LECTURE 8

Split Functions, Implicit Differentiation and Related Rates

$$\text{Implicit Differentiation} \leftrightarrow \frac{d}{dx}(f(y)) = \frac{d}{dy}(f(y))\frac{dy}{dx}.$$

Differentiating a relation between x and y implicitly with respect to t will produce a new relation between the rates of change $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

A split function is usually constructed from two or more differentiable component functions. To verify (or force) the differentiability of such a split function we simply need to first verify that the pieces join up (continuity) and then that they *join smoothly* (differentiability) by showing that the derivatives match up properly.

Example 1: Find all real values of a and b such that the function defined by

$$f(x) = \begin{cases} 1 - x^2, & x < 2; \\ ax + b, & x \geq 2. \end{cases}$$

is differentiable at $x = 2$.

A sketch:

Let $p(x) = 1 - x^2$ and $q(x) = ax + b$. It helps to name the pieces.

We first demand that the function be continuous at $x = 2$. That is the pieces must join, and hence $p(2) = q(2)$:

We next force the weld to be smooth! Thus we require that $p'(2) = q'(2)$. That is:

$p'(x) =$ $\rightarrow p'(2) =$

$q'(x) =$ $\rightarrow q'(2) =$

★ $a = -4$ $b = 5$ ★

Implicit Differentiation

Usually when you differentiate, your starting point is a nice clean function $y = f(x)$. But sometimes you need to start with a horrible messy relation instead, for example $x^2 + y^3 + 4y^2 = 3$. It can be difficult or even impossible to write y in terms of x. We can still find the derivative $\dfrac{dy}{dx}$ but need to use **implicit differentiation**. First a simple skill.

Example 2: If $\dfrac{3}{7} = \dfrac{3}{11} \times \dfrac{*}{*}$ what is $\dfrac{*}{*}$?

★ $\dfrac{11}{7}$ ★

Implicit differentiation is little more than the above trick!

Example 3: Find $\dfrac{dy}{dx}$ if $x^2 + y^3 + 4y^2 = 3$.

★ $\dfrac{-2x}{3y^2 + 8y}$ ★

Example 4: Find $\dfrac{dy}{dx}$ if $\sin(x) + e^y = \ln(y) + x^3$

$$\bigstar \quad \frac{3x^2y - y\cos(x)}{ye^y - 1} \quad \bigstar$$

Example 5: Find the equation of the tangent to $x^2y^5 + 3y - 2x = 3$ at the point $(0, 1)$.

$$\bigstar \quad 2x - 3y + 3 = 0 \quad \bigstar$$

Related Rates

Differentiating a relation between x and y implicitly with respect to t will produce a new relation between the rates of change $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

Example 6: Suppose that the surface area S (in m^2) of a human body is related to its weight W (in kg) by

$$S^3 = \frac{W^2}{512}$$

a) Bob weighs 64 kg. What is the surface area of his body?

b) Find a relation between $\frac{dS}{dt}$ and $\frac{dW}{dt}$.

c) Prove that if Bob's weight were to change in any way, the rate of change of his surface area would be $\frac{1}{48}$ the rate of change of his weight.

★ *a)* S=2 *b)* $3S^2\frac{dS}{dt} = \frac{W}{256}\frac{dW}{dt}$ *c) Proof* ★

Example 7: A spherical balloon is inflated at a rate of 100 $\mathrm{m}^3/\mathrm{sec}$. Determine the rate at which the radius is increasing when

a) $r = 5$m.

b) $V = 36\pi$ m^3.

Our first task is to find a relationship between the central variables which remains fixed throughout the entire process. This is of course the volume formula for a sphere:

$$V = \frac{4}{3}\pi r^3$$

$$\bigstar \quad a)\ \frac{1}{\pi}\ m/sec \quad b)\ \frac{100}{36\pi}\ m/sec \quad \bigstar$$

Error Estimates (Homework)

This topic was of enormous importance before the advent of calculators but is now a bit dated.

Recall that $\dfrac{\Delta y}{\Delta x} \approx \dfrac{dy}{dx}$ and hence $\Delta y \approx \dfrac{dy}{dx}\Delta x$. This gives us a way of estimating errors.

Example 8: Find an error estimate when approximating $\sqrt{9.001}$ by $\sqrt{9}$.

We have $x = 9$ and $\Delta x = 0.001$.

Let $y = \sqrt{x}$. Then $\dfrac{dy}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$.

Now $\Delta y \approx \dfrac{dy}{dx}\Delta x \to \Delta y \approx \dfrac{1}{2\sqrt{x}}\Delta x = \dfrac{1}{2\sqrt{9}}(0.001) = \dfrac{1}{6000}$.

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