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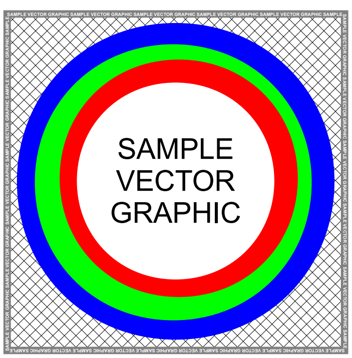
Thesis by

Insert Author Name Here

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy
in
Electrical Engineering and Computer Science



University Institute of College
Springfield, New York, USA

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Acknowledgments

Insert thesis acknowledgments here. Thesis acknowledgments typically include research advisers and mentors, thesis committee members, collaborators, and funding sources.

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Abstract

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Stern-Gerlach Experiments

3 Postulates

To discuss Stern-Gerlach experiments, we first consider the mathematical objects used to model physical systems and variables. By comparing the objects used in classical and quantum mechanics, we make sense of the first three postulates of quantum mechanics. Then, we compare standard and consistent descriptions of spin measurement, and their relation to the fourth and fifth postulates.

3.1 State Spaces and Physical Variables

Consider the spin (intrinsic angular momentum) of an electron. Our system is the electron, and the classical spin state is simply a vector $S = (S_x, S_y, S_z)$ in $\mathbb{R}^3$, where each component S_w is a physical variable that represents the magnitude of spin in the w direction. Consequently, the components of spin state are mutually independent, and they can take on any real value. In other words, the sample spaces for each component are compatible, continuous and infinitely large. Because of the compatibility of spin component sample spaces, succesive observations of S_x, S_y, and S_z uniquely determine the spin state.

Our physical variable of interest is the magnitude of spin in the z direction; clasically, this variable is modeled by a function of spin state: $S_z(S) = \langle z|S\rangle$. This function relies on the inner product of the state space $\mathbb{R}^3$ to determine the magnitude of the spin vector in the z direction only. In general, measurement of spin in three orthogonal directions uniquely determines the spin component in any direction w.

Experimental evidence raises issues with this classical description of spin states and S_z. First, only two values have ever been recorded, named *spin-up* and *spin-down*. These values are of equal magnitude $\frac{\hbar}{2}$ with positive and negative orientations, respectively. Second, the results of succesive measurements of a spin system impliy that the spin state is determined by spin along one axis only. Consequently, spin along more than one axis cannot be simultaneously determined; this is discussed further in section 3.2. To resolve these issues, we must change the mathematical objects used to represent system states and physical variables. Specifically, the sample space must restrict observed S_w values to spin up and spin down, and the inability of determining spin in more than one direction should be reflected through incompatible sample spaces.

The quantum mechanical system state is described by a normalized vector in a linear state space. In fact, this is the first postulate of quantum mechanics:

POSTULATE 1 *The state of a physical system is defined by specifying an abstract vector $|\psi\rangle$ in a Hilbert state space $\mathcal{H}$.*

For spin-$\frac{1}{2}$ systems, the state space consists of all linear combinations of spin-up and spin-down:

$$\mathcal{H} = \{\alpha \left|+\right\rangle + \beta \left|-\right\rangle\} \tag{3.1}$$

where $\alpha, \beta \in \mathbb{C}$

For ease of calculation, we only consider normalized states, as scalar multiples of any normalized state bear the same physical meaning. We require $\alpha^2 + \beta^2 = 1$.

Defining system states in this way has dramatic consequences. A state includes all information that can be known about a system. A classical spin state includes spin magnitude in x, y, and z. The quantum state is only capable of determining spin magintude in one general direction w; spin in other directions not parallel or antiparallel to w is undefined.

The second posulate of quantum mechanics states that physical variables are described by linear operators:

POSTULATE 2 *Every physical variable $\mathcal{A}$ is described by an operator A acting in $\mathcal{H}$.*

Justifying the second postulate is easiest when also considering the third postulate:

POSTULATE 3 *The only possible result of the measurement of a physical variable $\mathcal{A}$ is one of the eigenvalues of the corresponding operator A.*

The operator correlates elements of a finite sample space (eigenvalues) with particular system states (eigenstates). Consequently, a state can only be interpreted as having a definite variable value if it is an eigenstate of the variable's operator. To illustrate this, consider the operator representing S_z. Written in its own basis,

$$S_z \doteq \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

This operator correlates z spin-up $\left(S_z = \frac{\hbar}{2}\right)$ with

$$\left|\psi\right\rangle = \left|+\right\rangle_z \doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and z spin-down $\left(S_z = \frac{-\hbar}{2}\right)$ with

$$\left|\psi\right\rangle = \left|-\right\rangle_z \doteq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Similarly, the operator representing S_y written in the S_z basis is

$$S_y \doteq \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

This operator correlates y spin-up $\left(S_y = \frac{\hbar}{2}\right)$ with

$$|\psi\rangle = |+\rangle_y \doteq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

and y spin-down $\left(S_y = \frac{-\hbar}{2}\right)$ with

$$|\psi\rangle = |-\rangle_y \doteq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Operators for S_z and S_y share no common eigenstates, so no state can posses definite values for both variables. In general, operators for any spin component S_j, where j is some direction not parallel or antiparallel with z, share no common eigenstates with S_z; in other words, S_z and S_j have incompatible sample spaces. By representing physical variables with an operator rather than a function, sample spaces become quantized and can be incompatible with each other. These features are necessary for predicting experimental results.

The first three postulates designate the mathematical objects used to model physical system states and variables. Consistent quantum theory does not modify these postulates; we will see that all of the strange features unique to quantum systems are a consequence of these postulates.

3.2 Measurement

In standard quantum mechanics, the fifth postualte (known as the projection postulate) describes how a system changes upon measurement. Defined as "an interaction with a classical apparatus", measurement instantaneously changes the measured state to the eigenstate corresponding to the measruement result.

POSTULATE 5 *If the measurement of the physical variable $\mathcal{A}$ on the system in the state $|\psi\rangle$ gives the result a_n, the state of the system immediately after the measurement is the normalized projection*

$$|\psi\rangle' = \frac{P_n |\psi\rangle}{\sqrt{\langle\psi| P_n |\psi\rangle}}$$

onto the subspace associated with a_n.

Consider a measurement result for the z component of spin, S_z. We represent the result with n_z, which could be either spin-up or spin-down. $P^n{}_z = |n\rangle_z \, {}_z\langle n|$ is the projection operator for the state $|n\rangle_z$ corresponding to n_z. The new state is the projection of $|\psi\rangle$ onto $|n\rangle_z$, divided by the magnitude of that projection. The end result is that $|\psi\rangle$ becomes the normalized state $|n\rangle_z$. This process is known as *state collapse* or *wavefunction collapse*.

As an example, consider the system shown in 3.1. The first apparatus serves as a state

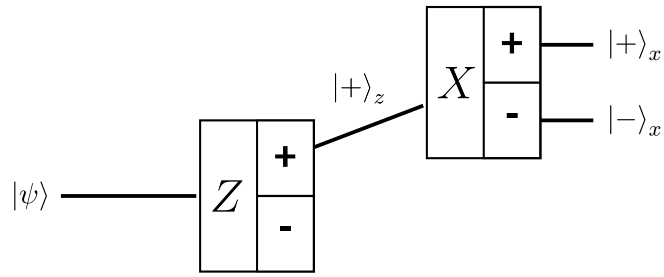

FIGURE 3.1 Demonstrating renormalizing upon measurment in standard quantum mechanics

preparation device, since we are only interested in particles exiting the spin-up output. Using the projection postulate, the state after the first measurement is

$$|\psi_{top}\rangle = \frac{P^z{}_+ |\psi\rangle}{\sqrt{\langle\psi| P^z{}_+ |\psi\rangle}} = |+\rangle_z$$

Similarly, the possible output states from the second apparatus are

$$|\psi_{top}\rangle = \frac{P^x{}_+ |+\rangle_z}{\sqrt{\langle+| P^x{}_+ |+\rangle_z}} = |+\rangle_x$$

$$|\psi_{bottom}\rangle = \frac{P^x{}_- |+\rangle_z}{\sqrt{\langle+| P^x{}_- |+\rangle_z}} = |-\rangle_x$$

In consistent quantum theory, what is meant by "measurement" in the projection postulate is itself modeled as a physical process. Therefore, no notion of "interaction with a classical apparatus" is required to describe how states evolve when their classical properties are recorded. To describe measurement, each Stern-Gerlach apparatus has its own detector state space, containing orthonormal states representing each measurement result. For a Stern-Gerlach apparatus, a detection state is defined by a particle being located at some spatially separated output. In general, a detection state is some classical indicator; examples include an apparatus needle pointing up, or a particle colliding with a screen in a distinguishable region.

Let the state space of the z apparatus be represented by

$$\mathcal{H}_{Dz} = \{|D_+\rangle_z, |D_-\rangle_z\}$$

where

$$_z\langle D_+|D_+\rangle_z = 1$$
$$_z\langle D_+|D_-\rangle_z = 0$$

$\mathcal{H}_{Dx}$ is similarly defined. Each detector state space is a subset of a global detector state space $\mathcal{H}_D$,

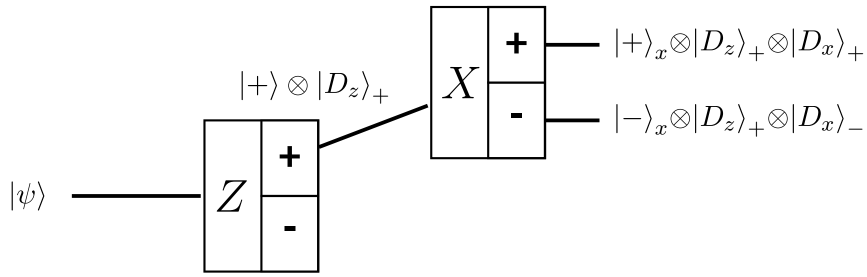

FIGURE 3.2 Demonstrating description of measurment as an abstract physical process in consistent quantum mechanics

so that we can require orthogonality of states in seperate dector spaces:

$$
\begin{aligned}
{}_z \langle D_n | D_m \rangle_x &= 0 \\
&\forall \, |D_n\rangle_z \in \mathcal{H}_{Dz} \\
&\forall \, |D_m\rangle_x \in \mathcal{H}_{Dx}
\end{aligned}
$$

The act of measurement is described by correlating detector states with quantum states. The system then evolves by

$$
\begin{aligned}
&V: \\
&|\psi\rangle = \left(\sum_n P^z{}_n |\psi\rangle \right) \otimes \left(\sum_m P_m |D\rangle \right) \mapsto \sum_n P^z{}_n |\psi\rangle \otimes |D_n\rangle
\end{aligned} \tag{3.2}
$$

where $P^z{}_n$ is the projection operator for the $n^{th} z$ spin eigenstate and P_m is the projection operator for the m^{th} detector state.

This evolution describes a change in the eigenstates of the system. Before, $|\psi\rangle$ was a superposition of the tensor products of any spin eigenstate and any detector state. Any detector state could be realized with any quantum state. Afterwards, $|\psi\rangle$ is a superposition of the tensor products of a spin eigenstate and one specific detector state; a detector state can be realized with one specific quantum state only.

3.2 illustrates this description of measurement. The state of a particle leaving each output is the tensor product of a spin eigenstate and a detection state. As the quantum state is realized in incompatible sample spaces (S_z and S_x), it loses information about its history; by posessing a definite S_x value, information about S_z is lost. However, the detector states are not impacted by these measurements, and the *event history* of the state is preserved.

3.3 The Born Rule

We begin by comparing standard and consistent descriptions of measuring a quantum states' spin along one axis using a Stern-Gerlach apparatus.

We accept the first three postulates of quantum mechanics.

1) All information known about a quantum mechanical system is represented by an abstract vector $|\psi\rangle$. This vector lives in a linear state space $\mathcal{H}$, which is the set of all possible states of the quantum system.
2) A physical observable of the system is represented by a linear operator A that acts on vectors in $\mathcal{H}$.
3) The only possible measurement results of a physical observable are the eigenvalues a_n of the corresponding observable A.

We accept the second and thirst postulates because they do not describe the operator's relationship to the measurement process. The second correlates a measurable quantity of the system to an operator, while the third describes the possible results. However, care must be taken when interpreting a "measurement result of a_n". We take this to mean that some record exists of the system behaving classically such that A must have been a_n.

The fourth and fifth postulates require more scrutiny. In standard quantum mechanics, the act of measurement plays a special role in assigning probabilities to measurement outcomes through the probability postulate, and in determmining the evolution of the state through the projection postulate.

The probability postulate assigns probabilities to each measurement result by taking the inner product of $|\psi\rangle$ and the eigenstate $|n\rangle$ corresponding to measurement result n:

$$\mathcal{P}_n = |\langle n|\psi\rangle|^2 \tag{3.3}$$

This postulate is also known as the *Born Rule*, which is usually presented in the language of wavefunctions. The probability that a system is found at position x is

$$\mathcal{P}_x = |\psi(x)|^2 \tag{3.4}$$

which follows from the wavefunction's definition as the inner product of the $|x\rangle$ eigenstate and ψ.

To compute probabilities of each outcome, we introduce an analog of the Born Rule: sum the magnitudes of each branch wavefunction that includes the corresponding detector state. This is accomplished by finding the trace of the projection operator of that detector state acting on the projection operator or the overall evolved state.

$$\mathcal{P}_n = Tr(P_n^D \cdot V|\psi\rangle\langle\psi| V^\dagger) \tag{3.5}$$

TODO: explain trace

In this simple Stern-Gerlach example, there is only one branch wavefunction correlated with each detector state. So, computing probabilities is done by finding the magnitude of each branch wavefunction.

TODO: compute probabilities, show it is equal to std QM

ChapterComplementarity

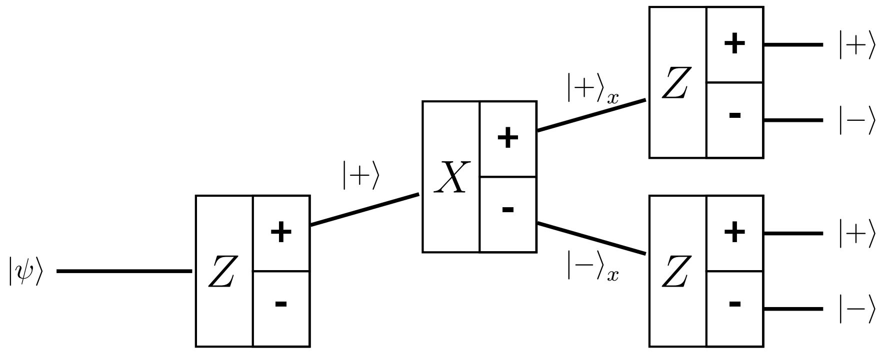

FIGURE 3.3 Demonstrating complementary measurments in standard quantum mechanics

We now compare standard and consistent quantum mechanics' treatment of the principle of complementarity. Arguably the most fundamental feature of quantum mechanics, the principle of complementarity states that a quantum system has pairs of physical observables which cannot be measured simultaneously. Components of spin on orthogonal axes are complementary properties, so we examine measurements of succesive Stern-Gerlach experiments. We will see how complementarity is a strange consequence of the standard postulates of quantum mechanics, while in consistent formulations, complementarity is itself a postulate from which strange consequences arise.

First, we compute the probabilities of each outcome using standard quantum mechanics. The first apparatus serves as a state preparation device with output $|+\rangle$. By the direction of the projection postulate, the state is renormalized upon each measurement. After measuring a property complementary to what is known (such as spin along x, knowing spin along z), any information known about the input state is lost; the input state instantaneously changes to the state corresponding to the observed quantity. Consequently, there is an equal probability of observing the final state as $|+\rangle$ or $|-\rangle$ at either final apparatus, even though the state was initially prepared as $|+\rangle$, since

$$
\begin{aligned}
\mathcal{P}_n &= |\,\langle + | + \rangle_x\,|^2 \\
&= |\,\langle - | + \rangle_x\,|^2 \\
&= |\,\langle + | - \rangle_x\,|^2 \\
&= |\,\langle - | - \rangle_x\,|^2 \\
&= \frac{1}{4}
\end{aligned}
$$

This contradiction with classical intuition is a direct result of the projection postulate. The act of measurement causes the system to shed properties previously recorded. TODO: describe measurement problem.

TODO: Demonstrate complementarity and single framework rule

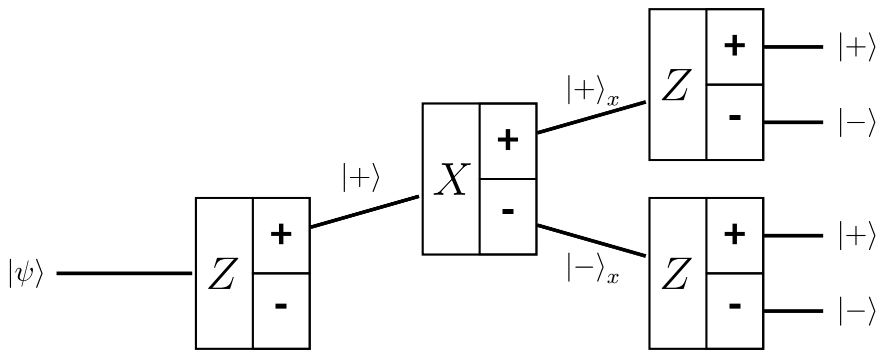

FIGURE 3.4 Demonstrating complementary measurments in consistent quantum mechanics

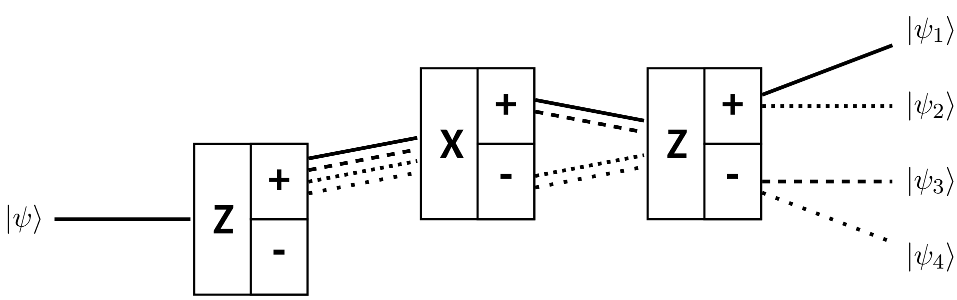

FIGURE 3.5 TODO: create section to discuss this example, and how it creates an inconsistent set of histories. Describe how the set can be made to be consistent

4 Insert Chapter Title Here

4.1 Introduction

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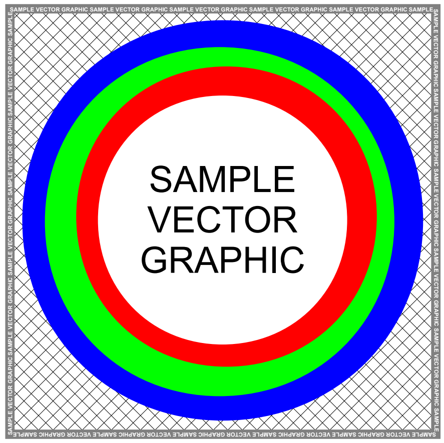

FIGURE 4.1 Insert the full caption here for this floating figure.

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Proofs of theorems are deferred to Section 4.5.

4.2 Some Examples

Sed commodo posuere pede. Mauris ut est. Ut quis purus. Sed ac odio. Sed vehicula hendrerit sem. Duis non odio. Morbi ut dui. Sed accumsan risus eget odio. In hac habitasse platea dictumst. Pellentesque non elit. Fusce sed justo eu urna porta tincidunt. Mauris felis odio, sollicitudin sed, volutpat a, ornare ac, erat. Morbi quis dolor. Donec pellentesque, erat ac sagittis semper, nunc dui lobortis purus, quis congue purus metus ultricies tellus. Proin et quam. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos hymenaeos. Praesent sapien turpis, fermentum vel, eleifend faucibus, vehicula eu, lacus.

TABLE 4.1 Insert the full caption here for this floating table.

Symbol	Definition
α	insert definition of α here, $\alpha \geq 1$
β	insert definition of β here, $\beta \geq 2$
γ	insert definition of γ here, $\gamma \geq 3$
δ	insert definition of δ here, $\delta \geq 4$

4.2.1 Examples of Figures and Tables

This is a reference to Figure 4.1. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Donec odio elit, dictum in, hendrerit sit amet, egestas sed, leo. Praesent feugiat sapien aliquet odio. Integer vitae justo. Aliquam vestibulum fringilla lorem. Sed neque lectus, consectetuer at, consectetuer sed, eleifend ac, lectus. Nulla facilisi. Pellentesque eget lectus. Proin eu metus. Sed porttitor. In hac habitasse platea dictumst. Suspendisse eu lectus. Ut mi mi, lacinia sit amet, placerat et, mollis vitae, dui. Sed ante tellus, tristique ut, iaculis eu, malesuada ac, dui. Mauris nibh leo, facilisis non, adipiscing quis, ultrices a, dui.

Here we say something about Figures 4.1 and 4.2. Note how the effect in Figure 4.2 is stronger that in Figure 4.1. Morbi luctus, wisi viverra faucibus pretium, nibh est placerat odio, nec commodo wisi enim eget quam. Quisque libero justo, consectetuer a, feugiat vitae, porttitor eu, libero. Suspendisse sed mauris vitae elit sollicitudin malesuada. Maecenas ultricies eros sit amet ante. Ut venenatis velit. Maecenas sed mi eget dui varius euismod. Phasellus aliquet volutpat odio. Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae; Pellentesque sit amet pede ac sem eleifend consectetuer. Nullam elementum, urna vel imperdiet sodales, elit ipsum pharetra ligula, ac pretium ante justo a nulla. Curabitur tristique arcu eu metus. Vestibulum lectus. Proin mauris. Proin eu nunc eu urna hendrerit faucibus. Aliquam auctor, pede consequat laoreet varius, eros tellus scelerisque quam, pellentesque hendrerit ipsum dolor sed augue. Nulla nec lacus.

We summarize our notation in Table 4.1. Suspendisse vitae elit. Aliquam arcu neque, ornare in, ullamcorper quis, commodo eu, libero. Fusce sagittis erat at erat tristique mollis. Maecenas sapien libero, molestie et, lobortis in, sodales eget, dui. Morbi ultrices rutrum lorem. Nam elementum ullamcorper leo. Morbi dui. Aliquam sagittis. Nunc placerat. Pellentesque tristique sodales est. Maecenas imperdiet lacinia velit. Cras non urna. Morbi eros pede, suscipit ac, varius vel, egestas non, eros. Praesent malesuada, diam id pretium elementum, eros sem dictum tortor, vel consectetuer odio sem sed wisi.

Table 4.2 summarizes our simulation results. Sed feugiat. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Ut pellentesque augue sed urna. Vestibulum diam eros, fringilla et, consectetuer eu, nonummy id, sapien. Nullam at lectus. In sagittis ultrices mauris. Curabitur malesuada erat sit amet massa. Fusce blandit. Aliquam erat volutpat. Aliquam

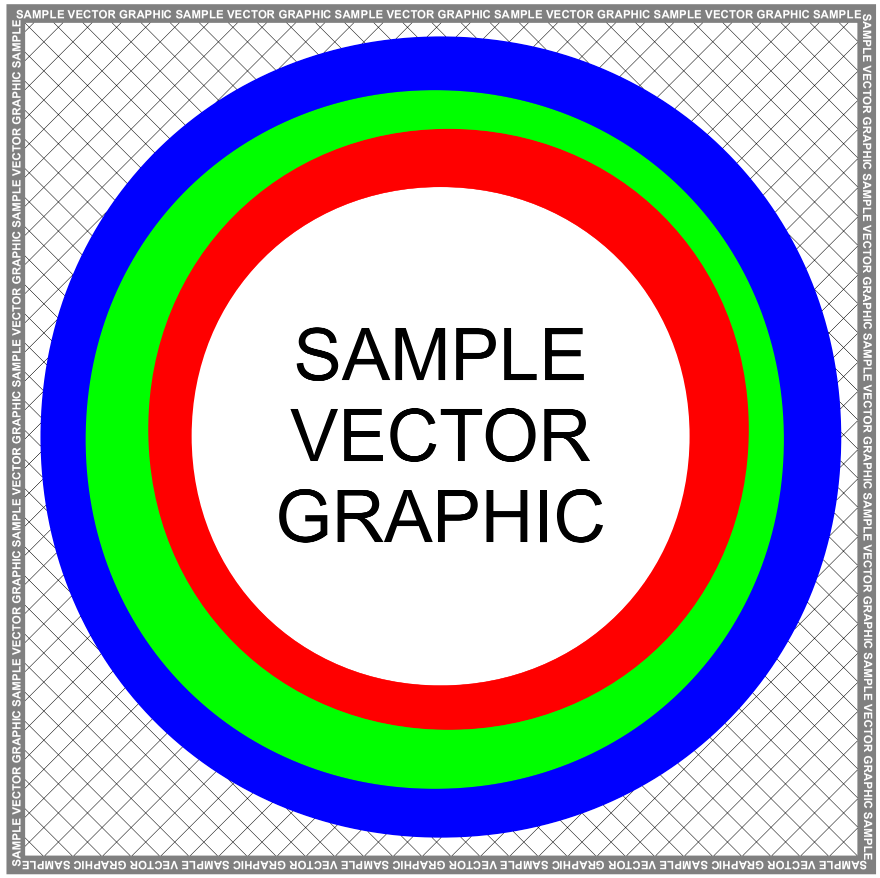

FIGURE 4.2 Insert the full caption here for this floating figure. The caption should provide sufficient context to interpret the figure. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris.

TABLE 4.2 Insert the full caption here for this floating table. The caption should provide sufficient context to interpret the table. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris.

Variable	Initial Value	Value at $t = 100$
c	0.012	3.456
δ	0.312	1.416
γ	0.042	3.252
h	0.012	3.353
c	0.012	4.446
δ	0.015	3.556
γ	0.612	6.656
h	0.072	7.456
c	0.018	8.756
δ	0.912	9.456
γ	0.092	5.956
h	0.012	2.326

euismod. Aenean vel lectus. Nunc imperdiet justo nec dolor.

4.2.2 Examples of Enumerated and Itemized Lists

Here are some citations [**?**, **?**, **?**, **?**, **?**, **?**]. The following is an enumerated list, or numbered list, with multiple levels:

1) First level item
2) First level item
    a) Second level item
    b) Second level item
        i) Third level item
            A) Fourth level item
            B) Fourth level item
        ii) Third level item
    c) Second level item
3) First level item

We draw your attention to items 1 and 3 in particular because they are very important in our study. The following is an itemized list, or unnumbered list, with multiple levels:

- First level item
- First level item

  - Second level item
  - Second level item
    - Third level item
      - Fourth level item
      - Fourth level item
    - Third level item
  - Second level item
- First level item

4.3 Some More Examples

According to [**?**], this behavior can be explained this way. Etiam euismod. Fusce facilisis lacinia dui. Suspendisse potenti. In mi erat, cursus id, nonummy sed, ullamcorper eget, sapien. Praesent pretium, magna in eleifend egestas, pede pede pretium lorem, quis consectetuer tortor sapien facilisis magna. Mauris quis magna varius nulla scelerisque imperdiet. Aliquam non quam. Aliquam porttitor quam a lacus. Praesent vel arcu ut tortor cursus volutpat. In vitae pede quis diam bibendum placerat. Fusce elementum convallis neque. Sed dolor orci, scelerisque ac, dapibus nec, ultricies ut, mi. Duis nec dui quis leo sagittis commodo.

4.3.1 Examples of Mathematical Expressions, Definitions, and Theorems

We have the following unnumbered mathematical equation:

$$E = mc^2.$$

On the other hand, the following is a numbered mathematical inequality:

$$x \leq \frac{\sum_{i=1}^{n} y^2 \cdot \mathbb{1}\left[y > 1\right]}{\int_{-\infty}^{\infty} x^3 \, \mathrm{d}z \cdot \binom{\alpha}{\beta}^{\frac{\left\lfloor \frac{a}{b} \right\rfloor}{\left\lceil \frac{c}{d} \right\rceil}}}. \tag{4.1}$$

Inequality (4.1) will be applied multiple times to prove our theorems, in a manner similar to [**?**, **?**]. We now introduce the following definition:

DEFINITION 4.1 (Name of Term Being Defined) This is the definition of the term, along with relevant conditions, trivial cases, exceptions, etc.

We can rewrite the result of [**?**, Theorem 2.5] in the following convenient form for our problem:

PROPOSITION 4.2 *For all $a, b, c \in \mathbb{Z}^+$, we have*

$$a^2 + b^3 \leq c^4.$$

Based on our numerical observations, we make the following conjecture about the upper bound:

CONJECTURE 4.3 *If $x \geq 3$ and $0 < y < x^2$, then for all $n \in \mathbb{Z}^+$,*

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n \leq T_{\text{all}}.$$

Here is a lemma that will be quite useful in deriving our results:

LEMMA 4.4 (Name of Lemma if any) *If $x, y, z \in \mathbb{Z}_0^+$, then $f(x+y+z) = 1$.*

Applying Lemma 4.4 to [**?**, Theorem 4.2] produces the following theorem:

THEOREM 4.5 (Name of Theorem if any) *If $x + y \geq z$, then*

$$\sum_{i=x}^{y} f(i) \leq z.$$

As a special case of Theorem 4.5, we have the following corollary:

COROLLARY 4.6 *If $x = 4$ and $y = z$, then $\sum_{i=x}^{y} f(i) = 5$.*

Aliquam lectus. Vivamus leo. Quisque ornare tellus ullamcorper nulla. Mauris porttitor pharetra tortor. Sed fringilla justo sed mauris. Mauris tellus. Sed non leo. Nullam elementum, magna in cursus sodales, augue est scelerisque sapien, venenatis congue nulla arcu et pede. Ut suscipit enim vel sapien. Donec congue. Maecenas urna mi, suscipit in, placerat ut, vestibulum ut, massa. Fusce ultrices nulla et nisl.

4.4 Conclusion and Future Work

Etiam ac leo a risus tristique nonummy. Donec dignissim tincidunt nulla. Vestibulum rhoncus molestie odio. Sed lobortis, justo et pretium lobortis, mauris turpis condimentum augue, nec ultricies nibh arcu pretium enim. Nunc purus neque, placerat id, imperdiet sed, pellentesque nec, nisl. Vestibulum imperdiet neque non sem accumsan laoreet. In hac habitasse platea dictumst. Etiam condimentum facilisis libero. Suspendisse in elit quis nisl aliquam dapibus. Pellentesque auctor sapien. Sed egestas sapien nec lectus. Pellentesque vel dui vel neque bibendum viverra. Aliquam porttitor nisl nec pede. Proin mattis libero vel turpis. Donec rutrum mauris et libero. Proin euismod porta felis. Nam lobortis, metus quis elementum commodo, nunc lectus elementum mauris, eget vulputate ligula tellus eu neque. Vivamus eu dolor.

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4.5 Proofs of Theorems

Remember to manually disable (and re-enable) updates to the table of contents (TOC), using

`\DisableTOCUpdates` and `\EnableTOCUpdates`,

if you want to omit subsections, tables, figures, etc., from the table of contents.

4.5.1 Proof of Lemma 4.4

Nulla mattis luctus nulla. Duis commodo velit at leo. Aliquam vulputate magna et leo. Nam vestibulum ullamcorper leo. Vestibulum condimentum rutrum mauris. Donec id mauris. Morbi molestie justo et pede. Vivamus eget turpis sed nisl cursus tempor. Curabitur mollis sapien condimentum nunc. In wisi nisl, malesuada at, dignissim sit amet, lobortis in, odio. Aenean consequat arcu a ante. Pellentesque porta elit sit amet orci. Etiam at turpis nec elit ultricies imperdiet. Nulla facilisi. In hac habitasse platea dictumst. Suspendisse viverra aliquam risus. Nullam pede justo, molestie nonummy, scelerisque eu, facilisis vel, arcu.

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■

4.5.2 Proof of Theorem 4.5

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The following lemma will be quite useful in deriving the theorem:

LEMMA 4.7 *If $a, b, c \in \mathbb{Z}$, then $g(a \cdot b \cdot c) \leq -1$.*

Proof of Lemma 4.7: Nulla non mauris vitae wisi posuere convallis. Sed eu nulla nec eros scelerisque pharetra. Nullam varius. Etiam dignissim elementum metus. Vestibulum faucibus, metus sit amet mattis rhoncus, sapien dui laoreet odio, nec ultricies nibh augue a enim. Fusce in ligula. Quisque at magna et nulla commodo consequat. Proin accumsan imperdiet sem. Nunc porta. Donec feugiat mi at justo. Phasellus facilisis ipsum quis ante. In ac elit eget ipsum pharetra faucibus. Maecenas viverra nulla in massa.

Nulla ac nisl. Nullam urna nulla, ullamcorper in, interdum sit amet, gravida ut, risus. Aenean ac enim. In luctus. Phasellus eu quam vitae turpis viverra pellentesque. Duis feugiat felis ut enim. Phasellus pharetra, sem id porttitor sodales, magna nunc aliquet nibh, nec blandit nisl mauris at pede. Suspendisse risus risus, lobortis eget, semper at, imperdiet sit amet, quam. Quisque scelerisque dapibus nibh. Nam enim. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Nunc ut metus. Ut metus justo, auctor at, ultrices eu, sagittis ut, purus. Aliquam aliquam.

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Applying Lemma 4.7 yields the following:

$$
\begin{multline}
A + B + C + D + E + F + \alpha + \beta + \gamma + \delta + \Gamma \\
\leq \Omega + \Sigma + \omega + \sigma + \Theta + \theta + \epsilon + S + T + U + V + W + X + Y + Z. \tag{4.2}
\end{multline}
$$

Finally, the desired result is obtained by substituting $A = b$ into (4.2). ■

4.6 Acknowledgment

Insert chapter acknowledgment here. Etiam suscipit aliquam arcu. Aliquam sit amet est ac purus bibendum congue. Sed in eros. Morbi non orci. Pellentesque mattis lacinia elit. Fusce molestie velit in ligula. Nullam et orci vitae nibh vulputate auctor. Aliquam eget purus. Nulla auctor wisi sed ipsum. Morbi porttitor tellus ac enim. Fusce ornare. Proin ipsum enim, tincidunt

in, ornare venenatis, molestie a, augue. Donec vel pede in lacus sagittis porta. Sed hendrerit ipsum quis nisl. Suspendisse quis massa ac nibh pretium cursus. Sed sodales. Nam eu neque quis pede dignissim ornare. Maecenas eu purus ac urna tincidunt congue.