A Spins-First Introduction to Consistent Histories and Decoherence

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Acknowledgments

TODO: insert acknowledgments

Abstract

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1 Introduction

Quantum mechanics is plagued by interpretational issues surrounding measurement. The standard description of measurment postualates a special type of dynamics in which a quantum system instantaneously evolves upon measurement. The conditions in which this postulate applies are not well defined, leading to confusion on the nature of measurement itself.

The predictions of the standard description of measurement can be reproduced using alternative descriptions of the measurement process. We study an alternative description that explains many, but not all, of the infamous measurement related problems. The *von Neumann measurement scheme* is used to describe how states evolve when being measured, and the *consistent* or *decoherent histories* interpretation of quantum mechanics is used to explain what is happening physically.

Numerous papers and books describe these concepts in detail TODO CITE, but they have yet to permeate far outside the quantum foundations community. A primary goal of this thesis is to introduce these concepts in a form more accessible to those new in their study of quantum foundations or physicists with other specialties. Following the lead of research in spins-first introductions to quantum mechancis in physics education TODO CITE, we introduce these new ideas in the context of the Stern-Gerlach experiment. Having only two degrees of freedom, spin-$\frac{1}{2}$ systems are the simplest possible. We explain all fundamental aspects of quantum mechanics within this context, as well as our proposed changes.

Descriptions of measuring spin-$\frac{1}{2}$ systems in this way are exemplified in works by Griffiths, Hohenberg, and Schlosshauer TODO CITE. However, they either require prior knowledge of concepts in quantum foundations, neglect implementing von Neumann measurement, or do not offer interpretational explanations. Our explanation of Stern-Gerlach experiments does all of these things. Another primary goal of this thesis is showing how to implement these ideas explicitly. In doing so, we introduce a unitary operator describing measurement specific to ideal measurement of spin-$\frac{1}{2}$ systems. Using the same tools, we describe how states decay in time through *decoherence*.

In addition to circumventing some measurement related paradoxes, our description of quantum mechanics contains other tangible advantages. We exemplify this by comparing the simulation of measurement in both frameworks. Existing code simulating sequential Stern-Gerlach measurements is used as a baseline TODO CITE, and relevant code is rewritten using our new formalism. The resulting program control flow becomes drastically simplified. We conclude by discussing existing and future experiments that may distinguish whether the standard or *relative states* formalisms correctly represent physical reality.

2 Stern-Gerlach Experiments

TODO: provide a brief explanation of the experimental setup and results. Discuss historical and pedagogical significance. Explain how measurement results correspond to the particle's localization in certain regions. This section is background information so that I can discuss the von Neumann measurement, consistent histories, and decoherence in the context of this experiment. I am saving it for later to focus on the introducing the new theory for now.

3 Postulates of Quantum Mechanics

We first consider the mathematical objects used to model physical systems and variables. By comparing the objects used in classical and quantum mechanics, we make sense of the first three postulates of quantum mechanics. Then, we compare the Copenhagen and von Neumann descriptions of measurement and their relation to the fourth and fifth postulates.

3.1 Physical Variables and State Spaces

3.1.1 Classical States

Consider the spin of an electron. Treating the electron as a classical system, its spin state is modeled by a vector $\vec{S} \in \mathbb{R}^3$:

$$\vec{S} = (S_x, S_y, S_z) \tag{3.1}$$

Each component S_{x_i} is a physical variable representing the magnitude of spin oriented in the $\hat{x}_i$ direction.

$\vec{S}$ has the capacity to determine spin in any direction using the inner product of the state space $\mathbb{R}^3$:

$$S_n(S) = \vec{S} \cdot \hat{n} \tag{3.2}$$

We see that in classical mechanics, physical variables are modeled using functions. Each function S_n maps a spin state [$\vec{S}$ to a real scalar representing the spin of the electron aligned along the $\hat{n}$ axis.

What makes classial mechanics more familiar to everyday experience boils down to intuitive but important properties of the state space $\mathbb{R}^3$:

- For any direction $\hat{n}$, $\vec{S}$ determines spin S_n
- S_n can be any real value

$\vec{S}$ determines spin in any direction because TODO. Consequently, the sample spaces for spin in any two directions $\hat{n}$ and $\hat{m}$ are *compatible*, meaning that S_n and S_m may be simultaneously determined. Spin states in $\mathbb{R}^3$ are interpreted physically as the electron posessing definite values for every S_n at some instant in time.

In addition to spin states determing all S_n, the state space allows S_n to take on any real value. There are no fundamental restrictions on which real numbers S_n could be; its sample space is continuous and infinitely large.

3.1.2 Quantum States

Measurements of electron spin show that the intuitive classical properties do not hold. Recall that only two magnitudes of spin have ever been measured. S_n is a *quantized* physical variable; its sample space is discrete and finite.

Second, the results of successive measurements of a spin system imply that $\vec{S}$ does not determine spin in some general direction S_n. Recall the results of succesively measuring spin in orthogonal directions discussed in (TODO ref). After measuring S_x, $\vec{S}$ appears to "forget" a previous measurement of S_z. All we may know about the system at one instant in time is spin in one direction. The inabilitiy to simultaneously determine spin in two independent directions $\hat{n}$, $\hat{m}$ should be reflected through S_n and S_m having *incompatible* sample spaces.

Electron spin measurements violate the intuitive classical state space properties mentioned in 3.1.1. In response, we must change the mathematical objects used to represent system states and physical variables. Specifically, the sample space of S_n must restrict observable values to spin up and spin down, and S_n and S_m should have incompatible sample spaces. In combination, the first three postulates of quantum mechanics takes care of these differences.

Quantum mechanics postulates that a system state is completely described by a normalized vector in a linear state space.

POSTULATE 1 *The state of a physical system is defined by specifying an abstract vector $|\psi\rangle$ in a Hilbert state space $\mathcal{H}$.*

For spin-$\frac{1}{2}$ systems such as electrons, the two-dimensional Hilbert space consists of all linear combinations of spin-up and spin-down:

$$\mathcal{H} = \{\alpha\,|+\rangle + \beta\,|-\rangle\} \tag{3.3}$$

where $\alpha, \beta \in \mathbb{C}$.

$\mathcal{H}$ is an abstract state space; components of $|\psi\rangle$ cannot be interpreted as physical variables as they are for the classical spin state S. So, we introduce physical meaning with more postulates.

The second posulate of quantum mechanics states that physical variables are described by linear operators:

POSTULATE 2 *Every physical variable $\mathcal{A}$ is described by an operator A acting in $\mathcal{H}$.*

Justifying the second postulate is easiest when also considering the third postulate:

POSTULATE 3 *The only possible result of the measurement of a physical variable $\mathcal{A}$ is one of the eigenvalues of the corresponding operator A.*

The operator correlates elements of a finite sample space (eigenvalues) with particular system states (eigenstates). Consequently, a state can only be interpreted as having a definite variable value if it is an eigenstate of that variable's operator. To illustrate this, consider the operator representing S_z. Written in the basis of its own eigenstates,

$$S_z \doteq \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.4}$$

This operator correlates z spin-up $\left(S_z = \frac{\hbar}{2}\right)$ with eigenstate

$$|\psi\rangle = |+\rangle_z \doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3.5}$$

and z spin-down $\left(S_z = \frac{-\hbar}{2}\right)$ with eigenstate

$$|\psi\rangle = |-\rangle_z \doteq \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.6}$$

Similarly, the operator representing S_y written in the S_z basis is

$$S_y \doteq \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{3.7}$$

This operator correlates y spin-up $\left(S_y = \frac{\hbar}{2}\right)$ with eigenstate

$$|\psi\rangle = |+\rangle_y \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} \tag{3.8}$$

and y spin-down $\left(S_y = \frac{-\hbar}{2}\right)$ with eigenstate

$$|\psi\rangle = |-\rangle_y \doteq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \tag{3.9}$$

Operators for S_z and S_y share no common eigenstates, so no state can posses definite values for both variables. In general, operators for any two spin components S_i and S_j do not share common eigenstates with each other; in other words, S_i and S_j have incompatible sample spaces.

By representing physical variables with operators rather than functions, sample spaces become quantized and may be incompatible with each other. These features are necessary for predicting the results of electron spin measurements.

The first three postulates designate the mathematical objects used to model physical system states and variables. The fundamental differences between classical and quantum systems are completely described by these postulates and their consequences.

3.1.3 Linearity

TODO: compare addition of S_x, S_y states and their interpretations. Introduce superposition states and coherence.

3.2 Copenhagen Description of Measurement

The fourth and fifth postulates constitute the Copenhagen description of measurement. This description is a key component of the standard interpretation of quantum mechanics, taught in textbooks and introductory quantum courses worldwide.

The probability postulate, also known as the *Born Rule*, assigns a probability distribution to the sample space of a physical variable.

POSTULATE 4 *When measuring physical variable A, the probability $\mathcal{P}(n)$ of obtaining result a_n corresponding to $|a_n\rangle$ is equal to*

$$\mathcal{P}(n) = |\langle_n a|\psi\rangle|^2 \tag{3.10}$$

The probabilities assigned to each state leaving the S_x Stern-Gerlach device in Figure 4.1 are

$$\mathcal{P}_{+_y} = |_y\langle+|+\rangle|^2 = \frac{1}{2} \tag{3.11}$$

$$\mathcal{P}_{-_y} = |_y\langle-|+\rangle|^2 = \frac{1}{2} \tag{3.12}$$

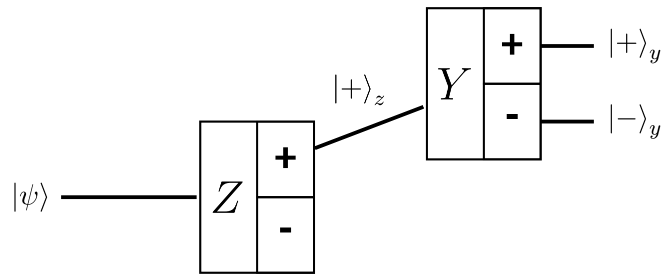

FIGURE 3.1 The Stern-Gerlach experiment as described by the standard measurement scheme. Notice that each measurement outcome is renormalized, so that information about the state prior to measurement is lost.

The spirit of the Born Rule is unchanged in consistent quantum theory. Differences are discussed in (TODO: ref future section).

The fifth postualte (known as the projection postulate) describes how a system evolves upon measurement. Contingent upon interaction of the system with a "classical apparatus", measurement instantaneously changes the state of the system to the eigenstate corresponding to the measurement result.

POSTULATE 5 *If the measurement of the physical variable $\mathcal{A}$ on the system in the state $|\psi\rangle$ gives the result a_n, the state of the system immediately after the measurement is the normalized projection*

$$|\psi\rangle' = \frac{P_n |\psi\rangle}{\sqrt{\langle\psi| P_n |\psi\rangle}} \tag{3.13}$$

onto the subspace associated with a_n.

${P^n}_z = |n\rangle_z \, {}_z\langle n|$ is the projection operator for the state $|n\rangle_z$ corresponding to n_z.

Consider a measurement result for the z component of spin, S_z. We represent the result with n_z, which could be either spin-up or spin-down. The new state is the normalized projection of $|\psi\rangle$ onto $|n\rangle_z$. In other words, $|\psi\rangle$ instantaneously becomes $|n\rangle_z$ upon measurement. This process is known as *state collapse* or *wavefunction collapse*.

As an example, consider the system shown in 3.1. The first apparatus serves as a state preparation device, since we are only interested in particles exiting the spin-up output. Using the projection postulate, the state after the first measurement is

$$|\psi_{top}\rangle = \frac{{P^z}_+ |\psi\rangle}{\sqrt{\langle\psi| {P^z}_+ |\psi\rangle}} = |+\rangle_z \tag{3.14}$$

Similarly, the possible output states from the second apparatus are

$$|\psi_{top}\rangle = \frac{P^y{}_+ \, |+\rangle_z}{\sqrt{{}_z\langle +| \, P^y{}_+ \, |+\rangle_z}} = |+\rangle_y \tag{3.15}$$

$$\tag{3.16}$$

$$|\psi_{bottom}\rangle = \frac{P^y{}_- \, |+\rangle_z}{\sqrt{{}_z\langle +| \, P^y{}_- \, |+\rangle_z}} = |-\rangle_y \tag{3.17}$$

TODO: carry out example calculations using Born rule

3.3 Dynamics

TODO: introduce 6th postulate (Schrodinger time evolution)

4 Measurement

Quantum mechanics is plagued by interpretational issues surrounding measurement. The state collapse mechanism described by the projection postulate applies upon "interaction with a classical measuring apparatus". Lacking a precise definition of such an apparatus, the role of the experimentalist in measurement interactions is easily inflated. This ambiguity makes quantum mechanics exploitable for justification of anthropocentric worldviews, found in both popular and scientific literature. For example, the von Neumann-Wigner interpretation of quantum mechanics asserts that

> "There exist external observers which cannot be treated within quantum mechanics, namely human (and perhaps animal) minds, which perform measurements on the brain causing wave function collapse." (Schreiber's description of the von Neumann–Wigner interpretation [1]).

This attitude towards quantum measurement prompted Einstein to ask his colleague if they believed that the moon existed only when they looked at it [2]. In this thesis, we assert that the moon does exist, even when not directly observed by a human. Instead, we allow a multitude of non-living systems to continuously "measure" the moon. This is done using the *von Neumann measurement scheme*, which describes measurement as a physical process involving two quatum systems, neither of which need be human or a "classical measuring apparatus". We describe this scheme first in the context of the Stern-Gerlach experiment to demonstrate its difference from the standard description.

4.1 von Neumann Measurement Scheme

In a mechanical theory, the equations of motion (or *dynamics*) describe how a state evolves with time. In classical Newtonian mechanics, this equation is given by Newton's law of motion $\vec{F} = m\vec{a}$. These dynamics are *unitary*, meaning that given a final state of a physical process, the corresponding initial state is recovered by applying the dynamics with time reversed.

In quantum mechaincs, the analgous dynamics are postulated by the Schrödinger equation. State collapse is an entirely seperate non-unitary dynamics. The dynamics to be employed depend upon whether or not the system is being "measured". Upon measurement, all previous information about the state is lost as the state instantaneously becomes an eigenstate of the measured variable.

Because this process cannot be reversed, state collapse injects time asymmetry into the very foundations of quantum mechanics.

Describing measurement as a unitary process is desirable for multiple reasons:

- With dynamics symmetric in time, the emergence of the "arrow of time" can be studied
- Humans and measurement apparatuses do not play a special role indescribale by the theory
- TODO: describe cosmology benefit.

Fortunately, such a description exists in the von Neumann measurement scheme. The measurement interaction is a unitary physical process, so we no longer have to assume separate dynamics during measurement as a fundamental component of quantum theory.

Considering the apparatus itself as a quantum system, an additional Hilbert space $\mathcal{H}_D$ is introduced. We call this the *detector* or *pointer space*, as it represents some classical indicator. $|\psi_D\rangle$ could represent an apparatus needle pointing up, or a particle colliding with a screen in some distinguishable region.

4.1.1 Example 1

Revisiting the example of measuring spin along the z axis, we define the pointer space as

$$
\begin{aligned}
&|\psi_D\rangle \in \mathcal{H}_D, \\
&|\psi_D\rangle = \alpha\,|+_D\rangle + \beta\,|-_D\rangle + \gamma\,|\varnothing_D\rangle
\end{aligned}
\tag{4.1}
$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ and

$$
\langle i_D | j_D \rangle = \delta_{i,j} \tag{4.2}
$$

The D subscript distinguishes pointer states from the spin states. Recalling the non-observability of superposition states TODO REF, we represent definite apparatus readings as pure pointer states. That is, $|+_D\rangle$ represents the particle's localization in the spin-up region of the analyzer, $|-_D\rangle$ the spin-down output region, and $|\varnothing_D\rangle$ anywhere else in space. $|\varnothing_D\rangle$ represents the state of the apparatus when it has not measured anything, referred to in literature as the "ready state" [**?**]. These states are mutually exclusive (reflected by requiring orthonormality) and exhaustive (every position in space is encompassed by some pointer state).

The total quantum system $|\psi\rangle$ is now a composite spin-pointer system. Naming our spin system $|\psi_s\rangle \in \mathcal{H}_s$, the composite state space is $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_D$.

The von Neumann scheme describes measurement as *entanglement* of pure pointer states and spin eigenstates. At the instant measurement begins t_0, the pointer state is $|\varnothing_D\rangle$ as the electron

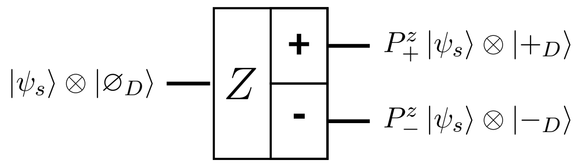

FIGURE 4.1 The Stern-Gerlach experiment as described by the von Neumann measurement scheme. Each measurement outcome corresponds to a term in the time-evolved state (TODO REF). Notice that the measurement interaction results in a branching structure, represented here as a tree graph with the apparatus as a node.

enters the magnetic field. At the instant measurement ends t_1, the pointer state is either $|+_D\rangle$ or $|-_D\rangle$, realized with spin-up and spin-down spin states respectively. This correlation is the desired result of unitary evolution.

We introduce a unitary operator that accomplishes this in explicit form:

$$\begin{aligned} U(t_0, t_1) = P^z_+ \otimes (\, |+_D\rangle \langle\varnothing_D| \;+\; |\varnothing_D\rangle \langle +_D| \;+\; |-_D\rangle \langle -_D| \,) \\ + P^z_- \otimes (\, |-_D\rangle \langle\varnothing_D| \;+\; |\varnothing_D\rangle \langle -_D| \;+\; |+_D\rangle \langle +_D| \,) \end{aligned} \tag{4.3}$$

This is the unitary operator accomplishing entanglement, as evident by $UU^\dagger = I$.

In general, the final state is

$$\begin{aligned} U(t_0, t_1) |\psi\rangle &= U(t_0, t_1) (|\psi_s\rangle \otimes |\varnothing_D\rangle) \\ &= P^z_+ |\psi_s\rangle \otimes |+_D\rangle \;+\; P^z_- |\psi_s\rangle \otimes |-_D\rangle \end{aligned} \tag{4.4}$$

This process is represented schematically in TOD ref; the initial state branches into two distinct outcomes, each represented by a term in the final state.

Notice that the final sum does not contain any terms representing incorrect correlations between spin and pointer states (such as $P^z_+ |\psi_s\rangle \otimes |-_D\rangle$). Consequently, the final state cannot be written as the tensor product of a state in $\mathcal{H}_s$ and a state in $\mathcal{H}_D$ (as the inital state was). This is the definition of entanglement; the von Neumann measurement scheme describes measurement as entanglement of the measured system and the apparatus.

In general, the measurement process is described by a linear map:

$$\begin{aligned} &U(t_0, t_1) : \\ &\qquad |\psi\rangle = \left(\sum_n P^z_n |\psi_s\rangle \right) \otimes |\varnothing_D\rangle \mapsto \sum_n (P^z_n |\psi_s\rangle \otimes |n_D\rangle) \end{aligned} \tag{4.5}$$

where $n = +, -$.

Notice that the initial state is a single tensor product, while the final state is a sum of tensor

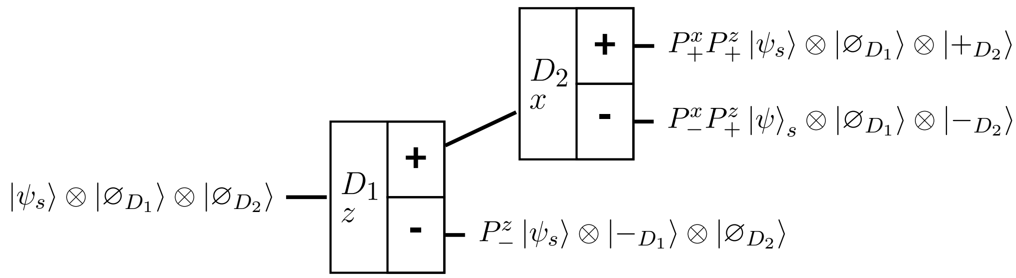

FIGURE 4.2 The Stern-Gerlach experiment as described by the von Neumann measurement scheme. Each measurement outcome corresponds to a term in the time-evolved state (Eq 4.4). Notice that the measurement interaction results in a branching structure, which is represented here visually with a tree graph.

products. The coherence initially present only in the spin state is extended to the composite spin-pointer system. TODO: elaborate, mention why this allows observation of quantum effects for multiple measurements.

4.1.2 Example 2

Now a case of consecutive measurements is considered. The von Neumann measurement scheme can be applied succesively; depending on the experimental setup, it is applied to either the entire resultant state, or only specific branches of it. In this example, spin along the x axis is only measured if we first measure spin-up along the z axis, so we only apply the measurement scheme to the term projecting onto $|+\rangle$.

Now that we have two apparatuses, the Hilbert space includes two pointer spaces: $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_{D_1} \otimes \mathcal{H}_{D_2}$. The dynamics are unchanged from TODO REF Example 1 for the first measurement, with the identity acting on the second apparatus to leave it unaffected:

$$\begin{aligned} U(t_0,t_1) = P_+^z &\otimes \left(|+_{D_1}\rangle\langle\varnothing_{D_1}| \;+\; |\varnothing_{D_1}\rangle\langle +_{D_1}| \;+\; |-_{D_1}\rangle\langle -_{D_1}|\right) \otimes I_{D_2} \\ + P_-^z &\otimes \left(|-_{D_1}\rangle\langle\varnothing_{D_1}| \;+\; |\varnothing_{D_1}\rangle\langle -_{D_1}| \;+\; |+_{D_1}\rangle\langle +_{D_1}|\right) \otimes I_{D_2} \end{aligned} \tag{4.6}$$

Now we determine the dynamics for the second measurement. We expect the dynamics to do two things: return the first apparatus to the ready state, and entangle the second apparatus with spin eigenstates. We return the first apparatus to the ready state as it is no longer measuring the system. $U(t_0,t_1)^\dagger$ reverses the entanglement, but conveniently, $U(t_0,t_1)$ is Hermitian $\left(U(t_0,t_1)^\dagger = U(t_0,t_1)\right)$. So, the D_1 component of the unitary operator for the second measurement is the same as that of the first measurement:

$$\begin{aligned} U(t_1,t_2)_{D_1} = P_+^z &\otimes \left(|+_{D_1}\rangle\langle\varnothing_{D_1}| \;+\; |\varnothing_{D_1}\rangle\langle +_{D_1}| \;+\; |-_{D_1}\rangle\langle -_{D_1}|\right) \otimes I_{D_2} \\ + P_-^z &\otimes \left(|-_{D_1}\rangle\langle\varnothing_{D_1}| \;+\; |\varnothing_{D_1}\rangle\langle -_{D_1}| \;+\; |+_{D_1}\rangle\langle +_{D_1}|\right) \otimes I_{D_2} \end{aligned} \tag{4.7}$$

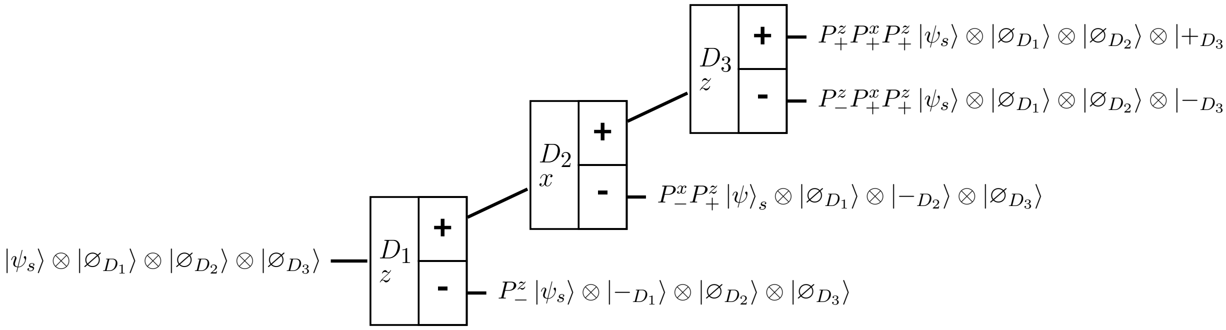

FIGURE 4.3 The Stern-Gerlach experiment as described by the von Neumann measurement scheme. Each measurement outcome corresponds to a term in the time-evolved state (Eq 4.4). Notice that the measurement interaction results in a branching structure, which is represented here visually with a tree graph.

For the D_2 component, we want to entangle pointer states with spin states only when the first apparatus has measured spin-up. In the branch where spin-down has been measured, we use the identity operator $I_s \otimes I_{D_2}$, representing the lack of a second measurement. In the branch where spin-up has been measured, we apply the measurement scheme again to entangle D_2 states with x spin eigenstates. The D_2 component of the unitary operator during second measurement is

$$\begin{aligned} U(t_1,t_2)_{D_2} = {} & P_+^x P_+^z \otimes I_{D_1} \otimes \left(\left|+_{D_2}\right\rangle \left\langle\varnothing_{D_2}\right| \,+\, \left|\varnothing_{D_2}\right\rangle \left\langle+_{D_2}\right| \,+\, \left|-_{D_2}\right\rangle \left\langle-_{D_2}\right| \right) \\ & + P_-^x P_+^z \otimes I_{D_1} \otimes \left(\left|-_{D_2}\right\rangle \left\langle\varnothing_{D_2}\right| \,+\, \left|\varnothing_{D_2}\right\rangle \left\langle-_{D_2}\right| \,+\, \left|+_{D_2}\right\rangle \left\langle+_{D_2}\right| \right) \\ & + P_-^z \otimes I_{D_1} \otimes I_{D_2} \end{aligned} \tag{4.8}$$

So the complete unitary operator for the second measurement is

$$\begin{aligned} U(t_1,t_2) &= U(t_1,t_2)_{D_1} U(t_1,t_2)_{D_2} \\ &= P_+^x P_+^z \otimes \left(\left|+_{D_1}\right\rangle \left\langle\varnothing_{D_1}\right| \,+\, \left|\varnothing_{D_1}\right\rangle \left\langle+_{D_1}\right| \,+\, \left|-_{D_1}\right\rangle \left\langle-_{D_1}\right| \right) \\ &\qquad \otimes \left(\left|+_{D_2}\right\rangle \left\langle\varnothing_{D_2}\right| \,+\, \left|\varnothing_{D_2}\right\rangle \left\langle+_{D_2}\right| \,+\, \left|-_{D_2}\right\rangle \left\langle-_{D_2}\right| \right) \\ &+ P_-^x P_+^z \otimes \left(\left|+_{D_1}\right\rangle \left\langle\varnothing_{D_1}\right| \,+\, \left|\varnothing_{D_1}\right\rangle \left\langle+_{D_1}\right| \,+\, \left|-_{D_1}\right\rangle \left\langle-_{D_1}\right| \right) \\ &\qquad \otimes \left(\left|-_{D_2}\right\rangle \left\langle\varnothing_{D_2}\right| \,+\, \left|\varnothing_{D_2}\right\rangle \left\langle-_{D_2}\right| \,+\, \left|+_{D_2}\right\rangle \left\langle+_{D_2}\right| \right) \\ &+ P_-^z \otimes \left(\left|-_{D_1}\right\rangle \left\langle\varnothing_{D_1}\right| \,+\, \left|\varnothing_{D_1}\right\rangle \left\langle-_{D_1}\right| \,+\, \left|+_{D_1}\right\rangle \left\langle+_{D_1}\right| \right) \\ &\qquad \otimes I_{D_2} \end{aligned} \tag{4.9}$$

4.1.3 Example 3

4.2 Measurement Problem

TODO describe which measurement problems we have solved, which ones we haven't.

5 Consistent Histories

The fourth postulate TODO REF makes probabilistic predictions of measurement results. The mathematics of this postulate still apply, but now that we are using the von Neumann measurement scheme, we no longer want to frame the calculation of probabilities in the context of the standard description of measurement. That is, the formalism can stay, but we need different words surrounding it to give it meaning. The *consistent* (or *decoherent*) *histories* interpretation of quantum mechanics modifies this postulate to make predictions about the more general *quantum history* rather than measurement results.

TODO: history about development. preview ordering. Griffiths provides a thorough set of logic/math, while Gell-Man/Hartle/Craig's work informs (and is informed by) cosmological applications. First section will reiterate Griffith's articuation of foundations of theory in Schrödinger picture, following will examine simplificatoins made by using Heisenberg picture as used by GMHC. So far, we have assumed the Schrödinger picture of time evolution, in which the unitary operator determined by the Schrödinger equations act on the quantum state. In the Heisenberg picture, the unitary operator acts on operators in the Hilbert space rather than states. TODO better explain this. Consistency conditions are expressed intuitively in the Schrödinger picture, and with simplifications in the Heisenberg picture. TODO finish this.

5.1 Properties, Events and Histories

A *quantum property* is a true or false statement about a physical variable. Recalling Example 1 TODO REF, we find through experiment that the system will posses one of two properties at the end of measurement:

- "Spin along the z axis is $\frac{\hbar}{2}$"
- "Spin along the z axis is $-\frac{\hbar}{2}$"

A property corresponds to a subspace of the Hilbert space. For example, the subspace corresponding to the spin-up property is only the state $|+\rangle$.

A *quantum event* is a system's possession of a property. An event is represented by the projection operator for the property's subspace. For the spin-up example, this operator is P_+^z.

A *quantum history* is a set of events at sequential times. Now that we have dropped the projection postulate, the TODO REF 6th postulate now completely describes how states evolve with time. A history is a finite set of events that necessarily ignores an infinite amount of insignificant events.

For example, consider the example of consecutive measurement in TODO REF. The history for measuring spin up in both analyzers is $\{P_+^z, P_+^x\}$.

5.2 Schrödinger Picture

A history, as a set of events, specifies possible states of $|\psi\rangle$ at multiple instances of time. Formally, this is no different than specifying possible states of a composite system consisting of a copy of $\mathcal{H}$ for each instant in time [**?**].

A *history Hilbert space* is such a composite system. Once again, the tensor product is employed to create a composite system; this time, the entire spin-pointer system is considered at different points in time. The history Hilbert space representing $|\psi\rangle$ at times $(t_0, t_1, t_2, ..., t_f)$ is

$$\mathcal{H}_h = \mathcal{H}_{t_0} \odot \mathcal{H}_{t_1} \odot \mathcal{H}_{t_2} \odot ... \odot \mathcal{H}_{t_f} \tag{5.1}$$

where $\odot$ is the ordinary tensor product, but denotes that the same quantum system $\mathcal{H}$ is considered at different times.

In this history Hilbert space, the state representing the system at times $(t_0, t_1, t_2, ..., t_f)$ is

$$|\psi_h\rangle = |\psi_{t_0}\rangle \odot |\psi_{t_1}\rangle \odot |\psi_{t_2}\rangle \odot ... \odot |\psi_{t_f}\rangle \tag{5.2}$$

where each component $|\psi_{t_i}\rangle$ is determined by the unitary dynamics experienced by the system up until that point, $U(t_0, t_i)$. In other words,

$$|\psi_h\rangle = |\psi_{t_0}\rangle \odot U(t_0, t_1)\,|\psi_{t_0}\rangle \odot U(t_0, t_2)\,|\psi_{t_0}\rangle \odot ... \odot U(t_0, t_f)\,|\psi_{t_0}\rangle \tag{5.3}$$

In this history Hilbert space, a history is represented by the tensor product of its events. That is, history $n = (P_0, P_1, P_2, ..., P_f)$ is represented by $P_n^h = P_0 \odot P_1 \odot P_2 \odot ... \odot P_f$.

5.2.1 Extending the Probability Postulate

The standard probability postulate TODO REF is written as the inner product of the system state $|\psi\rangle$ and an eigenstate of a physical variable $|a_n\rangle$. It can instead be written in terms of the projection operator for $|a_n\rangle$ by expanding the complex square.

$$\mathcal{P}(n) = |\,\langle a_n|\psi\rangle\,|^2 \tag{5.4}$$

$$= \langle\psi|a_n\rangle\,\langle a_n|\psi\rangle$$
$$= \langle\psi|\,P_n^a\,|\psi\rangle$$

To extend this postulate to make predictions about histories, we replace the system $|\psi\rangle$ with the history system $|\psi_h\rangle$ and the measurement projector P_n^a with the history projector P_n^h.

$$\mathcal{P}(h_n) = \langle\psi_h|\,P_n^h\,|\psi_h\rangle \tag{5.5}$$

5.2.1.1 Example 1

As shown in TODO REF, the measurement of an initial spin state $|\psi_s\rangle$ results in

$$\begin{aligned} |\psi_1\rangle &= U(t_0, t_1)\,|\psi_0\rangle \\ &= P_+^z\,|\psi_s\rangle \otimes |+_D\rangle \;+\; P_-^z\,|\psi_s\rangle \otimes |-_D\rangle \end{aligned} \tag{5.6}$$

In the history Hilbert space, the state representing the system before and after measurement is

$$\begin{aligned} |\psi_h\rangle &= |\psi_0\rangle \odot |\psi_1\rangle \\ &= (|\psi_s\rangle \otimes |\varnothing_D\rangle) \odot \left(P_+^z\,|\psi_s\rangle \otimes |+_D\rangle \;+\; P_-^z\,|\psi_s\rangle \otimes |-_D\rangle\right) \end{aligned} \tag{5.7}$$

The history for measuring spin-up is composed of the initial spin state with a ready detector, and an up spin state with an up detector:

$$P_+^h = \left(P_{\psi_s} \otimes P_\varnothing^D\right) \odot \left(P_+^z \otimes P_+^D\right) \tag{5.8}$$

Using the new probaility postulate, the probability of measuring spin-up is

$$\begin{aligned} \mathcal{P}(h_+) &= \langle\psi_h|\,P_+^h\,|\psi_h\rangle \\ &= \langle\psi_h|\left(P_{\psi_s} \otimes P_\varnothing^D\right) \odot \left(P_+^z \otimes P_+^D\right)|\psi_h\rangle \\ &= \langle\psi_h|\,(|\psi_s\rangle \otimes |\varnothing_D\rangle) \odot \left(P_+^z\,|\psi_s\rangle \otimes |+_D\rangle\right) \\ &= \langle\psi_s|\,P_+^z\,|\psi_s\rangle \end{aligned} \tag{5.9}$$

and we recover the prediction of the standard Born Rule.

5.2.2 Consistency Conditions

The third postulate TODO REF defines the subset of states corresponding to “measurement results”, and the fourth postulate makes predictions about these states only. Now that we have extended the Born Rule to make predictions about histories, we need to be careful about the context in which the predictions are made, as it is no longer postulated for us.

5.2.2.1 Example 1

The standard Born Rule makes predictions for measurements of spin-up and spin-down along the z axis. Since these states exhaust the possible outcomes, the probability of their measurement must sum to unity $(\mathcal{P}(+) + \mathcal{P}(-) = 1)$ TODO make notation match Ch3. We saw in TODO REF that the extended Born Rule reproduced these results.

However, the extended Born Rule goes on to make predictions about other outcomes. For example, we could change $|+\rangle$ to $|+_x\rangle$ and ask

$$\begin{aligned} \mathcal{P}(h_{x_+}) &= \langle\psi_h| \, P^h_{x_+} \, |\psi_h\rangle \\ &= \langle\psi_s| \, P^x_+ \, |\psi_s\rangle \end{aligned} \tag{5.10}$$

which is non-zero in general.

We could ask an infinite amount of similar questions, since the spin Hilbert space includes states representing spin-up along every direction in space TODO REF STATE SPACE SECTION. Consequently, our probabilities no longer sum to unity, which is not consistent with probaility theory.

The solution is to use *consistency conditions* to determine the sets of histories that are consistent with probability theory. The consistency conditions require that histories represent exhaustive and mutually exclusive outcomes:

$$P^{h\dagger}_i P^h_j = \delta_{i,j} P^h_i \tag{5.11}$$

$$\sum_i P^h_i = I_h \tag{5.12}$$

A set of histories satisfying these conditions is called a *consistent family* of histories. Once a consistent family is specified, we can use the extended Born Rule to calculate probabilities within that context. For our example, $\{P^h_+, P^h_-, P^h_0\}$ is a consistent family where

$$\begin{aligned} P^h_+ &= \left(P_{\psi_s} \otimes P^D_\varnothing\right) \odot \left(P^z_+ \otimes P^D_+\right) \\ P^h_- &= \left(P_{\psi_s} \otimes P^D_\varnothing\right) \odot \left(P^z_- \otimes P^D_-\right) \end{aligned} \tag{5.13}$$

$$P_0^h = I_h - P_+^h - P_-^h$$

P_0^h represents any history distinct from P_+^h and P_-^h. Showing this,

$$\begin{aligned} {P_0^h}^\dagger P_\pm^h &= (I_h - P_+^h - P_-^h)P_\pm^h \\ &= P_\pm^h - P_\pm^h \\ &= 0 \end{aligned} \tag{5.14}$$

Furthermore, P_+^h and P_-^h are distinct since

$$P_+^{D*} P_-^D = 0 \tag{5.15}$$

and it is shown that histories in this set are mutually exclusive.

Showing that the set is exhaustive,

$$\begin{aligned} P_+^h + P_-^h + P_0^h &= P_+^h + P_-^h + \left(I_h - P_+^h - P_-^h\right) \\ &= I_h \end{aligned} \tag{5.16}$$

Now that the consistency of the family is confirmed, we can use TODO REF Eq 5.5 to find probabilities for each history. Notice that P_0^h is included to make the set exhaustive; even though its probability of occurence is 0, its inclusion in the family enables such a prediction.

TODO: reference frameworks, complementarity

5.3 Heisenberg Picture

TODO: intro paragraph. Before, we applied dynamics to the state to find $|\psi_0\rangle$, $|\psi_1\rangle$, etc. Now, we apply dynamics to each operator in the Hilbert space. The operators representing events become

$$\bar{P_n} = U(t_0, t_n)P_n \tag{5.17}$$

and a history TODO REF becomes

$$\bar{h_n} = \left(\bar{P_0},\ \bar{P_1},\ \bar{P_2},\ ...,\ \bar{P_f}\right) \tag{5.18}$$

TODO: applying U to states causes loss of information compared to applying U to projectors. This lets us avoid history hilbert space.

In the non-history Hilbert space, a history is represented by a *class operator*

$$C_{h_n} = \bar{P}_0 \bar{P}_1 \bar{P}_2 ... \bar{P}_f \tag{5.19}$$

5.3.1 Extending the Probability Postulate

We again use the form of TODO REF EQs, replacing $|\psi_h\rangle$ with $|\psi\rangle$ and P_n^h with C_n:

$$\mathcal{P}(h_n) = \langle\psi| C_n |\psi\rangle \tag{5.20}$$

Since $\mathcal{P}(h_n)$ is a real scalar,

$$\mathcal{P}(h_n) = \mathcal{P}(h_n)^\dagger \tag{5.21}$$
$$\mathcal{P}(h_n) = \langle\psi| C_n^\dagger |\psi\rangle$$

so that the class operator projects in the same order as the events:

$$C_n^\dagger |\psi\rangle = \bar{P}_f ... \bar{P}_2 \bar{P}_1 \bar{P}_0 |\psi\rangle \tag{5.22}$$

We call $C_n^\dagger |\psi\rangle$ the *branch wave function* for the history h_n. Notice that in TODO REF FIGURES, the branch wavefunctions are represented by following a path from the initial state to some final outcome.

5.3.2 Consistency Conditions

Since histories are now represented by class operators rather than projectors into the history Hilbert space, the condition of mutually exclusive and exhaustive outcomes is

$$C_i^\dagger C_j = \delta_{i,j} C_i \tag{5.23}$$
$$\sum_i C_i = I \tag{5.24}$$

6 Decoherence

All remaining content are outlines/notes.

TODO: introduce decoherence, explain role played by vnms. Incomplete which path, Environment continuously monitors system. TODO: sometimes we must ignore the environment, results are inaccessible or ignored experimentally.

6.1 Density Matrices

As stated in TODO ref, $|\psi\rangle$ can no longer be written in the form $|\psi\rangle_s \otimes |\mathcal{X}\rangle_z$. Rather, $|\psi\rangle$ is a superposition of such states; quantum *coherency* has been extended from the spin system to the spin-pointer composite system.

In the case of environmental decoherence, we assume that this type of interaction while possessing no information about the pointer system. Since multiple pointer states may correspond to the same spin state, “ignoring the environment” now means that we must coarse grain out all possible environment states for each spin state. We can no longer “factor out” the environment subsystem.

Some definite environment state is realized; we just do not know which one. This uncertainty is classical in nature; it has nothing to do with any inherent quantum uncertainty. We are now dealing with a ”classical mixture” of superposition states.

Such a system is well represented by a *density matrix*. For a *pure state*, the representative density matrix is just the projection operator for that state

$$\rho = |\psi\rangle \langle\psi| \tag{6.1}$$

Recall that when considering the projection operator of some state, we can equivalently think of the subset of states in the Hilbert space into which it projects.

The density operator for the state after measurement is

$$\rho = \sum_{n,m} ({P^z}_n |\psi\rangle_s \otimes |\mathcal{X}_n\rangle_z) \cdot ({P^z}_m \langle\psi| \otimes \langle\mathcal{X}_m|) \tag{6.2}$$

$$\rho = \sum_{n,m} \langle n|\psi\rangle_s \langle m|\psi\rangle_s |n\rangle \langle m| \otimes |\mathcal{X}_n\rangle_z \langle\mathcal{X}_m| \tag{6.3}$$

6.2 Density matrix Born rule

The Born Rule can also be expressed in terms of *density matrices*. From (TODO: reference appendix), we know that an inner product can be written as the trace of the corresponding dyad. The Born Rule is the complex square of an inner product, so we should be able to assign probabilities to detection states by tracing over some corresponding tensor. We will manipulate the Born Rule to take this form, and then examine the resulting tensor.

Expanding the complex square,

$$\mathcal{P}(n) = (\langle D_n| V |\psi\rangle) \cdot (\langle D_n| V |\psi\rangle)^* \tag{6.4}$$

$$\mathcal{P}(n) = (\langle D_n| V |\psi\rangle) \cdot \left(\langle\psi| V^\dagger |D_n\rangle\right) \tag{6.5}$$

Rewriting the second inner product as the trace of a dyad,

$$\mathcal{P}(n) = (\langle D_n| V |\psi\rangle) \cdot Tr\left(|D_n\rangle \langle\psi| V^\dagger\right) f \tag{6.6}$$

The remaining inner product, like any, is a scalar. Since trace is a linear operator, we can scale any factor inside the operation by this inner product.

$$\mathcal{P}(n) = Tr\left(|D_n\rangle \cdot (\langle D_n| V |\psi\rangle) \cdot \left(\langle\psi| V^\dagger\right)\right) \tag{6.7}$$

Now we can rewrite $|D_n\rangle \langle D_n|$ as ${P^D}_n$, and simplify grouping:

$$\mathcal{P}(n) = Tr\left({P^D}_n \cdot \left(V |\psi\rangle \langle\psi| V^\dagger\right)\right) \tag{6.8}$$

TODO: discuss remaining object ${P^D}_n \cdot \left(V |\psi\rangle \langle\psi| V^\dagger\right)$ TODO: discuss coarse graining, conditional probabilities

7 Complementarity

We now discuss the the principle of complementarity in the context of standard and consistent quantum mechanics. Arguably the most fundamental feature of quantum mechanics, the principle of complementarity states that a quantum system has pairs of physical observables which cannot be measured simultaneously. The operators corresponding to pairs of complementary properties do not commute; that is, $[A, B] = AB - BA \neq 0$. Components of spin on orthogonal axes are complementary properties, so we examine measurements of succesive Stern-Gerlach experiments.

8 Simulation

Research for this section is complete. I expect writing this section to take 1-2 days of writing, to be done over spring break.

8.1 Standard Description

Using the Born Rule, we calculate the probabilities of observing each final state in Figure 4.1. The first apparatus serves as a state preparation device with output $|+\rangle$. By the direction of the projection postulate, the state is renormalized upon each measurement. After measuring a property complementary to what is known (such as spin along x, knowing spin along z), any information known about the input state is lost; the input state instantaneously changes to the state corresponding to the observed quantity. Consequently, there is an equal probability of observing the final state as $|+\rangle$ or $|-\rangle$ at either final apparatus, even though the state was initially prepared as $|+\rangle$, since

$$\begin{aligned}
\mathcal{P}_n &= |\langle +|+\rangle_y|^2 && (8.1)\\
&= |\langle -|+\rangle_y|^2 && (8.2)\\
&= |\langle +|-\rangle_y|^2 && (8.3)\\
&= |\langle -|-\rangle_y|^2 && (8.4)\\
&= \frac{1}{4} && (8.5)
\end{aligned}$$

TODO: make above separate equations for clarity. It appears that this contradiction with classical intuition is a direct result of the projection postulate. The act of measurement and ensuing state collapse causes the system to shed properties previously recorded.

8.2 Consistent Description

TODO: calculate probabilities using consistent Born Rule. Consistent quantum theory predicts the same loss of a definite S_z value, but for different reasons. The description of measurment in consistent histories does not postulate state collapse; rather, the system evolves through some Hamiltonian that correlates system and detector states. This implies that the measurement process has nothing to do with the principle of complementarity. We can trace the cause back to our

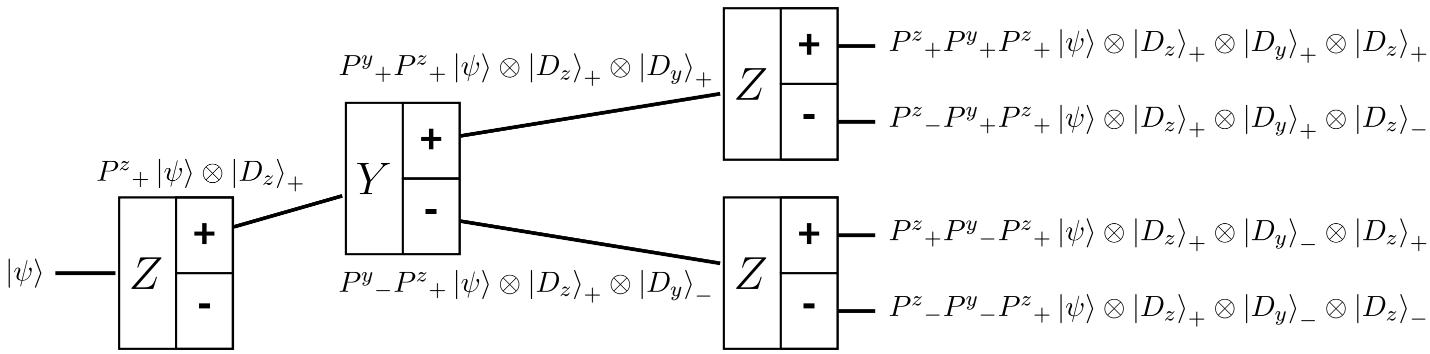

FIGURE 8.1 Demonstrating complementary measurments in consistent quantum mechanics

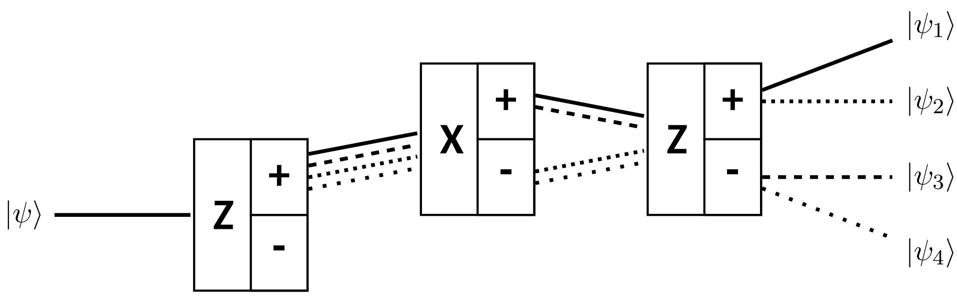

FIGURE 8.2 TODO: create section to discuss this example

definition of the state space. For a spin state, $|\psi\rangle$ is completely defined by spin-up or spin-down in a single direction w. The Hilbert space does not include states that could be interpreted as possesing a definite spin value in more than one direction. Consequently, the operators for spin in directions not parallel or antiparallel to each other share no eigenstates. It follows mathematically that these operators do not commute: TODO run through this math.

This description of complementary implies that the principle is a limitation inherent to the quantum state, rather than a consequence of the role of measurement. In consistent histories, this limitation is embodied by *the single framework rule*. There exists multiple ways in which a quantum system can be described, yet descriptions from only one of these *frameworks* can be meaningfully compared or combined.

TODO: discuss any structural or algorithmic changes resulting from rewriting spins simulation

9 Conclusion

Bibliography

[1] Z. Schreiber, *The Nine Lives of Schrödinger's Cat*, Masters Thesis, University of London (1994).

[2] A. Pais, "Einstein agnd the quantum theory", *Rev. Mod. Phys.* 51 (1979).