LECTURE 20
The Natural Log Function

The Limit Form of the Comparison Test: Suppose that f and g are non-negative and bounded on $[a, \infty]$ and $\displaystyle\lim_{x \longrightarrow \infty} \frac{f(x)}{g(x)} = L$ where $0 < L < \infty$. Then $\displaystyle\int_a^\infty f(x)\,dx$ and $\displaystyle\int_a^\infty g(x)\,dx$ will either both converge or both diverge.

The natural log function is defined as $\displaystyle \ln(x) = \int_1^x \frac{1}{t}\,dt$

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

The Limit Form of the Comparison Test

Recall from the last lecture that $p-$integrals of the form $\displaystyle\int_a^\infty \frac{1}{x^p}\,dx$ converge if $p > 1$ and diverge if $p \leq 1$. These simple integrals can then be used in comparison tests to attack more complicated improper integrals. Unfortunately a direct comparision cannot always be made. We then use the more subtle limit form of the comparison test.

The Limit Form of the Comparison Test:
Suppose that f and g are non-negative and bounded on $[a, \infty]$ and

$$\lim_{x \longrightarrow \infty} \frac{f(x)}{g(x)} = L$$

where $0 < L < \infty$. Then $\displaystyle\int_a^\infty f(x)\,dx$ and $\displaystyle\int_a^\infty g(x)\,dx$ will either both converge or both diverge.

★

That is f and g will have identical outcomes as improper integrals. The functions are locked together.

This test is remarkably effective. It allows us to convert intuitive arguments into solid mathematics.

Proof: See Printed Notes.

Example 1: Determine whether $\displaystyle\int_3^\infty \frac{2x^{11}+5}{x^{12}-7x^4+24}\,dx$ converges or diverges.

We would struggle to establish a comparison using direct inequalities here, and we certainly can't find the primitive. However by considering dominant terms it looks as though this integral is very similar to the diverging $p-$integral $\displaystyle\int_3^\infty \frac{1}{x}\,dx$.

So let $f(x) = \dfrac{2x^{11}+5}{x^{12}-7x^4+24}$ and $g(x) = \dfrac{1}{x}$. Then

$$\lim_{x\longrightarrow\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{2x^{11}+5}{x^{12}-7x^4+24} \times \frac{x}{1}$$

$$= \lim_{x\to\infty} \frac{2x^{12}+5x}{x^{12}-7x^4+24} = \lim_{x\to\infty} \frac{2+\cancel{\frac{5}{x^{11}}}}{1-\cancel{\frac{7}{x^8}}+\cancel{\frac{24}{x^{12}}}}$$

$$= 2 \quad , \quad 0 < 2 < \infty$$

Know $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges.

(diverging p-integral $p = 1 \leq 1$

$\therefore$ By limit form of comparison test

$\displaystyle\int_3^\infty \frac{2x^{11}+5}{x^{12}-7x^4+24}$ <u>also</u> <u>diverges</u>

★ The integral diverges by the limit form of the comparison test ★

This test should make sense. It is is saying that if $\displaystyle\lim_{x\longrightarrow\infty} \frac{f(x)}{g(x)}$ is finite and non-zero then the two functions have equal powers at infinity and hence will integrate up in a similar fashion.

Question: How do we know who is f and who is g ?

<u>Doesn't matter</u>

Example 2: Determine whether $\displaystyle\int_7^\infty \frac{8}{\sqrt{x^4-x^2-1}}\,dx$ converges or diverges.

$$\frac{8}{\sqrt{x^4-x^2-1}} \approx \frac{1}{\sqrt{x^4}} = \frac{1}{x^2}$$

and $\displaystyle\int_7^\infty \frac{1}{x^2}\,dx$ is a converging p−integral with $p = 2 > 1$. So we expect convergence!

Let $f(x) = \dfrac{8}{\sqrt{x^4-x^2-1}}$ and $g(x) = \dfrac{1}{x^2}$. Then

$$\lim_{x\longrightarrow\infty} \frac{f(x)}{g(x)} = \frac{8}{\sqrt{x^4-x^2-1}} \times \frac{x^2}{1} = \lim_{x\to\infty} \frac{8x^2}{\sqrt{x^4-x^2-1}}$$

$$= \lim_{x\to\infty} \sqrt{\frac{64x^4}{x^4-x^2-1}}$$

$$= \lim_{x\to\infty} \sqrt{\frac{64}{1-\cancel{\frac{1}{x^2}}-\cancel{\frac{1}{x^4}}}} = \sqrt{64} = 8$$

$$0 < 8 < \infty$$

$\displaystyle\int_7^\infty \frac{1}{x^2}\,dx$ is a converging p-integral $p = 2 > 1$

$\therefore \displaystyle\int_7^\infty \frac{8}{\sqrt{x^4-x^2-1}}\,dx$ also converges by limit form of the comparison test.

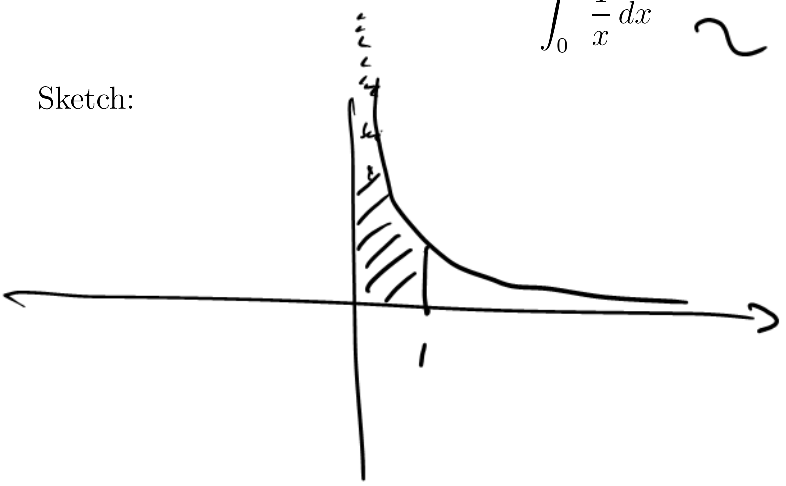
★ The integral converges by the limit form of the comparison test ★

We close this topic by noting that there is a second class of improper integrals. These involve integrating up to a vertical asymptote. For example

$$\int_0^1 \frac{1}{x}\,dx \sim \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \frac{1}{x}\,dx.$$

not in current Math1131 syllabus

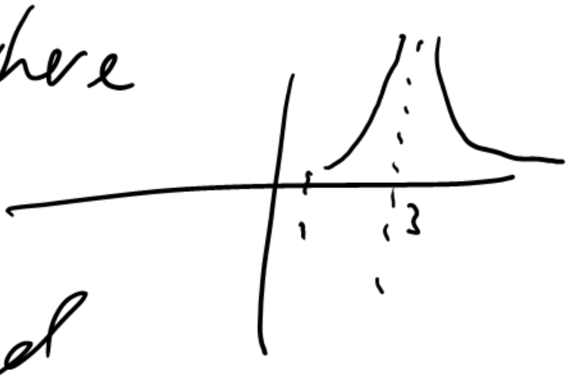
Sketch:



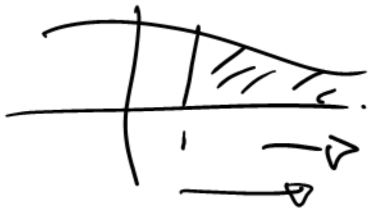
The problem here is not that the limits are infinite but rather that the function is infinite! We have a vertical tail rather than a horizontal tail.

This theory is approachable in much the same way, and strikes similar technical obstacles, however we do not examine these sort of improper integrals in Math1131.

Note also that situations like $\int_1^\infty f(x)\,dx$ where $f(x)$ has graph



will not be considered in Math1131. You can't just integrate across a vertical asymptote!

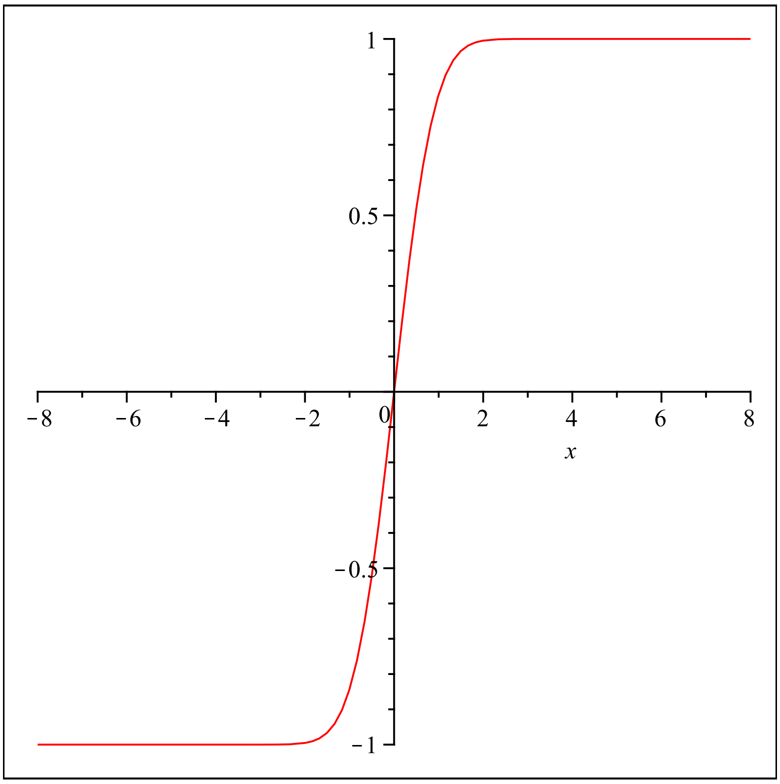


Functions Defined By Integrals

Whenever we cannot find the primitive of a function there is a temptation to use the integral to define a brand new function. For example $\int e^{-x^2}\,dx$ cannot be expressed in terms of standard functions prompting the definition of the error function $\mathrm{erf}(x)$:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt.$$

erf is a perfectly good function with enormous applications! It has a graph



$y = \mathrm{erf}(x)$

It also has all sorts of interesting properties.

Example 3: Prove that $\mathrm{erf}(x)$ is an odd function.

RTP $\mathrm{erf}(-x) = -\mathrm{erf}(x)$

$$\mathrm{erf}(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-t^2}\,dt$$

Let

$u = -t$

$du = -dt$

$t = 0 \to u = 0$

$t = -x \to u = --x = x$

$$\int = \int_0^x e^{-(-u)^2}(-du)$$

$$= -\int_0^x e^{-u^2}\,du$$

$$= -\int_0^x e^{-t^2}\,dt = -\mathrm{erf}(x)$$

$\therefore$ erf is odd. ★

Some of the most important functions in mathematics are defined by integrals. They include $\mathrm{Dilog}(x)$, $\mathrm{Chi}(x)$, $\mathrm{Shi}(x)$, $\mathrm{Si}(x)$, $\mathrm{Li}(x)$ and a host of others. You will not meet any of these until second year. A function that you all know very well however is the natural log function $y = \ln(x)$. It too is defined by an integral!

The Natural Log Function

You are all familiar with the natural logarithm function $y = \ln(x)$. Some of you will have seen the natural log function written as $y = \log_e(x)$. Both of these notations are quite common but your calculators probably have a ln button so we will stick to this notation.

You may not however have seen $y = \ln(x)$ properly defined as an integral:

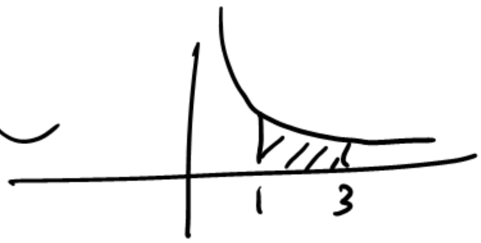
$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$

ie $x > 0$

This integral defines the ln function over the domain $(0, \infty)$. What does this mean?

Say you want to evaluate $\ln(3)$. You need to find $\displaystyle\int_1^3 \frac{1}{t}\,dt$. This is just a Riemann integral which can be if necessary knocked off using upper and lower sums. It is OK to define a function as an integral since we now have a nice tight definition of the integral.

The natural log function $y = \ln(x)$ has the following properties:



a) $\dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$.

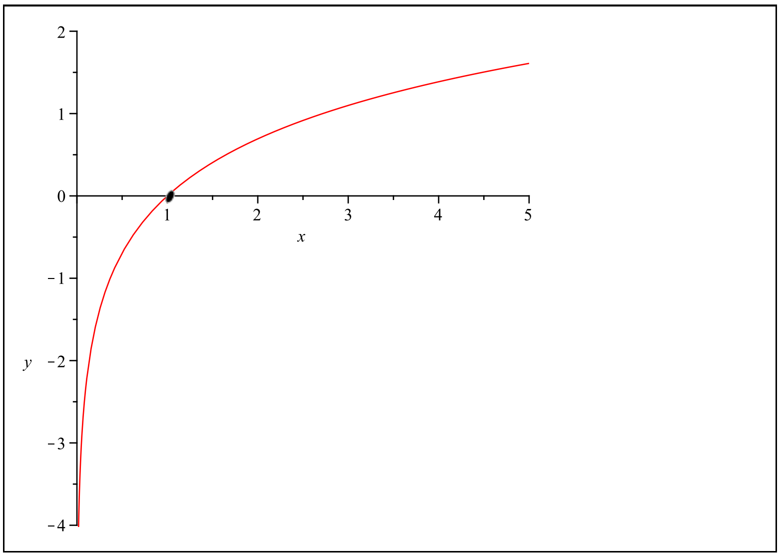
b) $\ln(1) = 0$.

c) $\ln(ab) = \ln(a) + \ln(b)$.

d) $\ln\left(\dfrac{a}{b}\right) = \ln(a) - \ln(b)$.

e) $\ln(a^r) = r\ln(a)$ for r rational.

The graph of the ln function is



Observe that the ln function is only defined for positive x and that $y = \ln(x)$ increases to ∞.

Let's take a look at some proofs:

a) $\dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$.

$$\frac{d}{dx}\ln x = \frac{d}{dx}\left[\int_1^x \frac{1}{t}\,dt\right] = \frac{1}{x}.$$

by 1st Fundamental th^m of calc.

b) $\ln(1) = 0$.

$$\ln(1) = \int_1^1 \frac{1}{t}\,dt = 0$$

c) $\ln(ab) = \ln(a) + \ln(b)$.

RHS$= \ln(a) + \ln(b) = \displaystyle\int_1^a \frac{1}{t}\,dt + \int_1^b \frac{1}{t}\,dt$

We make the substitution $u = at$ in the second integral

$$\int_1^b \frac{1}{t}\,dt \qquad \text{Let} \quad u = at \to t = \frac{u}{a}. \quad du = a\,dt$$

$$\int = \int_a^{ab} \frac{1}{\left(\frac{u}{\cancel{a}}\right)}\,\frac{du}{\cancel{a}} \qquad t = 1 \to u = a \quad t = b \to u = ab$$

$$= \int_a^{ab} \frac{du}{u} = \int_a^{ab} \frac{dt}{t}$$

$$\text{RHS} = \int_1^a \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt$$

$$= \int_1^{ab} \frac{1}{t}\,dt = \ln(ab)$$

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So we now have a different way of defining $\ln(x)$. But it is still the same boring old log function under calculus.

a) Find $\frac{d}{dx}x^2\ln(x)$.

$$\frac{d}{dx}(x^2 \ln x)$$

$$u = x^2 \rightarrow u' = 2x$$

$$v = \ln(x) \rightarrow v' = \frac{1}{x}.$$

$$(uv)' = u'v + v'u$$

$$= 2x\ln x + \frac{1}{x}\cdot x^2$$

$$= 2x\ln x + x$$

b) Find $\frac{d}{dx}\ln(\sin(x))$. $= \frac{1}{\sin(x)} \times \cos x$

$$= \frac{\cos x}{\sin x} = \cot(x)$$

$$\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)}\cdot f'(x).$$

c) Find $\frac{d}{dx}\ln\left(\frac{\sqrt{1+x^4}}{3x-1}\right)$.

Always use your log laws **before** the calculus!

$$y = \ln\left(\frac{\sqrt{1+x^4}}{3x-1}\right)$$

$$= \ln(\sqrt{1+x^4}) - \ln(3x-1)$$

$$= \ln((1+x^4)^{\frac{1}{2}}) - \ln(3x-1)$$

$$y = \frac{1}{2}\ln(1+x^4) - \ln(3x-1)$$

$$y' = \frac{1}{2}\cdot\frac{1}{1+x^4}\cdot 4x^3 - \frac{1}{3x-1}\cdot 3$$

$$= \frac{2x^3}{1+x^4} - \frac{3}{3x-1}$$

Since $\frac{d}{dx}\ln(x) = \frac{1}{x}$ we also have $\int \frac{1}{x}\,dx = \ln|x| + C$.

d) Find $\int \frac{1}{x^3}\,dx$. $= \int x^{-3}\,dx = \frac{x^{-2}}{-2} = -\frac{1}{2}\frac{1}{x^2} + C.$

e) Find $\int \frac{1}{x^2}\,dx$. $= \int x^{-2}\,dx = \frac{x^{-1}}{-1} = \frac{-1}{x} + C$

f) Find $\int \frac{1}{x}\,dx$. The above methods do not ~~work here!~~

$\int x^{-1}\,dx = \cancel{\frac{x^0}{0}}$ can't be!

$= \ln|x| + C.$ ✓

g) Find $\int \frac{x}{x^2+1}\,dx$.

Clearly the chain rule implies that $\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$.

Hence $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx. = \frac{1}{2}\ln|x^2+1| + C$$
$$= \frac{1}{2}\ln(x^2+1) + C$$

h) What about $\int \frac{1}{x^2+1}\,dx$?

$\int \frac{dx}{1+x^2} = \frac{1}{\cancel{2x}}\int \frac{2x\,dx}{1+x^2}$ XX (can't do this!!)

(Not a log integral!) ★

✓ $\tan^{-1}(x) + C$