MATH1131 Mathematics 1A – Algebra

Lecture 10: Complex Division and Conjugates

Lecturer: Sean Gardiner – sean.gardiner@unsw.edu.au

Based on slides by Jonathan Kress

Argand diagram

The set of complex numbers

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

is often represented as a plane called the
Argand diagram or complex plane:

Argand diagram

The set of complex numbers

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

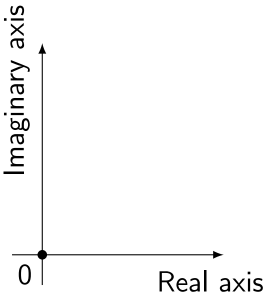
is often represented as a plane called the
Argand diagram or complex plane:

Argand diagram

The set of complex numbers

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

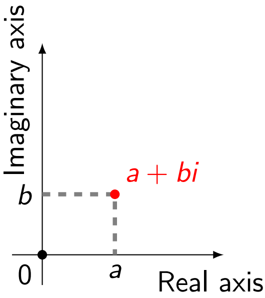
is often represented as a plane called the Argand diagram or complex plane:



The set of complex numbers

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

is often represented as a plane called the Argand diagram or complex plane:

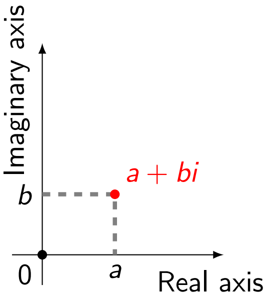


Argand diagram

The set of complex numbers

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

is often represented as a plane called the Argand diagram or complex plane:



If a and b are real numbers, then the complex number

$$z = a + bi$$

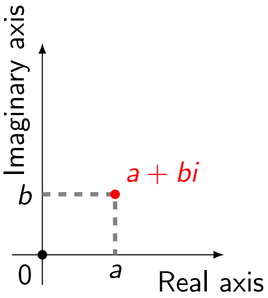
has real part a and imaginary part b.

Argand diagram

The set of complex numbers

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

is often represented as a plane called the Argand diagram or complex plane:



If a and b are real numbers, then the complex number

$$z = a + bi$$

has real part a and imaginary part b.

We write:

$$\text{Re}(z) = a \quad \text{and} \quad \text{Im}(z) = b.$$

Real and imaginary parts

Note that $z = \text{Re}(z) + \text{Im}(z)i$, where $\text{Re}(z) \in \mathbb{R}$ and $\text{Im}(z) \in \mathbb{R}$.

Real and imaginary parts

Note that $z = \text{Re}(z) + \text{Im}(z)i$, where $\text{Re}(z) \in \mathbb{R}$ and $\text{Im}(z) \in \mathbb{R}$.

We say $z \in \mathbb{C}$ is real if $z = \text{Re}(z) \;\Leftrightarrow\; \text{Im}(z) = 0$.

Real and imaginary parts

Note that $z = \text{Re}(z) + \text{Im}(z)i$, where $\text{Re}(z) \in \mathbb{R}$ and $\text{Im}(z) \in \mathbb{R}$.

We say $z \in \mathbb{C}$ is real if $z = \text{Re}(z) \iff \text{Im}(z) = 0$.

We say $z \in \mathbb{C}$ is imaginary if $z = \text{Im}(z)i \iff \text{Re}(z) = 0$.

Real and imaginary parts

Note that $z = \text{Re}(z) + \text{Im}(z)i$, where $\text{Re}(z) \in \mathbb{R}$ and $\text{Im}(z) \in \mathbb{R}$.

We say $z \in \mathbb{C}$ is real if $z = \text{Re}(z) \iff \text{Im}(z) = 0$.

We say $z \in \mathbb{C}$ is imaginary if $z = \text{Im}(z)i \iff \text{Re}(z) = 0$.

Examples

- 3 is real

Real and imaginary parts

Note that $z = \text{Re}(z) + \text{Im}(z)i$, where $\text{Re}(z) \in \mathbb{R}$ and $\text{Im}(z) \in \mathbb{R}$.

We say $z \in \mathbb{C}$ is real if $z = \text{Re}(z) \;\Leftrightarrow\; \text{Im}(z) = 0$.

We say $z \in \mathbb{C}$ is imaginary if $z = \text{Im}(z)i \;\Leftrightarrow\; \text{Re}(z) = 0$.

Examples

- 3 is real
- $3i$ is imaginary

Real and imaginary parts

Note that $z = \text{Re}(z) + \text{Im}(z)i$, where $\text{Re}(z) \in \mathbb{R}$ and $\text{Im}(z) \in \mathbb{R}$.

We say $z \in \mathbb{C}$ is real if $z = \text{Re}(z) \iff \text{Im}(z) = 0$.

We say $z \in \mathbb{C}$ is imaginary if $z = \text{Im}(z)i \iff \text{Re}(z) = 0$.

Examples

- 3 is real
- $3i$ is imaginary
- Both 3 and $3i$ are complex

Equality of complex numbers

Note: Two complex numbers are equal if and only if their real and imaginary parts are equal.

Equality of complex numbers

Note: Two complex numbers are equal if and only if their real and imaginary parts are equal.

Example

Find $a, b \in \mathbb{R}$ satisfying

$$(3 + 4i)(a + bi) = 23 + 14i.$$

Equality of complex numbers

Note: Two complex numbers are equal if and only if their real and imaginary parts are equal.

Example

Find $a, b \in \mathbb{R}$ satisfying

$$(3+4i)(a+bi) = 23+14i.$$

By assumption, $23 + 14i \quad = \quad (3+4i)(a+bi)$

Equality of complex numbers

Note: Two complex numbers are equal if and only if their real and imaginary parts are equal.

Example

Find $a, b \in \mathbb{R}$ satisfying

$$(3+4i)(a+bi) = 23+14i.$$

By assumption,
$$\begin{aligned} 23+14i &= (3+4i)(a+bi) \\ &= 3a+3bi+4ai+4bi^2 \end{aligned}$$

Note: Two complex numbers are equal if and only if their real and imaginary parts are equal.

Example

Find $a, b \in \mathbb{R}$ satisfying

$$(3+4i)(a+bi) = 23+14i.$$

By assumption,
$$\begin{aligned} 23+14i &= (3+4i)(a+bi) \\ &= 3a+3bi+4ai+4bi^2 \\ &= (3a-4b)+(4a+3b)i \end{aligned}$$

Equality of complex numbers

Note: Two complex numbers are equal if and only if their real and imaginary parts are equal.

Example

Find $a, b \in \mathbb{R}$ satisfying

$$(3+4i)(a+bi) = 23+14i.$$

By assumption,
$$\begin{aligned} 23+14i &= (3+4i)(a+bi) \\ &= 3a+3bi+4ai+4bi^2 \\ &= (3a-4b)+(4a+3b)i \end{aligned}$$

Comparing real and imaginary parts, we need:

$$3a-4b=23 \quad \text{and} \quad 4a+3b=14.$$

Equality of complex numbers

Note: Two complex numbers are equal if and only if their real and imaginary parts are equal.

Example

Find $a, b \in \mathbb{R}$ satisfying

$$(3+4i)(a+bi) = 23+14i.$$

By assumption,
$$\begin{aligned} 23+14i &= (3+4i)(a+bi) \\ &= 3a+3bi+4ai+4bi^2 \\ &= (3a-4b)+(4a+3b)i \end{aligned}$$

Comparing real and imaginary parts, we need:

$$3a-4b=23 \quad \text{and} \quad 4a+3b=14.$$

Solving these simultaneously gives the answer $a=5$ and $b=-2$.

Equality of complex numbers

Note: Two complex numbers are equal if and only if their real and imaginary parts are equal.

Example

Find $a, b \in \mathbb{R}$ satisfying

$$(3 + 4i)(a + bi) = 23 + 14i.$$

By assumption,
$$\begin{aligned} 23 + 14i &= (3 + 4i)(a + bi) \\ &= 3a + 3bi + 4ai + 4bi^2 \\ &= (3a - 4b) + (4a + 3b)i \end{aligned}$$

Comparing real and imaginary parts, we need:

$$3a - 4b = 23 \quad \text{and} \quad 4a + 3b = 14.$$

Solving these simultaneously gives the answer $a = 5$ and $b = -2$.

Note: We will see a neater way to solve this problem in a few slides.

Inverse of a complex number

Example

Find the multiplicative inverse of $2 + 5i$.

Inverse of a complex number

Example

Find the multiplicative inverse of $2 + 5i$.

We want to find the complex number $a + bi$ with $a, b \in \mathbb{R}$, such that

$$(2 + 5i)(a + bi) = 1.$$

Example

Find the multiplicative inverse of $2 + 5i$.

We want to find the complex number $a + bi$ with $a, b \in \mathbb{R}$, such that

$$(2 + 5i)(a + bi) = 1.$$

Expanding yields

$$(2a - 5b) + (5a + 2b)i = 1$$

Example

Find the multiplicative inverse of $2 + 5i$.

We want to find the complex number $a + bi$ with $a, b \in \mathbb{R}$, such that

$$(2 + 5i)(a + bi) = 1.$$

Expanding yields

$$(2a - 5b) + (5a + 2b)i = 1 = 1 + 0i,$$

so $2a - 5b = 1$ and $5a + 2b = 0$.

Example

Find the multiplicative inverse of $2 + 5i$.

We want to find the complex number $a + bi$ with $a, b \in \mathbb{R}$, such that

$$(2 + 5i)(a + bi) = 1.$$

Expanding yields

$$(2a - 5b) + (5a + 2b)i = 1 = 1 + 0i,$$

so $2a - 5b = 1$ and $5a + 2b = 0$.

Solving these simultaneously gives the answer $a = \dfrac{2}{29}$ and $b = -\dfrac{5}{29}$.

Example

Find the multiplicative inverse of $2 + 5i$.

We want to find the complex number $a + bi$ with $a, b \in \mathbb{R}$, such that

$$(2 + 5i)(a + bi) = 1.$$

Expanding yields

$$(2a - 5b) + (5a + 2b)i = 1 = 1 + 0i,$$

so $2a - 5b = 1$ and $5a + 2b = 0$.

Solving these simultaneously gives the answer $a = \dfrac{2}{29}$ and $b = -\dfrac{5}{29}$.

Hence

$$(2 + 5i)^{-1} = \frac{1}{2 + 5i} = \frac{2}{29} - \frac{5}{29}i.$$

Example

Find the multiplicative inverse of $2 + 5i$.

We want to find the complex number $a + bi$ with $a, b \in \mathbb{R}$, such that

$$(2 + 5i)(a + bi) = 1.$$

Expanding yields

$$(2a - 5b) + (5a + 2b)i = 1 = 1 + 0i,$$

so $2a - 5b = 1$ and $5a + 2b = 0$.

Solving these simultaneously gives the answer $a = \dfrac{2}{29}$ and $b = -\dfrac{5}{29}$.

Hence

$$(2 + 5i)^{-1} = \frac{1}{2 + 5i} = \frac{2}{29} - \frac{5}{29}i.$$

Note: We will see a neater way to solve this problem in a few slides.

Square roots

Example

Find the square roots of $-3 + 4i$.

Square roots

Example

Find the square roots of $-3 + 4i$.

We want to find the complex numbers $a + bi$ with $a, b \in \mathbb{R}$, such that

$$(a + bi)^2 = -3 + 4i.$$

Square roots

Example

Find the square roots of $-3 + 4i$.

We want to find the complex numbers $a + bi$ with $a, b \in \mathbb{R}$, such that

$$(a + bi)^2 = -3 + 4i.$$

Expanding yields

$$a^2 + 2abi + b^2 i^2 = (a^2 - b^2) + 2abi$$

Square roots

Example

Find the square roots of $-3 + 4i$.

We want to find the complex numbers $a + bi$ with $a, b \in \mathbb{R}$, such that

$$(a + bi)^2 = -3 + 4i.$$

Expanding yields

$$a^2 + 2abi + b^2 i^2 = (a^2 - b^2) + 2abi = -3 + 4i,$$

so $a^2 - b^2 = -3$ and $2ab = 4$.

Example

Find the square roots of $-3+4i$.

We want to find the complex numbers $a+bi$ with $a, b \in \mathbb{R}$, such that

$$(a+bi)^2 = -3+4i.$$

Expanding yields

$$a^2 + 2abi + b^2 i^2 = (a^2 - b^2) + 2abi = -3 + 4i,$$

so $a^2 - b^2 = -3$ and $2ab = 4$.

Substituting for a gives $b^4 - 3b^2 - 4 = (b^2 - 4)(b^2 + 1) = 0$,

Square roots

Example

Find the square roots of $-3+4i$.

We want to find the complex numbers $a+bi$ with $a, b \in \mathbb{R}$, such that

$$(a+bi)^2 = -3+4i.$$

Expanding yields

$$a^2 + 2abi + b^2 i^2 = (a^2 - b^2) + 2abi = -3 + 4i,$$

so $a^2 - b^2 = -3$ and $2ab = 4$.

Substituting for a gives $b^4 - 3b^2 - 4 = (b^2-4)(b^2+1) = 0$, so $b = \pm 2$ and $a = \pm 1$ respectively.

Square roots

Example

Find the square roots of $-3+4i$.

We want to find the complex numbers $a+bi$ with $a, b \in \mathbb{R}$, such that

$$(a+bi)^2 = -3+4i.$$

Expanding yields

$$a^2 + 2abi + b^2 i^2 = (a^2 - b^2) + 2abi = -3+4i,$$

so $a^2 - b^2 = -3$ and $2ab = 4$.

Substituting for a gives $b^4 - 3b^2 - 4 = (b^2-4)(b^2+1) = 0$, so $b = \pm 2$ and $a = \pm 1$ respectively.

Thus the square roots of $-3+4i$ are $1+2i$ and $-1-2i$.

Square roots

Example

Find the square roots of $-3+4i$.

We want to find the complex numbers $a+bi$ with $a, b \in \mathbb{R}$, such that

$$(a+bi)^2 = -3+4i.$$

Expanding yields

$$a^2 + 2abi + b^2i^2 = (a^2 - b^2) + 2abi = -3 + 4i,$$

so $a^2 - b^2 = -3$ and $2ab = 4$.

Substituting for a gives $b^4 - 3b^2 - 4 = (b^2 - 4)(b^2 + 1) = 0$, so $b = \pm 2$ and $a = \pm 1$ respectively.

Thus the square roots of $-3+4i$ are $1+2i$ and $-1-2i$.

Note: We will see a neater way to solve this problem in a few lectures, once we have seen the polar form.

Complex conjugate

Definition

Let $z = a + bi$ be a complex number with $a, b \in \mathbb{R}$. The complex conjugate of z is

$$\overline{z} = a - bi.$$

Complex conjugate

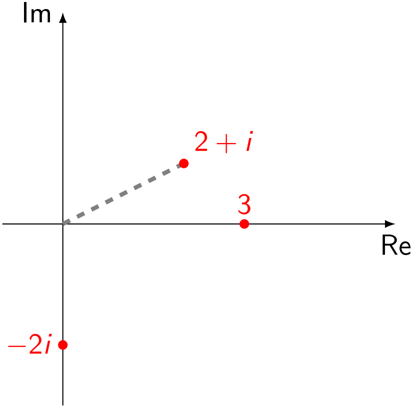
Definition

Let $z = a + bi$ be a complex number with $a, b \in \mathbb{R}$. The complex conjugate of z is

$$\overline{z} = a - bi.$$

Examples

$$\begin{aligned} \overline{2+i} &= \\ \overline{-2i} &= \\ \overline{3} &= \end{aligned}$$



Complex conjugate

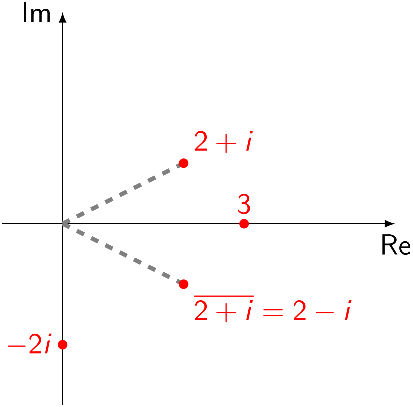
Definition

Let $z = a + bi$ be a complex number with $a, b \in \mathbb{R}$. The complex conjugate of z is

$$\overline{z} = a - bi.$$

Examples

$$\begin{aligned} \overline{2+i} &= 2 - i \\ \overline{-2i} &= \\ \overline{3} &= \end{aligned}$$



Complex conjugate

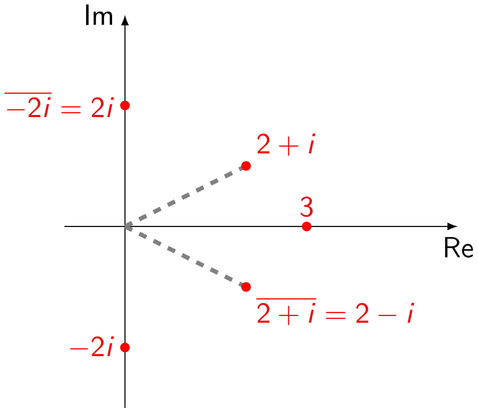
Definition

Let $z = a + bi$ be a complex number with $a, b \in \mathbb{R}$. The complex conjugate of z is

$$\overline{z} = a - bi.$$

Examples

$$\begin{aligned} \overline{2+i} &= 2 - i \\ \overline{-2i} &= 2i \\ \overline{3} &= \end{aligned}$$



Complex conjugate

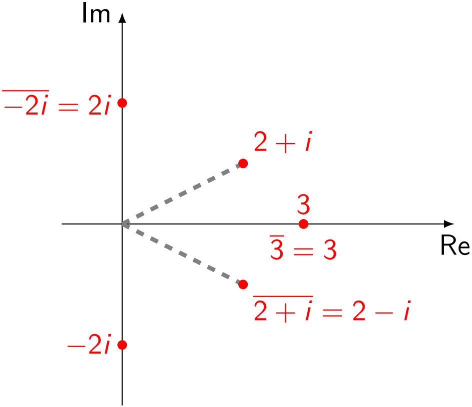
Definition

Let $z = a + bi$ be a complex number with $a, b \in \mathbb{R}$. The complex conjugate of z is

$$\overline{z} = a - bi.$$

Examples

$$\begin{aligned} \overline{2+i} &= 2 - i \\ \overline{-2i} &= 2i \\ \overline{3} &= 3 \end{aligned}$$



Inverse of a complex number – revisited

To simplify a complex fraction, multiply the top and bottom by the conjugate of the denominator.

Inverse of a complex number – revisited

To simplify a complex fraction, multiply the top and bottom by the conjugate of the denominator.

Example

Find the multiplicative inverse of $2 + 5i$.

Inverse of a complex number – revisited

To simplify a complex fraction, multiply the top and bottom by the conjugate of the denominator.

Example

Find the multiplicative inverse of $2 + 5i$.

We want to find the complex number $\dfrac{1}{2+5i}$ in the form $a + bi$ for $a, b \in \mathbb{R}$.

To simplify a complex fraction, multiply the top and bottom by the conjugate of the denominator.

Example

Find the multiplicative inverse of $2+5i$.

We want to find the complex number $\dfrac{1}{2+5i}$ in the form $a+bi$ for $a, b \in \mathbb{R}$.

$$\frac{1}{2+5i}$$

Inverse of a complex number – revisited

To simplify a complex fraction, multiply the top and bottom by the conjugate of the denominator.

Example

Find the multiplicative inverse of $2 + 5i$.

We want to find the complex number $\dfrac{1}{2+5i}$ in the form $a + bi$ for $a, b \in \mathbb{R}$.

$$\frac{1}{2+5i} = \frac{1}{2+5i} \times \frac{2-5i}{2-5i}$$

Inverse of a complex number – revisited

To simplify a complex fraction, multiply the top and bottom by the conjugate of the denominator.

Example

Find the multiplicative inverse of $2 + 5i$.

We want to find the complex number $\dfrac{1}{2+5i}$ in the form $a + bi$ for $a, b \in \mathbb{R}$.

$$
\begin{aligned}
\frac{1}{2+5i} &= \frac{1}{2+5i} \times \frac{2-5i}{2-5i} \\
&= \frac{2-5i}{2^2 - (5i)^2}
\end{aligned}
$$

Inverse of a complex number – revisited

To simplify a complex fraction, multiply the top and bottom by the conjugate of the denominator.

Example

Find the multiplicative inverse of $2 + 5i$.

We want to find the complex number $\dfrac{1}{2+5i}$ in the form $a + bi$ for $a, b \in \mathbb{R}$.

$$\begin{aligned} \frac{1}{2+5i} &= \frac{1}{2+5i} \times \frac{2-5i}{2-5i} \\ &= \frac{2-5i}{2^2-(5i)^2} \\ &= \frac{2-5i}{4+25} \end{aligned}$$

Inverse of a complex number – revisited

To simplify a complex fraction, multiply the top and bottom by the conjugate of the denominator.

Example

Find the multiplicative inverse of $2 + 5i$.

We want to find the complex number $\frac{1}{2+5i}$ in the form $a + bi$ for $a, b \in \mathbb{R}$.

$$\begin{aligned}
\frac{1}{2+5i} &= \frac{1}{2+5i} \times \frac{2-5i}{2-5i} \\
&= \frac{2-5i}{2^2-(5i)^2} \\
&= \frac{2-5i}{4+25} \\
&= \frac{2}{29} - \frac{5}{29}i
\end{aligned}$$

Inverse of a general complex number

Example

What is

$$\frac{1}{a+bi}$$

for any $a, b \in \mathbb{R}$ (not both 0)?

Inverse of a general complex number

Example

What is

$$\frac{1}{a+bi}$$

for any $a, b \in \mathbb{R}$ (not both 0)?

We use the same process:

Inverse of a general complex number

Example

What is

$$\frac{1}{a+bi}$$

for any $a, b \in \mathbb{R}$ (not both 0)?

We use the same process:

$$\frac{1}{a+bi} \quad = \quad \frac{1}{a+bi} \times \frac{a-bi}{a-bi}$$

Example

What is
$$\frac{1}{a+bi}$$
for any $a, b \in \mathbb{R}$ (not both 0)?

We use the same process:

$$\begin{aligned} \frac{1}{a+bi} &= \frac{1}{a+bi} \times \frac{a-bi}{a-bi} \\ &= \frac{a-bi}{a^2-(bi)^2} \end{aligned}$$

Inverse of a general complex number

Example

What is

$$\frac{1}{a+bi}$$

for any $a, b \in \mathbb{R}$ (not both 0)?

We use the same process:

$$\begin{aligned}
\frac{1}{a+bi} &= \frac{1}{a+bi} \times \frac{a-bi}{a-bi} \\
&= \frac{a-bi}{a^2-(bi)^2} \\
&= \frac{a-bi}{a^2+b^2}
\end{aligned}$$

Inverse of a general complex number

Example

What is

$$\frac{1}{a+bi}$$

for any $a, b \in \mathbb{R}$ (not both 0)?

We use the same process:

$$\begin{aligned}
\frac{1}{a+bi} &= \frac{1}{a+bi} \times \frac{a-bi}{a-bi} \\
&= \frac{a-bi}{a^2-(bi)^2} \\
&= \frac{a-bi}{a^2+b^2} \\
&= \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i
\end{aligned}$$

Division of complex numbers

We can use the same method for division by complex numbers.

Division of complex numbers

We can use the same method for division by complex numbers.

Example

Find $a, b \in \mathbb{R}$ satisfying

$$(3+4i)(a+bi) = 23+14i.$$

Division of complex numbers

We can use the same method for division by complex numbers.

Example

Find $a, b \in \mathbb{R}$ satisfying

$$(3+4i)(a+bi) = 23+14i.$$

Rearranging, we want to find $a+bi = \dfrac{23+14i}{3+4i}$.

Division of complex numbers

We can use the same method for division by complex numbers.

Example

Find $a, b \in \mathbb{R}$ satisfying

$$(3+4i)(a+bi) = 23+14i.$$

Rearranging, we want to find $a + bi = \dfrac{23+14i}{3+4i}$.

$$\frac{23+14i}{3+4i} = \frac{23+14i}{3+4i} \times \frac{3-4i}{3-4i}$$

Division of complex numbers

We can use the same method for division by complex numbers.

Example

Find $a, b \in \mathbb{R}$ satisfying

$$(3+4i)(a+bi) = 23+14i.$$

Rearranging, we want to find $a+bi = \dfrac{23+14i}{3+4i}$.

$$\begin{aligned}\frac{23+14i}{3+4i} &= \frac{23+14i}{3+4i} \times \frac{3-4i}{3-4i} \\ &= \frac{23\times 3 - 23\times 4i + 14\times 3i - 14\times 4i^2}{3^2-(4i)^2}\end{aligned}$$

We can use the same method for division by complex numbers.

Example

Find $a, b \in \mathbb{R}$ satisfying

$$(3+4i)(a+bi) = 23+14i.$$

Rearranging, we want to find $a+bi = \dfrac{23+14i}{3+4i}$.

$$\begin{aligned}
\frac{23+14i}{3+4i} &= \frac{23+14i}{3+4i} \times \frac{3-4i}{3-4i} \\
&= \frac{23\times 3 - 23\times 4i + 14\times 3i - 14\times 4i^2}{3^2-(4i)^2} \\
&= \frac{125-50i}{25}
\end{aligned}$$

Division of complex numbers

We can use the same method for division by complex numbers.

Example

Find $a, b \in \mathbb{R}$ satisfying

$$(3+4i)(a+bi) = 23+14i.$$

Rearranging, we want to find $a + bi = \dfrac{23+14i}{3+4i}$.

$$\begin{aligned}
\frac{23+14i}{3+4i} &= \frac{23+14i}{3+4i} \times \frac{3-4i}{3-4i} \\
&= \frac{23 \times 3 - 23 \times 4i + 14 \times 3i - 14 \times 4i^2}{3^2 - (4i)^2} \\
&= \frac{125-50i}{25} \\
&= 5-2i
\end{aligned}$$

Division of general complex numbers

Exercise

What is
$$\frac{c + di}{a + bi}$$
for $a, b, c, d \in \mathbb{R}$ with a and b not both zero?

Division of general complex numbers

Exercise

What is
$$\frac{c+di}{a+bi}$$
for $a, b, c, d \in \mathbb{R}$ with a and b not both zero?

We again use the same process:

Division of general complex numbers

Exercise

What is
$$\frac{c+di}{a+bi}$$
for $a, b, c, d \in \mathbb{R}$ with a and b not both zero?

We again use the same process:

$$\frac{c+di}{a+bi} = \frac{c+di}{a+bi} \times \frac{a-bi}{a-bi}$$

Division of general complex numbers

Exercise

What is
$$\frac{c+di}{a+bi}$$
for $a, b, c, d \in \mathbb{R}$ with a and b not both zero?

We again use the same process:

$$\begin{aligned} \frac{c+di}{a+bi} &= \frac{c+di}{a+bi} \times \frac{a-bi}{a-bi} \\ &= \frac{ac - bci + adi - bdi^2}{a^2 - (bi)^2} \end{aligned}$$

Division of general complex numbers

Exercise

What is
$$\frac{c+di}{a+bi}$$
for $a, b, c, d \in \mathbb{R}$ with a and b not both zero?

We again use the same process:

$$\begin{aligned}\frac{c+di}{a+bi} &= \frac{c+di}{a+bi} \times \frac{a-bi}{a-bi} \\ &= \frac{ac - bci + adi - bdi^2}{a^2 - (bi)^2} \\ &= \frac{(ac+bd) + (ad-bc)i}{a^2+b^2}\end{aligned}$$

Division of general complex numbers

Exercise

What is
$$\frac{c+di}{a+bi}$$
for $a, b, c, d \in \mathbb{R}$ with a and b not both zero?

We again use the same process:

$$\begin{aligned}
\frac{c+di}{a+bi} &= \frac{c+di}{a+bi} \times \frac{a-bi}{a-bi} \\
&= \frac{ac - bci + adi - bdi^2}{a^2 - (bi)^2} \\
&= \frac{(ac+bd) + (ad-bc)i}{a^2+b^2} \\
&= \frac{ac+bd}{a^2+b^2} - \frac{ad-bc}{a^2+b^2} i
\end{aligned}$$

Complex conjugate – Properties

Theorem

For all $z \in \mathbb{C}$,

- $\overline{(\overline{z})} = z$
- $\mathrm{Re}(z) = \dfrac{1}{2}(z + \overline{z})$ and $\mathrm{Im}(z) = \dfrac{1}{2i}(z - \overline{z})$
- $z\overline{z} = (\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2$, so $z\overline{z} \in \mathbb{R}$ and $z\overline{z} \geq 0$

Complex conjugate – Properties

Theorem

For all $z \in \mathbb{C}$,

- $\overline{(\overline{z})} = z$
- $\mathrm{Re}(z) = \dfrac{1}{2}(z + \overline{z}) \quad$ and $\quad \mathrm{Im}(z) = \dfrac{1}{2i}(z - \overline{z})$
- $z\overline{z} = (\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2$, so $z\overline{z} \in \mathbb{R}$ and $z\overline{z} \geq 0$

Proof

Let $z = a + bi \in \mathbb{C}$ where $a, b \in \mathbb{R}$. Then

$$\overline{(\overline{z})} = \overline{(\overline{a + bi})} = \overline{a - bi} = a + bi = z.$$

Complex conjugate – Properties

Theorem

For all $z \in \mathbb{C}$,

- $\overline{(\overline{z})} = z$
- $\text{Re}(z) = \dfrac{1}{2}(z + \overline{z})$ and $\text{Im}(z) = \dfrac{1}{2i}(z - \overline{z})$
- $z\overline{z} = (\text{Re}(z))^2 + (\text{Im}(z))^2$, so $z\overline{z} \in \mathbb{R}$ and $z\overline{z} \geq 0$

Proof

Let $z = a + bi \in \mathbb{C}$ where $a, b \in \mathbb{R}$. Then

$$z + \overline{z} = (a + bi) + (a - bi) = 2a = 2\text{Re}(z)$$

and

$$z - \overline{z} = (a + bi) - (a - bi) = 2bi = 2\text{Im}(z)i.$$

Hence $\text{Re}(z) = \dfrac{1}{2}(z + \overline{z})$ and $\text{Im}(z) = \dfrac{1}{2i}(z - \overline{z})$.

Complex conjugate – Properties

Theorem

For all $z \in \mathbb{C}$,

- $\overline{(\overline{z})} = z$
- $\text{Re}(z) = \frac{1}{2}(z + \overline{z})$ and $\text{Im}(z) = \frac{1}{2i}(z - \overline{z})$
- $z\overline{z} = (\text{Re}(z))^2 + (\text{Im}(z))^2$, so $z\overline{z} \in \mathbb{R}$ and $z\overline{z} \geq 0$

Proof

Let $z = a + bi \in \mathbb{C}$ where $a, b \in \mathbb{R}$. Then

$$z\overline{z} = (a + bi)(a - bi) = (a^2 - (bi)^2) = a^2 + b^2.$$

Since $a, b \in \mathbb{R}$, and the square of a real number is non-negative, it follows that $z\overline{z} \in \mathbb{R}$ and $z\overline{z} \geq 0$.

Complex conjugate – Properties

Theorem

For all $z, w \in \mathbb{C}$,

- $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{z-w} = \overline{z} - \overline{w}$
- $\overline{zw} = \overline{z}\ \overline{w}$ and $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$

Complex conjugate – Properties

Theorem

For all $z, w \in \mathbb{C}$,

- $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{z-w} = \overline{z} - \overline{w}$
- $\overline{zw} = \overline{z}\,\overline{w}$ and $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$

Proof

Let $z = a + bi \in \mathbb{C}$ and $w = c + di \in \mathbb{C}$ where $a, b, c, d \in \mathbb{R}$. Then

$$\begin{aligned} \overline{z+w} &= \overline{(a+bi)+(c+di)} \\ &= \overline{(a+c)+(b+d)i} \\ &= (a+c)-(b+d)i \\ &= (a-bi)+(c-di) \\ &= \overline{z}+\overline{w}, \text{ and} \end{aligned}$$

$$\begin{aligned} \overline{z-w} &= \overline{(a+bi)-(c+di)} \\ &= \overline{(a-c)+(b-d)i} \\ &= (a-c)-(b-d)i \\ &= (a-bi)-(c-di) \\ &= \overline{z}-\overline{w}. \end{aligned}$$

Complex conjugate – Properties

Theorem

For all $z, w \in \mathbb{C}$,

- $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{z-w} = \overline{z} - \overline{w}$
- $\overline{zw} = \overline{z}\ \overline{w}$ and $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$

Proof

Let $z = a + bi \in \mathbb{C}$ and $w = c + di \in \mathbb{C}$ where $a, b, c, d \in \mathbb{R}$. Then

$$\begin{aligned}
\overline{zw} &= \overline{(a+bi)(c+di)} \\
&= \overline{(ac-bd)+(ad+bc)i} \\
&= (ac-bd)-(ad+bc)i \\
&= (a-bi)(c-di) \\
&= \overline{z}\ \overline{w}.
\end{aligned}$$

It follows that $\overline{z} = \overline{\left(\frac{z}{w}\,w\right)} = \overline{\left(\frac{z}{w}\right)}\ \overline{w}$. Hence $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$.

Complex conjugate – Example

Example

Let $z, w \in \mathbb{C}$ with $z\bar{z} = w\overline{w}$. Prove that $\dfrac{z+w}{z-w}$ is imaginary.

Complex conjugate – Example

Example

Let $z, w \in \mathbb{C}$ with $z\bar{z} = w\overline{w}$. Prove that $\dfrac{z+w}{z-w}$ is imaginary.

To prove something is imaginary, show that its real part is 0:

Example

Let $z, w \in \mathbb{C}$ with $z\bar{z} = w\overline{w}$. Prove that $\dfrac{z+w}{z-w}$ is imaginary.

To prove something is imaginary, show that its real part is 0:

$$\operatorname{Re}\left(\frac{z+w}{z-w}\right)$$

Complex conjugate – Example

Example

Let $z, w \in \mathbb{C}$ with $z\bar{z} = w\overline{w}$. Prove that $\dfrac{z+w}{z-w}$ is imaginary.

To prove something is imaginary, show that its real part is 0:

$$\operatorname{Re}\left(\frac{z+w}{z-w}\right) = \frac{1}{2}\left(\frac{z+w}{z-w} + \overline{\left(\frac{z+w}{z-w}\right)}\right)$$

Example

Let $z, w \in \mathbb{C}$ with $z\overline{z} = w\overline{w}$. Prove that $\dfrac{z+w}{z-w}$ is imaginary.

To prove something is imaginary, show that its real part is 0:

$$\operatorname{Re}\left(\frac{z+w}{z-w}\right) = \frac{1}{2}\left(\frac{z+w}{z-w} + \overline{\left(\frac{z+w}{z-w}\right)}\right) = \frac{1}{2}\left(\frac{z+w}{z-w} + \frac{\overline{z}+\overline{w}}{\overline{z}-\overline{w}}\right)$$

Complex conjugate – Example

Example

Let $z, w \in \mathbb{C}$ with $z\overline{z} = w\overline{w}$. Prove that $\dfrac{z+w}{z-w}$ is imaginary.

To prove something is imaginary, show that its real part is 0:

$$\begin{aligned}
\mathrm{Re}\left(\frac{z+w}{z-w}\right) &= \frac{1}{2}\left(\frac{z+w}{z-w} + \overline{\left(\frac{z+w}{z-w}\right)}\right) = \frac{1}{2}\left(\frac{z+w}{z-w} + \frac{\overline{z}+\overline{w}}{\overline{z}-\overline{w}}\right) \\
&= \frac{1}{2}\left(\frac{(z+w)(\overline{z}-\overline{w}) + (\overline{z}+\overline{w})(z-w)}{(z-w)(\overline{z}-\overline{w})}\right)
\end{aligned}$$

Example

Let $z, w \in \mathbb{C}$ with $z\overline{z} = w\overline{w}$. Prove that $\dfrac{z+w}{z-w}$ is imaginary.

To prove something is imaginary, show that its real part is 0:

$$\begin{aligned}
\operatorname{Re}\left(\frac{z+w}{z-w}\right) &= \frac{1}{2}\left(\frac{z+w}{z-w} + \overline{\left(\frac{z+w}{z-w}\right)}\right) = \frac{1}{2}\left(\frac{z+w}{z-w} + \frac{\overline{z}+\overline{w}}{\overline{z}-\overline{w}}\right) \\
&= \frac{1}{2}\left(\frac{(z+w)(\overline{z}-\overline{w}) + (\overline{z}+\overline{w})(z-w)}{(z-w)(\overline{z}-\overline{w})}\right) \\
&= \frac{1}{2}\left(\frac{z\overline{z} - z\overline{w} + w\overline{z} - w\overline{w} + \overline{z}z - \overline{z}w + \overline{w}z - \overline{w}w}{(z-w)(\overline{z}-\overline{w})}\right)
\end{aligned}$$

Complex conjugate – Example

Example

Let $z, w \in \mathbb{C}$ with $z\overline{z} = w\overline{w}$. Prove that $\dfrac{z+w}{z-w}$ is imaginary.

To prove something is imaginary, show that its real part is 0:

$$
\begin{aligned}
\operatorname{Re}\left(\frac{z+w}{z-w}\right) &= \frac{1}{2}\left(\frac{z+w}{z-w} + \overline{\left(\frac{z+w}{z-w}\right)}\right) = \frac{1}{2}\left(\frac{z+w}{z-w} + \frac{\overline{z}+\overline{w}}{\overline{z}-\overline{w}}\right) \\
&= \frac{1}{2}\left(\frac{(z+w)(\overline{z}-\overline{w}) + (\overline{z}+\overline{w})(z-w)}{(z-w)(\overline{z}-\overline{w})}\right) \\
&= \frac{1}{2}\left(\frac{z\overline{z} - z\overline{w} + w\overline{z} - w\overline{w} + \overline{z}z - \overline{z}w + \overline{w}z - \overline{w}w}{(z-w)(\overline{z}-\overline{w})}\right) \\
&= \frac{1}{2}\left(\frac{2(z\overline{z} - w\overline{w})}{(z-w)(\overline{z}-\overline{w})}\right)
\end{aligned}
$$

Complex conjugate – Example

Example

Let $z, w \in \mathbb{C}$ with $z\overline{z} = w\overline{w}$. Prove that $\dfrac{z+w}{z-w}$ is imaginary.

To prove something is imaginary, show that its real part is 0:

$$\begin{aligned}
\operatorname{Re}\left(\frac{z+w}{z-w}\right) &= \frac{1}{2}\left(\frac{z+w}{z-w} + \overline{\left(\frac{z+w}{z-w}\right)}\right) = \frac{1}{2}\left(\frac{z+w}{z-w} + \frac{\overline{z}+\overline{w}}{\overline{z}-\overline{w}}\right) \\
&= \frac{1}{2}\left(\frac{(z+w)(\overline{z}-\overline{w}) + (\overline{z}+\overline{w})(z-w)}{(z-w)(\overline{z}-\overline{w})}\right) \\
&= \frac{1}{2}\left(\frac{z\overline{z} - z\overline{w} + w\overline{z} - w\overline{w} + \overline{z}z - \overline{z}w + \overline{w}z - \overline{w}w}{(z-w)(\overline{z}-\overline{w})}\right) \\
&= \frac{1}{2}\left(\frac{2(z\overline{z} - w\overline{w})}{(z-w)(\overline{z}-\overline{w})}\right) = 0 \quad (z\overline{z} = w\overline{w})
\end{aligned}$$

Complex conjugate – Example

Example

Let $z, w \in \mathbb{C}$ with $z\overline{z} = w\overline{w}$. Prove that $\dfrac{z+w}{z-w}$ is imaginary.

To prove something is imaginary, show that its real part is 0:

$$\begin{aligned}
\operatorname{Re}\left(\frac{z+w}{z-w}\right) &= \frac{1}{2}\left(\frac{z+w}{z-w} + \overline{\left(\frac{z+w}{z-w}\right)}\right) = \frac{1}{2}\left(\frac{z+w}{z-w} + \frac{\overline{z}+\overline{w}}{\overline{z}-\overline{w}}\right) \\
&= \frac{1}{2}\left(\frac{(z+w)(\overline{z}-\overline{w}) + (\overline{z}+\overline{w})(z-w)}{(z-w)(\overline{z}-\overline{w})}\right) \\
&= \frac{1}{2}\left(\frac{z\overline{z} - z\overline{w} + w\overline{z} - w\overline{w} + \overline{z}z - \overline{z}w + \overline{w}z - \overline{w}w}{(z-w)(\overline{z}-\overline{w})}\right) \\
&= \frac{1}{2}\left(\frac{2(z\overline{z} - w\overline{w})}{(z-w)(\overline{z}-\overline{w})}\right) = 0 \quad (z\overline{z} = w\overline{w})
\end{aligned}$$

So since its real part is 0, the expression is imaginary.