THE UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1131 Calculus

Section 2: - Limits.

As part of our study of functions, we need to be able to look at the behaviour of a given function $f(x)$ as x approaches some point a, or as x gets very large, i.e. approaches infinity.

For example, it is obvious that as $x \to 2$, we have $x^2 \to 4$, and that as $x \to \infty$ we have $\frac{1}{x} \to 0$.

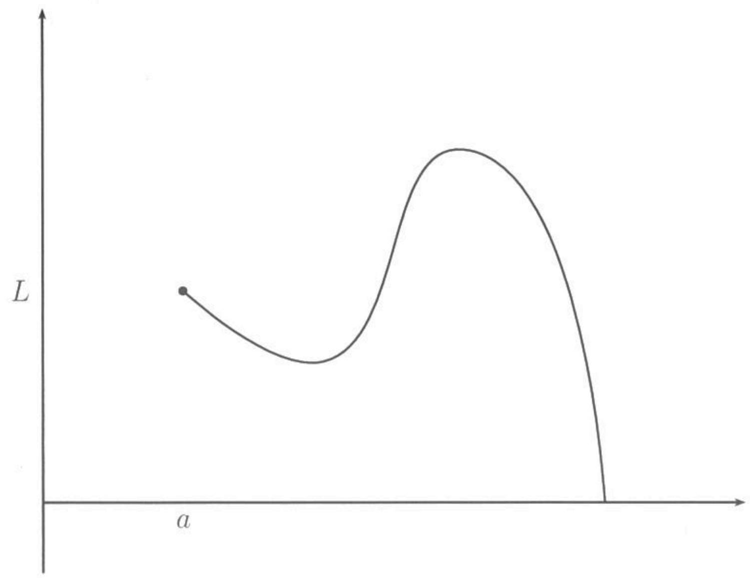
But what about $\frac{\sin x}{x}$ as $x \to 0$ or $\frac{\cos x}{x-\frac{\pi}{2}}$ as $x \to \frac{\pi}{2}$?

Left and Right-hand Limits:

We use the notation $x \to a$, to mean that x approaches the real number a either from the right or from the left. By this we mean that the distance between x and a can be made as small as we please. Note that this is not the same as saying that x actually takes the value a.

For some functions we need some further refinement of this idea.
For example, if $f(x) = \sqrt{x}$, then f is not defined for $x < 0$ and so we cannot talk about the limit $x \to 0$.

We will use the notation $x \to a^-$ and $x \to a^+$ to describe taking values of x 'close to a' from the **left** and values of x 'close to a' from the **right** respectively.



$\lim\limits_{x\to a^-} f(x)$ is called the **Left-hand limit** while $\lim\limits_{x\to a^+} f(x)$ is called the **Right-hand limit**.

Thus, for example, we can write $\lim\limits_{x\to 0^+} \sqrt{x} = 0$.

Ex: $\lim_{x\to 0^+} \frac{1}{x} \to \infty$ and $\lim_{x\to 0^-} \frac{1}{x} \to -\infty$.

Ex: Look at $f(x) = \frac{|x|}{x}$ near $x = 0$.

If $x>0$, $f(x) = \frac{|x|}{x} = \frac{x}{x} = 1$. So

$$\lim_{x\to 1^+} f(x) = 1.$$

If $x<0$, $f(x) = \frac{|x|}{x} = \frac{-x}{x} = -1$. Thus

$$\lim_{x\to 0^-} f(x) = -1.$$

Definition: If $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist and are equal to L, then we say that $\lim_{x\to a} f(x)$ exists and equals L.

Ex: Look at the limits, $\lim_{x\to 3} \frac{x^2-9}{x-3}$, and $\lim_{x\to 3} \frac{|x^2-9|}{x-3}$.

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3}$$

$$= \lim_{x\to 3} x+3 = 6.$$

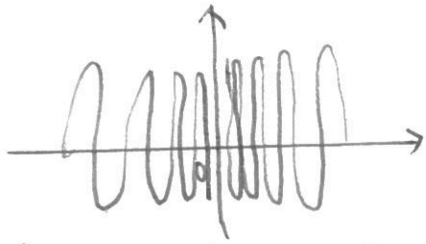
Now $\frac{|x^2-9|}{x-3} = \frac{|x-3|\,|x+3|}{x-3}$

If $x>3$, then $\frac{|x^2-9|}{x-3} = |x+3|$.

If $x<3$, then $\frac{|x^2-9|}{x-3} = -|x+3|$.

So $\lim_{x\to 3^+} \frac{|x^2-9|}{x-3} = 6$, and $\lim_{x\to 3^-} \frac{|x^2-9|}{x-3} = -6$.

Ex: Look at $\lim_{x\to 0} \cos(\frac{1}{x})$.



As $x \to 0$, $\cos(\frac{1}{x})$ oscilates between 1 and -1 faster and faster. There is no particular value that $\cos(\frac{1}{x})$ tends to as $x \to 0$. So this limit does not exist.

Ex: $\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$.

$$\frac{\sqrt{x+4}-2}{x} = \frac{(\sqrt{x+4}-2)(\sqrt{x+4}+2)}{x(\sqrt{x+4}+2)}$$

$$= \frac{x+4-4}{x(\sqrt{x+4}+2)} = \frac{1}{\sqrt{x+4}+2}.$$

So $\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x} = \frac{1}{4}$.

Rules for Limits:

We use the above ideas to construct a set of rules which will enable us to find limits without too much work.

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = \ell$, then as $x \to a$ we have:

(i) $f(x) \pm g(x) \to L \pm \ell$

(ii) $f(x).g(x) \to L\ell$

(iii) $\dfrac{f(x)}{g(x)} \to \dfrac{L}{\ell}$ provided $\ell \neq 0$.

(iv) $(f(x))^k \to L^k$ for any positive real number k, with $L > 0$.

Pinching Theorem:

The following theorem is very useful in finding limits.

Theorem: Suppose that f, g, h are three functions such that

(i) $f(x) \leq h(x) \leq g(x)$ for all x in some interval containing the point a

(ii) $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both **exist** and are *equal* to L,

then $\lim_{x\to a} h(x)$ exists and equals L.

In simple terms, the function h is *squeezed* or *pinched* between f and g.

Use the definition of limit to show that $\lim_{x\to 2} x^2 = 4$.

We want to make $|x^2-4| < \varepsilon$ by making x close to 2.

$|x^2-4| = |x+2|\,|x-2|$.

By making x close to 2, we can make $|x+2| < 5$, i.e. if $1<x<3$, or $|x-2|<1$

By making x close to 2, we can make $|x-2|$ as small as desired. So if $|x-2| < \min\{1, \varepsilon/5\}$, then $|x+2|\,|x-2| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon$, as needed.

Let $\varepsilon > 0$ be given. Set $\delta = \min\{1, \varepsilon/5\}$. Now if $0<|x-2|<\delta$, then $|x^2-4| = |x+2|\,|x-2| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon$. Hence

$\lim_{x\to 2} x^2 = 4$

Ex: $\lim_{x \to 0} x \sin \frac{1}{x}$.

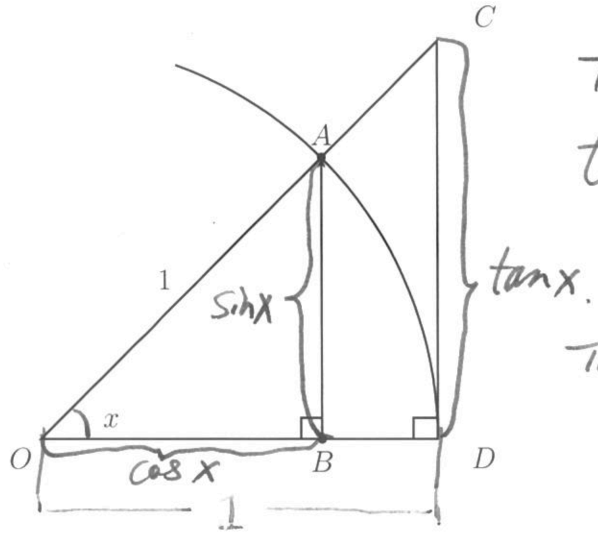
Since $-1 \le \sin\frac{1}{x} \le 1$, $x \sin\frac{1}{x}$ lies between $-x$ and x, both of which tend to zero as $x \to 0$. So by the pinching theorem, $\lim_{x \to 0} x \sin\frac{1}{x} = 0$.

An important limit:

Theorem:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Proof:



The area of the triangle OAB is $\frac{1}{2} \cos x \sin x$.

The area of the sector. OAD is $\frac{x}{2\pi} \pi = \frac{x}{2}$.

The area of the triangle OCD is $\frac{1}{2} \tan x$. Thus we have.

$$\frac{1}{2} \cos x \sin x \le \frac{x}{2} \le \frac{1}{2} \tan x. \quad \text{if } x \in [0, \tfrac{\pi}{2}]$$

This gives that $\cos x \le \frac{x}{\sin x} \le \frac{1}{\cos x}$ or

$$\cos x \le \frac{\sin x}{x} \le \frac{1}{\cos x}.$$ Now by the pinching theorem,

$$\lim_{x \to 0^+} \frac{\sin x}{x} = 1.$$

Using the fact that sin is an odd function, we can write

$$\lim_{x\to 0^-}\frac{\sin x}{x}=\lim_{x\to 0^+}\frac{\sin(-x)}{-x}=\lim_{x\to 0^+}\frac{\sin x}{x}=1.$$

Hence

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

Ex: Find $\lim_{x\to 0}\frac{\sin 3x}{2x}$.

$$\frac{\sin 3x}{2x}=\frac{3}{2}\cdot\frac{\sin(3x)}{(3x)}$$

So $$\lim_{x\to 0}\frac{\sin 3x}{2x}=\frac{3}{2}\lim_{x\to 0}\frac{\sin(3x)}{3x}=\frac{3}{2}.$$

Limits to Infinity:

To study the behaviour of functions for large positive (and negative) values of x, we need the concept of the *limit to infinity.*

It is intuitively obvious that $\frac{1}{x}\to 0$ as $x\to\infty$. For other rational functions, (and often when dealing with quotients), we use the rule:

Divide by the highest power of x in the denominator.

Ex: Find $\lim_{x\to\infty}\frac{3x^2+5x-2}{6x^2+8x-1}$.

$$\lim_{x\to\infty}\frac{3x^2+5x-2}{6x^2+8x-1}=\lim_{x\to\infty}\frac{3+\frac{5}{x}-\frac{2}{x^2}}{6+\frac{8}{x}-\frac{1}{x^2}}$$

$$=\frac{1}{2}$$

Ex: Find $\lim_{x\to\infty} \frac{x+\sin x}{x-\sin x}$

$$\lim_{x\to\infty} \frac{x+\sin x}{x-\sin x} = \lim_{x\to\infty} \frac{1+\frac{\sin x}{x}}{1-\frac{\sin x}{x}}$$

Since $\lim_{x\to\infty} \frac{\sin x}{x} = 0$, we have

$$\lim_{x\to\infty} \frac{x+\sin x}{x-\sin x} = 1.$$

Ex: Find $\lim_{x\to\infty} \frac{x+4}{x^2+3x+1}$.

$$\lim_{x\to\infty} \frac{x+4}{x^2+3x+1} = \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{4}{x^2}}{1+\frac{3}{x}+\frac{1}{x^2}}$$

$$= 0.$$

Ex: Find $\lim_{x\to\infty} \sqrt{x^2+5x} - x$.

$$\lim_{x\to\infty} \frac{(\sqrt{x^2+5x}-x)\cdot(\sqrt{x^2+5x}+x)}{\sqrt{x^2+5x}+x}$$

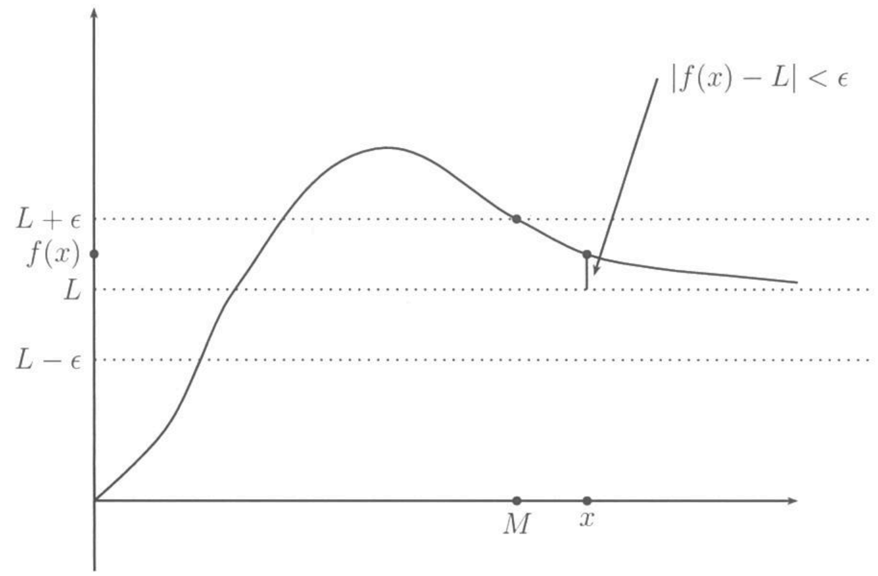
$$= \lim_{x\to\infty} \frac{5x}{\sqrt{x^2+5x}+x}$$

$$= \lim_{x\to\infty} \frac{5}{\sqrt{1+\frac{5}{x}}+1}$$

$$= \frac{5}{2}$$

Be extremely careful with these so-called 'indeterminate forms', i.e. limits which appear to have '$\frac{\infty}{\infty}$, $\frac{0}{0}$, $\infty - \infty$' and so on.

Geometric Interpretation:



The statement $\lim_{x \to \infty} f(x) = L$ can be interpreted geometrically as follows:

We draw an ϵ band about the line $y = L$, where we think of ϵ as a small positive real number. Then 'eventually', $f(x)$ moves inside the band and stays there forever, no matter how small ϵ is.

More formally,

Given $\epsilon > 0$, we can find an M, such that if $x > M$ we have $|f(x) - L| < \epsilon$.

Ex: Prove from the definition that $\lim_{x \to \infty} \frac{2x^2 + 3}{x^2} = 2$.

$$\frac{2x^2+3}{x^2} = 2 + \frac{3}{x^2}$$

So $\left| \frac{2x^2+3}{x^2} - 2 \right| = \left| \frac{3}{x^2} \right| = \frac{3}{x^2}$.

This quantity is less than any $\varepsilon > 0$, so long as $x > \sqrt{\frac{3}{\varepsilon}}$.

Now let $\varepsilon > 0$ be given. Set $M = \sqrt{\frac{3}{\varepsilon}}$.

If $x > M$, then

$$\left| \frac{2x^2+3}{x^2} - 2 \right| = \left| \frac{3}{x^2} \right| < \left| \frac{3}{\left(\sqrt{\frac{3}{\varepsilon}}\right)^2} \right| = \varepsilon.$$

Ex: Prove from the definition that $\lim_{x\to\infty} \frac{3x^2-4}{x^2+1} = 3$.

$$\frac{3x^2-4}{x^2+1} = \frac{3x^2+3-7}{x^2+1} = 3 - \frac{7}{x^2+1}.$$

Thus we want $\left|\frac{7}{x^2+1}\right| < \varepsilon$.

It suffices to have $\frac{7}{x^2} < \varepsilon$ or $x > \sqrt{\frac{7}{\varepsilon}}$.

Let $\varepsilon > 0$ be given. Set $M = \sqrt{\frac{7}{\varepsilon}}$. If $x > M$, then

$$\left|\frac{3x^2-4}{x^2+1} - 3\right| = \left|-\frac{7}{x^2+1}\right| = \frac{7}{x^2+1} < \frac{7}{x^2} < \frac{7}{(\sqrt{7/\varepsilon})^2} = \varepsilon.$$

(Note: You may be required to do problems of this sort in tests and exams.)

Continuity:

At school you learned that roughly speaking a function is continuous (cts) if you can draw it without taking your pen off the page. While this is an intuitively helpful way to think about it, what can be said about the continuity of the function

$$f(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

at the point $x = 0$?

We clearly need a more formal approach.

Definition: Let f be defined on some interval containing the point $x = a$.
We say that f is **continuous** (write 'cts') at $x = a$ iff $\lim_{x\to a} f(x) = f(a)$.

Note that this implies that the limit exists and equals the value of the function at the given point.

In practise we may need to check that each of the right-hand and left-hand limits exists and equals $f(a)$.

Hence in the above example:

$$f(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

we note that $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist, and so f cannot be cts at $x = 0$.

On the other hand, the function defined by

$$f(x) = \begin{cases} x\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

has $\lim_{x\to 0} x\sin\frac{1}{x} = 0$ (by the pinching theorem) and so f is continuous at $x = 0$.

Definition: If a function f is defined on an open interval I then we say that f is cts on I if it is cts at each point in I.

Most 'reasonable functions' are cts everywhere unless they have a good reason not to be!

Ex: Where is $f(x) = \frac{x}{x^2-9}$ cts?

$f(x)$ is defined on $\mathbb{R} \setminus \{\pm 3\}$.
In this domain, $f(x)$ is quotient of two continuous functions and the denominator is never zero. So $f(x)$ is continuous on $\mathbb{R} \setminus \{\pm 3\}$.

Algebra of Continuous Functions:

Suppose that $f(x), g(x)$ are cts at $x = a$, then

(i) $f(x) + g(x)$ and $f(x) - g(x)$ are cts at $x = a$.

(ii) $f(x).g(x)$ is cts at $x = a$.

(iii) $\frac{f(x)}{g(x)}$ is cts at $x = a$ provided $g(a) \neq 0$.

(iv) $(f(x))^k$ is cts at $x = a$ provided $k \in \mathbb{Q}$ and $(f(a))^k$ is defined.

Also

(v) If $g(x)$ is cts at $x = a$ and $f(x)$ is cts at $x = g(a)$, then $(f \circ g)(x)$ is cts at $x = a$.

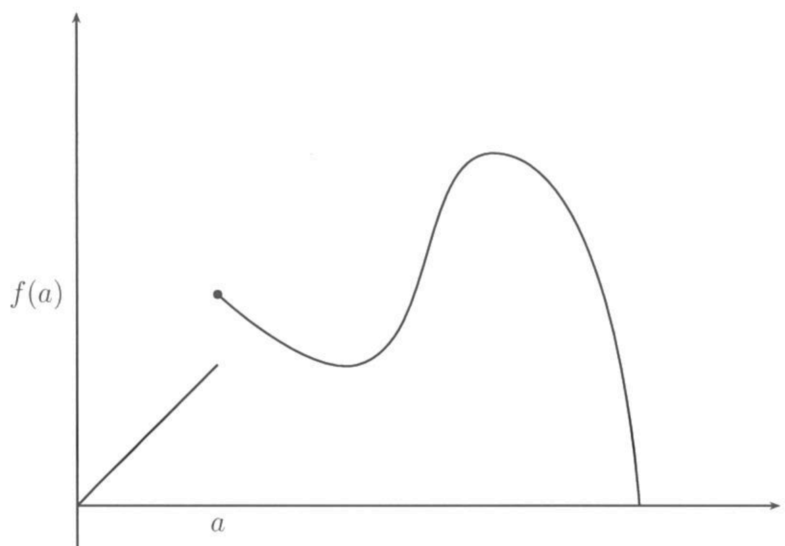
Ex: Since any polynomial is a finite combination of powers of x, it must be cts everywhere.

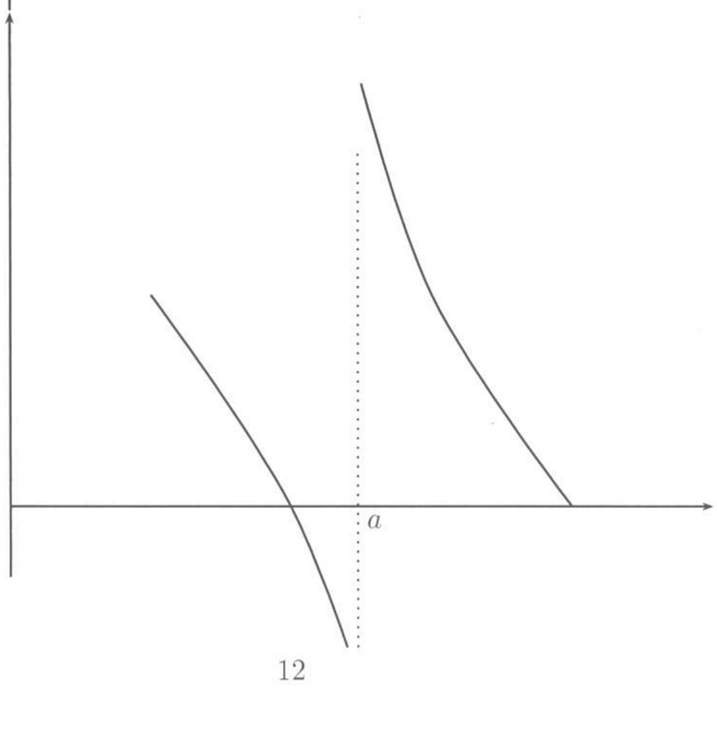
Types of Discontinuity:

We wish to classify the different types of discontinuity.

(i) **Essential Discontinuity**:

In this case, the limit of $f(x)$ as $x \to a$ does not exist.





The example in diagram 1, shows a **jump discontinuity**.

(ii) **Removable Discontinuity:**

In this case $f(x)$ does have a limit as $x \to a$, but it is not equal to $f(a)$, or $f(a)$ may not be defined.

Ex: $f(x) = \frac{x^2-9}{x-3}$ has a removable discontinuity at $x = 3$.

If we define

$$f_1(x) = \begin{cases} \frac{x^2-9}{x-3} & x \neq 3 \\ 6 & x = 3 \end{cases}$$

then $f_1(x)$ is continuous everywhere.

Ex: $f(x) = \frac{1}{x}$ has an essential discontinuity at $x = 0$.

Ex: $f(x) = \sin\frac{1}{x}$ has an essential discontinuity at $x = 0$.

Ex: Discuss $f(x) = \frac{(x-1)(x-2)}{(x^2-3x+2)(x+3)}$.

$f(x)$ is defined on $\mathbb{R} \setminus \{1, 2, -3\}$. and is continuous everywhere in its domain.

$\lim_{x \to 1} f(x) = \frac{1}{4}$ and $\lim_{x \to 2} f(x) = \frac{1}{5}$, upon noting $\frac{(x-1)(x-2)}{(x^2-3x+2)(x+3)} = \frac{1}{x+3}$.

So $f(x)$ has removable discontinuity at $x=1$ and $x=2$, but an essential discontinuity at $x=-3$.