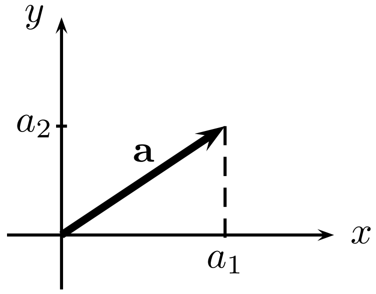
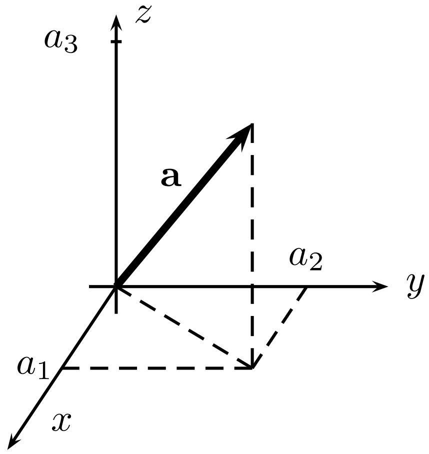
§2 Vector Geometry (2020T1: W2-We, W3-Tu-We-Th)

Length.





- The *length* of a vector $\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$ is $|\mathbf{a}| = \sqrt{a_1^2 + \cdots + a_n^2}$.
- The *distance* between two points A and B is the length of the vector $\overrightarrow{AB}$.
- **Properties of length.** For $\mathbf{a} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$,
  - $|\mathbf{a}|$ is a real number,
  - $|\mathbf{a}| \geq 0$,
  - $|\mathbf{a}| = 0$ if and only if $\mathbf{a} = \mathbf{0}$,
  - $|\lambda\mathbf{a}| = |\lambda||\mathbf{a}|$.

Exercise. Find the distance between the point A with coordinates $(2, 3, 1, 5)$ and the point B with coordinates $(-1, 3, 2, 4)$.

Example. Prove the property $|\lambda\mathbf{a}| = |\lambda||\mathbf{a}|$ for $\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and $\lambda \in \mathbb{R}$.

Proof:

$$
\begin{aligned}
|\lambda\mathbf{a}| &= \left| \begin{pmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{pmatrix} \right| && \text{definition of multiplication by a scalar} \\
&= \sqrt{(\lambda a_1)^2 + \cdots + (\lambda a_n)^2} && \text{definition of length} \\
&= |\lambda|\sqrt{a_1^2 + \cdots + a_n^2} && \text{property of real numbers} \\
&= |\lambda||\mathbf{a}| && \text{definition of length}
\end{aligned}
$$

- **Dot product.**
  - The *dot product* (*scalar product*) of two vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ is a number (scalar)

$$\boxed{\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + \cdots + a_n b_n = \sum_{j=1}^{n} a_j\, b_j}\,.$$

  - **Properties of dot product.** For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$,
    - $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$, and hence $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$.
    - $(\lambda \mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\lambda \mathbf{b}) = \lambda (\mathbf{a} \cdot \mathbf{b})$.
    - *Commutative law*: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.
    - *Distributive law*: $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$.

Exercise. Find the dot product of $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$.

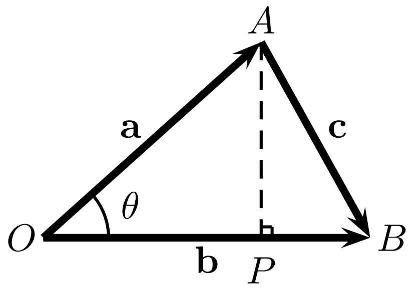
Example. Prove the distributive law of dot product $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ for $\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$.

Proof:

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 + c_1 \\ \vdots \\ b_n + c_n \end{pmatrix} && \text{definition of vector addition} \\
&= a_1(b_1 + c_1) + \cdots + a_n(b_n + c_n) && \text{definition of dot product} \\
&= (a_1 b_1 + a_1 c_1) + \cdots + (a_n b_n + a_n c_n) \\
& && \text{distributive law of real numbers} \\
&= (a_1 b_1 + \cdots + a_n b_n) + (a_1 c_1 + \cdots + a_n c_n) \\
& && \text{commutative and associative laws of real numbers} \\
&= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} && \text{definition of dot product}
\end{aligned}$$

Geometric interpretation of dot product in $\mathbb{R}^2$ and $\mathbb{R}^3$.

- Let OAB be a triangle in $\mathbb{R}^3$ with sides given by vectors $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$ and $\mathbf{c} = \overrightarrow{AB}$, and let *$\theta$ be the interior angle between $\mathbf{a}$ and $\mathbf{b}$*.



- Length of $\mathbf{c}$:

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\,\mathbf{a}\cdot\mathbf{b} \qquad (\spadesuit)$$

- Cosine rule:

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\,|\mathbf{a}|\,|\mathbf{b}|\cos\theta \qquad (\heartsuit)$$

- From $(\spadesuit)$ and $(\heartsuit)$, we conclude that

$$\boxed{\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta} \qquad \text{where} \quad 0 \le \theta \le \pi\,.$$

This is often used as the definition of dot product in physics and engineering.

Proof of $(\spadesuit)$: Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. Then

$$\begin{aligned}
|\mathbf{c}|^2 = |\mathbf{b}-\mathbf{a}|^2 &= \left|\begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}\right|^2 \\
&= (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2 \\
&= (a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2) - 2(a_1 b_1 + a_2 b_2 + a_3 b_3) \\
&= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\,\mathbf{a}\cdot\mathbf{b}\,.
\end{aligned}$$

Proof of $(\heartsuit)$: From the diagram, we see that APB is a right-angle triangle, so

$$\begin{aligned}
|\mathbf{c}|^2 = |\overrightarrow{AP}|^2 + |\overrightarrow{PB}|^2 &= (|\mathbf{a}|\sin\theta)^2 + (|\mathbf{b}| - |\mathbf{a}|\cos\theta)^2 \\
&= |\mathbf{a}|^2\sin^2\theta + |\mathbf{a}|^2\cos^2\theta + |\mathbf{b}|^2 - 2\,|\mathbf{a}|\,|\mathbf{b}|\cos\theta \\
&= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\,|\mathbf{a}|\,|\mathbf{b}|\cos\theta\,.
\end{aligned}$$

- **Geometric interpretation of dot product in $\mathbb{R}^n$.**
  - We define the *angle* θ between non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ by

$$\boxed{\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}} \qquad \text{where} \qquad 0 \le \theta \le \pi\,. \qquad (\clubsuit)$$

  - The *Cauchy-Schwarz inequality*: if $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ then

$$-|\mathbf{a}|\,|\mathbf{b}| \le \mathbf{a}\cdot\mathbf{b} \le |\mathbf{a}|\,|\mathbf{b}|\,.$$

    This result implies that $-1 \le \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} \le 1$, which ensures that the definition $(\clubsuit)$ makes sense.
  - *Minkowski's inequality* or *triangle inequality*: if $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ then

$$|\mathbf{a}+\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|\,.$$

    This follows easily from the Cauchy-Schwarz inequality.

Exercise. Find the angle between the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ -4 \\ -1 \end{pmatrix}$.

Exercise. Let ABC be a triangle with vertices $A(2,1,3)$, $B(3,0,-1)$ and $C(0,1,-2)$. Find the angle A.

- **Orthogonality.**
  - Two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ are said to be *orthogonal* if $\mathbf{a} \cdot \mathbf{b} = 0$.
  - In $\mathbb{R}^2$ or $\mathbb{R}^3$, two non-zero vectors are orthogonal if they are at right angles to each other. In this case, the vectors are said to be *perpendicular* or *normal* to each other.
  - An *orthonormal* set of vectors is a set of vectors which are unit length and mutually orthogonal.
  - If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthonormal set in $\mathbb{R}^n$ then

$$\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

  - For any vector $\mathbf{a}$, we use $\widehat{\mathbf{a}}$ to denote a vector of unit length in the direction of $\mathbf{a}$:

$$\widehat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}.$$

Note. The word "normal" has multiple mathematical meanings in different mathematical contexts!

Example. The three standard basis vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

of $\mathbb{R}^3$ form an orthonormal set. Indeed,

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = 1, \quad \mathbf{e}_2 \cdot \mathbf{e}_2 = 1, \quad \mathbf{e}_3 \cdot \mathbf{e}_3 = 1,$$

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = 0, \quad \mathbf{e}_1 \cdot \mathbf{e}_3 = 0, \quad \mathbf{e}_2 \cdot \mathbf{e}_3 = 0.$$

For any other vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, we have

$$\mathbf{a} \cdot \mathbf{e}_1 = a_1, \quad \mathbf{a} \cdot \mathbf{e}_2 = a_2, \quad \mathbf{a} \cdot \mathbf{e}_3 = a_3.$$

Thus we can write

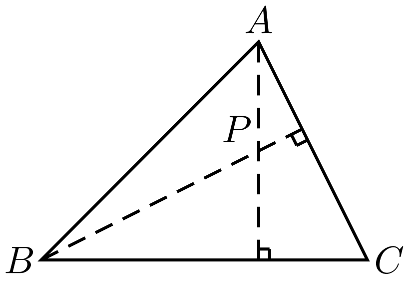
$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = (\mathbf{a} \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\mathbf{a} \cdot \mathbf{e}_2)\,\mathbf{e}_2 + (\mathbf{a} \cdot \mathbf{e}_3)\,\mathbf{e}_3.$$

Corresponding results also hold in $\mathbb{R}^n$.

Exercise. Show that the two vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ are orthogonal.

Exercise. Given $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, find the unit-length vectors $\widehat{\mathbf{u}}_1$, $\widehat{\mathbf{u}}_2$, $\widehat{\mathbf{u}}_3$, and show that they form an orthonormal set.

Exercise. Show that the three *altitudes* of a triangle are *concurrent*, i.e., they intersect at a point.



In a triangle ABC, suppose that the altitudes through A and B intersect at a point P as drawn above. Let $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, $\mathbf{c} = \overrightarrow{OC}$, $\mathbf{p} = \overrightarrow{OP}$. Show that P lies on the altitude through C.

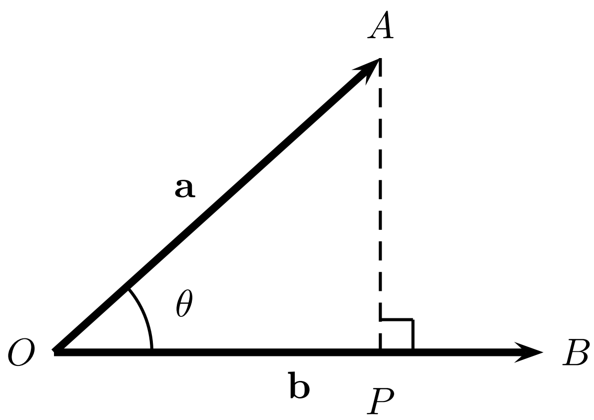
- **Projection.**
  - For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ with $\mathbf{b} \neq \mathbf{0}$, the *projection* of $\mathbf{a}$ on $\mathbf{b}$ is

$$\boxed{\mathrm{proj}_{\mathbf{b}}\mathbf{a} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b}}.$$

  - The *length of the projection* of $\mathbf{a}$ on $\mathbf{b}$ is

$$\boxed{|\mathrm{proj}_{\mathbf{b}}\mathbf{a}| = \frac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{b}|}}.$$

- **Geometric interpretation of projection in $\mathbb{R}^2$ and $\mathbb{R}^3$.**
  - Let $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$. Let P be the point along the line OB such that $\overrightarrow{PA}$ is perpendicular to $\overrightarrow{OB}$. Then $\mathrm{proj}_{\mathbf{b}}\mathbf{a}$ is the vector $\overrightarrow{OP}$.



  - The projection $\mathrm{proj}_{\mathbf{b}}\mathbf{a} = \overrightarrow{OP}$ is parallel to $\overrightarrow{OB}$, and we have

$$\overrightarrow{OP} = \frac{|\overrightarrow{OP}|}{|\overrightarrow{OB}|}\overrightarrow{OB} = \frac{|\mathbf{a}|\cos\theta}{|\mathbf{b}|}\mathbf{b} = \frac{|\mathbf{a}|\,|\mathbf{b}|\cos\theta}{|\mathbf{b}|^2}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}\,.$$

- **Geometric interpretation of projection in $\mathbb{R}^n$.**
  - The vector $\mathrm{proj}_{\mathbf{b}}\mathbf{a} = \lambda\mathbf{b}$ is the unique vector parallel to $\mathbf{b}$ such that

$$(\mathbf{a} - \lambda\mathbf{b})\cdot\mathbf{b} = 0 \iff \mathbf{a}\cdot\mathbf{b} - \lambda|\mathbf{b}|^2 = 0 \iff \lambda = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}.$$

Exercise. Find the projection of $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ on $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$.

Exercise. Let $\mathbf{a}$ and $\mathbf{b}$ be as given in the previous exercise. Write $\mathbf{a}$ as a sum of two vectors, one parallel to $\mathbf{b}$ and another perpendicular to $\mathbf{b}$.

Exercise. Find the length of the projection of $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ on $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$.

Notes. See Page 20 – distance between a point and a line in $\mathbb{R}^3$.

Cross product (only defined in $\mathbb{R}^3$).

- The *cross product* of two vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ in $\mathbb{R}^3$ is a vector

$$\boxed{\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2\, b_3 - a_3\, b_2 \\ a_3\, b_1 - a_1\, b_3 \\ a_1\, b_2 - a_2\, b_1 \end{pmatrix}}.$$

- *The cross product is only defined for vectors in three dimensions.*
- **Properties of cross product.** For $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$,
  - $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.
  - $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, i.e., the cross product is not commutative.
  - $\mathbf{a} \times (\lambda \mathbf{b}) = (\lambda \mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b})$.
  - *Distributive laws*:
    $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ and $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$.

Notes. There are several tricks to help remembering the formula for cross product. One trick is to write it in terms of a 3×3 *determinant* (see Chapter 5 later) and "expand it along the first column" which contains the the standard basis vectors:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{e}_1 & a_1 & b_1 \\ \mathbf{e}_2 & a_2 & b_2 \\ \mathbf{e}_3 & a_3 & b_3 \end{vmatrix} \\ &= \mathbf{e}_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} - \mathbf{e}_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} + \mathbf{e}_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (a_2 b_3 - a_3 b_2) - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (a_1 b_3 - a_3 b_1) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (a_1 b_2 - a_2 b_1) \\ &= \begin{pmatrix} a_2\, b_3 - a_3\, b_2 \\ a_3\, b_1 - a_1\, b_3 \\ a_1\, b_2 - a_2\, b_1 \end{pmatrix} \end{aligned}$$

Example.

$$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{vmatrix} \mathbf{e}_1 & 1 & 2 \\ \mathbf{e}_2 & 3 & 1 \\ \mathbf{e}_3 & 2 & -1 \end{vmatrix} = \mathbf{e}_1 \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} - \mathbf{e}_2 \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + \mathbf{e}_3 \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = \begin{pmatrix} -5 \\ 5 \\ -5 \end{pmatrix}$$

Exercise.

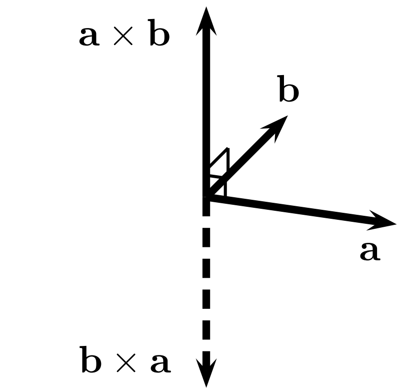
(a) $\begin{pmatrix}1\\3\\2\end{pmatrix} \times \begin{pmatrix}2\\1\\-1\end{pmatrix}$

(b) $\begin{pmatrix}2\\1\\-1\end{pmatrix} \times \begin{pmatrix}1\\3\\2\end{pmatrix}$

(c) $\begin{pmatrix}1\\-2\\3\end{pmatrix} \times \begin{pmatrix}4\\-5\\6\end{pmatrix}$

Exercise. Show that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0$ and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$, i.e., the cross product is perpendicular to both vectors.

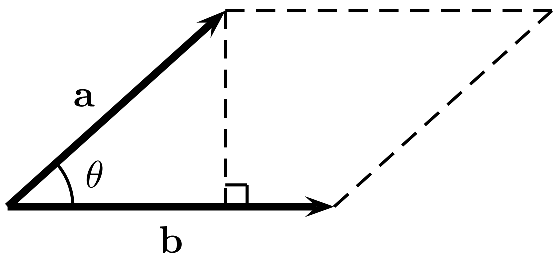
Geometric interpretation of cross product (only in $\mathbb{R}^3$).



- Following the "right-hand rule", the cross product $\mathbf{a}\times\mathbf{b}$ is a vector which is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{b}\times\mathbf{a}$ is a vector pointing in the opposite direction.
- Let *θ be the angle between $\mathbf{a}$ and $\mathbf{b}$*, with $0 \le \theta \le \pi$. Then

$$\boxed{|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta}\,. \qquad (\diamondsuit)$$

- Hence $\mathbf{a}\times\mathbf{b}$ is a vector of length $|\mathbf{a}|\,|\mathbf{b}|\sin\theta$ in the direction perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ as given by the right-hand rule. This is usually taken as the definition of cross product in physics and engineering.
- *Area of parallelogram* spanned by two vectors $\mathbf{a}$ and $\mathbf{b}$:



$$\text{area of parallelogram} = \text{base} \times \text{height} = |\mathbf{a}||\mathbf{b}|\sin\theta = |\mathbf{a}\times\mathbf{b}|\,.$$

- *Area of triangle* spanned by two vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\text{area of triangle} = \frac{1}{2}\,|\mathbf{a}\times\mathbf{b}|\,.$$

Proof of $(\diamondsuit)$: If $\mathbf{a}=\mathbf{0}$ or $\mathbf{b}=\mathbf{0}$ then both sides are 0 (θ is not defined).

For $\mathbf{a}\neq\mathbf{0}$ and $\mathbf{b}\neq\mathbf{0}$, we have $0\le\theta\le\pi$ and $\sin\theta\ge 0$. Thus it suffices to prove

$$|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2\,|\mathbf{b}|^2\sin^2\theta.$$

We have

$$
\begin{aligned}
|\mathbf{a} \times \mathbf{b}|^2 &= \left| \begin{pmatrix} a_2\, b_3 - a_3\, b_2 \\ a_3\, b_1 - a_1\, b_3 \\ a_1\, b_2 - a_2\, b_1 \end{pmatrix} \right|^2 \\
&= (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2 , \quad \text{and} \\
|\mathbf{a}|^2\, |\mathbf{b}|^2 \sin^2\theta &= |\mathbf{a}|^2\, |\mathbf{b}|^2\, (1 - \cos^2\theta) \\
&= |\mathbf{a}|^2\, |\mathbf{b}|^2 - |\mathbf{a}|^2\, |\mathbf{b}|^2 \cos^2\theta \\
&= |\mathbf{a}|^2\, |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\
&= (a_1^2 + a_2^2 + a_3^2)\, (b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2.
\end{aligned}
$$

Expanding both expressions shows that they are indeed equal.

Exercise. Find a vector which is perpendicular to both $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

Exercise. Find the area of the parallelogram spanned by the vectors $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

Exercise. Find the area of the triangle with vertices $(1, 1, 2)$, $(2, 1, -3)$ and $(3, 0, -1)$.

- **Scalar triple product (only defined in $\mathbb{R}^3$).**
  - The *scalar triple product* of three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in $\mathbb{R}^3$ is a number

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) .$$

  - **Properties of scalar triple product:**
    - $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, i.e., the dot and cross can be interchanged.
    - $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$, i.e., swapping two vectors changes sign.
    - $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{a}) \cdot \mathbf{b} = 0$, i.e., zero if two vectors are the same.

Notes. The scalar triple product can be written using the 3×3 determinant:

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\
&= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \\
&= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1
\end{aligned}$$

Exercise. Let $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$. Evaluate $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

Exercise. With the same vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ as above, obtain if possible:

(a) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

(b) $\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$

(c) $(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}$

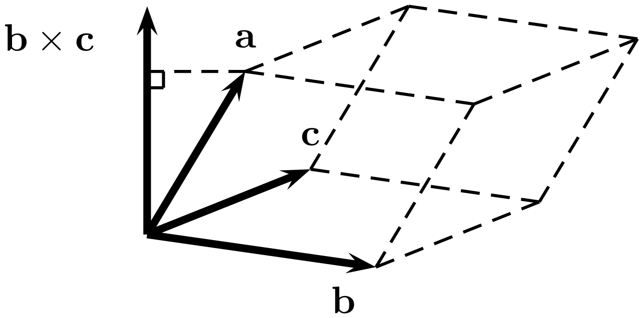
(d) $(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{c}$

(e) $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$

(f) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

Geometric interpretation of scalar triple product (only in $\mathbb{R}^3$).

- *Volume of parallelepiped* spanned by three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$:



$$\begin{aligned}\text{volume of parallelepiped} &= \text{base area} \times \text{height}\\ &= |\mathbf{b}\times\mathbf{c}| \times |\mathrm{proj}_{(\mathbf{b}\times\mathbf{c})}\mathbf{a}|\\ &= |\mathbf{b}\times\mathbf{c}| \times \frac{|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|}{|\mathbf{b}\times\mathbf{c}|}\\ &= |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|\end{aligned}$$

- *Volume of tetrahedron* spanned by three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$:

$$\text{volume of tetrahedron} = \frac{1}{6}\,|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$$

- To check if three vectors (or four points) are *coplanar*, it suffices to show that the parallelepiped spanned by the three vectors (or the tetrahedron with these four points as vertices) has volume 0, i.e., $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0$.

Exercise. Find the volume of the parallelepiped spanned by the three vectors $\begin{pmatrix}1\\2\\3\end{pmatrix}$, $\begin{pmatrix}-2\\3\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\\-2\end{pmatrix}$.

Exercise. Find the volume of the tetrahedron with vertices $(1, 2, 1)$, $(2, 3, -1)$, $(0, 1, -2)$, and $(-2, 1, 4)$.

Exercise. Show that the four points $(1, 2, 3)$, $(2, 2, 4)$, $(3, 3, 5)$ and $(1, 4, 3)$ are coplanar.

- **Equation of planes – parametric vector form.** A plane *through a point* with position vector $\mathbf{p}$ and *parallel to two vectors* $\mathbf{v}_1$ and $\mathbf{v}_2$ (non-zero and non-parallel) has *parametric vector form*

$$\mathbf{x} = \mathbf{p} + \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2, \qquad \lambda_1, \lambda_2 \in \mathbb{R}$$

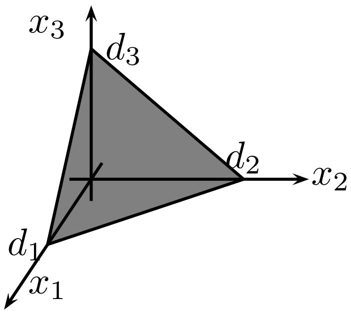
- **Equation of planes in $\mathbb{R}^3$ – Cartesian form.** A plane in $\mathbb{R}^3$ has *Cartesian form*

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = b$$

  - If $b = 0$, then the plane passes through the origin.
  - If $b \neq 0$, then we can divide the equation through by b, obtaining

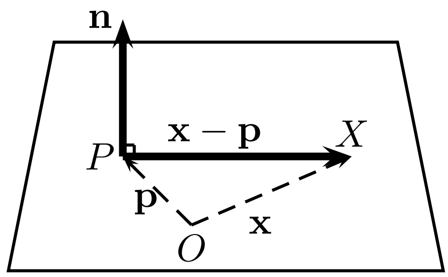
$$\frac{x_1}{d_1} + \frac{x_2}{d_2} + \frac{x_3}{d_3} = 1.$$

    The numbers d_1, d_2, and d_3 correspond to the intercepts on the x_1, x_2, and x_3 axes.



- **Equation of planes in $\mathbb{R}^3$ – point normal form.** A plane in $\mathbb{R}^3$ *through a point* with position vector $\mathbf{p}$ and *perpendicular to a vector* $\mathbf{n}$ (called a *normal vector* of the plane) has *point normal form*

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0 \qquad (\bigstar)$$



  - We can rewrite $(\bigstar)$ as $n_1 x_1 + n_2 x_2 + n_3 x_3 = n_1 p_1 + n_2 p_2 + n_3 p_3$. Thus the coefficients in the Cartesian form correspond to a normal vector

$$\mathbf{n} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

  - A normal vector $\mathbf{n}$ of a plane can be obtained by taking the cross product $\mathbf{v}_1 \times \mathbf{v}_2$ of two non-parallel vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ both parallel to the plane.

Exercise. Find the equation of the plane with intercepts 3, -2 and 6 on the three axes.

Exercise. Find a point normal form of the plane $2x_1 - 3x_2 + x_3 = 5$.

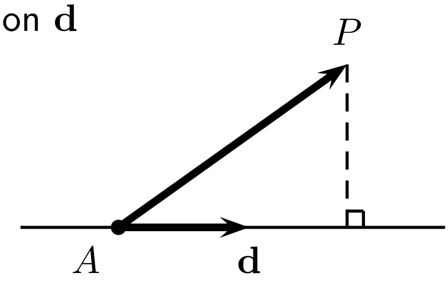
Exercise. Find a point normal form, the Cartesian form, and a parametric vector form of the equation of the plane which passes through the point $(1, 2, 3)$ and whose normal vector is $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$.

Exercise. Find a point normal form, the Cartesian form, and a parametric vector form of the plane which passes through the point $(1, 2, 3)$ and is parallel to the vectors $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

Exercise. Find a point normal form, the Cartesian form, and a parametric vector form of the plane which passes through the points $(1, 2, 3)$, $(2, 3, 1)$, and $(0, 3, -2)$.

Distance between a point and a line in $\mathbb{R}^3$. Given a point P and a line,

1. Identify one point A on the line and a direction vector $\mathbf{d}$ of the line
2. Form $\overrightarrow{AP}$ and the projection of $\overrightarrow{AP}$ on $\mathbf{d}$
3. Distance $= \sqrt{|\overrightarrow{AP}|^2 - |\mathrm{proj}_{\mathbf{d}}\overrightarrow{AP}|^2}$



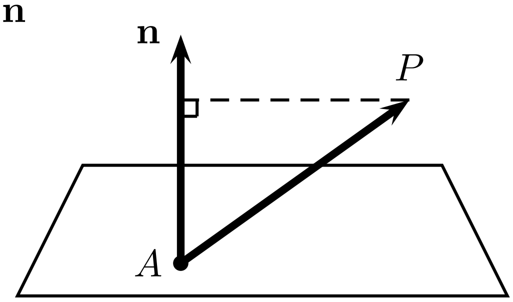
Exercise. Find the distance from the point $(1, 2, 3)$ to the line which passes through the points $(0, 1, 2)$ and $(2, 3, -1)$.

Exercise. Find the distance from the point $(2, 3, 1)$ to the line

$$\frac{x_1 - 2}{3} = \frac{x_3 + 1}{4}, \quad x_2 = 5.$$

Distance between a point and a plane in $\mathbb{R}^3$. Given a point P and a plane,

1. Identify one point A on the plane and a normal vector $\mathbf{n}$ of the plane
2. Form $\overrightarrow{AP}$ and the projection of $\overrightarrow{AP}$ on $\mathbf{n}$
3. Distance $= |\text{proj}_{\mathbf{n}}\overrightarrow{AP}|$



Exercise. Find the distance from the point $(1, 2, 3)$ to the plane $3x_1 - 2x_2 + x_3 = 5$.

Exercise. Find the distance from the point $(2, -2, 1)$ to the plane which passes through the points $(1, 2, 3)$, $(2, 0, -1)$, and $(1, 1, 4)$.