LECTURE 10
Extrema and L'Hopital's Rule

If $f'(c) = 0$ and $f''(c) < 0$ then $(c, f(c))$ is a local maximum.

If $f'(c) = 0$ and $f''(c) > 0$ then $(c, f(c))$ is a local minimum.

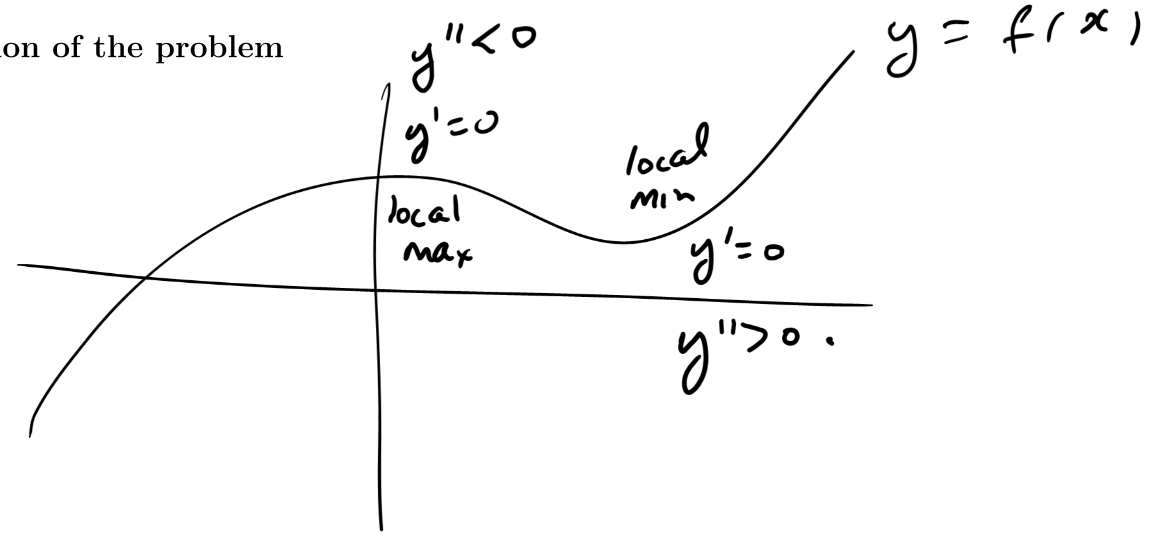
A continuous function on a closed interval will attain a global minimum and a global maximum. Furthermore these extrema will always occur at a critical point, that is a stationary point, an endpoint or a point of non-differentiability.

L'Hopitals Rule: Suppose that f and g are differentiable and that $\dfrac{f(a)}{g(a)}$ is the indeterminate form $\dfrac{"0"}{0}$ or $\dfrac{"\infty"}{\infty}$. Then $\displaystyle\lim_{x \longrightarrow a} \frac{f(x)}{g(x)} = \lim_{x \longrightarrow a} \frac{f'(x)}{g'(x)}$.

Maxima and Minima

Stationary points are some of the most important points on the graph of a function. The are found by setting $f'(x) = 0$ and categorised by considering the sign of the second derivative at the stationary point or alternatively the behaviour of the first derivative around the point in question.

Discussion of the problem



We are often faced with word problems where it is required to maximise or minimise a quantity of interest. Our approach is to:

a) Find a formula for the thing to be maximised or minimised.

b) Clean the formula up so that there is only one independent variable.

c) Hit it with the calculus. Don't forget to test for max/min!

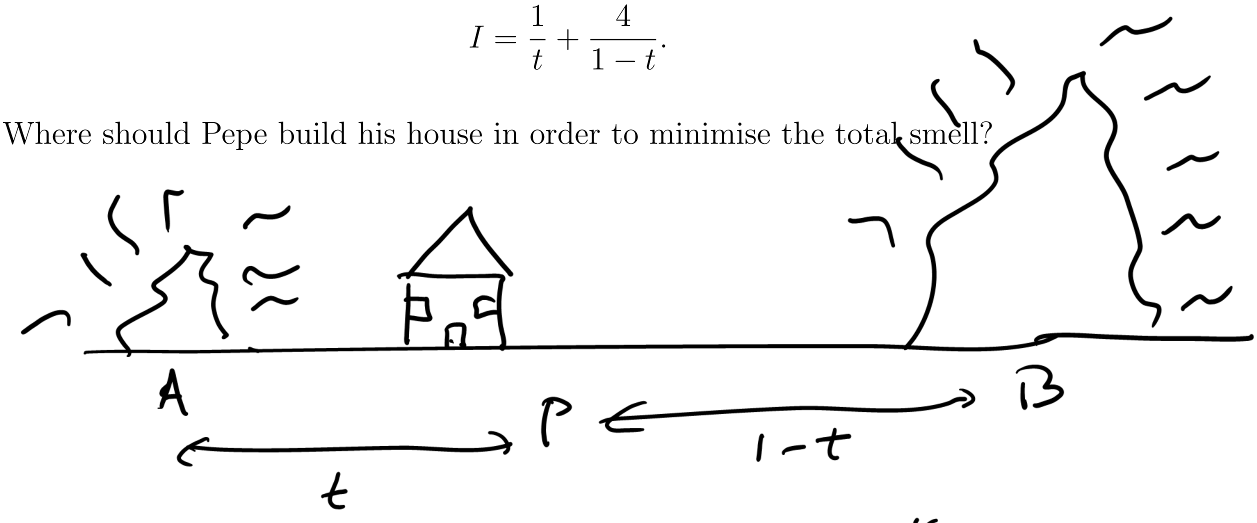
Example 1: Two garbage dumps A and B are 1km apart on a straight road. At a distance of x from dump A the intensity of the smell is given by $I_A = \frac{1}{x}$. Dump B smells four times as bad, so that at a distance of x from dump B the intensity of the smell is $I_B = \frac{4}{x}$.

Pepe wants to build his house at a point P between A and B. Suppose that P is a distance of t km from A.

a) Show that the total intensity of the smell at P is

$$I = \frac{1}{t} + \frac{4}{1-t}.$$

b) Where should Pepe build his house in order to minimise the total smell?



a) $I = \frac{1}{t} + \frac{4}{1-t} = t^{-1} + \frac{4}{1-t}$

dist A

dist B

$$\frac{dI}{dt} = -t^{-2} + \frac{\cancel{(1-t)(0)} - 4(-1)}{(1-t)^2} = -\frac{1}{t^2} + \frac{4}{(1-t)^2} = 0$$

(sf. pnts)

$$\frac{1}{t^2} = \frac{4}{(1-t)^2} \Rightarrow (1-t)^2 = 4t^2 \Rightarrow (1-t) = \pm 2t$$

$1-t = 2t$, $1 = 3t \Rightarrow t = \frac{1}{3}$.

$1-t = -2t$, $1 = -t$, $\cancel{t = -1}$

$$I'' = 2t^{-3} + \frac{0 - (4)(2)(1-t)^1(-1)}{(1-t)^4}$$

$= \frac{2}{t^3} + \frac{8}{(1-t)^3}$. Clearly $I''(\frac{1}{3}) > 0 \to \cup \to$ local min at $t = \frac{1}{3}$

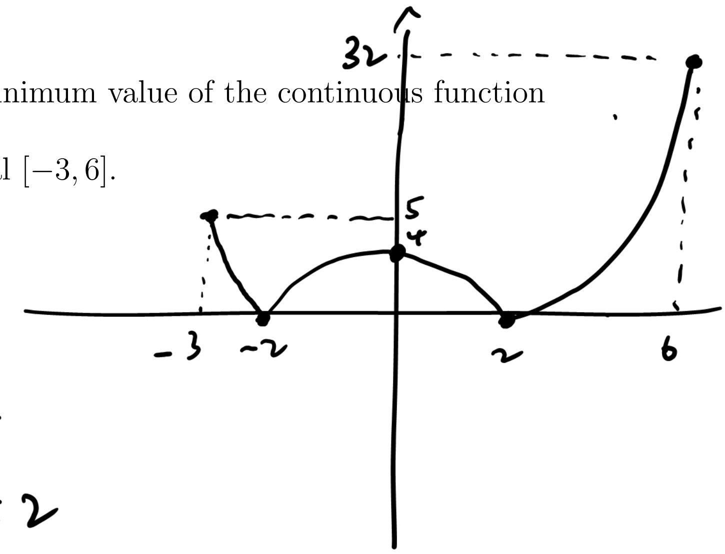
★ $\frac{1}{3}$ km from A ★

A continuous function on a closed interval will attain a global minimum and a global maximum. Furthermore these extrema will always occur at a critical point, that is a stationary point, an endpoint or a point of non-differentiability. When dealing with continuous functions over closed intervals don't forget to check out the cusps and endpoints when searching for maxima and minima !

Example 2: Find the maximum and minimum value of the continuous function

$f(x) = |\,x^2 - 4|$ over the closed interval $[-3, 6]$.

Sketch:



max = 32 @ $x = 6$
min = 0 @ $x = \pm 2$

(st. pnt. did not get used)

★ Max of 32 and a min of 0 ★

Note in the above example that the stationary point didn't even get used! The extrema for continuous function over closed intervals can come from three different sources! Stationary points, Non-differential points or endpoints.

Counting Zeros

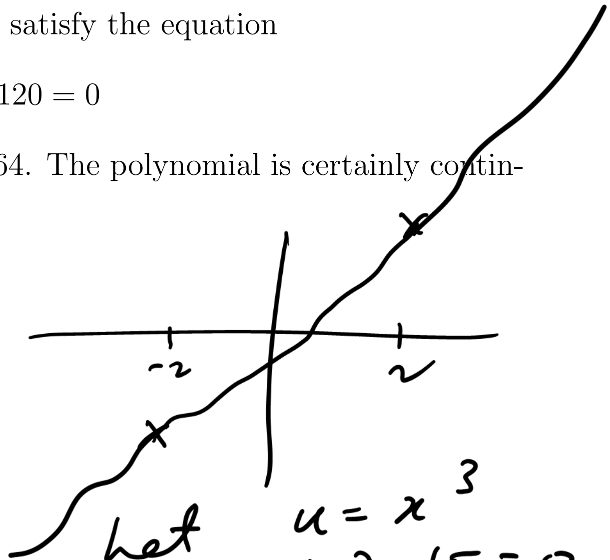
An analysis of stationary points for a polynomial can often reveal the number of times the polynomial cuts the x axis.

Example 3: Determine how many real numbers satisfy the equation

$$2x^7 + 7x^4 + 70x + 120 = 0$$

We first note that $p(2) = 628$ and $p(-2) = -164$. The polynomial is certainly continuous, thus by the IVT there is a zero in $[-2, 2]$.

Let us now try to find the stationary points:



$p'(x) = 14x^6 + 28x^3 + 70$

$= 0$ (st. pnts)

$\Rightarrow \underline{x}^6 + 2\underline{x}^3 + 5 = 0.$

no stationary points!

$p'(0) = 70 > 0 \Rightarrow$ increasing!

Let $u = x^3$

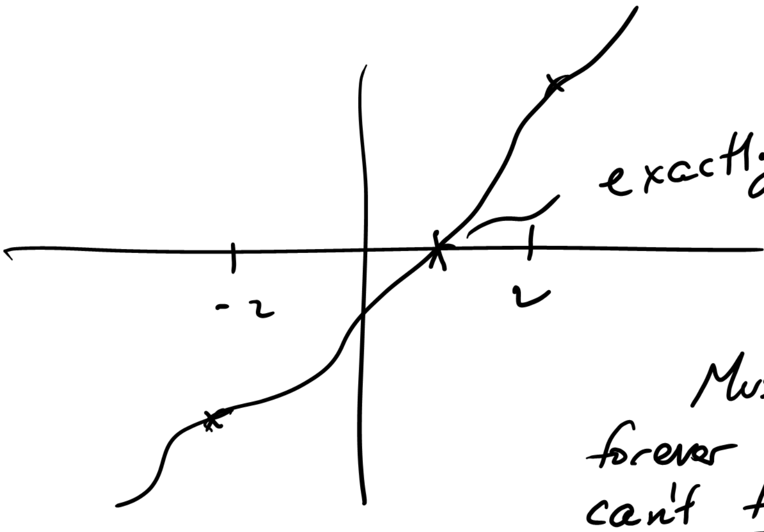
$u^2 + 2u + 5 = 0$

$\Delta = b^2 - 4ac = 4 - 20 < 0.$

$\therefore$ <u>no</u> real sol^{ns}.

The derivative is never equal to 0. Thus there are no turning points and the polynomial must be monotonically increasing or decreasing. Since $p'(0) = 70$ the polynomial is increasing over $\mathbb{R}$.

Sketch:



exactly one solution.

Must increase forever since it <u>can't</u> <u>turn</u>.

Clearly there is only one zero. Once you know all the stationary points for a well behaved function the number of zeros becomes quite clear. Keep in mind however that knowing how many there are in no way helps you to actually find them!

★ There is only one root ★

L'Hopital's rule

We close the lecture with a truly wonderful way of finding all sorts of limits called l'Hopital's rule.

Earlier in this course we had a look at limits and considered some specific methods of solution. L'Hopital's rule is a remarkable technique for evaluating limits that can be used to attack some of the limits that we have dealt with so far, together with new much more complicated examples. As a bonus it is a very easy rule to use even though it does involve a little differentiation.

L'Hopital's Rule: Suppose that f and g are differentiable and that $\dfrac{f(a)}{g(a)}$ is the indeterminate form $\dfrac{"0"}{0}$ or $\dfrac{"\infty"}{\infty}$. Then $\displaystyle\lim_{x\longrightarrow a}\frac{f(x)}{g(x)} = \lim_{x\longrightarrow a}\frac{f'(x)}{g'(x)}$.

This means that if you start with $\dfrac{"0"}{0}$ or $\dfrac{"\infty"}{\infty}$ then all you have to do is differentiate the top and differentiate the bottom and try again! If you then get an answer with the new problem then that is also the answer to the old problem!!

L'Hopital's rule also works with $\displaystyle\lim_{x\longrightarrow\infty}\frac{f(x)}{g(x)}$ and one sided limits $\displaystyle\lim_{x\longrightarrow a^{\pm}}\frac{f(x)}{g(x)}$.

The proof of l'Hopital's rule is in your printed notes and is not examinable. It essentially revolves around the fact that the ratio of function values may be identified with the ratio of their rates of change through the mean value theorem.

To start with let's revisit some of the examples from earlier in the course. We will do them the old way and then see how they can be handled with l'Hopital's rule. We denote the use of l'Hopital's rule by $\stackrel{l'h}{=}$.

Example 4: Evaluate each of the following limits the old way and then using l'Hopital's rule.

a) $\lim\limits_{x\longrightarrow 2} \dfrac{x^2-5x+6}{x^2-4}$ $= \text{"}\dfrac{0}{0}\text{"}$

b) $\lim\limits_{x\longrightarrow \infty} \dfrac{10x-4}{5x+7}$

$$\text{a) } \lim_{x\to 2} \frac{\cancel{(x-2)}(x-3)}{\cancel{(x-2)}(x+2)} = \lim_{x\to 2} \frac{x-3}{x+2} = \frac{2-3}{2+2} = \underline{\underline{\frac{-1}{4}}}$$

$$\lim_{x\to 2} \frac{x^2-5x+6}{x^2-4} = \text{"}\frac{0}{0}\text{"}$$

$$\underline{\underline{\stackrel{l'h}{}}} \lim_{x\to 2} \frac{2x-5}{2x} = \frac{4-5}{4} = -\frac{1}{4}$$

$$\text{b) } \lim_{x\to \infty} \frac{10x-4}{5x+7} = \text{"}\frac{\infty}{\infty}\text{"}$$

$$= \lim_{x\to \infty} \frac{10-\cancel{\frac{4}{x}}}{5+\cancel{\frac{7}{x}}} = \frac{10}{5} = \underline{\underline{2}}$$

<u>Via L'H</u>

$$\stackrel{l'h}{=} \lim_{x\to \infty} \frac{10}{5} = \underline{2}$$

$\bigstar$ $a)$ $-\dfrac{1}{4}$ $b)$ 2 $\bigstar$

Feel free to use l'Hopital's rule whenever you need to evaluate a limit. But keep in mind that you need to start with $\frac{"0"}{0}$ or $\frac{"\infty"}{\infty}$ before it can be applied. l'Hopital's rule does not work well on exotic examples involving the matching of left and right hand limits.

Lets take a look at all the tricks ! Keep in mind that l'Hopital's rule has nothing to do with the quotient rule rule for differentiation. We differentiate the top and bottom separately.

Example 5: Evaluate each of the following limits:

a) $\lim_{x\longrightarrow 0} \frac{e^{3x}-1}{\sin(x)}$ $= \frac{1-1}{0} = "\frac{0}{0}"$ ✓

$\overset{l'h}{=} \lim_{x\to 0} \frac{3e^{3x}}{\cos(x)} = \frac{3e^{0}}{\cos(0)} = \frac{3\times 1}{1} = 3$

b) $\lim_{x\longrightarrow 0} \frac{\sin(4x)}{\sin(7x)}$ $= "\frac{0}{0}"$

$\overset{l'h}{=} \lim_{x\to 0} \frac{4\cos(4x)}{7\cos(7x)} = \frac{4\times 1}{7\times 1} = \frac{4}{7}$

c) $\lim_{x\longrightarrow \infty} \frac{x^2}{e^{3x}}$ $= "\frac{\infty}{\infty}"$

$\overset{l'h}{=} \lim_{x\to\infty} \frac{2x}{3e^{3x}} = "\frac{\infty}{\infty}"$

$\overset{l'h}{=} \lim_{x\to\infty} \frac{2}{9e^{3x}} = \underline{0}$

$(x^{-1})' = -x^{-2} = -\frac{1}{x^2}$

d) $\lim_{x\longrightarrow 0^+} x\ln(x) = 0(-\infty)$

$= \lim_{x\to 0^+} \frac{\ln(x)}{\left(\frac{1}{x}\right)} = -\frac{\infty}{\infty}$

$\overset{l'h}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0^+} -\frac{1}{x}\times x^2$

$= \lim_{x\to 0^+} (-x) = 0$

e) $\lim_{x\longrightarrow\infty} (1+\frac{1}{x})^x$

This one is very tricky. Let $y = \lim_{x\longrightarrow\infty}(1+\frac{1}{x})^x$. Then

$$\ln(y) = \lim_{x\longrightarrow\infty} \ln(1+\frac{1}{x})^x = \lim_{x\longrightarrow\infty} x\ln(1+\frac{1}{x}) = \infty\times 0 = \lim_{x\longrightarrow\infty}\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}} = \frac{"0"}{0}$$

$$\overset{l'h}{=} \lim_{x\longrightarrow\infty}\frac{\frac{1}{1+\frac{1}{x}}\times \cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}} = \lim_{x\longrightarrow\infty}\frac{1}{1+\frac{1}{x}} = 1. \text{ Therefore } \ln(y) = 1 \longrightarrow y = e. = e$$

★ *a)* 3 *b)* $\frac{4}{7}$ *c)* 0 *d)* 0 *e)* e ★

Do be careful how you use the rule!! You must have at the start $\frac{"0"}{0}$ or $\frac{"\infty"}{\infty}$.

Example 6: Evaluate $\lim_{x\longrightarrow 3}\frac{x^2+1}{3x+5} = \frac{9+1}{9+5} = \frac{10}{14} = 5/7$

$\overset{l'h}{=} \lim_{x\to 3} \frac{2x}{3} = \frac{2\times 3}{3} = 2.$

★ $\frac{5}{7}$ ★

Example 7: Evaluate $\lim_{x\longrightarrow\infty}\frac{2x-\sin(x)}{5x-\sin(x)} = "\frac{\infty}{\infty}"$

$= \lim_{x\to\infty} \frac{2 - \cancel{\frac{\sin(x)}{x}}}{5 - \cancel{\frac{\sin(x)}{x}}} = 2/5 =$ correct answer.

★ $\frac{2}{5}$ ★

What went wrong??

$\overset{l'h}{=} \lim_{x\to\infty} \frac{2-\cos(x)}{5-\cos(x)} \overset{l'h}{=} \lim_{x\to\infty} \frac{\sin(x)}{\sin(x)} = 1 \overset{??}{=}$

$\neq \frac{0}{0}$ invalid

$\neq \frac{\infty}{\infty}$