School of Mathematics and Statistics Math1131-Algebra

Lec20: Properties of Determinants

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Determinants and inverses

Exercise 1. Use $\det(AB) = \det(A)\det(B)$ to prove that

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$



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$$AA^{-1} = I$$

$$\Rightarrow det(AA^{-1}) = det(I)$$

$$\Rightarrow det(A)det(A^{-1}) = I$$

$$\Rightarrow det(A^{-1}) = \frac{I}{det(A)}$$

$$det\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

$$= |x|x| = |x|$$



Properties of determinants

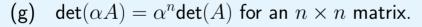


Properties of determinants

- (a) $det(A^T) = det(A)$
- (b) det(AB) = det(A)det(B)
- (c) $R_i \leftarrow R_i + \alpha R_j$ for $i \neq j$ does not change the determinant.
- $R_i \leftrightarrow R_j$ for $i \neq j$ changes the sign of the determinant.
- (e) $R_i \leftarrow \alpha R_i$ scales the determinant by α .
- If A has a zero row or column then det(A) = 0.



Some important consequences:





Note that it is α^n not α

- (h) $\det(A^{-1}) = 1/\det(A)$.
- Swapping two columns changes the sign of the determinant.
- Row operations can simplify the calculation of determinants.
- (k) A is invertible if and only $det(A) \neq 0$.
- If one row of A is a multiple of another row then det(A) = 0. (I)
- If one column of A is a multiple of another column then det(A) = 0. (m)



Exercise 2.

Suppose that A is a 3×3 matrix and det(A) = 5. Find det(B) if B is given by

1.
$$A^{T}$$
. $det(A^{T}) = det(A) = 5$

- 2. A with the first two rows swapped. det(B) = -det(A) = -5
- A with the first two rows swapped and the last two columns swapped. det (A) = (-1) (-1) dut A = 5
- 4. A with the second row multiplied by 7. $der(B) = 7 der(A) = 7 \times 5 = 35$

5.
$$2A$$
. $det(2A) = 2 \times 2 \times 2 \times dut A = 2 \times 5 = 8 \times 5 = 40$

 \boldsymbol{A} with the second row replaced by 3 times the first row.

$$dtB = 0$$

6. A with the second row replaced by 3 times the first row.

$$det B = O \qquad (A row is a rowling)$$
7. A^{-1} .

$$det (A^{-1}) = \frac{1}{5}$$

Exercise 3. Let
$$A = \begin{pmatrix} 1 & 3 \\ 2 & 8 \end{pmatrix}$$
.

- 1. Find det(A).
- 2. Check that $R_2 \leftarrow R_2 5R_1$ doesn't change the determinant.
- 3. Check that $C_2 \leftarrow C_2 + 4C_1$ doesn't change the determinant.
- 4. Show that $R_2 \leftarrow 3R_2 + 4R_1$ does change the determinant.



Exercise 3. Let
$$A = \begin{pmatrix} 1 & 3 \\ 2 & 8 \end{pmatrix}$$
.

1. $det(A) = \begin{vmatrix} 1 & 3 \\ 2 & 8 \end{vmatrix} = 1 \times 8 - 2 \times 3$
 $= 8 - 6 = 2$

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2.
$$\det \begin{pmatrix} 1 & 3 \\ -3 & -7 \end{pmatrix} = 1 \times (-7) - (-3) \times 3 = -7 + 9 = 2 = \det(A)$$

3.
$$\det\left(\frac{1}{2},\frac{7}{16}\right) = 1 \times 16 - 2 \times 7 = 16 - 14 = 2 = \det(A)$$

4.
$$det \begin{pmatrix} 1 & 3 \\ 10 & 36 \end{pmatrix} = 1 \times 36 - 10 \times 3 = 36 - 30 = 6 \neq 2 = det(A)$$



Exercise 3, continued.
$$A = \begin{pmatrix} 1 & 3 \\ 2 & 8 \end{pmatrix}$$
.

- 4. Show that $R_2 \leftarrow 3R_2 + 4R_1$ does change the determinant.
- 5. Find the determinant after swapping the rows of A.
- 6. Reduce A to echelon form and find the determinant.



Exercise 3, continued.
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- 4. Show that $R_2 \leftarrow 3R_2 + 4R_1$ does change the determinant.
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5.
$$det(\frac{28}{13}) = 2\times 3 - 1\times 8 = 6-8 = -2 = -det(A)$$

$$\sim$$
 $\left(\begin{array}{c} 113\\ 012 \end{array}\right)$

$$\det\left(\frac{13}{02}\right) = 1\times2 - 0\times3 = 2$$



Determinants and inverses



Determinants and inverses.

For a square matrix A, the following are equivalent:

- $det(A) \neq 0$
- \bullet A is invertible
- $A\overrightarrow{x} = \overrightarrow{0}$ has the unique solution $\overrightarrow{x} = \overrightarrow{0}$
- $A\overrightarrow{x} = \overrightarrow{b}$ has a unique solution for all $\overrightarrow{b} \in \mathbb{R}^n$.



The determinant is a detector of invertible matrices

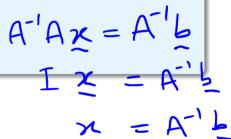
Determinants and inverses



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The determinant is a detector of invertible matrices



Exercise 4. Last lecture, we found that $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ does not have an inverse.

Verify this fact by checking that det(A) = 0.

$$0 \quad C_2 \leftarrow C_2 - C_1$$

$$C_3 \leftarrow C_3 - C_1$$



Exercise 5. For what values of
$$\alpha$$
 is $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ \alpha & 0 & 1 \end{pmatrix}$ invertible?

Strategy: $dvA \neq 0 \iff A$ is invalid.

And $dvA = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ \alpha & 0 & 1 \end{pmatrix}$ invalid.

Let's expand along R3

$$= \alpha \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} - \omega + \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

$$dut A = \times -5$$

$$\frac{\partial \omega}{\partial x} = x - 5$$
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Determinants with Maple



Exercise 6. Find
$$\begin{vmatrix} 1 & -1 & 0 & 3 \\ 2 & -2 & 6 & -1 \\ 4 & -2 & 1 & 7 \\ 3 & 5 & -7 & 2 \end{vmatrix}$$



Exercise 6. Find
$$\begin{vmatrix} 1 & -1 & 0 & 3 \\ 2 & -2 & 6 & -1 \\ 4 & -2 & 1 & 7 \\ 3 & 5 & -7 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 & 3 \\ -1 & -2 & -2 & 5 \\ 0 & 6 & 1 & -7 \\ 3 & -1 & 7 & 2 \end{vmatrix} R_2 \leftarrow R_2 + R_1$$

$$= \begin{vmatrix} 0 & 2 & 4 & 2 \\ 6 & 0 & 2 & 8 \\ 0 & 6 & 1 & -7 \\ 0 & -7 & -5 & -7 \end{vmatrix} = 1 \times \begin{vmatrix} 0 & 2 & 8 \\ 6 & 1 & -7 \\ -7 & -5 & -7 \end{vmatrix} = 2 \begin{vmatrix} 0 & 1 & 4 \\ 4 & 1 & -7 \\ -7 & -5 & -7 \end{vmatrix}$$

$$= 2 \left(0 \begin{vmatrix} 1 & -7 \\ -5 & -7 \end{vmatrix} - 6 \begin{vmatrix} 1 & 4 \\ -5 & -7 \end{vmatrix} + (-7) \begin{vmatrix} 1 & 4 \\ 1 & -7 \end{vmatrix} \right)$$

$$= 2 \left(-6 \left(-7 + 2a \right) - 7 \left(-7 - 4 \right) \right)$$

$$= 2 \left(-78 + 77 \right) = -2$$



Exercise 7. Find
$$\begin{vmatrix} 1 & 5 & 4 & 3 \\ 3 & -2 & 12 & -2 \\ 1 & -3 & 4 & 3 \\ 2 & -11 & 8 & 1 \end{vmatrix}$$



Exercise 8. Let $t \in (-\pi, \pi]$. For what values of t is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 + \cos t & 1 + \sin 2t \\ 1 & 1 + \sin t & 1 + \sin 2t \end{pmatrix}$$

invertible?



Exercise 8, continued. Let $t \in (-\pi, \pi]$. For what values of t is

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Determinant with Maple

```
> restart:
   with (LinearAlgebra):
> # Enter the matrices columnwise
  A := \langle \langle 1, 1, 1 \rangle | \langle 1, 1 + \cos(t), 1 + \sin(t) \rangle | \langle 1, 1 + \sin(2*t), 1 + \sin(2*t) \rangle >;
                                    A := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 + \cos(t) & 1 + \sin(2t) \\ 1 & 1 + \sin(t) & 1 + \sin(2t) \end{bmatrix}
> # Calculate the determinant of A
   Det of A := Determinant(A);
                               Det of A := \cos(t) \sin(2t) - \sin(2t) \sin(t)
> # Factorise the determinant of A
   factor (Det of A);
                                         \sin(2t)(\cos(t) - \sin(t))
> # A is invertible unless det(A) = 0
    solve (Det of A = 0, t);
                                                    \frac{\pi}{4}, 0
```



THE END

- Check online that your marks are correctly recorded.
- Read the information about the online final exam on Moodle!
- Make sure you book at time for your exam.
- Past papers with solutions are in the course pack and on Moodle. The 2019 past paper give you an idea of what to expect.
- A link to last semester's exam revision live stream will be posted online.
- Make sure you have a UNSW approved calculator.
- Please complete the myExperience surveys.
- A pre-exam consultation roster will be posted on Moodle.

Good luck!

