Some Notes about Imagination and Mathematics related to a conversation with Eva Brann

May 16, 2008

Abstract

These are the notes I prepared for myself to engage in a conversation with Eva Brann and roundtable discussion with the audience at the Phioloctetes Center in New York City on May 13, 2008. The title of the event was *Imagination and Mathematics: The Geometry of Thought*.

Contents

1 Mathematics in the air we breathe

Math, among the general public, has the reputation of being an abstruse subject: it takes training, hard work, and even beyond that, some people seem to insist on it requiring that funny word "aptitute." Let us start our discussion by charting an end-run around all this reputation, and heaviness, by focussing on some of the mathematical visions, understandings, and sensibilities that we all share, with seemingly no effort, no special talent, and that are all but invisible to us since we accept them so fully in our common language, culture, and imagination. What Eva and I hope will be a pleasant surprise is that some of this *mathematics in the air that we breathe* already represents *extremely* important leaps of the mathematical imagination, leaps that we all have done, without any special training.

I. We all are capable of turning almost anything into *geometry*

Without giving it a second thought we all, for example, visualize time as a geometric entity ("far in the future" is a perfectly familiar phrase to us) or as a marker of some well-defined activity as in the somewhat out of fashion "two cigarettes later." Moreover, we are comfortable when the exchange goes the other way; that is, where distance is measured by time: a day's journey, three light-years away, etc. The notion of *speed* is even more curiously accepted by us without a whimper: when our speedometer tells us that we are traveling at 60 miles per hour we are happy, even though there are some famous *Zeno paradoxes* lurking around this. Think of the large metaphorical leap we have taken when we have represented the flow of time as a *static* (after all) straight line. This is a simple leap, but a vastly important one, the *geometrization of time* leading us to our ubiquitous charts, charting the progress of (through time) of anything that can be quantified: temperature, humidity, salinity, or money, or all of these taken together. We geometrize, talking of peaks, and valleys, wells, saddles.

Figure 1:

(*Stock Chart for Apple on April 15*)

II. We all are capable of turning almost anything into *algebra*

For example, we happily turn geometry into numbers. For example, New York City: *eighth avenue and forty-second Street* describes a specific spot. We are willing to coordinatize, to calibrate, to quantify almost anything in terms of numbers, even Intelligence.

Each of these activities, *quantifying* and *geometrizing*, allow us to consider the "Thing" (quantified, or geometrized) via a different set of intuitions, and a different vocabulary. And it is this ability to shift (intuitions, vocabulary, viewpoint) that is quintessentially the mathematical imagination.

We will survey some of the different ways our imagination is exercised when we think about mathematics. But before we do this, it would be good to emphasize that "intuitions" (whatever this word means) play an enormous role in the mathematical imaginative process and they are very dependent upon the complexity of our psychological make-up. Moreover, these *intuitions* have fascinating limitations: flaws, if you wish: marvelous idiosyncratic constraints that color our experience of mathematics. Take, for example, our intuitions regarding the numerical"size" of things. We aren't very good at estimating size.

2 Estimating numbers

There are web games that allow us to practice (http://www.oswego.org/ocsd-web/games/Estimate/estimate.html). The cognitive psychologists study the curious traps that hinder us from being very accurate (are they intrinsic flaws in our innate wiring?). Here is a question (thanks to Amos Tversky!)

> Estimate (reasonably rapidly—don't whip out your calculator) the following number: *ten times nine times eight times seven times six times five times four times three times two times one.*

Do it silently: no one will know! Tversky asked some fifty "subjects" to do this, and the average number that they came up with was some number, call it X, probably not terrible far from your guess! Then Tversky approached a different group—fifty new subjects—and asked them a somewhat different question:

> Estimate (reasonably rapidly—don't whip out your calculator) the following number: *one times two times three times four times five times six times seven times eight times nine times ten.*

This time the average number that they came up with was some number Y *significantly smaller than* X!

Of course this "somewhat different question" has exactly the same answer as the first question, that is, $3,628,800$, which turns out to be larger than X (or Y) by a wide margin.

Why do you think there would be such a discrepancy? Do you think that you can train yourself out of falling into similar mis-estimations?

Having pointed out a weakness in our intuition, let us move to elements of strength.

3 How is it that the most simple geometric constructions can reveal so much?

> Do two parallelograms with the same base-length and the same height have the same area?

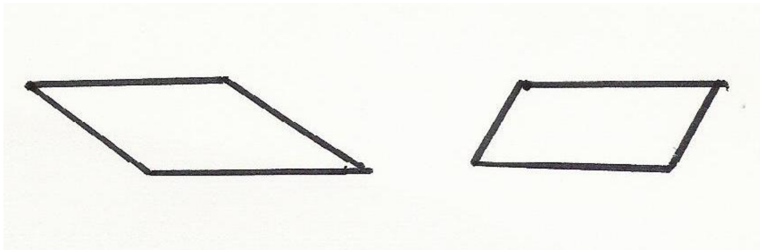

Figure 2:

(*Two parallelograms with the same height and same base-length*)

If you think of the right *construction*, i.e., drawing just the right line, you will immediately see (or fairly immediately see) that any two parallelograms of the same base and height have the *same* area. BUT, in fact, you'll also see more interesting mathematical questions raised by your construction.

For example, drawing this line will favor a very specific way—perhaps not the most general way—of dealing with the concept of *area*.

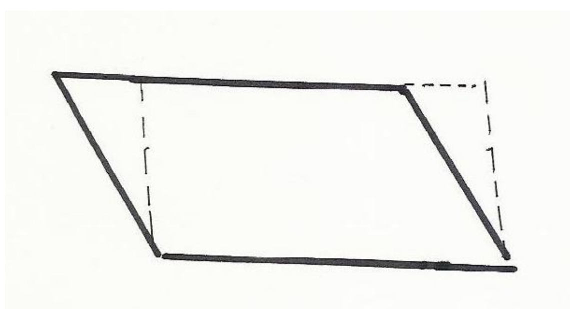

Figure 3:

(*Straightening out a parallelogram, by cutting and moving*)

Move the triangle, to get:

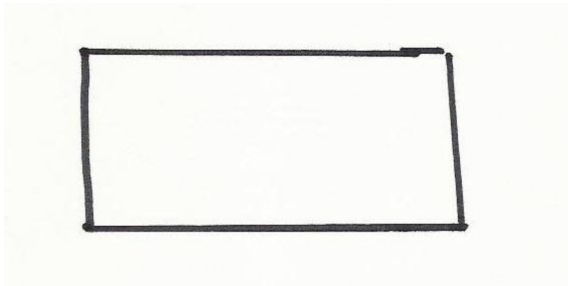

Figure 4:

(*The straightened out parallelogram*)

Note the various issues raised by this simple construction.

- We have dealt with the panoply of *all* parallelograms of a given base-length and given height by relating them all to a single exemplar, the rectangle, which now will act as "intermediary" between any two general parallelograms of the same base-length and height. The introduction of such an "intermediary," missing from the initial formulation of the problem is one of the many things at play here.

- The diagram above is—of course—misleading, as all diagrams are. It assumes that if you raise a perpendicular as depicted, from the acute vertex of the base, that perpendicular will actually 'hit' the(top) side of the parallelogram that is parallel to the base. Can you draw a parallelogram where this *doesn't happen*? What might you do in that case? (Answer: you need only rethink which of the sides of the parallelogram you wish to view as base, and which as side!)

- An interesting issue raised by the constructions is that the operation of cutting-into-pieces-and-moving-the-pieces-around suﬃces to obtain a rectangle from any such parallelogram. This raises a host of questions regarding the possible transmogrification of figures into other figures by cutting-and-moving.

- Perhaps the most important eﬀect of using the construction, perhaps, is that–although in the initial formulation of the question, we did not actually give a definition of the concept of area—we have quite elegantly replaced the question of *area* by a question about *cutting-and-moving*.

All this is a lot of stuff for a single construction to rake up!

4 On the magical power of definitions; how we phrase definitions matters immensely

It is a truism that the slightest change in our viewpoint can clarify, or obscure. There are decisive moments where we—wittingly or unwittingly—are framing our viewpoint. The moment when we define a concept is usual one.

Even when *logically* a paraphrase of a definition is immediately seen to be equivalent to some other way of phrasing the same definition, the harmless paraphrase can provide an insightful (for certain purposes) shift of viewpoint—or the reverse.

Here are two definitions of the same concept, the **angle-bisector at a vertex of a triangle:**

- The angle-bisector at a vertex v of a triangle is the straight line starting at v that bisects the angle at v and ends on the side of the triangle opposite v.

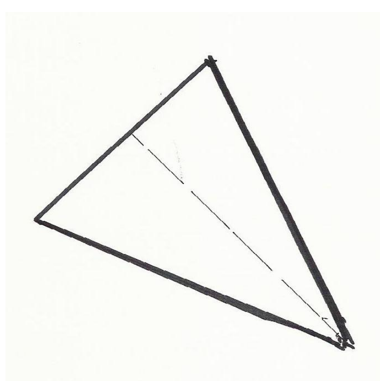

Figure 5:

(*The angle-bisector at a vertex*)

- The angle-bisector at a vertex v of a triangle is the set of points *equidistant* to the two edges of the triangle that touch the vertex v.

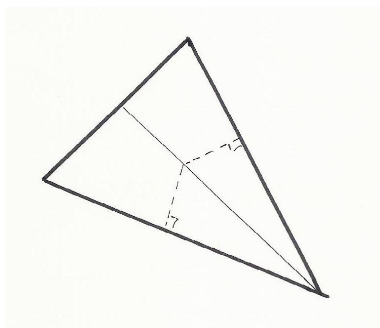

Figure 6:

(*The angle-bisector at a vertex from the vantage point of definition 2*)

Now, these two definitions, somewhat different from each other, are easily seen to be equivalent: they define the same concept. (If you note that the two lines forming the angle at v being bisected are brought one to the other under the operation of flipping around the angle-bisector, you'll immediately see that the points on the angle-bisector are equidistant from those two lines.) But the definitions underscore different things. The first definition focuses on the vertex v, while the second definition is phrased entirely in terms of the relationship the points on the angle-bisector have with the two lines of the triangle touching v. Vastly different focusses.

Now suppose you wanted to convince yourself that

The three angle-bisectors of a triangle intersect in one point.

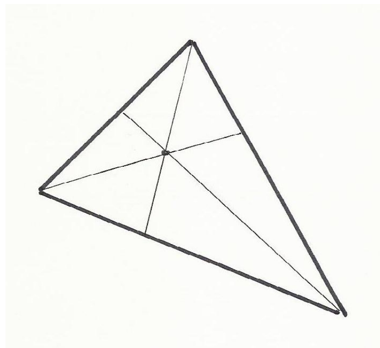

Figure 7:

Discuss: How the second definition "immediately" shows this, for if you have a point on the intersection of any *two* of the angle-bisectors it is equidistant to all *three sides* so it also lies on the third angle-bisector.

And how the first definition will, surely, allow us to prove the same result, but not see it in a flash[1]. Concepts have many different—visibly equivalent—definitions, but the suitable one (for a given issue) may utterly clarify the issue.

5 Symmetry, where you least expect it: the Monty Hall TV quiz game

Imagine that you are going to be a repeat guest on Monty Hall's quiz show. Monty's guest is ushered on the stage and confronts three closed doors. Behind two of them there is nothing. Behind the third there lies a bright shiny new Porsche. To get the prize (like the suitors for Portia in *Merchant of Venice*) you have to open the right door. Monty then asks his guest not to open, but merely to point to, the door he intends to open. The guest points. Monty then offers a helpful hint, and a further option. *The hint:* Monty opens one of the two doors that the guest has not pointed to, to reveal that behind the door Monty has just opened, there is nothing. *The*

[1] A very similar argument about viewpoints and proofs-that-can-be-seen-in-a-flash is made by Laurent Berger about the three mid-perpendiculars of the sides of a triangle intersecting at a point, a *mid-perpendicular* being alternatively defined as the locus of points equidistant to two vertices of a triangle. See *The Unravelers* to be published by A.K. Peters.

option: Monty then says to the guest: *Do you want to stick with your choice, and open the door you pointed to, or do you want to change your choice and open another door?* To be sure, if the guest chooses to open "another door" he or she will naturally open the "third door," i.e., the one that he has not pointed to and Monty has not opened.

OK, if you were a guest on Monty's show, what would you do: would you stick to your choice, or change your choice after Monty offered his hint, and why? (I find it slightly easier to think about this if I imagined that I was to be a repeat guest on his show, and wished to choose a uniform strategy: stick to my choice, or change my choice, for all appearances on his show.)

Three things are of interest here: what is the "appropriate" winning strategy? And what is an often-made mistake in reasoning about this question? And why is the mistake often made? For me, an interesting side-issue is symmetry.

There are many ways of understanding this; for example, in terms of response to the following "catechism:" Imagine that you are a repeat guest at Monty's show and have decided on one of the following two strategies:

- Stick to your guns. Pay no attention to Monty, and simply open the door that you initially pointed to.
- Always switch: Point to a door at random at the outset, but when Monty opens a second door, always choose the third door—the one to which you did *not* point, and which Monty did *not* open.

Call these strategies the *sticking-to-your-first-choice* strategy and the *switching strategy*.

Here, then, are the questions:

1. What is the probability of your initially pointing to the right door?
2. What is the probability of your guessing the right door if Monty never did his "thing" (opening a door to show it empty of the Porsche)?
3. What is the probability of your winning if you followed the *sticking-to-your-first-choice* strategy?

   Now note: In any one guest appearance, if you had initially pointed to the correct door, and switched, you would lose the Porsche; if you had initially pointed to an incorrect door, and switched, you would win the Porsche.

The switching strategy then converts any initial WIN to an official LOSS and any initial LOSS to an official WIN. So, in view of your answer to question (3),

4. What is the probability of your winning if you followed the switching strategy?

The symmetry in question is that the switching strategy, in effect, flips WINS to LOSSES, and LOSSES to WINS.