1. (10 points) Consider the following problem

Minimize $(x_1 - \frac{9}{4})^2 + (x_2 - 2)^2$

subject to $x_2 - x_1^2 \geq 0$

$x_1 + x_2 \leq 6$

$x_1, x_2 \geq 0$

write the KKT optimality conditions and verify that these conditions are true at the point $\hat{x} = (\frac{3}{2}, \frac{9}{4})^t$.

Answer:

First, Transfer Primal Feasibility to Dual Feasibility:

Minimize $f(x_1, x_2)$: $(x_1 - \frac{9}{4})^2 + (x_2 - 2)^2$

$g_1(x_1, x_2): {x_1}^2 - x_2 \leq 0$

$g_2(x_1, x_2): x_1 + x_2 - 6 \leq 0$

$g_3(x_1, x_2): -x_1 \leq 0$

$g_4(x_1, x_2): -x_2 \leq 0$

$\nabla f\,(x_1, x_2) = (2x_1 - \frac{9}{2}\,,\, 2x_2 - 4)$

$\nabla g_1(x_1, x_2) = (2x_1\,, -1)$

$\nabla g_2(x_1, x_2) = (1\,, 1)$

$\nabla g_3(x_1, x_2) = (-1\,, 0)$

$\nabla g_4(x_1, x_2) = (0\,, -1)$

Now, below is Dual Feasibility form for possible points:

$\nabla f + u_1 \nabla g_1 + u_2 \nabla g_2 + u_3 \nabla g_3 + u_4 \nabla g_4 = 0,\;\; u_1, u_2, u_3, u_4 \geq 0$

Complementary slackness: $u_1 g_1 = u_2 g_2 = u_3 g_3 = u_4 g_4 = 0$

$\hat{x} = (\frac{3}{2}, \frac{9}{4})^t$

For this Dual Feasibility at $\hat{x}$, Check **KKT optimality conditions:**

$\nabla f(\hat{x}) + u_1 \nabla g_1(\hat{x}) + u_2 \nabla g_2(\hat{x}) + u_3 \nabla g_3(\hat{x}) + u_4 \nabla g_4(\hat{x}) = 0,$

$u_1, u_2, u_3, u_4 \geq 0 \; and \; u_1 g_1(\hat{x}) = u_2 g_2(\hat{x}) = u_3 g_3(\hat{x}) = u_4 g_4(\hat{x}) = 0$

Because only $g_1(\hat{x}): (\frac{3}{2})^2 - \frac{9}{4} = 0$, I = {1}, set $u_2 = u_3 = u_4 = 0$

Then, $\begin{pmatrix} 2(\frac{3}{2}) - \frac{9}{2} \\ 2(\frac{9}{4}) - 4 \end{pmatrix} + u_1 \begin{pmatrix} 2(\frac{3}{2}) \\ -1 \end{pmatrix} + u_2 \nabla g_2 + u_3 \nabla g_3 + u_4 \nabla g_4 = \binom{0}{0},$

and solve $u_1 = \frac{1}{2}$. $\hat{x}$ Satisfies KKT optimality conditions for this Dual Feasibility as

well as f is circle (convex function), so f($\hat{x}$) = $\frac{5}{8}$ is optimal minimum solution.

2. (20 points) Please write C/C++ program of the steepest descent algorithm with termination criterion $\varepsilon = 0.000001$ to find the minimum of $f(x_1, x_2) = x_1(x_1 - 13) + x_2^2 + x_1(x_2 + 7)$ using *(0, 0)* as the initial solution. You are required to submit your source code, which should be able to compile at Linux system.

The result of minimum solution point is $(\widehat{x_1},\ \widehat{x_2})$ = (3.9999998, -1.9999994), and its gradient value is less than 0.000001.

The minimum value of f(x_1, x_2) is approximately -12.

$$\nabla \mathrm{f}(x_1, x_2) = (2x_1 + x_2 - 6, x_1 + 2x_2)$$

The minimum solution point is very close to (4, -2).

Besides, in line search part, the accuracy of threshold for finding minimum value along gradient direction is 1e-14 due to calculation error of x1*x2 consideration.

The simulation result at hera server:

```
[redacted]@hera:~/ece687

[[redacted]@hera ece687]$ ./hw3_q2.out
local minimum is at < 3, 0>,  norm (gradient):3
local minimum is at < 3, -1.5>,  norm (gradient):1.5
local minimum is at < 3.75, -1.5>,  norm (gradient):0.75
local minimum is at < 3.75, -1.875>,  norm (gradient):0.375
local minimum is at < 4, -1.875>,  norm (gradient):0.2795085
local minimum is at < 3.9440983, -1.9868034>,  norm (gradient):0.10292741
local minimum is at < 3.9740365, -1.9778443>,  norm (gradient):0.034971065
local minimum is at < 4.00064, -1.9942399>,  norm (gradient):0.014050947
local minimum is at < 3.9986828, -1.9976205>,  norm (gradient):0.0034511894
local minimum is at < 3.9988271, -1.9995683>,  norm (gradient):0.0019390487
local minimum is at < 3.9997911, -1.9994124>,  norm (gradient):0.00098109025
local minimum is at < 3.9997066, -1.9998933>,  norm (gradient):0.00048675626
local minimum is at < 3.9999474, -1.9998532>,  norm (gradient):0.00024463963
local minimum is at < 3.9999266, -1.9999735>,  norm (gradient):0.00012190377
local minimum is at < 3.9999868, -1.9999633>,  norm (gradient):6.1090031e-05
local minimum is at < 3.9999817, -1.9999934>,  norm (gradient):3.0499341e-05
local minimum is at < 3.9999967, -1.9999908>,  norm (gradient):1.5264816e-05
local minimum is at < 3.9999954, -1.9999983>,  norm (gradient):7.6273953e-06
local minimum is at < 3.9999992, -1.9999977>,  norm (gradient):3.8153588e-06
local minimum is at < 3.9999989, -1.9999996>,  norm (gradient):1.9071294e-06
local minimum is at < 3.9999998, -1.9999994>,  norm (gradient):9.5374691e-07
[result]: local minimum is at < 3.9999998, -1.9999994>, f(x):-12
```

Source code is attached in folder hw3_answer_V2.