School of Mathematics and Statistics
Math1131-Algebra

Lec14: Introduction to systems of linear equations

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The WHY slide

At School you learnt how to solve two-by-two systems of linear equations such as

$$\begin{aligned} 2x &+ 3y &= 1 \\ 4x &- 2y &= 2 \end{aligned}$$

How? Using

- **substitution**: Use one equation to write on variable in terms of the other one and then substitute.
- **elimination**: multiplying the equations by various numbers and adding or subtracting so as to eliminate one of the unknowns.

What does one do when faced with a system of 30 equations in 30 unknowns (such things turn up routinely in the applications of mathematics to the real world) ? *Ad hoc* manipulation of the equations is clearly out of the question.

The WHY slide

Even that would be hard:

$$\begin{array}{rcrcrcrcr} 2x_1 & + & 3x_2 & + & 2x_3 & + & 7x_4 & = & 1 \\ 4x_1 & - & 2x_2 & - & x_3 & + & 8x_4 & = & 2 \\ x_1 & + & 9x_2 & & & - & 2x_4 & = & -5 \end{array}$$

We need a SYSTEMATIC method for solving systems of linear equations in many variables.

This is provided by a method known as *Gaussian Elimination*.
It is one of the absolute fundamentals of the algebra course and will be used constantly in MATH1231 as well as in this course. You will need to learn it carefully and get lots of practice at it. Before we look at this method, however, we wish to look carefully at the geometric interpretation of solving such systems.

Linear equations

Example 4. *Linear equations* are equations like the following.

- $3x = 7$
- $7t = 3$
- $2a + 3b = 0$
- $-3x + y = 7$
- $2x + 3y + 5z = -1$
- $3x_1 - x_2 + 7x_3 - x_4 = 10$

A *linear equation* is an equation of the form :
"Sum of scalar multiples of some variables equals a constant",
i.e. ' "a linear combination of some variables equals a constant."

You should recognise equations of lines and planes amongst the equations above.

As we said, linear equations can be solved systematically by one of the most celebrated and important algorithms of all time: *Gaussian Elimination*.

Linear equations

Solving a single linear equation in one variable, such as,

$$3x = 7$$

is easy. The solution is

$$x = \frac{7}{3}.$$

But what about a linear equation in two variables, for example,

$$-3x + y = 7?$$

What does "solving" mean in this case?

Solving means to describe all possible solutions, or show that no solutions are possible.

$y = 3x + 7$ Let $\lambda = x$

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \lambda \in \mathbb{R}}$$

TIP! Yes, expressing the unknowns in terms or one or more parameters ***IS*** solving, since you are describing all the possible solutions.

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Linear equations in $\mathbb{R}^3$

Solve

$$2x + 3y + 5z = -2.$$

That is, describe all possible solutions ... and vector parametric form is the way to do that.

Let's choose $y = \lambda$ and $z = \mu$ and so

$$x = -1 - \frac{3}{2}\lambda - \frac{5}{2}\mu.$$

Hence

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -\frac{5}{2} \\ 0 \\ 1 \end{pmatrix}.$$

That is, we consider a vector parametric form of a plane to be the solution to the Cartesian equation of a plane.

REMEMBER

Expressing the unknowns in terms or one or more parameters ***IS*** solving, since you are describing all the possible solutions.
This is what we do when there is an infinite number of solutions.

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Linear equations in $\mathbb{R}^2$

In high school you also learnt to solve systems of linear equations in two variables (You probably called these simultaneous equations).
For example,

$$\begin{cases} x + 3y = 10 \\ 3x - 5y = 2 \end{cases}$$

You could solve this in one of two ways.

- **Substitution** : Use one equation to write x in terms of y and then substitute.
- **Elimination** : Subtract a multiple of one equation from the other to eliminate one variable.

We are going to concentrate on the method of elimination because it can be adapted into a powerful method called Gaussian elimination that works for with any number of linear equations in any number of variables.

Linear equations in $\mathbb{R}^2$

Before solving, let's think geometrically.

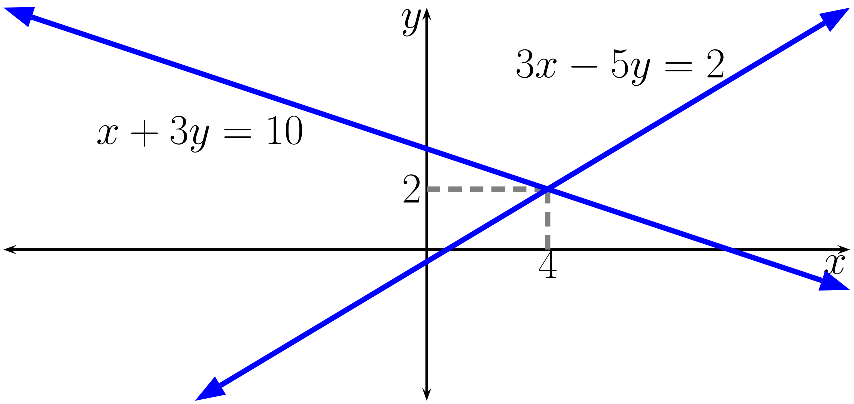
$$\begin{cases} x + 3y = 10 & (\ell_1) \\ 3x - 5y = 2 & (\ell_2) \end{cases}$$

$\vec{n}_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ $\vec{n}_2 \begin{pmatrix} 3 \\ -5 \end{pmatrix}$

Linear equations in $\mathbb{R}^2$

Before solving, let's think geometrically.

$$\begin{cases} x + 3y = 10 \\ 3x - 5y = 2 \end{cases}$$



Linear equations in $\mathbb{R}^2$

Solve the two linear equations in two variables algebraically.

$$
\begin{aligned}
x + 3y &= 10 \quad \times 3 & (1)\\
3x - 5y &= 2 & (2)
\end{aligned}
$$

Let's subtract 3 lots of equation (1) away from equation (2) giving

(2)−3(1)

$$
\begin{aligned}
x + 3y &= 10\\
-14y &= -28
\end{aligned}
$$

Linear equations in $\mathbb{R}^2$

Solve the two linear equations in two variables algebraically.

$$\begin{aligned} x + 3y &= 10 \qquad (1) \\ 3x - 5y &= 2 \qquad (2) \end{aligned}$$

Let's subtract 3 lots of equation (1) away from equation (2) giving

$$\begin{aligned} x + 3y &= 10 \\ -14y &= -28 \end{aligned}$$

Notice that the new second equation is easy to solve to find

$$y = 2$$

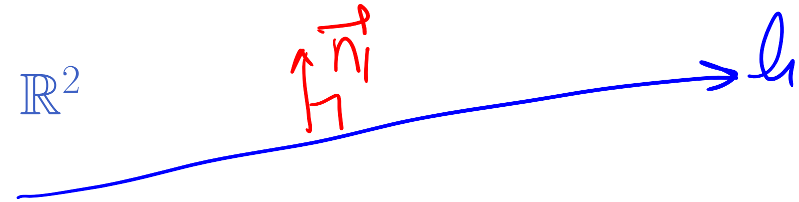
and then use this with the first equation to find

$$x = 10 - 3 \times 2 = 4.$$

So the solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

Linear equations in $\mathbb{R}^2$



Think geometrically about these equations:

$$\begin{aligned} 3x - 5y &= 10 \\ 3x - 5y &= 2. \end{aligned}$$

$\vec{n}_1 \begin{pmatrix} 3 \\ -5 \end{pmatrix}$

$\vec{n}_2 \begin{pmatrix} 3 \\ -5 \end{pmatrix}$

Linear equations in $\mathbb{R}^2$

Think geometrically about these equations:

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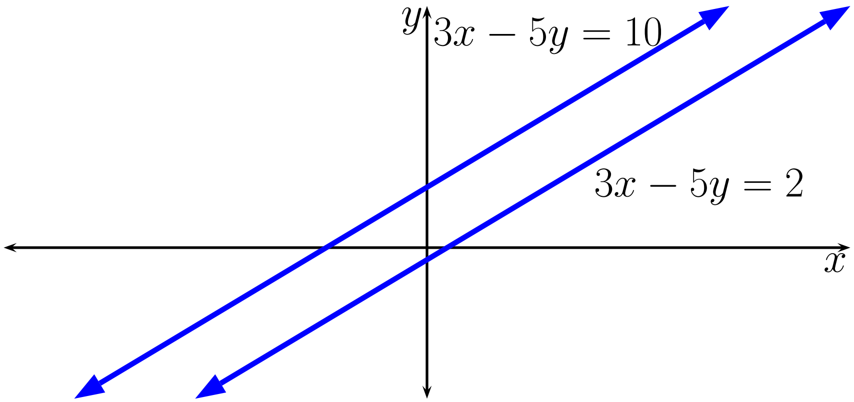

Linear equations in $\mathbb{R}^2$

Now try to solve algebraically.

$$\begin{aligned} 3x - 5y &= 10 && (1) \\ 3x - 5y &= 2. && (2) \end{aligned}$$

Subtracting equation (1) away from equation (2) gives

$$\begin{aligned} 3x - 5y &= 10 \\ 0x + 0y &= -8. \end{aligned}$$

There are no values of x and y that satisfy the second equation, so there are no solutions.

We say that the system of equations is *inconsistent*.

As we saw, equations (1) and (2) represent a pair of parallel lines.

Linear equations in $\mathbb{R}^2$

What about the following equations?

$$3x - 5y = 2 \quad (1)$$
$$6x - 10y = 4 \quad (2)$$

$\times 2$

The second equation is simply twice the first equation and so they represent the same line. The solutions are

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

Linear equations in $\mathbb{R}^2$

What about the following equations?

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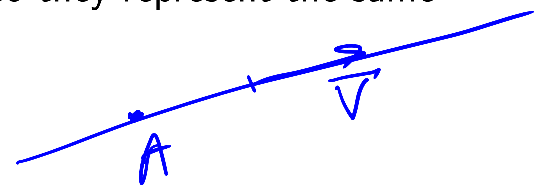
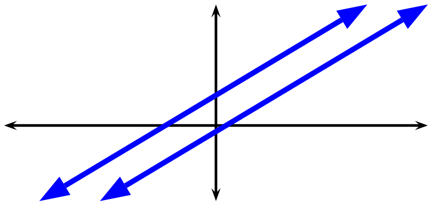


[Check this yourself!]

SUMMARY : Linear equations in $\mathbb{R}^2$

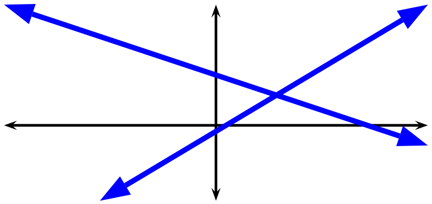
We found no solutions for

$$\begin{aligned} 3x - 5y &= 10 \\ 3x - 5y &= 2, \end{aligned}$$



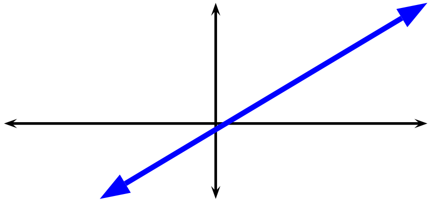
a unique solution for

$$\begin{aligned} x + 3y &= 10 \\ 3x - 5y &= 2 \end{aligned}$$



and infinitely many solutions for

$$\begin{aligned} 3x - 5y &= 2 \\ 6x - 10y &= 4. \end{aligned}$$



Linear equations in $\mathbb{R}^3$

What sort of solution do you expect from the intersection of the following two planes?

$$\begin{array}{lrcrcrcl} \Pi_1 & x & + & y & + & z & = & 5 \\ \Pi_2 & 3x & + & 4y & + & 7z & = & 20 \end{array}$$

$\underline{n}_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\underline{n}_2 \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}$

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Linear equations in $\mathbb{R}^3$

What sort of solution do you expect from the intersection of the following two planes?

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 5 \\ 3x & + & 4y & + & 7z & = & 20 \end{array} \quad \begin{array}{l} \times -3 \\ \times 1 \quad (2) \end{array}$$

These planes are not parallel, so we expect they intersect in a line.

To solve, subtract 3 times the first equation from the second equation to give

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 5 \\ & & y & + & 4z & = & 5 \end{array} \quad \begin{array}{l} (1) \\ (3) \end{array}$$

This new pair of equations has the same set of solutions as the original pair but they are easier to solve.

$y = 5 - 4z$

sub in (1) $\quad x = 5 - y - z = 5 - (5 - 4z) - z = 3z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} + z \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} \quad \text{for all } z \in \mathbb{R}$$

Linear equations in $\mathbb{R}^3$

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 5 \\ & & y & + & 4z & = & 5 \end{array}$$

To solve, start with the simpler (second) equation.
Let $z = \lambda$ and then from the second equation,

$$y = 5 - 4z = 5 - 4\lambda$$

and so the first equation becomes

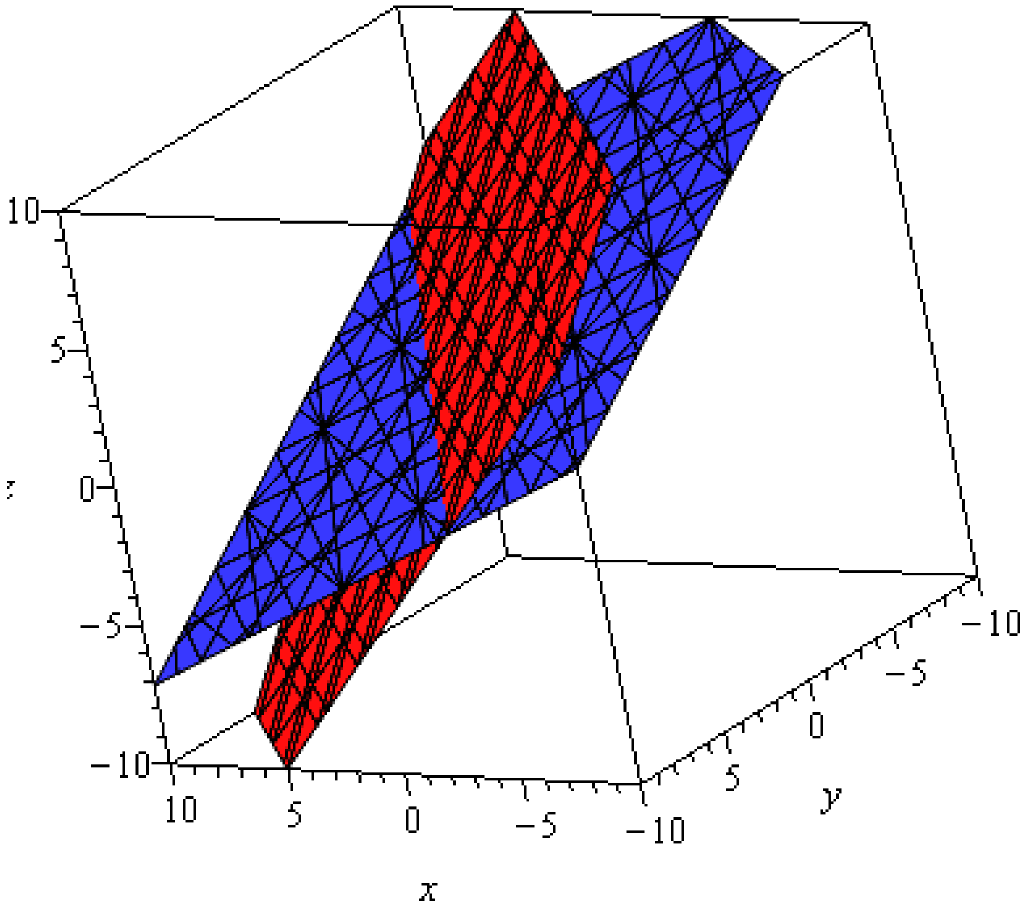
$$x + (5 - 4\lambda) + \lambda = 5 \implies x = 3\lambda.$$

That is,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}.$$

Working from the simplest and successively substituting is called *back substitution*.

Linear equations in $\mathbb{R}^3$



$$x + y + z = 5$$
$$3x + 4y + 7z = 20$$

Linear equations in $\mathbb{R}^3$

What sort of solution do you expect from the intersection of the following two planes?

$$\begin{array}{rcrcrcl} 2x & - & y & + & 4z & = & 5 \\ 4x & - & 2y & + & 8z & = & 12 \end{array}$$

$$\underset{\sim}{n}_1 = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$$

$$\underset{\sim}{n}_2 = \begin{pmatrix} 4 \\ -2 \\ 8 \end{pmatrix} = 2\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$$

Subtracting 2 times the first from the second gives

$$0 = 2.$$

There is no choice for (x, y, z) which makes this true.

Linear equations in $\mathbb{R}^3$

What sort of solution do you expect from the intersection of the following two planes?

$$\begin{array}{rcrcrcl} 2x & - & y & + & 4z & = & 5 \\ 4x & - & 2y & + & 8z & = & 12 \end{array}$$

Subtracting 2 times the first from the second gives

$$0 = 2.$$

There is no choice for (x, y, z) which makes this true.

The equations are inconsistent.

They represent parallel planes with no common points.

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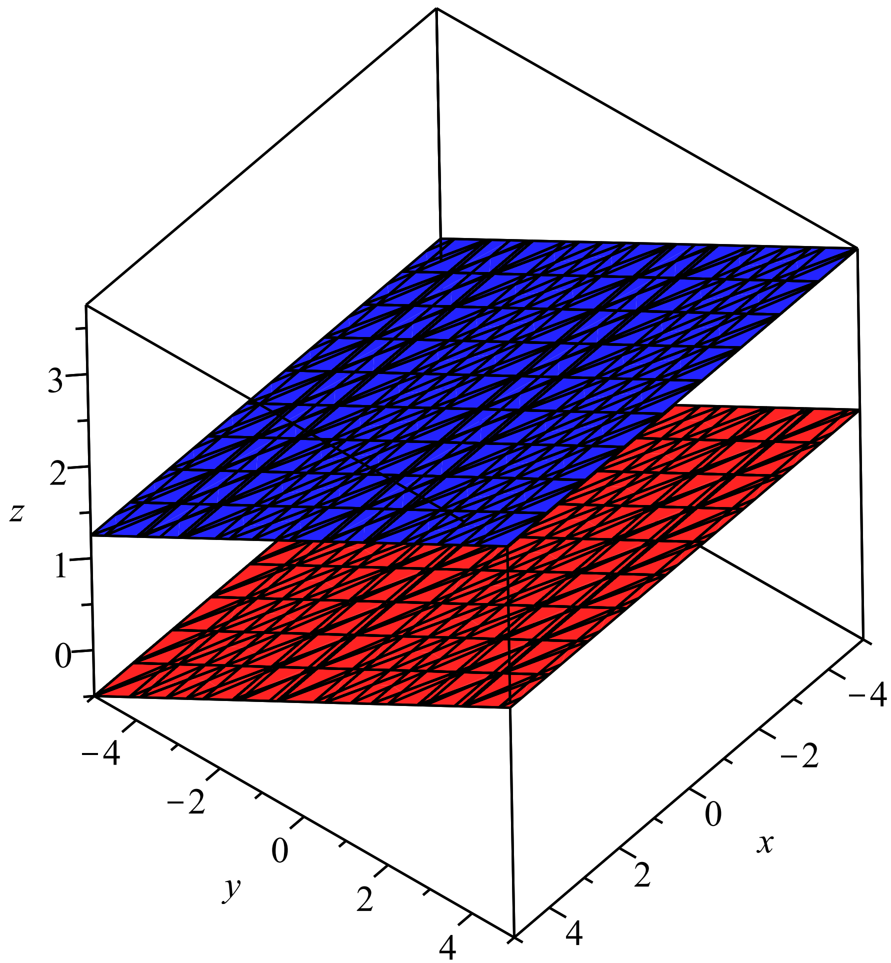
Linear equations in $\mathbb{R}^3$



$$2x - y + 4z = 5$$
$$4x - 2y + 8z = 12$$

Linear equations in $\mathbb{R}^3$

What sort of solution do you expect from the intersection of the following two planes?

$$\begin{array}{rcrcrcl} 2x & - & y & + & 4z & = & 5 \\ 4x & - & 2y & + & 8z & = & 10 \end{array}$$

Subtracting 2 times the first from the second gives

$$0 = 0.$$

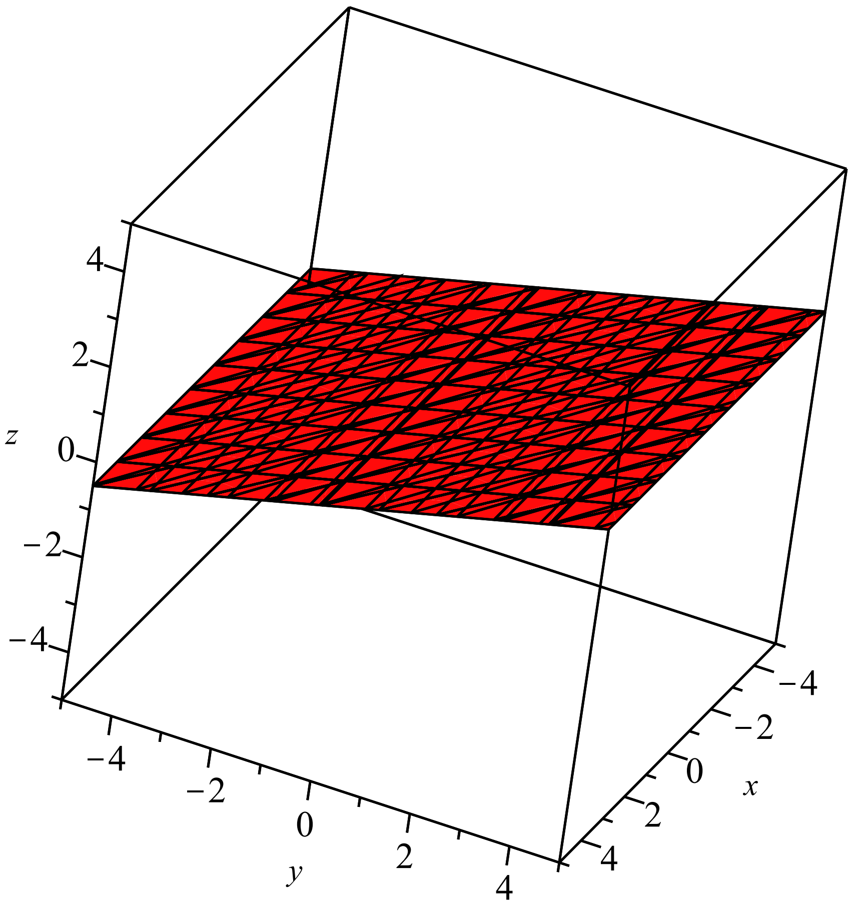
This is true for all (x, y, z) so the second equation puts no further restrictions.

These two equations represent the same plane.

So the solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$$

Linear equations in $\mathbb{R}^3$



$$2x - y + 4z = 5$$
$$4x - 2y + 8z = 10$$

Linear equations

$$\begin{aligned} x + y + 3z &= 4 \\ 2x + y + z &= 0 \\ x + 3y - z &= 6 \end{aligned}$$

Let's solve these equations.

$$\iff \begin{aligned} x + y + 3z &= 4 \\ -y - 5z &= -8 \\ x + 3y - z &= 6 \end{aligned}$$

(subtract $2\times$ row 1 from row 2)

$$\iff \begin{aligned} x + y + 3z &= 4 \\ y + 5z &= 8 \\ x + 3y - z &= 6 \end{aligned}$$

(multiply row 2 by -1)

$$\iff \begin{aligned} x + y + 3z &= 4 \\ y + 5z &= 8 \\ 2y - 4z &= 2 \end{aligned}$$

(subtract row 1 from row 3)

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Linear equations

$$\begin{aligned} x + y + 3z &= 4 \\ 2x + y + z &= 0 \\ x + 3y - z &= 6 \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} x + y + 3z &= 4 \\ y + 5z &= 8 \\ -14z &= -14 \end{aligned}$$

Both sets of equations have the same solutions but clearly the right hand set is simpler to work with.

We solve this set starting with the simplest (last) row which says

$$z = 1.$$

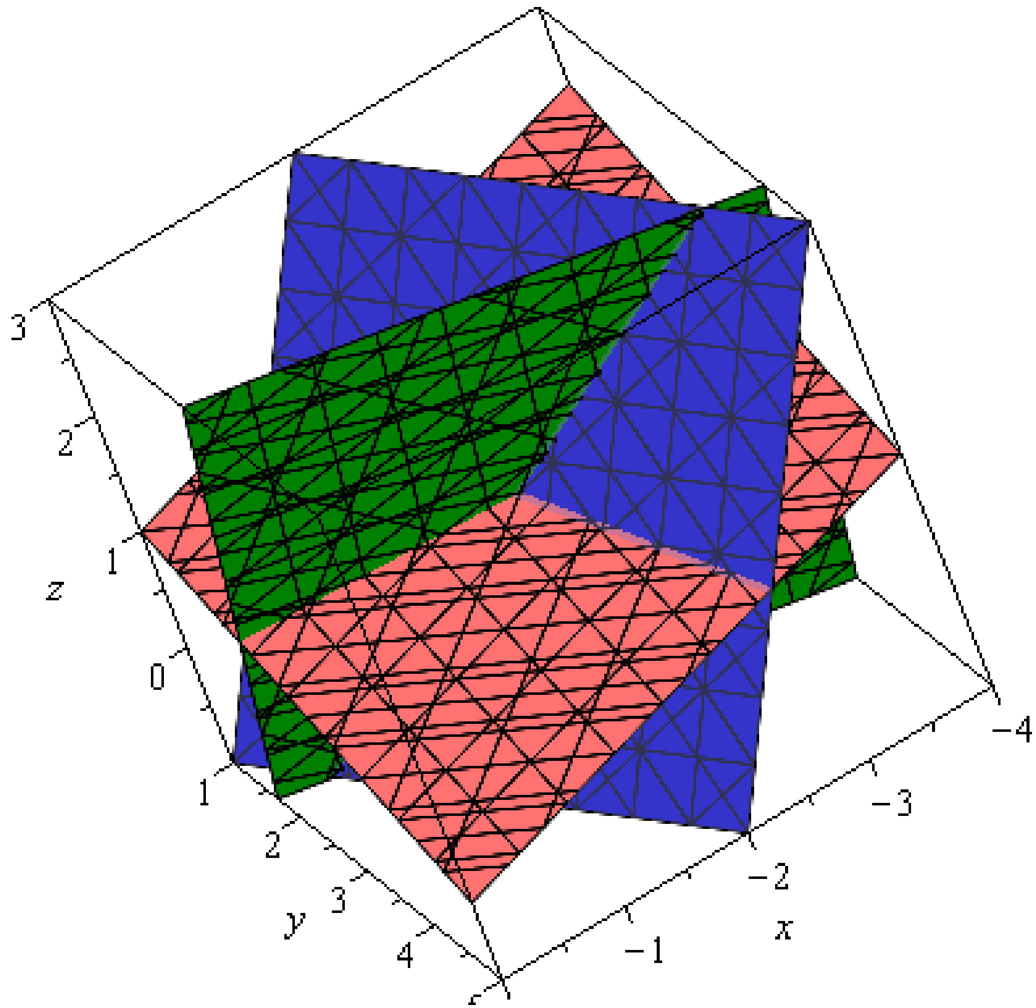
Next, substituting $z = 1$ in the next simplest row gives

$$y + 5 \times 1 = 8 \implies y = 3$$

and lastly substituting $z = 1$ and $y = 3$ into the remaining row gives

$$x + 3 + 3 \times 1 = 4 \implies x = -2.$$

Linear equations in $\mathbb{R}^3$



$$\begin{aligned} x + y + 3z &= 4 \\ 2x + y + z &= 0 \\ x + 3y - z &= 6 \end{aligned}$$

has solution

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$$

Augmented matrices

When we add and subtract equations, we don't really need to carry around the variable names. All we need is the coefficients. So, for the sake of efficiency, we represent a system of linear equations by an *augmented matrix*.

The system of linear equations

$$\begin{aligned} x + y + 3z &= 4 \\ 2x + y + z &= 0 \\ x + 3y - z &= 6 \end{aligned}$$

has augmented matrix

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 2 & 1 & 1 & 0 \\ 1 & 3 & -1 & 6 \end{array}\right) \end{array}.$$

Representing a system of equations by an augmented matrix.

In an an *augmented matrix*, each row represents one equation.
The coefficients of the variables are on the left of the vertical bar
and the constants in the right hand side of the equations go on the right of the vertical bar.

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Augmented matrices: Elementary row operations

We can perform the same manipulations now just using the coefficients written in an augmented matrix. We will call these *elementary row operations*.

We usually refer to row 1 as R_1 and row 2 as R_2, etc.

Example. We subtract $2R_1$ from R_2 and put the result into R_2.

$$\begin{array}{cl} \begin{array}{cccc} x & y & z & \end{array} & \\ \left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 2 & 1 & 1 & 0 \\ 1 & 3 & -1 & 6 \end{array}\right) & \begin{array}{l} \times -2 \\ \times 1 \\ \end{array} \end{array}$$

$$\iff \left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & -1 & -5 & -8 \\ 1 & 3 & -1 & 6 \end{array}\right) \qquad\qquad R_2 \leftarrow R_2 - 2R_1$$

Elementary row operations.

Elementary row operations allow you (1) to add a multiple of one row to another, (2) swap two rows or (3) scale one row, that is,

$$R_i \leftarrow R_i + \alpha R_j, \qquad R_i \leftrightarrow R_j \quad \text{or} \quad R_i \leftarrow \alpha R_i.$$

TIP!

"$R_2 \leftarrow R_2 - 2R_1$" is read "*$R_2$ is assigned the value $R_2 - 2R_1$*" or "*R_2 is replaced by $R_2 - 2R_1$*" or "*We put $R_2 - 2R_1$* into R_2".

Augmented matrices : Row Echelon Form

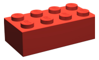
$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & -1 & -5 & -8 \\ 1 & 3 & -1 & 6 \end{array}\right)$$

$$\iff \left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & -1 & -5 & -8 \\ 0 & 2 & -4 & 2 \end{array}\right) \qquad R_3 \leftarrow R_3 - R_1$$

$$\iff \left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & 5 & 8 \\ 0 & 2 & -4 & 2 \end{array}\right) \qquad R_2 \leftarrow -R_2$$

$$\iff \left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & 5 & 8 \\ 0 & 0 & -14 & -14 \end{array}\right) \qquad R_3 \leftarrow R_3 - 2R_2$$

This final form where each row begins with more zeros than the previous row is called *Row Echelon Form*.



Row Echelon Form.
An augmented matrix is in *Row Echelon Form* if each row begins with more zeros than the previous row.

Augmented matrices

The augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & 5 & 8 \\ 0 & 0 & -14 & -14 \end{array}\right)$$

represents the system of equations

$$\begin{aligned} x + y + 3z &= 4 \\ y + 5z &= 8 \\ -14z &= -14 \end{aligned}$$

Which we can solve by back substitution as we did previously.