School of Mathematics and Statistics
Math1131-Algebra

Lec13: Fundamental Theorem of Algebra

Laure Helme-Guizon (Dr H)
Laure@unsw.edu.au
Jonathan Kress
j.kress@unsw.edu.au

Red-Centre, Rooms 3090 and 3073

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Complex polynomials

Complex polynomials = Polynomials with coefficients in $\mathbb{C}$

A *complex polynomial* $p(z)$ is a complex valued function of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

where $a_0, \ldots, a_n \in \mathbb{C}$ with $a_n \neq 0$ are the *coefficients* of $p(z)$.

- If the coefficients $a_0, \ldots, a_n$ are real numbers, then $p(z)$ is a *real polynomial*.
- $a_n x^n$ is the *leading term*; a_n is the *leading coefficient*. If $a_n = 1$ then $p(z)$ is *monic*.
- The *degree* of $p(z)$ is n.
- The degree of the zero polynomial $p(z) = 0$ is undefined.

Exercise 1. Which of the following functions are polynomials? " ✓" or " ✗ "

a) $f(z) = z^3 + z + 1$ ✓

b) $f(z) = z^5 + 6z^{-2} + 3z$ ✗

c) $f(z) = i\sqrt{3}z$ ✓

d) $f(z) = \sin z$ ✗

e) $f(z) = i$ ✓

f) $f(z) = e^z$ ✗

g) $f(z) = 2 + 3i$ ✓

h) $f(z) = 2z^3 - iz^2 + 4$ ✓

i) $f(z) = \dfrac{z+1}{z-1}$ ✗

Long Division and the Remainder Theorem

Long Division for polynomials.

Given polynomials $a(x), b(x)$, we can find polynomials $q(x)$ and $r(x)$ such that

$$a(x) = q(x)b(x) + r(x)$$

where the degree of the remainder $r(x)$ is less than the degree of the divisor $b(x)$.
(We are dividing a by b, so b is the *divisor*, q is the *quotient* and r is the *remainder*).

The remainder theorem

If the polynomial $p(z)$ is divided by the linear factor $z - \alpha$, then the remainder is simply the number $r = p(\alpha)$.

PROOF

Write $p(z) = q(z)(z - \alpha) + r(z)$.

The degree of $r(z)$ must be smaller than the degree of $z - \alpha$ which is 1, and so $r(z)$ must be constant.
Therefore, $r(z) = r(\alpha) = p(\alpha) - q(\alpha)(\alpha - \alpha) = p(\alpha)$.

The Remainder Theorem

REMEMBER

Dividing $p(z)$ by $z - \alpha$ gives the remainder $r = p(\alpha)$.

Exercise 5. Find the remainder when $z^3 - 6z^2 + 11z - 7$ is divided by $z - 4$.
Do this by both long division and using the Remainder Theorem.

$$
\begin{array}{r|l}
 & z^2 - 2z + 3 \quad \leftarrow \text{quotient} \\
\hline
z - 4 & z^3 - 6z^2 + 11z - 7 \\
 & -(z^3 - 4z^2) \\
\hline
 & -2z^2 + 11z \\
 & -(-2z^2 + 8z) \\
\hline
 & 3z - 7 \\
 & -(3z - 12) \\
\hline
 & 5 \quad \leftarrow \text{remainder}
\end{array}
$$

$$\underbrace{z^3 - 6z^2 + 11z - 7}_{a(z)} = \underbrace{(z^2 - 2z + 3)}_{q(z)}\underbrace{(z-4)}_{b(z)} + \underbrace{5}_{r(z)}$$

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Checking some of our answers with Maple

Remainder thm:

$r = p(\alpha) = p(4) = 4^3 - 6\times 4^2 + 11\times 4 - 7$

$= \boxed{5}$

```
> # Long division for polynomials

  a := z -> z^3 -6*z^2 +11*z -7;
  b := z -> z-4;
```

$$a := z \mapsto z^3 - 6z^2 + 11z - 7$$

$$b := z \mapsto z - 4$$

```
> # quotient when we divide a by b, the variable being z

  quo(a(z), b(z),z);
```

$$z^2 - 2z + 3$$

```
> # remainder when we divide a by b, the variable being z

  rem(a(z), b(z),z);
```

$$5$$

```
> # Using the remainder theorem

  a(4);
```

$$5$$

The Factor Theorem

Example 8. We have seen that $p(z) = z^2 + 1 = (z - i)(z + i)$ has roots i and $-i$, and its factors are $z - i$ and $z + i$.

The Factor Theorem

$z - \alpha$ is a factor of $p(z)$ if and only if $P(\alpha) = 0$.

PROOF Let r be the remainder of $p(z)$ when divided by $z - \alpha$. Then

$$\alpha \text{ is a root of } p(z)$$
$$\Updownarrow \text{ by definition of "root"}$$
$$p(\alpha) = 0$$
$$\Updownarrow$$
$$r = r(\alpha) = p(\alpha) = 0$$
$$\Updownarrow$$
$$z - \alpha \text{ is a factor of } p(z)$$

means $p = (z - \alpha)q$

"$\Longleftrightarrow$" means "if and only if"

$$a = bq + \cancel{r}$$
$$a = (z - \alpha)q + \cancel{r}$$

The Factor Theorem

Example 8. We have seen that $p(z) = z^2 + 1 = (z - i)(z + i)$ has roots i and $-i$, and its factors are $z - i$ and $z + i$.

The Factor Theorem

$z - \alpha$ is a factor of $p(z)$ if and only if $P(\alpha) = 0$.

PROOF Let r be the remainder of $p(z)$ when divided by $z - \alpha$. Then

$$\begin{gathered}\alpha \text{ is a root of } p(z)\\ \Updownarrow \\ p(\alpha) = 0 \\ \Updownarrow \\ r = r(\alpha) = p(\alpha) = 0 \\ \Updownarrow \\ z - \alpha \text{ is a factor of } p(z)\end{gathered}$$

$$p(z) = q(z)(z-\alpha) + r$$
$$p(\alpha) = q(\alpha)(\alpha-\alpha) + r$$
$$= 0 + r$$

Factorisation

Fundamental Theorem of Algebra.
Each complex polynomial of degree $n \geqslant 1$ has at least one complex root.

This leads to...

The Factorisation Theorem
Each complex polynomial $p(z)$ of degree $n \geqslant 1$ has a factorisation

$$p(z) = a_n(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_n),$$

where the n complex numbers $\alpha_1, \alpha_2, \dots \alpha_n$ are roots of $p(z)$.
The roots are not necessarily distinct.

Exercise 7. Factorise the following polynomials into complex linear factors (= degree one factors).

a) $p_1(z) = z^2 - 4$ b) $p_2(z) = 2z^3 + 2z^2 - 4z$ c) $p_3(z) = z^3 - 8i$

a) $p_1(z) = z^2 - 4 = (z+2)(z-2)$

b) $p_2(z) = 2z(z^2+z-2)$
$= 2z(z+2)(z-1)$

We will start at 1pm

UNSW SYDNEY

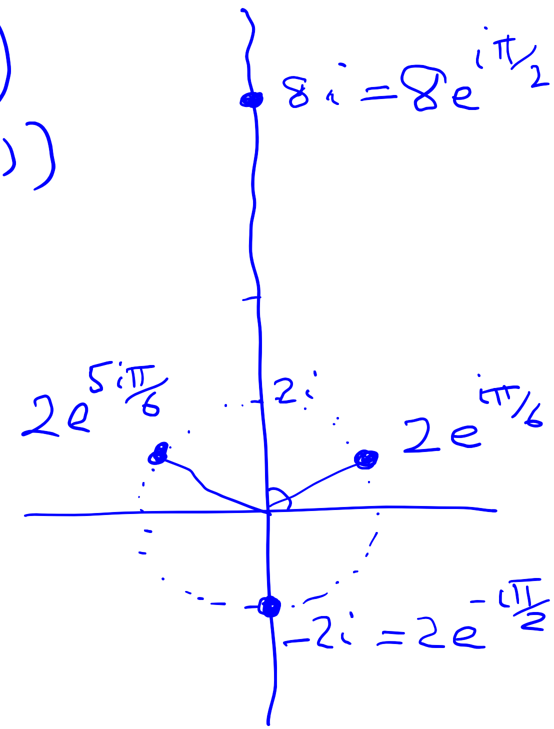
Exercise 7, continued.

Factorise the following polynomials into linear factors (= degree one factors).

a) $p_1(z) = z^2 - 4$ b) $p_2(z) = 2z^3 + 2z^2 - 4z$ c) $p_3(z) = z^3 - 8i$

$$z^3 = 8i$$

$$\begin{aligned} p_3(z) &= z^3 - 8i \\ &= (z-(-2i))(z-2e^{i\pi/6})(z-2e^{5i\pi/6}) \\ &= (z+2i)(z-(\sqrt{3}+i))(z-(-\sqrt{3}+i)) \end{aligned}$$



$$\frac{\pi}{6} + \frac{2\pi}{3} = \frac{5\pi}{6} \qquad \boxed{\frac{2\pi}{3}}$$

Factorising polynomials in $\mathbb{C}$ with Maple

```
>  # Example 7, part b
> p := z -> 2*z^3 +2*z^2 -4*z;
```

$$p := z \mapsto 2z^3 + 2z^2 - 4z$$

```
> factor(p(z));
```

$$2z(z+2)(z-1)$$

```
> # Example 7, part c
> p(z) := z^3 -8*I;
```

$$p(z) := z^3 - 8\,\mathrm{I}$$

```
> # Naive approach
  factor(p(z));
```

$$-(2\,\mathrm{I}z - z^2 + 4)(z + 2\,\mathrm{I})$$

```
> # The problem is that if the coefficients are all integers then 'factor'
  computes all irreducible factors with integer coefficients. Thus factor does
  not necessarily factor into linear factors.

> # All roots in a+ib form

  solve(p(z) = 0);
```

$$\mathrm{I} + \sqrt{3}, \mathrm{I} - \sqrt{3}, -2\,\mathrm{I}$$

```
> # In order to get linear factors, we force a factorisation using i and sqrt(3)
  factor(p(z), {I, sqrt(3)});
```

$$\left(-z + \mathrm{I} + \sqrt{3}\right)\left(-z + \mathrm{I} - \sqrt{3}\right)(z + 2\,\mathrm{I})$$

Real polynomials and conjugate roots

Real polynomials and conjugate roots

If α is a root of a real polynomial[a] $p(z)$, then its conjugate $\overline{\alpha}$ is also a root of this polynomial.

[a] which means all the coefficients are real numbers rather than complex numbers

PROOF Suppose that α is a root of a real polynomial

$$p(z) = a_n z^n + \ldots + a_1 z + a_0,$$

that is, $a_0, a_1, \ldots, a_n$ are real and $p(\alpha) = 0$. Then

$$\begin{aligned} p(\overline{\alpha}) &= a_n \overline{\alpha}^n + \ldots + a_1 \overline{\alpha} + a_0 \\ &= \overline{a_n}\, \overline{\alpha}^n + \ldots + \overline{a_1}\, \overline{\alpha} + \overline{a_0} \\ &= \overline{a_n \alpha^n + \ldots + a_1 \alpha + a_0} \\ &= \overline{p(\alpha)} \\ &= \overline{0} \\ &= 0. \end{aligned}$$

Hence $\overline{\alpha}$ is also a root of $p(z)$.

Real polynomials and conjugate roots

Suppose that the real polynomial $p(z)$ has non-real root α. Then $\overline{\alpha}$ is also a root and $z-\alpha$ and $z-\overline{\alpha}$ are factors of $p(z)$. Hence $p(z)$ has the quadratic factor

$$\begin{aligned}(z-\alpha)(z-\overline{\alpha}) &= z^2-(\alpha+\overline{\alpha})z+\alpha\overline{\alpha}\\ &= z^2-2\text{Re}(\alpha)z+|\alpha|^2.\end{aligned}$$

Since $\text{Re}(\alpha)$ and $|\alpha|^2$ are real, this quadratic factor is real.

As a consequence, every real polynomial can be factored into real linear and quadratic factors.

Exercise 8.

a) Express $z^6 - 1$ as a product of linear factors.

b) Express $z^6 - 1$ as a product of real linear and quadratic factors.

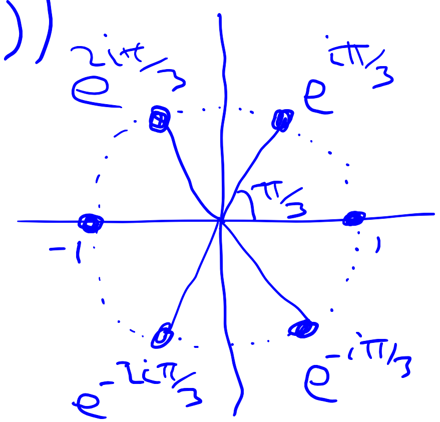
Assessed

a) $z^6 - 1 = (z-1)(z-(-1))(z-e^{i\pi/3})(z-e^{-i\pi/3})(z-e^{2i\pi/3})(z-e^{-2i\pi/3})$

$= (z-1)(z+1)\left(z-\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)\left(z-\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right)$

$\left(z-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)\left(z-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right)$

$z^6 = 1 = 1e^{i0}$



b) $z^6 - 1 = (z-1)(z+1)(z^2 - z + 1)(z^2 + z + 1)$

Complex polynomials and Maple: Ex 8

```
> p := z -> z^6-1:
> evalc([solve(z^6 = 1)]);
  # Straight brackets to store the answers as a list so we can apply
  'map' next
```

$$\left[1, -1, \frac{1}{2} - \frac{I\sqrt{3}}{2}, -\frac{1}{2} + \frac{I\sqrt{3}}{2}, \frac{1}{2} + \frac{I\sqrt{3}}{2}, -\frac{1}{2} - \frac{I\sqrt{3}}{2}\right]$$

```
> map(polar, %);
  #  '%' means 'previous result'
  # 'map' is used to apply 'polar' to each term of the previous list
```

$$\left[\mathrm{polar}(1, 0), \mathrm{polar}(1, \pi), \mathrm{polar}\left(1, -\frac{\pi}{3}\right), \mathrm{polar}\left(1, \frac{2\pi}{3}\right), \mathrm{polar}\left(1, \frac{\pi}{3}\right), \mathrm{polar}\left(1, -\frac{2\pi}{3}\right)\right]$$

```
> factor(p(z));
```

$$(z-1)\,(z+1)\,(z^2+z+1)\,(z^2-z+1)$$

```
> # If the coefficients are all integers then 'factor' computes all
  irreducible factors with integer coefficients. Thus 'factor' does not
  necessarily factor into linear factors.

> # To get linear factors, we force a factorisation using i and sqrt(3)
  factor(p(z), {I, sqrt(3)});
```

$$\frac{(z-1)\,(I\sqrt{3}-2z+1)\,(I\sqrt{3}+2z-1)\,(z+1)\,(I\sqrt{3}+2z+1)\,(I\sqrt{3}-2z-1)}{16}$$