Chapter 5: The Mean Value Theorem and Applications

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The Mean Value Theorem (MVT) is one of the most important results for establishing the theoretical structure of calculus.

Applications of the mean value theorem include

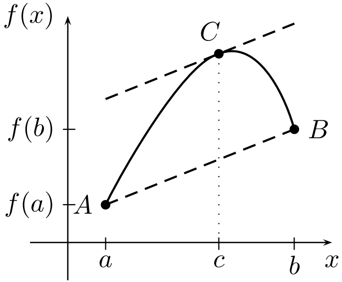
- identifying where a function is increasing or decreasing,
- identifying different types of stationary points,
- determining how many zeros a polynomial has,
- evaluating limits which are indeterminate forms of type $\frac{\infty}{\infty}$ and $\frac{0}{0}$,
- proving useful inequalities and
- estimating errors in approximations.

Mean Value Theorem (MVT)

The mean value theorem

Suppose that a function f is continuous on $[a, b]$ and differentiable on (a, b). Then, there exists at least one real number c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$



Note that the quotient on the left hand side of the equation is the average rate of change of f over the whole interval $[a, b]$, while $f'(c)$ is the instantaneous rate of change at c.

Thus, MVT says that for any (suitable) function, **there must be a point at which the instantaneous rate of change is equal to the average (mean) rate of change over a whole interval**.

Geometrically, MVT says that there is at least one point c where the tangent to the curve is parallel to the secant over the whole interval, that is, the line joining $(a, f(a))$ and $(b, f(b))$ (as they have the same slope).

Remark.

In the above theorem, it is required that f is continuous on the closed interval but differentiable only on the open interval!

Example. Find counterexamples which demonstrate that the continuity and differentiability requirements must be met.

Consider the function

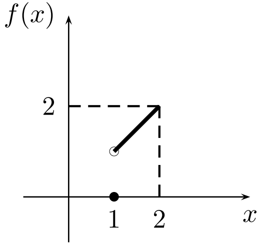
$$f(x) = \begin{cases} x, & 1 < x \leq 2 \\ 0, & x = 1. \end{cases}$$

We have that

$f'(x) = 1$, for every $1 < x < 2$.

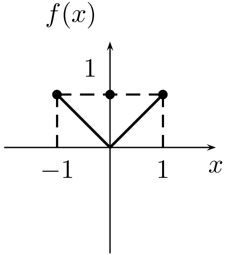
$$\frac{f(2) - f(1)}{2 - 1} = \frac{2 - 0}{1} = 2.$$

There is no $c \in (1, 2)$ such that $f'(c) = 2$.



So MVT does not work for discontinuous functions.

Consider the function
$f(x) = |x|$, for $-1 \le x \le 1$.
We have that
$f'(x) = 1$, for every $0 < x < 1$.
$f'(x) = -1$, for every $-1 < x < 0$.
The derivative $f'(0)$ does not exist.
$\dfrac{f(1) - f(-1)}{1 - (-1)} = \dfrac{0}{2} = 0.$
There is no $c \in (-1, 1)$ such that $f'(c) = 0$.



So MVT does not work for non-differentiable functions.

Remark. Note, however, that the conditions of MVT do not require the function to be differentiable at the endpoints a and b.

Example

Apply the MVT to the function f defined by $f(x) = x^5$ and $a = -1$, $b = 4$. Find the value(s) of c which satisfy the conclusion of the MVT.

Solution. Observe that f is continuous and differentiable everywhere.

Now, $f'(x) = 5x^4$ and, by MVT, there exist at least one $c \in (-1, 4)$ such that

$$f'(c) = \frac{f(4) - f(-1)}{4 - (-1)}.$$

This implies

$$5c^4 = \frac{4^5 - (-1)^5}{5} = \frac{1025}{5} = 205.$$

We finally obtain $c = \pm\sqrt[4]{41} \sim \pm 2.53$. Since $c \in (-1, 4)$, we take $c = 2.53$.

Remark. One can have more than one solution. E.g., change a to -2 and b to 3 and see what you get.

In this case it was not hard to calculate the specific value(s) of c. But the real point of the MVT is that it guarantees the existence of c, even if the necessary calculations are difficult or impossible.

Application: Proving inequalities using the MVT

Example. Show that

$$e^x > 1 + x \quad \text{for all} \quad x > 0.$$

Hint: Fix $x > 0$ and apply the MVT with $[a, b] = [0, x]$ for this fixed $x > 0$.

Application: Proving inequalities using the MVT

Example. Show that

$$e^x > 1 + x \quad \text{for all} \quad x > 0.$$

Hint: Fix $x > 0$ and apply the MVT with $[a, b] = [0, x]$ for this fixed $x > 0$.

Solution. Suppose that $x > 0$ and consider the closed interval $[0, x]$. We define the function $f : [0, x] \to \mathbb{R}$ by $f(t) = e^t$.
Now, f is continuous on $[0, x]$ and differentiable on $(0, x)$ so that MVT implies that there exists $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} \Longrightarrow e^c = \frac{e^x - 1}{x}.$$

A lower bound of $f'(c)$ on $(0, x)$ is given by $f'(c) = e^c > 1$.
We therefore conclude that $\frac{e^x - 1}{x} > 1$ or, equivalently, $e^x > x + 1$.

Example

General 'philosophy': Apply the MVT to an appropriate function f and find a lower or upper bound for $f'(c)$ on (a, b).

Example. It is known that any polynomial 'grows faster than' the logarithm. For, instance, show that

$$\ln x < x - 1 \quad \text{for all} \quad x > 1.$$

Example

General 'philosophy': Apply the MVT to an appropriate function f and find a lower or upper bound for $f'(c)$ on (a, b).

Example. It is known that any polynomial 'grows faster than' the logarithm. For, instance, show that

$$\ln x < x - 1 \quad \text{for all} \quad x > 1.$$

Solution. Suppose that $x > 1$ and consider the closed interval $[1, x]$. We define the function $f : [1, x] \to \mathbb{R}$ by $f(t) = \ln t$.
Now, f is continuous on $[1, x]$ and differentiable on $(1, x)$ so that MVT implies that there exists $c \in (1, x)$ such that

$$f'(c) = \frac{f(x) - f(1)}{x - 1} \Longrightarrow \frac{1}{c} = \frac{\ln x}{x - 1}.$$

An upper bound of $f'(c)$ on $(1, x)$ is given by $f'(c) = \frac{1}{c} < 1$, since $c > 1$.
We therefore conclude that $\frac{\ln x}{x-1} < 1$ or, equivalently, $\ln x < x - 1$.

Application: Error bounds

Question: How much bigger than $\frac{1}{2}$ can $\sin 31^\circ$ be?

Solution. So, we want to estimate

$$\sin 31^\circ - \sin 30^\circ = \sin 31^\circ - \frac{1}{2} = \sin \frac{31\pi}{180} - \sin \frac{\pi}{6}.$$

Let $f(x) = \sin x$, and choose the interval $[a, b]$ with $a = \frac{\pi}{6}$ and $b = \frac{31\pi}{180}$.

f is continuous on $\left[\frac{\pi}{6}, \frac{31\pi}{180}\right]$ and differentiable on $\left(\frac{\pi}{6}, \frac{31\pi}{180}\right)$.

Then by the MVT, there is some number $c \in (a, b)$ such that

$$\begin{gathered}
\frac{f(b) - f(a)}{b - a} = f'(c) \\
\frac{\sin b - \frac{1}{2}}{\pi/180} = \cos c \\
\sin \frac{31\pi}{180} - \frac{1}{2} = \frac{\pi \cos c}{180}.
\end{gathered}$$

Cont.

Now, since

$$\frac{\pi}{6} < c < \frac{31\pi}{180},$$

we have

$$\cos\frac{\pi}{6} > \cos c > \cos\frac{31\pi}{180},$$

because $\cos$ is a decreasing function between 0 and π. Also $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

Thus,

$$\begin{aligned}\sin\frac{31\pi}{180} - \frac{1}{2} &= \frac{\pi\cos c}{180} \\ &< \frac{\sqrt{3}}{2}\frac{\pi}{180} \\ &< 0.0152.\end{aligned}$$

Thus, the error of approximating $\sin 31^\circ$ by $\frac{1}{2}$ is at most 0.0152.

Application: The sign of a derivative

Definition

Let a function f be defined on an interval I. We say that

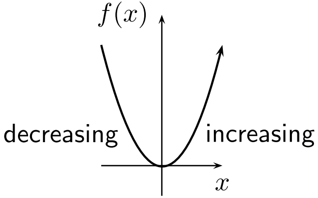
- f is increasing on I if for every two points x_1 and x_2 in I,

$$x_1 < x_2 \quad \text{implies that} \quad f(x_1) < f(x_2).$$

- f is decreasing on I if for every two points x_1 and x_2 in I,

$$x_1 < x_2 \quad \text{implies that} \quad f(x_1) > f(x_2).$$

Example. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is increasing on $[0, 10)$ and decreasing on $[-5, 0]$.



Theorem

Let f be differentiable on (a, b).

- If $f'(x) > 0$ for all x in (a, b) then f is increasing on $[a, b]$.
- If $f'(x) < 0$ for all x in (a, b) then f is decreasing on $[a, b]$.
- If $f'(x) = 0$ for all x in (a, b) then f is constant on $[a, b]$.

This is proven using the MVT!

Proof. Suppose that $f'(x) > 0$ for all x in (a, b) and choose two points x_1 and x_2 in $[a, b]$ such that $x_1 < x_2$.

Since f is differentiable on (a, b), it is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). Hence, by the MVT,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

for some c in (x_1, x_2).

Accordingly, $$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0.$$

The remaining two statements are proven in a similar manner.

Remarks:

A similar result applies on the infinite intervals $(-\infty, b]$, $[a, \infty)$ and $(-\infty, \infty)$.

The above theorem may be directly used to classify stationary points.

Example. Find and classify all stationary points of the function $f : \mathbb{R} \to \mathbb{R}$ whose derivative is given by

$$f'(x) = (x-4)(x-1)(x+5)^2.$$

Stationary points: Set $f'(x) = 0$.

The solutions are: $x = 4$, $x = 1$ and $x = -5$.

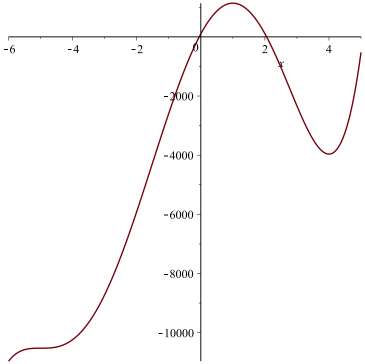
Classification: Investigate $f'(x)$ in a ‘small’ neighbourhood of any stationary point.

	-5^-	-5	-5^+	1^-	1	1^+	4^-	4	4^+
$x-4$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x-1$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$
$(x+5)^2$	$+$	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$f'(x)$	$+$	0	$+$	$+$	0	$-$	$-$	0	$+$
$f(x)$	$\nearrow$		$\nearrow$	$\nearrow$		$\searrow$	$\searrow$		$\nearrow$

Conclusion:

- $x = 4$: local minimum point
- $x = 1$: local maximum point
- $x = -5$: horizontal point of inflexion

Look at the graph: $>$plot$(x^5/5 + 5 * x^4/4 - 7 * x^3 - 85 * x^2/2 + 100 * x$, $x = -6..5)$



The second derivative and applications

Another (potential) method for classifying the stationary points of a function f involves the second derivative of f, which is denoted by

$$f'' \quad \text{or} \quad \frac{d^2y}{dx^2}, \quad \text{or} \quad y''$$

if we set $y = f(x)$.

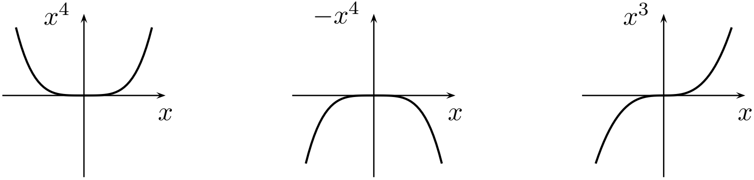
The second derivative test

Suppose that a function f is twice differentiable on (a, b) and that $c \in (a, b)$.

- If $f'(c) = 0$ and $f''(c) > 0$ then c is a local minimum point of f (🙂).
- If $f'(c) = 0$ and $f''(c) < 0$ then c is a local maximum point of f (🙁).

Proof. See the Calculus Notes.

Remark. If $f'(c) = f''(c) = 0$, **no** conclusion may be drawn!



- If $f(x) = x^4$ then $f'(0) = f''(0) = 0$ and there is a local minimum at 0.
- If $f(x) = -x^4$ then $f'(0) = f''(0) = 0$ and there is a local maximum at 0.
- If $f(x) = x^3$ then $f'(0) = f''(0) = 0$ and there is a horizontal point of inflexion at 0.

Hence if $f'(c) = f''(c) = 0$ then it is best to classify the stationary point c by examining the sign of the derivative on either side of c!

Example

Exercise. Find and classify the stationary points of the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x^3 - 6x^2 + 9x - 5.$$

Solution. $f'(x) = 3x^2 - 12x + 9 = 0$, whenever $(x-1)(x-3) = 0$.

So $x = 1$, and $x = 3$ are stationary points.

We compute the 2nd derivative: $f''(x) = 6x - 12$.

We have: $f''(1) < 0$, therefore, $x = 1$ is a local maximum.

Similarly, $f''(3) > 0$, therefore, $x = 3$ is a local minimum.

Critical points, maxima and minima

Question. How does one find global maxima or minima?

Definition

Suppose that f is defined on $[a, b]$. We say that a point c in $[a, b]$ is a critical point for f on $[a, b]$ if c satisfies one of the following properties:

- c is an endpoint a or b of the interval $[a, b]$,
- f is not differentiable at c,
- f is differentiable at c and $f'(c) = 0$ (i.e. a stationary point).

Theorem

Suppose that f is continuous on $[a, b]$. Then, f has a global maximum and global minimum on $[a, b]$. Moreover, the global maximum point and the global minimum point are both critical points for f on $[a, b]$.

Proof. As f is continuous on $[a, b]$, by the Max-Min Theorem, f attains a global max. and a global min. on $[a, b]$.

Let c be the global min. of f on $[a, b]$. Then, either $c = a$ or $c = b$, or $c \in (a, b)$.

If $c \in (a, b)$, then we have two possibilities:

- f is differentiable at c. Then, by Theorem 4.8.2 in Calculus Notes, we know that $f'(c) = 0$.
- f is not differentiable at c.

In all these cases, c is a critical point by definition.

Example

Exercise. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is given by the rule

$$f(x) = |x^2 - 3x - 4|.$$

Find the global maximum and global minimum values of f on the interval $[0, 5]$.

Example

Exercise. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is given by the rule

$$f(x) = |x^2 - 3x - 4|.$$

Find the global maximum and global minimum values of f on the interval $[0, 5]$.

Solution. We have

$$f(x) = |x+1| * |x-4| = \begin{cases} -(x+1)(x-4) & 0 \leq x \leq 4 \\ (x+1)(x-4) & 4 < x \leq 5. \end{cases}$$

Let us identify the critical points. First, 0 and 5 are critical points.

$$f'(x) = \begin{cases} -2x+3 & 0 \leq x < 4 \\ 2x-3 & 4 < x \leq 5. \end{cases}$$

Thus, another critical point is the root of $f'(x) = 0$, that is, $x = \frac{3}{2}$.
We note that f is not differentiable at $x = 4$, and thus x=4 is another critical point.

Example

Exercise. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is given by the rule

$$f(x) = |x^2 - 3x - 4|.$$

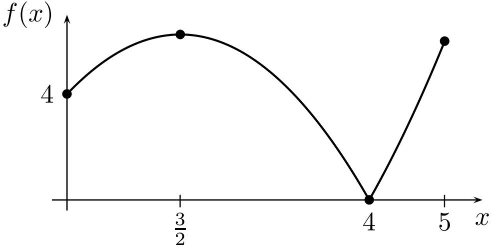
Find the global maximum and global minimum values of f on the interval $[0, 5]$.

So, the critical points are 0, $3/2$, 4 and 5.
To find the global max. and global min., we compute the values in the critical points:

$$f(0) = 4, \ f(4) = 0, \ f(3/2) = 6.25, \ f(5) = 6.$$

So, the global minimum is at $x = 4$ with the global min. value $f(4) = 0$.
The global maximum point is $x = 3/2$ with the global max. value $f(3/2) = 6.25$.
$x = 0$ and $x = 5$ are local min. and loc max., respectively.

Our result above can be also seen directly on the graph:



Application: Counting zeros

Using the information where the function is increasing and decreasing, coupled with the Intermediate Value Theorem (IVT), one can see how many times the graph intersects Ox and detect the intervals containing each zero.

Theorem

Suppose that

- f is continuous on $[a, b]$ and differentiable on (a, b),
- $f(a)$ and $f(b)$ have opposite signs,
- $f' > 0$ on (a, b) or $f' < 0$ on (a, b).

Then f has exactly one zero in (a, b).

Proof. Since f is continuous on a closed interval and 0 lies between $f(a)$ and $f(b)$, by the IVT, there exists $c \in (a, b)$ such that $f(c) = 0$.

Moreover, f cannot have more than one zero because it is either strickly increasing or decreasing on (a, b).

Example

Example. Determine the number of (real) zeros of

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = x^4 - x^3 - 3x^2 - 8x - 5$$

and give an approximate location for each zero.

Solution. f is continuous and differentiable everywhere. Differentiating, we have

$$f'(x) = 4x^3 - 3x^2 - 6x - 8 = (x-2)(4x^2 + 5x + 4).$$

Thus, $f'(2) = 0$ and $f'(x) < 0$ on $(-\infty, 2)$ and $f'(x) > 0$ on $(2, \infty)$.

- On $[-1, 2]$: f is decreasing and therefore can not have more than one zero on this interval.
  Also $f(-1) = 2$ and $f(2) = -25$, hence, by the above theorem, f has exactly one zero on $(-1, 2)$.
- On $[2, 4]$: f is increasing and therefore can not have more than one zero on this interval.
  Also $f(2) = -25$ and $f(4) = 107$, hence, by the above theorem, f has exactly one zero on $(2, 4)$.

Conclusion: f has two real zeros, one in the interval $(-1, 2)$ and one in $(2, 4)$.

General approach for finding zeros:

- Calculate f' and solve $f'(x) = 0$.
- Determine the intervals where f' is positive and negative.
- By the previous step, determine the intervals where f is monotonically increasing and decreasing.
- Evaluate f at the endpoints of each interval. If f changes sign on the interval, there is exactly one root on that interval.
  If it does not change sign, there are no roots on that interval.

Application: Antiderivatives

Remark. If we are given a formula stating the displacement of a particle at any time t, we can find the particle's velocity by differentiating the displacement formula.

If we are given the velocity and wish to find the displacement we have to do the reverse. The opposite of differentiation is called *antidifferentiation*.

Definition

Suppose that f is continuous on an open interval I. A function F is said to be an antiderivative (or a primitive) of f on I if

$$F'(x) = f(x) \quad \text{for all} \quad x \in I.$$

The process of finding an antiderivative of a function is called antidifferentiation.

Remark. It F the unique antiderivative of f?

Theorem

Suppose that f is a continuous function on an open interval I and that F and G are two antiderivatives of f on I. Then, there exists a real constant C such that

$$G(x) = F(x) + C$$

for all x in I.

Proof. Let H denote the function given by

$$H(x) = G(x) - F(x)$$

for all x in I. Then, H is differentiable on I and

$$\begin{aligned} H'(x) &= G'(x) - F'(x) \\ &= f(x) - f(x) \\ &= 0 \end{aligned}$$

for all x in I. Hence, there exists a constant C such that $H(x) = C$ for all x in I (Why?) so that

$$G(x) = F(x) + C, \quad \forall x \in I.$$

Some well-known antiderivatives are given below.

Function	Antiderivative
x^r, where r is rational and $r \neq -1$	$\dfrac{1}{r+1}x^{r+1} + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
e^{ax}	$\dfrac{1}{a}e^{ax} + C$
$\dfrac{1}{x}$	$\ln\lvert x\rvert + C$
$\dfrac{f'(x)}{f(x)}$	$\ln\lvert f(x)\rvert + C$

Application: L'Hôpital's rule

Question. What is the limit of the 'indeterminate expression'

$$\lim_{x\to 0} \frac{1-\cos x}{2x}, ?$$

L'Hôpital's rule

Suppose that f and g are both differentiable functions in a neighbourhood of some $a \in \mathbb{R}$ and that either one of the two following conditions hold:

- $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$
- $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$.

If

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} \quad \text{exists}$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Remark. The theorem also holds for

- limits as $x \to \infty$ or $x \to -\infty$
- one-sided limits (as $x \to a^+$ or $x \to a^-$).

L'Hôpital's rule is proved using the MVT!

Example. Determine the limit

$$\lim_{x \to 0} \frac{1 - \cos x}{2x}.$$

Solution. We have an inderterminate form $\frac{'0'}{0}$, hence:

$$\lim_{x \to 0} \frac{1 - \cos x}{2x} = \lim_{x \to 0} \frac{\sin x}{2} = 0.$$

by l'Hôpital's rule, since the final limit exists.

Remark. l'Hôpital's rule may be applied iteratively.

Example.

$$\lim_{x \to 0} \frac{x - \sin x}{x^2} = \lim_{x \to 0} \frac{1 - \cos x}{2x} = \text{... see above.}$$

Remark. It is important that the limit exists after a finite number of applications of l'Hôpital's rule!

Exercise. What is the limit

$$\lim_{x \to \infty} \frac{2x - \sin x}{3x + \sin x}$$

and why can l'Hôpital's rule **not** be applied?

Solution. We can not apply l'Hôpital's rule, since the limit $\lim\limits_{x \to \infty} \dfrac{2 - \cos x}{3 + \cos x}$ does not exist.

But we have

$$\lim_{x \to \infty} \frac{2x - \sin x}{3x + \sin x} = \lim_{x \to \infty} \frac{2 - \frac{\sin x}{x}}{3 + \frac{\sin x}{x}} = \frac{2}{3}.$$

Example

Example. Find $\lim_{x \to 0^+} x \ln x$.

Example

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Solution. We have

$$\lim_{x\to 0^+} x\,\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \left(\frac{\infty}{\infty}\right) = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0.$$

The use of L'Hôpital's rule was justified as the final limit exists.

Example

Example. Find $\displaystyle\lim_{x \to 0^+} \frac{3x^2 e^{4x}}{1 - \cos 5x}$.

Example

Example. Find $\lim_{x\to 0^+} \frac{3x^2e^{4x}}{1-\cos 5x}$.

Solution. Here $f(x) = 3x^2e^{4x}$ and $g(x) = 1 - \cos 5x$.
Note that $f(x) \to 0$ and $g(x) \to 0$ as $x \to 0^+$. So we can use L'Hôpital rule.
Now

$$f'(x) = 6xe^{4x} + 12x^2e^{4x} \quad \text{and} \quad g'(x) = 5\sin(5x).$$

But here $f'(x) \to 0$ and $g'(x) \to 0$ as $x \to 0^+$, so we differentiate again:

$$\begin{aligned} f''(x) &= 6e^{4x} + 24xe^{4x} + 24xe^{4x} + 48x^2e^{4x} \\ &= 6e^{4x} + 48xe^{4x} + 48x^2e^{4x} \\ g''(x) &= 25\cos(5x). \end{aligned}$$

This time $f''(x) \to 6$ and $g''(x) \to 25$ as $x \to 0^+$, so by L'Hôpital Rule we have

$$\lim_{x\to 0^+} \frac{3x^2e^{4x}}{1-\cos 5x} = \lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = \lim_{x\to 0^+} \frac{f''(x)}{g''(x)} = \frac{6}{25}.$$

Summary: What did we learn in this chapter?

- The mean value theorem (MVT, p. 3)
- Proving inequalities using the MVT (p. 8)
- Finding error bounds using the MVT (p.10)
- Finding if a function is increasing or decreasing (p. 13)
- Classify stationary points (p. 14 and 16)
- Critical points (p. 19)
- Critical points and absolute extrema (p. 20)
- Counting zeros (p. 23)
- Antiderivatives (p. 26)
- L'Hôpital's rule (p. 29)