Chapter 6: Inverse Functions

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Motivation

We often think of a function as a rule which takes in an input and assigns to it an output.

Usually we have a nice formula or recipe which tells us how to calculate the output for a given input.

Many hard and interesting problems go the other way: You know the output and you want to work out what the input must have been.

Example: Find all the x such that $f(x) = x^3 - 3x^2 + x - 4 = 0$. This is much harder than finding $f(x)$ for a given input!

The challenge is to reconstruct the 'input data' from the output information.

More abstractly, the problem is:

Given a function $f : A \to B$, if we set $y = f(x)$, under what circumstances is it possible to express x as a function of y, that is, to find a function $g : B \to A$ such that $x = g(y)$?

So, what we try to do is to "undo" the function f, that is to find a function g (called the inverse of f) such that $f(g(y)) = y$ and $g(f(x)) = x$.

The first things to worry about are:

1. Is it true that: for any $y \in B$ there is $x \in A$ such that $x = g(y)$ and $y = f(x)$?
2. If so, is this x unique?

In answering these questions it is vital that one considers not just the formula for f, but also what the domain of f is.

Example

Is it possible to invert the function $f : (-1, \infty) \to \mathbb{R}$ defined by $y = f(x) = \dfrac{2x}{x+1}$?

Solution. To find the function which "undoes" f, we need to express x as a function of y.

$$f(x) = y = \frac{2x}{x+1} \quad \text{so} \quad xy + y = 2x \quad , \quad x(y-2) = -y \quad \text{or} \quad x = \frac{y}{2-y}$$

Thus, we find the function $g : \mathrm{Range}(f) \to (-1, \infty)$, defined by $g(y) = \dfrac{y}{2-y}$.

Here we could "undo" f using the function g, with rule $y \mapsto \dfrac{y}{2-y}$, where y is in the range of f.

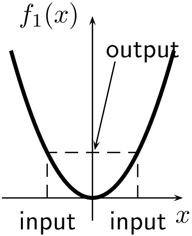
Standard example

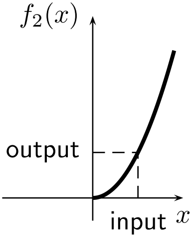
Consider the rule
$$y = x^2.$$
Whether any function defined by this rule is invertible depends on the domain:

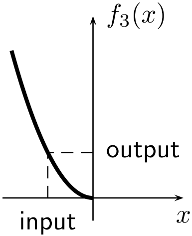
- $f_1 : \mathbb{R} \to \mathbb{R}, \qquad y = f_1(x) = x^2$
  The latter is not invertible since for any 'output' $y \neq 0$ there exist two 'inputs' $x = \sqrt{y}$ and $x = -\sqrt{y}$.

- $f_2 : [0, \infty) \to \mathbb{R}, \qquad y = f_2(x) = x^2$
  If we take into account that $\mathrm{Range}(f_2) = [0, \infty)$ then the inverse function is given by
  $$g_2 : [0, \infty) \to [0, \infty), \qquad x = g_2(y) = \sqrt{y}.$$

- $f_3 : (-\infty, 0] \to \mathbb{R}, \qquad y = f_3(x) = x^2$
  If we take into account that, again, $\mathrm{Range}(f_3) = [0, \infty)$ then the inverse function is given by
  $$g_3 : [0, \infty) \to (-\infty, 0], \qquad x = g_3(y) = -\sqrt{y}.$$







Remark. It is evident that it might be possible to construct an invertible function by restricting the domain of a given function.

Conclusion. The main criterion for invertibility is the existence of a **one-to-one correspondence** between 'inputs' and 'outputs'.

One-to-one functions

Idea. A function is one-to-one if every 'output' corresponds to a **unique** 'input'.

Definition

A function f is said to be one-to-one

$$\text{if} \quad f(x_1) = f(x_2) \quad \text{implies that} \quad x_1 = x_2$$

for all $x_1, x_2 \in \mathrm{Dom}(f)$.

Terminology. One-to-one functions are also called injective functions.

Remark. An 1-to-1 function is equivalently characterised by

$$\text{For every } x_1, x_2 \in \mathrm{Dom}(f), \text{if } x_1 \neq x_2, \text{then } f(x_1) \neq f(x_2).$$

"different points in the domain give different values in the codomain".

Easy example

Example.

Any linear function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$, $a \neq 0$, is one-to-one.

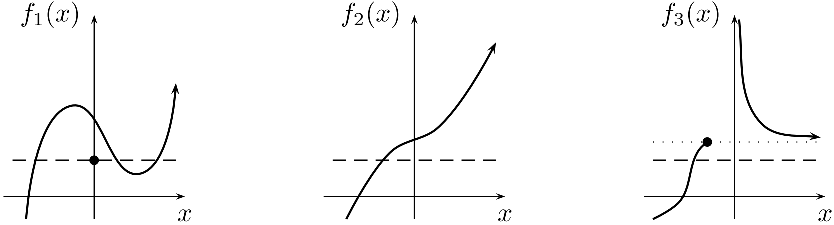
Indeed, $f(x_1) = f(x_2)$ gives $ax_1 + b = ax_2 + b$ which implies $x_1 = x_2$.

Remark. If $f : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$, then you can easily identify one-to-one functions by looking at the graph of f.

Useful test

The horizontal line test

Suppose that f is a real-valued function defined on some subset of $\mathbb{R}$. Then, f is one-to-one if and only if every horizontal line in the Cartesian plane intersects the graph of f at most once.



- f_1 is not one-to-one.
- f_2 is one-to-one.
- f_3 is one-to-one (even though it is neither strictly increasing nor strictly decreasing).

Theorem

Although not every one-to-one function is strictly increasing (or strictly decreasing), it is true that every strictly increasing function is one-to-one.

Theorem

If a function f is either strictly increasing or strictly decreasing then f is one-to-one.

(**Idea of proof.** If $x_1 \neq x_2$, say $x_1 < x_2$, then $f(x_1) < f(x_2)$, and thus $f(x_1) \neq f(x_2)$.)

This includes cases like:

1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ which is strictly increasing;
2. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3x - |x|$ which is strictly increasing.

Example

Example. Is the function $f : (-1, 1) \to \mathbb{R}$ defined by $f(x) = 3 + 2\tan\left(\dfrac{\pi}{2}x\right)$ one-to-one?

Solution. The function $\tan\left(\frac{\pi}{2}x\right)$ is continuous and differentiable at every point of the interval $(-1, 1)$. Therefore, the function f is also continuous and differentiable on $(-1, 1)$. We have

$$f'(x) = \frac{\pi}{\cos^2\left(\frac{\pi}{2}x\right)} > 0,$$

for every $x \in (-1, 1)$. Hence, by the theorem above, the function f is one-to-one.

Remark. Not every function whose derivative is only positive (or only negative) is one-to-one. For example,

$$\frac{d}{dx}\tan x = \sec^2 x \geq 1$$

but $\tan$ is not one-to-one on its maximal domain! The problem here is that $\mathrm{Dom}(\tan)$ has gaps.

Inverse functions

Theorem

Suppose that f is a one-to-one function. Then, there exists a unique function g satisfying

$$g(f(x)) = x \quad \text{for all} \quad x \in \mathrm{Dom}(f)$$

and

$$f(g(y)) = y \quad \text{for all} \quad y \in \mathrm{Range}(f).$$

Moreover,

$$\mathrm{Dom}(g) = \mathrm{Range}(f), \qquad \mathrm{Range}(g) = \mathrm{Dom}(f)$$

and g is one-to-one.

The theorem allows us to define the term inverse function.

Definition

Suppose that f is a one-to-one function. Then the inverse function of f is the unique function g given by the above theorem. The inverse function for f is often denoted by f^{-1}.

Inverse functions

Remark. If f^{-1} denotes the inverse function of a one-to-one function f then the relations in the above theorem may be expressed as

$$f^{-1}(f(x)) = x \quad \text{for all} \quad x \in \mathrm{Dom}(f)$$

and

$$f(f^{-1}(y)) = y \quad \text{for all} \quad y \in \mathrm{Range}(f)$$

so that f may also be interpreted as the inverse of the function f^{-1}.

Note. $f^{-1}(y)$ does NOT mean $1/f(y)$!

Remark. Since f^{-1} is a function just like any other function, we regard it as a function

$$x \mapsto f^{-1}(x)$$

so that we can graph f^{-1} in the usual manner.

Example. Determine f^{-1}, where $f : \mathbb{R} \to \mathbb{R}$, $\qquad f(x) = 4 - \frac{1}{3}x^3.$

Solution. First, since $f'(x) = -x^2 \leq 0$ for all $x \in \mathbb{R}$ (with $f'(x) = 0$ only when $x = 0$), f is decreasing and thus one-to-one.

Set

$$y = 4 - \frac{1}{3}x^3$$

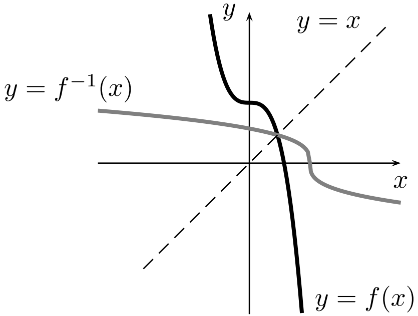
so that

$$x^3 = 3(4-y) \quad \text{and} \quad x = \sqrt[3]{12-3y}.$$

Hence, (interchanging x and y),

$$f^{-1} : \mathbb{R} \to \mathbb{R}, \qquad f^{-1}(x) = \sqrt[3]{12-3x}.$$

Inverse functions



Note that the graph of $y = f^{-1}(x)$ is the reflection of the graph of f in the line $y = x$.

Proof. The point (x, y) lies on the graph of f if $y = f(x)$. This is equivalent with $f^{-1}(y) = f^{-1}(f(x)) = x$.The latter is equivalent with the point (y, x) belonging to the graph of f^{-1}.

Example

Example. Given $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = -x^3 + 3x^2 + 24x - 13$, find all intervals I, as large as possible, such that $f : I \to \mathbb{R}$ has an inverse function.

Example

Example. Given $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = -x^3 + 3x^2 + 24x - 13$, find all intervals I, as large as possible, such that $f : I \to \mathbb{R}$ has an inverse function.

Solution. We need to find intervals I where f is one-to-one. As f is continuous on $\mathbb{R}$, just need to find intervals where f is either increasing or decreasing. As the given f is differentiable on $\mathbb{R}$, we just need to find intervals where $f'(x)$ has the same sign.

$$f'(x) = -3x^2 + 6x + 24 = -3(x^2 - 2x - 8) = -3(x+2)(x-4),$$

$$\begin{aligned} f'(x) &\leq 0 \quad \text{for } -\infty < x \leq -2 \\ f'(x) &\geq 0 \quad \text{for } \ -2 \leq x \leq 4 \\ f'(x) &\leq 0 \quad \text{for } \quad\ 4 \leq x < \infty. \end{aligned}$$

Thus

- f restricted to $I_1 = (-\infty, -2]$ has an inverse,
- f restricted to $I_2 = [-2, 4]$ has an inverse,
- f restricted to $I_3 = [4, \infty)$ also has an inverse.

The inverse function theorem

Question. If the derivative of an invertible function exists, under what circumstances is the inverse function also differentiable?

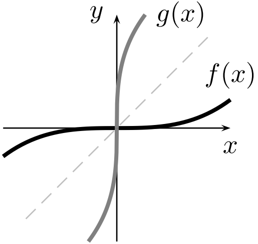
Subtlety. Consider the function

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = x^3.$$

Its inverse is given by

$$g : \mathbb{R} \to \mathbb{R}, \qquad g(x) = \sqrt[3]{x}$$

but g is not differentiable at $x = 0$ since $g'(x) = \frac{1}{3\sqrt[3]{x^2}}$!



Observation: The points at which the derivative of f vanishes must be excluded!

The inverse function theorem

Suppose that I is an open interval, $f : I \to \mathbb{R}$ is differentiable and

$$f'(x) \neq 0$$

for all x in I. Then,

- f is one-to-one and has an inverse function

$$g : \mathrm{Range}(f) \to \mathrm{Dom}(f)$$

- g is differentiable at all points in $\mathrm{Range}(f)$
- The derivative of g is given by

$$g'(y) = \frac{1}{f'(g(y))}$$

  for all $y \in \mathrm{Range}(f)$.

The inverse function theorem: proof

Proof.

- Since $f'(x) \neq 0$ on I, f is one-to-one (MVT!)...
- g is differentiable ... too hard! (not for MATH1131)
- Differentiation of

$$f(g(y)) = y$$

with respect to y yields

$$f'(g(y)) \times g'(y) = 1.$$

Since f' is never zero on I, we can divide by $f'(g(y))$ to obtain

$$g'(y) = \frac{1}{f'(g(y))}.$$

Remark. Once again, we usually write the derivative of the inverse function g as

$$g'(x) = \frac{1}{f'(g(x))}$$

for $x \in \mathrm{Range}(f)$.

Remark. Let us look again at the following examples:

1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ which is one-to-one (being strictly increasing), but where $f'(x)$ is sometimes zero $\to$ can not use the inverse function theorem!
2. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3x - |x|$ which is one-to-one (being strictly increasing), but not differentiable $\to$ can not use the inverse function theorem!

Examples

Example. Consider the function $f : (0, \infty) \to (0, \infty), \quad f(x) = x^3$.

f is differentiable and $f'(x) \neq 0$ on $(0, \infty)$, so by the inverse function theorem, its inverse is:

$$g : (0, \infty) \to (0, \infty), \qquad g(x) = \sqrt[3]{x}.$$

and,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{3[g(x)]^2} = \frac{1}{3x^{2/3}}$$

as expected.

Example

Previous example. Determine the derivative of the inverse of the function $f : (-1, 1) \to \mathbb{R}$ defined by

$$f(x) = 3 + 2\tan\left(\frac{\pi}{2}x\right)$$

at the point $f(x) = 3$.

Cont.

Cont.

Solution. We showed that f is one-to-one, and thus $g = f^{-1}$ exists. Calculate

$$f'(x) = \frac{\pi}{\cos^2\left(\frac{\pi}{2}x\right)} \quad \text{for } -1 < x < 1.$$

Now we can apply the Inverse Function Theorem to get

$$g'(x) = \frac{1}{f'\big(g(x)\big)} = \frac{1}{\pi}\cos^2\left(\frac{\pi g(x)}{2}\right).$$

We do not know $g(x)$ explicitly, so can go no further for general values of x. But, by chance, we observe that $f(0) = 3$ which means that

$$g\left(f(0)\right) = g(3) \quad \text{i.e.} \quad 0 = g(3),$$

so

$$g'(3) = \frac{1}{\pi}\cos^2(0) = \frac{1}{\pi}.$$

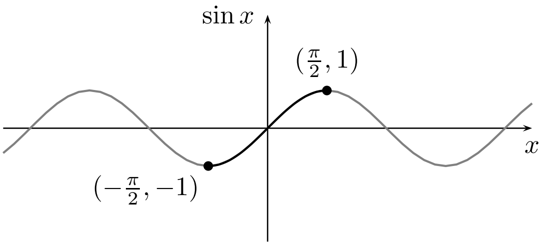
Applications to the trigonometric functions: $\sin^{-1}$

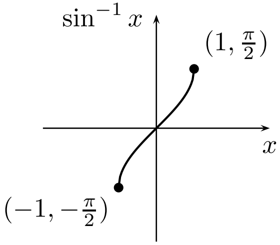
The inverse sine function. The function $\sin : \mathbb{R} \to \mathbb{R}$ is not a one-to-one function, and thus has no inverse. We consider thus the restricted sine function

$$\sin : [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \to [-1, 1].$$

This function is one-to-one (strictly increasing) and therefore has an inverse

$$\sin^{-1} : [-1, 1] \to [-\tfrac{\pi}{2}, \tfrac{\pi}{2}].$$





Derivative of $\sin^{-1}$

Since the derivative of $\sin x$ is not zero on $(-\frac{\pi}{2}, \frac{\pi}{2})$, by the Inverse Function Theorem, $\sin^{-1}\colon (-1,1) \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is differentiable and its derivative is given by

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\cos(\sin^{-1} x)}.$$

Using the identity $\cos^2 y + \sin^2 y = 1$, we have $\cos y = \pm\sqrt{1 - \sin(y)^2}$ with $y = \sin^{-1} x$.

Since $\cos$ is positive on $(-\frac{\pi}{2}, \frac{\pi}{2})$, we need $\cos y = +\sqrt{1 - \sin(y)^2}$.

Finally

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - \left(\sin(\sin^{-1} x)\right)^2}} = \frac{1}{\sqrt{1 - x^2}}$$

whenever $-1 < x < 1$.

Note. $\frac{d}{dx}(\sin^{-1} x) > 0$ on $(-1, 1)$ and thus $\sin^{-1}$ is an increasing function.

Applications to the trigonometric functions: $\cos^{-1}$, $\tan^{-1}$

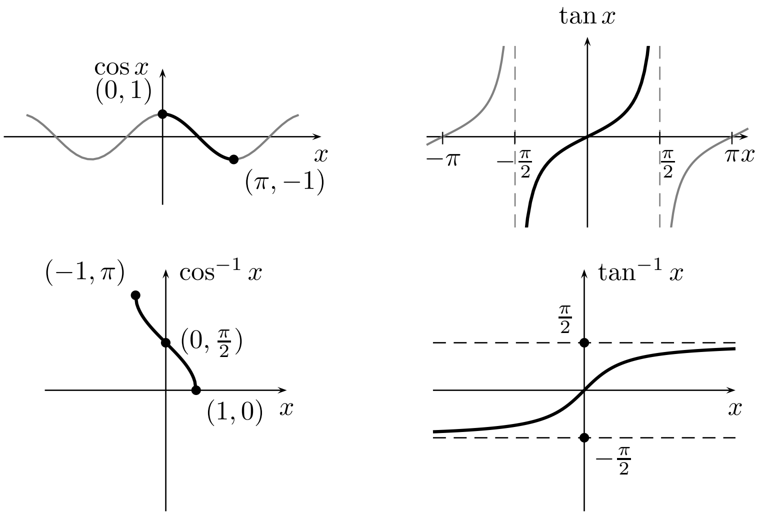


Table of inverse trigonometric functions

Function	Domain	Range	Derivative
$\sin$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$[-1, 1]$	$\dfrac{d}{dx}(\sin x) = \cos x$
$\sin^{-1}$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$
$\cos$	$[0, \pi]$	$[-1, 1]$	$\dfrac{d}{dx}(\cos x) = -\sin x$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$	$\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{1-x^2}}$
$\tan$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$(-\infty, \infty)$	$\dfrac{d}{dx}(\tan x) = \sec^2 x$
$\tan^{-1}$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$

Important Remark

Remark. Even though $\sin(\sin^{-1} x) = x$ for $x \in [-1, 1]$ (which is always the case, as $\sin^{-1}$ is not defined otherwise),

in general, $\sin^{-1}(\sin x) \neq x$, unless $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

(E.g., $0 = \sin(3\pi)$, but $3\pi \neq \sin^{-1}(0)$.)

Similarly, $\cos^{-1}(\cos x) \neq x$, unless $x \in [0, \pi]$.

and, $\tan^{-1}(\tan x) \neq x$, unless $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Example

Example. Determine

$$\cos\left(2\sin^{-1}\frac{3}{5}\right).$$

Example

Example. Determine

$$\cos\left(2\sin^{-1}\frac{3}{5}\right).$$

Solution.
We have

$$\begin{aligned}\cos\left(2\sin^{-1}\frac{3}{5}\right) &= 1-2\sin^2\left(\sin^{-1}\frac{3}{5}\right)\\ &= 1-2\left(\frac{3}{5}\right)^2 = \frac{7}{25}.\end{aligned}$$

Example

Example. Determine

$$\sin^{-1}\left(\sin\frac{5\pi}{6}\right).$$

Example

Example. Determine

$$\sin^{-1}\left(\sin\frac{5\pi}{6}\right).$$

Solution. Since $\frac{5\pi}{6}$ does not belong to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ we do not have $\sin^{-1}\left(\sin\frac{5\pi}{6}\right) = \frac{5\pi}{6}$.
We have

$$\begin{aligned}\sin^{-1}\left(\sin\frac{5\pi}{6}\right) &= \sin^{-1}\left(\sin\left(\pi - \frac{\pi}{6}\right)\right)\\ &= \sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}.\end{aligned}$$

Example

Show that $h(x) = \cos^{-1}(x) + \cos^{-1}(-x) = \pi$.

Example

Show that $h(x) = \cos^{-1}(x) + \cos^{-1}(-x) = \pi$.

Solution. To show that the function h is constant, we have to show that the derivative is zero.
We have

$$\begin{aligned} h'(x) &= \frac{d}{dx}\cos^{-1}(x) + \left(\frac{d}{du}\cos^{-1}(u)\right)\frac{du}{dx} \quad \text{where } u = -x \\ &= \frac{-1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-u^2}}(-1) \\ &= \frac{-1}{\sqrt{1-x^2}} - \frac{-1}{\sqrt{1-x^2}} = 0 \end{aligned}$$

Thus $h(x)$ is a constant, so we can work out the constant by looking at any value of x; $x = 0$ is good choice so

$$h(x) = h(0) = \cos^{-1}(0) + \cos^{-1}(-0) = \pi/2 + \pi/2 = \pi.$$

Example

Evaluate $\frac{d}{dx}\tan^{-1}(\sqrt{x^2-1})$.

Solution. Let $u(x)=\sqrt{x^2-1}$ and $g(u)=\tan^{-1}(u)$. Then

$$\tan^{-1}(\sqrt{x^2-1})=(g\circ u)(x),$$

i.e.,

$$\frac{d}{dx}\tan^{-1}(\sqrt{x^2-1})=g'(u(x))u'(x)=\frac{dg}{du}\frac{du}{dx}.$$

$$\frac{du}{dx}=\frac{d}{dx}\sqrt{x^2-1}=\frac{x}{\sqrt{x^2-1}}$$

and

$$\frac{dg}{du}=\frac{d}{du}\tan^{-1}(u)=\frac{1}{1+u^2}.$$

So,

$$\begin{aligned}\frac{d}{dx}\tan^{-1}(\sqrt{x^2-1})&=\frac{dg}{du}\frac{du}{dx}=\frac{1}{1+x^2-1}\frac{x}{\sqrt{x^2-1}}\\&=\frac{1}{x\sqrt{x^2-1}}.\end{aligned}$$

Summary: What did we learn in this chapter?

- One-to-one function (p. 7)
- Horizontal line test (p. 9)
- One-to-one and strictly increasing / decreasing (p. 10)
- Inverse function definition (p. 12)
- Inverse function theorem (p. 18)
- Inverse trigonometric functions (p. 24 and p. 26)
- Derivatives of inverse trigonometric functions (p. 27)