MATH1131 Mathematics 1A – Algebra

Lecture 10: Complex Division and Conjugates

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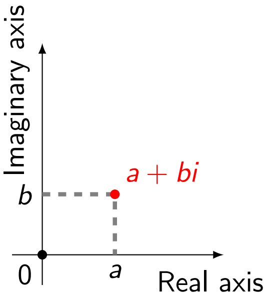
Based on slides by Jonathan Kress

Argand diagram

The set of complex numbers

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

is often represented as a plane called the Argand diagram or complex plane:



If a and b are real numbers, then the complex number

$$z = a + bi$$

has real part a and imaginary part b.

We write:

$$\mathrm{Re}(z) = a \quad \text{and} \quad \mathrm{Im}(z) = b.$$

Real and imaginary parts

Note that $z = \text{Re}(z) + \text{Im}(z)i$, where $\text{Re}(z) \in \mathbb{R}$ and $\text{Im}(z) \in \mathbb{R}$.

We say $z \in \mathbb{C}$ is real if $z = \text{Re}(z) \;\Leftrightarrow\; \text{Im}(z) = 0$.

We say $z \in \mathbb{C}$ is imaginary if $z = \text{Im}(z)i \;\Leftrightarrow\; \text{Re}(z) = 0$.

Examples

- 3 is real
- $3i$ is imaginary
- Both 3 and $3i$ are complex

Equality of complex numbers

Note: Two complex numbers are equal if and only if their real and imaginary parts are equal.

Example

Find $a, b \in \mathbb{R}$ satisfying

$$(3+4i)(a+bi) = 23+14i.$$

By assumption,
$$\begin{aligned} 23+14i &= (3+4i)(a+bi) \\ &= 3a+3bi+4ai+4bi^2 \\ &= (3a-4b)+(4a+3b)i \end{aligned}$$

Comparing real and imaginary parts, we need:

$$3a-4b=23 \quad \text{and} \quad 4a+3b=14.$$

Solving these simultaneously gives the answer $a=5$ and $b=-2$.

Note: We will see a neater way to solve this problem in a few slides.

Inverse of a complex number

Example

Find the multiplicative inverse of $2 + 5i$.

We want to find the complex number $a + bi$ with $a, b \in \mathbb{R}$, such that

$$(2 + 5i)(a + bi) = 1.$$

Expanding yields

$$(2a - 5b) + (5a + 2b)i = 1 = 1 + 0i,$$

so $\ 2a - 5b = 1\ $ and $\ 5a + 2b = 0$.

Solving these simultaneously gives the answer $a = \dfrac{2}{29}$ and $b = -\dfrac{5}{29}$.

Hence

$$(2 + 5i)^{-1} = \frac{1}{2 + 5i} = \frac{2}{29} - \frac{5}{29}i.$$

Note: We will see a neater way to solve this problem in a few slides.

Example

Find the square roots of $-3+4i$.

We want to find the complex numbers $a+bi$ with $a, b \in \mathbb{R}$, such that

$$(a+bi)^2 = -3+4i.$$

Expanding yields

$$a^2 + 2abi + b^2 i^2 = (a^2 - b^2) + 2abi = -3 + 4i,$$

so $a^2 - b^2 = -3$ and $2ab = 4$.

Substituting for a gives $b^4 - 3b^2 - 4 = (b^2-4)(b^2+1) = 0$, so $b = \pm 2$ and $a = \pm 1$ respectively.

Thus the square roots of $-3+4i$ are $1+2i$ and $-1-2i$.

Note: We will see a neater way to solve this problem in a few lectures, once we have seen the polar form.

Complex conjugate

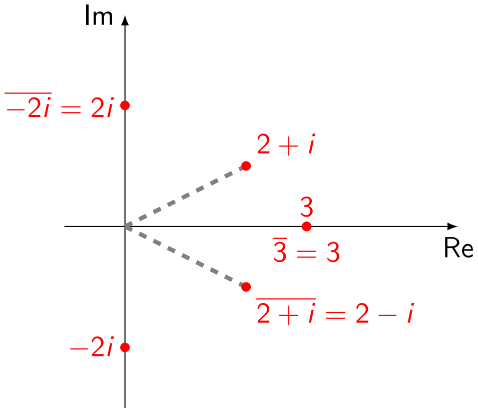
Definition

Let $z = a + bi$ be a complex number with $a, b \in \mathbb{R}$. The complex conjugate of z is

$$\overline{z} = a - bi.$$

Examples

$$\begin{aligned}
\overline{2+i} &= 2 - i \\
\overline{-2i} &= 2i \\
\overline{3} &= 3
\end{aligned}$$



Inverse of a complex number – revisited

To simplify a complex fraction, multiply the top and bottom by the conjugate of the denominator.

Example

Find the multiplicative inverse of $2 + 5i$.

We want to find the complex number $\dfrac{1}{2+5i}$ in the form $a + bi$ for $a, b \in \mathbb{R}$.

$$\begin{aligned}
\frac{1}{2+5i} &= \frac{1}{2+5i} \times \frac{2-5i}{2-5i} \\
&= \frac{2-5i}{2^2-(5i)^2} \\
&= \frac{2-5i}{4+25} \\
&= \frac{2}{29} - \frac{5}{29}i
\end{aligned}$$

Inverse of a general complex number

Example

What is

$$\frac{1}{a+bi}$$

for any $a, b \in \mathbb{R}$ (not both 0)?

We use the same process:

$$\begin{aligned}
\frac{1}{a+bi} &= \frac{1}{a+bi} \times \frac{a-bi}{a-bi} \\
&= \frac{a-bi}{a^2-(bi)^2} \\
&= \frac{a-bi}{a^2+b^2} \\
&= \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i
\end{aligned}$$

Division of complex numbers

We can use the same method for division by complex numbers.

Example

Find $a, b \in \mathbb{R}$ satisfying

$$(3+4i)(a+bi) = 23+14i.$$

Rearranging, we want to find $a + bi = \dfrac{23+14i}{3+4i}$.

$$\begin{aligned}
\frac{23+14i}{3+4i} &= \frac{23+14i}{3+4i} \times \frac{3-4i}{3-4i} \\
&= \frac{23\times 3 - 23\times 4i + 14\times 3i - 14\times 4i^2}{3^2-(4i)^2} \\
&= \frac{125-50i}{25} \\
&= 5-2i
\end{aligned}$$

Division of general complex numbers

Exercise

What is

$$\frac{c+di}{a+bi}$$

for $a, b, c, d \in \mathbb{R}$ with a and b not both zero?

We again use the same process:

$$\begin{aligned}\frac{c+di}{a+bi} &= \frac{c+di}{a+bi} \times \frac{a-bi}{a-bi} \\ &= \frac{ac - bci + adi - bdi^2}{a^2 - (bi)^2} \\ &= \frac{(ac+bd) + (ad-bc)i}{a^2+b^2} \\ &= \frac{ac+bd}{a^2+b^2} - \frac{ad-bc}{a^2+b^2} i\end{aligned}$$

Complex conjugate – Properties

Theorem

For all $z \in \mathbb{C}$,

- $\overline{(\overline{z})} = z$
- $\text{Re}(z) = \dfrac{1}{2}(z + \overline{z})$ and $\text{Im}(z) = \dfrac{1}{2i}(z - \overline{z})$
- $z\overline{z} = (\text{Re}(z))^2 + (\text{Im}(z))^2$, so $z\overline{z} \in \mathbb{R}$ and $z\overline{z} \geq 0$

Proof

Let $z = a + bi \in \mathbb{C}$ where $a, b \in \mathbb{R}$. Then

$$\overline{(\overline{z})} = \overline{(\overline{a + bi})} = \overline{a - bi} = a + bi = z.$$

Complex conjugate – Properties

Theorem

For all $z \in \mathbb{C}$,

- $\overline{(\overline{z})} = z$
- $\text{Re}(z) = \dfrac{1}{2}(z + \overline{z})$ and $\text{Im}(z) = \dfrac{1}{2i}(z - \overline{z})$
- $z\overline{z} = (\text{Re}(z))^2 + (\text{Im}(z))^2$, so $z\overline{z} \in \mathbb{R}$ and $z\overline{z} \geq 0$

Proof

Let $z = a + bi \in \mathbb{C}$ where $a, b \in \mathbb{R}$. Then

$$z + \overline{z} = (a + bi) + (a - bi) = 2a = 2\text{Re}(z)$$

and

$$z - \overline{z} = (a + bi) - (a - bi) = 2bi = 2\text{Im}(z)i.$$

Hence $\text{Re}(z) = \dfrac{1}{2}(z + \overline{z})$ and $\text{Im}(z) = \dfrac{1}{2i}(z - \overline{z})$.

Complex conjugate – Properties

Theorem

For all $z \in \mathbb{C}$,

- $\overline{(\overline{z})} = z$
- $\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$ and $\mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z})$
- $z\overline{z} = (\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2$, so $z\overline{z} \in \mathbb{R}$ and $z\overline{z} \geq 0$

Proof

Let $z = a + bi \in \mathbb{C}$ where $a, b \in \mathbb{R}$. Then

$$z\overline{z} = (a + bi)(a - bi) = (a^2 - (bi)^2) = a^2 + b^2.$$

Since $a, b \in \mathbb{R}$, and the square of a real number is non-negative, it follows that $z\overline{z} \in \mathbb{R}$ and $z\overline{z} \geq 0$.

Complex conjugate – Properties

Theorem

For all $z, w \in \mathbb{C}$,

- $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{z-w} = \overline{z} - \overline{w}$
- $\overline{zw} = \overline{z}\ \overline{w}$ and $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$

Proof

Let $z = a + bi \in \mathbb{C}$ and $w = c + di \in \mathbb{C}$ where $a, b, c, d \in \mathbb{R}$. Then

$$\begin{aligned}
\overline{z+w} &= \overline{(a+bi)+(c+di)} \\
&= \overline{(a+c)+(b+d)i} \\
&= (a+c)-(b+d)i \\
&= (a-bi)+(c-di) \\
&= \overline{z}+\overline{w}, \text{ and}
\end{aligned}
\qquad
\begin{aligned}
\overline{z-w} &= \overline{(a+bi)-(c+di)} \\
&= \overline{(a-c)+(b-d)i} \\
&= (a-c)-(b-d)i \\
&= (a-bi)-(c-di) \\
&= \overline{z}-\overline{w}.
\end{aligned}$$

Complex conjugate – Properties

Theorem

For all $z, w \in \mathbb{C}$,

- $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{z-w} = \overline{z} - \overline{w}$
- $\overline{zw} = \overline{z}\ \overline{w}$ and $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$

Proof

Let $z = a + bi \in \mathbb{C}$ and $w = c + di \in \mathbb{C}$ where $a, b, c, d \in \mathbb{R}$. Then

$$\begin{aligned}
\overline{zw} &= \overline{(a+bi)(c+di)} \\
&= \overline{(ac-bd)+(ad+bc)i} \\
&= (ac-bd)-(ad+bc)i \\
&= (a-bi)(c-di) \\
&= \overline{z}\ \overline{w}.
\end{aligned}$$

It follows that $\overline{z} = \overline{\left(\dfrac{z}{w}\,w\right)} = \overline{\left(\dfrac{z}{w}\right)}\,\overline{w}$. Hence $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$.

Complex conjugate – Example

Example

Let $z, w \in \mathbb{C}$ with $z\overline{z} = w\overline{w}$. Prove that $\dfrac{z+w}{z-w}$ is imaginary.

To prove something is imaginary, show that its real part is 0:

$$\begin{aligned}
\operatorname{Re}\left(\frac{z+w}{z-w}\right) &= \frac{1}{2}\left(\frac{z+w}{z-w} + \overline{\left(\frac{z+w}{z-w}\right)}\right) = \frac{1}{2}\left(\frac{z+w}{z-w} + \frac{\overline{z}+\overline{w}}{\overline{z}-\overline{w}}\right) \\
&= \frac{1}{2}\left(\frac{(z+w)(\overline{z}-\overline{w}) + (\overline{z}+\overline{w})(z-w)}{(z-w)(\overline{z}-\overline{w})}\right) \\
&= \frac{1}{2}\left(\frac{z\overline{z} - z\overline{w} + w\overline{z} - w\overline{w} + \overline{z}z - \overline{z}w + \overline{w}z - \overline{w}w}{(z-w)(\overline{z}-\overline{w})}\right) \\
&= \frac{1}{2}\left(\frac{2(z\overline{z} - w\overline{w})}{(z-w)(\overline{z}-\overline{w})}\right) = 0 \quad (z\overline{z} = w\overline{w})
\end{aligned}$$

So since its real part is 0, the expression is imaginary.