LECTURE 23
The Inverse Hyperbolic Trig Functions

$$\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$$

$$\cosh^{-1}(x) = \ln(x + \sqrt{x^2-1})$$

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1}(x) + C = \ln(x + \sqrt{x^2+1}) + C$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1}(x) + C = \ln(x + \sqrt{x^2-1}) + C$$

$$\int \frac{1}{1-x^2}\,dx = \tanh^{-1}(x) + C = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + C \quad \text{for} \quad |x| < 1$$

We also have the slightly more general

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}(\frac{x}{a}) + C = \ln(x + \sqrt{x^2+a^2}) + C$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}(\frac{x}{a}) + C = \ln(x + \sqrt{x^2-a^2}) + C$$

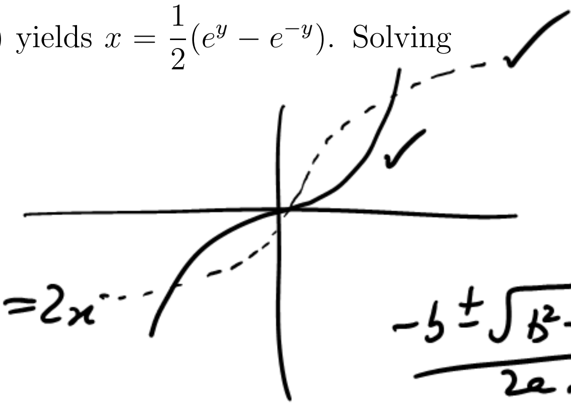
$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\tanh^{-1}(\frac{x}{a}) + C = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + C \quad \text{for} \quad |x| < a$$

All of these integrals will be made available to you in the final examination in a table of integrals.

In this final lecture we will invert the hyperbolic trigonometric functions of the previous lecture. Recall that $\sinh(x) = \dfrac{1}{2}(e^x - e^{-x})$ and that $\sinh : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing function. Hence $\sinh^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$ is well defined.

Claim: $\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$

Proof: Swapping the variables in $y = \dfrac{1}{2}(e^x - e^{-x})$ yields $x = \dfrac{1}{2}(e^y - e^{-y})$. Solving for y we obtain:



$$y = \sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$x = \frac{e^y - e^{-y}}{2} \Rightarrow e^y - e^{-y} = 2x$$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$e^{2y} - 1 = 2x e^y$$

$$e^{2y} - 2x e^y - 1 = 0 \qquad \text{Let } t = e^y$$

$$t^2 - 2xt - 1 = 0 \Rightarrow t = \frac{2x \pm \sqrt{4x^2 - 4(1)(-1)}}{2}$$

$$a = 1 \quad b = -2x \quad c = -1$$

$$\therefore \quad t = \frac{2x \pm \sqrt{4x^2+4}}{2} = \frac{2x \pm \sqrt{4}\sqrt{x^2+1}}{2}$$

$t = e^y > 0!!$

$$t = x \pm \sqrt{x^2+1} = x + \sqrt{x^2+1} \quad \sim t > 0$$

$$\therefore \quad e^y = x + \sqrt{x^2+1}$$

$$\therefore \quad y = \ln(x + \sqrt{x^2+1})$$

$$\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$$

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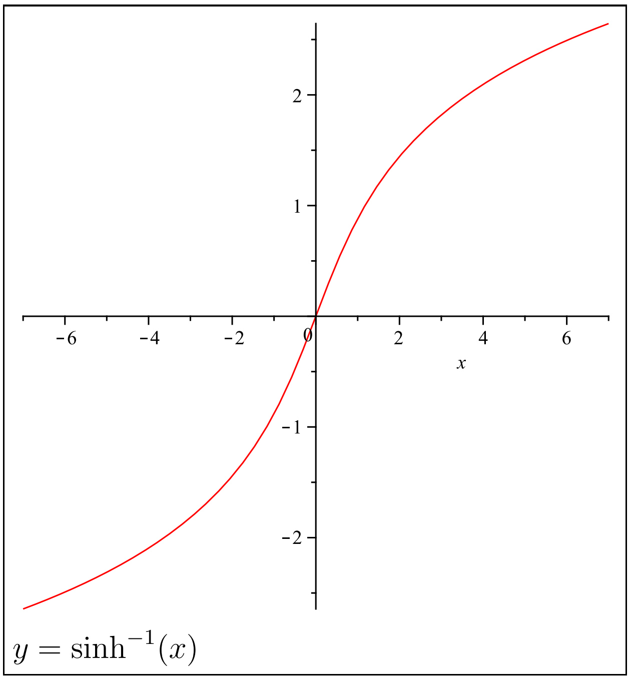
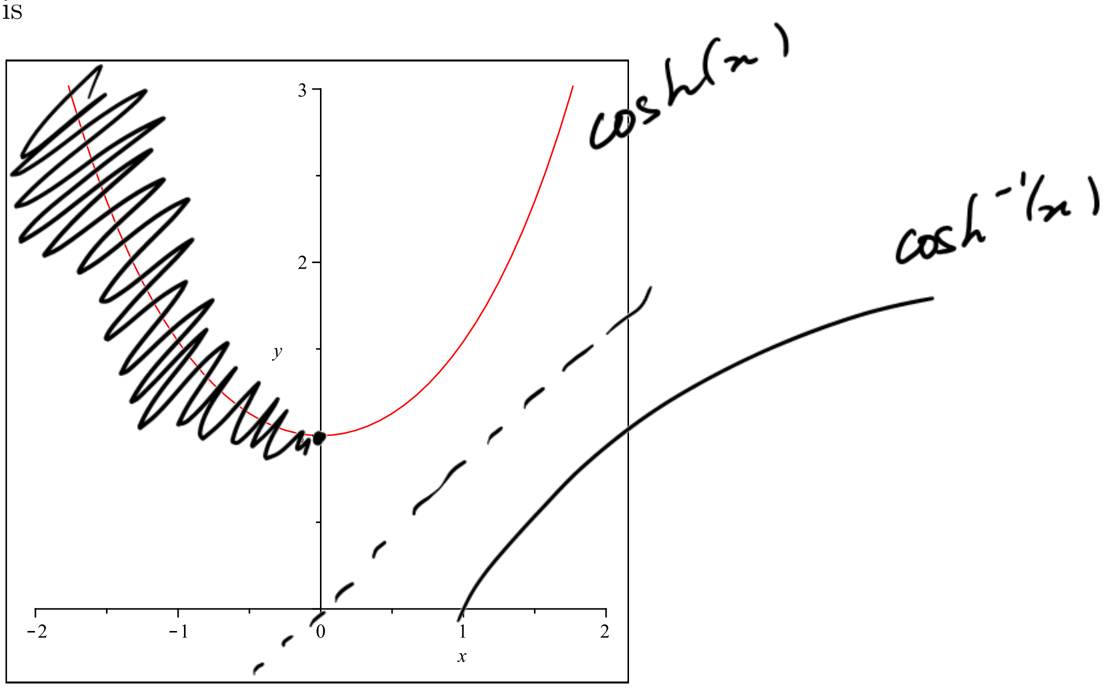
Note that the inverse trig functions have no alternative form. When talking about $y = \sin^{-1}(x)$ we have no choice but to write down $y = \sin^{-1}(x)$. But the hyperbolic trig functions are defined in term of the exponential function so it is not surprising that their inverses may be recast in term of the natural log function! Sometimes we will use $\sinh^{-1}(x)$, sometimes $\ln(x + \sqrt{x^2+1})$. It is nice to have a choice. Note also that almost all of the hyperbolic and inverse hyperbolic trig functions are built-in to the modern calculators.

The graph of $y = \sinh^{-1}(x)$ is



The cosh curve causes a little trouble as it fails the horizontal line test. Recall that the graph of $\cosh(x)$ is

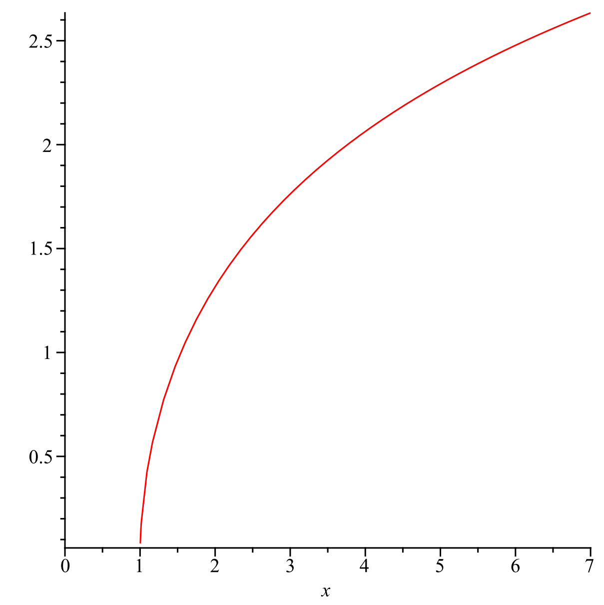


Here we first need to restrict the domain to $[0, \infty)$ so that the function becomes $1-1$ with a domain of $[0, \infty)$ and a range of $[1, \infty)$. So we delete the left hand half of the cosh graph and then invert.

A similar argument to the one above yields $\cosh^{-1}(x) : [1, \infty) \longrightarrow [0, \infty)$ given by

$$\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1}).$$

A sketch is

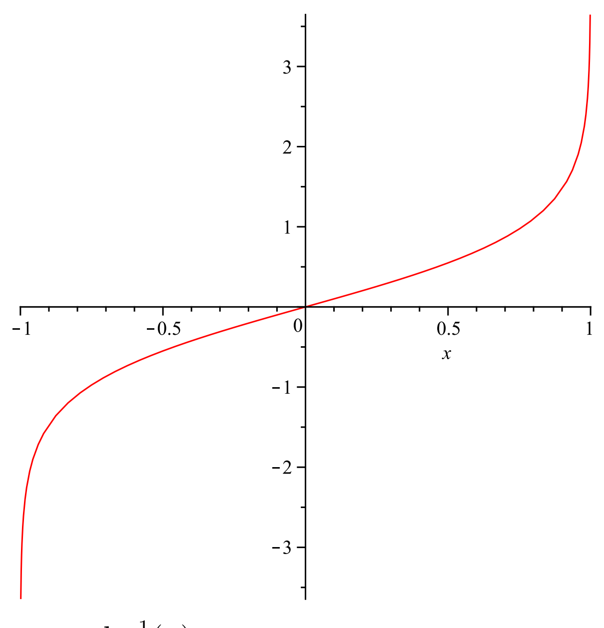


$y = \cosh^{-1}(x)$

Finally $\tanh(x) : \mathbb{R} \longrightarrow (-1, 1)$ is invertible with $\tanh^{-1}(x) : (-1, 1) \longrightarrow \mathbb{R}$ given by

$$\tanh^{-1}(x) = \frac{1}{2} \ln \left(\frac{1 + x}{1 - x} \right)$$

with a graph



$y = \tanh^{-1}(x)$

Example 1: Find $\cosh(\sinh^{-1}(\frac{1}{4}))$.

$\cosh^2 x - \sinh^2 x = 1$

Method 1 (Exact) : $\cosh^2(x) = 1 + \sinh^2(x) \longrightarrow \cosh(x) = \sqrt{1 + \sinh^2(x)}$

(we take the positive square root since $\cosh(x)$ is always positive). So

$$\cosh(\sinh^{-1}\left(\frac{1}{4}\right)) = \sqrt{1 + \sinh^2\left(\sinh^{-1}\left(\frac{1}{4}\right)\right)} = \sqrt{1 + \sinh\left(\sinh^{-1}\left(\frac{1}{4}\right)\right)\sinh\left(\sinh^{-1}\left(\frac{1}{4}\right)\right)}$$
$$= \sqrt{1 + (\frac{1}{4})^2} = \sqrt{1 + (\frac{1}{16})} = \sqrt{(\frac{17}{16})} = \frac{\sqrt{17}}{4}.$$

Method 2 (Calculator) : If an approximate answer is acceptable we can just use the calculator:

$\underbrace{\text{HYP cos}}_{\cosh}$ ($\underbrace{\text{SHIFT HYP sin}}_{\sinh^{-1}}$ $(1 \div 4)) = 1.030776406 \approx \frac{\sqrt{17}}{4}$.

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Example 2: Simplify $\cosh(2\cosh^{-1}(5))$.

We have $\cosh(2x) = \cosh^2(x) + \sinh^2(x) \longrightarrow$

$\cosh(2x) = \cosh^2(x) + (\cosh^2(x) - 1) = 2\cosh^2(x) - 1.$

So $\cosh(2\cosh^{-1}(5)) = 2\cosh^2(\cosh^{-1}(5)) - 1 = 2\cosh(\cosh^{-1}(5))\cosh(\cosh^{-1}(5)) - 1$

$= 2 \times 5 \times 5 - 1 = 49.$

Once again we can also simply crunch the numbers on a calculator:

$\underbrace{\text{HYP cos}}_{\cosh}(2\times\underbrace{\text{SHIFT HYP cos}}_{\cosh^{-1}}(5)) = 49.$

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We close the lecture and the course with the calculus of the inverse hyperbolic trig functions.

We have the following results:

a) $\dfrac{d}{dx}\sinh^{-1}(x) = \dfrac{1}{\sqrt{x^2+1}}$ ★ $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$ ★

b) $\dfrac{d}{dx}\cosh^{-1}(x) = \dfrac{1}{\sqrt{x^2-1}}$ ★ $\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$ ★

c) $\dfrac{d}{dx}\tanh^{-1}(x) = \dfrac{1}{1-x^2}$ ★ $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$ ★

Proof a:

$y = \sinh^{-1}(x) \longrightarrow x = \sinh(y) \longrightarrow$ $\dfrac{dx}{dy} = \cosh(y) = \sqrt{1+\sinh^2(y)}$

$= \sqrt{1+x^2}$

$\cosh^2 - \sinh^2 = 1$

$\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+x^2}} = \dfrac{1}{\sqrt{x^2+1}}$

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The proofs of b) and c) follow similar lines.

Example 3:

a) Find $\dfrac{d}{dx}\sinh^{-1}(x^3)$. $= \dfrac{1}{\sqrt{(x^3)^2+1}} \times (3x^2)$

$= \dfrac{3x^2}{\sqrt{x^6+1}}$

★ $\dfrac{3x^2}{\sqrt{x^6+1}}$ ★

b) Find $\dfrac{d}{dx}\{\ln(x)\cosh^{-1}(x)\}$.

$$= \frac{1}{x}\cosh^{-1}(x) + \frac{1}{\sqrt{x^2-1}}\cdot\ln(x)$$

$$\star \quad \frac{\ln(x)}{\sqrt{x^2-1}} + \frac{\cosh^{-1}(x)}{x} \quad \star$$

It follows from the above derivatives that:

a) $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1}(x)+C = \ln(x+\sqrt{x^2+1})+C$ $\quad\star\ \displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x)+C\ \star$

b) $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1}(x)+C = \ln(x+\sqrt{x^2-1})+C$ $\quad\star\ \displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = -\cos^{-1}(x)+C\ \star$

c) $\displaystyle\int \frac{1}{1-x^2}\,dx = \tanh^{-1}(x)+C = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)+C$ $\quad\star\ \displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x)+C\ \star$

We also have the slightly more general

d) $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}(\frac{x}{a})+C = \ln(x+\sqrt{x^2+a^2})+C$ $\quad\star\ \displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}(\frac{x}{a})+C\ \star$

e) $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}(\frac{x}{a})+C = \ln(x+\sqrt{x^2-a^2})+C$ $\quad\star\ \displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = -\cos^{-1}(\frac{x}{a})+C\ \star$

f) $\displaystyle\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\tanh^{-1}(\frac{x}{a})+C = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)+C \quad \text{for} \quad |x|<a$

$\star\ \displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}(\frac{x}{a})+C\ \star$

All of these will be available to you in the final examination in a table of integrals.

Example 4: Evaluate $\int \frac{dx}{\sqrt{9x^2+4}}$ typo

$$= \int \frac{dx}{\sqrt{9(x^2+\frac{4}{9})}}$$

$$= \int \frac{dx}{\sqrt{9}\sqrt{x^2+\frac{4}{9}}} = \frac{1}{3}\int \frac{dx}{\sqrt{x^2+4/9}}$$

a^2

$a = 2/3$

$$= \frac{1}{3}\sinh^{-1}\left(\frac{x}{2/3}\right)$$

$$= \frac{1}{3}\sinh^{-1}\left(\frac{3x}{2}\right) + C$$

= log form too

★ $\frac{1}{3}\sinh^{-1}(\frac{3x}{2})$ ★

Example 5: Evaluate $\int_0^{\frac{1}{2}} \frac{dx}{1-x^2}$

$$= \left[\tanh^{-1}(x)\right]_0^{\frac{1}{2}} = \tanh^{-1}\left(\tfrac{1}{2}\right) - \cancel{\tanh^{-1}(0)}$$

$$= \frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)$$

$$= \frac{1}{2}\ln\left(\frac{3/2}{\frac{1}{2}}\right)$$

$$= \underline{\frac{1}{2}\ln(3)}$$

★ $\frac{1}{2}\ln(3)$ ★

SOME FINAL INFORMATION

1. Please check online that all your marks are recorded correctly.

2. Read the school pages on additional assessment/special consideration so that you are fully aware of the rules that apply.

3. Past papers are on Moodle.

4. Make sure that you are aware of the format and date of the final exam. If in any doubt please consult the Moodle page or contact the first year office.

8. Please take the time to complete all online surveys regarding the administration and teaching of the course.

9. Check Moodle for consultation options during stuvac.

Good Luck!

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