A Spins-First Introduction to Consistent Histories and Decoherence

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Acknowledgments

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Abstract

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Contents

Acknowledgments **ii**

Abstract **iii**

1 Introduction **1**

2 Stern-Gerlach Experiments **3**

3 Postulates of Quantum Mechanics **4**
3.1 Physical Variables and State Spaces 4
3.1.1 Classical States 4
3.1.2 Quantum States 5
3.1.2.1 Postulate 1 (System States) 5
3.1.2.2 Postulate 2 (Physical Variables as Operators) 5
3.1.2.3 Postulate 3 (Observable Values) 6
3.1.3 Linearity 7
3.2 Copenhagen Description of Measurement 7
3.2.0.1 Postulate 4 (Probability Distribution of Measurement Outcomes) 7
3.2.0.2 Postulate 5 (Collapse Dynamics) 8
3.3 Dynamics 9
3.3.0.1 Postulate 6 (Unitary Dynamics) 9

4 Measurement **10**
4.1 Measurement Problem 10
4.2 von Neumann Measurement Scheme 10

5 Consistent Histories **13**
5.1 Events and Histories 13
5.2 Consistency Conditions 13

6 Decoherence **15**
6.1 Density Matrices 15

7 Probabilities **16**
7.1 Density matrix Born rule 16
7.2 Consistent Histories 16

8 Complementarity **18**

9 Wigner's Friend **19**
9.1 Standard Description 19
9.2 Consistent Description 19

List of Figures

3.1 Insert an abbreviated caption here to show in the List of Figures 8

4.1 Insert an abbreviated caption here to show in the List of Figures 12

8.1 Insert an abbreviated caption here to show in the List of Figures 18

9.1 Insert an abbreviated caption here to show in the List of Figures 20
9.2 Insert an abbreviated caption here to show in the List of Figures 20

1 Introduction

In addition to circumventing some measurement related paradoxes, our description of quantum mechanics contains other tangible advantages. We show that when simulating sequential Stern-Gerlach experiments, the program control flow becomes drastically simplified. This is a consequence of incorporating the branching structure of quantum mechanics directly into the data structure representing the quantum system, which follows naturally from the objects resulting from von Neumann measurement. Without this, the branching structure manifests through recursive loops in the program, adding unnecessary complexity.

Descriptions of measuring spin-$\frac{1}{2}$ systems in the consistent histories approach are exemplified in works by Griffiths and Hohenberg (TODO: cite). However, they either require prior knowledge of concepts in quantum foundations, or neglect implementing von Neumann measurement. We describe these concepts while developing our model of the Stern-Gerlach experiment, and show that the role of the apparatus described in von Neumann measurement must be included for a consistent description of the interaction. Additionally, von Nemann measurement is typically described using a map of states before and after the interaction (TODO: cite Schlosshauer); we introduce an explicit unitary operator that accomlishes this mapping.

In this thesis, we abandon the projection postulate, and instead adopt the *von Neumann measurement scheme* to reproduce the measurement statistics of successive Stern-Gerlach experiments. The behavior described by this scheme is permitted by all other postulates, and resolves several paradoxes that plague quantum mechanics (TODO: ref to future sections). This scheme is central to the study of *decoherence*, which describes the quantum to classical transition. The von Neumann description of measurement produces the same statistics without reference to the fifth postulate. Rather than postulating state collapse, we describe our experimental setup through an interseting Hamiltonian, through which the system evolves unitarily. Measurement is itself modeled as a physical process, rather than postulated non-unitary dynamics.

Describing measurement in this way motivates the employment of an interpretation of quantum mechanics which does not require state collapse. The *consistent histories* interpretation assigns physical meaning to the mathematical objects used to model quantum systems, and prescribes strict rules of reasoning to be used for them. The result is that a *quantum history* becomes the object for which quantum theory makes predictions, and it can be used to elegantly describe the sequential events regarding the system. It is one of many interpretations employing the *relative states* formalism, which makes von Neumann measurement central. Understanding how quantum states decay in time is accomplished by studying *decoherence*, which is also expressed in terms of the von Neumann measurement scheme.

This is accomplished by modeling the measurement apparatus itself as a quantum system and redefining the measurement interaction. This extension of quantum mechanics is necessary for answering questions in cosmology, effectively allowing state collapse to occur in the early universe [1].

Numerous papers and books describe these concepts in detail, but they have yet to permeate far outside the quantum foundations community. The goal of this thesis, then, is to introduce these concepts in a more accessible form. Following the lead of research in physics education purporting the effectiveness of a spins-first introduction to quantum mechanics, we introduce the consistent histories approach using spin-$\frac{1}{2}$ systems. Having only two degrees of freedom, these are the simplest possible systems, and all fundamental aspects of quantum mechanics can explained in the context of the Stern-Gerlach experiment.

In conclusion, we examine the *Wigner's friend* experiment in which the standard and von Neumann descriptions of measurement make different predictions. We discuss existing and future experiments that may distingush which formalism correctly represents physical reality.

2 Stern-Gerlach Experiments

TODO: provide a brief explanation of the experimental setup and results. Discuss historical and pedagogical significance. This section is background information so that I can discuss the von Neumann measurement, consistent histories, and decoherence in the context of this experiment. I am saving it for later to focus on the introducing the new theory for now.

3 Postulates of Quantum Mechanics

We first consider the mathematical objects used to model physical systems and variables. By comparing the objects used in classical and quantum mechanics, we make sense of the first three postulates of quantum mechanics. Then, we compare the Copenhagen and von Neumann descriptions of measurement and their relation to the fourth and fifth postulates.

3.1 Physical Variables and State Spaces

3.1.1 Classical States

Consider the spin of an electron. Treating the electron as a classical system, its spin state is modeled by a vector $S \in \mathbb{R}^3$:

$$S = (S_x, S_y, S_z) \tag{3.1}$$

Each component S_{x_i} is a physical variable representing the magnitude of spin oriented in the $\hat{x}_i$ direction.

S has the capacity to determine spin in any direction using the inner product of the state space $\mathbb{R}^3$:

$$S_n(S) = S \cdot \hat{n} \tag{3.2}$$

We see that in classical mechanics, physical variables are modeled using functions. Each function S_n maps a spin state S to a real scalar representing the spin of the electron aligned along the $\hat{n}$ axis.

What makes classial mechanics more familiar to everyday experience boils down to intuitive but important properties of the state space $\mathbb{R}^3$:

- For any direction $\hat{n}$, S determines spin S_n
- S_n can be any real value

S determines spin in any direction because TODO. Consequently, the sample spaces for spin in any two directions $\hat{n}$ and $\hat{m}$ are *compatible*, meaning that S_n and S_m may be simultaneously determined. Spin states in $\mathbb{R}^3$ are interpreted physically as the electron posessing definite values for every S_n at some instant in time.

In addition to spin states determing all S_n, the state space allows S_n to take on any real value. There are no fundamental restrictions on which real numbers S_n could be; its sample space is continuous and infinitely large.

3.1.2 Quantum States

Measurements of electron spin show that the intuitive classical properties do not hold. Recall that only two magnitudes of spin have ever been measured. S_n is a *quantized* physical variable; its sample space is discrete and finite.

Second, the results of successive measurements of a spin system imply that S does not determine spin in some general direction S_n. Recall the results of succesively measuring spin in orthogonal directions discussed in (TODO ref). After measuring S_x, S appears to "forget" a previous measurement of S_z. All we may know about the system at one instant in time is spin in one direction. The inabilitiy to simultaneously determine spin in two independent directions $\hat{n}$, $\hat{m}$ should be reflected through S_n and S_m having *incompatible* sample spaces.

Electron spin measurements violate the intuitive classical state space properties mentioned in 3.1.1. In response, we must change the mathematical objects used to represent system states and physical variables. Specifically, the sample space of S_n must restrict observable values to spin up and spin down, and S_n and S_m should have incompatible sample spaces. In combination, the first three postulates of quantum mechanics takes care of these differences.

Quantum mechanics postulates that a system state is completely described by a normalized vector in a linear state space.

POSTULATE 1 *The state of a physical system is defined by specifying an abstract vector $|\psi\rangle$ in a Hilbert state space $\mathcal{H}$.*

For spin-$\frac{1}{2}$ systems such as electrons, the two-dimensional Hilbert space consists of all linear combinations of spin-up and spin-down:

$$\mathcal{H} = \{\alpha\,|+\rangle + \beta\,|-\rangle\} \tag{3.3}$$

where $\alpha, \beta \in \mathbb{C}$.

$\mathcal{H}$ is an abstract state space; components of $|\psi\rangle$ cannot be interpreted as physical variables as they are for the classical spin state S. So, we introduce physical meaning with more postulates.

The second posulate of quantum mechanics states that physical variables are described by linear operators:

POSTULATE 2 *Every physical variable $\mathcal{A}$ is described by an operator A acting in $\mathcal{H}$.*

Justifying the second postulate is easiest when also considering the third postulate:

POSTULATE 3 *The only possible result of the measurement of a physical variable $\mathcal{A}$ is one of the eigenvalues of the corresponding operator A.*

The operator correlates elements of a finite sample space (eigenvalues) with particular system states (eigenstates). Consequently, a state can only be interpreted as having a definite variable value if it is an eigenstate of that variable's operator. To illustrate this, consider the operator representing S_z. Written in the basis of its own eigenstates,

$$S_z \doteq \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.4}$$

This operator correlates z spin-up $\left(S_z = \frac{\hbar}{2}\right)$ with eigenstate

$$|\psi\rangle = |+\rangle_z \doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3.5}$$

and z spin-down $\left(S_z = \frac{-\hbar}{2}\right)$ with eigenstate

$$|\psi\rangle = |-\rangle_z \doteq \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.6}$$

Similarly, the operator representing S_y written in the S_z basis is

$$S_y \doteq \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{3.7}$$

This operator correlates y spin-up $\left(S_y = \frac{\hbar}{2}\right)$ with eigenstate

$$|\psi\rangle = |+\rangle_y \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} \tag{3.8}$$

and y spin-down $\left(S_y = \frac{-\hbar}{2}\right)$ with eigenstate

$$|\psi\rangle = |-\rangle_y \doteq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \tag{3.9}$$

Operators for S_z and S_y share no common eigenstates, so no state can posses definite values for both variables. In general, operators for any two spin components S_i and S_j do not share common eigenstates with each other; in other words, S_i and S_j have incompatible sample spaces.

By representing physical variables with operators rather than functions, sample spaces become quantized and may be incompatible with each other. These features are necessary for predicting the results of electron spin measurements.

The first three postulates designate the mathematical objects used to model physical system states and variables. The fundamental differences between classical and quantum systems are completely described by these postulates and their consequences.

3.1.3 Linearity

TODO: compare addition of S_x, S_y states and their interpretations. Introduce superposition states and coherence.

3.2 Copenhagen Description of Measurement

The fourth and fifth postulates constitute the Copenhagen description of measurement. This description is a key component of the standard interpretation of quantum mechanics, taught in textbooks and introductory quantum courses worldwide.

The probability postulate assigns a probability distribution to the sample space of a physical variable.

POSTULATE 4 *When measuring physical variable A, the probability $\mathcal{P}(n)$ of obtaining result a_n corresponding to $|a\rangle_n$ is equal to*

$$\mathcal{P}(n) = |\ {}_n\langle a|\psi\rangle\ |^2 \tag{3.10}$$

This postulate is also known as the *Born Rule*, which is sometimes presented in the language of wavefunctions. The probability that a system is found at position x is the magnitude of the

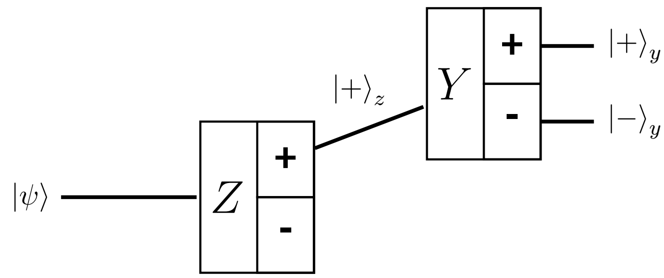

FIGURE 3.1 Demonstrating renormalizing upon measurment in standard quantum mechanics

wavefunction at that point:

$$\mathcal{P}_x = |\psi(x)|^2 \tag{3.11}$$

The probabilities assigned to each state leaving the S_x Stern-Gerlach device in Figure 4.1 are

$$\mathcal{P}_{+_y} = |_y \langle +|+\rangle|^2 = \frac{1}{2} \tag{3.12}$$

$$\mathcal{P}_{-_y} = |_y \langle -|+\rangle|^2 = \frac{1}{2} \tag{3.13}$$

The spirit of the Born Rule is unchanged in consistent quantum theory. Differences are discussed in (TODO: ref future section).

The fifth postualte (known as the projection postulate) describes how a system evolves upon measurement. Contingent upon interaction of the system with a “classical apparatus”, measurement instantaneously changes the state of the system to the eigenstate corresponding to the measurement result.

POSTULATE 5 *If the measurement of the physical variable $\mathcal{A}$ on the system in the state $|\psi\rangle$ gives the result a_n, the state of the system immediately after the measurement is the normalized projection*

$$|\psi\rangle' = \frac{P_n |\psi\rangle}{\sqrt{\langle\psi| P_n |\psi\rangle}} \tag{3.14}$$

onto the subspace associated with a_n.

$P^n{}_z = |n\rangle_z \, {}_z\langle n|$ is the projection operator for the state $|n\rangle_z$ corresponding to n_z.

Consider a measurement result for the z component of spin, S_z. We represent the result with n_z, which could be either spin-up or spin-down. The new state is the normalized projection of $|\psi\rangle$ onto $|n\rangle_z$. In other words, $|\psi\rangle$ instantaneously becomes $|n\rangle_z$ upon measurement. This process is known as *state collapse* or *wavefunction collapse*.

As an example, consider the system shown in 3.1. The first apparatus serves as a state preparation device, since we are only interested in particles exiting the spin-up output. Using the projection postulate, the state after the first measurement is

$$|\psi_{top}\rangle = \frac{P^z{}_+ |\psi\rangle}{\sqrt{\langle\psi| P^z{}_+ |\psi\rangle}} = |+\rangle_z \tag{3.15}$$

Similarly, the possible output states from the second apparatus are

$$|\psi_{top}\rangle = \frac{P^y{}_+ |+\rangle_z}{\sqrt{{}_z\langle+| P^y{}_+ |+\rangle_z}} = |+\rangle_y \tag{3.16}$$

$$\tag{3.17}$$

$$|\psi_{bottom}\rangle = \frac{P^y{}_- |+\rangle_z}{\sqrt{{}_z\langle+| P^y{}_- |+\rangle_z}} = |-\rangle_y \tag{3.18}$$

TODO: carry out example calculations using Born rule

3.3 Dynamics

TODO: introduce 6th postulate (Schrodinger time evolution)

4 Measurement

TODO

4.1 Measurement Problem

Quantum mechanics is plagued by interpretational issues surrounding measurement. The state collapse mechanism described by the TODO REF fifth postulate occurs upon “interaction with a classical measuring apparatus”. Lacking a precise definition of such an apparatus, the role of the experimentalist in measurement interactions is easily inflated. This ambiguity makes quantum mechanics exploitable for justification of anthropocentric worldviews, found in both popular and scientific literature:

> “The human observer constitutes the final link in the chain of observational processes, and the properties of any atomic object can only be understood in terms of the object’s interaction with the observer.” [2]

> “There exist external observers which cannot be treated within quantum mechanics, namely human (and perhaps animal) minds, which perform measurements on the brain causing wave function collapse.” (Schreiber’s description of the von Neumann–Wigner interpretation [3])

This attitude towards quantum measurement prompted Einstein to ask his colleague if he believed that the moon exists only when they looked at it [4]. In this thesis, we assert that the moon does exist, even when not directly observed by a human. Instead, we allow a multitude of non-living systems to continuously “measure” the moon.

Furthermore, state collapse introduces dynamics entirely seperate from the unitary dynamics postulated by the Schrödinger equation. The dynamics to be employed depend upon whether or not the system is being “measured”.

4.2 von Neumann Measurement Scheme

Using the *von Neumann measurement scheme*, we describe the measurement interaction as a unitary physical process permitted by the Schrödinger equation. We no longer have to assume separate dynamics during measurement as a fundamental component of quantum theory.

To describe the apparatus as a quantum system, a *pointer* Hilbert space is introduced to represent each measurement result. For a Stern-Gerlach apparatus, a pointer state is defined by the localization of the particle at spatially separated output regions. In general, a pointer state is some classical indicator; examples include an apparatus needle pointing up, or a particle colliding with a screen in a distinguishable region.

Let the pointer state space of the z apparatus be represented by

$$H^z{}_{\mathcal{X}} = \{\alpha \left|\mathcal{X}_+\right\rangle_z + \beta \left|\mathcal{X}_-\right\rangle_z + \gamma \left|\mathcal{X}_{ready}\right\rangle_z\} \tag{4.1}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ and

$$_z \left\langle \mathcal{X}_i | \mathcal{X}_j \right\rangle_z = \delta_{i,j} \tag{4.2}$$

Recalling the non-observability of superposition states TODO REF, we represent definite apparatus readings as pure pointer states. That is, $\left|\mathcal{X}_+\right\rangle_z$ represents the particle's localization in the spin-up region of the analyzer, $\left|\mathcal{X}_-\right\rangle_z$ the spin-down output region, and $\left|\mathcal{X}_{ready}\right\rangle_z$ anywhere else in space. These states are mutually exclusive (reflected by requiring orthonormality) and exhaustive (every position in space is encompassed by some pointer state). TODO REF addresses the question of the preferred basis problem.

We now consider $|\psi\rangle$ as the state of a composite spin-pointer system. We call our spin system $|\psi\rangle_s \in \mathcal{H}_s$, so that the composite state space is $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}^z_{\mathcal{X}}$.

The von Neumann scheme describes measurement as *entanglement* between pure pointer states and spin eigenstates. At the instant measurement begins t_0, the pointer state is $|\mathcal{X}_{ready}\rangle$ as the electron enters the magnetic field. At the instant measurement ends t_1, the pointer state is either $|\mathcal{X}_+\rangle$ or $|\mathcal{X}_-\rangle$, realized with spin-up and spin-down spin states respectively. This correlation is the desired result of unitary evolution.

Introducing the unitary operator that accomplishes this in explicit form,

$$\begin{aligned} U(t_0, t_1) = P^z{}_+ \otimes (&|\mathcal{X}_+\rangle \langle \mathcal{X}_{ready}| + |\mathcal{X}_{ready}\rangle \langle \mathcal{X}_+| + |\mathcal{X}_-\rangle \langle \mathcal{X}_-|) \\ +P^z{}_- \otimes (&|\mathcal{X}_-\rangle \langle \mathcal{X}_{ready}| + |\mathcal{X}_{ready}\rangle \langle \mathcal{X}_-| + |\mathcal{X}_+\rangle \langle \mathcal{X}_+|) \end{aligned} \tag{4.3}$$

In general, the final state is

$$U(t_0, t_1) |\psi\rangle = U(t_0, t_1) (|\psi\rangle_s \otimes |\mathcal{X}_{null}\rangle_z) \tag{4.4}$$

$$= P^z{}_+ |\psi\rangle_s \otimes |\mathcal{X}_+\rangle_z + P^z{}_- |\psi\rangle_s \otimes |\mathcal{X}_-\rangle_z \tag{4.5}$$

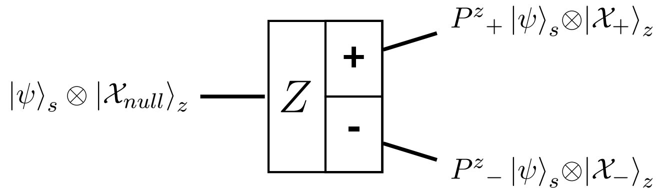

FIGURE 4.1 Schematic Diagram of von Neumann Measurement

Notice that the final sum does not contain any terms representing incorrect correlations between spin and pointer states (such as $\left|+\right\rangle_s \otimes \left|\mathcal{X}_{-z}\right\rangle$). Consequently, the final state cannot be written as the tensor product of a state in $\mathcal{H}_s$ and a state in $\mathcal{H}^z_{\mathcal{X}}$ (as the inital state was). This is the definition of *entanglement*; the von Neumann measurement scheme describes measurement as entanglement of the measured system and the apparatus.

In general, this process is described by a linear map:

$$U(t_0, t_1) : \\ \left|\psi\right\rangle = \left(\sum_n P^z{}_n \left|\psi\right\rangle_s\right) \otimes \left|\mathcal{X}_{ready}\right\rangle_z \mapsto \sum_n \left(P^z{}_n \left|\psi\right\rangle_s \otimes \left|\mathcal{X}_n\right\rangle_z\right) \tag{4.6}$$

Notice that the initial state is a single tensor product, while the final state is a sum of tensor products. The coherence initially present only in the spin state is extended to the composite spin-pointer system. TODO: mention why this allows observation of quantum effects for multiple measurements.

By discarding the projection postulate, no notion of an undefined "interaction with a classical apparatus" is required to describe how states evolve when their properties are recorded. The various paradoxes and interpretational issues associated with state collapse are entirely circumvented.

5 Consistent Histories

The TODO REF fourth postulate makes probabilistic predictions of measurement results. Now that we are using the von Neumann measurement scheme, we no longer want to frame the calculation of probabilities in the context of the Copenhagen definition of measurement. The *Consistent Histories* interpretation of quantum mechanics modifies the Born Rule to make predictions about *quantum histories* rather than measurement results.

5.1 Events and Histories

A *quantum event* represents a system's possession of a *quantum property* at a given time. For example, measuring spin-up in a Stern-Gerlach analyzer corresponds to the spin system possessing the property of "having spin magnitude $\frac{\hbar}{2}$ along the z axis" at time t_1. A property corresponds to a subspace of the Hilbert space; for the example of one spin measurement, this subspace is only the state $|+\rangle$. So, we represent an event with an operator projecting into that subspace; for measuring spin up, this operator is ${P^z}_+$.

A *quantum history* is a collection of events at sequential times [**?**]. Once again, the tensor product is employed to represent a composite system; this time, the spin-pointer system is considered at different points in time. The history Hilbert space for a single spin measurement is defined by

$$\bar{\mathcal{H}} = \mathcal{H}_{t_0} \odot \mathcal{H}_{t_1} \tag{5.1}$$

where $\odot$ is the ordinary tensor product, but denotes that the same quantum system H is considered at different times.

Histories are represented by projectors into this Hilbert space. For example, measuring spin-up in the z direction followed by measuring spin-down in the x direction would be represented by the history $Y = {P^z}_+ \odot {P^x}_-$.

5.2 Consistency Conditions

Consistent quantum theory does not assign probabilities to every history projecting into $\bar{\mathcal{H}}$. Rather, predictions are only made within the context of a set (or *family*) of histories satisfying consistency conditions. Just like our pointer system, histories in a consistent family are mutually exclusive

$$Y_i Y_j = 0 \tag{5.2}$$

and exhaustive

$$\sum_i \mathcal{P}(Y_i) = 1 \tag{5.3}$$

Once a consistent family of histories is found, we can use this new Born Rue to calculate the probability of a particular history's occurence.

is that we now compute the inner product of the time evolved state, $V\,|\psi\rangle$, and the detection state representing the result we are interested in, $|D_n\rangle$. So, we assign probabilities with the following: The probability $\mathcal{P}(n)$ of observing a result a_n at a detector D is equal to

$$\mathcal{P}(n) = |\,\langle D_n|V|\psi\rangle\,|^2 \tag{5.4}$$

By taking the inner product with a detector state, we sum the probability amplitudes of all the histories that include that detection state. Histories that include the other spin outcome are orthogonal, and contribute nothing to this sum.

Figure 4.1 illustrates this description of measurement. As physical variables with incompatible sample spaces are recorded (S_z and S_y), $|\psi\rangle$ loses information about its history; by posessing a definite S_y value, nothing can be said about S_z. However, the detector states are not impacted by these measurements, and the *event history* of the state is preserved. Our system now belongs to a *history Hilbert space* defined by $\widetilde{\mathcal{H}} = \mathcal{H} \otimes D_z \otimes D_y$.

Each pure state in $\widetilde{\mathcal{H}}$ represents a *history* (or sequence of events). Because of the orthonormality conditions imposed on detector states, these histories satisfy *consistency conditions*; that is, they are disjoint and their magnitudes sum to 1. Together, these pure states are a *consistent family of histories*, meaning that they constitue an exhaustive and disjoint set of event sequences.

6 Decoherence

All remaining content are rough notes.

TODO: introduce decoherence, explain role played by vnms. Incomplete which path, Environment continuously monitors system. TODO: sometimes we must ignore the environment, results are inaccessible or ignored experimentally.

6.1 Density Matrices

As stated in TODO ref, $|\psi\rangle$ can no longer be written in the form $|\psi\rangle_s \otimes |\mathcal{X}\rangle_z$. Rather, $|\psi\rangle$ is a superposition of such states; quantum *coherency* has been extended from the spin system to the spin-pointer composite system.

In the case of environmental decoherence, we consider this type of interaction while possessing no information on the pointer system. Since multiple pointer states may correspond to the same spin state, ”ignoring the environment” now means that we must coarse grain out all possible environment states for each spin state. We can no longer “factor out” the environment subsystem.

Some environment state is realized; we just do not know which one. This uncertainty is classical in nature; it has nothing to do with any inherent quantum uncertainty. We are now dealing with a ”classical mixture” of superposition states.

Such a system is well represented by a *density matrix*. For a *pure state*, the representative density matrix is just the projection operator for that state

$$\rho = |\psi\rangle \langle\psi| \tag{6.1}$$

Recall that when considering the projection operator of some state, we can equivalently think of the subset of states in the Hilbert space into which it projects.

The density operator for the state after measurement is

$$\rho = \sum_{n,m} ({P^z}_n |\psi\rangle_s \otimes |\mathcal{X}_n\rangle_z) \cdot ({P^z}_m \langle\psi| \otimes \langle\mathcal{X}_m|) \tag{6.2}$$

$$\rho = \sum_{n,m} \langle n|\psi\rangle_s \langle m|\psi\rangle_s |n\rangle \langle m| \otimes |\mathcal{X}_n\rangle_z \langle\mathcal{X}_m| \tag{6.3}$$

7 Probabilities

7.1 Density matrix Born rule

The Born Rule can also be expressed in terms of *density matrices*. From (TODO: reference appendix), we know that an inner product can be written as the trace of the corresponding dyad. The Born Rule is the complex square of an inner product, so we should be able to assign probabilities to detection states by tracing over some corresponding tensor. We will manipulate the Born Rule to take this form, and then examine the resulting tensor.

Expanding the complex square,

$$\mathcal{P}(n) = (\langle D_n| V |\psi\rangle) \cdot (\langle D_n| V |\psi\rangle)^* \tag{7.1}$$

$$\mathcal{P}(n) = (\langle D_n| V |\psi\rangle) \cdot \left(\langle\psi| V^\dagger |D_n\rangle\right) \tag{7.2}$$

Rewriting the second inner product as the trace of a dyad,

$$\mathcal{P}(n) = (\langle D_n| V |\psi\rangle) \cdot Tr\left(|D_n\rangle \langle\psi| V^\dagger\right) f \tag{7.3}$$

The remaining inner product, like any, is a scalar. Since trace is a linear operator, we can scale any factor inside the operation by this inner product.

$$\mathcal{P}(n) = Tr\left(|D_n\rangle \cdot (\langle D_n| V |\psi\rangle) \cdot \left(\langle\psi| V^\dagger\right)\right) \tag{7.4}$$

Now we can rewrite $|D_n\rangle \langle D_n|$ as ${P^D}_n$, and simplify grouping:

$$\mathcal{P}(n) = Tr\left({P^D}_n \cdot \left(V |\psi\rangle \langle\psi| V^\dagger\right)\right) \tag{7.5}$$

TODO: discuss remaining object ${P^D}_n \cdot \left(V |\psi\rangle \langle\psi| V^\dagger\right)$ TODO: discuss coarse graining, conditional probabilities

7.2 Consistent Histories

The von Neumann measurement scheme describes the measurement interaction between the electron spin system and the Stern-Gerlach apparatus as an entanglement between spin eigenstates and pointer states. TODO: This leads to the expected dynamics without state collapse.

How do we discuss probabilities of sequential measurements? Issue with first postulate: a state contains all information known.

8 Complementarity

We now discuss the the principle of complementarity in the context of standard and consistent quantum mechanics. Arguably the most fundamental feature of quantum mechanics, the principle of complementarity states that a quantum system has pairs of physical observables which cannot be measured simultaneously. The operators corresponding to pairs of complementary properties do not commute; that is, $[A, B] = AB - BA \neq 0$. Components of spin on orthogonal axes are complementary properties, so we examine measurements of succesive Stern-Gerlach experiments.

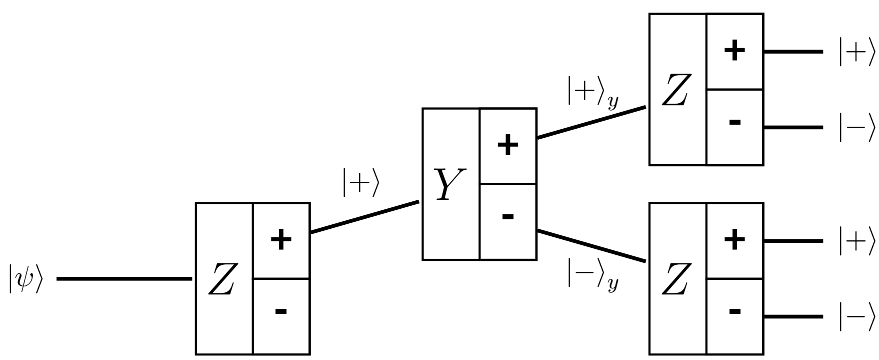

FIGURE 8.1 Demonstrating complementary measurments in standard quantum mechanics

9 Wigner's Friend

9.1 Standard Description

Using the Born Rule, we calculate the probabilities of observing each final state in Figure 4.1. The first apparatus serves as a state preparation device with output $|+\rangle$. By the direction of the projection postulate, the state is renormalized upon each measurement. After measuring a property complementary to what is known (such as spin along x, knowing spin along z), any information known about the input state is lost; the input state instantaneously changes to the state corresponding to the observed quantity. Consequently, there is an equal probability of observing the final state as $|+\rangle$ or $|-\rangle$ at either final apparatus, even though the state was initially prepared as $|+\rangle$, since

$$\begin{aligned}
\mathcal{P}_n &= |\langle +|+\rangle_y|^2 && (9.1)\\
&= |\langle -|+\rangle_y|^2 && (9.2)\\
&= |\langle +|-\rangle_y|^2 && (9.3)\\
&= |\langle -|-\rangle_y|^2 && (9.4)\\
&= \frac{1}{4} && (9.5)
\end{aligned}$$

TODO: make above separate equations for clarity. It appears that this contradiction with classical intuition is a direct result of the projection postulate. The act of measurement and ensuing state collapse causes the system to shed properties previously recorded.

9.2 Consistent Description

TODO: calculate probabilities using consistent Born Rule. Consistent quantum theory predicts the same loss of a definite S_z value, but for different reasons. The description of measurment in consistent histories does not postulate state collapse; rather, the system evolves through some Hamiltonian that correlates system and detector states. This implies that the measurement process has nothing to do with the principle of complementarity. We can trace the cause back to our definition of the state space. For a spin state, $|\psi\rangle$ is completely defined by spin-up or spin-down in a single direction w. The Hilbert space does not include states that could be interpreted as possesing a definite spin value in more than one direction. Consequently, the operators for spin in directions

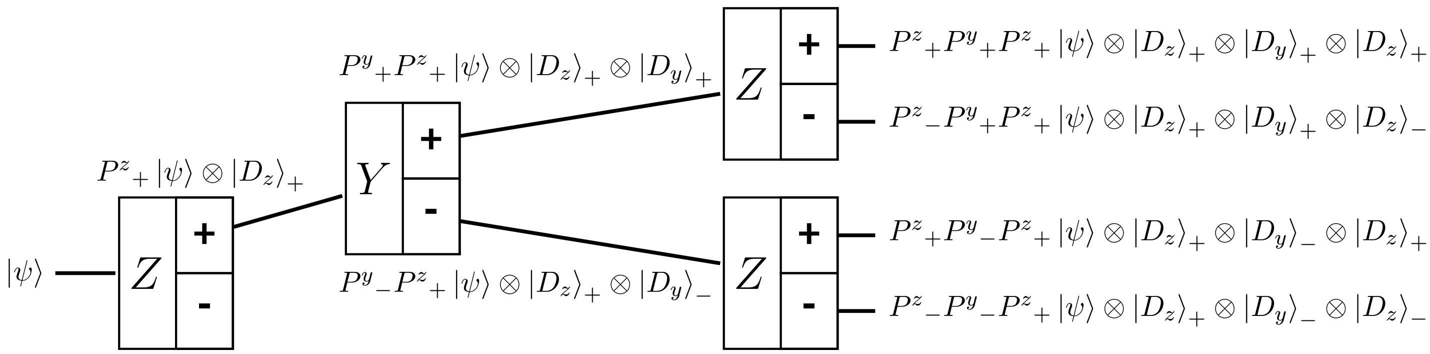

FIGURE 9.1 Demonstrating complementary measurments in consistent quantum mechanics

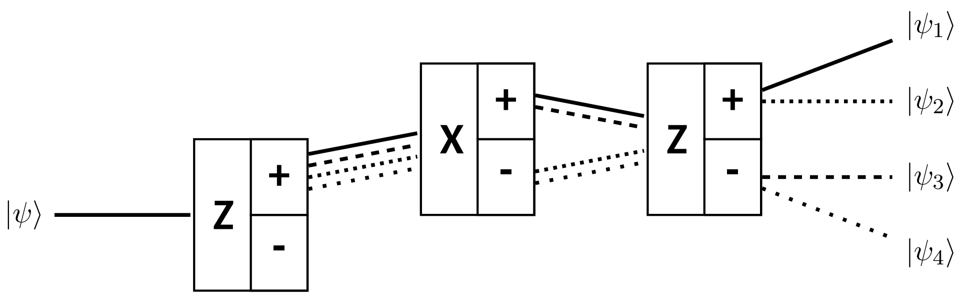

FIGURE 9.2 TODO: create section to discuss this example

not parallel or antiparallel to each other share no eigenstates. It follows mathematically that these operators do not commute: TODO run through this math.

This description of complementary implies that the principle is a limitation inherent to the quantum state, rather than a consequence of the role of measurement. In consistent histories, this limitation is embodied by *the single framework rule*. There exists multiple ways in which a quantum system can be described, yet descriptions from only one of these *frameworks* can be meaningfully compared or combined.

TODO: discuss any structural or algorithmic changes resulting from rewriting spins simulation

TODO: discuss any structural or algorithmic changes resulting from rewriting spins simulation

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