MATH1131 CALCULUS REVISION LECTURE 1
REVISION ON SETS AND INEQUALITIES

$$\mathbb{N} = \{0, 1, 2, 3, 4 \dots\}$$

$$\mathbb{Z} = \{\dots -4, -3, -2, -1, 0, 1, 2, 3, 4 \dots\}$$

$$\mathbb{Q} = \{\frac{p}{q} \text{ where } p, q \in \mathbb{Z} \text{ with } q \neq 0\}$$

$\mathbb{R}$ is the set of all real numbers. (the real line)

(a, b) represents the open interval $a < x < b$

$[a, b]$ represents the closed interval $a \leq x \leq b$

This revision sheet is to be completed by you **before** your first lecture on Monday Week 1, 4-5pm. The solutions are available on Moodle. All of my Math1131 Calculus lectures in Weeks 1,2 and 3 will have complete solutions posted on Moodle immediately after the lecture, to accommodate students still overseas.

After Week 3 students will need to attend class to see the theory worked.

Please bring a printout of the relevant lecture to each Math1131 calculus lecture hour. Administration of the calculus strand is outlined at the start of the Lecture 2 notes. Please read these instructions carefully and bring a printout of Lecture 2 to the first lecture hour.

Most of you have already seen some Calculus in high school and we will now build on that theory in a number of interesting ways. Note that our approach at university tends to be a little more formal and rigorous than that used in the schools.

We start with some notation which will streamline our future presentation.

Interval Notation

The backbone of all of your mathematical study has of course been the concept of a number. Starting with the counting numbers

$$\mathbb{N} = \{0, 1, 2, 3, 4 \dots\}$$

we progress to the integers

$$\mathbb{Z} = \{\dots -4, -3, -2, -1, 0, 1, 2, 3, 4 \dots\}$$

the rationals

$$\mathbb{Q} = \{\frac{p}{q} \text{ where } p, q \in \mathbb{Z} \text{ with } q \neq 0\}$$

and finally the set of all real numbers $\mathbb{R}$ which includes such curious creatures as π and $\sqrt{2}$.

Example 1: Graph each of the following sets on the number line:

$$\text{a) } \{x \in \mathbb{Z} : -3.5 < x \leq 3.5\}$$

$$\text{b) } \{x \in \mathbb{R} : -3.5 < x \leq 3.5\}$$

★

It is important to be able to specify subsets of the real line without resorting to the use of variables. To do this we use what is called interval notation.

$$(a, b) \quad \text{represents the open interval} \quad a < x < b$$

and

$$[a, b] \quad \text{represents the closed interval} \quad a \leq x \leq b.$$

Generally speaking "(" and ")" is an instruction to exclude the endpoint while "[" and "]" tells you to include the endpoint.

Example 2: Express each of the following sets in interval notation:

a) $\{x \in \mathbb{R} : 1 < x < 3\}$.

b) $\{x \in \mathbb{R} : 1 \leq x \leq 3\}$.

c) $\{x \in \mathbb{R} : 1 < x \leq 3\}$.

d) $\{x \in \mathbb{R} : 1 \leq x < 3\}$.

$\bigstar$

Note that we often write $\{x \in \mathbb{R} : 1 \leq x < 3\}$ simply as $1 \leq x < 3$.

Example 3: Write out the following intervals using inequalities and sketch on the real numberline:

a) $(-2, 7]$.

b) $[3, \infty)$.

$\bigstar$

Note that ∞ never gets a "]" as it is not a real number and hence cannot be included in the interval.

Unions and Intersections

We now revise some very old theory on unions and intersections.

Given two sets A and B:

$A \cup B$ (read as A union B) corresponds to A **or** B.

$A \cap B$ (read as A intersection B) corresponds to A **and** B.

A^c or $\overline{A}$ (read as A complement) corresponds to **not** A.

Example 4: Display each of the following on a Venn diagram:

a) $A \cup B$.

b) $A \cap B$.

c) A^c.

d) $A \cap B^c$.

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Sketching Polynomials

It is a trivial task to sketch polynomials if they are presented to you in factored form. All you need to remember is that all odd powers $n = 3, 5, 7, \ldots$ will meet the x axis just like a cubic and all the even powers $n = 2, 4, 6, \ldots$ will bounce just like a quadratic. If ever in doubt a simple strategy is to just plot a few points.

Example 5: Sketch each of the following polynomials:

a) $y = (x-1)(x+3)$.

b) $y = (x-3)(5-x)(x+4)$.

c) $y = (x-4)(x+3)^{22}(5-x)^{12}(x-6)^{17}$.

Example 6: Use the sketch in c) above to solve the inequality

$$(x-4)(x+3)^{22}(5-x)^{12}(x-6)^{17} \geq 0$$

★ $(-\infty, 4] \cup [6, \infty) \cup \{5\}$ ★

Inequalities

We adopt a range of different techniques when solving inequalities. Each of the following is slightly different.

Example 7: Solve each of the following inequalities. Sketch your solution on the number line and express your solution in interval notation:

a) $-3x + 1 < 16$.

b) $x^2 \leq 5x - 6$.

c) $\dfrac{4}{x-1} \leq x + 2$.

d) $|5x - 4| > 16$.

e) $|x - 1| > 5 - 2x$.

★ $a)$ $x > -5$ $b)$ $[2, 3]$ $c)$ $[-3, 1) \cup [2, \infty)$ $d)$ $(-\infty, -\dfrac{12}{5}) \cup (4, \infty)$ $e)$ $x > 2$ ★

MATH1131 CALCULUS LECTURE 2
FUNCTIONS

A function $f : A \to B$ is a rule which assigns each $x \in A$ to exactly one element $y \in B$.

$$(f \circ g)(x) = f(g(x))$$

Hello and welcome to Math1131 Calculus.

• You have two Math1131 Calculus lectures in weeks 1,3,5,8,10 and 11, and three lectures in weeks 2,4,7 and 9, as per your timetable. Classes run from Weeks 1-11.

• Note that all classes will run from 5 minutes past the hour to 5 minutes to the hour.

• You do not need to purchase a text book.

• The Math1131 Moodle page contains my skeleton calculus lecture notes. My notes are significantly different from the School's calculus notes on Moodle, and you may wish to also check out the School notes, to consolidate the material after each lecture.

• **Please bring printouts of my notes to each lecture, as you will need to fill out the notes by hand, in class**.

• The exists a preliminary revision lecture plus solutions on Moodle, which should be read in conjunction with this lecture.

• My calculus lectures in Weeks 1,2 and 3 will have complete solutions posted on Moodle immediately after the lecture. From Week 4 onwards students will need to attend class to see the material being worked.

• From the next lecture, students using vertical screens will be asked to sit in the back row. Due to privacy regulations, student photos and/or video recordings of my lectures are not permitted. Candidates seeking exemption due to medical concerns should contact me directly, either by e-mail, or at the end of class.

• Audio recordings of lectures will appear on Moodle. No video recordings of my lectures will be available, as all the work will be done on the board. Students may however access video recordings and lecture notes from other streams.

• Read the course outline carefully to ensure that you are completely familiar with the administrative structure, and assessment details of the course.

• I will finish a little early today so that students with specific questions may have them personally addressed.

FUNCTIONS

The abstract concept of a function is absolutely crucial for our later analysis of calculus. Indeed both integration and differentiations are process which are almost always applied to functions.

Definition: A function $f : A \to B$ is a rule which assigns each $x \in A$ to exactly one element $y \in B$.

The set A is referred to as the domain of f and denoted by $\text{Dom}(f)$, the set B is the co-domain of f and denoted by $\text{Codom}(f)$ and finally the set of all y values produced by the function is call the range of f and denoted by $\text{Range}(f)$.

The co-domain is an announcement of everything that the function **might** produce. The range is what actually **is** produced.

The mental image to have of a function is a little machine which takes x's as input and spits out y's as output.

Remember also that every function has a graph and the graph often reveals its essential features.

Example 1: Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 + 3$. Sketch a graph of f and find $\text{Dom}(f)$, $\text{Codom}(f)$ and $\text{Range}(f)$. Evaluate $f(7)$ and find $f(w - 5)$.

★ $\mathbb{R}$, $\mathbb{R}$, $[3, \infty)$, 52, $w^2 - 10w + 28$ ★

We can also artificially restrict the domain of a function if we need to:

Example 2: $f : (-1, 5) \to \mathbb{R}$ be given by $f(x) = x^2 + 3$. Sketch a graph of f and find Dom(f), Codom(f) and Range(f).

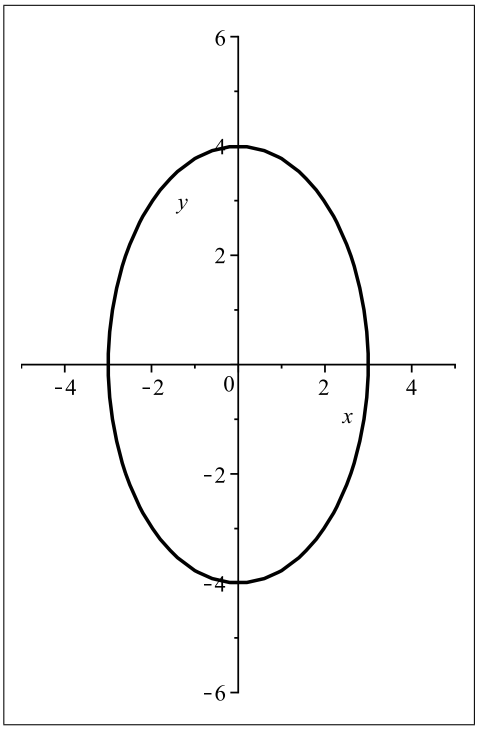
$\bigstar$ $(-1, 5)$, $\mathbb{R}$, $[3, 28)$ $\bigstar$

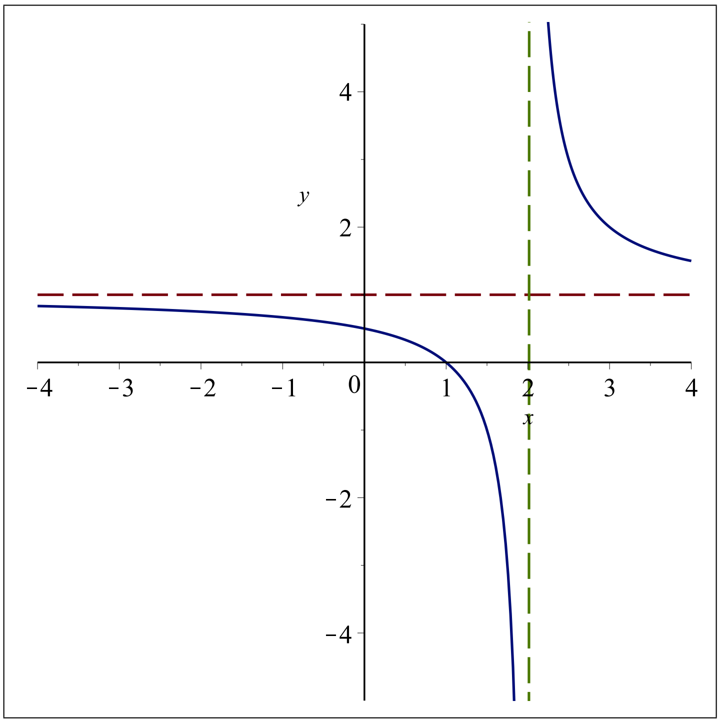
Question: What is the value of $f(7)$ in Example 2 above?

A small technicality before we continue. If a rule produces multiple y values for a single x value we call it a relation rather than a function. This is not the end of the world, however we do prefer the rigid mapping that springs from a definition of a function. It is always a worry to be picking over multiple possible y-values for relations which are not functions however we can cope with it if necessary.

Vertical Line Test: A relation is a function if a vertical line crosses its graph at most once.

Example 3: Identify which of the following are graphs of functions. In each case write down the domain and range.

a) 

b) 

★

When trying to find the domain of a function two issues to consider are that you may not take the square root of a negative number or divide by zero. Ranges are often determined intuitively or from a sketch.

Example 4: Find the domain of each of the following functions:

a) $f(x) = \dfrac{x-2}{x^2-9}$.

b) $f(x) = \sqrt{(x-5)^4(1-x)^3}$.

c) $f(x) = \dfrac{x-5}{x-5}$. What is the range of this function?

★

Example 5: Consider the function f defined to be

$$f(x) = \begin{cases} -x^2 - 1 \ , & x \leq 0; \\ 3x \ , & x > 0. \end{cases}$$

a) Sketch f.
b) Find $f(5)$.
c) Write down the domain and range of f.

★

Functions may be added, subtracted, multiplied and divided in the usual way. A new operation however is that of **composition** $\circ$. It is defined as

Definition: $(f \circ g)(x) = f(g(x))$.

Example 6: Let $f(x) = x^2$ and $g(x) = \dfrac{1}{x-5}$. Find $f \circ g$ and $g \circ f$.

$$\bigstar \quad (f \circ g)(x) = \left(\frac{1}{x-5}\right)^2, \quad (g \circ f)(x) = \frac{1}{x^2-5} \quad \bigstar$$

Observe from the above that in general $f \circ g \neq g \circ f$.

You must be able to sketch all the standard functions and be aware of their domains and ranges. See your printed notes for a comprehensive list of sketches:

Example 7: **HOMEWORK**

Sketch each of the following stating the domain and range in each case :

a) $y = x^2$.

b) $y = x^3$.

c) $y = \sqrt{x}$.

d) $y = e^x$.

e) $y = \ln(x)$.

f) $y = \sin(x)$.

g) $y = \cos(x)$.

h) $y = |x|$.

★

Later on the immense machinery of calculus will be used to sketch graphs. For the moment however we consider intercepts, vertical asymptotes (division by zero) and horizontal asymptotes (behaviour for large x). Sometimes we simply manipulate standard graphs.

LECTURE 3
Limit of Functions at Infinity Part 1

$$\pm\frac{x^2}{a^2} \pm \frac{y^2}{a^2} = 1 \text{ is an ellipse with two ++ and a hyperbola with one +.}$$

$\displaystyle\lim_{x\longrightarrow\infty} \frac{f(x)}{g(x)}$ can be attacked by dividing top and bottom by the dominant term.

$\displaystyle\frac{"0"}{0}$, $\displaystyle\frac{"\infty"}{\infty}$ and $"\infty - \infty"$ are called indeterminate forms.

If $\displaystyle\lim_{x\longrightarrow\infty} f(x) = L$ then $y = L$ is a horizontal asymptote to the graph of $y = f(x)$.

In this lecture we will look carefully at the concept of a limit as $x \to \infty$. But first a few sketches of relations. Ideally we like to start with nice clean functions $y = f(x)$. But sometimes we begin with a horrible tangled mess in the x and y variables.

For example

$$x^2y^3 + e^x \sin(y) = \ln(x^3 + y^4) + \frac{1}{xy}$$

These relations are usually not functions and it is often algebraically impossible to express y in terms of x. It is best not to even try! Later on we will see that calculus may still be used on such exotic creatures through the process of implicit differentiation. For the moment let's just have a look at the graph of a few standard relations.

Example 1: Sketch the graph of each of the following relations. Find the domain of the relation in each case:

a) $x^2+y^2 = 1$ b) $\dfrac{x^2}{9}+\dfrac{y^2}{16} = 1$ c) $\dfrac{x^2}{9}-\dfrac{y^2}{16} = 1$ d) $-\dfrac{x^2}{9}+\dfrac{y^2}{16} = 1$ e) $-\dfrac{x^2}{9}-\dfrac{y^2}{16} = 1$.

★

The limit of a function at infinity $\lim_{x \longrightarrow \infty} f(x)$

We now examine the behaviour of functions for large values of x through the limit $\lim_{x \longrightarrow \infty} f(x)$. Infinity is not really a number so we cannot just evaluate the function as normal. These limits can be done by inspection by considering dominance in numerator and denominator or a little more formally by simply dividing everything by the dominant term. Much like differentiation there is also an extremely formal approach (from first principles) which also must be mastered.

Note that if $\lim_{x \longrightarrow \infty} f(x) = L$ it means that $y = L$ is a horizontal asymptote for the function. That is for large x the function will settle down to $y = L$.

Consider the problem $\lim_{x \longrightarrow \infty} \frac{12x^2 - 2x + 1}{4x^2 + 7x + 6}$.

This is asking "What happens to $\frac{12x^2 - 2x + 1}{4x^2 + 7x + 6}$ when x gets really big".

There are a number of possible answers. Maybe the function also gets big, maybe it gets really small or perhaps it approaches a non-zero number. There is no ∞ button on your calculator so lets try evaluating $\frac{12x^2 - 2x + 1}{4x^2 + 7x + 6}$ at something really big.

At $x = 100$ $\quad \frac{12x^2 - 2x + 1}{4x^2 + 7x + 6} \approx 2.943$

At $x = 10000$ $\quad \frac{12x^2 - 2x + 1}{4x^2 + 7x + 6} \approx 2.999$

It looks like $\lim_{x \longrightarrow \infty} \frac{12x^2 - 2x + 1}{4x^2 + 7x + 6} = 3$. But how do we prove this?

We have a number of methods. First a really sloppy approach:

Method 1 (only look at powerful terms in the numerator and denominator)

Method 2 (divide top and bottom by the dominant term)

Method 3 (Next Lecture: A formal attack)

Some terminology

If you "imagine" putting infinity into $\dfrac{12x^2 - 2x + 1}{4x^2 + 7x + 6}$ you get $\dfrac{\text{“}\infty\text{”}}{\infty}$.

We call $\dfrac{\text{“}\infty\text{”}}{\infty}$ an indeterminate form, as it has no real meaning.

Other indeterminate forms are $\dfrac{\text{“}0\text{”}}{0}$, 0^{∞}, ∞^{0}, , $\infty - \infty$, and $0 \times \infty$.

Note that $\dfrac{\text{“}0\text{”}}{0}$ and $\dfrac{\text{“}\infty\text{”}}{\infty}$ must not naively be assigned a value of 1 !!
It is generally true however that $\dfrac{0}{\infty}$ can be evaluated as 0 and $\dfrac{\infty}{0}$ is unbounded.

We have a host of tricks to knock these off.

Remember to keep the $\lim\limits_{x\longrightarrow\infty}$ in your argument to the very end, that is until you actually take the limit! Marks are often lost for letting $\lim\limits_{x\longrightarrow\infty}$ disappear too early. Let's take a look:

Example 2: Evaluate each of the following limits.

a) $\lim\limits_{x\longrightarrow\infty} \dfrac{1}{x^3}$

b) $\lim\limits_{x\longrightarrow\infty} \dfrac{x^3}{1}$

c) $\lim\limits_{x\longrightarrow\infty} \dfrac{10x^3 - 2x^2 + 8x + 1}{5x^3 + 12x^2 + 11x - 2}$

d) $\lim\limits_{x\longrightarrow\infty} \dfrac{10x^3 - 2x^2 + 8x + 1}{5x^4 + 12x^2 + 11x - 2}$

e) $\lim\limits_{x \longrightarrow \infty} \dfrac{10x^4 - 2x^2 + 8x + 1}{5x^3 + 12x^2 + 11x - 2}$

f) $\lim\limits_{x \longrightarrow \infty} \dfrac{12 + x - 16x^2}{8x^2 - 2x + 5}$

g) $\lim\limits_{x \longrightarrow \infty} \dfrac{4x - 3}{\sqrt{25x^2 + 24x + 23}}$

h) $\lim\limits_{x \longrightarrow \infty} \dfrac{x^2 - 3x + \sin(x)}{25x^2 + \cos(33x) + 1}$

i) $\lim\limits_{x \longrightarrow \infty} \dfrac{x^2 - 3x + 1000000000000000000000000000000000000000000000000000\sin(x)}{25x^2 + 1000000000000000000000000000000000000000000000000000000000\cos(33x) + 1}$

j) $\lim_{x \longrightarrow \infty} \sqrt{x^2 - 8x} - x$

This is "$\infty - \infty$". There is a special trick for this one!

Let's first cheat: $x = 10000 \Longrightarrow \sqrt{x^2 - 8x} - x = -4.0008$. Looks like the limit is -4.

k) $\lim_{x \longrightarrow \infty} \dfrac{20e^x + 3x^2 + \sin(x)}{5e^x + 25x^2 + 1}$

Always remember that for large x, e^x is more powerful than ANY polynomial.

l) $\lim_{x \longrightarrow \infty} \sin(x)$

m) $\lim_{x \longrightarrow \infty} e^{-x} \sin(x)$

n) $\lim_{x \longrightarrow \infty} \sin\left(\dfrac{1}{x}\right)$

★

LECTURE 4
Limit of Functions at Infinity Part 2

$\lim\limits_{x \longrightarrow \infty} f(x) = L$ if for every (small) positive real number ϵ there is a (large) real number M with the property that if $x > M$ then $|f(x) - L| < \epsilon$.

Limits of the form $\lim\limits_{x \longrightarrow a} f(x)$ are best attacked via factorisation.

$\lim\limits_{x \longrightarrow a^-} f(x)$ is the limit of $f(x)$ as x approaches a from **the left**

$\lim\limits_{x \longrightarrow a^+} f(x)$ is the limit of $f(x)$ as x approaches a from **the right**

$\lim\limits_{x \longrightarrow a} f(x)$ exists if and only if $\lim\limits_{x \longrightarrow a^-} f(x)$ and $\lim\limits_{x \longrightarrow a^+} f(x)$ both exist and are equal.

Just like differentiation from first principles we also have a formal definition of $\lim\limits_{x \longrightarrow \infty} f(x)$. It is crucial that we understand these fundamental definitions as they provide a solid foundation for the analysis. We can't just keep waving our hands around! If we want you to evaluate a limit formally we will always give you a clear warning.

Definition: $\lim\limits_{x \longrightarrow \infty} f(x) = L$ if for every (small) positive real number ϵ there is a (large) real number M with the property that if $x > M$ then $|f(x) - L| < \epsilon$.

Discussion

Note that $|f(x) - L| < \epsilon$ simply means that the difference between $f(x)$ and L is really really small. The limit definition is then simply saying that any degree of closeness between $f(x)$ and L can be generated by choosing x to be sufficiently large.

For $\lim\limits_{x \longrightarrow \infty} f(x)$ to be equal to L the function must eventually $(x > M)$ get into (and stay in!) an ϵ band of L.

Example 1: Consider $\lim_{x \longrightarrow \infty} \dfrac{10x-4}{5x+7}$.

a) Show that the value of the limit is $L = 2$.

b) Find M so that $f(x)$ is within $\dfrac{1}{1000}$ of its limit whenever $x > M$.

c) Does b) prove that $\lim_{x \longrightarrow \infty} \dfrac{10x-4}{5x+7} = 2$?

d) Prove from the limit definition that $\lim_{x \longrightarrow \infty} \dfrac{10x-4}{5x+7} = 2$.

That is given $\epsilon > 0$ find M with the property that if $x > M$ then $|f(x) - L| < \epsilon$.

---

a)

b)

So $\left|\dfrac{10x-4}{5x+7} - 2\right| \leq \dfrac{1}{1000}$ provided we choose $x > 3598.6$. Note that we are not really trying to solve $\left|\dfrac{10x-4}{5x+7} - 2\right| \leq \dfrac{1}{1000}$. We are just trying to find an M with the property that the inequality is true for all $x > M$. This value of M is not unique and any $M > 3598.6$ would also work

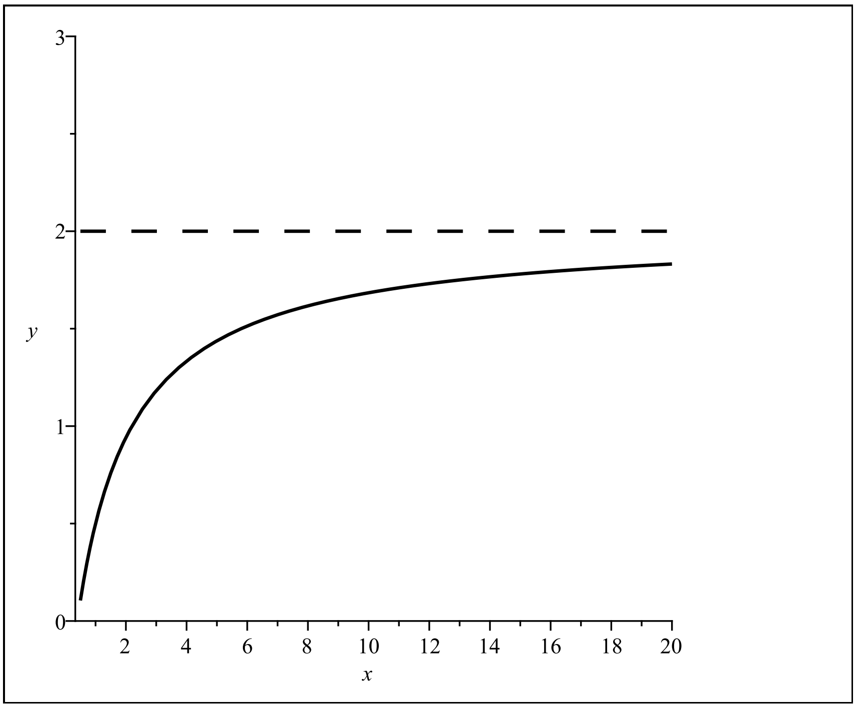
c) NO!!

d)

$$\bigstar \quad M = \frac{18}{5\epsilon} - \frac{7}{5}. \quad \bigstar$$

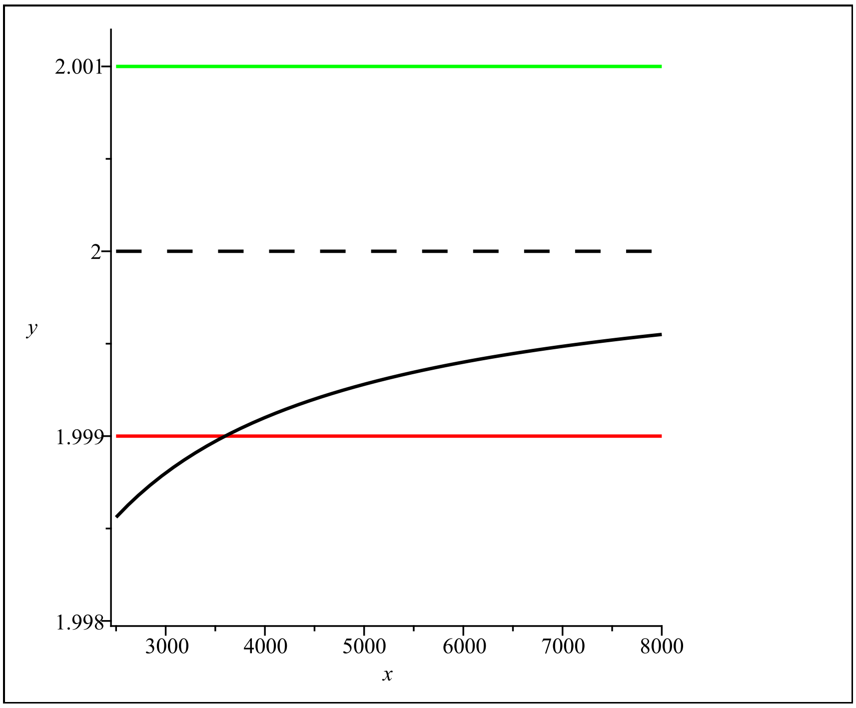
Note that since the above M depends on ϵ we sometimes write $M(\epsilon) = \dfrac{18}{5\epsilon} - \dfrac{7}{5}$.

Let's take a look at the graph for the situation of part b) where $\epsilon = \dfrac{1}{1000} = 0.001$. First the the function and its limit of $L = 2$ as a horizontal asymptote:



Next a band (red to green) of length $\pm\dfrac{1}{1000}$ either side of the limit $L = 2$.

Observe below that after $M = 3598.6$ the function is clearly locked within this limit band.



Example 2: Find $\lim_{x \longrightarrow \infty} \dfrac{\sin(x)}{x^2 + e^x}$ and prove from the limit definition that your answer is correct. Find a value of M for $\epsilon = \dfrac{1}{100}$ and sketch the behaviour of the function and its limit.

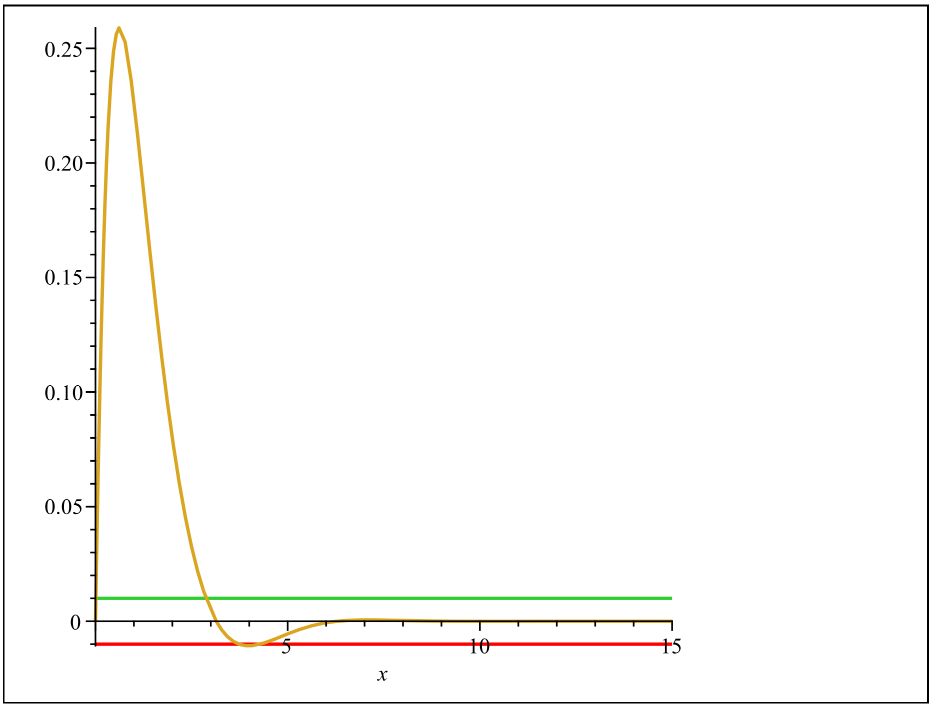
Clearly $\lim_{x \longrightarrow \infty} \dfrac{\sin(x)}{x^2 + e^x} =$

So $L = 0$. We now examine $|f(x) - L| = \left| \dfrac{\sin(x)}{x^2 + e^x} - 0 \right| =$

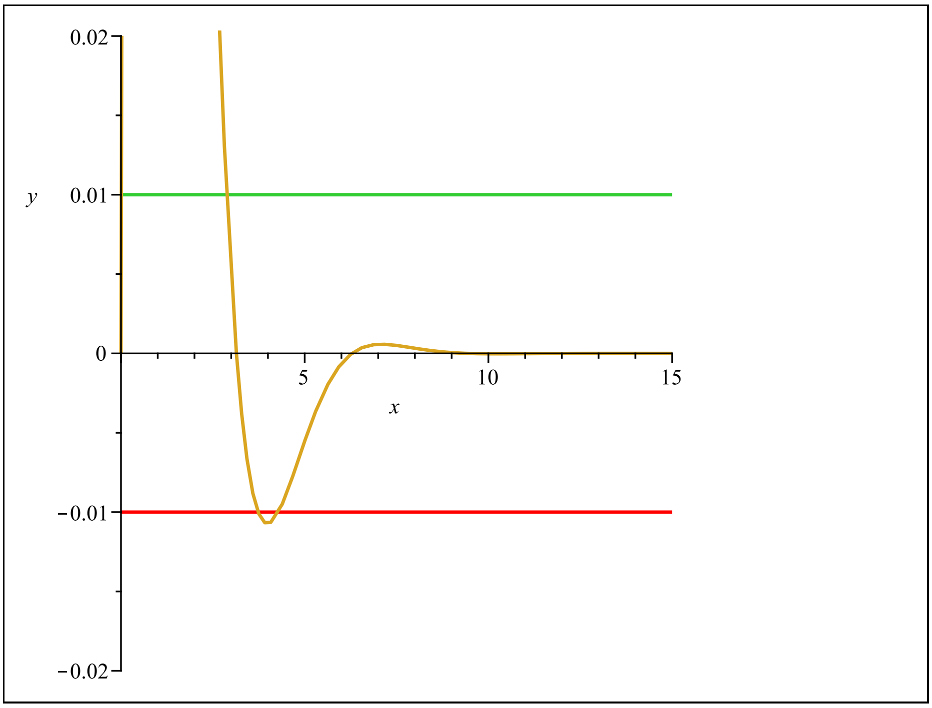
So we have $|f(x) - L| < \dfrac{1}{x^2}$. we therefore only need $\dfrac{1}{x^2} < \epsilon$. And hence:

★ $M = 10$ ★

Let's take a look at the graph:



Observe below that after $M = 10$ the function is clearly within its $\frac{1}{100}$ limit band although a smaller value of $M = 5$ would also do!!



Note that above we have most definitely **NOT** proven that $\lim_{x \longrightarrow \infty} \frac{\sin(x)}{x^2 + e^x} = 0$! The full proof would demand that we use a general small ϵ rather than the specific $\frac{1}{100}$.

Limit of Functions at a Point

There are certain situations where a function fails to be defined **at** a point but it is perfectly happy **near** the point. We then use $\lim\limits_{x \longrightarrow a} f(x)$ to get a feeling for the behaviour of the function.

Consider the following table of values for $f(x) = \dfrac{x^2 - 25}{x - 5}$ near $x = 5$.

x	4.9	4.99	5	5.01	5.1
y	9.9	9.99	?	10.01	10.1

It is clear that the function is trying to get to 10 at $x = 5$ even though it is undefined there. We write

$$\lim_{x \longrightarrow 5} \frac{x^2 - 25}{x - 5} = 10$$

and say

"The limit as x approaches 5 of $\dfrac{x^2 - 25}{x - 5}$ is 10".

Example 3: Find $\lim\limits_{x \longrightarrow 5} \dfrac{x^2 - 25}{x - 5}$ and hence sketch the function $y = \dfrac{x^2 - 25}{x - 5}$.

★

So for limits as x approaches a finite value our main technique is to factorise top and bottom. Always check first that you are facing the indeterminate form "$\frac{0}{0}$"

Example 4: Evaluate each of the following limits

a) $\lim\limits_{x \longrightarrow 2} \dfrac{x^2 - 5x + 6}{x^2 - 4}$

b) $\lim\limits_{x \longrightarrow 1} \dfrac{x^2 + 1}{x^2 + 4}$

$\bigstar \quad a) \; -\dfrac{1}{4} \quad b) \; \dfrac{2}{5} \quad \bigstar$

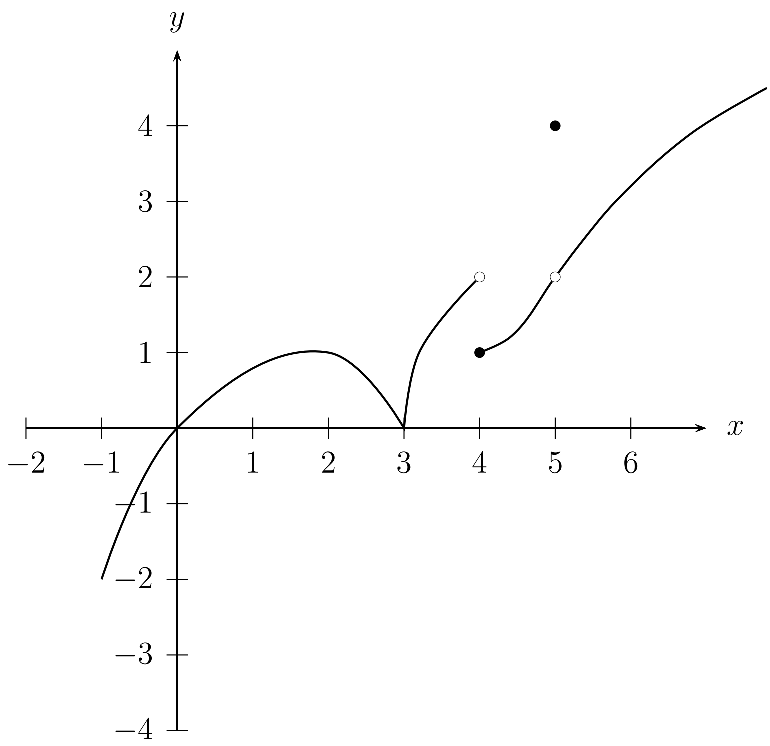
In order to define these limits a little more formally we require the concept of a one-sided limit:

$\lim\limits_{x \longrightarrow a^-} f(x)$ is the limit of $f(x)$ as x approaches a from **the left**

$\lim\limits_{x \longrightarrow a^+} f(x)$ is the limit of $f(x)$ as x approaches a from **the right**

We can then say that the full limit $\lim\limits_{x \longrightarrow a} f(x)$ formally exists if and only if $\lim\limits_{x \longrightarrow a^-} f(x)$ and $\lim\limits_{x \longrightarrow a^+} f(x)$ both exist and are equal.

Example 5: Consider the graph of $y = f(x)$ presented below:



For each of the following either evaluate the given quantity or explain why it does not exist:

a) $\lim\limits_{x \longrightarrow 4^-} f(x)$ b) $\lim\limits_{x \longrightarrow 4^+} f(x)$

c) $\lim\limits_{x \longrightarrow 4} f(x)$ d) $\lim\limits_{x \longrightarrow 5} f(x)$

e) $f(5)$ f) $\lim\limits_{x \longrightarrow 6} f(x)$

g) $\lim\limits_{x \longrightarrow 3} f(x)$

LECTURE 5
Continuity and the Pinching Theorems

Suppose that $f(x) \le g(x) \le h(x) \quad \forall x \in (b, \infty)$ and $\lim_{x \longrightarrow \infty} f(x) = \lim_{x \longrightarrow \infty} h(x) = L$. Then $\lim_{x \longrightarrow \infty} g(x) = L$.

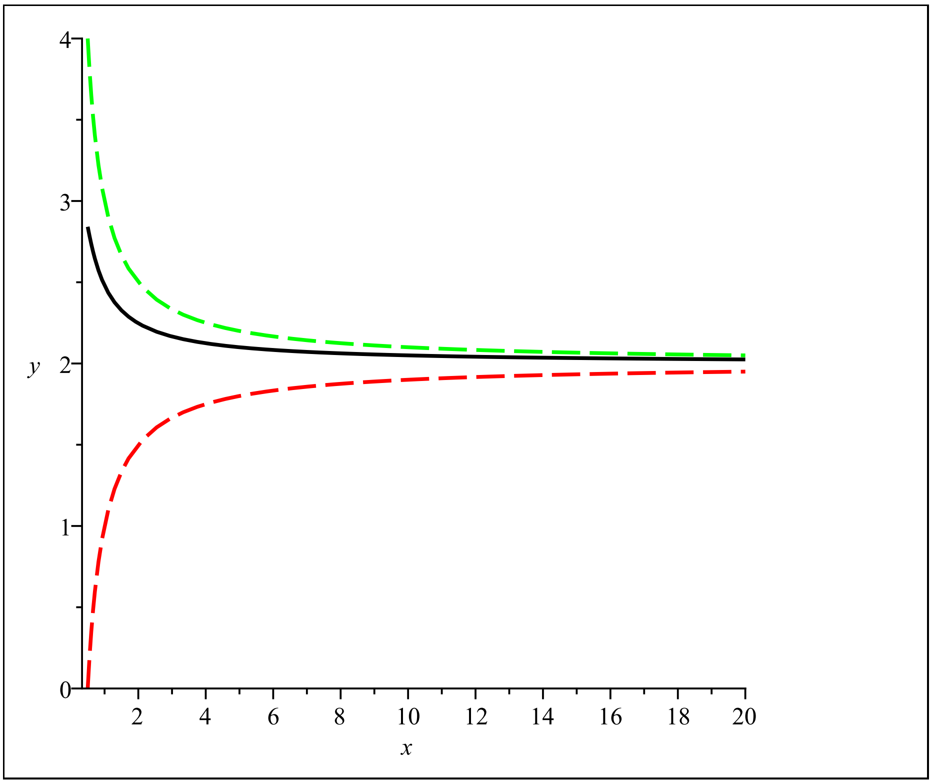
Let I be an open interval containing a point a and suppose that $f(x) \le g(x) \le h(x) \quad \forall x \in I$ (except perhaps at a). Assume also that $\lim_{x \longrightarrow a} f(x) = \lim_{x \longrightarrow a} h(x) = L$. Then $\lim_{x \longrightarrow a} g(x) = L$.

A function f is said to be continuous at $x = a$ if $\lim_{x \longrightarrow a} f(x) = f(a)$. Else it is discontinuous.

We now turn to two formal results called "Pinching Theorems" which will help us to pin down certain limits. Note that these theorems are also called Sandwich Theorems or Squeeze theorems. But first some new notation:

We have two special symbols which we use as abbreviations. Firstly $\forall$ is to be read as "for all" and also $\exists$ mean there exists. These two just make it a little easier to write down the mathematics. Now the Pinching theorem:

Pinching Theorem: Suppose that $f(x) \le g(x) \le h(x) \quad \forall x \in (b, \infty)$ and $\lim_{x \longrightarrow \infty} f(x) = \lim_{x \longrightarrow \infty} h(x) = L$. Then $\lim_{x \longrightarrow \infty} g(x) = L$.



This should make sense! In the above h is green (top) and f is red (bottom). Both

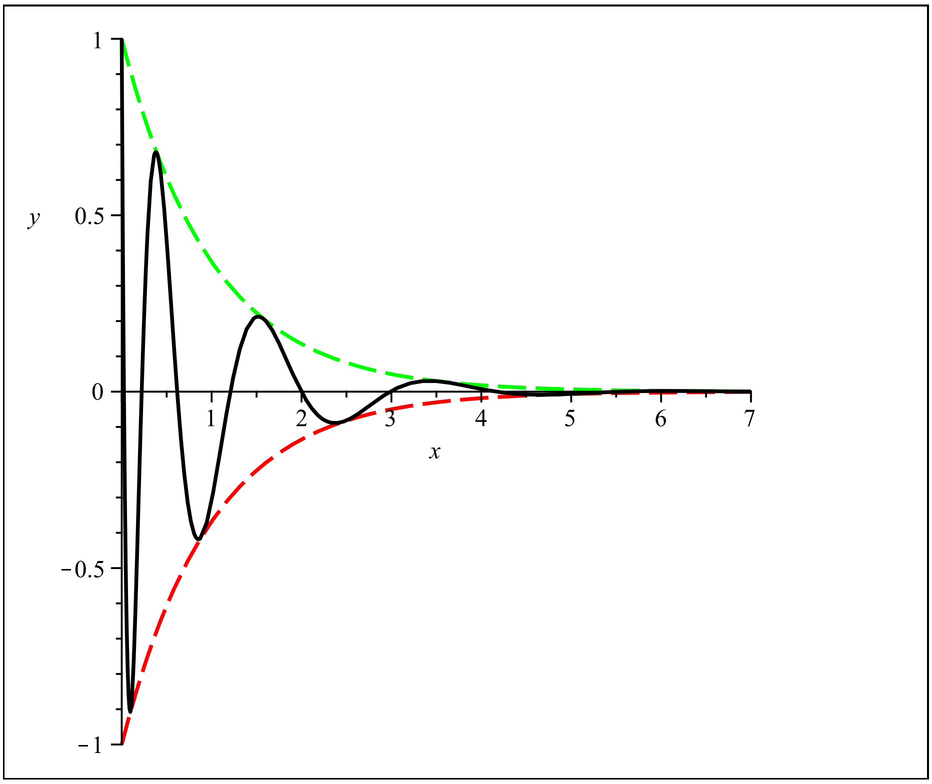
h and f approach 2 as $x \to \infty$. Poor old g (black) is pinched between h and f and as a consequence has no choice but to also approach 2 as $x \to \infty$.

We usually warn you if we expect a formal use of the pinching theorem.

Example 1: Use the pinching theorem to find $\lim\limits_{x \longrightarrow \infty} e^{-x}\cos(10\sqrt{x})$.

We begin with the simple observation that $-1 \le \cos(10\sqrt{x}) \le 1$. Then

Therefore by the pinching theorem $\lim\limits_{x \longrightarrow \infty} e^{-x}\cos(10\sqrt{x}) = 0$



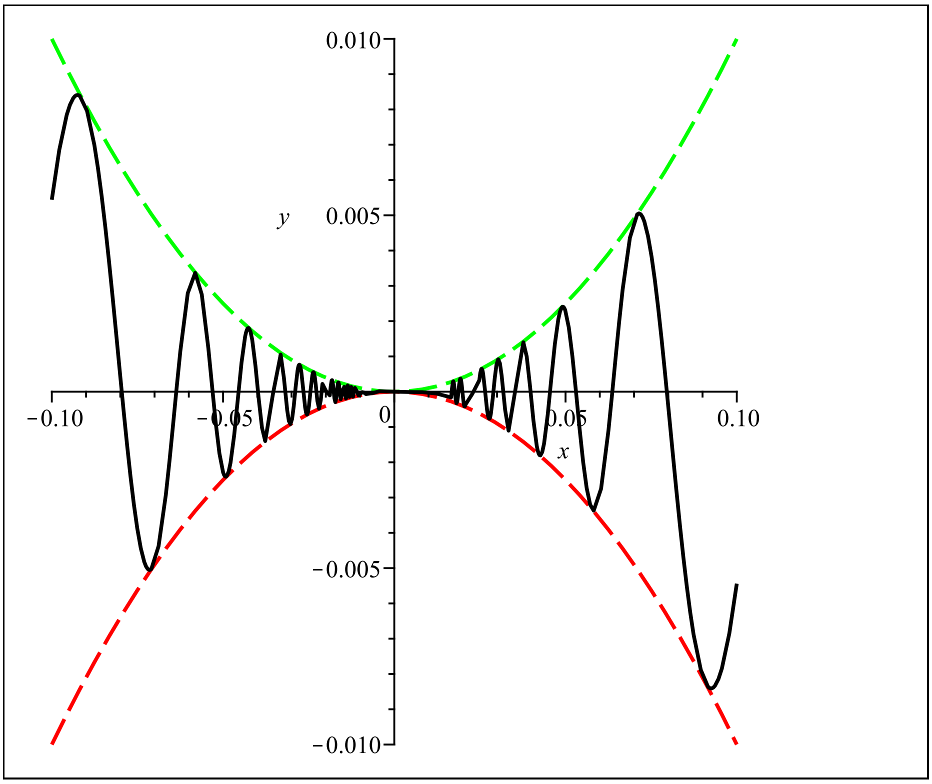
★

There is also a finite version of the pinching theorem:

Pinching Theorem: Let I be an open interval containing a point a and suppose that $f(x) \leq g(x) \leq h(x) \quad \forall x \in I$ (except perhaps at a). Assume also that $\lim_{x \longrightarrow a} f(x) = \lim_{x \longrightarrow a} h(x) = L$. Then $\lim_{x \longrightarrow a} g(x) = L$.

Example 2: Use the pinching theorem to show $\lim_{x \longrightarrow 0} x^2 \sin(1/x) = 0$.

★



Before examining the concept of continuity lets do a little revision on the existence of limits:

Example 3:

a) Find $\lim\limits_{x \longrightarrow 5^+} \dfrac{|\,x-5\,|}{x-5}$.

b) Find $\lim\limits_{x \longrightarrow 5^-} \dfrac{|\,x-5\,|}{x-5}$.

c) Does $\lim\limits_{x \longrightarrow 5} \dfrac{|\,x-5\,|}{x-5}$ exist ?

The best way to do these absolute value questions is to express the function in piecemeal form and draw a simple graph of the situation:

★

Example 4: Does $\lim_{x \longrightarrow 2} \dfrac{|\, x^2 - 4 \,|}{x - 2}$ exist ?

★

Continuity

We prefer our functions to be continuous. Abrupt jumps in graphs are a worry as the function then varies over large ranges in infinitesimal time intervals. For example one of the greatest concerns when trading in the stock-market are shares which close at one price on Friday afternoon and open at a drastically different price at the resumption of trading on Monday morning, leaving the trader with no chance to respond.

We will present a formal definition of continuity soon however for the moment you may think of a continuous function as one whose graph may be sketched without taking your pen off the paper.

Two different types of discontinuities are:

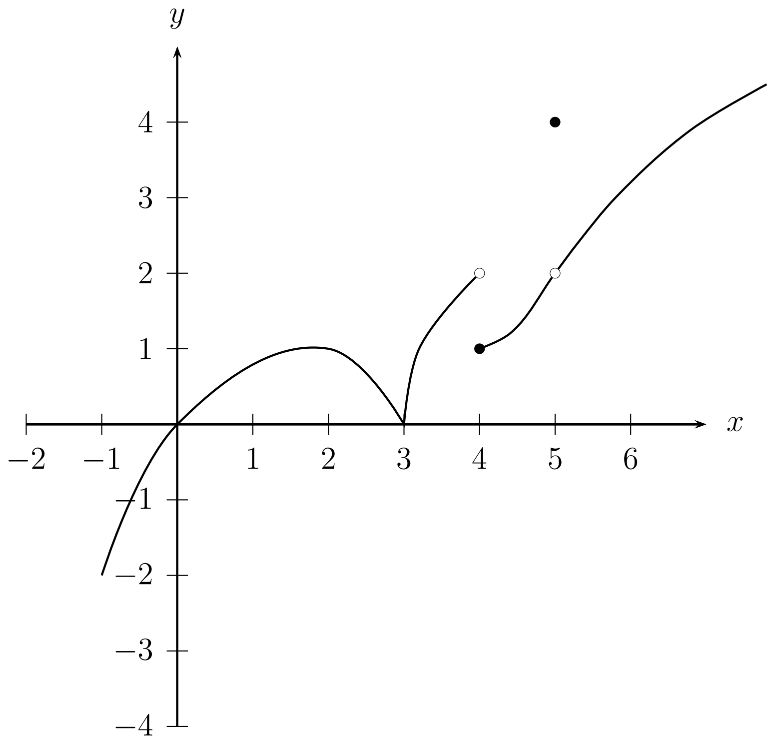
(I) A jump discontinuity:

(II) A removable discontinuity:

Definition: A function f is said to be continuous at a point $x = a$ in its domain if $\lim\limits_{x \longrightarrow a} f(x) = f(a)$. Else it is discontinuous at $x = a$.

What this means is that a function is continuous at $x = a$ if its behaviour **near** a perfectly matches its behaviour **at** a. Note that for continuity we need two things! The limit must exist and also be equal to the function value. You also have the intuitive idea of the pen lifting off the paper to help you out.

Example 5: Consider the graph of $y = f(x)$ presented below:



By considering appropriate limits determine whether the function is continuous at $x = 4, 5$ and 3. Classify the discontinuities.

$x = 4$:

$x = 5$:

$x = 3$:

★

Example 6: Let $g(x) = \begin{cases} \dfrac{x^2 - 49}{x - 7}, & x \neq 7; \\ \\ \alpha, & x = 7 \end{cases}$

Sketch the graph of g and find α so that g is continuous at $x = 7$.

★

Example 7: Let $f(x) = \dfrac{x^2 - 49}{x - 7}$. Write down dom($f$) and find the values of x for which f discontinuous.

This function is continuous over its domain but not continuous over $\mathbb{R}$. We would **not** classify $x = 7$ as a point of discontinuity since $x = 7$ is not in the domain of the function f.

★ dom(f)=all real x except $x = 7$. ★

Example 8: Let $f(x) = \begin{cases} \dfrac{|\,x^2-4\,|}{x-2}, & x \neq 2; \\ \\ \alpha, & x = 2 \end{cases}$

Is there an α for which f is continuous at $x = 2$?

Recall that we showed in Example 4 that $\lim\limits_{x \longrightarrow 2} f(x)$ does not exist since the left and the right hand limits do not agree. That is the end of continuity at $x = 2$! No α will fix the situation.

★

LECTURE 6
Properties of Continuous Functions

The Intermediate Value Theorem: Suppose that f is a continuous function on the closed interval $[a, b]$. Then if z lies between $f(a)$ and $f(b)$ there will be at least one real number c in $[a, b]$ with the property that $z = f(c)$.

The Maximum-Minimum Theorem: A continuous function on a closed interval will always attain both a maximum and a minimum value over the interval.

Continuous functions are very well behaved. Most of the elementary functions including all polynomial functions, e^x, $\ln(x)$, $\sin(x)$ and $\cos(x)$ are continuous over their domains. In this lecture we will examine the features that make continuous functions so appealing.

Recall that a function f is said to be continuous at $x = a$ if $\lim\limits_{x \longrightarrow a} f(x) = f(a)$. What this means is that a function is continuous at $x = a$ if its behaviour **near** a perfectly matches its behaviour **at** a. Note that for continuity we need two things! The limit must exist and also be equal to the function value. You also have the intuitive idea of the pen lifting off the paper to help you out.

First note that the sum, product and difference of continuous functions is clearly again continuous.

Example 1: Is the quotient of continuous functions over $\mathbb{R}$ always continuous over $\mathbb{R}$?

★ No ★

Piecemeal functions often have continuity problems, however it is also possible to carefully patch the pieces together in a continuous fashion.

Example 2: Let $f(x) = \begin{cases} x^2, & x \leq 3; \\ \\ -5x + b, & x > 3 \end{cases}$

Sketch the situation and find the value of b which makes f continuous.

Note that f is clearly continuous everywhere except at $x = 3$ so we only need to consider what's happening at $x = 3$. Now

★ $b = 24$ ★

The Intermediate Value Theorem: Suppose that f is a continuous function on the closed interval $[a, b]$. Then if z lies between $f(a)$ and $f(b)$ there will be at least one real number c in $[a, b]$ with the property that $z = f(c)$.

The proof of this theorem is surprisingly difficult and will not be examined.

Lets draw a picture of the situation. Sketch a continuous graph over a random closed interval $[a, b]$ in the x axis.

Note that a closed interval $[a, b\,]$ takes the form $a \leq x \leq b$ and MUST include the endpoints. Thus $[a, b) \equiv \{x \in \mathbb{R} \,|\, a \leq x < b\}$, $(a, b] \equiv \{x \in \mathbb{R} \,|\, a < x \leq b\}$ and $(a, b) \equiv \{x \in \mathbb{R} \,|\, a < x < b\}$ are all NOT closed intervals.

The intermediate value theorem should make a lot of sense! All it says is that continuous functions map closed intervals on the x axis over to closed intervals on the y axis with no possibility of gaps or holes.

An immediate corollary (consequence) is that if a continuous function $f(x)$ changes sign over a closed interval $[a, b]$ then the interval will contain a solution to the equation $f(x) = 0$.

Example 3: Prove that the equation $e^x = x^2 + 4$ has a solution somewhere in the closed interval $1 \leq x \leq 3$.

First note that $e^x = x^2 + 4 \leftrightarrow e^x - x^2 - 4 = 0$. So we let $f(x) = e^x - x^2 - 4$ and note that f is clearly continuous since all of its components are continuous. (always mention continuity!!) Now:

$f(1) \approx -2.28$ and $f(3) = 7.08$.

Since $-2.28 \leq 0 \leq 7.08$ it follows from the intermediate value theorem that there is $c \in [1, 3]$ such that $f(c) = 0$, as required.

★

Question: Could there be more than one such c ?

★

Question: Does the intermediate value theorem help you to find c ?

★ Not at all ★

Note also that there is $c \in [1, 3]$ such that $f(c) = 6$, though this isn't quite as interesting.

Be careful to make sure that your calculator is in radian mode if you do an example like the one above involving any of the trig functions.

Example 4: Let $f(x) = \begin{cases} x, & 2 \leq x \leq 3; \\ x+5, & 3 < x \leq 4 \end{cases}$ be a function defined over the closed interval $[2, 4]$.

Sketch:

Then $f(2) = 2$ and $f(4) = 9$. Noting that $2 \leq 7 \leq 9$ is there a $c \in [2, 4]$ such that $f(c) = 7$? Does this example violate the intermediate value theorem?

★ NO. The function is not continuous so the theorem does not apply. ★

We now turn to the Maximum Minimum Theorem for continuous functions. First a careful definition.

Definition: Suppose that f is defined on a closed interval $[a, b]$.

(a) We say that a point c in $[a, b]$ is an *absolute minimum point* for f on $[a, b]$ if $f(c) \leq f(x)$ for all x in $[a, b]$. The corresponding value $f(c)$ is called the *absolute minimum value* of f on $[a, b]$. If f has an absolute minimum point on $[a, b]$ then we say that *f attains its minimum on* $[a, b]$.

(b) We say that a point d in $[a, b]$ is an *absolute maximum point* for f on $[a, b]$ if $f(x) \leq f(d)$ for all x in $[a, b]$. The corresponding value $f(d)$ is called the *absolute maximum value* of f on $[a, b]$. If f has an absolute maximum point on $[a, b]$ then we say that *f attains its maximum on* $[a, b]$.

An absolute maximum point and an absolute minimum point are sometimes referred to as a *global maximum point* and a *global minimum point.*

All we are really saying above is that c is an *absolute maximum point* of f on $[a, b]$ if $f(c)$ is bigger than or equal to any other y value over the interval. Similarly for *absolute minimum point.*

There are a number of crucial facts to observe. Firstly the definitions of absolute maxima and minima have nothing to do with calculus! We use derivatives as a testing mechanism later but the definitions themselves do not require such sophisticated concepts.

Secondly absolute maxima or minima of a function may or may not exist over an interval. We see in the next theorem that if f is continuous then the function will always attain its max and min over any closed interval.

The Maximum-Minimum Theorem: A continuous function on a closed interval will always attain both a maximum and a minimum value over the interval.

Sketch:

Continuous functions over a closed interval will always achieve both a maximum value and a minimum value. Note that these do not have to be a turning points!!

Example 5: Consider $f(x) = |2x - 5|$ over the closed interval $[-1, 3]$.

a) Is f continuous over the interval?
b) How many stationary points does f have?
c) What is the maximum and minimum value of f over $[-1, 3]$?

★ *a*) Yes *b*) None *c*) Max=7 and Min=0 ★

Example 6: Let $f(x) = x^2 + 4$ over the interval $-1 < x < 3$.

a) Is f continuous over the interval?

b) What is the maximum and minimum value of f over $(-1, 3)$?

c) Does this violate the Max/Min Theorem?

★ *a)* Yes *b)* Max D.N.E. and Min=4 *c)* No, The interval is not closed ★

Example 7: Consider $f(x) = \begin{cases} -x, & -1 \leq x < 2; \\ 0, & 2 \leq x \leq 3 \end{cases}$ over the closed interval $[-1, 3]$.

a) Is f continuous over the interval?

b) What is the maximum and minimum value of f over $[-1, 3]$?

c) Does this violate the Max/Min Theorem?

★ *a)* No *b)* Max=1 and Min D.N.E *c)* No, The function is not continuous ★

Observe from the above example that we need both continuity and for the interval to be closed to draw the conclusions of The Max/Min theorem.

Example 8: Sketch the graph of a discontinuous function f defined over the open interval $(-1, 3)$ with the property that f attains both a maximum and a minimum value over the interval. Does this violate the max/min theorem ?

$\star$ *No* $\star$

LECTURE 7
Differentiable Functions

$$\frac{dy}{dx} = f'(x) = \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h}$$

If $f'(x) > 0$ for all x in an interval (a, b) then f is increasing on (a, b).

If $f'(x) < 0$ for all x in an interval (a, b) then f is decreasing on (a, b).

If $f'(x) = 0$ for all x in an interval (a, b) then f is constant on (a, b).

If $y = f(x)g(x)$ then $\dfrac{dy}{dx} = f'(x)g(x) + g'(x)f(x)$.

If $y = \dfrac{f(x)}{g(x)}$ then $\dfrac{dy}{dx} = \dfrac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$.

If $y = f(g(x))$ then $\dfrac{dy}{dx} = f'(g(x))g'(x)$.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

$$\frac{d}{dx}(\tan(x)) = \sec^2(x)$$

$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$$

* Note that we must use radians when dealing with the calculus of trig functions.

We turn now to differentiation and the calculus. This deals with the central problem of calculating the gradient of a function at any point. First developed by Gottfried Leibniz and Sir Isaac Newton in the late 1600's the derivative provides us with the perfect tool for calculating instantaneous rates of change.

Almost every application of mathematics to the physical sciences involves calculus in one way or another, so we will make a very detailed and formal attack on the theory. I will assume that you are already familiar with the basics of calculus from your high school studies. If not you will need to devote some extra time to make sure you understand this lecture, which is essentially just revision of your high school calculus.

When faced with the problem of calculating the gradient of $y = f(x)$, Newton's great realization was that it was possible to quickly and accurately calculate the gradient function $\frac{dy}{dx}$. The gradient function is called the derivative of $y = f(x)$ and is also often denoted by $f'(x)$. The formal limit definition of the derivative

$$\frac{dy}{dx} = f'(x) = \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h}$$

is rarely used! Instead we calculate $\frac{dy}{dx}$ through an increasingly sophisticated system of algorithms!

The term "gradient" is also referred to as slope or rate of change of the function. If the gradient (that is derivative) is positive the curve is increasing. A negative derivative signals a decreasing function and a zero derivative is usually an indication of a local max or min.

Example 1: Find the gradient of $y = 7x^2 - 5x + 3$ at the point where $x = 2$.

Using the facts that $\frac{d}{dx}(x^n) = nx^{n-1}$ and that the process of differentiation respects linearity we can easily find $\frac{dy}{dx}$:

★

Example 2: Find the derivative of the function $y = 7x^2 - 5x + 3$ from the previous example from first principles.

Discussion:

★

Example 3: Find the derivative for each of the following:

a) $y = 8x^2 + 2\sqrt{x} + \dfrac{1}{x^3} + 4$

b) $y = e^x \sin(x)$

c) $y = \dfrac{2t+1}{3t-2}$

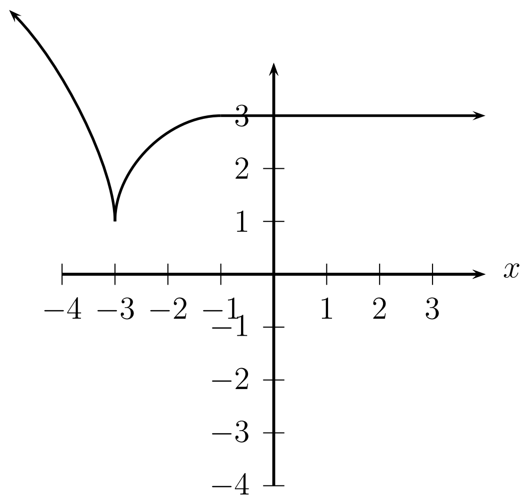
d) $y = (\ln(x) + 1)^{14}$

★

Example 4: For the following graph of $y = f(x)$ determine the value(s) of x for which:

a) $\dfrac{dy}{dx}$ is positive b) $\dfrac{dy}{dx}$ is negative

c) $\dfrac{dy}{dx}$ is zero d) $\dfrac{dy}{dx}$ is undefined



★

Note that the function above is continuous but not differentiable. Differentiability is a much stronger condition that continuity! A differentiable function is always continuous but a continuous function is not always differentiable (consider the absolute value function). Note also that any point where a function is discontinuous will automatically be a point of non-differentiability.

Example 5: A function $y = f(x)$ defined over the interval $-1 \leq x \leq 5$ has the the following 5 properties:

a) $f'(x) < 0$ for $-1 < x < 1$.

b) $f'(1) = 0$.

c) $f'(x) > 0$ for $1 < x < 3$.

d) $f'(3)$ is undefined; and

e) $f'(x) = 0$ for $3 < x < 5$.

Draw possible sketch of the graph of f.

★

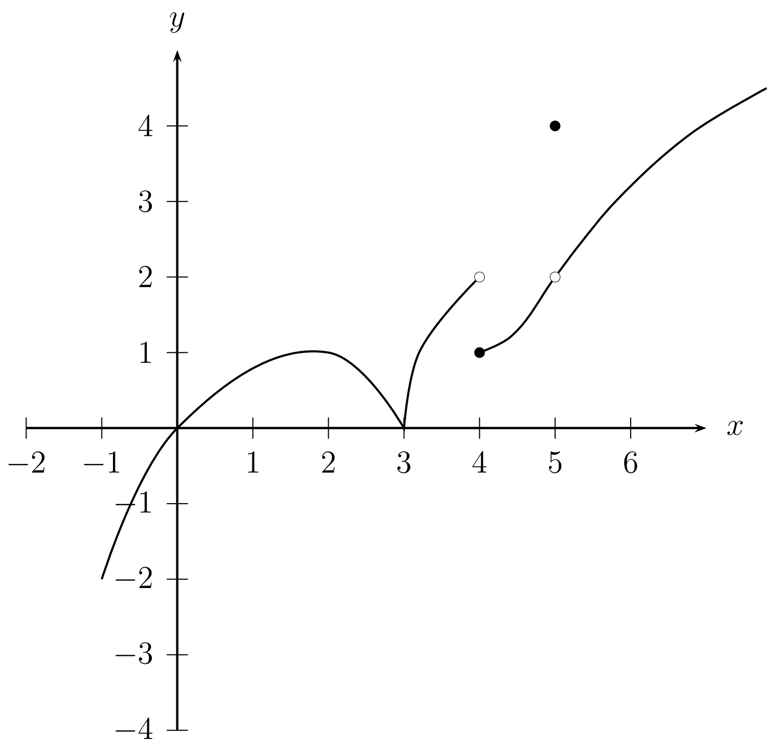
Example 6: A function $y = f(x)$ has a tangent at the point P on its graph. Sketch an example where:

a) The tangent meets the curve at more than one point.

b) The tangent cuts across the curve at P.

★

Example 7: Consider the graph of $y = f(x)$ presented below:



a) For which value(s) of x is the function discontinuous.

b) For which value(s) of x is the function non-differentiable.

★

Example 8: At which point(s) on the curve $y = x^3$ is the tangent parallel to the line $y = 12x + 13$?

★

Example 9: Let $f(x) = \begin{cases} \cos(x), & x < 0; \\ 3x + 1, & x \geq 0. \end{cases}$

a) Is f continuous at $x = 0$?
b) Is f differentiable at $x = 0$?

A Sketch:

a)

b) Let $p(x) = \cos(x)$ and $q(x) = 3x + 1$.

$p'(x) =$ $\quad\quad\quad \to p'(0) =$

$q'(x) =$ $\quad\quad\quad \to q'(0) =$

★

LECTURE 8

Split Functions, Implicit Differentiation and Related Rates

$$\text{Implicit Differentiation} \leftrightarrow \frac{d}{dx}(f(y)) = \frac{d}{dy}(f(y))\frac{dy}{dx}.$$

Differentiating a relation between x and y implicitly with respect to t will produce a new relation between the rates of change $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

A split function is usually constructed from two or more differentiable component functions. To verify (or force) the differentiability of such a split function we simply need to first verify that the pieces join up (continuity) and then that they *join smoothly* (differentiability) by showing that the derivatives match up properly.

Example 1: Find all real values of a and b such that the function defined by

$$f(x) = \begin{cases} 1 - x^2, & x < 2; \\ \\ ax + b, & x \geq 2. \end{cases}$$

is differentiable at $x = 2$.

A sketch:

Let $p(x) = 1 - x^2$ and $q(x) = ax + b$. It helps to name the pieces.

We first demand that the function be continuous at $x = 2$. That is the pieces must join, and hence $p(2) = q(2)$:

We next force the weld to be smooth! Thus we require that $p'(2) = q'(2)$. That is:

$p'(x) =$ $\qquad\qquad \to p'(2) =$

$q'(x) =$ $\qquad\qquad \to q'(2) =$

★ $a = -4$ $\qquad b = 5$ ★

Implicit Differentiation

Usually when you differentiate, your starting point is a nice clean function $y = f(x)$. But sometimes you need to start with a horrible messy relation instead, for example $x^2 + y^3 + 4y^2 = 3$. It can be difficult or even impossible to write y in terms of x. We can still find the derivative $\dfrac{dy}{dx}$ but need to use **implicit differentiation**. First a simple skill.

Example 2: If $\dfrac{3}{7} = \dfrac{3}{11} \times \dfrac{*}{*}$ what is $\dfrac{*}{*}$?

★ $\dfrac{11}{7}$ ★

Implicit differentiation is little more than the above trick!

Example 3: Find $\dfrac{dy}{dx}$ if $x^2 + y^3 + 4y^2 = 3$.

★ $\dfrac{-2x}{3y^2 + 8y}$ ★

Example 4: Find $\dfrac{dy}{dx}$ if $\sin(x) + e^y = \ln(y) + x^3$

$$\bigstar \quad \frac{3x^2y - y\cos(x)}{ye^y - 1} \quad \bigstar$$

Example 5: Find the equation of the tangent to $x^2y^5 + 3y - 2x = 3$ at the point $(0, 1)$.

$$\bigstar \quad 2x - 3y + 3 = 0 \quad \bigstar$$

Related Rates

Differentiating a relation between x and y implicitly with respect to t will produce a new relation between the rates of change $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

Example 6: Suppose that the surface area S (in m^2) of a human body is related to its weight W (in kg) by

$$S^3 = \frac{W^2}{512}$$

a) Bob weighs 64 kg. What is the surface area of his body?

b) Find a relation between $\frac{dS}{dt}$ and $\frac{dW}{dt}$.

c) Prove that if Bob's weight were to change in any way, the rate of change of his surface area would be $\frac{1}{48}$ the rate of change of his weight.

★ *a)* S=2 *b)* $3S^2\frac{dS}{dt} = \frac{W}{256}\frac{dW}{dt}$ *c) Proof* ★

Example 7: A spherical balloon is inflated at a rate of 100 m^3/sec. Determine the rate at which the radius is increasing when

a) $r = 5\text{m}$.

b) $V = 36\pi\ \text{m}^3$.

Our first task is to find a relationship between the central variables which remains fixed throughout the entire process. This is of course the volume formula for a sphere:

$$V = \frac{4}{3}\pi r^3$$

$$\bigstar \quad a)\ \frac{1}{\pi}\ m/sec \quad b)\ \frac{100}{36\pi}\ m/sec \quad \bigstar$$

Error Estimates (Homework)

This topic was of enormous importance before the advent of calculators but is now a bit dated.

Recall that $\dfrac{\Delta y}{\Delta x} \approx \dfrac{dy}{dx}$ and hence $\Delta y \approx \dfrac{dy}{dx}\Delta x$. This gives us a way of estimating errors.

Example 8: Find an error estimate when approximating $\sqrt{9.001}$ by $\sqrt{9}$.

We have $x = 9$ and $\Delta x = 0.001$.

Let $y = \sqrt{x}$. Then $\dfrac{dy}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$.

Now $\Delta y \approx \dfrac{dy}{dx}\Delta x \to \Delta y \approx \dfrac{1}{2\sqrt{x}}\Delta x = \dfrac{1}{2\sqrt{9}}(0.001) = \dfrac{1}{6000}$.

★

LECTURE 9

The Mean Value Theorem

The Mean Value Theorem: Suppose that f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is at least one real number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

We now turn to one of the central theorems in calculus the Mean Value Theorem.

Suppose that you travel to Wollongong from Sydney and your average speed on the journey is 73 km/hr. The Mean Value Theorem guarantees that there will be at least on time on your journey where your speedometer reading is also 73. In other words you can't have an average speed of 73 km/hr without at least once traveling at 73 km/hr. Makes sense!

The Mean Value Theorem: Suppose that f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is at least one real number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Discussion and Sketch:

$\dfrac{f(b) - f(a)}{b - a}$ is the average rate of change of f over the entire interval $[a, b]$ while $f'(c)$ is the instantaneousness rate of change of f at $x = c$.

★

The jumping around from open to closed intervals is not really all that important but must be mentioned. It stems from the problem that the end of of an arc has no tangent.

You must be able to accurately state the mean value theorem for the exams ! The proof (available in your printed notes) is not examinable. Lets take a look at the theorem in action.

Example 1: Find c which satisfies the conclusions of the mean value theorem for $f(x) = x^3$ on the closed interval $[1, 4]$.

Sketch:

★ $c = \sqrt{7} \approx 2.6$ ★

Example 2: Show that $f(x) = |\,x|$ over the closed interval $[-1, 4]$ does not satisfy the conclusion of the mean value theorem. Explain.

★

One the the sneaky applications of the mean value theorem is the verification of certain inequalities. The next two examples are from your printed notes but are done slightly differently:

Example 3: Use the mean value theorem to prove that $\ln(x) < x - 1$ whenever $x > 1$.

We will show that $x - 1 - \ln(x) > 0$ whenever $x > 1$.

Note that it would also be OK to show that $\ln(x) - x + 1 < 0$ but it is a little easier to show that something is positive rather than negative.

Consider $f(t) = t - 1 - \ln(t)$ over the closed interval $[1, x]$.

Note that we have introduced the variable t for the function to avoid overloading on the $x's$.

Then f is continuous on the closed interval $[1, x]$ and differentiable on the open interval $(1, x)$ so the mean value theorem may be applied.

That is there is $c \in (1, x)$ such that $\dfrac{f(x) - f(1)}{x - 1} = f'(c)$
Now

$f(x) = x - 1 - \ln(x)$

$f(1) = 1 - 1 - 0 = 0$

$f'(t) = 1 - \dfrac{1}{t} \to f'(c) = 1 - \dfrac{1}{c}$.

Therefore $\dfrac{(x - 1 - \ln(x)) - 0}{x - 1} = 1 - \dfrac{1}{c} \to x - 1 - \ln(x) = (1 - \dfrac{1}{c})(x - 1)$.

So we have that $x - 1 - \ln(x) = (1 - \dfrac{1}{c})(x - 1)$.

Therefore all we need to do now is prove that $(1 - \dfrac{1}{c})(x - 1) > 0$.

Well $(x - 1)$ is clearly not negative since $x > 1$. Also

$c > 1 \longrightarrow \dfrac{1}{c} < 1 \longrightarrow -\dfrac{1}{c} > -1 \longrightarrow 1 - \dfrac{1}{c} > 0$.

So $(1 - \dfrac{1}{c})(x - 1) > 0$.

Thus $x - 1 - \ln(x) > 0$ whenever $x > 1$ as required.

★

These are quite tricky. Let's do another one:

Example 4: Use the mean value theorem to prove that $\sqrt{x+4}-2 < \frac{x}{4}$ whenever $x > 0$.

We will show that $\frac{x}{4} - \sqrt{x+4} + 2 > 0$ whenever $x > 0$.

Consider $f(t) = \frac{t}{4} - \sqrt{t+4} + 2$ over the closed interval $[0, x]$.

Then f is continuous on the closed interval $[0, x]$ and differentiable on the open interval $(0, x)$ so the mean value theorem may be applied.

That is there is $c \in (0, x)$ such that $\frac{f(x) - f(0)}{x - 0} = f'(c)$. Now

$f(x) = \frac{x}{4} - \sqrt{x+4} + 2$ so $f(0) = 0$.

$f'(t) = \frac{1}{4} - \frac{1}{2}\frac{1}{\sqrt{t+4}} \longrightarrow f'(c) = \frac{1}{4} - \frac{1}{2}\frac{1}{\sqrt{c+4}}$.

Therefore via the M.V.T. there exists $c \in (0, x)$ such that:

$$\frac{(\frac{x}{4} - \sqrt{x+4} + 2) - (0)}{x - 0} = \frac{1}{4} - \frac{1}{2}\frac{1}{\sqrt{c+4}} \longrightarrow \frac{x}{4} - \sqrt{x+4} + 2 = x(\frac{1}{4} - \frac{1}{2}\frac{1}{\sqrt{c+4}}).$$

So all we need to show is that $x(\frac{1}{4} - \frac{1}{2}\frac{1}{\sqrt{c+4}}) > 0$.

Well $x > 0$. So we only need to verify that

$$\frac{1}{4} - \frac{1}{2}\frac{1}{\sqrt{c+4}} > 0 \quad \text{if} \quad c > 0.$$

Now $\frac{1}{4} - \frac{1}{2}\frac{1}{\sqrt{c+4}} > 0 \iff \frac{1}{4} > \frac{1}{2}\frac{1}{\sqrt{c+4}} \iff 1 > \frac{2}{\sqrt{c+4}}$

$\iff 1 > \frac{4}{c+4} \iff c + 4 > 4 \iff c > 0$.

Thus $\frac{x}{4} - \sqrt{x+4} + 2 > 0$ whenever $x > 0$ as required.

★

Example 5: By considering $f(t) = \sin(t)$ over the interval $[x, y]$ show that

$$|\sin(y) - \sin(x)| \leq y - x \text{ for all } x, y \in \mathbb{R}, x < y.$$

We begin by noting that $f(t) = \sin(t)$ is continuous over $[x, y]$ and differentiable over (x, y). Hence by the M.V.T. there exists $c \in (x, y)$ such that $\frac{f(y) - f(x)}{y - x} = f'(c)$.

Hence

Your printed notes present a detailed revision of the theory of maxima and minima at this stage and you are encouraged to read the section carefully before the next lecture. An important note that I wish to stress however, is that the definitions of maxima and minima, increasing and decreasing **have nothing to do with calculus!!!** These are simple concepts that can be easily explained to a five year old child. We certainly use calculus as tools in this arena but the concepts themselves do not require the sledgehammer of the derivative. Lets have a look at the definition of an increasing function.

Definition: A function f is increasing on an interval I if for every two points a and b in I

$$a < b \longrightarrow f(a) < f(b)$$

In other words it's going up!

Example 6: Explain why $f(x) = x^3$ is increasing over every interval.

But doesn't $y = x^3$ have a stationary point at $x = 0$? That is a zero derivative?

It certainly has but that doesn't stop the function from increasing (going up) everywhere!

It is important to get the direction of the theorems correct.

For an open interval I we have

Theorem: If $f'(x) > 0$ for all $x \in I$ then f is increasing on I.

That is, a function with a positive derivative is increasing. This is **NOT** saying that an increasing function has a positive derivative! We have seen that $y = x^3$ is an increasing function without a positive derivative!

We close the lecture with a proof of Theorem 5.5.3. In this proof we are trying to link the concept of increasing with the concept of a positive derivative. It is situations like this where the MVT saves the day.

Proof:
Let $a, b \in I$ be such that $a < b$. We must show that $f(a) < f(b)$.

From the fact that f is differentiable on I and Theorem 4.5.1, f is continuous on $[a, b]$ and differentiable on (a, b). Invoking the mean value theorem there is $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

That is

$$f(b) - f(a) = (b - a)f'(c)$$

But $(b - a) > 0$ and $f'(c) > 0$ so $f(b) - f(a) > 0$ and hence $f(a) < f(b)$ as required.

★

LECTURE 10
Extrema and L'Hopital's Rule

If $f'(c) = 0$ and $f''(c) < 0$ then $(c, f(c))$ is a local maximum.

If $f'(c) = 0$ and $f''(c) > 0$ then $(c, f(c))$ is a local minimum.

A continuous function on a closed interval will attain a global minimum and a global maximum. Furthermore these extrema will always occur at a critical point, that is a stationary point, an endpoint or a point of non-differentiability.

L'Hopitals Rule: Suppose that f and g are differentiable and that $\dfrac{f(a)}{g(a)}$ is the indeterminate form $\dfrac{"0"}{0}$ or $\dfrac{"\infty"}{\infty}$. Then $\displaystyle\lim_{x\longrightarrow a} \frac{f(x)}{g(x)} = \lim_{x\longrightarrow a} \frac{f'(x)}{g'(x)}$.

Maxima and Minima

Stationary points are some of the most important points on the graph of a function. The are found by setting $f'(x) = 0$ and categorised by considering the sign of the second derivative at the stationary point or alternatively the behaviour of the first derivative around the point in question.

Discussion of the problem

We are often faced with word problems where it is required to maximise or minimise a quantity of interest. Our approach is to:

a) Find a formula for the thing to be maximised or minimised.

b) Clean the formula up so that there is only one independent variable.

c) Hit it with the calculus. Don't forget to test for max/min!

Example 1: Two garbage dumps A and B are 1km apart on a straight road. At a distance of x from dump A the intensity of the smell is given by $I_A = \frac{1}{x}$. Dump B smells four times as bad, so that at a distance of x from dump B the intensity of the smell is $I_B = \frac{4}{x}$.

Pepe wants to build his house at a point P between A and B. Suppose that P is a distance of t km from A.

a) Show that the total intensity of the smell at P is

$$I = \frac{1}{t} + \frac{4}{1-t}.$$

b) Where should Pepe build his house in order to minimise the total smell?

★ $\frac{1}{3}$ km from A ★

A continuous function on a closed interval will attain a global minimum and a global maximum. Furthermore these extrema will always occur at a critical point, that is a stationary point, an endpoint or a point of non-differentiability. When dealing with continuous functions over closed intervals don't forget to check out the cusps and endpoints when searching for maxima and minima !

Example 2: Find the maximum and minimum value of the continuous function

$f(x) = |\, x^2 - 4|$ over the closed interval $[-3, 6]$.

Sketch:

★ Max of 32 and a min of 0 ★

Note in the above example that the stationary point didn't even get used! The extrema for continuous function over closed intervals can come from three different sources! Stationary points, Non-differential points or endpoints.

Counting Zeros

An analysis of stationary points for a polynomial can often reveal the number of times the polynomial cuts the x axis.

Example 3: Determine how many real numbers satisfy the equation

$$2x^7 + 7x^4 + 70x + 120 = 0$$

We first note that $p(2) = 628$ and $p(-2) = -164$. The polynomial is certainly continuous, thus by the IVT there is a zero in $[-2, 2]$.

Let us now try to find the stationary points:

The derivative is never equal to 0. Thus there are no turning points and the polynomial must be monotonically increasing or decreasing. Since $p'(0) = 70$ the polynomial is increasing over $\mathbb{R}$.

Sketch:

Clearly there is only one zero. Once you know all the stationary points for a well behaved function the number of zeros becomes quite clear. Keep in mind however that knowing how many there are in no way helps you to actually find them!

★ There is only one root ★

L'Hopital's rule

We close the lecture with a truly wonderful way of finding all sorts of limits called l'Hopital's rule.

Earlier in this course we had a look at limits and considered some specific methods of solution. L'Hopital's rule is a remarkable technique for evaluating limits that can be used to attack some of the limits that we have dealt with so far, together with new much more complicated examples. As a bonus it is a very easy rule to use even though it does involve a little differentiation.

L'Hopital's Rule: Suppose that f and g are differentiable and that $\dfrac{f(a)}{g(a)}$ is the indeterminate form $\dfrac{"0"}{0}$ or $\dfrac{"\infty"}{\infty}$. Then $\displaystyle\lim_{x\longrightarrow a}\frac{f(x)}{g(x)}=\lim_{x\longrightarrow a}\frac{f'(x)}{g'(x)}$.

This means that if you start with $\dfrac{"0"}{0}$ or $\dfrac{"\infty"}{\infty}$ then all you have to do is differentiate the top and differentiate the bottom and try again! If you then get an answer with the new problem then that is also the answer to the old problem!!

L'Hopital's rule also works with $\displaystyle\lim_{x\longrightarrow\infty}\frac{f(x)}{g(x)}$ and one sided limits $\displaystyle\lim_{x\longrightarrow a^{\pm}}\frac{f(x)}{g(x)}$.

The proof of l'Hopital's rule is in your printed notes and is not examinable. It essentially revolves around the fact that the ratio of function values may be identified with the ratio of their rates of change through the mean value theorem.

To start with let's revisit some of the examples from earlier in the course. We will do them the old way and then see how they can be handled with l'Hopital's rule. We denote the use of l'Hopital's rule by $\overset{l'h}{=}$.

Example 4: Evaluate each of the following limits the old way and then using l'Hopital's rule.

a) $\displaystyle\lim_{x\longrightarrow 2} \frac{x^2-5x+6}{x^2-4}$

b) $\displaystyle\lim_{x\longrightarrow \infty} \frac{10x-4}{5x+7}$

★ $a) \quad -\dfrac{1}{4} \qquad b) \quad 2$ ★

Feel free to use l'Hopital's rule whenever you need to evaluate a limit. But keep in mind that you need to start with $\frac{"0"}{0}$ or $\frac{"\infty"}{\infty}$ before it can be applied. l'Hopital's rule does not work well on exotic examples involving the matching of left and right hand limits.

Lets take a look at all the tricks ! Keep in mind that l'Hopital's rule has nothing to do with the quotient rule rule for differentiation. We differentiate the top and bottom separately.

Example 5: Evaluate each of the following limits:

a) $\lim\limits_{x \longrightarrow 0} \dfrac{e^{3x} - 1}{\sin(x)}$

b) $\lim\limits_{x \longrightarrow 0} \dfrac{\sin(4x)}{\sin(7x)}$

c) $\lim\limits_{x \longrightarrow \infty} \dfrac{x^2}{e^{3x}}$

d) $\lim_{x \longrightarrow 0^+} x\ln(x)$

e) $\lim_{x \longrightarrow \infty} (1+\frac{1}{x})^x$

This one is very tricky. Let $y = \lim_{x \longrightarrow \infty} (1+\frac{1}{x})^x$. Then

$$\ln(y) = \lim_{x \longrightarrow \infty} \ln(1+\frac{1}{x})^x = \lim_{x \longrightarrow \infty} x\ln(1+\frac{1}{x}) = \infty \times 0 = \lim_{x \longrightarrow \infty} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}} = \frac{"0"}{0}$$

$$\overset{l'h}{=} \lim_{x \longrightarrow \infty} \frac{\frac{1}{1+\frac{1}{x}} \times \frac{-1}{x^2}}{\frac{-1}{x^2}} = \lim_{x \longrightarrow \infty} \frac{1}{1+\frac{1}{x}} = 1. \text{ Therefore } \ln(y) = 1 \longrightarrow y = e.$$

$\bigstar$ *a)* 3 *b)* $\frac{4}{7}$ *c)* 0 *d)* 0 *e)* e $\bigstar$

Do be careful how you use the rule!! You must have at the start $\frac{"0"}{0}$ or $\frac{"\infty"}{\infty}$.

Example 6: Evaluate $\lim_{x \longrightarrow 3} \frac{x^2+1}{3x+5}$

$\bigstar$ $\frac{5}{7}$ $\bigstar$

Example 7: Evaluate $\lim_{x \longrightarrow \infty} \frac{2x-\sin(x)}{5x-\sin(x)}$

$\bigstar$ $\frac{2}{5}$ $\bigstar$

What went wrong??

LECTURE 11

Inverse Functions

To find f^{-1} swap and solve.

$$(f^{-1} \circ f)(x) = x \qquad \text{and} \qquad (f \circ f^{-1})(x) = x \ .$$

Dom(f)=Range(f^{-1}) and Range(f)=Dom(f^{-1})

The graph of f^{-1} is the graph of f reflected in the line $y = x$.

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

The concept of a function is all about transformation. For example the incredibly simple function $y = f(x) = 2x + 5$ transforms $x = 1$ to $y = 7$ and $x = 4$ to $y = 13$. Whenever there is change however, we are also interested in undoing that change. This is accomplished through the use of the inverse function f^{-1} whose sole job is to undo whatever f did. That is $f^{-1}(7) = 1$ and $f^{-1}(13) = 4$. We can find the equation for f^{-1} by swapping y and x and solving for y. Note that in general $f^{-1} \neq \dfrac{1}{f}$!!.

Example 1: Find a formula for f^{-1} for the function $y = f(x) = 2x + 5$ defined above. Check that $f^{-1}(7) = 1$ and $f^{-1}(13) = 4$.

★

Fact: $(f^{-1} \circ f)(x) = x$ for all $x \in \text{Dom}(f)$.

Discussion:

Example 2: Check that $(f^{-1} \circ f)(x) = x$ for f in example 1.

★

Example 3: Find a formula for f^{-1} for the function $y = f(x) = 3e^{2x}$ and check that $(f^{-1} \circ f)(x) = x$.

★ $f^{-1}(x) = \frac{1}{2}\ln(\frac{x}{3})$ ★

We do however have a technical problem when it comes to inverses. Consider the function $y = g(x) = x^2$. Then $g(-3) = 9$ and $g(3) = 9$.

What is $g^{-1}(9)$? Is it 3 or -3. Clearly its both and a function must never give us such a choice! What this means is that for $y = g(x) = x^2$, g^{-1} is a relation rather than a function. This is not the end of the world but we would prefer that this didn't happen.

Definition: A function f is said to be 1-1 if

$$f(a) = f(b) \to a = b.$$

We will see later that if f is 1-1 then it will always have a unique inverse function f^{-1}.

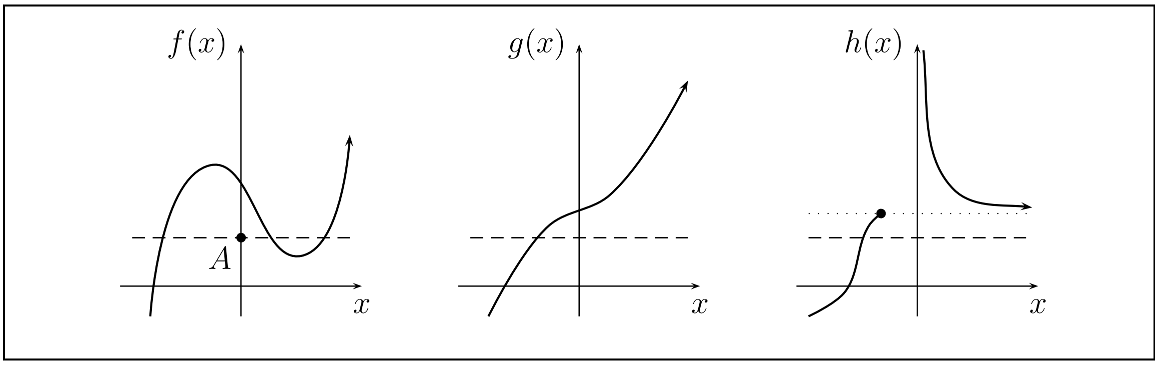
Example 4: Prove that $y = f(x) = 2x + 5$ is 1-1 and that $y = g(x) = x^2$ is not 1-1.

$\bigstar$

1-1 functions may also be identified graphically via the horizontal line test. Recall that the vertical line test established whether or not a relation was a function. The horizontal line test works in much the same way and tests whether or not a function has an inverse:

The Horizontal Line Test: A function f is 1-1 if and only if every horizontal line cuts the graph of f at most once.

Consider the functions graphed below.



f is not one-to-one because the dotted horizontal line passing through the point A cuts the graph of f more than once;

g is one-to-one (in fact, since g is increasing, every horizontal line can cut the graph of g graph no more than once);

h is also one-to-one (even though it is not always increasing).

Fact: A 1-1 function f (that is a function which passes the horizontal line test) will have a unique inverse function f^{-1}.

Some other facts regarding inverse functions:

- Dom(f)=Range(f^{-1}) and Range(f)=Dom(f^{-1})

- The graph of f^{-1} is the graph of f reflected in the line $y = x$.

- $(f^{-1} \circ f)(x) = x$ for all $x \in \text{Dom}(f)$.

Example 5: Consider the function $y = f(x) = x^2 + 5$.

a) Explain why the function f^{-1} does not exist.

b) Restrict Dom(f) so that f becomes a 1-1 function g which has an inverse.

c) If $f = g$?

d) Sketch the restricted function g and its inverse g^{-1} on the same set of axes.

e) Write down Dom(g), Range(g), Dom(g^{-1}) and Range(g^{-1}) in interval notation.

f) Find g^{-1}.

g) Show that $(g^{-1} \circ g)(x) = x$.

★

Example 6: Let $f(x) = 2x^5 + x^3 + x - 10$. Prove that f has an inverse function.

We have a problem here! The sketch is unclear and it is difficult to prove algebraically that f is 1-1. But we have one extra trick:

Fact: An increasing or decreasing function is 1-1.

Discussion:

Now $f'(x) =$

★

We close with the an important result which helps us to find the derivative of the inverse:

Fact: If f is differentiable and has an inverse f^{-1} then the derivative of the inverse $(f^{-1})'$ is given by

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

In other words the derivative of the inverse is one over the derivative of the original function evaluated at the inverse.

Proof: We start with $f(f^{-1}(x)) = x$. Differentiating both sides and using the chain rule yields:

★

Example 7: Let $f(x) = 3x + \cos(x)$. Show that f^{-1} exists on $\mathbb{R}$ and without actually finding f^{-1} evaluate $(f^{-1})'(1)$.

Since $f'(x) = 3 - \sin(x)$ we have $f'(x) > 0$ for all $x \in \mathbb{R}$ and hence f is an increasing function implying that f has an inverse.

Note also that the Range of f is $\mathbb{R}$ which is in turn the Domain of f^{-1}.

Now $(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))}$

What is $f^{-1}(1)$????

$\bigstar \quad \frac{1}{3} \quad \bigstar$

LECTURE 12
Inverse Trigonometric Functions

$$\sin^{-1} : [-1,1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\cos^{-1} : [-1,1] \longrightarrow [0,\pi]$$

$$\tan^{-1} : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\sin^{-1}(-x) = -\sin^{-1}(x)$$

$$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$$

$$\tan^{-1}(-x) = -\tan^{-1}(x)$$

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$$

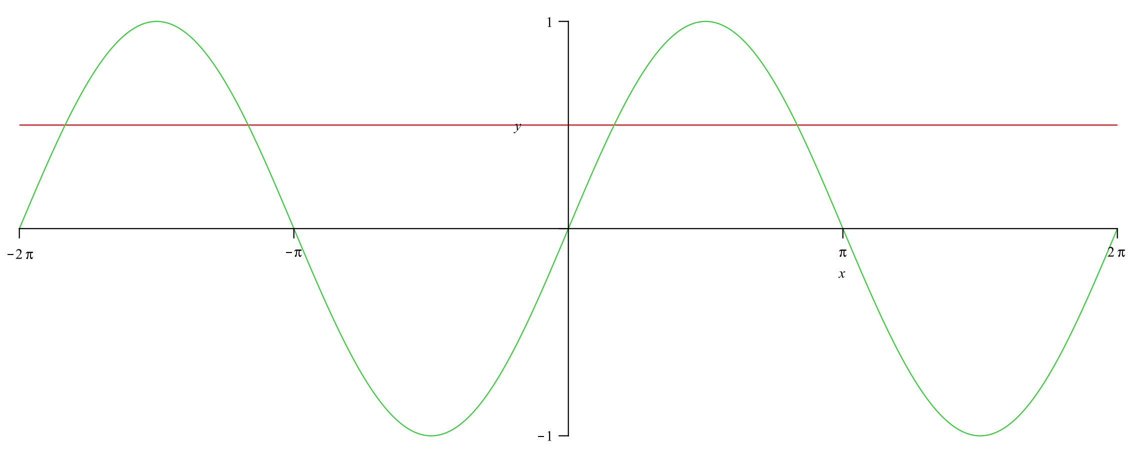
$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$$

We will now use the constructions of the previous lecture on the inverse trig functions. Let's analyse the sine curve with a view to constructing its inverse $\sin^{-1}(x)$.

We know that sin : angles$\rightarrow$ numbers and hence $\sin^{-1}$: numbers$\rightarrow$ angles. But the sine curve fails the horizontal line test dreadfully!

$$y = \sin(x)$$

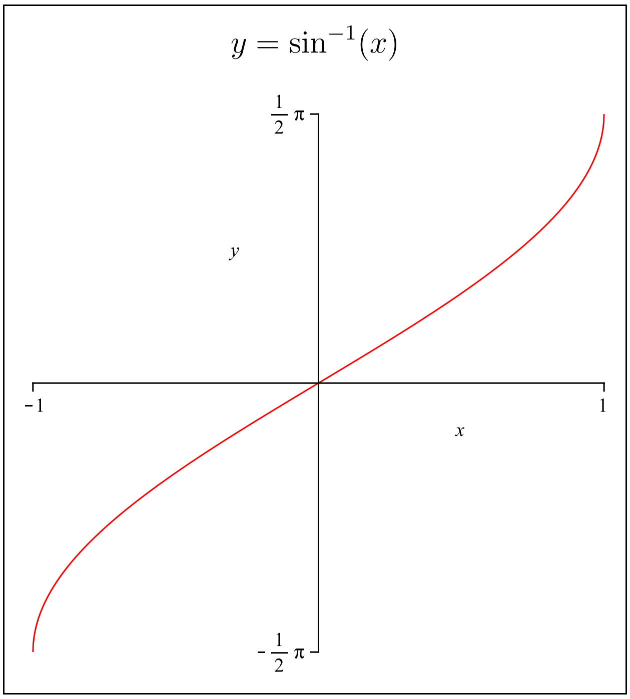


We have
$$\frac{1}{2} = \sin(\frac{\pi}{6}) = \sin(\frac{5\pi}{6}) = \sin(\frac{-11\pi}{6}) = \sin(\frac{-7\pi}{6})\dots$$
What do we mean by $\sin^{-1}(\frac{1}{2})$? Well it's the angle whose sine is $\frac{1}{2}$. But which one?

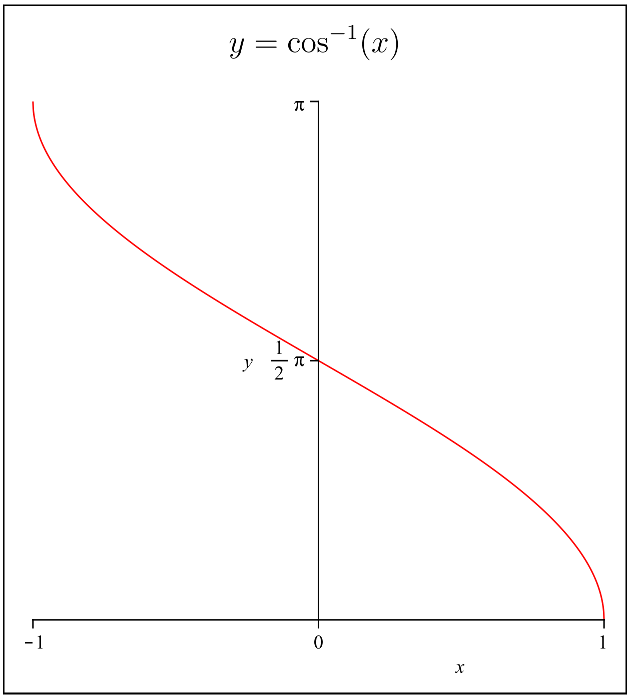
Lets trim up the graph:

★

Hence the graph of $y = \sin^{-1}(x)$ is:

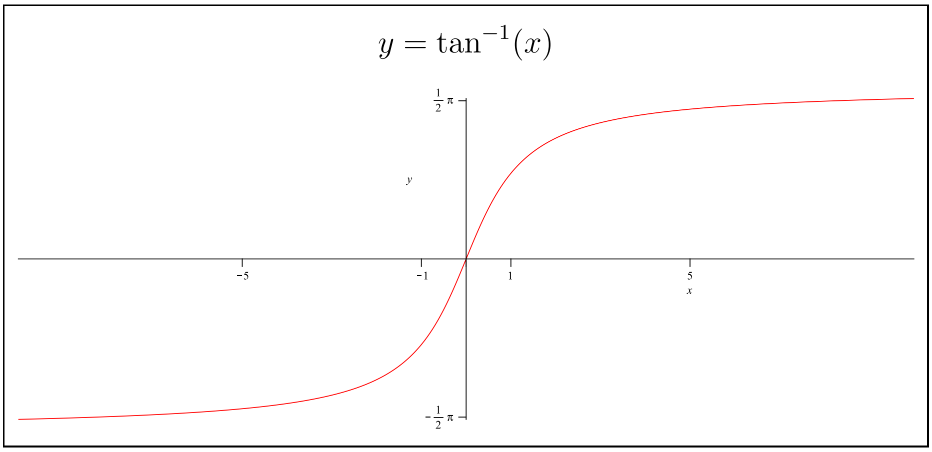


Always remember that $\sin^{-1} : [-1,1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
Similarly we have:



Always remember that $\cos^{-1} : [-1, 1] \longrightarrow [0, \pi]$

Finally



Always remember that $\tan^{-1} : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

When dealing with the inverse trig functions always be very careful with domain and range! Some other facts of intetrest which may be used:

$$\sin^{-1}(-x) = -\sin^{-1}(x)$$

$$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$$

$$\tan^{-1}(-x) = -\tan^{-1}(x)$$

Example 1: Evaluate each of the following:

a) $\sin^{-1}(\frac{1}{\sqrt{2}}) =$

b) $\cos^{-1}(-\frac{1}{2}) =$

c) $\sin^{-1}(\sin(\frac{2\pi}{3})) =$

d) $\cos^{-1}(\cos(\frac{2\pi}{3})) =$

e) $\cos(\sin^{-1}(\frac{3}{7})) =$

★

Observe that

$$\sin(\sin^{-1}(x)) = \cos(\cos^{-1}(x)) = \tan(\tan^{-1}(x)) = x \quad \textbf{always!!}$$

$$\sin^{-1}(\sin(x)) = \cos^{-1}(\cos(x)) = \tan^{-1}(\tan(x)) = x \quad \textbf{sometimes.}$$

Example 2: Sketch the graph of $y = 3\sin^{-1}(2x)$ and hence write down its domain and range.

★

Despite their elaborate definitions the inverse trig functions are just functions! Hence we should be able to differentiate them.

Facts:

$$a) \quad \frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$b) \quad \frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$c) \quad \frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

Discussion:

★

Proof a:

Method 1:

Method 2: Using $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$

★

Example 3: Find the derivative of each of the following:

a) $y = \sin^{-1}(x^7 + 5x)$.

b) $y = \ln(x)\cos^{-1}(x)$.

c) $y = \dfrac{\tan^{-1}(x)}{6x}$.

★

Parametrically Defined Curves

We sometimes define relations between x and y in terms of a third party called a parameter. The advantage of this approach is that all concerns become focused on a single object, the parameter rather than a multiplicity of other variables. You have already seen the power of parameters in the algebra strand where lines and planes in space are defined in parametric vector form.

Example 4: Prove that the circle $x^2 + y^2 = 25$ can be written parametrically as

$$\begin{cases} x = 5\cos(\theta) \\ y = 5\sin(\theta) \end{cases}$$

★

Other parametrically defines curves are:

Conic section	Cartesian equation	Parametric equation
Parabola	$4ay = x^2$	$x(t) = 2at$ $y(t) = at^2$
Circle	$x^2 + y^2 = a^2$	$x(t) = a\cos t$ $y(t) = a\sin t$
Ellipse	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	$x(t) = a\cos t$ $y(t) = b\sin t$
Hyperbola	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$x(t) = a\sec t$ $y(t) = b\tan t$

• Note that **any** function may be rewritten parametrically in many different ways.

Example 5: The function

$y = x^3 + 7$ may be expressed as $\begin{cases} x = t \\ y = t^3 + 7 \end{cases}$ or $\begin{cases} x = e^t \\ y = e^{3t} + 7 \end{cases}$

★

• Note also that it is often (but not always) possible to recover the Cartesian equation of a parametrically defined curve.

Example 6: Find the Cartesian equation of $\begin{cases} x = 3t - 1 \\ y = 9t^2 - 6t + 8 \end{cases}$

★ $y = x^2 + 7$ ★

Even though we do not have a direct relationship, it is still possible to find $\dfrac{dy}{dx}$ through the use of parametric differentiation.

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}}$$

Example 7: Suppose that a curve $\mathcal{C}$ is defined as $x = t^2 - 1$ and $y = \dfrac{3}{t}$.

a) Find a Cartesian relation between x and y.

b) Which point on the curve corresponds to $t = 6$?

c) Using parametric differentiation find $\dfrac{dy}{dx}$ at the point $(8, 1)$?

★ *a)* $y^2 = \dfrac{9}{x+1}$ *b)* $(35, \dfrac{1}{2})$ *c)* $-\dfrac{1}{18}$ ★

Example 8: Suppose that a curve is define parametrically by

$$x = t + \cos(t) \quad \text{and} \quad y = t^4 + 2t + 5.$$

a) Find a Cartesian relation between x and y.

b) Find the equation of the tangent to the curve at the point $(1, 5)$.

c) What is $\dfrac{d^2y}{dx^2}$?

$$\bigstar \quad a) \quad Impossible \quad b) \quad y = 2x + 3 \quad c) \quad \frac{12t^2(1-\sin(t)) + (4t^3+2)\cos(t)}{(1-\sin(t))^3} \quad \bigstar$$

LECTURE 13
Curve Sketching

A function f is said to be **odd** if $f(-x) = -f(x)$ over its domain.

A function f is said to be **even** if $f(-x) = f(x)$ over its domain.

To sketch an unknown graph of a function $y = f(x)$ we use a checklist to investigate its structure. We find:

✓ the y intercept by setting $x = 0$.
✓ the x intercept(s) by solving $f(x) = 0$ for x.
✓ whether or not the function is odd or even.
✓ (V.A.) the vertical asymptotes (usually a consequence of division by zero).
✓ (H.A.) the behaviour of the function for large $|x|$ by considering $\lim\limits_{x \longrightarrow \pm\infty} f(x)$.
(H.A. will determine horizontal and oblique asymptotes).
✓ the position and nature of any stationary points (this is crucial).
✓ !! If stuck plot points !!

Odd and Even Functions

A function f is said to be **odd** if $f(-x) = -f(x)$ over its domain.
Examples of odd functions are $y = x$, $y = x^3$,$y = x^5$, $y = \sin(x)$ and $y = \sin^{-1}(x)$.

Odd functions exhibit skew symmetry in their graphs.

Example 1:

★

A function f is said to be **even** if $f(-x) = f(x)$ over its domain.

Examples of even functions are $y = 1$, $\quad y = x^2$, $\quad y = x^4$ and $y = \cos(x)$.

Even functions exhibit symmetry about the y axis in their graphs.

Example 2:

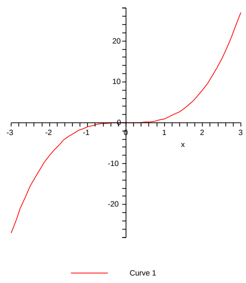
★

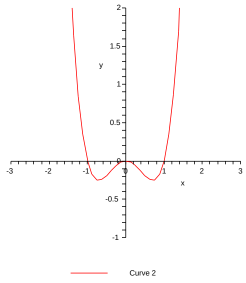
odd×odd=even, odd×even=odd, even×even=even.

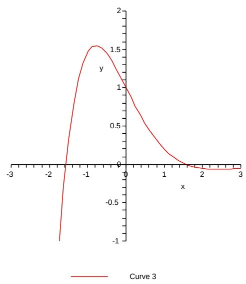
odd±odd=odd, even±even=even, even±odd=neither.

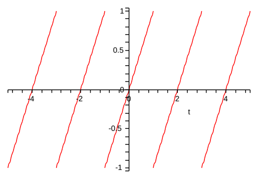
$$\int_{-a}^{a} odd = 0, \quad \int_{-a}^{a} even = 2\int_{0}^{a} even.$$

Example 3: Identify each of the following functions as odd, even or neither:

a) 

b) 

c) 

d) 

★

Example 4: Identify each of the following functions as odd, even or neither:

i) $f(x) = \cos(3x)$

ii) $f(x) = x\cos(x)$

iii) $f(x) = \sin^2(x)$

iv) Prove your answer in iii) from the definition.

★

Example 5: Prove that the derivative of an even function is an odd function.

★

We now turn to the problem of sketching an unknown function. This can be quite tricky and our approach is to piece together a graph using a host of different clues as to the shape of the curve. Our checklist is to find:

✓ the y intercept by setting $x = 0$.

✓ the x intercept(s) by solving $f(x) = 0$ for x.

✓ whether or not the function is odd or even.

✓ (V.A.) the vertical asymptotes (usually a consequence of division by zero).

✓ (H.A.) the behaviour of the function for large $|x|$ by considering $\lim_{x \longrightarrow \pm\infty} f(x)$.
(H.A. will determine horizontal and oblique asymptotes).

✓ the position and nature of any stationary points (this is crucial).

✓ !! If stuck plot points !!

Let's have a look at a batch of examples.

Example 6: Sketch the graph of $y = f(x) = x^3 - 3x$.

★

Example 7: Sketch the graph of $y = f(x) = \dfrac{2x-1}{x+1}$.

★

Example 8: Consider the function $f(x) = \dfrac{x^2 - 5x + 29}{x - 4}$.

a) Sketch the graph of $y = f(x)$.

b) Hence find the domain and range of f.

c) By considering your sketch find all values of k for which the equation

$$x^2 - (5 + k)x + (29 + 4k) = 0$$

has exactly one solution.

d) **Homework:** Check your answer to c) by using the discriminant.

$\bigstar$ $(-1,-7)$ is a local max and $(9,13)$ is a local min. $\bigstar$

$\bigstar$ $\mathrm{Dom(f)} = \{x \in \mathbb{R} : x \neq 4\}$ $\mathrm{Range(f)} = [13,\infty) \cup (-\infty,-7]$ $\bigstar$

$\bigstar$ $k = -7, 13$ $\bigstar$

Example 9: Sketch the graph of $y = f(x) = e^{-x}\sin(x)$.

We can just use common sense on this one!

$\bigstar$

LECTURE 14
Polar Coordinates (r, θ)

$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

$$r = \sqrt{x^2 + y^2}$$

$$\tan(\theta) = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{r\cos(\theta) + \dfrac{dr}{d\theta}\sin(\theta)}{-r\sin(\theta) + \dfrac{dr}{d\theta}\cos(\theta)}$$

We have already seen in the complex number section of the algebra strand that it is sometimes advantageous to abandon the rectangular coordinate system (x, y) and replace it with polars (r, θ) .

This opens up a new world when sketching curves and also helps in the analysis of "round" objects such as circles, spirals and cardioids.

Let us begin by defining polar coordinates (r, θ) and proving the transformation equations above. Note that r is simply the distance from the point to the origin and θ is the angle to the positive x axis.

Discussion:

This should all sound very familiar! r is just $|\,z\,|$ and θ is nothing but $\mathrm{Arg}(z)$.

We do demand that $r \geq 0$ but are relaxed about θ. You can use $-\pi < \theta \leq \pi$ or $0 \leq \theta < 2\pi$. Sometimes we even let $\theta \geq 0$.

Example 1: Sketch the following points given in polar coordinates and convert each point over to Cartesian form:

a) $A(3, \frac{\pi}{4})$.

b) $B(1, \frac{2\pi}{3})$.

★

Example 2: Sketch the following points given in Cartesian coordinates and convert each point over to polar form:

c) $C(\sqrt{3}, 1)$.

d) $D(-1, -1)$.

★

Polar graphs are weird and wonderful objects. To sketch a polar graph expressed in terms of r and θ we almost never retreat to Cartesians and rarely use any calculus. Let's begin with some simple examples:

Example 3: Sketch each of the following polar equations in the $x - y$ plane.

a) $r = 5$.

b) $\theta = \dfrac{\pi}{4}$.

---

a)

Note that $r = 5 \rightarrow r^2 = 25 \rightarrow x^2 + y^2 = 25$ which makes sense! It's a circle.

b)

★

For more complicated graphs we have a procedure to follow:

1) Check for symmetry about the x axis. If replacing θ by $-\theta$ has no impact then the graph is symmetric about the x axis. That is the top is the same as the bottom.

2) Check for symmetry about the y axis. If replacing θ by $\pi - \theta$ has no impact then the graph is symmetric about the y axis. That is the left is the same as the right.

Two identities to keep in mind are

$\cos(-\theta) = \cos(\theta)$. Symmetry about the x axis.

$\sin(\pi - \theta) = \sin(\theta)$. Symmetry about the y axis.

3) Our final step is to plot a detailed table of θ vs r, usually using a calculator.

We can then use this table (and an appropriate scale if necessary) to step out all the angles and measure the appropriate distances along the rays. The final sketch is then simply a question of joining the dots and using the symmetry.

Example 4: Sketch in the $x - y$ plane the graph of $r = 2\cos(\theta)$ for $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$.

The restriction on θ guarantees that $r \ge 0$.

Sketching $y = 2\cos(x)$ will be awarded zero marks!! It's a polar plot!

$\theta \to -\theta$:

$\theta \to (\pi - \theta)$:

We only need a table from 0 to 90°. Symmetry will take care of the rest.

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
r	2	1.97	1.88	1.73	1.53	1.29	1	.68	.35	0

Note that we can convert this to Cartesians! This is usually **not** a good idea.

★

Example 5: Sketch in the $x - y$ plane the graph of $r = 1 - \sin(\theta)$ for $-\pi < \theta \leq \pi$.

$\theta \to -\theta$:

$\theta \to (\pi - \theta)$:

We only need a table from -90 to $90°$. Symmetry will take care of the rest.

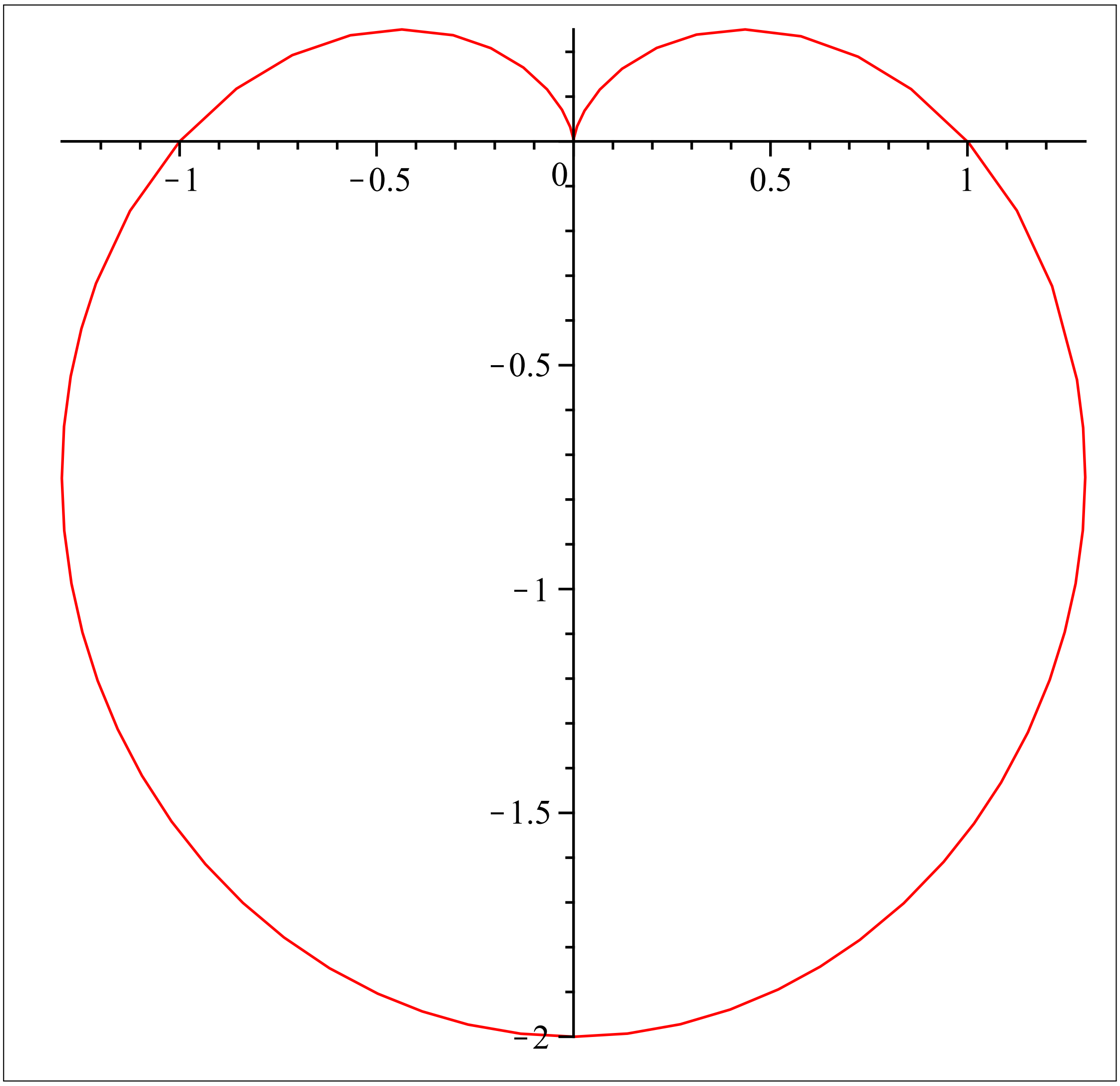
θ	$-90°$	$-60°$	$-45°$	$-30°$	$0°$	$30°$	$45°$	$60°$	$90°$
r	2	1.87	1.71	1.50	1	0.5	.29	0.13	0

For obvious reasons this curve is called a cardioid.

★

Let's see what Maple does with this sketch:

```
>plot([1-sin(theta),theta,theta=-Pi..Pi],coords=polar);
```



★

Example 6: Sketch in the $x - y$ plane the graph of $\quad r = |\sin(2\theta)| \quad$ for $0 < \theta \leq 2\pi$.

$\theta \rightarrow -\theta$:

$\theta \rightarrow (\pi - \theta)$:

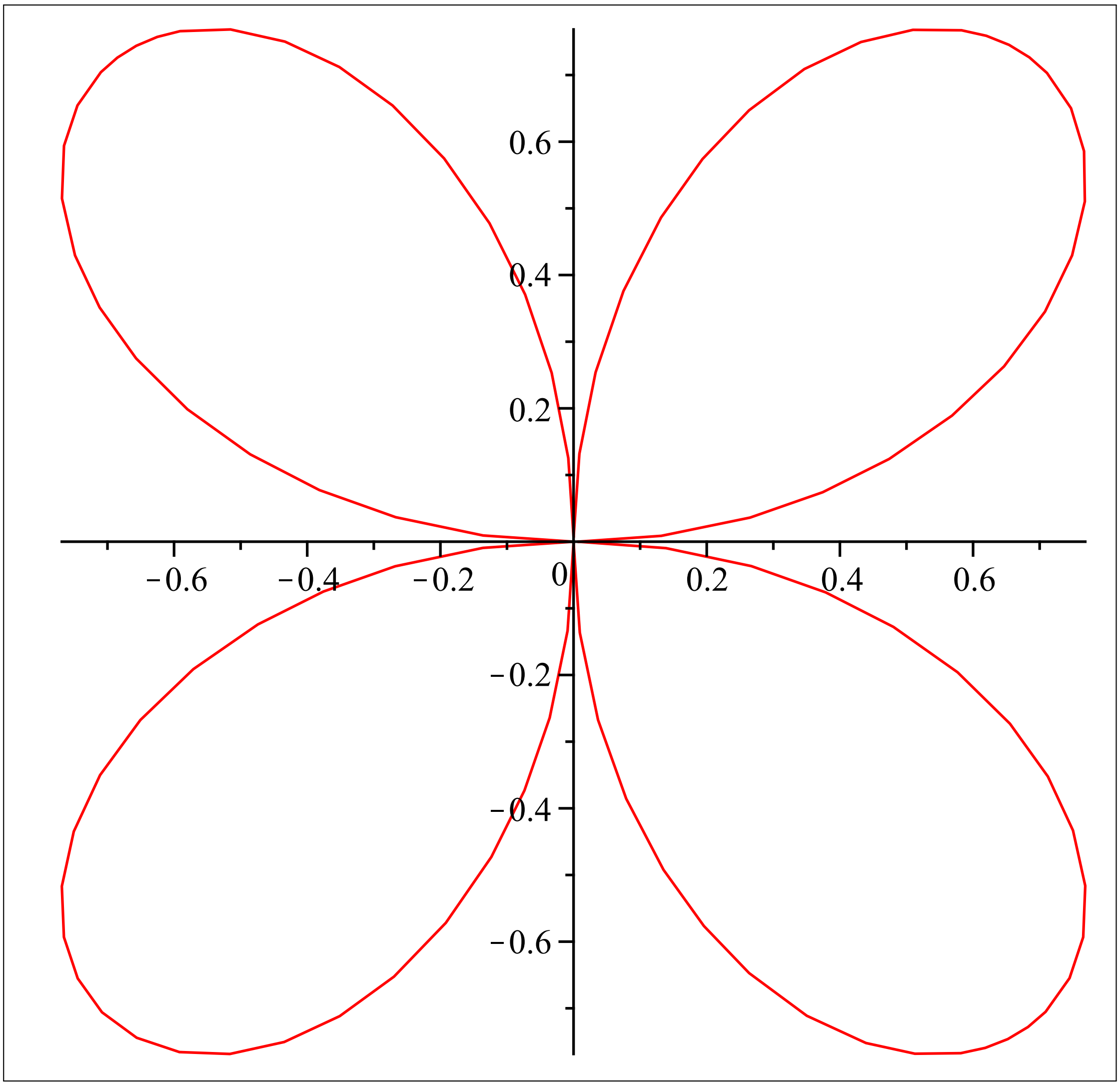
Due to the double symmetry we only need a table from 0 to 90°.

θ	0°	10°	20°	30°	40°	45°	50°	60°	70°	80°	90°
r	0	.34	.64	.87	.98	1	.98	0.87	0.64	.34	0

★

Let's see what Maple does with this sketch:

```
>plot([abs(sin(2*theta)),theta,theta=0..2*Pi],coords=polar);
```



★

We close with a discussion of differentiation as it applies to polar curves. Note that despite their exotic shapes polar sketches are still curves, hence they have tangents and gradients. The first important thing to realise is that $\dfrac{dy}{dx} \neq \dfrac{dr}{d\theta}$. The situation is much more complicated than that!

Fact: If r is a function of θ then

$$\frac{dy}{dx} = \frac{r\cos(\theta) + \dfrac{dr}{d\theta}\sin(\theta)}{-r\sin(\theta) + \dfrac{dr}{d\theta}\cos(\theta)}$$

Proof:

$\bigstar$

Example 7: Find the equation of the tangent to the curve $r = 1 - \sin(\theta)$ at $\theta = \dfrac{\pi}{3}$.

We already have a sketch of this from Example 5.

Now $r = 1 - \sin(\dfrac{\pi}{3}) = 1 - \dfrac{\sqrt{3}}{2}$ so the polar coordinates are $(r, \theta) = (1 - \dfrac{\sqrt{3}}{2}, \dfrac{\pi}{3})$.

Also $\dfrac{dr}{d\theta} = -\cos(\theta)$. So

$$\frac{dy}{dx} = \frac{r\cos(\theta) + \dfrac{dr}{d\theta}\sin(\theta)}{-r\sin(\theta) + \dfrac{dr}{d\theta}\cos(\theta)} = \frac{(1-\sin(\theta))\cos(\theta) + (-\cos(\theta))\sin(\theta)}{-(1-\sin(\theta))\sin(\theta) + (-\cos(\theta))\cos(\theta)}$$

Substituting $\theta = \dfrac{\pi}{3}$ yields

$\dfrac{(\frac{1}{2} - \frac{\sqrt{3}}{4}) - \frac{1}{2}\frac{\sqrt{3}}{2}}{(\frac{3}{4} - \frac{\sqrt{3}}{2}) - \frac{1}{4}} = 1$ amazingly. Hence the gradient of the tangent is 1 at $\theta = \dfrac{\pi}{3}$.

We now need to find a point (x, y).

$x = r\cos(\theta) = (1 - \dfrac{\sqrt{3}}{2})(\dfrac{1}{2}) = \dfrac{1}{2} - \dfrac{\sqrt{3}}{4}$.

$y = r\sin(\theta) = (1 - \dfrac{\sqrt{3}}{2})(\dfrac{\sqrt{3}}{2}) = \dfrac{\sqrt{3}}{2} - \dfrac{3}{4}$.

So the point is $(x, y) = (\dfrac{1}{2} - \dfrac{\sqrt{3}}{4}, \dfrac{\sqrt{3}}{2} - \dfrac{3}{4})$.

Hence the equation of the tangent is $(y - (\dfrac{\sqrt{3}}{2} - \dfrac{3}{4})) = 1(x - (\dfrac{1}{2} - \dfrac{\sqrt{3}}{4}))$ $\bigstar$

LECTURE 15
Riemann Sums

Suppose that a function f is bounded on $[a, b]$. If there exists a *unique* real number I such that $\underline{S}_{\mathcal{P}_n}(f) \leq I \leq \overline{S}_{\mathcal{P}_n}(f)$ for *every* partition $\mathcal{P}_n$ of $[a, b]$, then we say that f is *Riemann integrable* on the interval $[a, b]$. If f is Riemann integrable, then this unique real number I is called the *definite integral of f from a to b* and we write

$$I = \int_a^b f(x)\, dx.$$

The function f is called the *integrand* of the definite integral, while the points a and b are called the *limits* of the definite integral.

If $f \geq 0$ then I is the area bounded by f and the x axis from $x = a$ to $x = b$.

If f is bounded and piecewise continuous on the interval $[a, b]$ then f is Reimann integrable on $[a, b]$.

We turn now to the other half of Calculus, the central problem of calculating areas under curves. You have already seen plenty of integration techniques in your high school studies. Here however, we are are interested in the structural underpinnings of the theory of integration. In this lecture we will formally define the Reimann Integral. In the following lecture we will examine how the Reimann integral may be efficiently calculated. To motivate discussions we will take a very careful look at the area of the region bounded by $y = x^2$ and the x axis from $x = 0$ to $x = 1$.

Sketch:

★

The evaluation of this area is not an easy task! It is not a standard object such as a circle or a triangle so we cannot just use a formula. Our strategy is to approximate the area by partitioning the x interval from 0 to 1 into n equal segments.

This partition will be denoted by

$$\mathcal{P}_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}\right\}$$

.

To get a feeling for the process let us partition the interval up into 5 pieces

$$\mathcal{P}_5 = \left\{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\right\}$$

We then estimate the area by building rectangles off each of the subintervals. We start by deliberately underestimating the area by choosing as the height of each rectangle the smallest y-value over each subinterval to produce the lower Reimann sum $\underline{S}_{\mathcal{P}_5}$

Sketch (the lower Reimann sum $\underline{S}_{\mathcal{P}_5}$):

★

Clearly this lower Reimann sum is given by:

$\underline{S}_{\mathcal{P}_5} =$

$$\frac{1}{5}\times 0^2+\frac{1}{5}\times(\frac{1}{5})^2+\frac{1}{5}\times(\frac{2}{5})^2+\frac{1}{5}\times(\frac{3}{5})^2+\frac{1}{5}\times(\frac{4}{5})^2 = (\frac{1}{5})^3\left\{0^2+1^2+2^2+3^2+4^2\right\} = 0.24.$$

We can also deliberately overestimate the area by choosing the largest y value over each subinterval to produce the upper Reimann sum $\overline{S}_{\mathcal{P}_5}$

Sketch (the upper Reimann sum $\overline{S}_{\mathcal{P}_5}$):

★

Clearly this upper Reimann sum is given by:

$\overline{S}_{\mathcal{P}_5} =$

$$\frac{1}{5}\times(\frac{1}{5})^2+\frac{1}{5}\times(\frac{2}{5})^2+\frac{1}{5}\times(\frac{3}{5})^2+\frac{1}{5}\times(\frac{4}{5})^2+\frac{1}{5}\times(\frac{5}{5})^2 = (\frac{1}{5})^3\left\{1^2+2^2+3^2+4^2+5^2\right\} = 0.44.$$

It follows that the true area A is such that $0.24 \leq A \leq 0.44$. This is pretty much useless! We can refine the approximation by taking more rectangles and making them thinner. If we partition $[0,1]$ into 50 equal subintervals then we obtain $0.32 \leq A \leq 0.34$, which is an improvement. But to appreciate the full force of Reimann integration we take an arbitrary number of rectangles, say n. Then

$$\mathcal{P}_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}\right\}$$

We then underestimate the true area:

Sketch (the lower Reimann sum $\underline{S}_{\mathcal{P}_n}$):

★

Clearly this lower Reimann sum is given by:

$\underline{S}_{\mathcal{P}_n} =$

$\frac{1}{n} \times 0^2 + \frac{1}{n} \times (\frac{1}{n})^2 + \frac{1}{n} \times (\frac{2}{n})^2 + \frac{1}{n} \times (\frac{3}{n})^2 + \cdots + \frac{1}{n} \times (\frac{n-2}{n})^2 + \frac{1}{n} \times (\frac{n-1}{n})^2 =$
$(\frac{1}{n})^3 \left\{0^2 + 1^2 + 2^2 + \cdots + (n-2)^2 + (n-1)^2\right\}.$

It can be shown by induction (you will be given these formulae in an examination) that

$$0^2 + 1^2 + 2^2 + \cdots + (n-2)^2 + (n-1)^2 = \frac{1}{6}n(n-1)(2n-1) = \frac{1}{6}(2n^3 - 3n^2 + n).$$

Thus $\underline{S}_{\mathcal{P}_n} = (\frac{1}{n})^3 \frac{1}{6}(2n^3 - 3n^2 + n) = \frac{2n^3 - 3n^2 + n}{6n^3}$

We also overestimate the true area:

Sketch (the upper Reimann sum $\overline{S}_{\mathcal{P}_n}$):

★

Clearly this upper Reimann sum is given by:

$\overline{S}_{\mathcal{P}_n} =$

$\frac{1}{n} \times (\frac{1}{n})^2 + \frac{1}{n} \times (\frac{2}{n})^2 + \frac{1}{n} \times (\frac{3}{n})^2 + \cdots + \frac{1}{n}(\frac{n-1}{n})^2 + \frac{1}{n} \times (\frac{n}{n})^2 =$
$(\frac{1}{n})^3 \left\{1^2 + 2^2 + \cdots + (n-1)^2 + (n)^2\right\}.$

It can be shown by induction that

$$1^2 + 2^2 + \cdots + (n-1)^2 + (n)^2 = \frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}(2n^3 + 3n^2 + n).$$

Thus $\overline{S}_{\mathcal{P}_n} = (\frac{1}{n})^3 \frac{1}{6}(2n^3 + 3n^2 + n) = \dfrac{2n^3 + 3n^2 + n}{6n^3}$.

Therefore $\dfrac{2n^3 - 3n^2 + n}{6n^3} \leq A \leq \dfrac{2n^3 + 3n^2 + n}{6n^3}$

As a check letting $n = 5$ yields $0.24 \leq A \leq 0.44$ which agrees with the previous analysis.

Letting $n = 1000$ (that's one thousand thin little rectangles each of width $\frac{1}{1000}$) we get the true area A satisfying

$$0.3328335 \leq A \leq 0.3338335$$

But why not take infinitely many rectangles?! That is let $n \to \infty$.

We have $\lim\limits_{n \longrightarrow \infty} \dfrac{2n^3 + 3n^2 + n}{6n^3} = \dfrac{2}{6} = \dfrac{1}{3}$ and $\lim\limits_{n \longrightarrow \infty} \dfrac{2n^3 - 3n^2 + n}{6n^3} = \dfrac{2}{6} = \dfrac{1}{3}$. Thus

$$\frac{1}{3} \leq A \leq \frac{1}{3}$$

That is $A = \dfrac{1}{3}$.

So that's Reimann integration. We partition the interval into n subintervals and pin the true area between the upper and lower Reimann sums. We let $n \to \infty$ and if the upper and lower sums both converge to a common value we denote that value by $\int_a^b f(x)dx$ and refer to it as the definite integral. The area of a positive function f is then given by $A = \int_a^b f(x)dx$.

Discussion: Why do we use the notation $\int_a^b f(x)dx$?

★

More formally

> Suppose that a function f is bounded on $[a, b]$. If there exists a **UNIQUE** real number I such that $\underline{S}_{\mathcal{P}_n}(f) \leq I \leq \overline{S}_{\mathcal{P}_n}(f)$ for *every* partition $\mathcal{P}_n$ of $[a, b]$, then we say that f is *Riemann integrable* on the interval $[a, b]$. If f is Riemann integrable, then this unique real number I is called the *definite integral* of f from a to b and we write
>
> $$I = \int_a^b f(x)\, dx.$$
>
> The function f is called the *integrand* of the definite integral, while the points a and b are called the *limits* of the definite integral.
>
> If $f \geq 0$ then I is the area bounded by f and the x axis from $x = a$ to $x = b$.

The clear problem here is that the evaluation of the Reimann integral is a tortuous application of summation and limits. In the next lecture, we will use the Fundamental Theorem of Calculus to establish a dramatic shortcut. But first some technicalities.

Not all functions are Reimann integrable!

Example 1: Let $f(x) = \begin{cases} 2, & x \in \mathbb{Q}; \\ 7, & x \notin \mathbb{Q}. \end{cases}$

Sketch the function and by considering upper and lower sums for any partition $\mathcal{P}_n$ explain why the Reimann integral $\displaystyle\int_0^3 f(x)\,dx$ does not exist.

★

Fortunately most reasonable functions are integrable.

Fact: If f is bounded and piecewise continuous (that is discontinuous at only a finite number of points) on the interval $[a, b]$ then f is Reimann integrable on $[a, b]$.

So all bounded continuous functions are integrable and we can also tolerate a few discontinuities as well.

Question: Is $\displaystyle\int_{-1}^{1} \frac{1}{x}\,dx$ well defined?

★

Properties of the Reimann Integral

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \qquad \text{for } a \leq c \leq b$$

$$f \geq 0 \Rightarrow \int_a^b f(x)\,dx \geq 0$$

$$f \geq g \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx \qquad \text{for } \alpha \in \mathbb{R}$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

An important thing to understand is that the definition of the Reimann integral has **NOTHING** to do with calculus! If you look at this entire lecture I have not once mentioned the derivative, the primitive or the anti-derivative. But clearly from your previous studies you know that the calculation the integral is all about calculus? The crucial link between the definition and its implementation will be made in the next lecture through the Fundamental Theorem of Calculus, one of the most amazing results in mathematics.

LECTURE 16
The Fundamental Theorems of Calculus

The First Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$$

The Second Fundamental Theorem of Calculus:

$$\int_a^b f(x)dx = F(b) - F(a) \quad \text{where } F \text{ is a primitive of } f.$$

SOME BASIC INTEGRALS

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^{ax} dx = \frac{e^{ax}}{a} + C$$

$$\int \sin\, ax\, dx = -\frac{\cos ax}{a} + C$$

$$\int \cos ax\, dx = \frac{\sin ax}{a} + C$$

$$\int \sec^2 ax\, dx = \frac{\tan ax}{a} + C$$

In the last lecture we defined the Reimann integral and then evaluated a few simple integrals using an excruciating limiting process of upper and lower sums of rectangles. In this lecture we will show that the machinery of calculus may be used to establish a dramatic short cut to the evaluation of integrals.

We first define the primitive of a function. Given a function $y = f(x)$ a primitive of f is another function F with the property that $F' = f$.

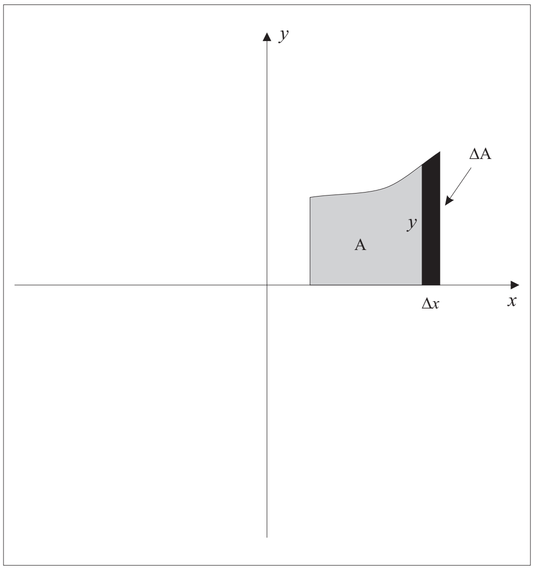
Thus a primitive of $3x^2$ is x^3 **because** $\frac{d}{dx}(x^3) = 3x^2$. The process of finding primitives is literally differentiation in reverse! We do of course have a slight technical problem in that $\frac{d}{dx}(x^3 + 7) = 3x^2$ as well. The most general form of the primitive of $3x^2$ is then $x^3 + C$ since the constant of integration C disappears when differentiated. What do primitives have to do with integration? There is certainly no obvious connection until we have the two Fundamental Theorems of Calculus:

The First Fundamental Theorem of Calculus: Let f be a continuous function defined on $[a, b]$ and define a new function F by

$$F(x) = \int_a^x f(t)\,dt$$

Then F is both continuous and differentiable on $[a, b]$ and $F' = f$.

Remember that (initially at least) the Reimann integral has nothing to do with the calculus. What the first fundamental theorem says is that if you differentiate an integral you get back to where you started. In other words integration is the opposite of differentiation and hence the process of integration can be managed by simply using primitives. A formal proof of the theorem is in your notes. Lets us have a look at an intuitive argument.



Let A be the area accumulated underneath the curve $y = f(x)$ from a to x, that is $A = \int_a^x f(t)\,dt$. Consider the extra increment of area ΔA. This region is approximately a rectangle so $\Delta A = y\Delta x = f(x)\Delta x$. Hence $\dfrac{\Delta A}{\Delta x} = f(x)$. Letting $\Delta x \to 0$ we have $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ which is the first fundamental theorem. It follows that we need to 'antidifferentiate' $f(x)$ to find A.

Just a small note on the equation $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ before we continue. Students are often confused by the presence of the variable t in this equation. Keep in mind however that t is really only a dummy variable and can be replaced with anything (except x). We can't use $\dfrac{d}{dx}\displaystyle\int_a^x f(x)\,dx = f(x)$ as the variable x will then be both a variable of integration and a limit, leading to all sorts of notational problems.

So the first fundamental theorem says that integration and antidifferentiation are essentially equivalent. Indeed we write the primitive of $f(x)$ as the indefinite integral $\int f(x)\,dx$. The second fundamental theorem provides us with a way of exploiting this identification to actually calculate areas under curves.

The Second Fundamental Theorem of Calculus: Let f be a continuous function defined on $[a, b]$ and let F be a primitive of f. Then

$$\int_a^b f(x)dx = \Big[F(x)\Big]_a^b = F(b) - F(a).$$

This is an amazing result. Calculation of the area under a curve over an interval $[a, b]$ depends only upon the value of the primitive at the endpoints of the interval! Lets take a look at how we use these theorems:

Example 1: Show that $\dfrac{d}{dx}\displaystyle\int_0^x t^2\,dt = x^2$ without using the first fundamental theorem of calculus.

★

The example above shows that the first fundamental theorem says something very simple! If you differentiate an integral you go around in a big circle with the only difference being a change of variable at the end.

Recall how hard it was to find the area under $y = x^2$ from $x = 0$ to $x = 1$ in the last lecture. The second fundamental theorem now provides a wonderful shortcut! But you still need to be a little careful.

Example 2: Evaluate the area of the region bounded by $y = x^2$ and the x axis from $x = 0$ to $x = 1$.

★

So much faster!

The above theorems put the processes of integration on a firm footing. You have however already seen a lot of integration theory in high school. Let's recall that theory before moving on to some more abstract questions.

Example 3: Evaluate $\int_0^2 1 - x^2\,dx$.

$$\star \quad -\frac{2}{3} \quad \star$$

Example 4: Find the area bounded by $y = 1 - x^2$ and the x axis from $x = 0$ to $x = 2$.

This is a different question! Whenever you are asked for area you must sketch the curve! The integral gets a little confused if the region is below the x axis and will assign a negative value to the area. You need to separate this section out and take its absolute value.

$$\star \quad 2 \quad u^2 \quad \star$$

We will not spend too much time on the evaluation of integrals here as this is a major topic extensively covered in the high school courses. An important point to keep in mind however is that integration is a touchy process. We have no general rules such as the product rule or the quotient rule for integration and many many functions cannot be integrated at all! Our main technique is to use a table of integrals (see page 1). Such a table will be available to you in your final exam. Integrals without limits are referred to as indefinite integrals and need a $+C$. Integrals with limits are called definite integrals and do not need a constant of integration.

Example 5: Evaluate

a) $\displaystyle\int_{\ln(5)}^{\ln(7)} e^{2x}\ dx$

b) $\displaystyle\int_{0}^{\frac{\pi}{6}} \sin(3x)\ dx$

$\bigstar \quad a)\ 12 \quad b)\ \dfrac{1}{3} \quad \bigstar$

We ask some fairly tricky questions involving the fundamental theorems of calculus. These always take the shape of the derivative of an integral:

Example 6: Evaluate

$$\frac{d}{dx}\int_3^x t^4\,dt.$$

This is trivial! By the first fundamental theorem the answer is x^4. Let's actually evaluate the integral to see what happens:

Note that the 3 can be replaced with any number and the result will be the same.

★ x^4 ★

Example 7: Use the first fundamental theorem of calculus to evaluate

$$\frac{d}{dx}\int_7^x \cos(t^3)\,dt.$$

We cannot do this integral! It is impossible. But that doesn't stop us:

★ $\cos(x^3)$ ★

Example 8: Use the first fundamental theorem of calculus to evaluate

$$\frac{d}{dx}\int_7^{x^5} \cos(t^3)\,dt.$$

This is now complicated by the strange upper limit! Our approach is to admit defeat on the integral and hope that we can still wriggle out of the problem.

Let a primitive of $f(t) = \cos(t^3)$ be $F(t)$.

We can't work out what $F(t)$ is, but we do know that $F'(t) = f(t) = \cos(t^3)$. So:

$$\int_7^{x^5} \cos(t^3)\,dt =$$

$$\frac{d}{dx}\int_7^{x^5} \cos(t^3)\,dt =$$

★ $\cos(x^{15})5x^4$ ★

It is clear from the above example that we have the following structure:

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = (\text{sub in upper limit} \times \text{upper}') - (\text{sub in lower limit} \times \text{lower}').$$

Example 9: Use the first fundamental theorem of calculus to evaluate

$$\frac{d}{dx}\int_{x^2}^{x^4} \sec\left(\frac{7}{1+t}\right)\,dt.$$

$$\bigstar \quad 4x^3 \sec(\frac{7}{1+x^4}) - 2x\sec(\frac{7}{1+x^2}) \quad \bigstar$$

If you are the sort of person that likes to memorize and use formulas you may use:

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v)v' - f(u)u'$$

Note also that if the examiner doesn't mention the first fundamental theorem in the question then you should mention it in your answer.

Example 10: Evaluate

$$\frac{d}{dx}\int_{\sin(x)}^{e^{3x}} \ln(1-z^2)\,dz.$$

$$\bigstar \quad 3\ln(1-e^{6x})e^{3x} - \ln(\cos^2(x))\cos(x) \quad \bigstar$$

Example 11: Evaluate

$$\frac{d}{d\alpha}\int_{\alpha^{30}}^{\ln\alpha} \sin(1+\beta^4)\, d\beta.$$

$$\bigstar \quad \sin(1+(\ln\alpha)^4)(\frac{1}{\alpha}) - \sin(1+\alpha^{120})(30\alpha^{19}) \quad \bigstar$$

When doing these sort of questions you must not fall into the trap of actually trying to evaluate the integral. Even simple little integrals can turn out to be impossible. For example it can be shown that

$$\int_0^x e^{-t^2}\, dt$$

cannot be expressed in terms of standard functions. Our response to these integrals is to admit defeat and just give them a name. Thus for example

$$\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt$$

is called $\mathrm{erf}(x)$ and the exotic erf function sits quite happily with all of its boring mates sin, cos, ln etc. There are many functions whose definition takes the form of an unachievable integral. Examples are

$$\mathrm{li}(x) = \int_2^x \frac{1}{\ln(t)}\, dt$$

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t}\, dt$$

$$\mathrm{FresnelC}(x) = \int_0^x \cos(\frac{\pi}{2}t^2)\, dt$$

You do not need to know these definition for Math1131 but you may come across these functions in your future studies. There are lots of strange functions in mathematics.

LECTURE 17
Integration by Substitution

A substitution of variable can often dramatically simplify an integral.

When using the method of substitution to evaluate $\int_a^b f(x)dx$ don't forget to also substitute the limits and the increment dx.

$$\text{If } f \text{ is odd then } \int_{-a}^{a} f(x)\,dx = 0.$$

$$\text{If } f \text{ is even then } \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

SOME BASIC INTEGRALS

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^{ax} dx = \frac{e^{ax}}{a} + C$$

$$\int \sin\, ax\,dx = -\frac{\cos ax}{a} + C$$

$$\int \cos ax\,dx = \frac{\sin ax}{a} + C$$

$$\int \sec^2 ax\,dx = \frac{\tan ax}{a} + C$$

In this lecture we will develop some new integration techniques. But first a revision example on the fundamental theorems.

Example 1: Use the first fundamental theorem of calculus to find f if

$$\int_0^x t f(t)\,dt = \int_x^0 (t-1) f(t)\,dt + x^3.$$

$$\bigstar \quad f(x) = \frac{3x^2}{2x-1} \quad \bigstar$$

Integration by Substitution

It is generally much harder to integrate functions than it is to differentiate them. A disturbing issue is that many functions have primitives which cannnot be expressed in terms of standard functions. This means that these functions can never be integrated via the second fundamental theorem of calculus. For example $\int e^{x^2}\,dx$ cannot be done regardless of how much mathematics you know. Looks simple but it's impossible! We do however have many tricks which help us to find the integrals of complicated functions.

The process of substitution through a change of variable can often clarify the nature of an integral. Substitutions do not solve the problem, but rather change the question into a format where the path to solution is a little clearer. When making a substitution always remember to take care of:

1) The function.

2) The increment dx.

3) The limits if they exist.

Example 2: Find $\int (2x+7)^{11} dx$ by using the substitution $u = 2x + 7$.

★ $\dfrac{(2x+7)^{12}}{24} + C$ ★

Note that it is crucial to return to the original variable. Note also that the increment dx plays a central role and should always be included when integrating by substitution.

We can always check our answer by differentiating.

★

We will not always give you the substitution so you need to develop the skill of deciding what is appropriate. Generally we substitute away the piece of the integral which is causing the most concern.

Be aware that the method of substitution is not always appropriate and that there are plenty of integrals where no substitution will work!

Example 3: Find $\displaystyle\int x^2 \cos(x^3)\ dx$

$$\bigstar \quad \frac{1}{3}\sin(x^3) + C \quad \bigstar$$

Example 4: Find $\displaystyle\int \frac{\sin(\ln x)}{x}\ dx$ and check your answer via differentiation.

$$\bigstar \quad -\cos(\ln x) + C \quad \bigstar$$

Example 5: Find $\displaystyle\int \frac{e^t}{1+e^t}\,dt$

$$\bigstar \quad \ln(1+e^t) \quad \bigstar$$

When there are limits involved the process is much the same, but you must make sure to take care of the limits of integration.

Example 6: Evaluate $\displaystyle\int_3^4 x(x-3)^{10}\,dx$

You should never present an integral with more than one variable in it. It is either all x or all u.

$$\bigstar \quad \frac{47}{132} \quad \bigstar$$

Note that if there are limits there is no need to return to the original variable as the final answer is a number.

Example 7: Evaluate $\displaystyle\int_0^1 \frac{2x}{(x+4)^2}dx$

$$\bigstar \quad 2\ln(\frac{5}{4}) - \frac{2}{5} \quad \bigstar$$

Sometimes substitutions do their work in very mysterious ways.

Example 8: Use the substitution $u = \dfrac{\pi}{2} - x$ to evaluate

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin(x)}{\sin(x) + \cos(x)}\, dx$$

$$\bigstar \quad \frac{\pi}{4} \quad \bigstar$$

Remember that the variable of integration is always a dummy variable:

$$\int_0^1 x^2\, dx = \int_0^1 t^2\, dt = \int_0^1 q^2\, dq = \dots$$

Fact: If f is odd then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0.$

Fact: If f is even then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$

You should always consider the issue of oddness and evenness when you are faced with a symmetric integral from $-a$ to a !

Why does this work?

★

Example 9: Evaluate $\displaystyle\int_{-1}^{1} \frac{x^8 \sin(x)}{\sqrt{1+x^2+x^4}}\,dx.$

★

LECTURE 18
Integration by Parts

$$\int u dv = uv - \int v du$$

SOME BASIC INTEGRALS

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int \frac{1}{x}\, dx = \ln|x| + C$$

$$\int e^{ax} dx = \frac{e^{ax}}{a} + C$$

$$\int \sin\, ax\, dx = -\frac{\cos ax}{a} + C$$

$$\int \cos ax\, dx = \frac{\sin ax}{a} + C$$

$$\int \sec^2 ax\, dx = \frac{\tan ax}{a} + C$$

One of your fundamental differentiation techniques is the product rule, which enables you to differentiate the product of two or more standard functions. Integration is a much trickier process and with integration we do not have the luxury of global processes such as the product, quotient and chain rules. However under certain circumstances we can integrate a product of two functions using a technique called integration by parts.

Parts does not have the wide ranging application of the product rule but it is a very effective technique for integrals of the type $\int$ (little polynomial) $\times$ (nice function). Parts can also be tweaked to knock off other types of integrals.

If faced with an integral of a product we assign one of the functions to be u and the other to be dv. Then the integration by parts formula is:

$$\int u dv = uv - \int v du$$

Proof

★

Example 1: Find $\displaystyle\int 3xe^{4x}\,dx$.

This is a typical parts question. Observe the little poly $3x$ and the nice function e^{4x}. Note that, unlike the technique of substitution the increment dx plays no effective role in the parts process.

$$\bigstar \quad \frac{3}{4}xe^{4x} - \frac{3}{16}e^{4x} + C \quad \bigstar$$

Note that, depending upon how you finished up, you may have produced

$$\pm\frac{3}{4}C \quad \text{or} \quad \pm\frac{3}{16}C \quad \text{or} \quad \pm 3C. \text{ All of these are equivalent to } \; +C.$$

Let us check that the solution $\displaystyle\int 3xe^{4x}\,dx = \frac{3}{4}xe^{4x} - \frac{3}{16}e^{4x} + C$ is correct!

$$\frac{d}{dx}\left(\frac{3}{4}xe^{4x} - \frac{3}{16}e^{4x} + C\right) =$$

$\bigstar$

As a general rule we let u be the little poly and dv be the other function although you will soon see that there are many exceptions to this rule. What if we get them around the wrong way?

$\bigstar$

Let's have a go at another one:

Example 2: Find $\displaystyle\int 5t\cos(9t)\,dt$.

Do not be put off by the t variable. All variables in an integral are dummy variables.

Note also that when calculating v from dv don't fuss with $+C$. The constant comes much later.

$\bigstar \quad \dfrac{5}{9}t\sin(9t) + \dfrac{5}{81}\cos(9t) + C \quad \bigstar$

Sometimes we need to use parts multiple times!

Example 3: Find $\displaystyle\int x^2 \cos(x)\, dx$.

$\bigstar \quad x^2 \sin(x) + 2x\cos(x) - 2\sin(x) + C \quad \bigstar$

If the integral is definite (that is it has limits) then you must keep in mind that the first half of the parts equation has already been integrated so it has square brackets and limits while the second half still needs to be done.

Example 4: Find $\displaystyle\int_0^1 2xe^{3x}\,dx$.

$$\bigstar \quad \frac{4}{9}e^3 + \frac{2}{9} \quad \bigstar$$

When the product involves the $\ln(x)$ function we often assign u to be $\ln(x)$.

Example 5: Find $\displaystyle\int x^2 \ln(x)\,dx$.

$$\bigstar \quad \frac{x^3}{3}\ln(x) - \frac{1}{9}x^3 + C \quad \bigstar$$

Curiously we don't always need two functions to use parts. We can artificially multiply by 1 to get parts going. . . . . . but this rarely works!

Example 6: Find $\displaystyle\int \ln(x)\,dx$.

$$\bigstar \quad x\ln(x) - x + C \quad \bigstar$$

Example 7: Find $\displaystyle\int_0^{\sqrt{3}} \tan^{-1}(x)\,dx$.

$$\bigstar \quad \frac{\sqrt{3}\pi}{3} - \ln(2) \quad \bigstar$$

So we can now integrate a wider class of functions. Keep in mind however that most integrals cannot be done!! We finish off with a very strange trick.

Example 8: Find $\int e^x \cos(x)\,dx$.

Let $I = \int e^x \cos(x)\,dx$. It will be soon clear why we need to name the integral. Then

$u = e^x \longrightarrow du = e^x$.

$dv = \cos(x) \longrightarrow v = \sin(x)$.

$$I = \int udv = uv - \int vdu = e^x \sin(x) - \int \sin(x)e^x dx = e^x \sin(x) - J \tag{1}$$

where $J = \int \sin(x)e^x dx$.

In this example it surprisingly doesn't matter which function you call u and which dv but you have to stick to your choice for e^x over the entire solution.

Let us now consider the second integral $J = \int \sin(x)e^x dx$.

$u = e^x \longrightarrow du = e^x$.

$dv = \sin(x) \longrightarrow v = -\cos(x)$.

$$\int udv = uv - \int vdu$$

$$J = -e^x \cos(x) - \int -\cos(x)e^x dx = -e^x \cos(x) + \int \cos(x)e^x dx = -e^x \cos(x) + I.$$

It is a little disturbing to see our question reappear in the answer but this is good news!

Substituting back into (1) yields

$$I = e^x \sin(x) - (-e^x \cos(x) + I) \longrightarrow I = e^x \sin(x) + e^x \cos(x) - I.$$

$$\longrightarrow 2I = e^x \sin(x) + e^x \cos(x) \longrightarrow I = \frac{1}{2}e^x(\sin(x) + \cos(x)) + C.$$

$$\bigstar \quad \frac{1}{2}e^x(\sin(x) + \cos(x)) + C \quad \bigstar$$

Integrals of the type: $\int \text{exponential} \times \{\sin \;\text{ or }\; \cos\}$ can be done in the above manner.

Homework Investigation:

What happens if we swap u and dv when calculating J? Try it and see.

LECTURE 19
Improper Integrals

$$\int_a^\infty f(x)\,dx = \lim_{R\longrightarrow\infty}\int_a^R f(x)\,dx$$

The improper integral $\displaystyle\int_a^\infty \frac{1}{x^p}\,dx$ is called a $p-$integral.

A $p-$integral converges if $p > 1$ and it diverges if $p \leq 1$.

(The Comparison Test) Suppose that f and g are integrable functions and that $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

i) $\displaystyle\int_a^\infty g(x)\,dx$ converges $\Longrightarrow \displaystyle\int_a^\infty f(x)\,dx$ converges.

ii) $\displaystyle\int_a^\infty f(x)\,dx$ diverges $\Longrightarrow \displaystyle\int_a^\infty g(x)\,dx$ diverges.

Up until this point we have always integrated functions between two finite limits a and b. We now investigate the interesting possibility of letting b become ∞ to produce what is called an **improper** integral $\displaystyle\int_a^\infty f(x)\,dx$. As you may well imagine this causes lots of problems! The picture of the situation is

Your initial reaction may be that the region has an infinite tail, therefore its area is infinite. Remarkably this is not necessarily the case. It is possible for the area $\displaystyle\int_a^\infty f(x)\,dx$ to be finite! We then say that the improper integral converges. Else we say that the improper integral diverges.

How can something infinite generate something finite? You have already seen this happen with GP's where the infinite series $\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}+\dfrac{1}{16}+\ldots\ldots$ converges to a finite value of 1. The key to evaluating improper integrals is to make effective use of limits. We make the following definition

$$\int_a^\infty f(x)\,dx = \lim_{R\longrightarrow\infty}\int_a^R f(x)\,dx$$

Example 1: Show that the improper integral $\displaystyle\int_2^\infty \frac{1}{x^4}\,dx$ converges.

Lets take a look at a sketch of $\displaystyle\int_2^\infty \frac{1}{x^4}\,dx$

The sketch is deceptive and doesn't really tell us anything. Our method is to abandon $\displaystyle\int_2^\infty \frac{1}{x^4}\,dx$ and consider instead $\displaystyle\int_2^R \frac{1}{x^4}\,dx$. The new sketch is

This is easily evaluated.

Now we simply let $R\longrightarrow\infty$. Whatever happens happens. If we get a finite number the integral converges. Else the integral diverges.

$\bigstar$ The improper integral converges to $\dfrac{1}{24}$ $\bigstar$

Question: What can we now say about $\displaystyle\int_7^\infty \frac{1}{x^4}\,dx$?

Remember to always have as the final step of your direct calculation of an improper integral the clear evaluation of a limit!

The next example shows you that the situation is fairly complex.

Example 2: Show that the improper integral $\displaystyle\int_2^\infty \frac{1}{x}\,dx$ diverges.

Lets take a look at a sketch of $\displaystyle\int_2^\infty \frac{1}{x}\,dx$

We abandon $\displaystyle\int_2^\infty \frac{1}{x}\,dx$ and consider instead $\displaystyle\int_2^R \frac{1}{x}\,dx$. The new sketch is

This looks the same as before! But the improper integral is drastically different.

Letting $R \longrightarrow \infty$.

★ The improper integral diverges ★

Question: What can we now say about $\displaystyle\int_7^\infty \frac{1}{x}\,dx$?

It will also diverge. The convergence/divergence of an improper integral $\displaystyle\int_a^\infty f(x)\,dx$ is completely determined by its tail. The lower limit a is of little consequence

So why did the first integral converge and the second diverge? The fundamental difference is that $\frac{1}{x^4}$ goes to zero faster than $\frac{1}{x}$.

For example at $x = 5$, $\frac{1}{x^4} = \frac{1}{3125}$ while $\frac{1}{x} = \frac{1}{5}$ only.

This is the key. For $\int_a^\infty f(x)\,dx$ to converge, the function $f(x)$ certainly needs to go to 0 as $x \longrightarrow \infty$. But this is not enough. The function f needs to go to zero fast enough, to avoid the accumulation of an infinite amount of area.

We have a way of measuring all of this.

Definition: The improper integral $\int_a^\infty \frac{1}{x^p}\,dx$ is called a $p-$integral.

Fact: A $p-$integral converges if $p > 1$ and diverges if $p \leq 1$.

Thus for example $\int_3^\infty \frac{1}{x^7}\,dx$ and $\int_1^\infty \frac{1}{x^3}\,dx$ both converge (getting to zero fast).

But $\int_1^\infty \frac{1}{x}\,dx$ and $\int_4^\infty \frac{1}{\sqrt{x}}\,dx$ both diverge (too slow to 0).

We will use $p-$integrals extensively later on when using comparison tests. There are many improper integrals which are not however $p-$integrals:

Example 3: Determine whether the following integrals converge or diverge.

a) $\int_0^\infty e^{-2x}\,dx$

b) $\int_0^\infty \cos(x)\,dx$

★ *a)* Converges to $\frac{1}{2}$ *b)* Diverges ★

We now turn to the problem of convergence and divergence for improper integrals where we cannot actually find a primitive. Under these circumstance we try to establish whether or not they converge (without actually evaluating a final answer) by using strategic comparisons.

(The Comparison Test)

Suppose that f and g are integrable functions and that $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

i) $\displaystyle\int_a^\infty g(x)\,dx$ converges $\Longrightarrow \displaystyle\int_a^\infty f(x)\,dx$ converges.

Discussion:

ii) $\displaystyle\int_a^\infty f(x)\,dx$ diverges $\Longrightarrow \displaystyle\int_a^\infty g(x)\,dx$ diverges.

Discussion:

Note carefully however that $\displaystyle\int_a^\infty g(x)\,dx$ diverging gives you **NO** information!!

Similarly $\displaystyle\int_a^\infty f(x)\,dx$ converging gives you **NO** information!!

To use the comparison test **your** integral must be smaller than a known converging integral or bigger than a known diverging integral, usually a $p-$integral.

When making comparisons remember that making the bottom of a fraction bigger will make the fraction smaller and vice versa. So for example $\frac{3}{7} > \frac{3}{10}$ because the denominator is larger on the right.

Example 4: Determine whether or not the following improper integrals converge or diverge by using the comparison test.

Note that there is no chance of actually evaluating the integral in either case. But the question is not one of evaluation! You are only being asked to investigate whether or not the integral converges.

a) $\displaystyle\int_3^{\infty} \frac{4}{x^2 + e^x + 7}\, dx$

★ The improper integral converges ★

b) $\displaystyle\int_{10}^{\infty} \frac{\cos(x) + 6}{\sqrt{x} - 2}\, dx$

★ The improper integral diverges ★

Just two small notes to finish off.

• Firstly what do we do with $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$? The answer is quite simple. The improper integral $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ converges if and only if $\displaystyle\int_{0}^{\infty} f(x)\,dx$ and $\displaystyle\int_{-\infty}^{0} f(x)\,dx$ **both** converge.

So for example $\displaystyle\int_{-\infty}^{\infty} x\,dx$ is a divergent integral since both $\displaystyle\int_{0}^{\infty} x\,dx$ and $\displaystyle\int_{-\infty}^{0} x\,dx$ diverge. It is divergent even though $\displaystyle\int_{-R}^{R} x\,dx = 0$ for all R.

• Secondly we are often in a position where we suspect that an integral is divergent but can't quite get the inequalities pointing in the right direction.

For example consider $\displaystyle\int_{5}^{\infty} \frac{1}{x+1}\,dx$. We know that $\displaystyle\int_{5}^{\infty} \frac{1}{x}\,dx$ is a divergent $p-$integral and that $\displaystyle\frac{1}{x+1} < \frac{1}{x}$ but this doesn't help! The inequality is pointing in the wrong direction for the comparison test. In the next lecture we will facilitate these intuitive attacks by using the more subtle **limit form** of the comparison test.

LECTURE 20
The Natural Log Function

The Limit Form of the Comparison Test: Suppose that f and g are non-negative and bounded on $[a, \infty]$ and $\lim\limits_{x \longrightarrow \infty} \dfrac{f(x)}{g(x)} = L$ where $0 < L < \infty$. Then $\displaystyle\int_a^\infty f(x)\,dx$ and $\displaystyle\int_a^\infty g(x)\,dx$ will either both converge or both diverge.

The natural log function is defined as $\ln(x) = \displaystyle\int_1^x \frac{1}{t}\,dt$

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

The Limit Form of the Comparison Test

Recall from the last lecture that $p-$integrals of the form $\displaystyle\int_a^\infty \frac{1}{x^p}\,dx$ converge if $p > 1$ and diverge if $p \leq 1$. These simple integrals can then be used in comparison tests to attack more complicated improper integrals. Unfortunately a direct comparision cannot always be made. We then use the more subtle limit form of the comparison test.

The Limit Form of the Comparison Test:
Suppose that f and g are non-negative and bounded on $[a, \infty]$ and

$$\lim_{x \longrightarrow \infty} \frac{f(x)}{g(x)} = L$$

where $0 < L < \infty$. Then $\displaystyle\int_a^\infty f(x)\,dx$ and $\displaystyle\int_a^\infty g(x)\,dx$ will either both converge or both diverge.

★

That is f and g will have identical outcomes as improper integrals. The functions are locked together.

This test is remarkably effective. It allows us to convert intuitive arguments into solid mathematics.

Proof: See Printed Notes.

Example 1: Determine whether $\displaystyle\int_3^\infty \frac{2x^{11}+5}{x^{12}-7x^4+24}\,dx$ converges or diverges.

We would struggle to establish a comparison using direct inequalities here, and we certainly can't find the primitive. However by considering dominant terms it looks as though this integral is very similar to the diverging $p-$integral $\displaystyle\int_3^\infty \frac{1}{x}\,dx$.

So let $f(x) = \dfrac{2x^{11}+5}{x^{12}-7x^4+24}$ and $g(x) = \dfrac{1}{x}$. Then

$$\lim_{x\longrightarrow\infty} \frac{f(x)}{g(x)} =$$

★ The integral diverges by the limit form of the comparison test ★

This test should make sense. It is is saying that if $\displaystyle\lim_{x\longrightarrow\infty} \frac{f(x)}{g(x)}$ is finite and non-zero then the two functions have equal powers at infinity and hence will integrate up in a similar fashion.

Question: How do we know who is f and who is g ?

Example 2: Determine whether $\displaystyle\int_7^\infty \frac{8}{\sqrt{x^4 - x^2 - 1}}\,dx$ converges or diverges.

$$\frac{8}{\sqrt{x^4 - x^2 - 1}} \approx \frac{1}{\sqrt{x^4}} = \frac{1}{x^2}$$

and $\displaystyle\int_7^\infty \frac{1}{x^2}\,dx$ is a converging $p-$integral with $p = 2 > 1$. So we expect convergence!

Let $f(x) = \dfrac{8}{\sqrt{x^4 - x^2 - 1}}$ and $g(x) = \dfrac{1}{x^2}$. Then

$$\lim_{x \longrightarrow \infty} \frac{f(x)}{g(x)} =$$

★ The integral converges by the limit form of the comparison test ★

We close this topic by noting that there is a second class of improper integrals. These involve integrating up to a vertical asymptote. For example

$$\int_0^1 \frac{1}{x}\, dx$$

Sketch:

The problem here is not that the limits are infinite but rather that the function is infinite! We have a vertical tail rather than a horizontal tail.

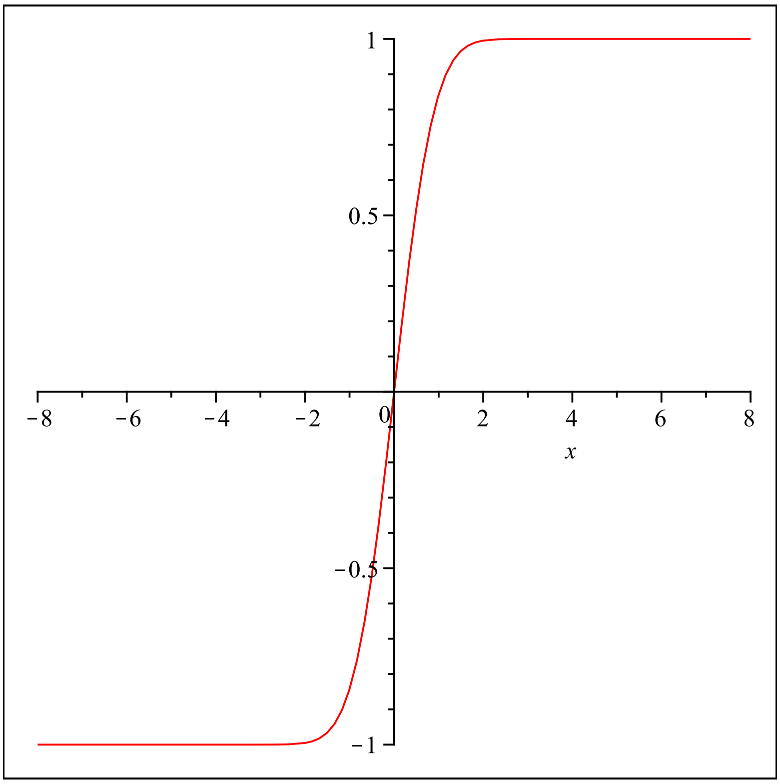
This theory is approachable in much the same way, and strikes similar technical obstacles, however we do not examine these sort of improper integrals in Math1131.

Functions Defined By Integrals

Whenever we cannot find the primitive of a function there is a temptation to use the integral to define a brand new function. For example $\int e^{-x^2}\,dx$ cannot be expressed in terms of standard functions prompting the definition of the error function $\mathrm{erf}(x)$:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt.$$

erf is a perfectly good function with enormous applications! It has a graph



It also has all sorts of interesting properties.

Example 3: Prove that $\mathrm{erf}(x)$ is an odd function.

★

Some of the most important functions in mathematics are defined by integrals. They include $\mathrm{Dilog}(x), \mathrm{Chi}(x),\ \mathrm{Shi}(x), \mathrm{Si}(x), \mathrm{Li}(x)$ and a host of others. You will not meet any of these until second year. A function that you all know very well however is the natural log function $y = \ln(x)$. It too is defined by an integral!

The Natural Log Function

You are all familiar with the natural logarithm function $y = \ln(x)$. Some of you will have seen the natural log function written as $y = \log_e(x)$. Both of these notations are quite common but your calculators probably have a ln button so we will stick to this notation.

You may not however have seen $y = \ln(x)$ properly defined as an integral:

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$

This integral defines the ln function over the domain $(0, \infty)$. What does this mean?

Say you want to evaluate $\ln(3)$. You need to find $\displaystyle\int_1^3 \frac{1}{t}\,dt$. This is just a Riemann integral which can be if necessary knocked off using upper and lower sums. It is OK to define a function as an integral since we now have a nice tight definition of the integral.

The natural log function $y = \ln(x)$ has the following properties:

a) $\dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$.

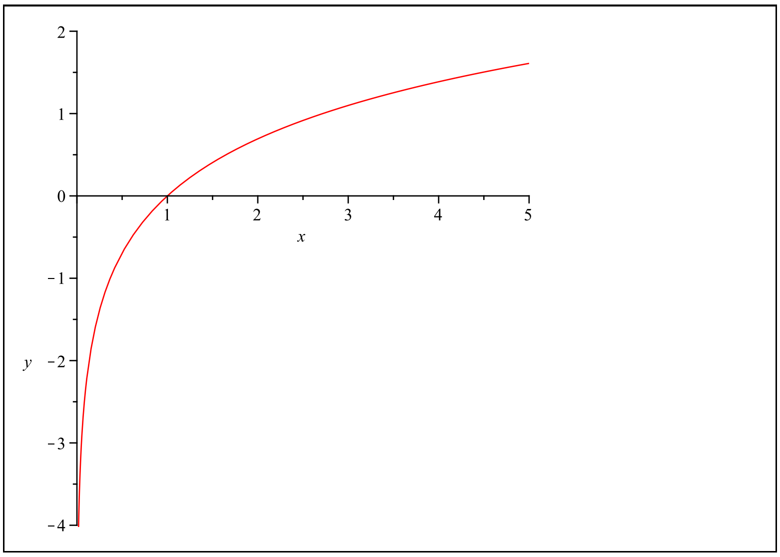
b) $\ln(1) = 0$.

c) $\ln(ab) = \ln(a) + \ln(b)$.

d) $\ln\left(\dfrac{a}{b}\right) = \ln(a) - \ln(b)$.

e) $\ln(a^r) = r\ln(a)$ for r rational.

The graph of the ln function is



Observe that the ln function is only defined for positive x and that $y = \ln(x)$ increases to ∞.

Let's take a look at some proofs:

a) $\dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$.

b) $\ln(1) = 0$.

c) $\ln(ab) = \ln(a) + \ln(b)$.

RHS$= \ln(a) + \ln(b) = \displaystyle\int_1^a \frac{1}{t}\,dt + \int_1^b \frac{1}{t}\,dt$

We make the substitution $u = at$ in the second integral

★

So we now have a different way of defining $\ln(x)$. But it is still the same boring old log function under calculus.

a) Find $\dfrac{d}{dx}x^2\ln(x)$.

b) Find $\dfrac{d}{dx}\ln(\sin(x))$.

c) Find $\dfrac{d}{dx}\ln\left(\dfrac{\sqrt{1+x^4}}{3x-1}\right)$.

Always use your log laws **before** the calculus!

Since $\dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$ we also have $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$.

d) Find $\displaystyle\int \frac{1}{x^3}\,dx$.

e) Find $\displaystyle\int \frac{1}{x^2}\,dx$.

f) Find $\displaystyle\int \frac{1}{x}\,dx$. The above methods do not work here!

g) Find $\displaystyle\int \frac{x}{x^2+1}\,dx$.

Clearly the chain rule implies that $\dfrac{d}{dx}\ln(f(x)) = \dfrac{f'(x)}{f(x)}$.

Hence $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

h) What about $\displaystyle\int \frac{1}{x^2+1}\,dx$?

★

LECTURE 21
The exponential function e^x

The inverse of $\ln(x)$ is e^x.

$$e^{\ln(x)} = \ln(e^x) = x.$$

$$\frac{d}{dx}e^x = e^x.$$

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C.$$

$$e^x e^y = e^{x+y}.$$

$$\frac{d}{dx}(a^x) = a^x \ln(a).$$

$$\int a^x\,dx = \frac{1}{\ln(a)}a^x.$$

You will recall from the previous lecture that the natural log function $y = \ln(x)$ is an increasing function. Hence it is $1-1$ and thus invertible. The inverse of $\ln(x)$ is without doubt the most important function in all of mathematics....the exponential function $y = e^x$.

The irrational real number e is approximately 2.71828 and can be defined in many ways:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$

or

$$e = \lim_{n \longrightarrow \infty} (1 + \frac{1}{n})^n$$

What makes e^x such a fascinating function is the simple fact $\frac{d}{dx}e^x = e^x$. It is its own derivative! No other function has this remarkable property. The exponential function e^x is immune to calculus!

The graphs of the two functions are reflected in $y = x$ in the usual manner.

Sketch:

Observe that $\mathrm{Dom}(\ln(x)) = (0, \infty) = \mathrm{Range}(e^x)$ and $\mathrm{Range}(\ln(x)) = \mathbb{R} = \mathrm{Dom}(e^x)$.

Both functions are increasing however the exponential function grows with enormous strength while the natural log function increases very weakly.

Further properties of the two functions are:

a) $e^{\ln(x)} = x$. This is just $(f^{-1} \circ f)(x) = x$.

b) $\ln(e^x) = x$. This is just $(f \circ f^{-1})(x) = x$.

c) $\dfrac{d}{dx} e^x = e^x$.

d) $\dfrac{d}{dx} e^{f(x)} = e^{f(x)} f'(x)$.

e) $\displaystyle\int e^x \, dx = e^x$.

f) $\displaystyle\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C$.

g) $e^x e^y = e^{x+y}$.

Note that g) indicates that the inverse of $\ln(x)$ actually has something to do with exponentials!!

Proofs:

a) and b) This is just the definition of the inverse function!

c) We start with $\ln(e^x) = x$ and differentiate both sides with respect to x.

d) This is just the chain rule.

e) follows from c)

f) Exercise.

g) $e^{x+y} = e^{\ln(e^x) + \ln(e^y)} = e^{\ln(e^x e^y)} = e^x e^y$.

★

Example 1:

a) Evaluate $\displaystyle\int_{\ln(2)}^{\ln(5)} e^{3x}\,dx$.

b) Solve $2^x = 9$.

c) Find $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

d) Find $\dfrac{d}{dx}(x^3 e^{5x})$.

$\bigstar \quad a)\ \ 39 \quad b)\ \ \dfrac{\ln(9)}{\ln(2)} \approx 3.17 \quad c)\ \ 2e^{\sqrt{x}} + C \quad d)\ \ e^{5x}\{3x^2 + 5x^3\} \quad \bigstar$

The functions 2^x, 3^x and 7^x are also exponential functions. Why do we obsess about e^x ? Only e^x is equal to its own derivative! So what is the derivative of 3^x? To answer this question we use what is called logarithmic differentiation. This is simply taking the log of both sides before differentiating implicitly. This works well to eliminate troublesome exponentials.

Example 2: Find $\frac{d}{dx}(7^x)$.

★ $7^x \ln(7)$ ★

It follows from the same argument that

$$\frac{d}{dx}(a^x) = a^x \ln(a),$$

and hence after integrating both sides with respect to x

$$\int a^x \, dx = \frac{1}{\ln(a)} a^x.$$

Proof:

★

Example 3: Use the above facts to find:

a) $\frac{d}{dx}(4^x) =$

b) $\frac{d}{dx}(e^x) =$

c) $\frac{d}{dx}(e^5 + \ln 7) =$

d) $\int 6^x \, dx =$

★

The process of logarithmic differentiation is a versatile tool, handy whenever exponents are blocking your path:

Example 4: Use logarithmic differentiation to find $\dfrac{dy}{dx}$ for $y = \dfrac{x\sqrt{x^2+1}}{x^2-1}$

$$\bigstar \quad \frac{dy}{dx} = \frac{x\sqrt{x^2+1}}{x^2-1}\left\{\frac{1}{x} + \frac{x}{x^2+1} - \frac{2x}{x^2-1}\right\} \quad \bigstar$$

Example 5: Find $\dfrac{dy}{dx}$ for $y = x^x$

$$\bigstar \quad y = x^x\{1 + \ln(x)\} \quad \bigstar$$

The same log tricks can also be used on limits. Simply give the limit a name and then log both sides:

Example 6: Evaluate $\lim\limits_{x \longrightarrow \infty} x^{\frac{1}{x}}$

$\bigstar \quad 1 \quad \bigstar$

Example 7: Evaluate $\lim\limits_{x \longrightarrow \infty} (1 + \dfrac{1}{x})^x$

$\bigstar \quad e \quad \bigstar$

LECTURE 22
The Hyperbolic Trig Functions

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\frac{d}{dx}\sinh(x) = \cosh(x)$$

$$\frac{d}{dx}\cosh(x) = \sinh(x)$$

$$\frac{d}{dx}\tanh(x) = \mathrm{sech}^2(x)$$

$$\cosh^2(x) - \sinh^2(x) = 1$$

$$1 - \tanh^2(x) = \mathrm{sech}^2(x)$$

$$\sinh(2x) = 2\sinh(x)\cosh(x)$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x)$$

$$\tanh(2x) = \frac{2\tanh(x)}{1 + \tanh^2(x)}$$

$$\int \cosh(ax+b)\,dx = \frac{1}{a}\sinh(ax+b) + C$$

$$\int \sinh(ax+b)\,dx = \frac{1}{a}\cosh(ax+b) + C$$

$$\int \mathrm{sech}^2(ax+b)\,dx = \frac{1}{a}\tanh(ax+b) + C$$

We close Math1131 Calculus with a pair of lectures examining a fascinating class of functions known as the hyperbolic trigonometric functions. These functions are defined in terms of the exponential function e^x but remarkably look, feel and taste almost exactly like trig functions. They are fake trig functions!

Their names are

$$f(x) = \sinh(x) \quad \text{(pronounced } \textit{shine} \text{ of } x\text{)}$$

$$f(x) = \cosh(x) \quad \text{(pronounced } \textit{cosh} \text{ of } x\text{)}$$

$$f(x) = \tanh(x) \quad \text{(pronounced } \textit{than} \text{ of } x \text{ or } \textit{tanch} \text{ of } x\text{).}$$

The definitions are:

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$

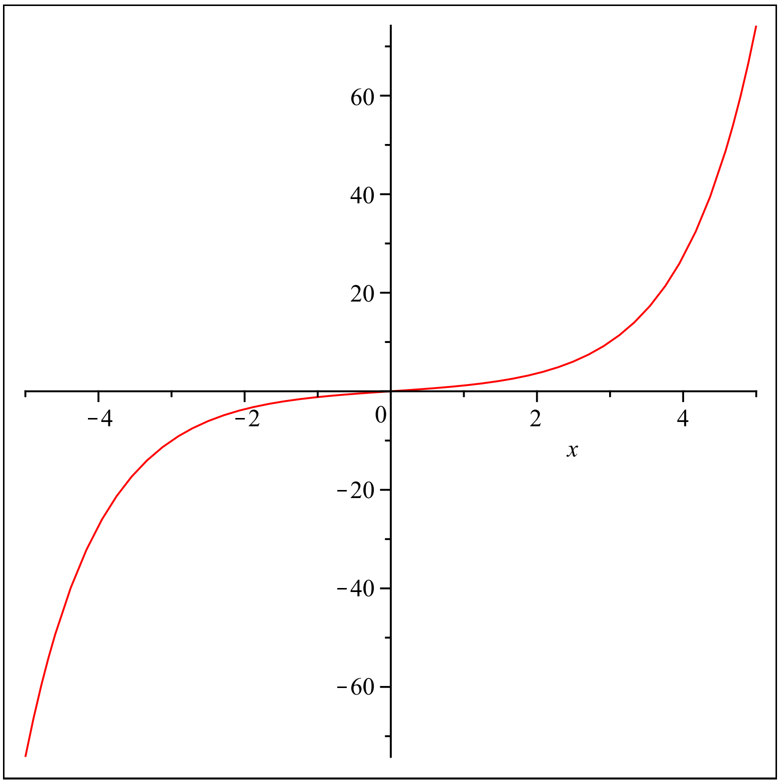
and

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

The extra h stands for hyperbolic and we will soon see the amazing connections with the trig functions. But first some graphs and properties.

$f(x) = \sinh(x)$

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$



$\sinh(x)$ is an odd function.

$\sinh(x)$ is an increasing function.

$\sinh(0) = 0$.

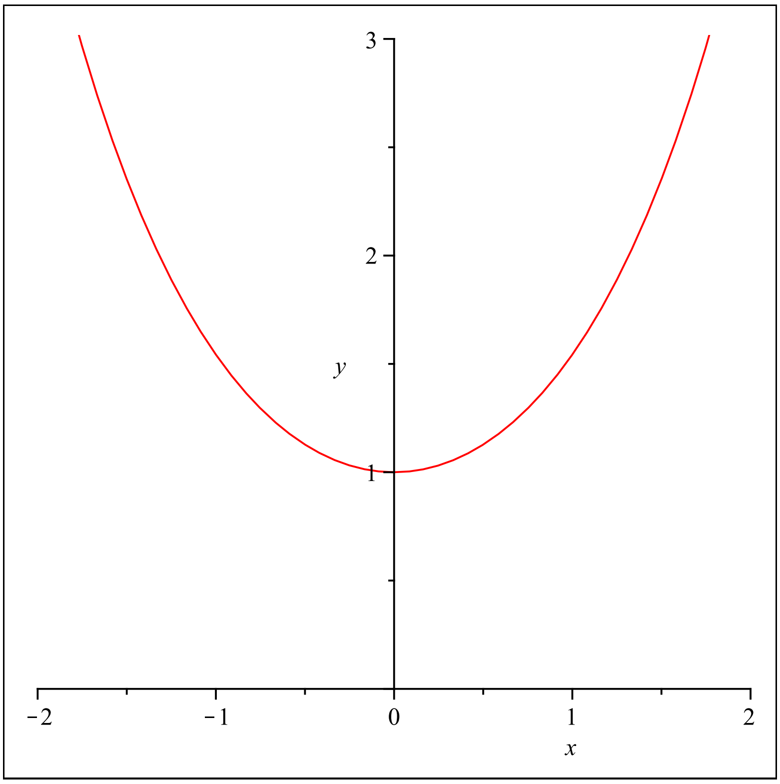
$\mathrm{Dom}(\sinh(x))=\mathbb{R}$.

$\mathrm{Range}(\sinh(x))=\mathbb{R}$.

For large $x \quad \sinh(x) \approx \dfrac{e^x}{2}$.

$f(x) = \cosh(x)$

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$



$\cosh(x)$ is an even function.

$\cosh(0) = 1$.

Dom($\cosh(x)$)=$\mathbb{R}$.

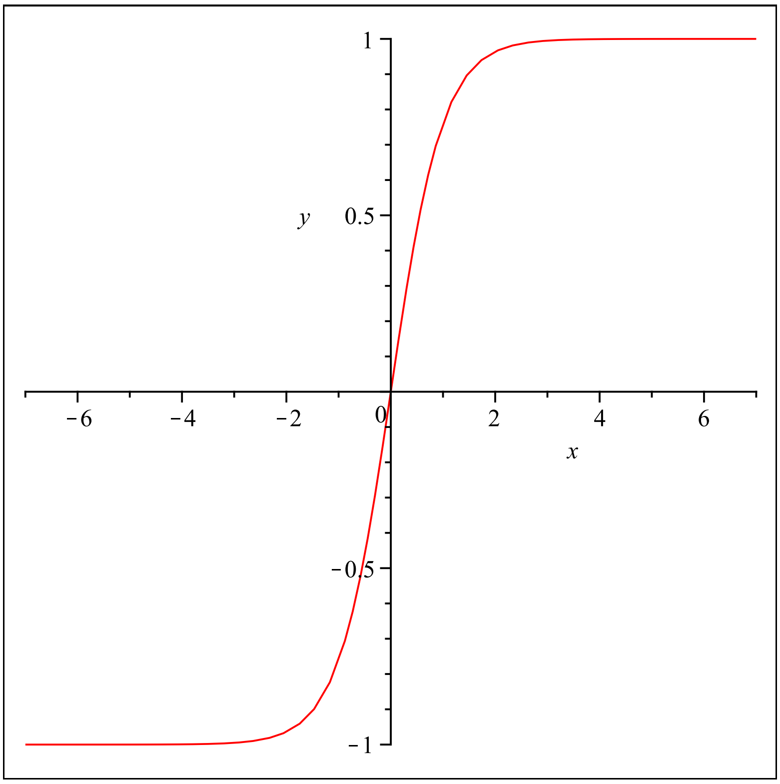
Range($\cosh(x)$)=$[1, \infty]$.

For large x $\quad \cosh(x) \approx \dfrac{e^x}{2}$.

The cosh curve is the shape of a hanging rope. Look at the hanging telegraph cables next time you are out on the road, they are all cosh curves!

$f(x) = \tanh(x)$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$



$\tanh(x)$ is an odd function.

$\tanh(x)$ is an increasing function

$\tanh(0) = 0.$

Dom($\tanh(x)$)=$\mathbb{R}$.

Range($\tanh(x)$)=$(-1, 1)$.

For large x $\quad \tanh(x) \approx 1.$

We also have some minor hyperbolic trig functions defined as:

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\operatorname{cosech}(x) = \frac{1}{\sinh(x)}$$

and

$$\coth(x) = \frac{1}{\tanh(x)}$$

Let's now take a look at the properties of these functions. They mimic almost exactly those of the trig functions. In the list below the corresponding trig property follows the hyperbolic result. You should not only know all of the results, you should also be able to prove them. The proofs are generally trivial.

a) $\frac{d}{dx}\sinh(x) = \cosh(x)$ $\quad\star\ \frac{d}{dx}\sin(x) = \cos(x)\ \star$

b) $\frac{d}{dx}\cosh(x) = \sinh(x)$ $\quad\star\ \frac{d}{dx}\cos(x) = -\sin(x)\ \star$

c) $\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$ $\quad\star\ \frac{d}{dx}\tan(x) = \sec^2(x)\ \star$

d) $\cosh^2(x) - \sinh^2(x) = 1$ $\quad\star\ \cos^2(x) + \sin^2(x) = 1\ \star$

e) $1 - \tanh^2(x) = \operatorname{sech}^2(x)$ $\quad\star\ 1 + \tan^2(x) = \sec^2(x)\ \star$

f) $\sinh(2x) = 2\sinh(x)\cosh(x)$ $\quad\star\ \sin(2x) = 2\sin(x)\cos(x)\ \star$

g) $\cosh(2x) = \cosh^2(x) + \sinh^2(x)$ $\quad\star\ \cos(2x) = \cos^2(x) - \sin^2(x)\ \star$

h) $\tanh(2x) = \dfrac{2\tanh(x)}{1 + \tanh^2(x)}$ $\quad\star\ \tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}\ \star$

i) $\displaystyle\int \cosh(ax+b)\,dx = \frac{1}{a}\sinh(ax+b) + C$ $\quad\star\ \int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C\ \star$

j) $\displaystyle\int \sinh(ax+b)\,dx = \frac{1}{a}\cosh(ax+b) + C$ $\quad\star\ \int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C\ \star$

k) $\displaystyle\int \operatorname{sech}^2(ax+b)\,dx = \frac{1}{a}\tanh(ax+b) + C$ $\quad\star\ \int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C\ \star$

It is clear from the above properties that the hyperbolic trig functions do indeed behave very much like the real trig functions! But keep your eye on those negatives!

Let's prove some of the results. The proofs will involve the exponential definitions and a little algebra.

Proof b): $\dfrac{d}{dx}\cosh(x) = \sinh(x)$

★

Proof d): $\cosh^2(x) - \sinh^2(x) = 1$

★

Proof f): $\sinh(2x) = 2\sinh(x)\cosh(x)$

★

Proof i): $\displaystyle\int \cosh(ax+b)\,dx = \frac{1}{a}\sinh(ax+b) + C$

★

We finish the lecture with some typical questions on the hyperbolic trig functions. Note that some calculators allow you to evaluate the hyperbolic trig functions by using a HYP button.

Example 1: Find $\cosh(2)$ correct to 3 decimal places.

Method 1: $\cosh(2) = \dfrac{e^2 + e^{-2}}{2} = 3.762$ or

Method 2: HYP $\cos 2 = 3.762$

★

Note that, despite the names, the hyperbolic trig functions don't actually have anything to do with the real trig functions. So when using a calculator to evaluate them, it doesn't matter in the slightest whether your calculator is in radian or degree mode.

Example 2: Find the derivative of $f(x) = x^3 \cosh(x)$.

★ $3x^2 \cosh(x) + x^3 \sinh(x)$ ★

Example 3: Find the derivative of $y = \sinh(x^7 + 1)$.

★ $7x^6 \cosh(x^7 + 1)$ ★

Example 4: Find the gradient of $y = \dfrac{\tanh(x)}{3x - 2}$ at $x = 0$.

★ $-\dfrac{1}{2}$ ★

Example 5: Evaluate $\displaystyle\int_0^{\frac{\ln(3)}{5}} \sinh(5x)\,dx$

$$\bigstar \quad \frac{2}{15} \quad \bigstar$$

Example 6: Find $\displaystyle\int x\cosh(x)\,dx$

$$\bigstar \quad x\sinh(x) - \cosh(x) + C \quad \bigstar$$

Example 7: Suppose that $x = 3\sinh(t)$ and $y = 4\cosh(t)$. Find a Cartesian relation between x and y.

$$\bigstar \quad \frac{y^2}{16} - \frac{x^2}{9} = 1. \quad \text{A hyperbola.} \quad \bigstar$$

LECTURE 23
The Inverse Hyperbolic Trig Functions

$$\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$$

$$\cosh^{-1}(x) = \ln(x + \sqrt{x^2-1})$$

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1}(x) + C = \ln(x + \sqrt{x^2+1}) + C$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1}(x) + C = \ln(x + \sqrt{x^2-1}) + C$$

$$\int \frac{1}{1-x^2}\,dx = \tanh^{-1}(x) + C = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + C \quad \text{for} \quad |x| < 1$$

We also have the slightly more general

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}(\frac{x}{a}) + C = \ln(x + \sqrt{x^2+a^2}) + C$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}(\frac{x}{a}) + C = \ln(x + \sqrt{x^2-a^2}) + C$$

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\tanh^{-1}(\frac{x}{a}) + C = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + C \quad \text{for} \quad |x| < a$$

All of these integrals will be made available to you in the final examination in a table of integrals.

In this final lecture we will invert the hyperbolic trigonometric functions of the previous lecture. Recall that $\sinh(x) = \dfrac{1}{2}(e^x - e^{-x})$ and that $\sinh : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing function. Hence $\sinh^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$ is well defined.

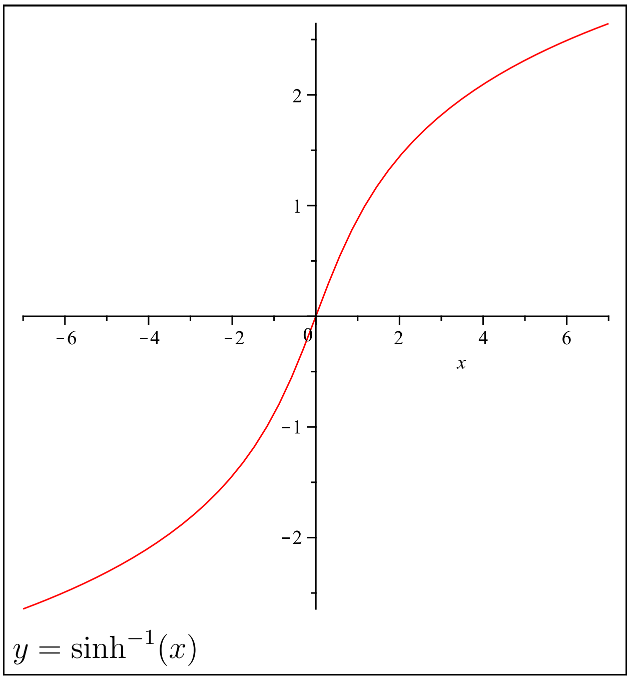
Claim: $\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$

Proof: Swapping the variables in $y = \dfrac{1}{2}(e^x - e^{-x})$ yields $x = \dfrac{1}{2}(e^y - e^{-y})$. Solving for y we obtain:

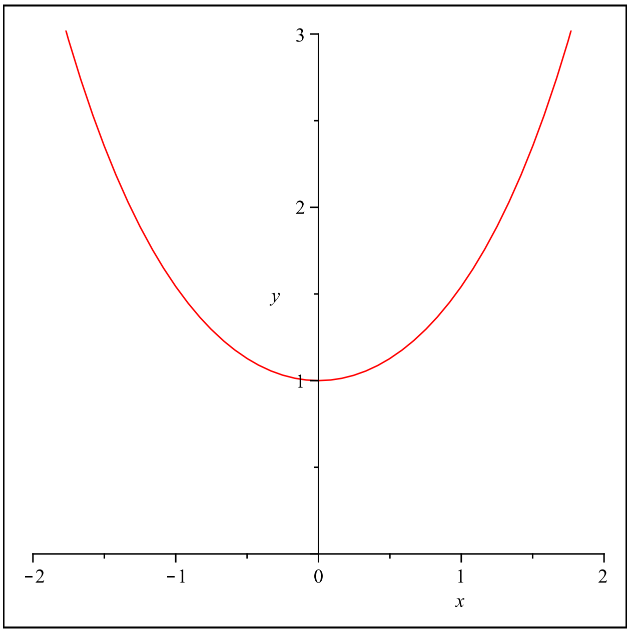
★

Note that the inverse trig functions have no alternative form. When talking about $y = \sin^{-1}(x)$ we have no choice but to write down $y = \sin^{-1}(x)$. But the hyperbolic trig functions are defined in term of the exponential function so it is not surprising that their inverses may be recast in term of the natural log function! Sometimes we will use $\sinh^{-1}(x)$, sometimes $\ln(x + \sqrt{x^2+1})$. It is nice to have a choice. Note also that almost all of the hyperbolic and inverse hyperbolic trig functions are built-in to the modern calculators.

The graph of $y = \sinh^{-1}(x)$ is



The cosh curve causes a little trouble as it fails the horizontal line test. Recall that the graph of $\cosh(x)$ is

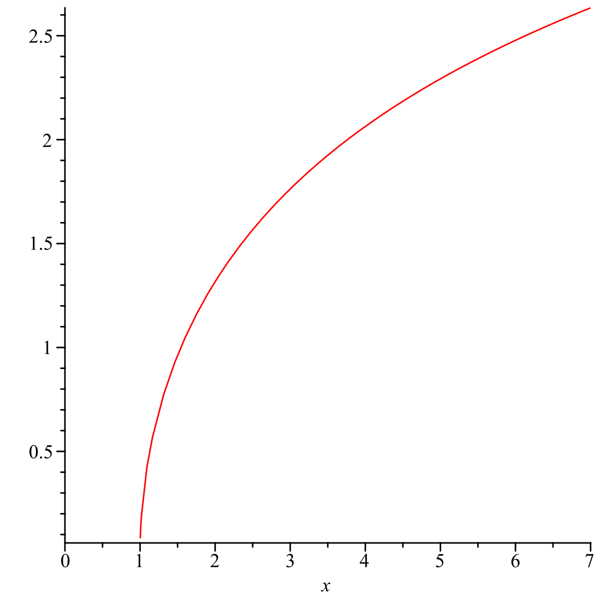


Here we first need to restrict the domain to $[0, \infty)$ so that the function becomes $1 - 1$ with a domain of $[0, \infty)$ and a range of $[1, \infty)$. So we delete the left hand half of the cosh graph and then invert.

A similar argument to the one above yields $\cosh^{-1}(x) : [1, \infty) \longrightarrow [0, \infty)$ given by

$$\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1}).$$

A sketch is

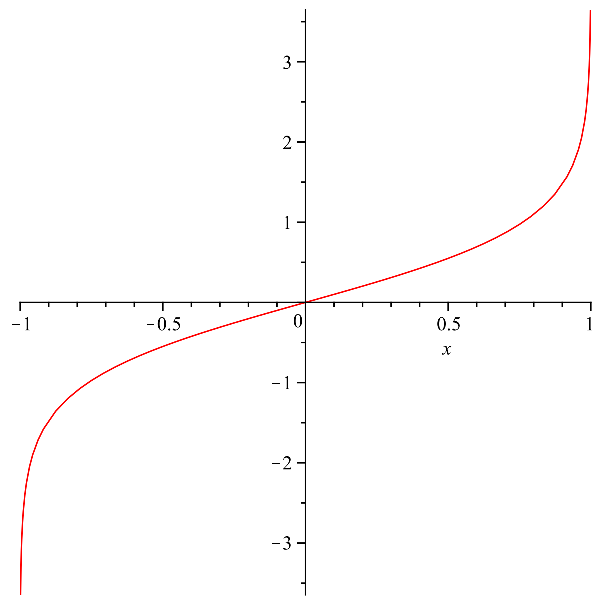


$y = \cosh^{-1}(x)$

Finally $\tanh(x) : \mathbb{R} \longrightarrow (-1, 1)$ is invertible with $\tanh^{-1}(x) : (-1, 1) \longrightarrow \mathbb{R}$ given by

$$\tanh^{-1}(x) = \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right)$$

with a graph



$y = \tanh^{-1}(x)$

Example 1: Find $\cosh(\sinh^{-1}(\frac{1}{4}))$.

Method 1 (Exact) : $\cosh^2(x) = 1 + \sinh^2(x) \longrightarrow \cosh(x) = \sqrt{1 + \sinh^2(x)}$

(we take the positive square root since $\cosh(x)$ is always positive). So

$$\cosh(\sinh^{-1}\left(\frac{1}{4}\right)) = \sqrt{1 + \sinh^2\left(\sinh^{-1}\left(\frac{1}{4}\right)\right)} = \sqrt{1 + \sinh\left(\sinh^{-1}\left(\frac{1}{4}\right)\right)\sinh\left(\sinh^{-1}\left(\frac{1}{4}\right)\right)}$$
$$= \sqrt{1 + (\frac{1}{4})^2} = \sqrt{1 + (\frac{1}{16})} = \sqrt{(\frac{17}{16})} = \frac{\sqrt{17}}{4}.$$

Method 2 (Calculator) : If an approximate answer is acceptable we can just use the calculator:

HYP cos (SHIFT HYP sin $(1 \div 4)) = 1.030776406 \approx \frac{\sqrt{17}}{4}$.

★

Example 2: Simplify $\cosh(2\cosh^{-1}(5))$.

We have $\cosh(2x) = \cosh^2(x) + \sinh^2(x) \longrightarrow$

$\cosh(2x) = \cosh^2(x) + (\cosh^2(x) - 1) = 2\cosh^2(x) - 1$.

So $\cosh(2\cosh^{-1}(5)) = 2\cosh^2(\cosh^{-1}(5)) - 1 = 2\cosh(\cosh^{-1}(5))\cosh(\cosh^{-1}(5)) - 1$

$= 2 \times 5 \times 5 - 1 = 49$.

Once again we can also simply crunch the numbers on a calculator:

HYP cos($2\times$SHIFT HYP cos (5))= 49.

★

We close the lecture and the course with the calculus of the inverse hyperbolic trig functions.

We have the following results:

a) $\dfrac{d}{dx}\sinh^{-1}(x) = \dfrac{1}{\sqrt{x^2+1}}$ $\qquad\star$ $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$ $\star$

b) $\dfrac{d}{dx}\cosh^{-1}(x) = \dfrac{1}{\sqrt{x^2-1}}$ $\qquad\star$ $\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$ $\star$

c) $\dfrac{d}{dx}\tanh^{-1}(x) = \dfrac{1}{1-x^2}$ $\qquad\star$ $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$ $\star$

Proof a:

$y = \sinh^{-1}(x) \longrightarrow x = \sinh(y) \longrightarrow$

$\star$

The proofs of b) and c) follow similar lines.

Example 3:

a) Find $\dfrac{d}{dx}\sinh^{-1}(x^3)$.

$\star \quad \dfrac{3x^2}{\sqrt{x^6+1}} \quad \star$

b) Find $\dfrac{d}{dx}\{\ln(x)\cosh^{-1}(x)\}$.

$$\star \quad \frac{\ln(x)}{\sqrt{x^2-1}} + \frac{\cosh^{-1}(x)}{x} \quad \star$$

It follows from the above derivatives that:

a) $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1}(x) + C = \ln(x + \sqrt{x^2+1}) + C$ $\quad\star \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C \quad \star$

b) $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1}(x) + C = \ln(x + \sqrt{x^2-1}) + C$ $\quad\star \int \frac{1}{\sqrt{1-x^2}}\,dx = -\cos^{-1}(x) + C \quad \star$

c) $\displaystyle\int \frac{1}{1-x^2}\,dx = \tanh^{-1}(x) + C = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + C$ $\quad\star \int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C \quad \star$

We also have the slightly more general

d) $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}(\frac{x}{a}) + C = \ln(x + \sqrt{x^2+a^2}) + C$ $\quad\star \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}(\frac{x}{a}) + C \quad \star$

e) $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}(\frac{x}{a}) + C = \ln(x + \sqrt{x^2-a^2}) + C$ $\quad\star \int \frac{1}{\sqrt{a^2-x^2}}\,dx = -\cos^{-1}(\frac{x}{a}) + C \quad \star$

f) $\displaystyle\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\tanh^{-1}(\frac{x}{a}) + C = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + C \quad \text{for} \quad |x| < a$

$$\star \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + C \quad \star$$

All of these will be available to you in the final examination in a table of integrals.

Example 4: Evaluate $\int \frac{dx}{\sqrt{9x^2+4}}\,dx$

$\bigstar\quad \frac{1}{3}\sinh^{-1}(\frac{3x}{2})\quad \bigstar$

Example 5: Evaluate $\int_0^{\frac{1}{2}} \frac{dx}{1-x^2}$

$\bigstar\quad \frac{1}{2}\ln(3)\quad \bigstar$

SOME FINAL INFORMATION

1. Please check online that all your marks are recorded correctly.

2. Read the school pages on additional assessment/special consideration so that you are fully aware of the rules that apply.

3. Past papers are on Moodle.

4. The final exam is 2 hours long with three questions.

5. Make sure you turn up at the right time in the right location. Check your exam timetable!!

6. Please start a new book for each of the 3 questions.

7. Make sure your calculator has a UNSW APPROVED sticker (available from the School of Mathematics office) or you will not be allowed to use it during the exam.

8. Please take the time to complete all online surveys regarding the administration and teaching of the course.

9. A consultation roster will shortly be posted on Moodle.

Good Luck!

Milan Pahor