Chapter 2: Limits

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MATH1131

UNSW

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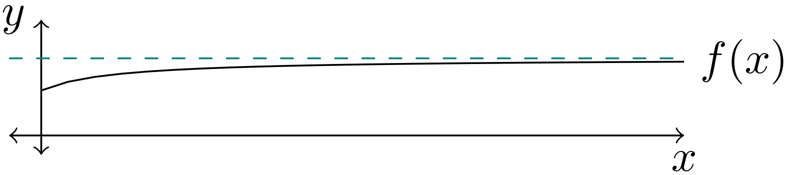
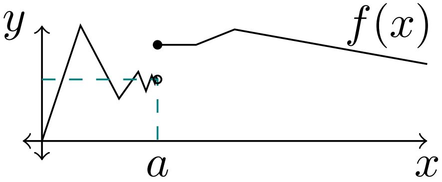
What do we study in this chapter?

Limit is the fundamental concept in calculus.There are two main types of limits:

Limits at ∞. *What is the long term behaviour of the function f?*



Limits at a point. *What is the local behaviour of f for x near some point $a \in \mathbb{R}$?*



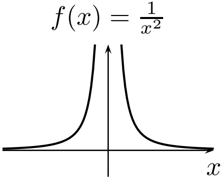
Limits of functions at infinity: Informal definition

- We say that $f(x)$ has **limit L as x goes to ∞** if $f(x)$ gets closer and closer to L as x gets greater and greater. In this case, we write

$$\lim_{x\to\infty} f(x) = L,$$

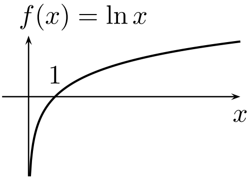
or

$$f(x) \to L \quad \text{as} \quad x \to \infty.$$



- If $f(x)$ gets 'arbitrarily large' (that is, 'approaches' ∞) as x tends to ∞, then we say also that the limit does not exist and we write

$$f(x) \to \infty \quad \text{as} \quad x \to \infty.$$



Remark: We do not write $\lim_{x\to\infty} f(x) = \infty$ since ∞ is not a real number.

Example

Why do we believe that

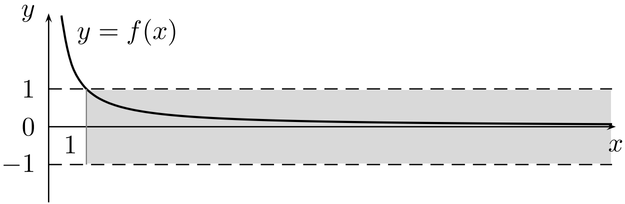
$$\lim_{x\to\infty} f(x) = 0 \quad \text{for} \quad f(x) = \frac{1}{x}\,?$$

Consider the distance between $f(x)$ and 0 denoted by

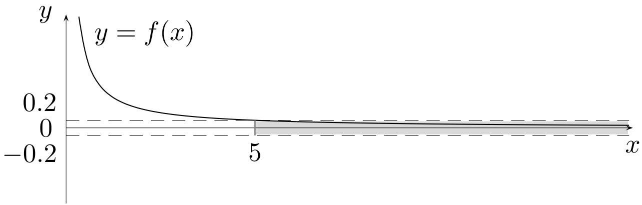
$$\operatorname{error}(x) = |f(x) - 0|.$$

Facts.

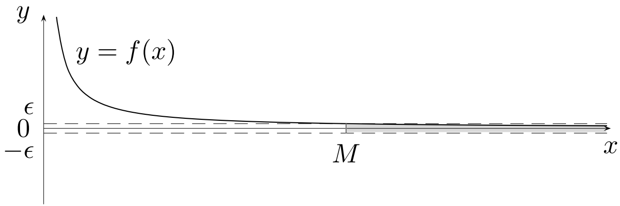
- $\operatorname{error}(x) < 1$ whenever $x > 1$.



- $\text{error}(x) < 0.2$ whenever $x > 5$.



- $\text{error}(x) < 0.1$ whenever $x > 10$.
- $\text{error}(x) < 0.01$ whenever $x > 100$.
- $\text{error}(x) < 0.0001$ whenever $x > 10000$.
- Set $\epsilon = 1/M$. Then, $\text{error}(x) < \epsilon$ whenever $x > M$.



Definition

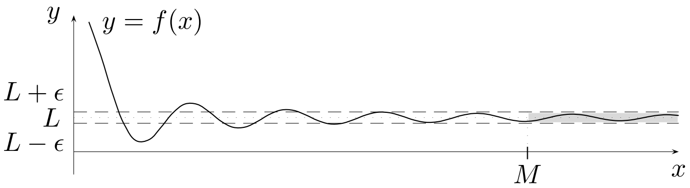
Let f be a function defined on some interval (b, ∞) and let L be a real number. We say that

$$\lim_{x \to \infty} f(x) = L$$

if

for every $\epsilon > 0$, there exists a real number M such that

$$\text{if } x > M \quad \text{then} \quad |f(x) - L| < \epsilon.$$



Remarks.

- $|f(x) - L| < \epsilon \iff -\epsilon < f(x) - L < \epsilon \iff f(x) \in (L - \epsilon, L + \epsilon)$
- The number M depends on ϵ, and in general, the smaller the value of ϵ is, the larger the value of M.

Proving that $\lim_{x\to\infty} f(x) = L$ using the limit definition

To show that $\lim_{x\to\infty} f(x) = L$ using the definition we need to give a recipe for finding an M that works for different ϵ.

If $x > M_\epsilon$ **then** $L - \epsilon < f(x) < L + \epsilon$**.**

Example. Prove that

$$\lim_{x\to\infty} \frac{2x+3}{x+5} = 2.$$

Proof. We consider the distance (we called it "error" earlier)

$$\begin{aligned} |f(x) - L| &= \left|\frac{2x+3}{x+5} - 2\right| = \left|\frac{2x+3-2x-10}{x+5}\right| = \left|\frac{-7}{x+5}\right| \\ &= \frac{7}{x+5} \qquad \text{for } x > -5 \\ &< \frac{7}{x} \qquad \text{[to make algebra simpler later on]} \end{aligned}$$

In summary,

$$|f(x) - L| < \frac{7}{x}.$$

This inequality gives an upper bound for the distance between $f(x)$ and L!

Accordingly,

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad \frac{7}{x} < \epsilon.$$

The latter condition is equivalent to

$$x > \frac{7}{\epsilon}$$

and hence if we set

$$M = \frac{7}{\epsilon}$$

then

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad x > M.$$

Remark. Note that the value of M is not unique. For example, in the above we used the upper bound $7/x$, but if instead we use $7/(x+1)$, then $7/(x+1) < \epsilon$ implies $M = 7/\epsilon - 1$.

Remark. In the preceding example it was easy to find an upper bound for $|f(x) - L|$. For most problems it is not even possible to exactly solve $|f(x) - L| < \epsilon$, and when it is, it usually gives a really messy formula for M.

General strategy. Given ϵ, we need to find a number M such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad x > M.$$

The number M can be found by following the procedure below.

1. Find a good upper bound for $|f(x) - L|$.
2. Find a simple condition on x such that this upper bound is less than ϵ.
3. Use this condition to state an appropriate value for M (in terms of ϵ).

Remarks

- As mentioned before, in general, M depends on ϵ but it is not uniquely defined.
- The definition of the limit does NOT require to specify M for a given ϵ! It requires to show (**to prove**) that such an M exists!!!
- The definition of the limit does not tell you what the limit is.
- The definition may be used to prove theorems which allow you to justify methods of finding limits.
- Applying the definition to verify an educated guess for a limit is usually the last resort.
- Make use of the theorems unless you are specifically asked to apply the definition.

Basic rules for limits

Elementary rules

- If f is a constant function, that is, $f(x) = c$ for all x, then

$$\lim_{x \to \infty} f(x) = c.$$

- If $f(x) \to \infty$ as $x \to \infty$ then

$$\lim_{x \to \infty} \frac{1}{f(x)} = 0.$$

These are intuitively obvious and give limits such as

$$\lim_{x \to \infty} \frac{2}{x^2} = 0, \qquad \lim_{x \to \infty} e^{-x} = \lim_{x \to \infty} \frac{1}{e^x} = 0.$$

Theorem

Suppose that

$$\lim_{x\to\infty} f(x) = a, \qquad \lim_{x\to\infty} g(x) = b$$

for some functions f and g. Then

1. $\lim_{x\to\infty} [f(x) + g(x)] = a + b$
2. $\lim_{x\to\infty} [f(x) - g(x)] = a - b$
3. $\lim_{x\to\infty} [f(x)g(x)] = ab$
4. $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \dfrac{a}{b}$ provided that $b \neq 0$.

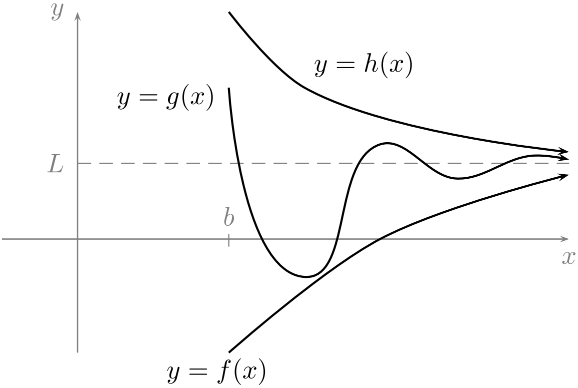
Example

Determine the limit of

$$f(x) = \frac{3 + \dfrac{1}{3x^3}}{5 - e^{-x}} \qquad \text{as } x \to \infty.$$

The pinching theorem: informally

Assume that two functions f and h have the same limit as $x \to \infty$ and the graph of a function g lies between the graphs of f and h (if x is large enough). Then, g has the same limit as f and h.



The pinching theorem: formal statement

Theorem

Suppose that f, g and h are three functions such that

$$f(x) \le g(x) \le h(x)$$

on an interval (b, ∞) for some $b \in \mathbb{R}$ and

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} h(x) = L.$$

Then

$$\lim_{x \to \infty} g(x) = L.$$

Remark. The case $x \to -\infty$ can be handled in a similar manner.

! Examples

(1) Determine the limit of

$$g(x) = \frac{\cos x}{x}$$

as $x \to \infty$.

! Examples

(2) Show that

$$\lim_{x\to\infty} e^{-2x}\sin(5x) = 0.$$

Limits of the form $f(x)/g(x)$

Suppose that we want to calculate a limit of the form

$$\lim_{x \to \infty} \frac{f(x)}{g(x)},$$

where both $f(x)$ and $g(x)$ tend to ∞ as $x \to \infty$.

Problem. We cannot apply the preceding rules since f and g do not have limits.

Idea. Divide both f and g by the leading term, that is the fastest growing term appearing in the denominator g (if it exists).

Examples

Example. Find the following limit (if it exists): $\lim_{x\to\infty} \dfrac{6x^3 - 4\sin x}{\cos 3x + 5x - x^3}$

Solution. The leading term in this example is x^3, therefore, we divide both numerator and denominator by x^3. We have

$$\lim_{x\to\infty} \frac{6x^3 - 4\sin x}{\cos 3x + 5x - x^3} = \lim_{x\to\infty} \frac{6 - 4\frac{\sin x}{x^3}}{\frac{\cos 3x}{x^3} + \frac{5}{x^2} - 1}$$

$$= \frac{6 - 4\displaystyle\lim_{x\to\infty} \frac{\sin x}{x^3}}{\displaystyle\lim_{x\to\infty} \frac{\cos 3x}{x^3} + \lim_{x\to\infty} \frac{5}{x^2} - 1}.$$

Since $\lim_{x\to\infty} \dfrac{5}{x^2} = 0$, $\lim_{x\to\infty} \dfrac{\cos 3x}{x^3} = 0$ and $\lim_{x\to\infty} \dfrac{\sin x}{x^3} = 0$ by the pinching theorem (show at home!) we obtain that

$$\lim_{x\to\infty} \frac{6x^3 - 4\sin x}{\cos 3x + 5x - x^3} = \frac{6}{-1} = -6.$$

Example. Find the following limit (if it exists)

$$\lim_{x \to \infty} \frac{x^2 + 5x}{\sqrt{x^4 + 1} - 4x}$$

Example: rational functions

Let $m < n$ be positive integers and $a_m \neq 0$ and $b_n \neq 0$ real numbers. Find the following limit (if it exists)

$$L = \lim_{x \to \infty} \frac{a_m x^m + \cdots + a_1 x + a_0}{b_n x^n + \cdots + b_1 x + b_0}.$$

Solution. We divide both the numerator and denominator by the highest power of x in the denominator, that is, x^n (recall $n > m$). We obtain

$$\begin{aligned}
L &= \lim_{x \to \infty} \frac{a_m \frac{x^m}{x^n} + \cdots + a_1 \frac{x}{x^n} + a_0 \frac{1}{x^n}}{b_n \frac{x^n}{x^n} + \cdots + b_1 \frac{x}{x^n} + b_0 \frac{1}{x^n}} \\
&= \lim_{x \to \infty} \frac{a_m \frac{1}{x^{n-m}} + \cdots + a_1 \frac{1}{x^{n-1}} + a_0 \frac{1}{x^n}}{b_n + \cdots + b_1 \frac{1}{x^{n-1}} + b_0 \frac{1}{x^n}} \\
&= \frac{a_m \lim\limits_{x \to \infty} \dfrac{1}{x^{n-m}} + \cdots + a_1 \lim\limits_{x \to \infty} \dfrac{1}{x^{n-1}} + a_0 \lim\limits_{x \to \infty} \dfrac{1}{x^n}}{b_n + \cdots + b_1 \lim\limits_{x \to \infty} \dfrac{1}{x^{n-1}} + b_0 \lim\limits_{x \to \infty} \dfrac{1}{x^n}} = 0
\end{aligned}$$

since $\lim\limits_{x \to \infty} \dfrac{1}{x^k} = 0$ for any $k \geq 1$

Question. What is the limit when $m = n$? $m > n$?

Limits of the form $\sqrt{f(x)} - \sqrt{g(x)}$

Idea. We divide and multiply by the factor $\sqrt{f(x)} + \sqrt{g(x)}$. Then we arrive at limits of the previous type.

Example. Determine the limit of

$$f(x) = \sqrt{x^2 + 2x} - \sqrt{x^2 - 1}$$

as $x \to \infty$.

Solution. We have

$$
\begin{aligned}
f(x) &= \sqrt{x^2+2x} - \sqrt{x^2-1} \\
&= \frac{\left(\sqrt{x^2+2x} - \sqrt{x^2-1}\right)\left(\sqrt{x^2+2x} + \sqrt{x^2-1}\right)}{\sqrt{x^2+2x} + \sqrt{x^2-1}} \\
&= \frac{(\sqrt{x^2+2x})^2 - (\sqrt{x^2-1})^2}{\sqrt{x^2+2x} + \sqrt{x^2-1}} \\
&= \frac{2x+1}{\sqrt{x^2+2x} + \sqrt{x^2-1}} \quad \text{divide both numerotor and denomiator by x} \\
&= \frac{2+\frac{1}{x}}{\sqrt{1+\frac{2}{x}} + \sqrt{1-\frac{1}{x^2}}}.
\end{aligned}
$$

Hence, the limit $\lim_{x\to\infty} f(x)$ exists and

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2+\frac{1}{x}}{\sqrt{1+\frac{2}{x}} + \sqrt{1-\frac{1}{x^2}}} = 1.$$

Exercise. Does

$$\lim_{x\to\infty} \sqrt{x^4 - x^3} - \sqrt{x^4 + 1}$$

exist?

Note: Indeterminate forms

The following limits have the form "$\frac{\infty}{\infty}$" but each displays a very different limiting behaviour as $x \to \infty$:

- $\dfrac{x^2}{x} \to \infty$
- $\dfrac{x}{x^2} \to 0$
- $\dfrac{2x^2}{x^2} \to 2$

Since we cannot determine in advance what kind of limiting behaviour something of the form "$\frac{\infty}{\infty}$" has, we say that "$\frac{\infty}{\infty}$" is an indeterminate form.

Other types of indeterminate forms are

- "$\frac{0}{0}$"
- "$\infty - \infty$"
- "$0 \times \infty$"

Limits of functions at a point

Informally: $\lim_{x\to a} f(x) = L$ means that the closer and closer x gets to a, the closer and closer $f(x)$ gets to L.

Example (high school limits!).

$$\begin{aligned}
\lim_{x\to 1}\frac{x^3-1}{x^2-1} &= \lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} \\
&= \lim_{x\to 1}\frac{x^2+x+1}{x+1} \textbf{(not a "}\frac{0}{0}\textbf{" form)} \\
&= \frac{\lim_{x\to 1} x^2+x+1}{\lim_{x\to 1} x+1} \\
&= \frac{3}{2}.
\end{aligned}$$

But what about $\lim_{x\to 1}\frac{|x^3-1|}{x^2-1}$?

Left-hand, right-hand and two-sided limits

Let $f(x) = \dfrac{|x^3 - 1|}{x^2 - 1}$.

- If $x > 1$ then $x^3 - 1 > 0$ so $f(x) = \dfrac{x^3 - 1}{x^2 - 1} = \dfrac{x^2 + x + 1}{x + 1} \approx \dfrac{3}{2}$ for x near 1.
- If $x < 1$ then $x^3 - 1 < 0$ so $f(x) = -\dfrac{x^3 - 1}{x^2 - 1} = -\dfrac{x^2 + x + 1}{x + 1} \approx -\dfrac{3}{2}$ for x near 1.

In this case $\lim_{x \to 1} f(x)$ does not exist. The value of $f(x)$ does not get closer and closer to a single number as x approaches closer and closer to 1.

However, if you only sneak up on 1 from the right, $f(x)$ gets closer and closer to $\dfrac{3}{2}$, and if you only sneak up on 1 from the left, $f(x)$ gets closer and closer to $-\dfrac{3}{2}$.

We say that f has a **right hand limit** at 1 and write $\lim_{x \to 1^+} f(x) = \dfrac{3}{2}$.

Similarly, this f also has a **left hand limit** at 1: $\lim_{x \to 1^-} f(x) = -\dfrac{3}{2}$.

Notation. Let f be a function defined on an an open interval containing a.

- left hand limit at a, $\lim_{x\to a^-} f(x) = L_1$: $f(x)$ gets "closer and closer" to L_1 when x gets "closer and closer" to a from the left.
- right hand limit at a, $\lim_{x\to a^+} f(x) = L_2$: $f(x)$ gets "closer and closer" to L_2 when x gets "closer and closer" to a from the right.

Example. What happens to $f(x) = \frac{1}{x}$ when x approaches 0?

Definition

Let f be defined on an open interval containing a. If $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^-} f(x)$ both exist and are equal to L, that is,

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L,$$

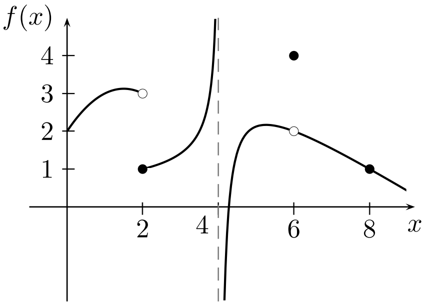
then we say that the limit of $f(x)$ as $x \to a$ exists and is equal to L, and we write

$$\lim_{x\to a} f(x) = L.$$

If any of these conditions fails, than we say that $\lim_{x\to a} f(x)$ does **not** exist.

Example

Consider the function f whose graph is shown below.



With reference to this graph, discuss the behaviour of $f(x)$ when x is near the points 2, 4, 6 and 8... What is so special about the above function at $x = 8$?

Rules for limits at a point

Theorem

Suppose that f and g are defined in an interval containing a (but not necessarily at a) and that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist. Then

1. $\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$,
2. $\lim_{x\to a}(f-g)(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$,
3. $\lim_{x\to a}(fg)(x) = \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right)$,
4. $\lim_{x\to a}(f/g)(x) = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, as long as $\lim_{x\to a} g(x) \neq 0$.

Remark. All these rules also apply for right and left hand limits.

Polynomials

- If $f(x) = c$ (a constant), then $\lim_{x \to a} f(x) = c$.
- If $g(x) = x$, then $\lim_{x \to a} g(x) = a$.
- Every polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is made up from combining 'f' and 'g' above a finite number of times, so by the theorem on the last slide

$$\lim_{x \to a} p(x) = p(a).$$

For example, $\lim_{x \to a}(x^2 + 3) = \left(\lim_{x \to a} x\right)^2 + \lim_{x \to a} 3 = a^2 + 3$.

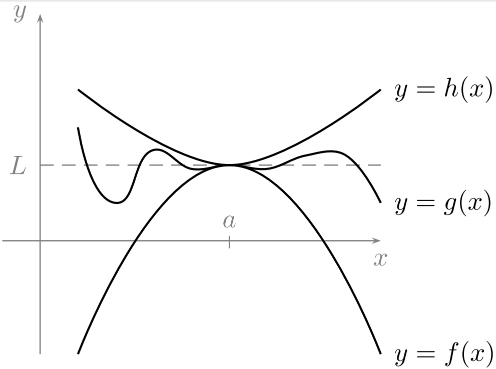
The Pinching Theorem for limit at a point

Theorem

Suppose that f, g, h are defined on an open interval I containing a (except possibly at a), and that

$$f(x) \leq g(x) \leq h(x), \qquad x \in I,\ x \neq a.$$

If $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x)$ exists and equals L too.



Examples

Example 1. The Pinching Theorem can be used to prove the well-known formula (see guided tutorial problem)

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Example 2. Find $\displaystyle\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$.

Limits and continuous functions

Definition

Let f be defined on some open interval containing the point a. We say that f is continuous at a if

$$\lim_{x \to a} f(x) = f(a);$$

otherwise we say that f is discontinuous at a.

If f is continuous at every point of its domain, we simply say that f is continuous.

So, to check that f is continuous at a or not you have to answer:

- Is f defined at a?
- Does $\lim_{x \to a} f(x)$ exist (check left and right hand limits)t?
- Is $\lim_{x \to a} f(x)$ equal to $f(a)$?

Example on slide 29. The function f is continuous everywhere except at $x = 2$ and $x = 6$.

Note that $x = 4$ is not part of the domain of f and hence asking whether or not f is continuous at $x = 4$ does not make any sense.

Continuity of elementary functions

- Polynomials, $\sin$, $\cos$ and $\exp$ are continuous functions everywhere;
- Rational functions, $\tan$ and $\ln$ are continuous on their domain of definition;
- Thus, limits (one-sided or two-sided) involving these elementary functions are easy to compute: just evaluate the function at the given point!

Remark. Continuity is a deep property for a function to have...

See the next Chapter!

Limits at a point and composition of functions

Theorem

If $\lim_{x \to a} f(x) = L$ and g is continuous at L then

$$\lim_{x \to a} g(f(x)) = g(\lim_{x \to a} f(x)).$$

If the functions f and g are continuous everywhere, then

$$\lim_{x \to a} g(f(x)) = g(\lim_{x \to a} f(x)).$$

Example

Let

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = e^{\sin^2 x + 3\sin x - 1}.$$

Find

$$\lim_{x \to \pi/2} f(x).$$

Solution. Let $h(x) = e^{x^2+3x-1}$ and $g(x) = \sin x$. Thus, we have

$$f(x) = h(g(x)).$$

Now, since h and g are continuous everywhere, we have

$$\lim_{x\to\pi/2} f(x) = \lim_{x\to\pi/2} h(g(x)) = h\left(\lim_{x\to\pi/2} g(x)\right) = h(g(\pi/2)) = h(1) = e^3.$$

!Exercise

Discuss behaviour of f as $x \to 2$ if

$$f(x) = \begin{cases} \frac{|x^2-4|}{x-2} & \text{for } x \neq 2 \\ 3 & \text{for } x = 2. \end{cases}$$

Is f continuous at 2?

Solution. Firstly, let's break up this split function a bit more...

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & x \leq -2 \\ \frac{-(x^2-4)}{x-2}, & -2 < x < 2 \\ 3, & x = 2 \\ \frac{x^2-4}{x-2}, & x > 2. \end{cases}$$

Summary: What did we learn in this chapter?

Limit at ∞

- Formal definition (p. 6)
- Limit rules (+, -, *, /) (p. 11)
- Pinching theorem (p. 15)
- Inderterminate form $f(x)/g(x)$ (p. 18)
- Inderterminate form $\sqrt{f(x)} - \sqrt{g(x)}$ (p. 22)

Limit at a point a

- Left-hand and right-hand limits (p. 28)
- Limit rules (+, -, *, /) (p. 31)
- Pinching theorem (p. 33)
- Limits and continuity (p. 35)
- Composition of limits (p. 37)