}{2} e^{-N^2} + \frac{1}{2}$$

$$= \frac{1}{2}.$$

$\therefore \int_0^\infty x e^{-x^2}\,dx$

converges (to $\frac{1}{2}$)





Ex: Look at $\int_0^\infty \frac{1}{1+x^2}\,dx$.

$$= \lim_{N\to\infty} \int_0^N \frac{1}{1+x^2}\,dx$$

$$= \lim_{N\to\infty} \tan^{-1}x \Big|_0^N$$

$$= \lim_{N\to\infty} \tan^{-1}N$$

$$= \frac{\pi}{2}$$

$\therefore$ integral converges (to $\frac{\pi}{2}$)

Ex: Look at $\int_3^\infty \frac{x^2}{1+x^3}\,dx$.

$$= \lim_{N\to\infty} \int_3^N \frac{x^2}{1+x^3}\,dx$$

$3x^2$

$$= \lim_{N\to\infty} \frac{1}{3}\int_3^N \frac{3x^2}{1+x^3}\,dx$$

$$= \lim_{N\to\infty} \frac{1}{3}\ln(1+x^3)\Big|_3^N$$

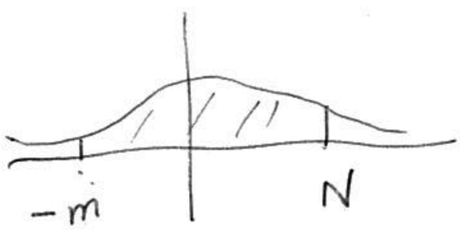
$$= \lim_{N\to\infty} \frac{1}{3}\ln(1+N^3) - \frac{1}{3}\ln 28$$

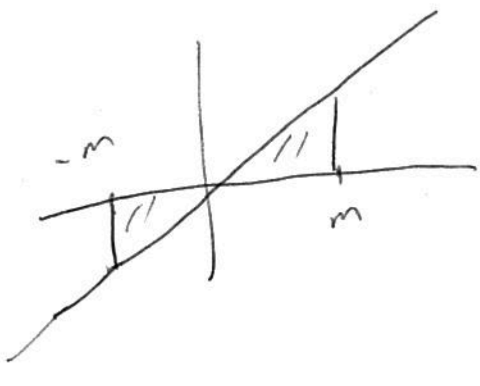
$$\to \infty$$

$\therefore$ integral diverges

Note that $\int_{-\infty}^{\infty} f(x)\,dx$ converges if and only if $\lim_{M\to\infty}\lim_{N\to\infty}\int_{-M}^{N} f(x)\,dx$ exists, where the two limits are done separately.

Hence for example, $\int_{-\infty}^{\infty} x\,dx$ does not exist even though $\int_{-M}^{M} f(x)\,dx = 0$ for every real number M.





Revision

$$I = \int \frac{x^3}{\sqrt{4-x^2}}\,dx. = \int \frac{x^2 \; x\,du}{\sqrt{4-x^2}}$$

Put $4 - x^2 = u^2$

$$-\cancel{2}x \frac{dx}{du} = \cancel{2}u$$

$$-x \frac{dx}{du} = u$$

$$I = \int -\frac{(4-u^2)\,u\,du}{u}$$

$$= \int u^2 - 4\,du$$

$$= \frac{u^3}{3} - 4u + C$$

$$= \frac{(4-x^2)^{\frac{3}{2}}}{3} - 4(4-x^2)^{\frac{1}{2}} + C$$

$$I = \int x^3 \ln x \; dx.$$

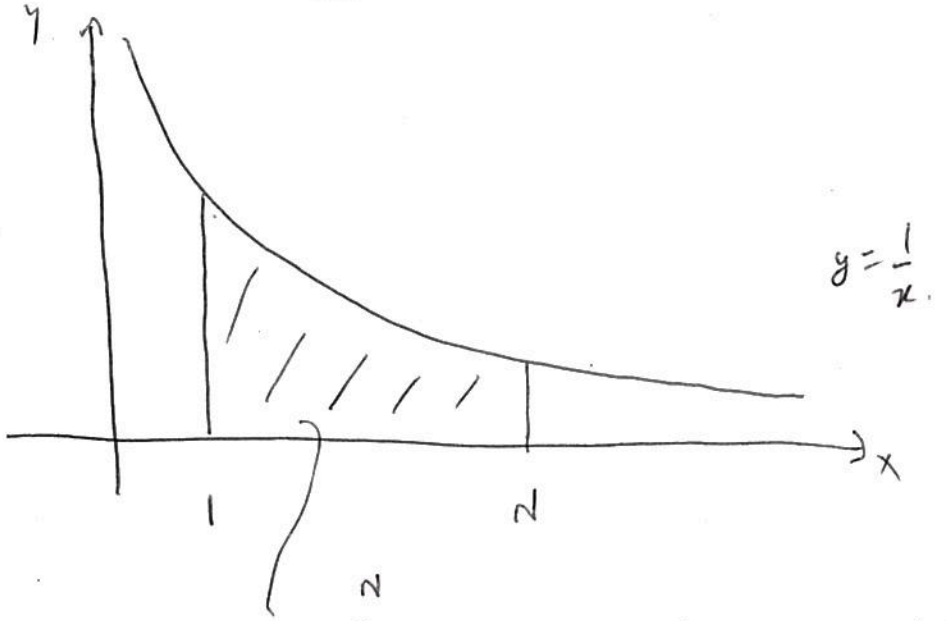
$$u = \ln x \qquad \frac{dv}{dx} = x^3$$

$$u' = \frac{1}{x} \qquad v = \frac{x^4}{4}$$

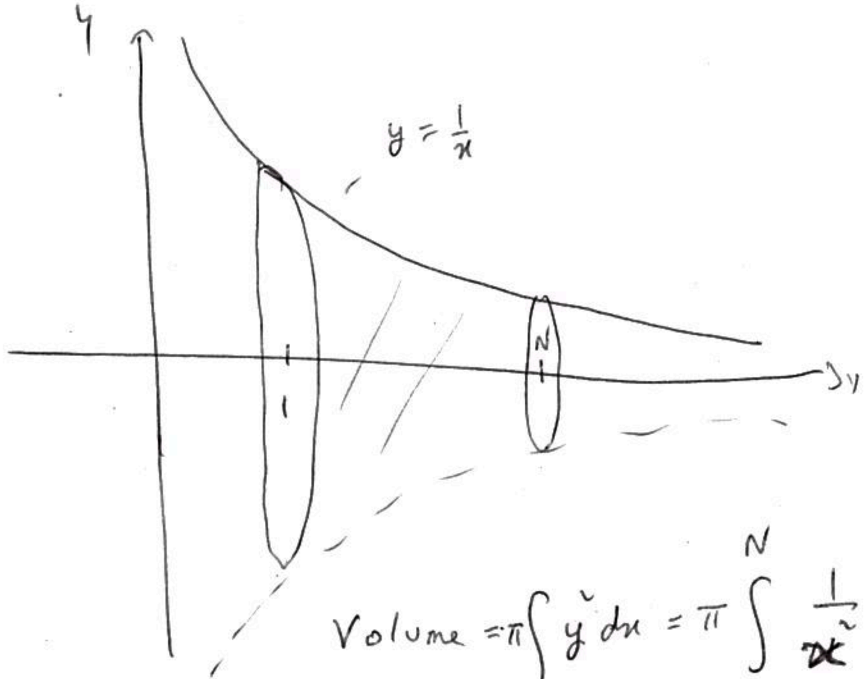
$$I = \frac{x^4}{4}\ln x - \int \frac{x^3}{4}\,dx$$

$$= \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$$

Torricelli's Paradox (1608 - 1647)



$$\text{Area} = \int_1^N \frac{1}{x}\,dx = \ln N \to \infty \quad \text{as } N \to \infty$$



$$\text{Volume} = \pi \int y^2\,dx = \pi \int_1^N \frac{1}{x^2}\,dx$$

$$= \pi \left(-\frac{1}{x} \right|_1^N = \pi\left(1 - \frac{1}{N}\right)$$

$$\to \pi \qquad \text{as } N \to \infty$$

x^{-p}.

Theorem: (*p*-test) The integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p > 1$ and diverges for $p \leq 1$.

Proof: $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$

$$= \lim_{N\to\infty} \int_1^N \frac{1}{x^p}\,dx = \lim_{N\to\infty} \frac{x^{-p+1}}{1-p}\Big|_1^N$$

$$= \lim_{N\to\infty} \frac{1}{1-p}\left[\frac{1}{N^{p-1}} - 1\right]$$

~~This exists~~ ~~$p \to 1$~~

If $p > 1$ this limit exists $\& = \dfrac{1}{p-1}$

If $p < 1$ — — — does not exist

If $p = 1$, we have $\displaystyle\int_1^\infty \frac{1}{x}\,dx \to \infty$.

This result can be referred to as the *p*-test.

Comparison Test:

In many applications, we are often not interested (or are unable to find) the value to which a convergent improper integral converges. We often are only interested in whether or not a given improper integral converges. To deal with this question, we try to develop some tests for convergence that do not rely on us actually finding the limit of the integral. This is done by trying to bound the integral by other integrals which we can easily evaluate.

Theorem: Suppose that $0 \leq f(x) \leq g(x)$ for all $x \in [a, \infty)$, and $\displaystyle\int g(x)\,dx$ converges. Then

$$\int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx,$$

i.e. $\displaystyle\int_a^\infty f(x)\,dx$ converges.

On the other hand, if $0 \leq f(x) \leq g(x)$ for all $x \in [a, \infty)$, and $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then $\displaystyle\int_a^\infty g(x)\,dx$ diverges.

Ex: Look at $\displaystyle\int_2^\infty \frac{1}{x^4+1}\,dx$

For large x, $\dfrac{1}{x^4+1} \sim \dfrac{1}{x^4}$

$\&$ $\displaystyle\int_2^\infty \frac{1}{x^4}\,dx$ converges by p-test

So, $0 \leq \underset{f}{\dfrac{1}{x^4+1}} \leq \underset{g}{\dfrac{1}{x^4}}$

$\&$ $\displaystyle\int_2^\infty \frac{1}{x^4}\,dx$ converges by p-test

$\therefore$ $\displaystyle\int_2^\infty \frac{1}{1+x^4}\,dx$ converges by comparison test

Ex: Look at $\displaystyle\int_2^\infty \frac{1}{\sqrt{x^4+2x+1}}\,dx$

[Discovery: $\dfrac{1}{\sqrt{x^4+2x+1}} \sim \dfrac{1}{x^2}$ for large x

& $\displaystyle\int_2^\infty \frac{1}{x^2}\,dx$ conv.]

Now

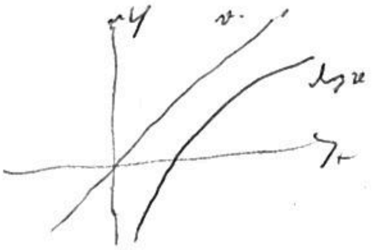
$$0 \le \frac{1}{\sqrt{x^4+2x+1}} \le \frac{1}{x^2}$$

& $\displaystyle\int_2^\infty \frac{1}{x^2}\,dx$ conv. by p-test, $p=2$

So $\displaystyle\int_2^\infty \frac{1}{\sqrt{x^4+2x+1}}\,dx$ conv. by comparison

Ex: Look at $\displaystyle\int_2^\infty \frac{1}{\log x}\,dx.$

~~Extra~~ $\log x < x$ for $x > 1$



so $\displaystyle\int_2^\infty \frac{1}{\ln x}\,dx > \int_2^\infty \frac{1}{x}\,dx$

& $\displaystyle\int_2^\infty \frac{1}{x}\,dx$ diverges

by p-test.

$\therefore \displaystyle\int_2^\infty \frac{1}{\ln x}\,dx$ diverges to ∞.

There is an equivalent result called the **limit form of the comparison test** which says:

Theorem: Suppose that f and g are non-negative and bounded on $[a,\infty)$. If $\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)}$ exists and is NON-ZERO then the integrals $\displaystyle\int_a^\infty f(x)\,dx$ and $\displaystyle\int_a^\infty g(x)\,dx$ either BOTH converge OR BOTH diverge.

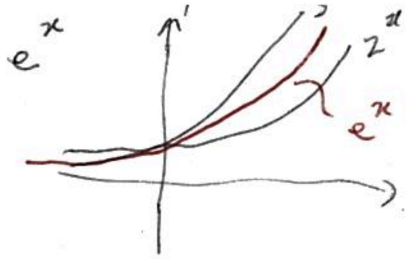
Ex: Look at $\displaystyle\int_2^\infty \frac{1}{x^4+1}\,dx.$

$\dfrac{1}{x^4+1} \sim \dfrac{1}{x^4}$ for large x.

$$\lim_{x\to\infty} \frac{\frac{1}{x^4+1}}{\frac{1}{x^4}} = \lim_{x\to\infty} \frac{x^4}{x^4+1} = 1$$

Also $\displaystyle\int_2^\infty \frac{1}{x^4}\,dx$ conv. by p-test

$\therefore \displaystyle\int_2^\infty \frac{1}{1+x^4}\,dx$ converges by L.C.T

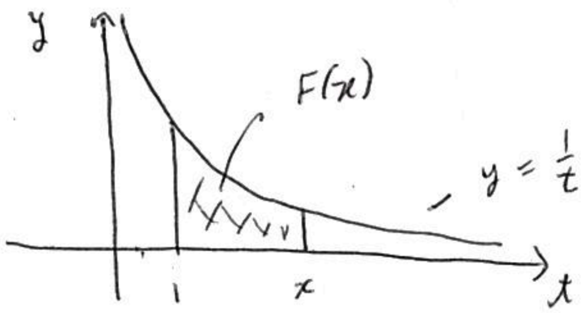


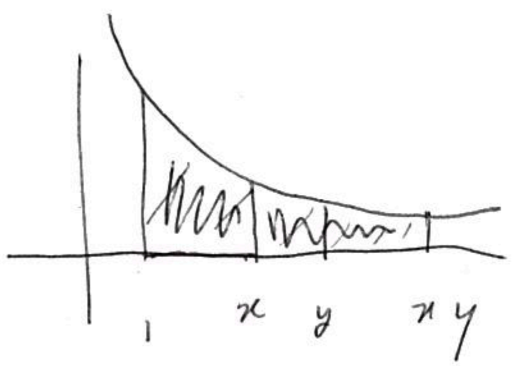
THE UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1131 Calculus
Section 9: - Logarithms and Exponentials.

In this section we assume you are familiar with the rules for logarithms and exponentials and concentrate on some of the underlying theory.

We begin with the problem of finding the area under the curve $y = \frac{1}{t}$ from 1 up to some point x.

Define $F(x) = \displaystyle\int_1^x \frac{1}{t}\, dt$.





Theorem: F has the following properties:

(i) $F(1) = 0$

(ii) If $0 < x < 1$ then $F(x) < 0$.

(iii) $F(xy) = F(x) + F(y)$ and $F(\frac{x}{y}) = F(x) - F(y)$.

(iv) $F(x^n) = nF(x)$ for all integers n.

$t = ux$

Proof: (i) and (ii) are obvious from the definition. For (iii), write

$\frac{dt}{du} = x$.

$$F(xy) = \int_1^{xy} \frac{1}{t}\, dt = \int_1^{x} \frac{1}{t}\, dt + \int_x^{xy} \frac{1}{t}\, dt.$$

$\displaystyle\int_1^y \frac{x}{u\cancel{x}}\, du = \int_1^y \frac{1}{u}\, du$

Make the change of variable $\boxed{t = ux}$ in the second integral to obtain

$$F(xy) = \int_1^{x} \frac{1}{t}\, dt + \int_1^{y} \frac{1}{t}\, dt = F(x) + F(y).$$

The remaining results now follow easily from (iii).

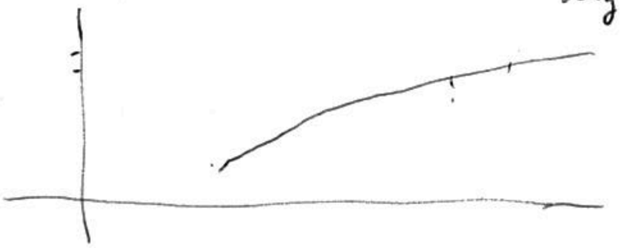
The function F behaves like a *logarithm* and so we rename the function F by $F(x) = \log_e(x) = \log x = \ln x$.

logarismoss naturalis

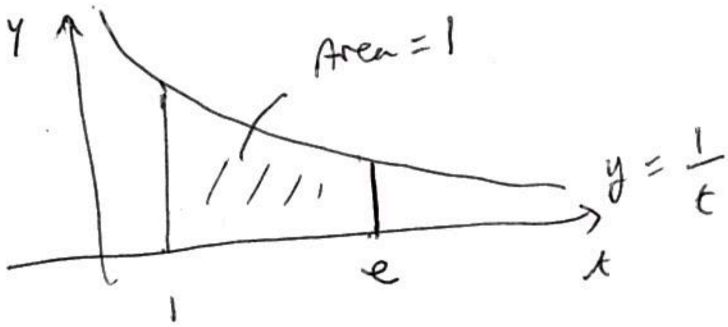
$$\log x = \int_1^x \frac{1}{t}\,dt$$

By the Fundamental Theorem of Calculus we have $\frac{d}{dx}(\log x) = \frac{1}{x}$, for all real $x > 0$.

Ex: Find $\frac{d}{dx}\log(\log(x))$ $= \dfrac{1}{\log x} \cdot \dfrac{1}{x} = \dfrac{1}{x \log x}$



Note that at some point x_0, the area under the curve $\frac{1}{t}$ from 1 to x_0 must reach 1, (by continuity), so we denote that point by the number e (after Euler). e has the value 2.7182818284... and is irrational.

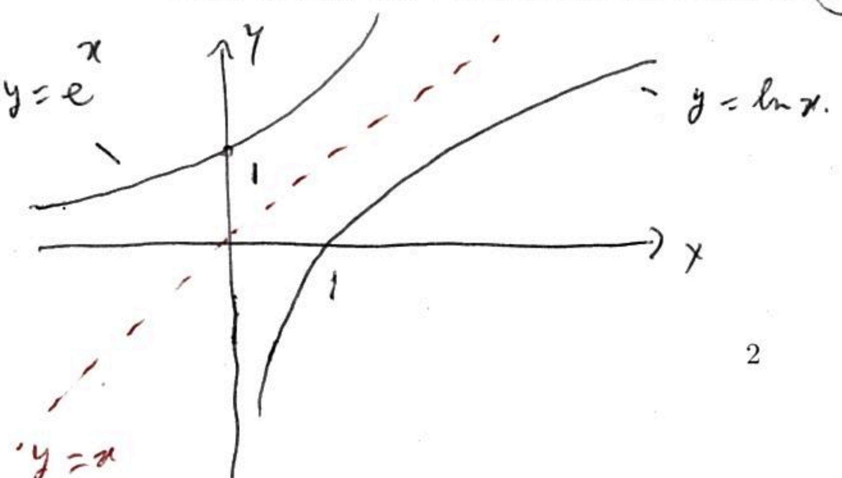


We showed earlier that $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ and next Session we shall show that $e =$

$e = \underset{0!}{1} + \underset{1!}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots..$

Exponential Function

Since the logarithmic function is continous and increasing for $x > 0$, it has an inverse. Also, since log is a logarithmic function with base e, its inverse is an exponential function which we write as e^x. This is sometimes written as exp(x).



Hence we know that $\log e^x = x$ for all real x and $e^{\log x} = x$, for all real $x > 0$.

Corollary If $y = e^x$ then $\frac{dy}{dx} = e^x$.

Proof: $y = e^x$

$\ln y = x$

$$\frac{dx}{dy} = \frac{1}{y} = \frac{1}{e^x}$$

$$\frac{dy}{dx} = e^x.$$

General Exponentials:

If $a \in \mathbb{R}^+$, can we make sense of a^x?

Definition: If $a \in \mathbb{R}^+$ define a^x to be $e^{x \log a}$.

Ex: Find $2^{\sqrt{3}} = e^{\sqrt{3} \ln 2}$

Ex: $5^x = e^{x \ln 5}$

Theorem: $\dfrac{d}{dx}a^x = a^x \log a$, $a > 0$

Proof:

$y = a^x$

$= e^{x \ln a}$

$$\frac{dy}{dx} = \ln a \, e^{x \ln a}$$

$= \ln a . a^x$

Ex: Find the derivative of 5^x.

$$\frac{d}{dx} 5^{x} = \ln 5 . 5^{x}$$

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \ln x}$$

$$= e^{\lim_{x \to 0^+} x \ln x} = e^0 = 1$$

Now $\lim_{x \to 0^+} x \ln x$

$$= \lim_{x \to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)} \quad \text{"}\frac{\infty}{\infty}\text{"}$$

$$\stackrel{L'Hop}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} -x = 0$$

Ex: Find $\lim_{x \to 0^+} x^x$ and hence sketch the curve $y = x^x$ by finding any stationary points.

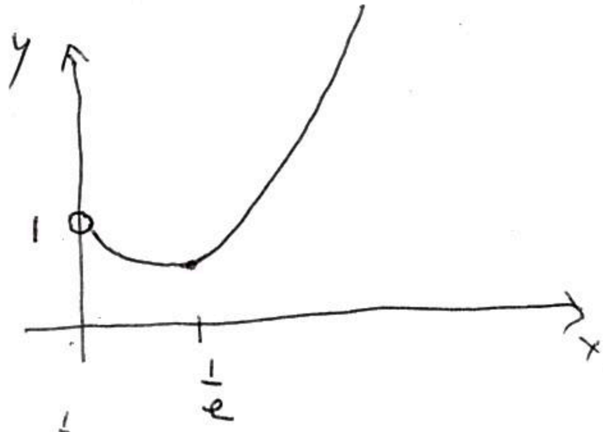
$$y = x^x = e^{x \ln x}$$

$$\frac{dy}{dx} = e^{x \ln x} . \left(x . \frac{1}{x} + \ln x\right)$$

$$= x^x (1 + \ln x) = 0 \text{ for a s.p.}$$

so $\ln x = -1 \Rightarrow x = \frac{1}{e}$ (local min)

$y = \left(\frac{1}{e}\right)^{\frac{1}{e}} \simeq 0.69$



Integrals:

Ex: Find a. $\int xe^{5x^2}\, dx$

b. $\int \tan\theta \, d\theta$,

c. $\int_0^\infty e^{-x} \cos x \, dx$

a) $$\int x e^{5x^2} dx = \frac{1}{10} \int 10x e^{5x^2} dx$$

$$= \frac{1}{10} e^{5x^2} + c$$

b) $$\int \tan\theta \, d\theta = -\int \frac{-\sin\theta}{\cos\theta} d\theta$$

$$= -\ln|\cos\theta| + c$$

$$= \ln|\sec\theta| + c$$

d. $\int \sec\theta \, d\theta. = \int \dfrac{\sec\theta\,(\sec\theta + \tan\theta)}{(\sec\theta + \tan\theta)} \, d\theta$

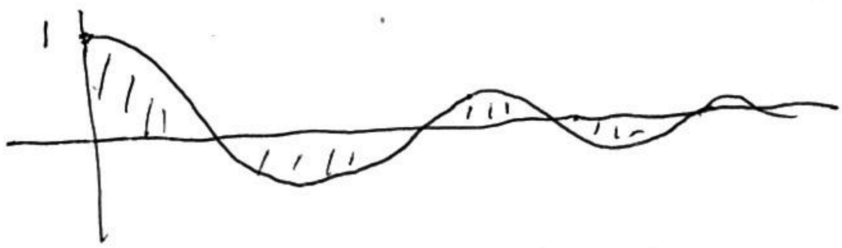
$$= \int \frac{\sec\theta\tan\theta + \sec^2\theta}{\sec\theta + \tan\theta} \, d\theta = \ln|\sec\theta + \tan\theta| + c$$

Logarithmic Differentiation:

Ex: Differentiate $y = \dfrac{\sqrt{x^2+x-1}}{\sqrt[3]{x^4+1}}$. $\quad \ln y = \ln\sqrt{x^2+x-1} - \ln\sqrt[3]{x^4+1}$

$$\ln y = \frac{1}{2}\ln(x^2+x-1) - \frac{1}{3}\ln(x^4+1)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x+1}{2(x^2+x-1)} - \frac{4x^3}{3(x^4+1)}, \quad \frac{dy}{dx} = \frac{\sqrt{x^2+x-1}}{\sqrt[3]{x^4+1}}\left(\frac{2x+1}{2(x^2+x-1)} - \frac{4x^3}{3(x^4+1)}\right)$$



c) $\displaystyle\int_0^\infty e^{-x}\cos x\,dx = \lim_{N\to\infty}\int_0^N e^{-x}\cos x\,dx.$

$$= \lim_{N\to\infty} e^{-x}\left(\frac{1}{2}\sin x - \frac{1}{2}\cos x\right)\Big|_0^N$$

$$= \lim_{N\to\infty} e^{-N}\left(\frac{1}{2}\sin N - \frac{1}{2}\cos N\right) + \frac{1}{2} = \frac{1}{2}$$

$$\int e^{-x}\cos x\,dx$$

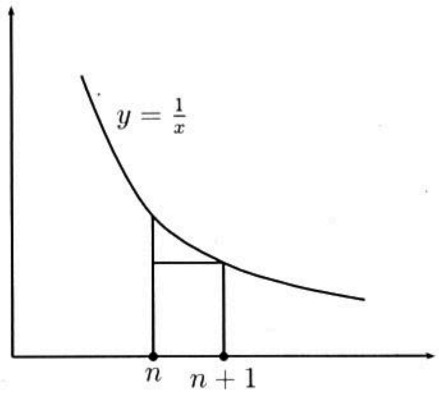
$$= e^{-x}(A\cos x + B\sin x)$$

$$\frac{d}{dx}e^{-x}(A\cos x + B\sin x) = e^{-x}\begin{pmatrix}-A\sin x + B\cos x \\ -A\cos x - B\sin x\end{pmatrix} = e^{-x}\cos x$$

$$\Rightarrow \begin{array}{l} B-A=1 \\ -A-B=0 \\ A=-\frac{1}{2},\ B=\frac{1}{2}\end{array}$$

An Interesting Limit: (If time).

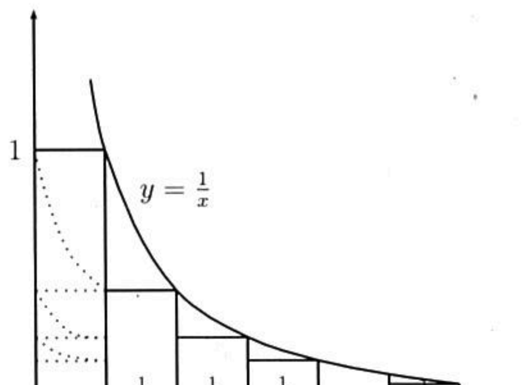
Let $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$, then $a_{n+1} - a_n = \frac{1}{n+1} - \log(n+1) + \log n$. Consider the area under the curve $y = \frac{1}{x}$ from $x = n$ to $x = n+1$.



Clearly,

$$\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x}\,dx = \log(n+1) - \log n$$

hence $a_{n+1} < a_n$, and so $\{a_n\}$ is a decreasing sequence.
In the diagram, we can see that



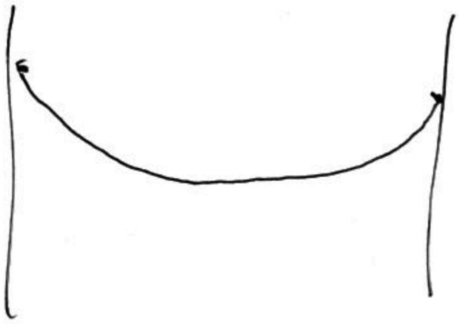
$$\int_1^2 \frac{1}{x}\,dx - \frac{1}{2} + \int_2^3 \frac{1}{x}\,dx - \frac{1}{3} + \cdots + \int_{n-1}^n \frac{1}{x}\,dx - \frac{1}{n}$$
$$= \int_1^n \frac{1}{x}\,dx - \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

gives the sum of the shaded areas, but this is clearly less than 1. Subtracting 1 from both sides we have

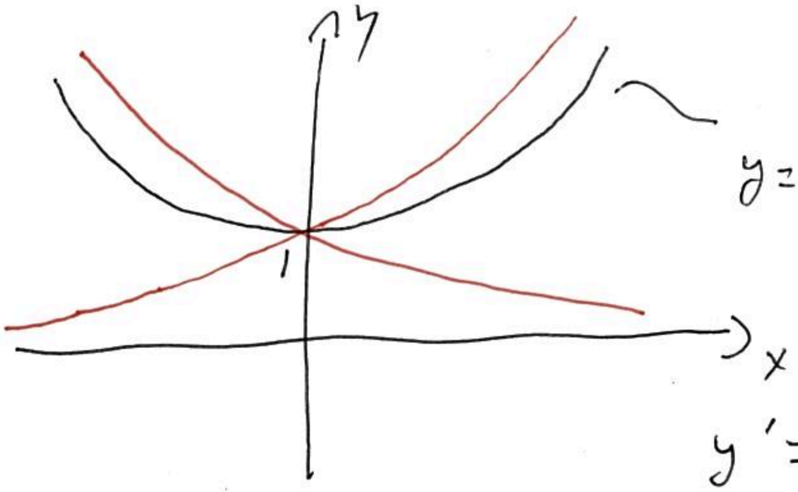
$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n > 0$$

and so the sequence is bounded below by 0. We can thus conclude that a_n converges to a limit. This number is written as γ and is called Euler's constant. It is approximately, 0.5772156649.... Little is known about it. It is not even known whether or not it is irrational!

$$\lim_{n \to \infty} 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n = \gamma$$



Bernoulli



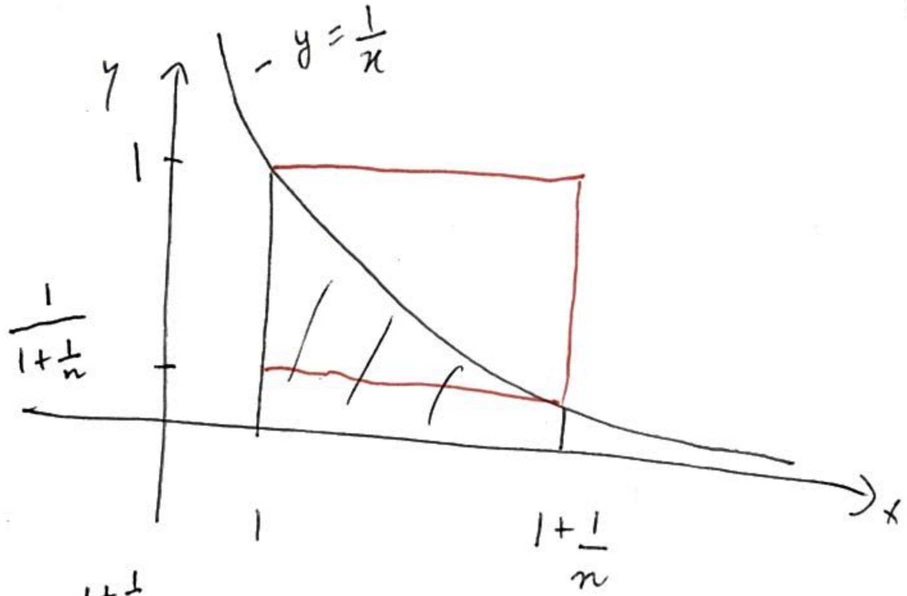
$$y = \frac{e^{x} + e^{-x}}{2} = \cosh x$$

$$y' = \frac{e^{x} - e^{-x}}{2} = \sinh x$$

'shine x'

$x(4u:s)$

$$g = \frac{x}{x\cancel{y}:s}$$



$$A = \int_1^{1+\frac{1}{n}} \frac{1}{x}\,dx = \ln\left(1+\frac{1}{n}\right)$$

$$\frac{1}{n}\cdot\frac{1}{1+\frac{1}{n}} \leq \ln\left(1+\frac{1}{n}\right) \leq \frac{1}{n}\cdot 1$$

$$\frac{1}{1+\frac{1}{n}} \leq n\ln\left(1+\frac{1}{n}\right) \leq 1$$

$$\frac{1}{1+\frac{1}{n}} \leq \ln\left(1+\frac{1}{n}\right)^n \leq 1$$

As $n \to \infty$, by Pinching Th

$$\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = 1$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

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MATH1131 Calculus

Section 10: - Hyperbolic Functions.

A number of centuries ago, one of the Bernoulli brothers asked the question: What curve do you get get when a piece of chain or rope is held loosely under gravity. It had been tacitly assumed that the curve must be a parabola, but Bernoulli showed that it wasn't, and indeed discovered a 'new' function which came to known as $\cosh x$. This function is not really 'new' since it is a combination of exponentials, but has enough important properties to be given its own special name.

Definition: We define the functions:

(i) $\sinh x = \dfrac{e^x - e^{-x}}{2}$

(ii) $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

(iii) $\tanh x = \dfrac{\sinh x}{\cosh x}$.

These are known as the **hyperbolic functions**. The name comes from the identity $\cosh^2 x - \sinh^2 x = 1$, so one can parametrize the hyperbola, $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ by $x = a\cosh t, y = b\sinh t$, in the same way that the trig. functions parametrize the circle (or ellipse).

The names are reminiscent of the trigonometric functions since they have properties that are very similar to the trigonometric functions. One can similarly define cosech x, sech x, coth x as the reciprocals of the above.

Simple differentiation yields:

$$\frac{d}{dx}\sinh x = \cosh x, \quad \frac{d}{dx}\cosh x = \sinh x, \quad \frac{d}{dx}\tanh x = \operatorname{sech}^2 x.$$

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right)$$

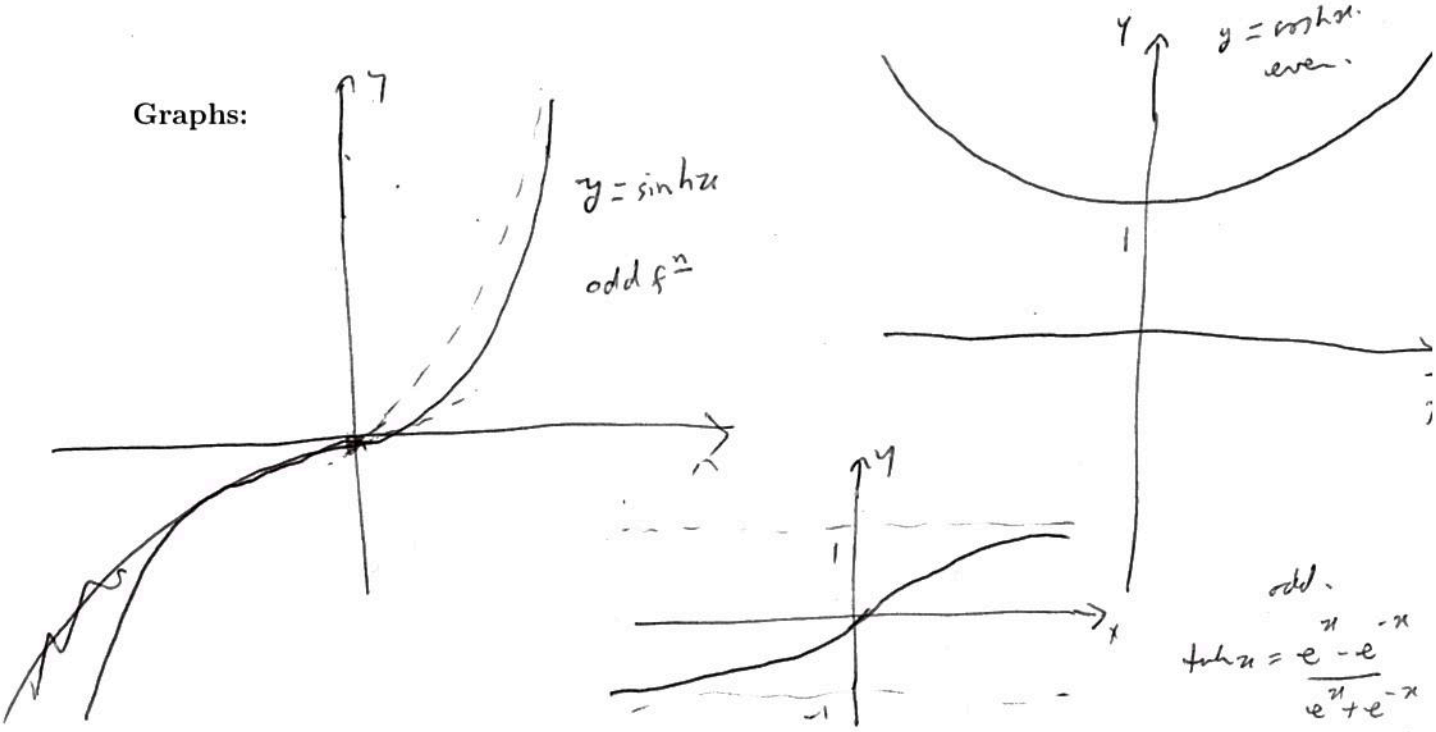
$$= \frac{e^x - e^{-x}}{2} = \sinh x.$$

$$\frac{d}{dx}(\tanh x)$$

$$= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)$$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

Graphs:



Identities: As with the trig functions, the hyperbolic functions satisfy a number of identities. We have already mentioned above

$$\cosh^2 x - \sinh^2 x = 1,$$

Proof: $\cosh^2 x - \sinh^2 x$

$= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$

$= \frac{1}{4}\left[e^{2x} + e^{-2x} + 2 - (e^{2x} + e^{-2x} - 2)\right]$

$= 1.$

Ex: Derive the formula: $\cosh 2\theta = \cosh^2\theta + \sinh^2\theta$.

RHS $= \cosh^2\theta + \sinh^2\theta$

$= \left(\frac{e^{\theta} + e^{-\theta}}{2}\right)^2 + \left(\frac{e^{\theta} - e^{-\theta}}{2}\right)^2 = \frac{1}{4}\left[2e^{2\theta} + 2e^{-2\theta}\right]$

$= \frac{e^{2\theta} + e^{-2\theta}}{2} = \cosh 2\theta.$

Recall that

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$\therefore \sin(i\theta) = \frac{e^{-\theta}-e^{\theta}}{2i} = i\left(\frac{e^{\theta}-e^{-\theta}}{2}\right) = i\sinh\theta$$

$$\therefore \sin(i\theta) = i\sinh\theta$$

$$/\!/ly \quad \cos(i\theta) = \cosh\theta$$

Connection with Trig. functions

Recall that

$$\therefore \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{i}\sinh(i\theta) \qquad \theta \leftrightarrow -i\theta$$

$$\therefore \sinh(i\theta) = i\sin\theta \Leftrightarrow -\sinh\theta =$$

$$\& \ \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$= \cosh(i\theta)$$

$$\therefore \cosh(i\theta) = \cos\theta$$

eg. $\cos^2\theta + \sin^2\theta = 1$.

replace θ with $i\theta$

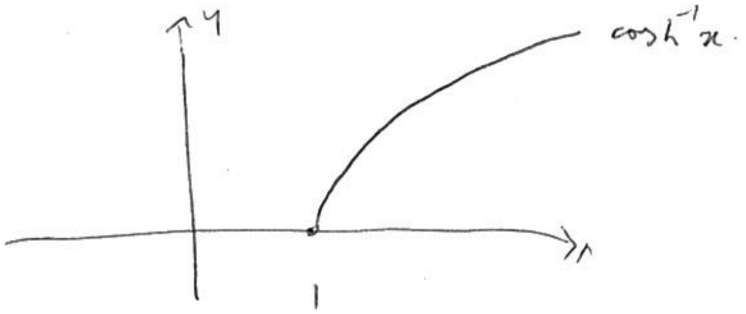
$$\cosh^2\theta + (i\sinh\theta)^2 = 1$$

$$\cosh^2\theta - \sinh^2\theta = 1$$

etc

Inverse Hyperbolic Functions:

The functions $\sinh x$ and $\tanh x$ are increasing on $\mathbb{R}$ and so have inverses written as $\sinh^{-1} x$ and $\tanh^{-1} x$ respectively.
The function $\cosh x$ is not one-to-one, but if we restrict its domain to $x \geq 0$ then we can define an inverse, $\cosh^{-1} x$ for this section of the curve.



Ex: Find a. $\cosh(\sinh^{-1}(\frac{3}{4}))$ b. $\cosh(2\cosh^{-1} 2)$.

a) Let $\alpha = \sinh^{-1}\frac{3}{4} \Rightarrow \sinh\alpha = \frac{3}{4}$

$$\cosh^2\alpha - \sinh^2\alpha = 1$$

$$\Rightarrow \cosh^2\alpha = 1 + \frac{9}{16} = \frac{25}{16}$$

$$\cosh\alpha = \frac{5}{4}.$$

b) Let $\alpha = \cosh^{-1} 2$

$$\Rightarrow \cosh\alpha = 2$$

$$\cosh^2\alpha - \sinh^2\alpha = 1$$

$$\Rightarrow \sinh^2\alpha = 3$$

$$\sinh\alpha = \sqrt{3} \quad (\alpha > 0)$$

$$\cosh(2\alpha) = \cosh^2\alpha + \sinh^2\alpha$$

$$= 4 + 3 = 7.$$

Inverse Hyperbolic and Logarithmic Functions:

Since the hyperbolic functions are defined in terms of exponentials, one would expect that there is some relationship between the inverse hyperbolic functions and the logarithmic functions.

Indeed, if we let $y = \cosh^{-1} x$ then

$\cosh y = x.$

$x = \frac{e^y + e^{-y}}{2} \Rightarrow e^y + e^{-y} = 2x.$

$e^{2y} - 2xe^y + 1 = 0$

$(e^y - x)^2 = x^2 - 1$

$e^y = x + \sqrt{x^2-1}$

$y = \cosh^{-1} x = \ln(x + \sqrt{x^2-1})$

N.B. we take +ve sq. root

(since eg. $\cosh^{-1} 2 \simeq 1.3169$

$\ln(2+\sqrt{3}) \simeq 1.3169$

$\ln(2-\sqrt{3}) \simeq -1.31$)

$(x - \sqrt{x^2-1}) \frac{(x+\sqrt{x^2-1})}{x+\sqrt{x^2-1}}$

$\frac{x^2-(x^2-1)}{x+\sqrt{x^2-1}} = \frac{1}{x+\sqrt{x^2-1}} < 1$ but $\cosh^{-1} x > 0$

so ln is neg.

$$\cosh^{-1} x = \log(x + \sqrt{x^2 - 1}).$$

There are similar formulae for the other inverse hyperbolic functions.

You do not need to memorize these formulae, but you might, for example, be asked to re-derive them.

Derivatives:

Let $y = \sinh^{-1} x$ then $\quad ch^2 - sh^2 = 1$

$x = \sinh y$

$\frac{dx}{dy} = \cosh y = \sqrt{1 + \sinh^2 y}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$

(c.f. $\frac{d}{dx}(\sin^{-1} x)$

$= \frac{1}{\sqrt{1-x^2}}$

Similarly, $\dfrac{d}{dx}(\cosh^{-1} x) = \dfrac{1}{\sqrt{x^2-1}}$, and $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$.

These formulae immediately lead to the integrals:

$$\int \frac{1}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a} + C, \quad \int \frac{1}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C, \quad \int \frac{1}{a^2-x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C.$$

Of these, the first only is worth commiting to memory.

Ex: Find $\displaystyle\int_0^1 \frac{1}{\sqrt{x^2+4}}\,dx$.

$= \sinh^{-1}\left(\frac{x}{2}\right)\Big|_0^1 = \sinh^{-1}\left(\frac{1}{2}\right)$

Ex: Find $\displaystyle\int_0^1 \frac{1}{\sqrt{x^2+4x+13}}\,dx$.

$= \displaystyle\int_0^1 \frac{1}{\sqrt{(x+2)^2+9}}\,dx = \sinh^{-1}\left(\frac{x+2}{3}\right)\Big|_0^1 = \sinh^{-1} 1 - \sinh^{-1}\frac{2}{3}$.

Ex: Find $\displaystyle\int \frac{x}{\sqrt{x^4+1}}\,dx$. (Put $u = x^2$ first).

Put $u = x^2$

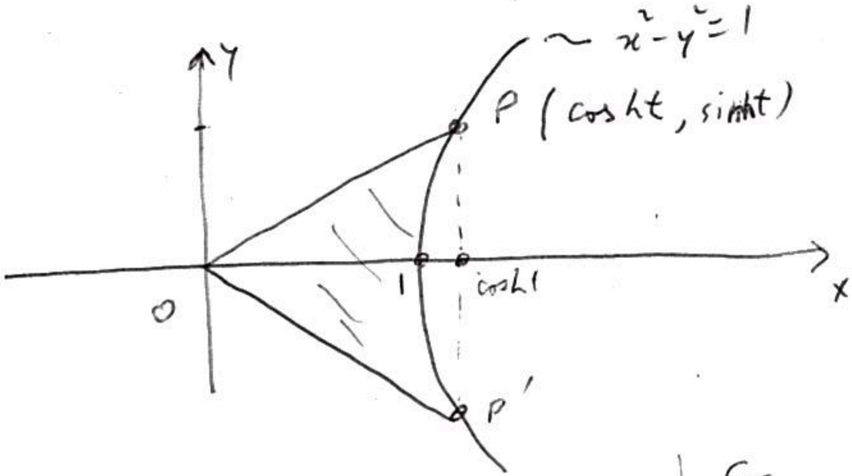
$\dfrac{du}{dx} = 2x$.

$I = \dfrac{1}{2}\displaystyle\int \frac{du}{\sqrt{u^2+1}}$

$= \dfrac{1}{2}\sinh^{-1} u + C$

$= \dfrac{1}{2}\sinh^{-1}(x^2) + C$

Ex: As mentioned above the hyperbolic functions can be used to parametrize the hyperbola. In fact, if we parametrize the point $P(x, y)$ on the hyperbola $x^2 - y^2 = 1$, by $x = \cosh t, y = \sinh t$, then the area bounded by the lines OP, OP' and the curve, where P' is the reflection of P in the x-axis, is t.



Let $A(t)$ be the shaded area

then

$$A(t) = \underbrace{\cosh t \sinh t}_{\text{area of } \Delta OPP'} - 2\int^{\cosh t} \sqrt{x^2-1}\, dx$$

Note that using the F.T.C., we can differentiate both sides

$$A'(t) = \cosh^2 t + \sinh^2 t - 2\sqrt{\cosh^2 t - 1} \cdot \sinh t$$

$$= \cosh^2 t + \sinh^2 t - 2\sinh^2 t$$

$$= \cosh^2 t - \sinh^2 t$$

$$= 1 \quad (!)$$

So

$$A(t) = t + C$$

now $A(0) = 0$ (see picture)

$\therefore C = 0$

$\therefore A(t) = t$