Chapter 4: Differentiable Functions

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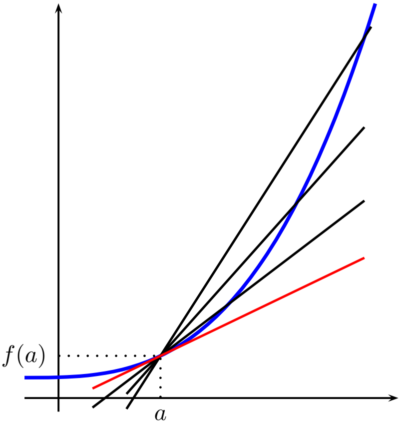
MATH1131

UNSW

Term 1 2020

Gradients of tangents



The ratio

$$\frac{f(a+h)-f(a)}{h}$$

is the slope of the secant passing through the points $(a, f(a))$ and $(a+h, f(a+h))$.

The limit (if it exists)

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

is the slope (or gradient) of the tangent line passing through the point $(a, f(a))$.

Example

Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$. Find the gradient of the tangent to the graph of f when $x = 2$.

Solution. Suppose that $h \neq 0$. Then

$$f(2+h) = (2+h)^3 = 8 + 12h + 6h^2 + h^3$$

and so

$$\begin{aligned} \frac{f(2+h) - f(2)}{h} &= \frac{8 + 12h + 6h^2 + h^3 - 8}{h} \\ &= 12 + 6h + h^2 \\ &\xrightarrow[h \to 0]{} 12. \end{aligned}$$

We thus have

$$\text{Gradient of tangent at } 2 = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = 12.$$

Now, there is nothing special about $x = 2$, the same works for any point x. Thus, if we replace 2 by x above we obtain

$$\begin{aligned}\text{Gradient of tangent at } x &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2.\end{aligned}$$

Thus, the gradient of the tangent at every point $x \in \mathbb{R}$ is $3x^2$.

This looks already familiar:

From high-school you know that the derivative of $f(x) = x^3$ is

$$f'(x) = 3x^2.$$

Differentiable functions

Definition

Suppose that f is defined on some open interval containing the point x. If

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists then it is called the derivative (or slope) of f at x and we say that f is differentiable at x.

The derivative of f at x is denoted by

$$f'(x) \quad \text{or} \quad \frac{df}{dx}(x) \quad \text{or} \quad \frac{d}{dx}f(x).$$

Remark. The ratio

$$\frac{f(x+h) - f(x)}{h}$$

is called the difference quotient for f at the point x.

Easy example

Find the derivative of $f : [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$ using the definition of derivative.

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Solution. Suppose that $h \neq 0$. Then, for any $x > 0$, we have

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \frac{1}{\sqrt{x+h} + \sqrt{x}} \to \frac{1}{2\sqrt{x}} \quad \text{as } h \to 0.\end{aligned}$$

Thus, $f'(x) = \lim_{x \to 0} \dfrac{f(x+h) - f(x)}{h} = \dfrac{1}{2\sqrt{x}}$.

Exercise.

Is the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ differentiable at the point 0?

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Solution. We need to investigate

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h}.$$

We have

$$\frac{|h|}{h} = \begin{cases} \frac{h}{h}, & h > 0 \\ \frac{-h}{h}, & h < 0 \end{cases} = \begin{cases} 1, & h > 0 \\ -1, & h < 0. \end{cases}$$

Thus,

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = 1, \quad \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = -1.$$

Since the left- and right-hand limits are different, the limit does not exist and thus f is not differentiable at 0.

Harder example

Let $f(x) = \sin(x^3)$. Find (using the definition of derivative) $f'(2)$.

Suppose that $h \neq 0$. Then

$$f(2+h) = \sin\big((2+h)^3\big) = \sin(8 + 12h + 6h^2 + h^3)$$

and so

$$\begin{aligned}\frac{f(2+h) - f(2)}{h} &= \frac{\sin(8 + 12h + 6h^2 + h^3) - \sin(8)}{h} \\ &\to ???\end{aligned}$$

That limit looks hard!

What calculus provides us with is a set of simple rules for

- Recognising which functions are differentiable.
- Finding the derivatives 'symbolically'.

As before you

- Use the definition to prove that a few very simple functions (e.g. constants, $f(x) = x$ or $f(x) = \sin x$) are differentiable.
- Prove differentiation rules to deal with most more complicated functions.
- Only resort to the definition when the differentiation rules don't apply.

Example 1. If $f(x) = c$ (constant), then, for all $x \in \mathbb{R}$, $f'(x) = 0$.

Proof.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0.$$

Example 2. If $f(x) = x$, then, for all $x \in \mathbb{R}$, $f'(x) = 1$.

Proof.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h) - x}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$$

Exercise 3. Use the binomial theorem to show that if $f(x) = x^n$ with $n \in \mathbb{Z}^+$, then $f'(x) = nx^{n-1}$.

Hard example. Let $f(x) = \sin x$. By using $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$ (see Calculus Notes, Chap 2, problem 14), show that $f'(x) = \cos x$.

Solution. To do this you need the identity

$$\sin(A+B) = \sin A \cos B + \cos A \sin B.$$

Using this we see that if $h \neq 0$ then the difference quotient is:

$$\begin{aligned}\frac{\sin(x+h) - \sin x}{h} &= \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h}\end{aligned}$$

Now you know that $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$ and need to show: $\lim_{h\to 0} \frac{\cos(h) - 1}{h} = 0$.
Then using these we see that

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} = \cos x$$

.

To prove that $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$:

$$\begin{aligned}
\frac{\cos h - 1}{h} &= \frac{\cos h - 1}{h} . \frac{\cos h + 1}{\cos h + 1} \\
&= \frac{\cos^2 h - 1}{h(\cos h + 1)} \\
&= \frac{-\sin^2 h}{h(\cos h + 1)} \qquad \text{since } \cos^2 h + \sin^2 h = 1 \\
&= \frac{-\sin h}{h} \cdot \frac{\sin h}{\cos h + 1} \to -1 \times 0 = 0 \quad \text{as } h \to 0.
\end{aligned}$$

Finally we showed that:

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \cos x$$

With these simple functions being differentiable (e.g. constants, $f(x) = x$ or $f(x) = \sin x$), we could construct and prove the differentiability of many more complicated functions...

Theorem

If f is differentiable at a then f is continuous at a.

Proof. Assume that f is differentiable at a, and we need to show that f is continuous at a, that is, $\lim_{x\to a} f(x) = f(a)$, equivalent to $\lim_{h\to 0} f(a+h) = f(a)$ (using $x = a + h$).

We have

$$\begin{aligned} \lim_{h\to 0} f(a+h) - f(a) &= \lim_{h\to 0} \left(\frac{f(a+h)-f(a)}{h} \times h \right) \\ &= \left(\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \right) \times \lim_{h\to 0} h \\ &= f'(a) \times 0 = 0 \qquad \text{since } f \text{ is differentiable at } a. \end{aligned}$$

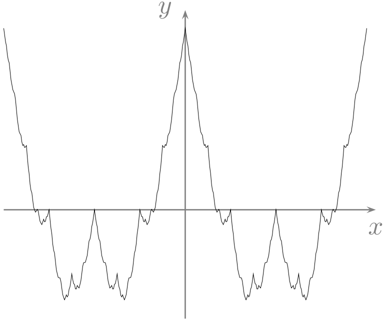
Accordingly, $\lim_{h\to 0} f(a+h) = f(a)$, hence f is continuous at a.

Corollary.

If f is not continuous at a then it is not differentiable at a.

Remark

Differentiability is a much stronger property than continuity. While every differentiable function is continuous, there exist functions that are continuous everywhere but differentiable nowhere: e.g. he *Weierstrass function*.



In fact, functions that are differentiable everywhere are a very rare breed, even among the continuous functions.

Rules for differentiation

Many differentiable functions may be constructed via addition, subtraction, multiplication and division ...

Theorem

Suppose that f and g are differentiable functions at x. Then,

- $(f+g)'(x) = f'(x) + g'(x)$
- $(cf)'(x) = cf'(x)$, where c is a constant
- $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ (product rule)
- $(f/g)'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$ (quotient rule)

  provided that $g(x) \neq 0$.

... and function composition

The chain rule

Suppose that g is differentiable at the point x and f is differentiable at the point $g(x)$. Then $f \circ g$ is differentiable at x and

$$(f \circ g)'(x) = (f(g(x)))' = f'(g(x))g'(x) \quad \text{(chain rule)}$$

Or if $y = f(g)$ and $u = g(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$.

Exercise. Use the product rule and induction on n to prove that $\dfrac{d}{dx}x^n = nx^{n-1}$ for $n \in \mathbb{N}$.

Example. If g is differentiable at the point x, then

$$(g(x)^n)' = ng(x)^{n-1}g'(x) \quad \forall n \in \mathbb{N}.$$

Indeed, it follows directly from the chain rule since $g(x)^n = (f \circ g)(x)$ where $f(x) = x^n$.

Proofs of differentiation rules

Proof of product rule. Suppose that f and g are differentiable at the point x. The difference quotient of fg at x gives

$$\begin{aligned}
&\frac{(fg)(x+h)-(fg)(x)}{h} \\
&\qquad = \frac{f(x+h)g(x+h)-f(x)g(x)}{h}\left(+\frac{g(x+h)f(x)}{h}-\frac{g(x+h)f(x)}{h}\right) \\
&\qquad = g(x+h)\frac{f(x+h)-f(x)}{h}+f(x)\frac{g(x+h)-g(x)}{h}.
\end{aligned}$$

Since the function g is differentiable at the point x, it is continuous at that point (we have just proved that)! Hence,

$$g(x+h) \xrightarrow[h\to 0]{} g(x).$$

Accordingly,

$$\frac{(fg)(x+h)-(fg)(x)}{h} \quad \xrightarrow[h\to 0]{} \quad g(x)f'(x)+f(x)g'(x).$$

Proofs of the other differentiation rules are found in most undergraduate calculus textbooks, but you should try to prove them yourself!

Table of derivatives of some standard functions

When not specified, you can find the derivatives *symbolically* without going back to the definition of derivative...

$f(x)$	$f'(x)$
C (constant)	0
x^n $(n \in \mathbb{Z})$	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
e^x	e^x
$\ln x$	$1/x$

Exercise. Find the derivative of $f(x) = \left[\sin\left(\dfrac{x}{x^2+1}\right)\right]^2$.

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Exercise. Find the derivative of $f(x) = \left[\sin\left(\frac{x}{x^2+1}\right)\right]^2$.

Solution. We have

$$\begin{aligned}
f'(x) &= \left(\left[\sin\left(\frac{x}{x^2+1}\right)\right]^2\right)' \\
&= 2\left[\sin\left(\frac{x}{x^2+1}\right)\right]\left(\sin\left(\frac{x}{x^2+1}\right)\right)' && \text{chain rule} \\
&= 2\sin\left(\frac{x}{x^2+1}\right)\cos\left(\frac{x}{x^2+1}\right)\left(\frac{x}{x^2+1}\right)' && \text{chain rule} \\
&= \sin\left(\frac{2x}{x^2+1}\right)\frac{1\cdot(x^2+1) - x\cdot 2x}{(x^2+1)^2} && \text{quotient rule} \\
&= \sin\left(\frac{2x}{x^2+1}\right)\frac{1-x^2}{(x^2+1)^2}
\end{aligned}$$

Implicit differentiation

Idea. On use of the chain rule, determine the derivative of a function which is implicitly defined.

Example. Determine the tangent line at the point $(2, 1)$ to the curve defined by

$$x^4 - x^2y^2 + y^4 = 13$$

Solution. This equation defines y implicitly as a function of x. Write

$$x^4 - x^2y(x)^2 + y(x)^4 = 13$$

and differentiate both sides with respect to x:

$$4x^3 - 2xy(x)^2 - 2x^2y(x)y'(x) + 4y(x)^3y'(x) = 0.$$

Evaluate at $(x, y) = (2, 1)$ and solve for $y'(2)$ (the slope of the tangent at (2,1)):

$$32 - 4 - 8y'(2) + 4y'(2) = 0 \quad \Rightarrow \quad y'(2) = 7.$$

Equation of tangent line at $(2, 1)$:

$$y - 1 = 7(x - 2).$$

Exercise

Suppose that y is a function of x, and that x and y are related by the formula $y^3 + 3xy^2 + 3x^2y + y + 5 = 0$. Find $\dfrac{dy}{dx}$ (in terms of x and y).

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Differentiating both sides of

$$0 = y^3 + 3xy^2 + 3x^2y + y + 5$$

with respect to x gives

$$\begin{aligned}
\frac{d}{dx}0 &= \frac{d}{dx}\left(y^3 + 3xy^2 + 3x^2y + y + 5\right) \\
0 &= \frac{d}{dx}y^3 + \left(3y^2 + 3x\frac{d}{dx}y^2\right) + \left(6xy + 3x^2\frac{dy}{dx}\right) + \frac{dy}{dx} + 0 \\
&= 3y^2\frac{dy}{dx} + 3y^2 + 6xy\frac{dy}{dx} + 6xy + 3x^2\frac{dy}{dx} + \frac{dy}{dx} \\
&= \left(3y^2 + 6xy + 3x^2 + 1\right)\frac{dy}{dx} + 3y^2 + 6xy.
\end{aligned}$$

and so rearranging,

$$\frac{dy}{dx} = -\frac{3y^2 + 6xy}{3y^2 + 6xy + 3x^2 + 1}.$$

Differentiability of split functions

Example. Suppose that $f : (0, \infty) \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 4\sqrt{x} & \text{if} \quad 0 < x \leq 1 \\ bx^2 + c & \text{if} \quad x > 1, \end{cases}$$

Find all values of b and c such that f is (i) continuous at $x = 1$; (ii) differentiable at $x = 1$.

Solution. (i) Since continuity is a prerequisite for differentiability we investigate continuity first by studying what happens to $f(x)$ as $x \to 1$. We have

$$\lim_{x \to 1^-} f(x) = 4 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = b + c \quad \text{and} \quad f(1) = 4 \ ,$$

so f is continuous whenever $b + c = 4$, and there are many b and c which satisfy this equation.

Cont.

(ii) For f to be differentiable at 1 it must first be continuous at 1 and so we need $b + c = 4$. Moreover, f is differentiable at $x = 1$ if the limit

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

exists.

For 'split functions' like this, you need to be rather careful about what happens **at the join** (in our case $x = 1$). Whether or not the function is differentiable at the split point can be determined by calculating left- and right-hand limits:

$$\begin{aligned}
\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} &= \lim_{h \to 0^-} \frac{4\sqrt{1+h} - 4}{h} \\
&= \lim_{h \to 0^-} 4\frac{\sqrt{1+h} - 1}{h} \cdot \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} \\
&= \lim_{h \to 0^-} \frac{4}{\sqrt{1+h} + 1} \\
&= 2
\end{aligned}$$

Cont.

... and

$$\begin{aligned}
\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0^+} \frac{b(1+h)^2+c-4}{h} \\
&= \lim_{h\to 0^+} \frac{b+2bh+bh^2+c-4}{h} \\
&= \lim_{h\to 0^+} 2b+bh \quad \text{since } b+c=4 \\
&= 2b.
\end{aligned}$$

Thus, f is differentiable at 1 if $2b=2$, that is $b=1$.

Since $b+c=4$ for continuity, this implies $c=3$.

Final answer: Thus, the function f is continuous and differentiable at $x=1$ if $b=1$ and $c=3$.

However, in many cases, this discussion may be avoided by using the following theorem:

Theorem

Suppose that a is a fixed real number and that a function f is defined by the rule

$$f(x) = \begin{cases} p(x) & \text{if} \quad x \geq a \\ q(x) & \text{if} \quad x < a, \end{cases}$$

where p and q are continuous and differentiable in some interval containing a. If f is continuous at a and $p'(a) = q'(a)$ then f is differentiable at $x = a$.

Remark. Note that the requirement of f being continuous at a means that

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a),$$

which is equivalent to demanding that $p(a) = q(a)$ since p and q are continuous at a.

Example

Let us consider the previous function $f : (0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 4\sqrt{x} & \text{if} \quad 0 < x \leq 1 \\ bx^2 + c & \text{if} \quad x > 1, \end{cases}$$

where b and c are real numbers, and we want to see when this function is differentiable at $x = 1$.

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where b and c are real numbers, and we want to see when this function is differentiable at $x = 1$.

Solution. In our case, both functions $p(x) = 4\sqrt{x}$ and $q(x) = bx^2 + c$ are continuous and differentiable everywhere.
Now, f is continuous at $x = 1$ if $p(1) = q(1)$, that is, if $b + c = 4$.
We have $p'(x) = \frac{2}{\sqrt{x}}$ and $q'(x) = 2bx$. Hence $p'(1) = 2$ and $q'(1) = 2b$.
From the previous theorem, whenever $b + c = 4$, f is differentiable at $x = 1$ if $p'(1) = q'(1)$, that is, when $b = 1 \implies c = 3$.

Derivatives and function approximation

By definition, if a function f is differentiable at a, we have

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

so that

$$f'(a) \approx \frac{f(x) - f(a)}{x - a}$$

if x is 'sufficiently close' to a. Thus,

$$f(x) \approx f(a) + f'(a)(x - a)$$

and the right-hand side may be regarded as an 'approximation' of $f(x)$ in a neighbourhood of $x = a$.

Examples

It is easy to see that $\frac{1}{100} = 0.01$. But what is $\frac{1}{103}$?

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Solution. Let $f(x) = \frac{1}{x}$. Since 103 is close to 100, we use the approximation formula

$$f(x) \approx f(a) + f'(a)(x-a), \quad \text{with} \quad a = 100,\ x = 103.$$

We obtain $\dfrac{1}{103} \approx \dfrac{1}{100} - \dfrac{1}{100^2}(103-100) = 0.01 - 0.0003 = 0.0097.$

Derivatives and rates of change

Many physical processes involve quantities (such as temperature, volume, concentration, velocity) that change with time but may not be independent of each other. Their rates of change may then be obtained by careful application of the chain rule or implicit differentiation.

Example. A spherical balloon is being inflated and its radius is increasing at a constant rate of 6 mm/sec. At what rate is its volume increasing when the radius of the balloon is 20 mm?

Let $V(t)$ be the volume of the balloon and $r(t)$ be its radius at time t. We are told that $\frac{dr}{dt} = 6$, and we need to find $\frac{dV}{dt}$ when $r = 20$.

Alternatively, let $V(r)$ be the volume of the balloon as a function of its radius r given by

$$V(r) = \frac{4}{3}\pi r^3$$

so that

$$\frac{dV}{dr} = 4\pi r^2.$$

Cont.

Then, the chain rule implies that

$$\frac{dV}{dt} = \frac{dV}{dr}\,\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Since $\frac{dr}{dt} = 6$, we have at $r = 20$:

$$\frac{dV}{dt} = 4\pi(20)^2 \times 6 = 9600\pi$$

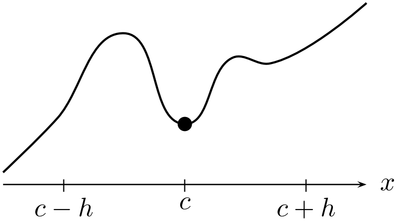
Hence, the volume is increasing at a rate of 9600π mm^3/sec when the radius is 20 mm.

The above example illustrates an approach to solving such problems.

1. Define variables for the quantities involved.
2. Write down what is known in terms of these variables and their derivatives.
3. Write down what you need to find in terms of these variables and their derivatives.
4. Write down anything else you know that relates the variables (for example, a volume or area formula).
5. Use the chain rule (or implicit differentiation) to find the relevant derivative.

Local maximum, local minimum and stationary points

In this section, we begin to develop a systematic approach to locating maxima and minima. A complete approach will be presented in the next chapter.



Local minimum point c

That is, for all x sufficiently close to c, $f(c) \leq f(x)$.

Definition

Let f be defined on some interval I.

- We say that a point c in I is a local minimum point if there exists an $h > 0$ such that
$$f(c) \leq f(x) \quad \text{for all} \quad x \in (c-h, c+h)$$
- We say that a point d in I is a local maximum point if there exists an $h > 0$ such that
$$f(x) \leq f(d) \quad \text{for all} \quad x \in (d-h, d+h)$$

With $(c-h, c+h)$ and $(d-h, d+h)$ two subsets of I.

Theorem

Suppose that f is defined on (a, b) and has a local maximum or minimum point at c for some c in (a, b). If f is differentiable at c then $f'(c) = 0$.

Proof. Suppose that f has a local maximum $c \in (a, b)$ and f is differentiable at c. To show that $f'(c) = 0$ we look at the quotient

$$\frac{f(c+h) - f(c)}{h}.$$

Since c is a local maximum, we have $f(c+h) \leq f(c)$ for h sufficiently close to 0.

Now, if $h > 0$ is sufficiently small, we have

$$\frac{f(c+h) - f(c)}{h} \leq 0 \Longrightarrow \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0.$$

if $h < 0$ is sufficiently small, we have

$$\frac{f(c+h) - f(c)}{h} \geq 0 \Longrightarrow \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0.$$

Since f is differentiable at c, these both limits must exist and be equal to $f'(c)$. Thus, we obtain $0 \leq f'(c) \leq 0$, which implies $f'(c) = 0$.

A similar argument works if c is a local minimum.

The converse of Theorem is false! That is, you can easily have $f'(c) = 0$ with f having neither a local max nor a local min at c. For instance, take $f(x) = x^3$.

Of course it is easy to give examples of functions where the local extreme points occur at points of nondifferentiability ($f(x) = |x|$).

Nonetheless, our method for hunting down local extreme points will involve reducing the possible points to consider from the infinite set (a, b) to a finite set.

Definition

If a function f is differentiable at a point c and $f'(c) = 0$ then c is called a stationary point of f.

Example

Find all the stationary, maximum and minimum points of the function $f : [-3, 4] \to \mathbb{R}$ defined by

$$f(x) = 4x^5 - 5x^4 - 40x^3 - 2$$

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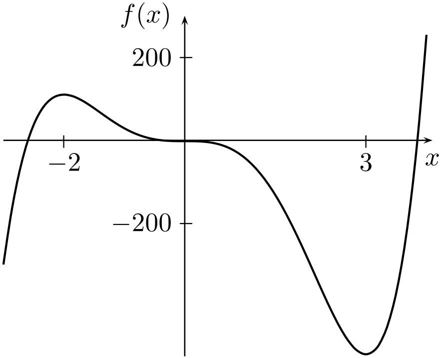

Differentiation yields

$$\begin{aligned} f'(x) &= 20x^4 - 20x^3 - 120x^2 \\ &= 20x^2(x^2 - x - 6) \\ &= 20x^2(x+2)(x-3). \end{aligned}$$

The stationary points are $-2, 0, 3$. From the graph of f:

- $x = 3$: local minimum point
- $x = -2$: local maximum point
- $x = 0$: stationary point, but neither a max. or a min. point (point of inflection)

Thus, although local minima and maxima of a function occur at stationary points, it is not true that every stationary point is a local max. or min. point.

Next chapter will show how to identify the different types of stationary points... application of the Mean Value Theorem!

Summary: What did we learn in this chapter?

- Definition of derivative via tangent (p. 5)
- Continuity and differentiability (p. 13)
- Rules for differentiation (+, -, *, /, chain rule) (p.15 and 16)
- Revision of symbolic derivatives (p. 18)
- Implicit differentiation (p. 19)
- Differentiability of split functions (p. 21 and p.24)
- Function approximation (p. 26)
- Rate of change (p. 28)
- Stationary point and local max / min (p. 33 and p. 34)