MATH1131 Mathematics 1A – Algebra

Lecture 11: Polar Form for Complex Numbers

Lecturer: Sean Gardiner – sean.gardiner@unsw.edu.au

Based on slides by Jonathan Kress

Polar form of a complex number

The Cartesian form of a complex number with real part x and imaginary part y is

$$z = x + yi.$$

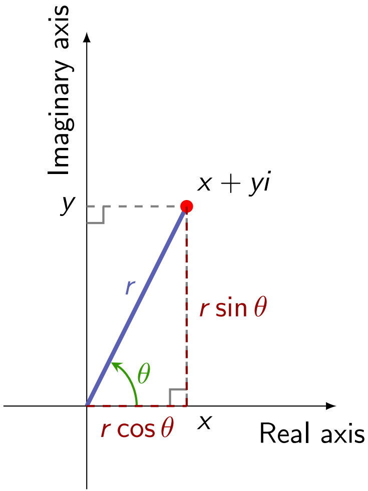
We can also describe z by its distance r from the origin and its angle θ from the positive real axis as shown.

Simple trigonometry shows that

$$x = r\cos\theta, \quad y = r\sin\theta$$

and

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}.$$



Modulus and argument

We call r the modulus of $z = x + iy$ and denote it $|z|$:

$$|z| = \sqrt{x^2 + y^2}$$

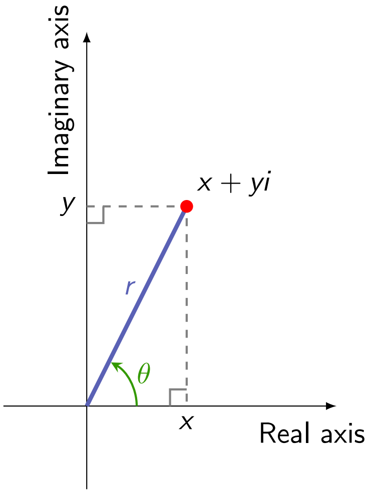
We call θ an argument of $z = x + iy$ and denote it $\arg(z)$:

$$\tan(\arg(z)) = \frac{y}{x}$$

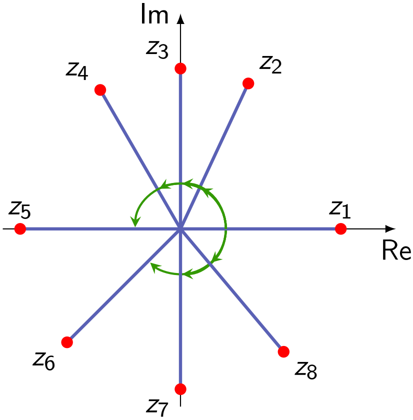
Note for any z there are many possible arguments that differ by multiples of 2π.

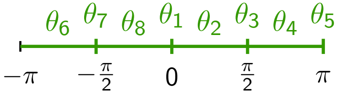
The principal argument of z is denoted $\text{Arg}(z)$ and satisfies:

$$-\pi < \text{Arg}(z) \leq \pi.$$



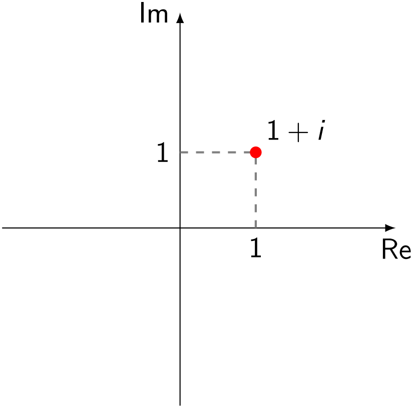
Principal argument





Example

Plot $1+i$ on an Argand diagram and find its modulus and principal argument.



Modulus:

$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$$

Argument:

$$\tan(\mathrm{Arg}(1+i)) = \tfrac{1}{1} = 1$$

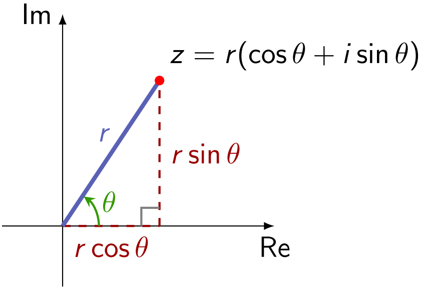
From the diagram,

$$0 < \mathrm{Arg}(1+i) < \tfrac{\pi}{2}.$$

So $\mathrm{Arg}(1+i) = \dfrac{\pi}{4}$.

Polar form

If $|z| = r$ and $\arg(z) = \theta$, then $\quad x = r\cos\theta \quad$ and $\quad y = r\sin\theta$.



So for $z = x + iy$, we have

$$z = r(\cos\theta + i\sin\theta).$$

We call this the polar form of a complex number.

Note that r must always be non-negative.

Examples

Example

Plot each of the following complex numbers on an Argand diagram and find its polar form.

(a) $1+i$

(b) $-1+\sqrt{3}i$

(c) $-3-3i$

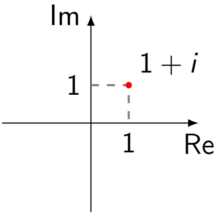
(d) $-3-4i$

(e) $4i$

(f) 5

(g) -5

(h) 0



$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$$

$\tan(\operatorname{Arg}(1+i)) = \frac{1}{1} = 1,$

and $0 < \operatorname{Arg}(1+i) < \dfrac{\pi}{2}$.

So $\operatorname{Arg}(1+i) = \dfrac{\pi}{4}$.

$$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Example

Plot each of the following complex numbers on an Argand diagram and find its polar form.

(a) $1+i$

(b) $-1+\sqrt{3}i$

(c) $-3-3i$

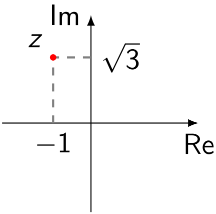
(d) $-3-4i$

(e) $4i$

(f) 5

(g) -5

(h) 0



$$|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$$

$$\tan(\mathrm{Arg}(z)) = \frac{\sqrt{3}}{-1} = -\sqrt{3},$$

and $\dfrac{\pi}{2} < \mathrm{Arg}(z) < \pi$.

So $\mathrm{Arg}(z) = \dfrac{2\pi}{3}$.

$$-1+\sqrt{3}i = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

Example

Plot each of the following complex numbers on an Argand diagram and find its polar form.

(a) $1+i$

(b) $-1+\sqrt{3}i$

(c) $-3-3i$

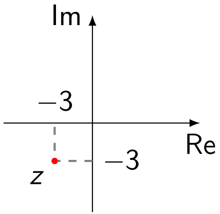
(d) $-3-4i$

(e) $4i$

(f) 5

(g) -5

(h) 0



$$|z| = \sqrt{(-3)^2 + (-3)^2} = 3\sqrt{2}$$

$\tan(\mathrm{Arg}(z)) = \frac{-3}{-3} = 1,$

and $-\pi < \mathrm{Arg}(z) < -\dfrac{\pi}{2}.$

So $\mathrm{Arg}(z) = -\dfrac{3\pi}{4}.$

$$-3 + -3i = 3\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$$

Examples

Example

Plot each of the following complex numbers on an Argand diagram and find its polar form.

(a) $1+i$

(b) $-1+\sqrt{3}i$

(c) $-3-3i$

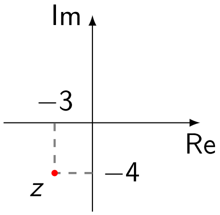
(d) $-3-4i$

(e) $4i$

(f) 5

(g) -5

(h) 0



$$|z| = \sqrt{(-3)^2 + (-4)^2} = 5$$

$$\tan(\mathrm{Arg}(z)) = \tfrac{-4}{-3} = \tfrac{4}{3},$$

and $-\pi < \mathrm{Arg}(z) < -\dfrac{\pi}{2}$.

So $\mathrm{Arg}(z) = -\pi + \tan^{-1}\left(\dfrac{4}{3}\right)$.

$$-3 + -4i = 5\left(\cos\alpha + i\sin\alpha\right),$$

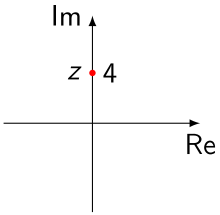
where $\alpha = -\pi + \tan^{-1}\left(\dfrac{4}{3}\right)$

Examples

Example

Plot each of the following complex numbers on an Argand diagram and find its polar form.

(a) $1 + i$
(b) $-1 + \sqrt{3}i$
(c) $-3 - 3i$
(d) $-3 - 4i$
(e) $4i$
(f) 5
(g) -5
(h) 0



$$|z| = 4$$

$$\text{Arg}(z) = \frac{\pi}{2}$$

$$4i = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

Examples

Example

Plot each of the following complex numbers on an Argand diagram and find its polar form.

(a) $1+i$

(b) $-1+\sqrt{3}i$

(c) $-3-3i$

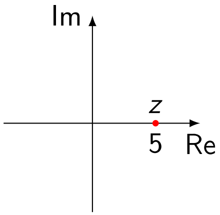
(d) $-3-4i$

(e) $4i$

(f) 5

(g) -5

(h) 0



$|z| = 5$

$\text{Arg}(z) = 0$

$$5 = 5\left(\cos 0 + i \sin 0\right)$$

Examples

Example

Plot each of the following complex numbers on an Argand diagram and find its polar form.

(a) $1 + i$

(b) $-1 + \sqrt{3}i$

(c) $-3 - 3i$

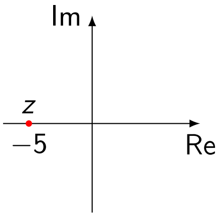
(d) $-3 - 4i$

(e) $4i$

(f) 5

(g) -5

(h) 0



$|z| = 5$

$\text{Arg}(z) = \pi$

$-5 = 5\,(\cos\pi + i\sin\pi)$

Examples

Example

Plot each of the following complex numbers on an Argand diagram and find its polar form.

(a) $1+i$

(b) $-1+\sqrt{3}i$

(c) $-3-3i$

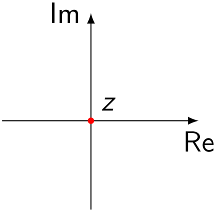
(d) $-3-4i$

(e) $4i$

(f) 5

(g) -5

(h) 0



$|z| = 0$

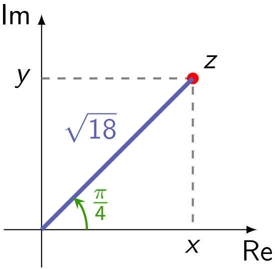
$\mathrm{Arg}(z)$ is undefined

0 has no standard polar form.

Examples

Example

Sketch $z = \sqrt{18}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$ in the complex plane and write z in Cartesian form.

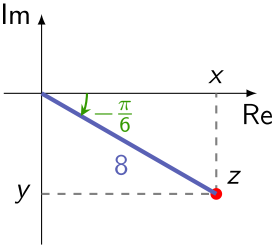


$$\begin{aligned} z &= \sqrt{18}\cos\left(\tfrac{\pi}{4}\right) + \sqrt{18}i\sin\left(\tfrac{\pi}{4}\right) \\ &= \sqrt{18} \times \frac{1}{\sqrt{2}} + \sqrt{18}i \times \frac{1}{\sqrt{2}} \\ &= 3 + 3i \end{aligned}$$

Examples

Example

Sketch $z = 8\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$ in the complex plane and write z in Cartesian form.



$$
\begin{aligned}
z &= 8\cos\left(-\frac{\pi}{6}\right) + 8i\sin\left(-\frac{\pi}{6}\right) \\
&= 8 \times \frac{\sqrt{3}}{2} + 8i \times -\frac{1}{2} \\
&= 4\sqrt{3} - 4i
\end{aligned}
$$

Example

Example

Find the polar form of $w = -7\left(\sin\left(-\frac{\pi}{3}\right) + i\cos\left(-\frac{\pi}{3}\right)\right)$.

The modulus cannot be negative in polar form. So rewriting:

$$\begin{aligned}
w &= -7\left(\sin\left(-\frac{\pi}{3}\right) + i\cos\left(-\frac{\pi}{3}\right)\right)\\
&= -7\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)\\
&= 7\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)\\
&= 7\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)
\end{aligned}$$