MATH1131 Mathematics 1A – Algebra

Lecture 20: Finding Inverses and Determinants

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Based on slides by Jonathan Kress

Inverses of 2×2 matrices

Definition

The inverse of the general 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is given by

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$$\begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 8 & -2 \\ -3 & 1 \end{pmatrix}.$$

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Since

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

we have

A is invertible if and only if $\det(A) \neq 0$.

Example

For each of the following matrices, determine if it is invertible and if so, find its inverse.

$$A = \begin{pmatrix} 3 & 9 \\ 2 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} -3 & -1 \\ 2 & 6 \end{pmatrix}$$

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Unfortunately we can't easily learn general formulae for the inverses of matrices of size larger than 2×2. Instead, we first need to consider how elementary row operations are related to matrix multiplication.

Row operations by matrix multiplication

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$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix},$$

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then EA is A with rows 2 and 4 swapped. That is,

$$EA = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 13 & 14 & 15 & 16 \\ 9 & 10 & 11 & 12 \\ 5 & 6 & 7 & 8 \end{pmatrix}.$$

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then EA is A with R_3 replaced by $2R_3$. That is,

$$EA = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 18 & 20 & 22 & 24 \\ 13 & 14 & 15 & 16 \end{pmatrix}.$$

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then EA is A with R_2 replaced by $R_2 + 3R_4$. That is,

$$EA = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 44 & 48 & 52 & 56 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}.$$

Matrix inverses

Suppose A is a square matrix, and the sequence of row operations that reduces it to Reduced Row Echelon Form (RREF) has corresponding matrices $E_1, E_2, \dots E_k$.

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where U is some RREF matrix. (Note that since matrix multiplication is not commutative, we must multiply on the left.)

If $U = I$, then we know A is invertible, and in fact

$$A^{-1} = E_k \dots E_2 E_1.$$

Conversely, if A is invertible, then the system of equations $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

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So when A is further reduced to RREF, each leading entry will become 1, and all other entries will become 0.

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- If $U \neq I$, A is not invertible.

Example

Find the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$.

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First write $(A|I)$, and row-reduce the left matrix:

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So A row-reduces to I in RREF, which means A is invertible.

Matrix inverses – Examples

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\cdots} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{5}{6} & -\frac{1}{6} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right)$$

So A row-reduces to I in RREF, which means A is invertible.

Furthermore, the resultant right-hand matrix is the inverse of A, so we have found

$$A^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{5}{6} & -\frac{1}{6} \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Matrix inverses – Examples

Example

Try to find the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

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First write $(A|I)$, and row-reduce the left matrix:

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Matrix inverses – Examples

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$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - 7R_1]{R_2 \to R_2 - 4R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right)$$

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Matrix inverses – Examples

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At this point we know A will not row-reduce to become I, since its REF contains a zero row.

Matrix inverses – Examples

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So A is not invertible.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - 7R_1]{R_2 \to R_2 - 4R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right)$$

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At this point we know A will not row-reduce to become I, since its REF contains a zero row.

So A is not invertible.

It would be useful if we could decide if a matrix is not invertible before carrying out these steps. What we would like to have is something similar to the 2×2 determinant, but for larger matrices.

We have seen how to find the determinant of a 2×2 matrix. For a 1×1 matrix, the determinant is just the single entry itself. We now look at larger matrices.

Determinants

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Definition

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For example, if

$$A = \begin{pmatrix} 1 & 4 & 6 \\ 1 & -8 & 7 \\ 5 & 9 & 0 \end{pmatrix},$$

we have

$$|A_{23}|$$

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$$|A_{23}| = \begin{vmatrix} 1 & 4 \\ 5 & 9 \end{vmatrix} = 1 \times 9 - 5 \times 4 = -11.$$

Determinants

Definition

The determinant of an $n \times n$ matrix A with entries a_{ij} is written as $\det(A)$ or $|A|$, and is given by:

$$\det(A) = a_{11}|A_{11}| - a_{12}|A_{12}| + a_{13}|A_{13}| - \cdots + (-1)^{n+1}a_{1n}|A_{1n}|$$

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Notice in particular that $\det(I) = 1$ no matter the size of I.

This is because

$$\begin{aligned} \det(I_n) &= 1|I_{n-1}| - 0 + \cdots + 0 \\ &= 1|I_{n-2}| - 0 + \cdots + 0 \\ &= \cdots \\ &= 1. \end{aligned}$$

Determinants – Example

Example

Find

$$\det \begin{pmatrix} 6 & 8 & 9 \\ 5 & 2 & 1 \\ 7 & 4 & 3 \end{pmatrix}.$$

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$$\det \begin{pmatrix} 6 & 8 & 9 \\ 5 & 2 & 1 \\ 7 & 4 & 3 \end{pmatrix} \quad = \quad 6\det\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} - 8\det\begin{pmatrix} 5 & 1 \\ 7 & 3 \end{pmatrix} + 9\det\begin{pmatrix} 5 & 2 \\ 7 & 4 \end{pmatrix}$$

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In fact, we can also expand along any other row or any column.

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The determinant is the sum of terms of the form $(-1)^{i+j} a_{ij} |A_{ij}|$ along any row or column.

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The determinant is the sum of terms of the form $(-1)^{i+j}a_{ij}|A_{ij}|$ along any row or column.

It's easy to remember the sign for each coefficient from the following checkerboard pattern:

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Determinants – Examples

Example

Find

$$\det \begin{pmatrix} 6 & 8 & 9 \\ 5 & 2 & 1 \\ 7 & 4 & 3 \end{pmatrix}$$

by expanding along a row other than the top row or along a column.

Determinants – Examples

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Find
$$\det\begin{pmatrix}6 & 8 & 9\\ 5 & 2 & 1\\ 7 & 4 & 3\end{pmatrix}$$
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Expanding down the middle column:

$$\det\begin{pmatrix}6 & 8 & 9\\ 5 & 2 & 1\\ 7 & 4 & 3\end{pmatrix}$$

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Expanding down the middle column:

$$\det\begin{pmatrix}6 & 8 & 9\\ 5 & 2 & 1\\ 7 & 4 & 3\end{pmatrix} \quad = \quad -8\det\begin{pmatrix}5 & 1\\ 7 & 3\end{pmatrix} + 2\det\begin{pmatrix}6 & 9\\ 7 & 3\end{pmatrix} - 4\det\begin{pmatrix}6 & 9\\ 5 & 1\end{pmatrix}$$

Determinants – Examples

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Determinants – Examples

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Find the determinant of

$$\begin{pmatrix} 4 & 5 & 8 & -1 \\ 2 & 0 & -2 & -3 \\ 0 & 0 & 2 & 0 \\ 8 & 0 & -4 & -5 \end{pmatrix}$$

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Determinants – Examples

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Find the determinant of

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Determinants – Examples

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Determinants – Examples

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&= 2\left(-5\det \begin{pmatrix} 2 & -3 \\ 8 & -5 \end{pmatrix} + 0 - 0 \right) \\
&= 2 \times (-5) \times 14 = -140.
\end{aligned}$$

Determinants – Examples

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We expanded along the third row, and then the second column, because they contained the most zeros.

Example

Find the determinant of

$$\begin{pmatrix} 4 & 1 & 7 & -1 \\ 0 & 2 & -2 & -3 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Determinants – Examples

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Determinants – Examples

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Determinants – Examples

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&= 4 \left(2 \det \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} - 0 + 0 \right) \\
&= 4 \times 2 \times (-3)
\end{aligned}$$

Determinants – Examples

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Determinants – Examples

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$$
\begin{aligned}
\det\begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\
0 & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\
0 & 0 & a_{33} & a_{34} & \cdots & a_{3n} \\
0 & 0 & 0 & a_{44} & \cdots & a_{4n} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & a_{nn}
\end{pmatrix}
&= a_{11}\det\begin{pmatrix}
a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\
0 & a_{33} & a_{34} & \cdots & a_{3n} \\
0 & 0 & a_{44} & \cdots & a_{4n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{nn}
\end{pmatrix} + 0 + \cdots + 0 \\
&= \cdots \\
&= a_{11}a_{22}a_{33}a_{44}\cdots a_{nn}.
\end{aligned}
$$