LECTURE 21

The exponential function e^x

The inverse of $\ln(x)$ is e^x.

$$e^{\ln(x)} = \ln(e^x) = x.$$

$$\frac{d}{dx}e^x = e^x.$$

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C.$$

$$e^x e^y = e^{x+y}.$$

$$\frac{d}{dx}(a^x) = a^x \ln(a).$$

$$\int a^x\,dx = \frac{1}{\ln(a)}a^x.$$

You will recall from the previous lecture that the natural log function $y = \ln(x)$ is an increasing function. Hence it is $1-1$ and thus invertible. The inverse of $\ln(x)$ is without doubt the most important function in all of mathematics....the exponential function $y = e^x$.

The irrational real number e is approximately 2.71828 and can be defined in many ways:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

or

$$e = \lim_{n \longrightarrow \infty}(1 + \frac{1}{n})^n$$

What makes e^x such a fascinating function is the simple fact $\frac{d}{dx}e^x = e^x$. It is its own derivative! No other function has this remarkable property. The exponential function e^x is immune to calculus!

The graphs of the two functions are reflected in $y = x$ in the usual manner.

Sketch:

Observe that $\text{Dom}(\ln(x)) = (0, \infty) = \text{Range}(e^x)$ and $\text{Range}(\ln(x)) = \mathbb{R} = \text{Dom}(e^x)$.

Both functions are increasing however the exponential function grows with enormous strength while the natural log function increases very weakly.

Further properties of the two functions are:

a) $e^{\ln(x)} = x.$ This is just $(f^{-1} \circ f)(x) = x$.

b) $\ln(e^x) = x.$ This is just $(f \circ f^{-1})(x) = x$.

c) $\dfrac{d}{dx} e^x = e^x.$

d) $\dfrac{d}{dx} e^{f(x)} = e^{f(x)} f'(x).$

e) $\displaystyle\int e^x \, dx = e^x.$

f) $\displaystyle\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C.$

g) $e^x e^y = e^{x+y}.$

Note that g) indicates that the inverse of $\ln(x)$ actually has something to do with exponentials!!

Proofs:

a) and b) This is just the definition of the inverse function!

c) We start with $\ln(e^x) = x$ and differentiate both sides with respect to x.

d) This is just the chain rule.

e) follows from c)

f) Exercise.

g) $e^{x+y} = e^{\ln(e^x) + \ln(e^y)} = e^{\ln(e^x e^y)} = e^x e^y.$

★

Example 1:

a) Evaluate $\displaystyle\int_{\ln(2)}^{\ln(5)} e^{3x}\,dx$.

b) Solve $2^x = 9$.

c) Find $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

d) Find $\displaystyle\frac{d}{dx}(x^3 e^{5x})$.

$$\bigstar \quad a)\quad 39 \quad b)\quad \frac{\ln(9)}{\ln(2)} \approx 3.17 \quad c)\quad 2e^{\sqrt{x}} + C \quad d)\quad e^{5x}\{3x^2 + 5x^3\} \quad \bigstar$$

The functions 2^x, 3^x and 7^x are also exponential functions. Why do we obsess about e^x ? Only e^x is equal to its own derivative! So what is the derivative of 3^x? To answer this question we use what is called logarithmic differentiation. This is simply taking the log of both sides before differentiating implicitly. This works well to eliminate troublesome exponentials.

Example 2: Find $\frac{d}{dx}(7^x)$.

★ $7^x \ln(7)$ ★

It follows from the same argument that

$$\frac{d}{dx}(a^x) = a^x \ln(a),$$

and hence after integrating both sides with respect to x

$$\int a^x \, dx = \frac{1}{\ln(a)} a^x.$$

Proof:

★

Example 3: Use the above facts to find:

a) $\frac{d}{dx}(4^x) =$

b) $\frac{d}{dx}(e^x) =$

c) $\frac{d}{dx}(e^5 + \ln 7) =$

d) $\int 6^x \, dx =$

★

The process of logarithmic differentiation is a versatile tool, handy whenever exponents are blocking your path:

Example 4: Use logarithmic differentiation to find $\dfrac{dy}{dx}$ for $y = \dfrac{x\sqrt{x^2+1}}{x^2-1}$

$$\bigstar \quad \frac{dy}{dx} = \frac{x\sqrt{x^2+1}}{x^2-1}\left\{\frac{1}{x} + \frac{x}{x^2+1} - \frac{2x}{x^2-1}\right\} \quad \bigstar$$

Example 5: Find $\dfrac{dy}{dx}$ for $y = x^x$

$$\bigstar \quad y = x^x\{1 + \ln(x)\} \quad \bigstar$$

The same log tricks can also be used on limits. Simply give the limit a name and then log both sides:

Example 6: Evaluate $\lim_{x \longrightarrow \infty} x^{\frac{1}{x}}$

$\bigstar \quad 1 \quad \bigstar$

Example 7: Evaluate $\lim_{x \longrightarrow \infty} (1 + \frac{1}{x})^x$

$\bigstar \quad e \quad \bigstar$