School of Mathematics and Statistics
Math1131-Algebra

Lec20: Properties of Determinants

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2020 Term 1

Determinants and inverses

Exercise 1. Use $\det(AB) = \det(A)\det(B)$ to prove that

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

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$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

$$AA^{-1} = I$$

$$\Rightarrow \det(AA^{-1}) = \det(I)$$

$$\Rightarrow \det(A)\det(A^{-1}) = 1$$

$$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$

$$\det\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

$$= 1 \times 1 \times 1 = 1$$

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Properties of determinants

Properties of determinants

REMEMBER

(a) $\det(A^T) = \det(A)$

(b) $\det(AB) = \det(A)\det(B)$

(c) $R_i \leftarrow R_i + \alpha R_j$ for $i \neq j$ does not change the determinant.

(d) $R_i \leftrightarrow R_j$ for $i \neq j$ changes the sign of the determinant.

(e) $R_i \leftarrow \alpha R_i$ scales the determinant by α.

(f) If A has a zero row or column then $\det(A) = 0$.

Some important consequences:

REMEMBER

(g) $\det(\alpha A) = \alpha^n \det(A)$ for an $n \times n$ matrix. ⚠ Note that it is α^n not α

(h) $\det(A^{-1}) = 1/\det(A)$.

(i) Swapping two columns changes the sign of the determinant.

(j) Row operations can simplify the calculation of determinants.

(k) A is invertible *if and only* $\det(A) \neq 0$.

(l) If one row of A is a multiple of another row then $\det(A) = 0$.

(m) If one column of A is a multiple of another column then $\det(A) = 0$.

Determinants

Exercise 2.

Suppose that A is a 3×3 matrix and $\det(A) = 5$. Find $\det(B)$ if B is given by

1. A^T. $\quad \det(A^T) = \det(A) = 5$

2. A with the first two rows swapped. $\quad \det(B) = -\det(A) = -5$

3. A with the first two rows swapped and the last two columns swapped.

   $\det(A) = (-1)(-1)\det A = 5$

4. A with the second row multiplied by 7. $\quad \det(B) = 7\det A = 7 \times 5 = 35$

5. $2A$. $\quad \det(2A) = \underbrace{2 \times 2 \times 2}\times \det A = 2^3 \times 5 = 8 \times 5 = 40$

6. A with the second row replaced by 3 times the first row. $\quad \begin{pmatrix} R_1 \\ 3R_1 \\ R_3 \end{pmatrix}$

   $\det B = 0$ (A row is a multiple of another one)

7. A^{-1}.

   $\det(A^{-1}) = \frac{1}{5}$

UNSW SYDNEY

Determinants: Checking the rules on an example

Exercise 3. Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 8 \end{pmatrix}$.

1. Find $\det(A)$.
2. Check that $R_2 \leftarrow R_2 - 5R_1$ doesn't change the determinant.
3. Check that $C_2 \leftarrow C_2 + 4C_1$ doesn't change the determinant.
4. Show that $R_2 \leftarrow 3R_2 + 4R_1$ does change the determinant.

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1. $\det(A) = \begin{vmatrix} 1 & 3 \\ 2 & 8 \end{vmatrix} = 1 \times 8 - 2 \times 3$
$= 8 - 6 = 2$

2. $\det \begin{pmatrix} 1 & 3 \\ -3 & -7 \end{pmatrix} = 1 \times (-7) - (-3) \times 3 = -7 + 9 = 2 = \det(A)$

3. $\det \begin{pmatrix} 1 & 7 \\ 2 & 16 \end{pmatrix} = 1 \times 16 - 2 \times 7 = 16 - 14 = 2 = \det(A)$

4. $\det \begin{pmatrix} 1 & 3 \\ 10 & 36 \end{pmatrix} = 1 \times 36 - 10 \times 3 = 36 - 30 = 6 \neq 2 = \det(A)$

Determinants: Checking the rules on an example

Exercise 3, continued. $A = \begin{pmatrix} 1 & 3 \\ 2 & 8 \end{pmatrix}$.

4. Show that $R_2 \leftarrow 3R_2 + 4R_1$ does change the determinant.
5. Find the determinant after swapping the rows of A.
6. Reduce A to echelon form and find the determinant.

Determinants: Checking the rules on an example

Exercise 3, continued. $A = \begin{pmatrix} 1 & 3 \\ 2 & 8 \end{pmatrix}$.

4. Show that $R_2 \leftarrow 3R_2 + 4R_1$ does change the determinant.
5. Find the determinant after swapping the rows of A.
6. Reduce A to echelon form and find the determinant.

5. $\det\begin{pmatrix} 2 & 8 \\ 1 & 3 \end{pmatrix} = 2\times 3 - 1\times 8 = 6-8 = -2 = -\det(A)$

6. $\begin{pmatrix} \boxed{1} & 3 \\ 2 & 8 \end{pmatrix}$ $R_2 \leftarrow R_2 - 2R_1$

$\rightsquigarrow \begin{pmatrix} \boxed{1} & 3 \\ 0 & \boxed{2} \end{pmatrix}$

$\det\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} = 1\times 2 - 0\times 3 = 2$

Determinants and inverses

Determinants and inverses.

For a square matrix A, the following are equivalent:

- $\det(A) \neq 0$
- A is invertible
- $A\overrightarrow{\boldsymbol{x}} = \overrightarrow{\mathbf{0}}$ has the unique solution $\overrightarrow{\boldsymbol{x}} = \overrightarrow{\mathbf{0}}$
- $A\overrightarrow{\boldsymbol{x}} = \overrightarrow{\boldsymbol{b}}$ has a unique solution for all $\overrightarrow{\boldsymbol{b}} \in \mathbb{R}^n$.

REMEMBER

TIP!

The determinant is a detector of invertible matrices

Determinants and inverses

REMEMBER

Determinants and inverses.

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$$A^{-1}A\underset{\sim}{x} = A^{-1}\underset{\sim}{b}$$

$$I\underline{x} = A^{-1}\underline{b}$$

$$\underline{x} = A^{-1}\underline{b}$$

TIP!

The determinant is a detector of invertible matrices

Determinants

Exercise 4. Last lecture, we found that $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ does not have an inverse.

Verify this fact by checking that $\det(A) = 0$.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \overset{①}{=} \begin{vmatrix} 1 & 1 & 2 \\ 4 & 1 & 2 \\ 7 & 1 & 2 \end{vmatrix}$$

$$\overset{②}{=} 0$$

① $C_2 \leftarrow C_2 - C_1$
$C_3 \leftarrow C_3 - C_1$

② C_3 is a multiple of C_2 $\Leftrightarrow$ $\det = 0$

Determinants

Exercise 5. For what values of α is $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ \alpha & 0 & 1 \end{pmatrix}$ invertible?

① strategy: $\det A \neq 0 \iff A$ is invertible

② $\det A = \begin{vmatrix} +1 & 2 & -1 \\ -3 & 1 & 0 \\ +\alpha & -0 & +1 \end{vmatrix}$

let's expand along R_3

$$= \alpha \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} - 0 + 1 \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

$$\boxed{\det A = \alpha - 5}$$

③ Ans: $\boxed{A \text{ is invertible iff } \alpha \neq 5}$

Determinants with Maple

```
> # exercise 5
> with(LinearAlgebra):
> # Enter the matrices columnwise

  A := < <1,3,alpha>|<2,1,0>|<-1, 0, 1> >;
```

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ \alpha & 0 & 1 \end{bmatrix}$$

```
> # Calculate the determinant of A

  Det_of_A := Determinant(A);
```

$$Det\_of\_A := -5 + \alpha$$

Determinants

Exercise 6. Find $\begin{vmatrix} 1 & -1 & 0 & 3 \\ 2 & -2 & 6 & -1 \\ 4 & -2 & 1 & 7 \\ 3 & 5 & -7 & 2 \end{vmatrix}$

Determinants

Exercise 6. Find $\begin{vmatrix} 1 & -1 & 0 & 3 \\ 2 & -2 & 6 & -1 \\ 4 & -2 & 1 & 7 \\ 3 & 5 & -7 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 & 3 \\ -1 & -2 & -2 & 5 \\ 0 & 6 & 1 & -7 \\ 3 & -1 & 7 & 2 \end{vmatrix}$ $\quad R_2 \leftarrow R_2 + R_1$, $\quad R_4 \leftarrow R_4 - 3R_1$

$$= \begin{vmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & 2 & 8 \\ 0 & 6 & 1 & -7 \\ 0 & -7 & -5 & -7 \end{vmatrix} = 1 \times \begin{vmatrix} 0 & 2 & 8 \\ 6 & 1 & -7 \\ -7 & -5 & -7 \end{vmatrix} = 2 \begin{vmatrix} 0 & 1 & 4 \\ 6 & 1 & -7 \\ -7 & -5 & -7 \end{vmatrix}$$

$$= 2\left(0 \begin{vmatrix} 1 & -7 \\ -5 & -7 \end{vmatrix} - 6 \begin{vmatrix} 1 & 4 \\ -5 & -7 \end{vmatrix} + (-7) \begin{vmatrix} 1 & 4 \\ 1 & -7 \end{vmatrix} \right)$$

$$= 2\left(-6(-7+20) - 7(-7-4) \right)$$

$$= 2(-78+77) = -2$$

Determinants

Exercise 7. Find $\begin{vmatrix} 1 & 5 & 4 & 3 \\ 3 & -2 & 12 & -2 \\ 1 & -3 & 4 & 3 \\ 2 & -11 & 8 & 1 \end{vmatrix}$

Determinants

Exercise 8. Let $t \in (-\pi, \pi]$. For what values of t is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1+\cos t & 1+\sin 2t \\ 1 & 1+\sin t & 1+\sin 2t \end{pmatrix}$$

invertible?

Determinants

Exercise 8, continued. Let $t \in (-\pi, \pi]$. For what values of t is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1+\cos t & 1+\sin 2t \\ 1 & 1+\sin t & 1+\sin 2t \end{pmatrix}$$

invertible?

Determinant with Maple

```
> restart;
  with(LinearAlgebra):
> # Enter the matrices columnwise
  A := < <1,1,1> |  <1,1+cos(t),1+sin(t)> |  <1,1+sin(2*t),1+sin(2*t)> >;
```

$$A := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1+\cos(t) & 1+\sin(2\,t) \\ 1 & 1+\sin(t) & 1+\sin(2\,t) \end{bmatrix}$$

```
> # Calculate the determinant of A
  Det_of_A := Determinant(A);
```

$$Det\_of\_A := \cos(t)\,\sin(2\,t) - \sin(2\,t)\,\sin(t)$$

```
> # Factorise the determinant of A
  factor(Det_of_A);
```

$$\sin(2\,t)\,(\cos(t) - \sin(t))$$

```
> # A is invertible unless det(A) = 0
   solve(Det_of_A = 0,t);
```

$$\frac{\pi}{4}, 0$$

THE END

- Check online that your marks are correctly recorded.
- Read the information about the online final exam on Moodle!
- Make sure you book at time for your exam.
- Past papers with solutions are in the course pack and on Moodle. The 2019 past paper give you an idea of what to expect.
- A link to last semester's exam revision live stream will be posted online.
- Make sure you have a UNSW approved calculator.
- Please complete the myExperience surveys.
- A pre-exam consultation roster will be posted on Moodle.

Good luck!