MATH1131 Mathematics 1A – Algebra

Lecture 13: Complex Roots and Powers

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Based on slides by Jonathan Kress

Powers

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So $s^n = r$ and $n\phi = \theta + 2k\pi$ for some $k \in \mathbb{Z}$.

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So the 5^{th} roots of $1 + \sqrt{3}i$ are:

$$2^{\frac{1}{5}}\, e^{i\frac{\pi}{15}},\ 2^{\frac{1}{5}}\, e^{i\frac{7\pi}{15}},\ 2^{\frac{1}{5}}\, e^{i\frac{13\pi}{15}},\ 2^{\frac{1}{5}}\, e^{-i\frac{11\pi}{15}},\ \text{and } 2^{\frac{1}{5}}\, e^{-i\frac{\pi}{3}}.$$

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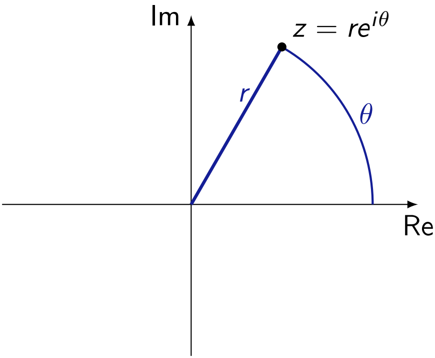

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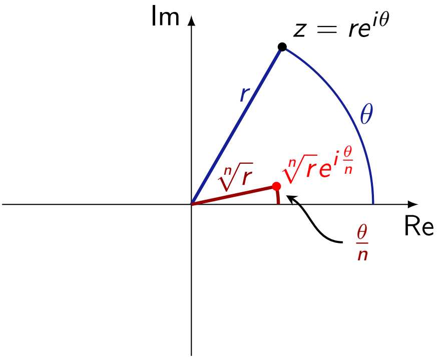

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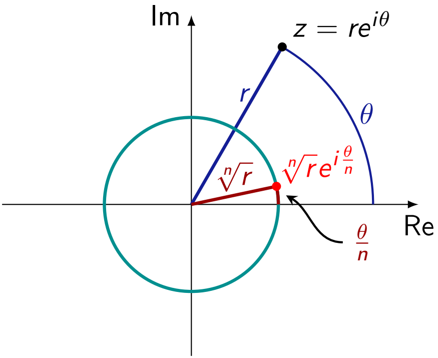

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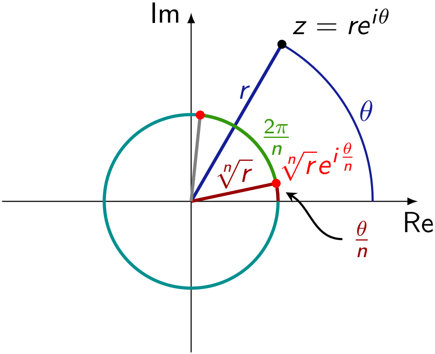

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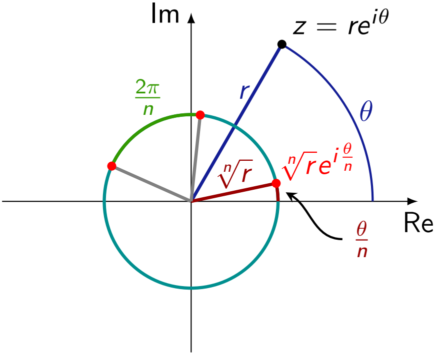

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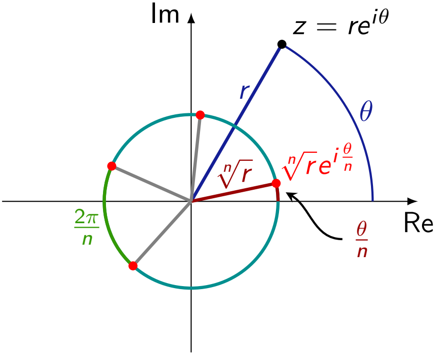

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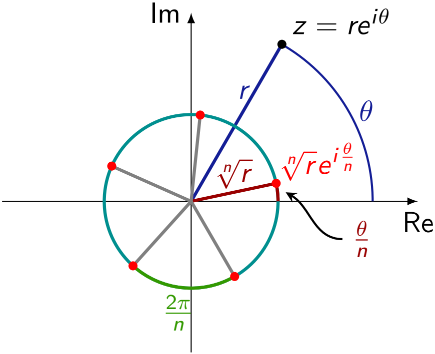

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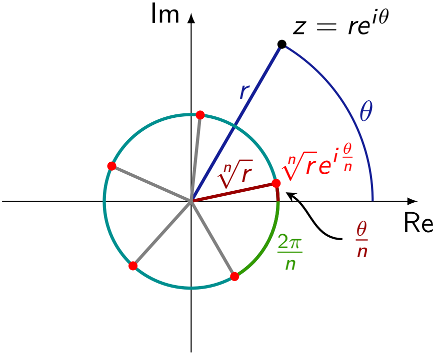

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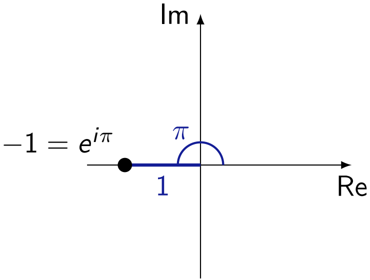

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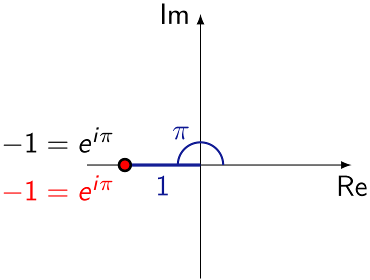

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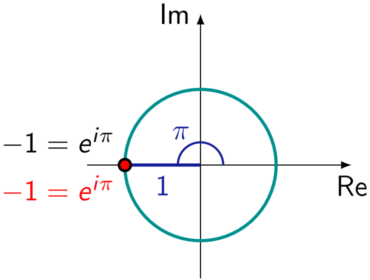

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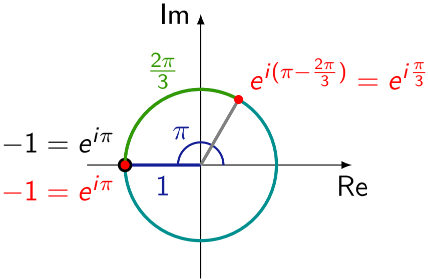

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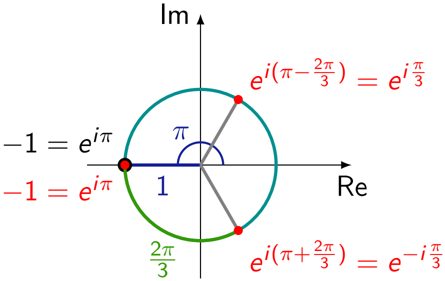

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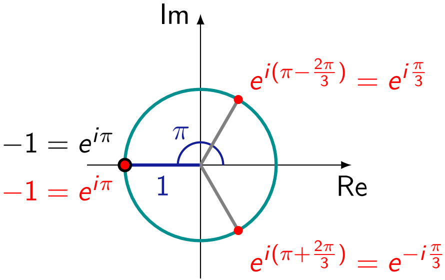

So the three 3rd roots of -1 are: -1, $e^{i\frac{\pi}{3}}$ and $e^{-i\frac{\pi}{3}}$.

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$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & \vdots & & & &
\end{array}$$

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&\quad = 1\times(-2)^7 i^0 + 7\times(-2)^6 i^1 + 21\times(-2)^5 i^2 + 35\times(-2)^4 i^3 + \\
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\end{aligned}$$

Binomial theorem

Example

Find $(-2+i)^7$.

The corresponding row in Pascal's triangle is:

$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$

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&\quad = -128 \times 1 + 448 \times i - 672 \times (-1) + 560 \times (-i) + \\
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The Binomial Theorem is particularly useful when expressing $\cos(n\theta)$ and $\sin(n\theta)$ in terms of powers of $\cos\theta$ and $\sin\theta$, and vice versa.

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Example

Use De Moivre's Theorem to find formulae for $\cos(4\theta)$ and $\sin(4\theta)$ in terms of powers of $\sin\theta$ and $\cos\theta$.

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Expanding the right-hand side:

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So equating real and imaginary parts, we find

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So

$$\int \sin^3\theta \, d\theta = \int \left(-\frac{1}{4}(\sin(3\theta) - 3\sin\theta)\right) d\theta = \frac{1}{12}\cos(3\theta) - \frac{3}{4}\cos\theta.$$