A Spins-First Introduction to Consistent Histories and Decoherence

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TODO: insert acknowledgments

Abstract

Standard quantum mechanics makes foundational assumptions describing the measurement process. We show that postulating state collapse artifically limits the scope of quantum theory, motivating a unitary description of measurement. In the context of Stern-Gerlach experiments, we explain measurement as entanglement of the measured spin system, the measured position system, and the measuring apparatus. To interpret the entangled state, the consistent histories approach is used to make probabilistic predictions. To do this, the environment must also be entangled with the system, playing the role of a record keeper. The differing methods used by Griffiths and Gell-Mann/Hartle to incorporate the environment is articulated. We conclude by exemplifying tangible advantages of this approach by simplifying code that simulates consecutive Stern-Gerlach measurements.

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1 Introduction

Quantum mechanics is plagued by interpretational issues surrounding measurement. The postulated "state collapse" mechanism describes the evolution of systems upon "interaction with a classical measuring apparatus". While the mechanism's predictions agree with experiment, Lacking a precise definition of such an apparatus, the role of the experimentalist in measurement interactions is easily inflated.

In this thesis, we present relies on less fundamental assumptions.

Quantum mechanics is plagued by interpretational issues surrounding measurement. The standard description of measurment postualates a special type of dynamics in which a quantum system instantaneously evolves upon measurement. The conditions in which this postulate applies are not well defined, leading to confusion on the nature of measurement itself.

The predictions of the standard description of measurement can be reproduced using alternative descriptions of the measurement process. We study an alternative description that explains many, but not all, of the infamous measurement related problems. The *von Neumann measurement scheme* is used to describe how states evolve when being measured, and the *consistent* or *decoherent histories* interpretation of quantum mechanics is used to explain what is happening physically.

Numerous papers and books describe these concepts in detail TODO CITE, but they have yet to permeate far outside the quantum foundations community. A primary goal of this thesis is to introduce these concepts in a form more accessible to those new in their study of quantum foundations or physicists with other specialties. Following the lead of research in spins-first introductions to quantum mechancis in physics education TODO CITE, we introduce these new ideas in the context of the Stern-Gerlach experiment. Having only two degrees of freedom, spin-$\frac{1}{2}$ systems are the simplest possible. We explain all fundamental aspects of quantum mechanics within this context, as well as our proposed changes.

Descriptions of measuring spin-$\frac{1}{2}$ systems in this way are exemplified in works by Griffiths, Hohenberg, and Schlosshauer TODO CITE. However, they either require prior knowledge of concepts in quantum foundations, neglect implementing von Neumann measurement, or do not offer interpretational explanations. Our explanation of Stern-Gerlach experiments does all of these things. Another primary goal of this thesis is showing how to implement these ideas explicitly. In doing so, we introduce a unitary operator describing measurement specific to ideal measurement of spin-$\frac{1}{2}$ systems. Using the same tools, we describe how states decay in time through *decoherence*.

In addition to circumventing some measurement related paradoxes, our description of quantum mechanics contains other tangible advantages. We exemplify this by comparing the simulation of measurement in both frameworks. Existing code simulating sequential Stern-Gerlach measurements is used as a baseline TODO CITE, and relevant code is rewritten using our new formalism. The resulting program control flow becomes drastically simplified. We conclude by discussing existing and future experiments that may distinguish whether the standard or *relative states* formalisms correctly represent physical reality.

The Stern-Gerlach is the archetypal experiment for quantum measurement.

In addition to its historical significance,

2 Stern-Gerlach Experiments

TODO: provide a brief explanation of the experimental setup and results. Discuss historical and pedagogical significance. Explain how measurement results correspond to the particle's localization in certain regions. This section is background information so that I can discuss the von Neumann measurement, consistent histories, and decoherence in the context of this experiment. I am saving it for later to focus on the introducing the new theory for now.

3 Postulates of Quantum Mechanics

We first consider the mathematical objects used to model physical systems and variables. By comparing the objects used in classical and quantum mechanics, we make sense of the first three postulates of quantum mechanics. Then, we compare the Copenhagen and von Neumann descriptions of measurement and their relation to the fourth and fifth postulates.

3.1 Physical Variables and State Spaces

3.1.1 Classical States

Consider the spin of an electron. Treating the electron as a classical system, its spin state is modeled by a vector $\vec{S} \in \mathbb{R}^3$:

$$\vec{S} = (S_x, S_y, S_z) \tag{3.1}$$

Each component S_{x_i} is a physical variable representing the magnitude of spin oriented in the $\hat{x}_i$ direction.

$\vec{S}$ has the capacity to determine spin in any direction using the inner product of the state space $\mathbb{R}^3$:

$$S_n(S) = \vec{S} \cdot \hat{n} \tag{3.2}$$

We see that in classical mechanics, physical variables are modeled using functions. Each function S_n maps a spin state [$\vec{S}$ to a real scalar representing the spin of the electron aligned along the $\hat{n}$ axis.

What makes classial mechanics familiar to everyday experience boils down to intuitive but important properties of the state space $\mathbb{R}^3$:

- For any direction $\hat{n}$, $\vec{S}$ determines spin S_n
- S_n can be any real value

$\vec{S}$ determines spin in any direction because TODO. Consequently, the sample spaces for spin in any two directions $\hat{n}$ and $\hat{m}$ are *compatible*, meaning that S_n and S_m may be simultaneously determined. Spin states in $\mathbb{R}^3$ are interpreted physically as the electron posessing definite values for every S_n at some instant in time.

In addition to spin states determing all S_n, the state space allows S_n to take on any real value. There are no fundamental restrictions on which real numbers S_n could be; its sample space is continuous and infinitely large.

3.1.2 Quantum States

Measurements of electron spin show that the intuitive classical properties do not hold. Recall that only two magnitudes of spin have ever been measured. S_n is a *quantized* physical variable; its sample space is discrete and finite.

Second, the results of successive measurements of a spin system imply that $\vec{S}$ does not determine spin in some general direction S_n. Recall the results of succesively measuring spin in orthogonal directions discussed in TODO REF. After measuring S_x, $\vec{S}$ appears to "forget" a previous measurement of S_z. All we may know about the system at one instant in time is spin in one direction. The inabilitiy to simultaneously determine spin in two independent directions $\hat{n}$, $\hat{m}$ should be reflected through S_n and S_m having *incompatible* sample spaces.

Electron spin measurements violate the intuitive classical state space properties mentioned in 3.1.1. In response, we must change the mathematical objects used to represent system states and physical variables. Specifically, the sample space of S_n must restrict observable values to spin up and spin down, and S_n and S_m should have incompatible sample spaces. In combination, the first three postulates of quantum mechanics take care of these differences.

Quantum mechanics postulates that a system's state is completely described by a normalized vector in a linear state space.

POSTULATE 1 *The state of a physical system is defined by specifying an abstract vector $|\psi\rangle$ in a Hilbert state space $\mathcal{H}$.*

For spin-$\frac{1}{2}$ systems such as electrons, the two-dimensional Hilbert space consists of all linear combinations of spin-up and spin-down:

$$|\psi\rangle \in \mathcal{H}\,|\psi\rangle\,\{\alpha\,|+\rangle + \beta\,|-\rangle\}$$

where $\alpha, \beta \in \mathbb{C}$.

$\mathcal{H}$ is an abstract state space; components of $|\psi\rangle$ cannot be interpreted as physical variables as they are for the classical spin state S. So, we introduce physical meaning with more postulates.

The second posulate of quantum mechanics states that physical variables are described by linear operators:

POSTULATE 2 *Every physical variable $\mathcal{A}$ is described by an operator A acting in $\mathcal{H}$.*

Justifying the second postulate is easiest when also considering the third postulate:

POSTULATE 3 *The only possible result of the measurement of a physical variable $\mathcal{A}$ is one of the eigenvalues of the corresponding operator A.*

The operator correlates elements of a finite sample space (eigenvalues) with particular system states (eigenstates). Consequently, a state can only be interpreted as having a definite variable value if it is an eigenstate of that variable's operator. To illustrate this, consider the operator representing S_z. Written in the basis of its own eigenstates,

$$S_z \doteq \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.3}$$

This operator correlates z spin-up $\left(S_z = \frac{\hbar}{2}\right)$ with eigenstate

$$|\psi\rangle = |+_z\rangle \doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3.4}$$

and z spin-down $\left(S_z = \frac{-\hbar}{2}\right)$ with eigenstate

$$|\psi\rangle = |-_z\rangle \doteq \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.5}$$

Here, the subscript z specifies that the state represents spin-up along the z axis.

Similarly, we write the operator representing S_x in the S_z basis:

$$S_x \doteq \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{3.6}$$

This operator correlates x spin-up $\left(S_x = \frac{\hbar}{2}\right)$ with eigenstate

$$|\psi\rangle = |+_x\rangle \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{3.7}$$

and x spin-down $\left(S_x = \frac{-\hbar}{2}\right)$ with eigenstate

$$|\psi\rangle = |-_x\rangle \doteq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{3.8}$$

Operators for S_z and S_x share no common eigenstates, so no state can posses definite values for both variables. In general, operators for any two spin components S_i and S_j do not share common eigenstates with each other; in other words, S_i and S_j have incompatible sample spaces. S_i and S_j are called *complementary* variables.

By representing physical variables with operators rather than functions, sample spaces become quantized and may be incompatible with each other. These features are necessary for predicting the results of electron spin measurements.

The first three postulates designate the mathematical objects used to model physical system states and variables. The fundamental differences between classical and quantum systems are completely described by these postulates and their consequences.

3.1.3 Linearity

TODO: compare addition of S_x, S_y states and their interpretations. Introduce superposition states and coherence.

3.2 Copenhagen Description of Measurement

The fourth and fifth postulates constitute the Copenhagen description of measurement. This description, taught in textbooks and introductory quantum courses, is a key component of the standard interpretation of quantum mechanics.

The probability postulate, also known as the *Born Rule*, assigns a probability distribution to the sample space of a physical variable.

POSTULATE 4 *When measuring physical variable A, the probability $\mathcal{P}(n)$ of obtaining result a_n corresponding to $|a_n\rangle$ is equal to*

$$\mathcal{P}(n) = |\langle a_n|\psi\rangle|^2 \tag{3.9}$$

The spirit of the Born Rule is unchanged in consistent quantum theory. Differences are discussed in (TODO: ref future section).

The fifth postualte (known as the projection postulate) describes how a system evolves upon

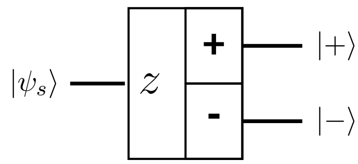

FIGURE 3.1 Stern-Gerlach Experiment 1 as described by the Copenhagen description of measurement. Each measurement outcome

measurement. Contingent upon interaction of the system with a "classical apparatus", measurement instantaneously changes the state of the system to some eigenstate of the variable being measured.

POSTULATE 5 *If the measurement of the physical variable $\mathcal{A}$ on the system in the state $|\psi\rangle$ gives the result a_n, the state of the system immediately after the measurement is the normalized projection*

$$|\psi'\rangle = \frac{P_n |\psi\rangle}{\sqrt{\langle\psi| P_n |\psi\rangle}} \tag{3.10}$$

onto the subspace associated with a_n.

3.2.3 Experiment 1

Consider a measurement result for the z component of spin. By the third postulate, the result is either either spin-up or spin-down $\left(S_z = \pm\frac{\hbar}{2}\right)$. By the projection postulate, the state evolves to the normalized projection of $|\psi\rangle$ onto $|\pm\rangle$. In other words, $|\psi\rangle$ instantaneously becomes $|\pm\rangle$ upon measurement. This process is known as *state collapse* or *wavefunction collapse*. The spatial separation of these outcomes is represented by Figure 3.1

The Born Rule gives the probabilities of measuring spin-up and spin-down as a function of initial spin state:

$$\mathcal{P}(\pm) = |\langle\pm|\psi_s\rangle|^2 \tag{3.11}$$

3.2.4 Experiment 2

Now we consecutively measure spin as shown in Figure 3.2. The first apparatus serves as a state preparation device, since we are only interested in particles exiting the spin-up output. Using the projection postulate, the state after the first measurement is

$$|\psi'\rangle = \frac{P_+^z |\psi\rangle}{\sqrt{\langle\psi| P_+^z |\psi\rangle}} = |+_z\rangle \tag{3.12}$$

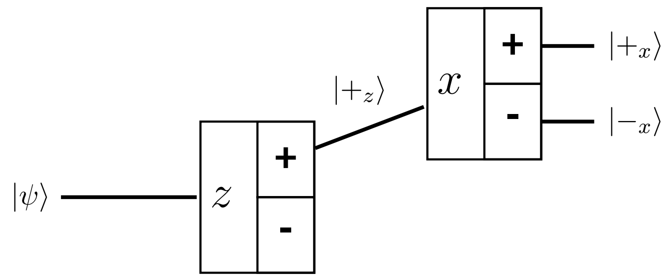

FIGURE 3.2 The Stern-Gerlach experiment as described by the standard measurement scheme. Notice that each measurement outcome is renormalized, so that information about the state prior to measurement is lost.

Similarly, the possible output states from the second apparatus are

$$|\psi''\rangle = \frac{P_+^x |+_z\rangle}{\sqrt{\langle +_z| P_+^x |+_z\rangle}} = |+_x\rangle \tag{3.13}$$

or

$$|\psi''\rangle = \frac{P_-^x |+_z\rangle}{\sqrt{\langle +_z| P_-^x |+_z\rangle}} = |-_x\rangle \tag{3.14}$$

Because we ignore spin-down particles from the first measurement, we are certain that $|\psi'\rangle = |+\rangle$. The probabilities assigned to each state leaving the S_x Stern-Gerlach device are

$$\mathcal{P}(+_x) = |\langle +_x|+_z\rangle|^2 = \frac{1}{2} \tag{3.15}$$

$$\mathcal{P}(-_x) = |\langle -_x|+_z\rangle|^2 = \frac{1}{2} \tag{3.16}$$

3.3 Dynamics

In a mechanical theory, the equations of motion (or *dynamics*) describe how a state evolves with time. In classical Newtonian mechanics, this is given by Newton's law of motion

$$\vec{F} = m\vec{a}. \tag{3.17}$$

These dynamics are *unitary*, meaning that given the final state of a physical process, the corresponding initial state is recovered by applying the dynamics with time reversed. The dynamics can be represented by a one-to-one map from initial to final states.

The projection postulate describes one type of dynamics, which apply only during measurement. When applied, all information about the initial state is lost as the state instantaneously becomes an eigenstate of the measured variable. The map from initial to final states is not one-to-one; “collapse dynamics” are non-unitary.

Quantum theory postulates an another type of dynamics that is analogous to Newton’s law of motion. These dynamics are unitary, and apply at all times (not just during measurement).

POSTULATE 6 *TODO write Schrödinger equation*

4 Measurement

TODO: chapter preview State collapse requires extra assumption, ambiguous defs lead to interepretation issues, arrow of time

4.1 Issues with State Collapse

The projection postulate introduces foundational assumptions to describe the measurement process. The principle of Occam's razor says that, in general, a theory is strengthened by making as few assumptions as possible. In classical mechanics, there are no foundational assumptions made to describe measurement; this motivates the pursuit to describe quantum measurement without using the projection postulate.

Furthermore, the projection postulate relies on ambiguous definitions. State collapse occurs upon "interaction with a classical measuring apparatus", yet there is no specification of what makes a system classical. Classical systems are not described by the theory, yet they play a fundamental role in the measurement process.

TODO: describe interpretational issues

Because the measurement process cannot be reversed, state collapse injects time asymmetry into the foundations of quantum mechanics. TODO: discuss arrow of time.

The issues with interpretation of state collapse and non-unitary dynamics in general are indicators that collapse dynamics are formulated with ignorance of some underlying process. To begin describing this process, we discard the projection postulate and describe measurement using dynamics permitted by the Schrödinger equation.

Describing measurement as a unitary process is desirable for multiple reasons:

- With dynamics symmetric in time, the emergence of the "arrow of time" can be studied
- Humans and measurement apparatuses do not play a special role indescribale by the theory
- TODO: describe cosmology benefit.
- interpretational issues with state collapse go away
- less fundamental assumptions

Fortunately, such a description is possible using the von Neumann measurement scheme.

4.2 von Neumann Measurement Scheme

In the discussion of Stern-Gerlach experiments, the position of the electron played an implicit role in measurement. In exemplifying use of the projection posulate TODO, we define a measurement result as the localization of the electron in the spin-up or spin-down regions. The primary measurement is that of position, which is used to imply the spin state, yet the position system is never formalized.

Our goal is to use the von Neumann measurement scheme to formalize the correlation of position and spin eigenstates we observe in Stern-Gerlach experiments. We start by representing the electron with a composite spin-position system,

$$\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_\mathcal{X} \tag{4.1}$$

4.2.1 Experiment 1

We revisit the example of one Stern-Gerlach measurement of spin along the z axis. We name position states of interest; $|\varnothing_\mathcal{X}\rangle$ represents the localization at the beginning of the analyzer (which we call the "ready" position), while $|+_\mathcal{X}\rangle$ and $|-_\mathcal{X}\rangle$ represent localization at the spin-up and spin-down outputs, respectively (reference TODO REF FIG). Numerous papers detail these position states using Gaussian wave packets TODO REF, but for simplicity we leave these abstracted. We will say that $\{|+_\mathcal{X}\rangle, |-_\mathcal{X}\rangle, |\varnothing_\mathcal{X}\rangle\}$ are mutually orthonormal.

We introduce an operator that correlates these position states with spin eigenstates in explicit form:

$$\begin{aligned} U(t_0, t_1) = P_+^z \otimes (\, |+_\mathcal{X}\rangle \langle\varnothing_\mathcal{X}| \;+\; |\varnothing_\mathcal{X}\rangle \langle +_\mathcal{X}| \;+\; |-_\mathcal{X}\rangle \langle -_\mathcal{X}| \,) \\ + P_-^z \otimes (\, |-_\mathcal{X}\rangle \langle\varnothing_\mathcal{X}| \;+\; |\varnothing_\mathcal{X}\rangle \langle -_\mathcal{X}| \;+\; |+_\mathcal{X}\rangle \langle +_\mathcal{X}| \,) \end{aligned} \tag{4.2}$$

Starting with a general spin state, the final state is

$$\begin{aligned} U(t_1, t_0)\,|\psi\rangle &= U(t_1, t_0)\,(|\psi_s\rangle \otimes |\varnothing_\mathcal{X}\rangle) \\ &= P_+^z\,|\psi_s\rangle \otimes |+_\mathcal{X}\rangle \;+\; P_-^z\,|\psi_s\rangle \otimes |-_\mathcal{X}\rangle \end{aligned} \tag{4.3}$$

At the instant measurement begins t_0, the position state is $|\varnothing_\mathcal{X}\rangle$ as the electron enters the magnetic field. At the instant measurement ends t_1, the position state is either $|+_\mathcal{X}\rangle$ or $|-_\mathcal{X}\rangle$, realized with spin-up and spin-down spin states respectively. Notice that the final sum does not contain any terms representing incorrect correlations between spin and position states (such as $P_+^z\,|\psi_s\rangle \otimes |-_\mathcal{X}\rangle$). Consequently, the final state cannot be written as the tensor product of a state

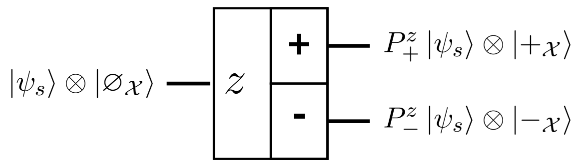

FIGURE 4.1 The Stern-Gerlach experiment as described by the von Neumann measurement scheme. Each measurement outcome corresponds to a term in the time-evolved state (Equation 4.3). Notice that the measurement interaction results in a branching structure, represented here as a tree graph with the apparatus as a node.

in $\mathcal{H}_s$ and a state in $\mathcal{H}_{\mathcal{X}}$ (as the inital state was). This is the definition of *entanglement*; the von Neumann measurement scheme describes the measurement process as entanglement. $U(t_1, t_0)$ is *the* unitary operator accomplishing the desired correlation, as evident by $UU^\dagger = I$.

The von Neumann scheme is usually written as a linear map:

$$\begin{aligned} &U(t_1, t_0) : \\ &\qquad |\psi\rangle = \left(\sum_n P_n^z |\psi_s\rangle\right) \otimes |\varnothing_{\mathcal{X}}\rangle \mapsto \sum_n \left(P_n^z |\psi_s\rangle \otimes |n_{\mathcal{X}}\rangle\right) \end{aligned} \tag{4.4}$$

where $n = +, -$.

Notice that the initial state is a single tensor product, while the final state is a sum of tensor products. The coherence initially present only in the spin state is extended to the composite spin-momentum system. This process is represented schematically in TOD ref; the initial state branches into two distinct outcomes, each represented by a term in the final state.

By describing measurement as a unitary process, we have eliminated many aspects of the quantum measurement problem. TODO cite specific examples. However, there are two overarching problems with interpretation of the final state of the von Neumann measurement scheme. These are called the *preferred basis problem* and the *problem of outcomes* in literature [**?**] [**?**]. The problem of outcomes remains an open research question, and potential solutions lay outside the scope of this thesis. The preferred basis problem can be solved using the tools developed so far, so we show how it may be solved by including the apparatus in our measurement model.

4.3 Preferred Basis Problem

The preferred basis problem arises from the ability to write the final state in Equation 4.4 using different bases:

$$|\psi'\rangle = \sum_n \left(P_n^z |\psi_s\rangle \otimes |n_{\mathcal{X}}\rangle\right) = \sum_{n'} \left(P_{n'}^z |\psi_s\rangle \otimes |n'_{\mathcal{X}}\rangle\right) = \ldots \tag{4.5}$$

That $\{|n_s\rangle\}$ and $\{|n_{\mathcal{X}}\rangle\}$ are orthognal sets means that $|\psi'\rangle$ is a biorthongal system. The biorthogonal decomposition theorem states that alternate bases $\{|n_s'\rangle\}$ and $\{|n_{\mathcal{X}}'\rangle\}$ exist when $\langle n_s|\psi_s\rangle$ are not all distinct [**?**]. We exemplify such a case using experiment 1.

4.3.1 Experiment 1

Consider setting the initial spin state to spin-up in the x direction:

$$|\psi_s\rangle = |+^x\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}} \tag{4.6}$$

The final state by Equation 4.2 is

$$|\psi_f\rangle = \frac{|+\rangle \otimes |+_{\mathcal{X}}\rangle + |-\rangle \otimes |-_{\mathcal{X}}\rangle}{\sqrt{2}} \tag{4.7}$$

Similar to the S_x eigenstates, we define orthonormal position states

$$\begin{aligned} |+^x_{\mathcal{X}}\rangle &= \frac{|+_{\mathcal{X}}\rangle + |-_{\mathcal{X}}\rangle}{\sqrt{2}} \\ |-^x_{\mathcal{X}}\rangle &= \frac{|+_{\mathcal{X}}\rangle - |-_{\mathcal{X}}\rangle}{\sqrt{2}} \end{aligned} \tag{4.8}$$

so the final state can be written

$$|\psi'\rangle = |+^x\rangle \otimes |+^x_{\mathcal{X}}\rangle \tag{4.9}$$

TODO REF matches the form TODO REF; it appears that the measurement process of spin along the z axis has entangled position states with spin states along the x axis. If we regard the measurement process as such an entangelement, the von Neumann measurement scheme violates the princple of complementarity by simultaneously measuring S_z and S_x. We know that our experimental setup was configured to measure S_z, but nothing in the theory singles out S_z as the *preferred basis*.

4.4 Inselection

To solve this problem, we need only to look back to the original phrasing of the von Neumann measurement scheme, where measurement is described as the entanglement of a microscopic system with a macroscopic measuring apparatus [1]. Many descriptions of non-unitary Stern-Gerlach measurement consider the electron position system as the apparatus itself. This is a reasonable abstraction, as the position of the electron is used to "read off" the result of the measurement. However, this approach is misleading, because it conflates two distinct physical systems; the apparatus, and the position system belonging to the electron. By labeling the position system as the "apparatus", the degree of freedom corresponding to the actual apparatus is effectively ignored.

Newton's third law asserts that the force exerted on the electron by the apparatus magnet is paired with a force exerted on the magnet by the electron. This motivates the definition of apparatus states $\{|+_a\rangle, |-_a\rangle, |\varnothing_a\rangle\}$ representing the effect of positive, negative, and zero torques on the magnet, respectively. Unlike the position states, these states need not be mutually orthongonal. We expect the torque exerted on macroscopic magnets by an electron to be small, so that

$$\langle n_a|m_a\rangle \approx 1 \neq 1 \tag{4.10}$$

where $n, m = +, -, \varnothing$. In other words, these states are nearly indistinguishable, but not exactly the same.

We now formalize the spin-position-apparatus interaction, similar to how the implied spin-position correlation in the projection postulate was formalized.

$$U(t_1, t_0):$$
$$|\psi\rangle = \left(\sum_n P_n^z |\psi_s\rangle\right) \otimes |\varnothing_{\mathcal{X}}\rangle \otimes |\varnothing_a\rangle \mapsto \sum_n \left(P_n^z |\psi_s\rangle \otimes |n_{\mathcal{X}}\rangle \otimes |n_a\rangle\right) \tag{4.11}$$

While systems in the form Equation 4.4 do not generally have unique decompositions, systems with three or more components do by the triorthongal decomposition theorem [**?**]. Furthermore, so long as two components are expanded in orthongal bases, the third component need only be expanded in a non-colinear basis. In other words, because $\{n_s\}$ and $\{n_{\mathcal{X}}\}$ are orthongal sets, $\{n_a\}$ need only be linearly independent. The small difference between states $\langle n_a|m_a\rangle \neq 1$ satistfies this requirement.

Using this result, we cannot write the final state in another basis:

$$|\psi'\rangle = \sum_n \left(P_n^z |\psi_s\rangle \otimes |n_{\mathcal{X}}\rangle \otimes |n_a\rangle\right) \neq \sum_{n'} \left(P_{n'}^z |\psi_s\rangle \otimes |n'_{\mathcal{X}}\rangle \otimes |n'_a\rangle\right) \tag{4.12}$$

Since this is the only possible way to write $|\psi'\rangle$, the preferred basis has been chosen by including the apparatus system. The uniqueness of the decomposition is dependent on the orthogonality of $|n_s\rangle$ states, the orthogonality of $|n_{\mathcal{X}}\rangle$ states, and the non-colinearity of $|n_a\rangle$ states. The apparatus states only need to satisfy $\langle n_a|m_a\rangle \neq 1$, which is a much looser condition than requiring mutual orthogonality of apparatus states. Misidentifying the position system as the apparatus imposes strict

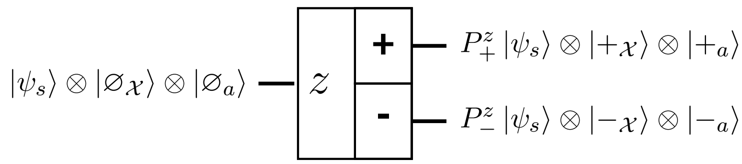

FIGURE 4.2 The most complete description of Experiment 1 presented, including position and apparatus degres of freedom. The seemingly redundant correlation of both position and apparatus states to spin states validifies the abstraction of position states as apparatus. However, formalizing the interaction with the apparatus provides a more complete description that resolves the preferred basis problem.

restrictions on the apparatus that truly belong to the position system.

4.4.1 Experiment 1

Our system is now composed of spin, position, and apparatus systems $H = H_s \otimes H_{\mathcal{X}} \otimes H_a$. The unitary operator satisfying Equation 4.11 is

$$\begin{aligned} U(t_1, t_0) = P^z_+ &\otimes \left(|+_{\mathcal{X}}\rangle \langle\varnothing_{\mathcal{X}}| \,+\, |\varnothing_{\mathcal{X}}\rangle \langle +_{\mathcal{X}}| \,+\, |-_{\mathcal{X}}\rangle \langle -_{\mathcal{X}}| \right) \\ &\otimes \left(|+_a\rangle \langle\varnothing_a| \,+\, |\varnothing_a\rangle \langle +_a| \,+\, |-_a\rangle \langle -_a| \right) \\ + P^z_- &\otimes \left(|-_{\mathcal{X}}\rangle \langle\varnothing_{\mathcal{X}}| \,+\, |\varnothing_{\mathcal{X}}\rangle \langle -_{\mathcal{X}}| \,+\, |+_{\mathcal{X}}\rangle \langle +_{\mathcal{X}}| \right) \\ &\otimes \left(|-_a\rangle \langle\varnothing_a| \,+\, |\varnothing_a\rangle \langle -_a| \,+\, |+_a\rangle \langle +_a| \right) \end{aligned} \tag{4.13}$$

To shorten this expression, we define the "entanglement operator"

$$E^i_\pm = |\pm_i\rangle \langle\varnothing_i| \,+\, |\varnothing_i\rangle \langle \pm_i| \,+\, |\mp_i\rangle \langle \mp_i| \tag{4.14}$$

Note that E is Hermitian $\left(E^\dagger = E\right)$ and unitary $\left(E^\dagger E = I\right)$. The unitary operator is now

$$\begin{aligned} U(t_1, t_0) = P^z_+ \otimes E^{\mathcal{X}}_+ \otimes E^a_+ \\ + P^z_- \otimes E^{\mathcal{X}}_- \otimes E^a_- \end{aligned} \tag{4.15}$$

For a general initial spin state, this produces final state

$$|\psi'\rangle = P^z_+ |\psi_s\rangle \otimes |+_{\mathcal{X}}\rangle \otimes |+_a\rangle \,+\, P^z_- |\psi_s\rangle \otimes |-_{\mathcal{X}}\rangle \otimes |-_a\rangle \tag{4.16}$$

This experiment is visualized schematically in Figure 4.2.

4.5 Consecutive Measurements

Now that

TODO: introduce assumptions of consecutive measurement.

The von Neumann measurement scheme can be applied succesively.

4.5.1 Example 2

Now that we have two apparatuses, the Hilbert space includes two pointer spaces: $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_\mathcal{X} \otimes \mathcal{H}_{a_1} \otimes \mathcal{H}_{a_2}$. We also add two new coarse grained position states for the second apparatus. The position states are now $\{\left|+_{x_1}\right\rangle, \left|-_{x_1}\right\rangle, \left|+_{x_2}\right\rangle, \left|-_{x_2}\right\rangle, \left|\varnothing_\mathcal{X}\right\rangle\}$, where $\left|\varnothing_\mathcal{X}\right\rangle$ is any position not in either apparatus' spin-up or spin-down region. TODO: figure.

The dynamics are unchanged from TODO REF Example 1 for the first measurement, with the identity acting on the second apparatus to leave it unaffected:

$$\begin{aligned} U(t_1, t_0)_{a_1} = {} & P^z_+ \otimes E^{x_1}_+ \otimes E^{a_1}_+ \otimes I_{a_2} \\ & + P^z_- \otimes E^{x_1}_- \otimes E^{a_1}_- \otimes I_{a_2} \end{aligned}$$

Now we determine the dynamics for the second measurement. We expect the dynamics to do two things: revers the entanglements from the first measurement, and entangle the second apparatus with spin and position eigenstates. We return the first apparatus to the ready state, as it is no longer measuring the system and no torque is exerted on the magnet, and return position back to $\left|\varnothing_\mathcal{X}\right\rangle$ as the electron leaves the analyzer. $U(t_1, t_0)^\dagger$ reverses the entanglement, but conveniently, $U(t_1, t_0)$ is Hermitian. So, the a_1 component of the unitary operator for the second measurement is the same as that of the first measurement:

$$U(t_2, t_1)_{a_1} = U(t_1, t_0) \tag{4.17}$$

For the a_2 component, we want to entangle pointer states with position and spin states only when the first apparatus has measured spin-up. In the branch where spin-down has been measured, we use the identity operator $I = I_s \otimes I_\mathcal{X} \otimes I_{a_1} \otimes I_{a_2}$ to represent the lack of a second measurement. In the branch where spin-up has been measured, we apply the measurement scheme again. The a_2 component of the unitary operator during second measurement is

$$\begin{aligned} U(t_2, t_1)_{a_2} = {} & P^x_+ P^z_+ \otimes E^{x_2}_+ \otimes I_{a_1} \otimes E^{a_2}_+ \\ & + P^x_- P^z_+ \otimes E^{x_2}_- \otimes I_{a_1} \otimes E^{a_2}_- \\ & + P^z_- \otimes I_\mathcal{X} \otimes I_{a_1} \otimes I_{a_2} \end{aligned} \tag{4.18}$$

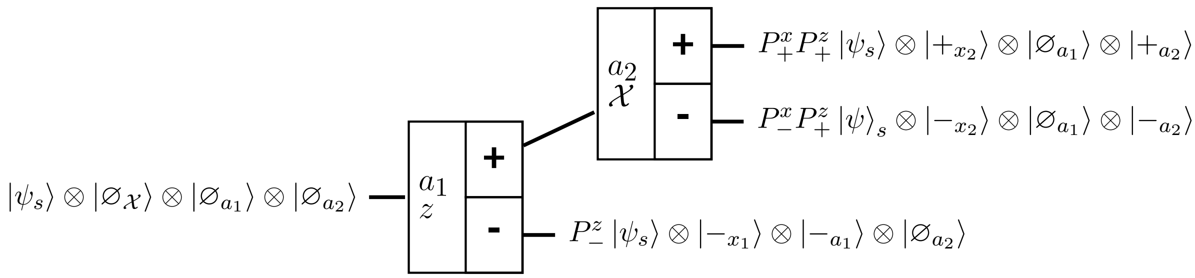

FIGURE 4.3 TODO: caption

So the complete unitary operator for the second measurement is

$$
\begin{aligned}
U(t_2,t_1) &= U(t_2,t_1)_{a_2} U(t_2,t_1)_{a_1} \\
&= P^x_+ P^z_+ \otimes E^{x_2}_+ E^{x_1}_+ \otimes E^{a_1}_+ \otimes E^{a_2}_+ \\
&+ P^x_- P^z_+ \otimes E^{x_2}_- E^{x_1}_+ \otimes E^{a_1}_+ \otimes E^{a_2}_- \\
&+ P^z_- \otimes E^{x_1}_- \otimes E^{a_1}_- \otimes I_{a_2}
\end{aligned}
\tag{4.19}
$$

and the final state is represented in Figure TODO REF.

5 Consistent Histories

The fourth postulate TODO REF makes probabilistic predictions of measurement results. The mathematics of this postulate still apply, but now that we are using the von Neumann measurement scheme, we can no longer frame the calculation of probabilities in the context of the standard description of measurement. That is, the formalism can stay, but we need different words surrounding it to give it meaning. The *consistent* (or *decoherent*) *histories* interpretation of quantum mechanics modifies this postulate to make predictions about the more general *quantum history* rather than measurement results.

TODO: history about development. preview ordering. Griffiths introduced the logic/math, while Gell-Man/Hartle/Craig's work informs (and is informed by) cosmological applications. First section will reiterate Griffith's articuation of foundations of theory in Schrödinger picture, following will examine simplifications made by using Heisenberg picture as used by GMHC.

So far, we have assumed the Schrödinger picture of time evolution, in which the unitary operator acts on the quantum state. In the Heisenberg picture, the unitary operator acts on operators in the Hilbert space rather than states. TODO better explain this. Consistency conditions are expressed intuitively in the Schrödinger picture, and with simplifications in the Heisenberg picture. TODO finish this.

5.1 Properties, Events and Histories

A *quantum property* is a true or false statement about a physical variable. Recalling Example 1 TODO REF, we find through experiment that the system will posses one of two properties at the end of measurement:

- "Spin along the z axis is $\frac{\hbar}{2}$"
- "Spin along the z axis is $-\frac{\hbar}{2}$"

A property corresponds to a subspace of the Hilbert space. For example, the subspace corresponding to the spin-up property is only the state $|+\rangle$.

A *quantum event* is a system's possession of a property. An event is represented by the projection operator for the property's subspace. For the spin-up example, this operator is P_+^z.

A *quantum history* is a set of events at sequential times. Now that we have dropped the projection postulate, the TODO REF sixth postulate now completely describes how states evolve with time. A history is a finite set of events that necessarily ignores an infinite amount of insignificant events.

For example, consider the example of consecutive measurement in TODO REF. The history for measuring spin up in both analyzers is $\{P_+^z, P_+^x\}$.

5.2 Schrödinger Picture

A history, as a set of events, specifies possible system states at multiple instances of time. Formally, this is no different than specifying possible states of a composite system consisting of a copy of $\mathcal{H}$ for each instant in time [2]. This motivates the definition of a *history Hilbert space*. Once again, the tensor product is employed to create a composite system; this time, the entire spin-pointer system is considered at different points in time. The history Hilbert space representing $|\psi\rangle$ at times $(t_0, t_1, t_2, ..., t_f)$ is

$$\mathcal{H}_h = \mathcal{H}_{t_0} \odot \mathcal{H}_{t_1} \odot \mathcal{H}_{t_2} \odot ... \odot \mathcal{H}_{t_f} \tag{5.1}$$

where $\odot$ is the ordinary tensor product, but denotes that the same quantum system $\mathcal{H}$ is considered at different times.

In this history Hilbert space, the state representing the system at times $(t_0, t_1, t_2, ..., t_f)$ is

$$|\psi_h\rangle = |\psi_{t_0}\rangle \odot |\psi_{t_1}\rangle \odot |\psi_{t_2}\rangle \odot ... \odot |\psi_{t_f}\rangle \tag{5.2}$$

where each component $|\psi_{t_i}\rangle$ is determined by the unitary dynamics experienced by the system up until that point, $U(t_i, t_0)$. In other words,

$$|\psi_h\rangle = |\psi_{t_0}\rangle \odot U(t_1, t_0)\,|\psi_{t_0}\rangle \odot U(t_2, t_0)\,|\psi_{t_0}\rangle \odot ... \odot U(t_f, t_0)\,|\psi_{t_0}\rangle \tag{5.3}$$

In this history Hilbert space, a history is represented by the tensor product of its events. That is, history $n = (P_0, P_1, P_2, ..., P_f)$ is represented by $P_n^h = P_0 \odot P_1 \odot P_2 \odot ... \odot P_f$.

5.2.1 Extending the Probability Postulate

The standard probability postulate TODO REF is written as the inner product of the system state $|\psi\rangle$ and an eigenstate of a physical variable $|a_n\rangle$. It can instead be written in terms of the projection operator for $|a_n\rangle$ by expanding the complex square.

$$\mathcal{P}(n) = |\,\langle a_n|\psi\rangle\,|^2 \tag{5.4}$$

$$
\begin{aligned}
&= \langle\psi|a_n\rangle \langle a_n|\psi\rangle \\
&= \langle\psi| P_n^a |\psi\rangle
\end{aligned}
$$

To extend this postulate to make predictions about histories, we replace the system $|\psi\rangle$ with the history system $|\psi_h\rangle$ and the measurement projector P_n^a with the history projector P_n^h.

$$
\mathcal{P}(h_n) = \langle\psi_h| P_n^h |\psi_h\rangle \tag{5.5}
$$

5.2.1.1 Example 1

As shown in TODO REF, the measurement of an initial spin state $|\psi_S\rangle$ results in

$$
\begin{aligned}
|\psi_1\rangle &= U(t_1, t_0) |\psi_0\rangle \\
&= P_+^{S_z} |\psi_S\rangle \otimes |+_\mathcal{X}\rangle \otimes |+_A\rangle \ + \ P_-^{S_z} |\psi_S\rangle \otimes |-_\mathcal{X}\rangle \otimes |-_A\rangle
\end{aligned} \tag{5.6}
$$

In the history Hilbert space, the state representing the system before and after measurement is

$$
\begin{aligned}
|\psi_h\rangle &= |\psi_0\rangle \odot |\psi_1\rangle \\
&= (|\psi_S\rangle \otimes |\varnothing_\mathcal{X}\rangle \otimes |\varnothing_A\rangle) \odot \left(P_+^{S_z} |\psi_S\rangle \otimes |+_\mathcal{X}\rangle \otimes |+_A\rangle \ + \ P_-^{S_z} |\psi_S\rangle \otimes |-_\mathcal{X}\rangle \otimes |-_A\rangle\right)
\end{aligned} \tag{5.7}
$$

The history for measuring spin-up is composed of the projector for an initial spin state with a ready position and apparatus, and the projector for an up spin state with an up position and apparatus:

$$
\begin{aligned}
P_+^h &= P_\varnothing \odot P_+ \\
&= \left(P_{\psi_S}^{S_z} \otimes P_\varnothing^\mathcal{X} \otimes P_\varnothing^A\right) \odot \left(P_+^{S_z} \otimes P_+^\mathcal{X} \otimes P_+^A\right)
\end{aligned} \tag{5.8}
$$

Using the new probaility postulate, the probability of measuring spin-up is

$$
\begin{aligned}
\mathcal{P}(h_+) &= \langle\psi_h| P_+^h |\psi_h\rangle \\
&= \langle\psi_h| P_\varnothing \odot P_+ |\psi_h\rangle \\
&= \langle\psi_h| (|\psi_S\rangle \otimes |\varnothing_\mathcal{X}\rangle \otimes |\varnothing_A\rangle) \odot \left(P_+^{S_z} |\psi_S\rangle \otimes |+_\mathcal{X}\rangle \otimes |+_A\rangle\right) \\
&= \langle\psi_S| P_+^{S_z} |\psi_S\rangle
\end{aligned} \tag{5.9}
$$

and we recover the prediction of the standard Born Rule.

5.2.2 Consistency Conditions

The third postulate TODO REF defines the subset of states corresponding to "measurement results", and the fourth postulate makes predictions about these states only. Now that we have extended the Born Rule to make predictions about histories, we need to be careful about the context in which the predictions are made, as it is no longer postulated for us.

5.2.2.1 Example 1

The standard Born Rule makes predictions for measurements of spin-up and spin-down along the z axis. Since these states exhaust possible outcomes, the probabilities of their measurement must sum to unity

$$(\mathcal{P}(+) + \mathcal{P}(-) = 1) \tag{5.10}$$

We saw in TODO REF that the extended Born Rule reproduced the probabilities of the standard Born Rule. However, the extension goes on to make predictions about other outcomes. For example, we could ask about the probability that spin-up in the x direction is measured using the history

$$\begin{aligned} P^h_{+_x} &= P_\varnothing \odot P_{+_x} \\ &= \left(P^{S_z}_{\psi_S} \otimes P^{\mathcal{X}}_\varnothing \otimes P^A_\varnothing\right) \odot \left(P^{S_x}_+ \otimes P^{\mathcal{X}}_+ \otimes P^A_+\right) \end{aligned} \tag{5.11}$$

which results in

$$\begin{aligned} \mathcal{P}(h_{+_x}) &= \langle\psi_h| \, P^h_{+_x} \, |\psi_h\rangle \\ &= \langle\psi_s| \, P^{S_x}_+ \, |\psi_s\rangle \end{aligned} \tag{5.12}$$

which is non-zero in general. We could ask an infinite amount of similar questions, since the spin Hilbert space includes states representing spin-up along every direction in space TODO REF STATE SPACE SECTION. Consequently, our probabilities no longer sum to unity, which is not consistent with probaility theory.

The solution is to use *consistency conditions* to determine the sets of histories that are consistent with probability theory. The consistency conditions require that histories represent exhaustive and mutually exclusive outcomes:

$$P^{h\dagger}_i P^h_j = \delta_{i,j} P^h_i \tag{5.13}$$

$$\sum_i P_i^h = I_h \tag{5.14}$$

A set of histories satisfying these conditions is called a *consistent family* of histories. Once a consistent family is specified, we can use the extended Born Rule to calculate probabilities within that context. For our example, $\{P_+^h, P_-^h, P_0^h\}$ is a consistent family where

$$\begin{aligned} P_+^h &= P_\varnothing \odot P_+ \\ P_-^h &= P_\varnothing \odot P_- \\ P_0^h &= I_h - P_+^h - P_-^h \end{aligned} \tag{5.15}$$

P_0^h represents any history distinct from P_+^h and P_-^h. Showing this,

$$\begin{aligned} P_0^{h\dagger} P_\pm^h &= (I_h - P_+^h - P_-^h) P_\pm^h \\ &= P_\pm^h - P_\pm^h \\ &= 0 \end{aligned} \tag{5.16}$$

Furthermore, P_+^h and P_-^h are distinct since

$$P_+^{\mathcal{X}\dagger} P_-^{\mathcal{X}} = 0 \tag{5.17}$$

so all histories in this set are mutually exclusive.

Showing that the set is exhaustive,

$$\begin{aligned} P_+^h + P_-^h + P_0^h &= P_+^h + P_-^h + \left(I_h - P_+^h - P_-^h\right) \\ &= I_h \end{aligned} \tag{5.18}$$

Now that the consistency of the family is confirmed, we can use TODO REF to find probabilities for each history. Notice that P_0^h is included to make the set exhaustive; even though its probability of occurence is 0, its inclusion in the family enables such a prediction.

5.2.3 Interpretation

TODO: describe how environment is implied, records every component of system at all times

5.3 Heisenberg Picture

TODO: describe environment as perfect record keeper

Before, we applied dynamics to the state to find $|\psi_0\rangle$, $|\psi_1\rangle$, etc. Now, we apply dynamics to each operator in the Hilbert space. The operators representing events become

$$\bar{P}_n = U(t_n, t_0)P_n \tag{5.19}$$

and a history TODO REF becomes

$$\bar{h_n} = \left(\bar{P}_0,\ \bar{P}_1,\ \bar{P}_2,\ ...,\ \bar{P}_f\right) \tag{5.20}$$

In the Hilbert space, a history is represented by a *class operator*

$$C_{h_n} = \bar{P}_0\bar{P}_1\bar{P}_2...\bar{P}_f \tag{5.21}$$

5.3.1 Extending the Probability Postulate

We again use the form of TODO REF EQs, replacing $|\psi_h\rangle$ with $|\psi\rangle$ and P_n^h with C_n:

$$\mathcal{P}(h_n) = \langle\psi|\, C_n\, |\psi\rangle \tag{5.22}$$

Since $\mathcal{P}(h_n)$ is a real scalar,

$$\begin{aligned} \mathcal{P}(h_n) &= \mathcal{P}(h_n)^\dagger \\ \mathcal{P}(h_n) &= \langle\psi|\, C_n^\dagger\, |\psi\rangle \end{aligned} \tag{5.23}$$

so that the class operator projects in the same order as the events:

$$C_n^\dagger\, |\psi\rangle = \bar{P}_f...\bar{P}_2\bar{P}_1\bar{P}_0\, |\psi\rangle \tag{5.24}$$

We call $C_n^\dagger\, |\psi\rangle$ the *branch wave function* for the history h_n. Notice that in TODO REF FIGURES, the branch wavefunctions are represented by following a path from the initial state to some final outcome.

5.3.2 Consistency Conditions

Since histories are now represented by class operators rather than projectors into the history Hilbert space, the condition of mutually exclusive and exhaustive outcomes is

$$C_i^\dagger C_j = \delta_{i,j}C_i \tag{5.25}$$

$$\sum_i C_i = I \tag{5.26}$$

5.3.3 Example 2

TODO REF FIG, INTRO PARAGRAPH

6 Decoherence

TODO: exemplify how tracing out degrees of freedom results in mixed state, discuss what happens if environment is not a perfect record keeper.

TODO: discuss any structural or algorithmic changes resulting from rewriting spins simulation

7 Conclusion

Bibliography

[1] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik*, Julius Springer (1932).

[2] R. Griffiths, *Consistent Quantum Theory*, Cambridge University Press (2002).

7.1 Einselection vs. Inselection

We could solve the preferred basis problem by including the environment rather than the apparatus. Such an approach is called superselection. However, the apparatus must exist by Newton's third law, and it solves the problem without reference to the environment. So, it seems reasonable that the configuration of the measurement fixes the basis (reflected in dynamics), as the basis we observe is a property of the apparatus. We will see that the role of the environment is to record the history TODO.