THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 2 2017

MATH1131
MATHEMATICS 1A

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS IS APPENDED TO THE PAPER

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 1

1. i) Determine the value of each of the following limits or explain why the limit does not exist.

a) $\displaystyle\lim_{x\to\infty} \frac{7x^2+5}{16x^2+13x-2}$

b) $\displaystyle\lim_{x\to\infty} \frac{x^{2017}}{e^{0.001x}}$

c) $\displaystyle\lim_{x\to 1} \frac{|x^2-1|}{x-1}$

d) $\displaystyle\lim_{x\to\infty} \left(1+\frac{5}{x}\right)^{2x}$

ii) Find all values a and b (if any) such that the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x) = \begin{cases} \cos 2x + 1 & x \geqslant 0 \\ ax + b & x < 0. \end{cases}$$

is both continuous and differentiable at 0.

iii) Let $y = x^{\sin x}$. Find $\dfrac{dy}{dx}$.

iv) Evaluate the following integrals.

a) $I_1 = \displaystyle\int_1^2 xe^{2x}\,dx$

b) $I_2 = \displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

v) Show that the

$$e^x + 1 = 5\cos x$$

has at least one positive solution.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 2

2. i) Let

$$y = \int_0^{\sin x} \frac{1}{1+t^2}\, dt.$$

a) Find $\dfrac{dy}{dx}$.

b) Find all stationary points for y on the interval $(0, \pi)$.

c) For each stationary point, determine whether it is a local maximum, a local minimum, or neither.

ii) A lighthouse with a rotating beacon is located in the ocean 2 kilometers from the shore. The beacon rotates at a constant rate of 5 revolutions per minute. Assume the shoreline is straight, and let P be the point on the coastline which is closest to the lighthouse. At any time t, let X be the point where the beacon's beam of light hits the shoreline, and x the distance from X to P. How fast is x changing when X is 3 kilometers away from P?

iii) Determine which, if any, of the following improper integrals converge Give reasons for your answers.

a) $\displaystyle\int_2^{\infty} \frac{1}{x\ln(x)}\, dx,$

b) $\displaystyle\int_1^{\infty} \frac{\sin^2 x}{x(1+\sqrt[3]{x})}\, dx.$

iv) a) Show that

$$\cosh x + \sinh x = e^x.$$

b) Show that

$$(\cosh x + \sinh x)^3 = \cosh(3x) + \sinh(3x).$$

v) Consider the polar curve

$$r = 4 - \sin\theta.$$

a) Find all points at which the tangent is horizontal or vertical.

b) Show that the curve is symmetric under reflection in the y-axis.

c) Sketch the curve.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 3

3. i) Let $z = \alpha + 2i$ and $w = 2 - 3i$, where α is a real number.

a) Evaluate $z\bar{w}$.

b) Calculate $\dfrac{z}{w}$, expressing your answer in Cartesian form $a + ib$.

ii) Let z be a complex number with $|z| = 2$ and $\mathrm{Arg}(z) = \dfrac{\pi}{3}$.

a) Write down the polar form of z.

b) Hence or otherwise evaluate $\left(1 + \sqrt{3}i\right)^{3001}$ expressing your answer in Cartesian form.

iii) a) Find all the sixth roots of unity, expressing your answers in Cartesian form.

b) Write $z^6 - 1$ as a product of real linear and irreducible real quadratic factors.

iv) a) Use De Moivre's theorem to write $\sin 3\theta$ as a sum of powers of $\sin\theta$.

b) Hence or otherwise find one solution to the equation

$$4x^3 - 3x + \frac{1}{\sqrt{2}} = 0.$$

(Your solution must be expressed exactly, and not in decimal form).

v) The system of linear equations

$$\begin{array}{rcrcrcl} x & + & y & - & z & = & 0 \\ 2x & + & y & + & 2z & = & 0 \end{array}$$

has infinitely many solutions.

a) Use Gaussian elimination to find the general solution to the system in terms of a parameter.

b) Hence, or otherwise, write down the point normal form of the plane with parametric vector equation

$$\mathbf{x} = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 4

4. i) Sketch the following region S on the Argand diagram:

$$S = \left\{ z \in \mathbb{C} : \ \mathrm{Im}(z) \le 1 \text{ and } -\frac{\pi}{3} \le \ \mathrm{Arg}(z) \le \frac{\pi}{3} \right\}.$$

ii) Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & \sqrt{3} \\ 0 & \sqrt{2} & 0 \\ -\sqrt{3} & 0 & -1 \end{pmatrix}.$$

a) Find $\det(A)$.

b) Use the following Maple output to help you answer the question below.

```
> A := <<1,0,sqrt(3)>|<0,sqrt(2),0>|<-sqrt(3),0,-1>>;
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$$A := \begin{pmatrix} 1 & 0 & \sqrt{3} \\ 0 & \sqrt{2} & 0 \\ -\sqrt{3} & 0 & -1 \end{pmatrix}$$

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> A^5/4;
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$$\begin{pmatrix} 1 & 0 & \sqrt{3} \\ 0 & \sqrt{2} & 0 \\ -\sqrt{3} & 0 & -1 \end{pmatrix}$$

α) [**2 marks**] Show that $A^4 = 4I$, giving reasons.

β) [**1 mark**] Hence express A^{-1} as a scalar multiple of A^3.

iii) Let

$$P = \begin{pmatrix} 1 & -1 & 5 \\ 2 & 1 & -3 \\ 2 & 0 & 4 \end{pmatrix}, \qquad Q = \begin{pmatrix} 2 & 1 \\ 3 & 0 \\ 1 & -1 \end{pmatrix}.$$

a) Find $Q^T P$.

b) Write down a 3×3 matrix D such that the rows of DP are $2\mathbf{r}_1$, $\mathbf{r}_2$ and $-\mathbf{r}_3$, where $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ are the rows of P.

iv) Find the shortest distance between the point P with position vector $\mathbf{p} = \begin{pmatrix} 7 \\ 4 \\ 6 \end{pmatrix}$ and the plane

$$x + 2y + 3z = 5.$$

v) Let α be a real parameter and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$ be a vector in $\mathbb{R}^4$. Consider the following system of linear equations in x_1, x_2, x_3, x_4.

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & + & 3x_3 & - & x_4 & = & b_1 \\ 2x_1 & - & x_2 & & & + & 2x_4 & = & b_2 \\ x_1 & - & 2x_2 & + & \alpha x_3 & + & 3x_4 & = & b_3 \\ & & 3x_2 & + & 6x_3 & - & \alpha x_4 & = & b_4 \end{array}$$

a) Find all possible values of α such that the system has a unique solution for all choices of $\mathbf{b}$.

b) Find conditions on $\mathbf{b}$ that ensure the system has a solution for all choices of α.

vi) Given that P and Q are invertible $n \times n$ matrices and that Q is symmetric, simplify $(P^TQ)^T(QP)^{-1}$.

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BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln|k|$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$

$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$

$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$

$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$

$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$

$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$

$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$

$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\begin{aligned}\int \frac{dx}{a^2 - x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, \qquad |x| < a\\ &= \frac{1}{a}\coth^{-1}\frac{x}{a} + C, \qquad |x| > a > 0\\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, \qquad x^2 \neq a^2\end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$