MATH1131 Mathematics 1A – Algebra

Lecture 1: Geometric Vectors

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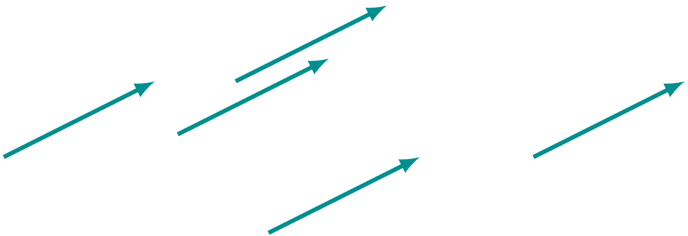
Based on slides by Jonathan Kress

Geometric Vectors

A vector is a quantity that has both magnitude (i.e. size) and direction. Quantities with these properties arise often in areas such as Physics (displacement; velocity; force...).

Geometrically, a vector can be thought of as an arrow with a length and a direction. The relative position of the vector does not matter.

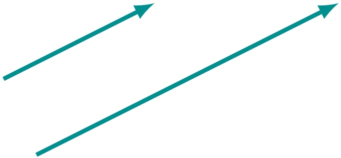
Hence all these vectors are the same:



... because they all have the same length and direction.

Geometric Vectors

These vectors are different:



... because they have different lengths.

And these vectors are different:

... because they have different directions.

Real numbers have magnitude but not direction (in the same sense). In comparison to vectors, we call real numbers scalars.

To distinguish vector quantities from scalar quantities we use different notation for vectors and scalars.

- Scalars are denoted with plain letters such as x , y , α , β , ...
- Vectors are written as letters in boldface (when typed), or with an arrow drawn over them, or with a tilde drawn under them:

$$\mathbf{u}\ ,\quad \overrightarrow{u}\ ,\quad \utilde{u}$$

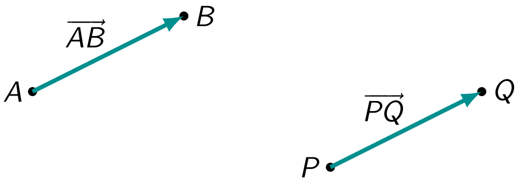
Note: Always be sure to distinguish vectors by underlining or over-arrowing them when handwritten.

Other notation:

- The length of a vector is denoted with vertical bars:

$$|\mathbf{u}| = \text{the length of } \mathbf{u}$$

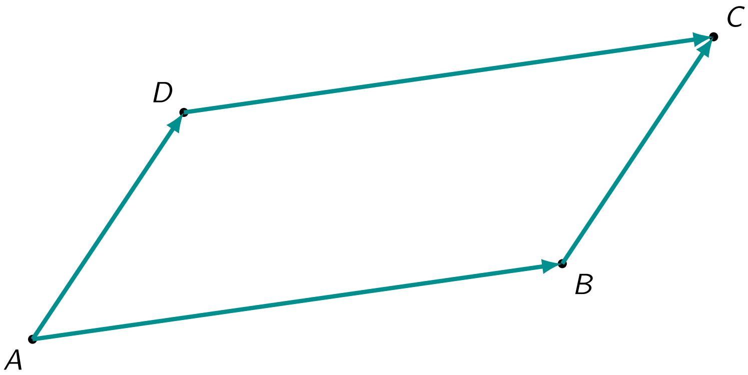
The vector that points from point A to point B is denoted $\overrightarrow{AB}$.



In the picture above, the vector that points from A to B is the same as the vector that points from P to Q.

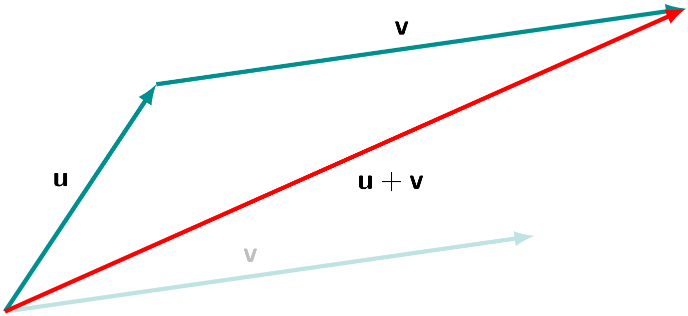
Since vectors' positions do not matter, we have that $\overrightarrow{AB} = \overrightarrow{PQ}$.

In the parallelogram below, which vectors are equal?



$$\overrightarrow{AB} = \overrightarrow{DC} \qquad \text{and} \qquad \overrightarrow{AD} = \overrightarrow{BC}$$

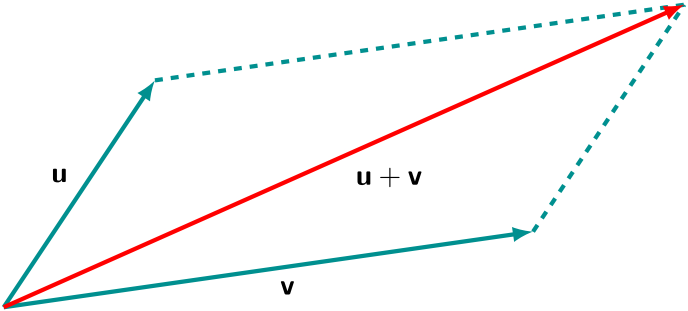
To geometrically add two vectors, we place them tip to tail and form a new vector that points from the tail of the first vector to the tip of the second.



So given $\mathbf{u}$ and $\mathbf{v}$, to find $\mathbf{u} + \mathbf{v}$, we move the tail of $\mathbf{v}$ to the tip of $\mathbf{u}$ and then complete the triangle.

This method of addition is known as the triangle law of addition.

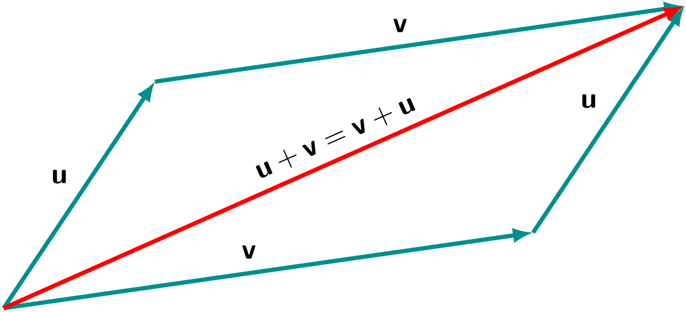
An alternative interpretation of vector addition uses the parallelogram with the two vectors $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides. Then $\mathbf{u}+\mathbf{v}$ is the vector that points from the tails of the two vectors to the furthest vertex.



This method of addition is known as the parallelogram law of addition.

Obviously, the two definitions are equivalent.

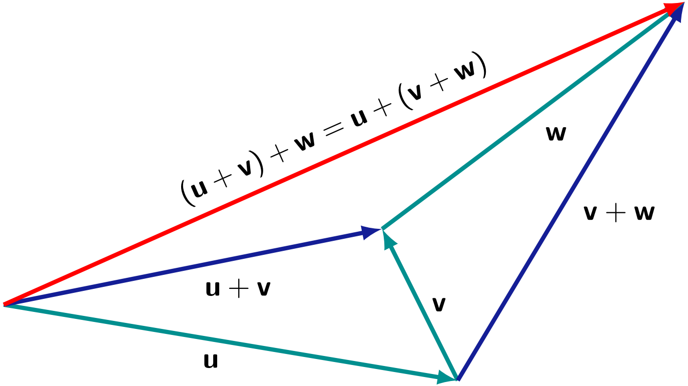
Vector addition happens to have many of the properties we take for granted with normal scalar addition.



Vector addition is commutative:

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

so we don't have to worry about the order in which vectors are added.



Vector addition is associative:

$$(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$$

so it is safe to write $\mathbf{u}+\mathbf{v}+\mathbf{w}$.

There is a special vector called the zero vector that has length 0 and no direction. It can't be drawn as an arrow, but it is still a valid vector.

The zero vector is written as $\mathbf{0}$ or $\underset{\sim}{0}$ or $\vec{0}$.

The zero vector added to any vector leaves the vector unchanged:

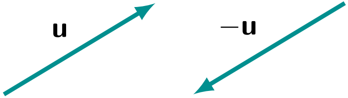
$$\mathbf{u} + \mathbf{0} = \mathbf{u}$$

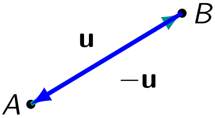
Note: Never write the zero vector simply as 0, or it could be confused with the zero scalar!

For each vector $\mathbf{u}$ there is another vector called its negative, denoted by $-\mathbf{u}$ ("minus u"). It has the property that:

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}.$$

For example,



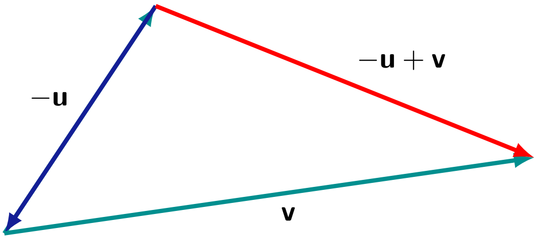


Note:

- $\mathbf{u}$ and $-\mathbf{u}$ have the same length (that is, $|\mathbf{u}| = |-\mathbf{u}|$), but opposite directions.
- In general, $\overrightarrow{BA} = -\overrightarrow{AB}$.

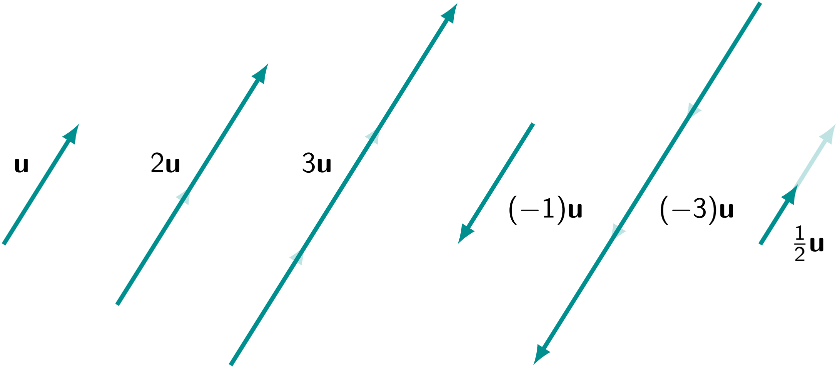
To subtract vectors we add the negative:

$$\mathbf{v} - \mathbf{u} = \mathbf{v} + -\mathbf{u}$$



So $\mathbf{v} - \mathbf{u}$ is the vector that points from the tip of $\mathbf{u}$ to the tip of $\mathbf{v}$.

Scalar multiplication is another operation on vectors. Multiplying a vector by a positive scalar changes its length but not its direction. Multiplying a vector by a negative scalar changes its length and reverses its direction.



For any scalar λ and vector $\mathbf{v}$, the scalar multiple $\lambda\mathbf{v}$ is the vector that has length $|\lambda||\mathbf{v}|$ and

- the same direction as $\mathbf{v}$ when $\lambda > 0$, or
- the opposite direction to $\mathbf{v}$ when $\lambda < 0$.

Note:

- For $\lambda = 0$: $\quad 0\mathbf{v} = \mathbf{0}$
  since $0\mathbf{v}$ has length $|0||\mathbf{v}| = 0$, and no direction.
- For $\lambda = 1$: $\quad 1\mathbf{v} = \mathbf{v}$
  since $1\mathbf{v}$ has the same length as $\mathbf{v}$ $\quad (|1||\mathbf{v}| = |\mathbf{v}|)$,
  and the same direction as $\mathbf{v}$.
- For $\lambda = -1$: $\quad (-1)\mathbf{v} = -\mathbf{v}$
  since $(-1)\mathbf{v}$ has the same length as $-\mathbf{v}$ $\quad (|-1||\mathbf{v}| = |\mathbf{v}|)$,
  but the opposite direction to $\mathbf{v}$.

Definition

Two vectors are said to be parallel if they have the same or the opposite direction.

In fact, two vectors $\mathbf{u}$ and $\mathbf{v}$ have the same or the opposite direction if and only if they are scalar multiples of each other:

Clearly if one is a scalar multiple of the other, then the two vectors have the same or the opposite direction.

Conversely, if the two vectors have the same or the opposite direction, then we can write one as a scalar multiple of the other. For example,

$$\mathbf{u} = \begin{cases} \frac{|\mathbf{u}|}{|\mathbf{v}|}\mathbf{v} & \text{if they have the same direction} \\ -\frac{|\mathbf{u}|}{|\mathbf{v}|}\mathbf{v} & \text{if they have the opposite direction} \end{cases}$$

This means two vectors are parallel if one is a non-zero scalar multiple of the other.

For any vectors $\mathbf{u}$ and $\mathbf{v}$, and scalars λ and μ, the following properties hold:

Associative law of multiplication by a scalar:

$$\lambda(\mu\mathbf{u}) = (\lambda\mu)\mathbf{u}$$

Scalar distributive law:

$$(\lambda + \mu)\mathbf{u} = \lambda\mathbf{u} + \mu\mathbf{u}$$

Vector distributive law:

$$\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$$

We can use these laws to simplify vector expressions.

Example

Simplify $3(2\mathbf{u} - \mathbf{v}) + (\mathbf{u} - 2\mathbf{v})$.

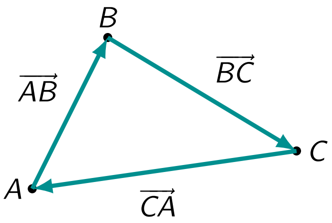
$$
\begin{aligned}
&3(2\mathbf{u} - \mathbf{v}) + (\mathbf{u} - 2\mathbf{v}) \\
&= 3(2\mathbf{u} + (-1)\mathbf{v}) + (\mathbf{u} + (-1)(2\mathbf{v})) && \text{(Definition of subtraction)} \\
&= (3(2\mathbf{u}) + 3((-1)\mathbf{v})) + (\mathbf{u} + (-1)(2\mathbf{v})) && \text{(Vector distributivity)} \\
&= (6\mathbf{u} + (-3)\mathbf{v}) + (\mathbf{u} + (-2)\mathbf{v}) && \text{(Scalar associativity)} \\
&= (6\mathbf{u} + \mathbf{u}) + ((-3)\mathbf{v} + (-2)\mathbf{v}) && \text{(Associativity and commutativity)} \\
&= (6 + 1)\mathbf{u} + ((-3) + (-2))\mathbf{v} && \text{(Scalar distributivity)} \\
&= 7\mathbf{u} - 5\mathbf{v} && \text{(Definition of subtraction)}
\end{aligned}
$$

In practice, we simply write

$$3(2\mathbf{u} - \mathbf{v}) + (\mathbf{u} - 2\mathbf{v}) = 6\mathbf{u} - 3\mathbf{v} + \mathbf{u} - 2\mathbf{v} = 7\mathbf{u} - 5\mathbf{v}.$$

Example

For any three points A, B, and C, what is $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$?



Here are our three points.

$\overrightarrow{AB}$ takes us from A to B, then $\overrightarrow{BC}$ takes us from B to C, and $\overrightarrow{CA}$ takes us from C back to A.

So in total we have not moved at all.

This means $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$.