A Spins-First Introduction to Consistent Histories and Decoherence

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Abstract

Standard quantum mechanics makes foundational assumptions describing the measurement process. We show that postulating state collapse artifically limits the scope of quantum theory, motivating a unitary description of measurement. In the context of Stern-Gerlach experiments, we explain measurement as entanglement of the measured spin system, the measured position system, and the measuring apparatus. To interpret the entangled state, the consistent histories approach is used to make probabilistic predictions. To do this, the environment must also be entangled with the system, playing the role of a record keeper. The differing methods used by Griffiths and Gell-Mann/Hartle to incorporate the environment is articulated. We conclude by exemplifying tangible advantages of this approach by simplifying code that simulates consecutive Stern-Gerlach measurements.

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1 Introduction

Quantum mechanics is plagued by interpretational issues surrounding measurement. The standard description of measurment postualates a special type of dynamics in which a quantum system instantaneously evolves upon measurement. The conditions in which this postulate applies are not well defined, leading to confusion on the nature of measurement itself.

The predictions of the standard description of measurement can be reproduced using alternative descriptions of the measurement process. We study an alternative description that explains many, but not all, of the infamous measurement related problems. The *von Neumann measurement scheme* is used to describe how states evolve when being measured, and the *consistent* or *decoherent histories* interpretation of quantum mechanics is used to explain what is happening physically.

Numerous papers and books describe these concepts in detail TODO CITE, but they have yet to permeate far outside the quantum foundations community. A primary goal of this thesis is to introduce these concepts in a form more accessible to those new in their study of quantum foundations or physicists with other specialties. Following the lead of research in spins-first introductions to quantum mechancis in physics education TODO CITE, we introduce these new ideas in the context of the Stern-Gerlach experiment. Having only two degrees of freedom, spin-$\frac{1}{2}$ systems are the simplest possible. We explain all fundamental aspects of quantum mechanics within this context, as well as our proposed changes.

Descriptions of measuring spin-$\frac{1}{2}$ systems in this way are exemplified in works by Griffiths, Hohenberg, and Schlosshauer TODO CITE. However, they either require prior knowledge of concepts in quantum foundations, neglect implementing von Neumann measurement, or do not offer interpretational explanations. Our explanation of Stern-Gerlach experiments does all of these things. Another primary goal of this thesis is showing how to implement these ideas explicitly. In doing so, we introduce a unitary operator describing measurement specific to ideal measurement of spin-$\frac{1}{2}$ systems. Using the same tools, we describe how states decay in time through *decoherence*.

In addition to circumventing some measurement related paradoxes, our description of quantum mechanics contains other tangible advantages. We exemplify this by comparing the simulation of measurement in both frameworks. Existing code simulating sequential Stern-Gerlach measurements is used as a baseline TODO CITE, and relevant code is rewritten using our new formalism. The resulting program control flow becomes drastically simplified. We conclude by discussing existing and future experiments that may distinguish whether the standard or *relative states* formalisms correctly represent physical reality.

The Stern-Gerlach is the archetypal experiment for quantum measurement.
In addition to its historical significance,

2 Stern-Gerlach Experiments

TODO: provide a brief explanation of the experimental setup and results. Discuss historical and pedagogical significance. Explain how measurement results correspond to the particle's localization in certain regions. This section is background information so that I can discuss the von Neumann measurement, consistent histories, and decoherence in the context of this experiment. I am saving it for later to focus on the introducing the new theory for now.

3 Postulates of Quantum Mechanics

We first consider the mathematical objects used to model physical systems and variables. By comparing the objects used in classical and quantum mechanics, we make sense of the first three postulates of quantum mechanics. Then, we compare the Copenhagen and von Neumann descriptions of measurement and their relation to the fourth and fifth postulates.

3.1 Physical Variables and State Spaces

3.1.1 Classical States

Consider the spin of an electron. Treating the electron as a classical system, its spin state is modeled by a vector $\vec{S} \in \mathbb{R}^3$:

$$\vec{S} = (S_x, S_y, S_z) \tag{3.1}$$

Each component S_{x_i} is a physical variable representing the magnitude of spin oriented in the $\hat{x}_i$ direction.

$\vec{S}$ has the capacity to determine spin in any direction using the inner product of the state space $\mathbb{R}^3$:

$$S_n(S) = \vec{S} \cdot \hat{n} \tag{3.2}$$

We see that in classical mechanics, physical variables are modeled using functions. Each function S_n maps a spin state [$\vec{S}$ to a real scalar representing the spin of the electron aligned along the $\hat{n}$ axis.

What makes classial mechanics familiar to everyday experience boils down to intuitive but important properties of the state space $\mathbb{R}^3$:

- For any direction $\hat{n}$, $\vec{S}$ determines spin S_n
- S_n can be any real value

$\vec{S}$ determines spin in any direction because TODO. Consequently, the sample spaces for spin in any two directions $\hat{n}$ and $\hat{m}$ are *compatible*, meaning that S_n and S_m may be simultaneously determined. Spin states in $\mathbb{R}^3$ are interpreted physically as the electron posessing definite values for every S_n at some instant in time.

In addition to spin states determing all S_n, the state space allows S_n to take on any real value. There are no fundamental restrictions on which real numbers S_n could be; its sample space is continuous and infinitely large.

3.1.2 Quantum States

Measurements of electron spin show that the intuitive classical properties do not hold. Recall that only two magnitudes of spin have ever been measured. S_n is a *quantized* physical variable; its sample space is discrete and finite.

Second, the results of successive measurements of a spin system imply that $\vec{S}$ does not determine spin in some general direction S_n. Recall the results of succesively measuring spin in orthogonal directions discussed in TODO REF. After measuring S_x, $\vec{S}$ appears to "forget" a previous measurement of S_z. All we may know about the system at one instant in time is spin in one direction. The inabilitiy to simultaneously determine spin in two independent directions $\hat{n}$, $\hat{m}$ should be reflected through S_n and S_m having *incompatible* sample spaces.

Electron spin measurements violate the intuitive classical state space properties mentioned in 3.1.1. In response, we must change the mathematical objects used to represent system states and physical variables. Specifically, the sample space of S_n must restrict observable values to spin up and spin down, and S_n and S_m should have incompatible sample spaces. In combination, the first three postulates of quantum mechanics take care of these differences.

Quantum mechanics postulates that a system's state is completely described by a normalized vector in a linear state space.

POSTULATE 1 *The state of a physical system is defined by specifying an abstract vector $|\psi\rangle$ in a Hilbert state space $\mathcal{H}$.*

For spin-$\frac{1}{2}$ systems such as electrons, the two-dimensional Hilbert space consists of all linear combinations of spin-up and spin-down:

$$|\psi\rangle \in \mathcal{H}\, |\psi\rangle \left\{\alpha\, |+_{S_z}\rangle + \beta\, |-_{S_z}\rangle\right\}$$

where $\alpha, \beta \in \mathbb{C}$.

$\mathcal{H}$ is an abstract state space; components of $|\psi\rangle$ cannot be interpreted as physical variables as they are for the classical spin state S. So, we introduce physical meaning with more postulates.

The second posulate of quantum mechanics states that physical variables are described by linear operators:

POSTULATE 2 *Every physical variable $\mathcal{A}$ is described by an operator A acting in $\mathcal{H}$.*

Justifying the second postulate is easiest when also considering the third postulate:

POSTULATE 3 *The only possible result of the measurement of a physical variable $\mathcal{A}$ is one of the eigenvalues of the corresponding operator A.*

The operator correlates elements of a finite sample space (eigenvalues) with particular system states (eigenstates). Consequently, a state can only be interpreted as having a definite variable value if it is an eigenstate of that variable's operator. To illustrate this, consider the operator representing S_z. Written in the basis of its own eigenstates,

$$S_z \doteq \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.3}$$

This operator correlates z spin-up $\left(S_z = \frac{\hbar}{2}\right)$ with eigenstate

$$|\psi\rangle = |+_{S_z}\rangle \doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3.4}$$

and z spin-down $\left(S_z = \frac{-\hbar}{2}\right)$ with eigenstate

$$|\psi\rangle = |-_{S_z}\rangle \doteq \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.5}$$

Here, the subscript z specifies that the state represents spin-up along the z axis.

Similarly, we write the operator representing S_x in the S_z basis:

$$S_x \doteq \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{3.6}$$

This operator correlates x spin-up $\left(S_x = \frac{\hbar}{2}\right)$ with eigenstate

$$|\psi\rangle = |+_{S_x}\rangle \doteq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{3.7}$$

and x spin-down $\left(S_x = \frac{-\hbar}{2}\right)$ with eigenstate

$$|\psi\rangle = |-_{S_x}\rangle \doteq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{3.8}$$

Operators for S_z and S_x share no common eigenstates, so no state can posses definite values for both variables. In general, operators for any two spin components S_i and S_j do not share common eigenstates with each other; in other words, S_i and S_j have incompatible sample spaces. S_i and S_j are called *complementary* variables.

By representing physical variables with operators rather than functions, sample spaces become quantized and may be incompatible with each other. These features are necessary for predicting the results of electron spin measurements.

The first three postulates designate the mathematical objects used to model physical system states and variables. The fundamental differences between classical and quantum systems are completely described by these postulates and their consequences.

3.1.3 Linearity

TODO: compare addition of S_x, S_y states and their interpretations. Introduce superposition states and coherence.

3.2 Copenhagen Description of Measurement

The fourth and fifth postulates constitute the Copenhagen description of measurement. This description, taught in textbooks and introductory quantum courses, is a key component of the standard interpretation of quantum mechanics.

The probability postulate, also known as the *Born Rule*, assigns a probability distribution to the sample space of a physical variable.

POSTULATE 4 *When measuring physical variable A, the probability $\mathcal{P}(n)$ of obtaining result a_n corresponding to $|a_n\rangle$ is equal to*

$$\mathcal{P}(n) = |\langle a_n|\psi\rangle|^2 \tag{3.9}$$

The spirit of the Born Rule is unchanged in consistent quantum theory. Differences are discussed in (TODO: ref future section).

The fifth postulate (known as the projection postulate) describes how a system evolves upon

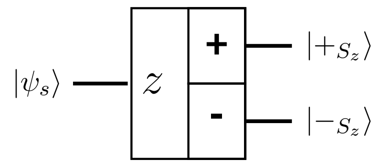

FIGURE 3.1 Stern-Gerlach Experiment 1 as described by the Copenhagen description of measurement. Each measurement outcome

measurement. Contingent upon interaction of the system with a “classical apparatus”, measurement instantaneously changes the state of the system to some eigenstate of the variable being measured.

POSTULATE 5 *If the measurement of the physical variable $\mathcal{A}$ on the system in the state $|\psi\rangle$ gives the result a_n, the state of the system immediately after the measurement is the normalized projection*

$$|\psi'\rangle = \frac{P_n\,|\psi\rangle}{\sqrt{\langle\psi|\,P_n\,|\psi\rangle}} \tag{3.10}$$

onto the subspace associated with a_n.

3.2.3 Experiment 1

Consider a measurement result for the z component of spin. By the third postulate, the result is either either spin-up or spin-down $\left(S_z = \pm\frac{\hbar}{2}\right)$. By the projection postulate, the state evolves to the normalized projection of $|\psi\rangle$ onto $|\pm\rangle$. In other words, $|\psi\rangle$ instantaneously becomes $|\pm\rangle$ upon measurement. This process is known as *state collapse* or *wavefunction collapse*. The spatial separation of these outcomes is represented by Figure 3.1

The Born Rule gives the probabilities of measuring spin-up and spin-down as a function of initial spin state:

$$\mathcal{P}(\pm) = |\,\langle\pm|\psi_s\rangle\,|^2 \tag{3.11}$$

3.2.4 Experiment 2

Now we consecutively measure spin as shown in Figure 3.2. The first apparatus serves as a state preparation device, since we are only interested in particles exiting the spin-up output. Using the projection postulate, the state after the first measurement is

$$|\psi'\rangle = \frac{P_+^{S_z}\,|\psi\rangle}{\sqrt{\langle\psi|\,P_+^{S_z}\,|\psi\rangle}} = |+_{S_z}\rangle \tag{3.12}$$

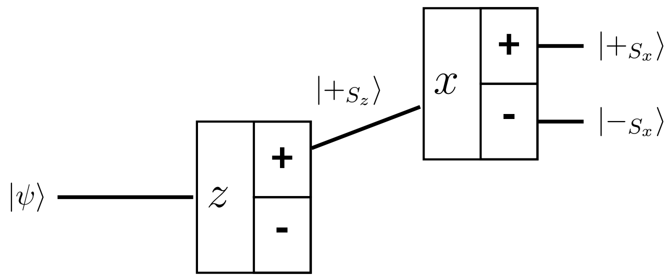

FIGURE 3.2 The Stern-Gerlach experiment as described by the standard measurement scheme. Notice that each measurement outcome is renormalized, so that information about the state prior to measurement is lost.

Similarly, the possible output states from the second apparatus are

$$|\psi''\rangle = \frac{P_{+}^{S_x}|+_{S_z}\rangle}{\sqrt{\langle +_z| P_{+}^{S_x}|+_{S_z}\rangle}} = |+_{S_x}\rangle \tag{3.13}$$

or

$$|\psi''\rangle = \frac{P_{-}^{S_x}|+_{S_z}\rangle}{\sqrt{\langle +_z| P_{-}^{S_x}|+_{S_z}\rangle}} = |-_{S_x}\rangle \tag{3.14}$$

Because we ignore spin-down particles from the first measurement, we are certain that particles entering the second analyzer are in the spin-up state. The probabilities assigned to each state leaving the S_x Stern-Gerlach device are

$$\mathcal{P}(+_x) = |\langle +_{S_x}|+_{S_z}\rangle|^2 = \frac{1}{2} \tag{3.15}$$

$$\mathcal{P}(-_x) = |\langle -_{S_x}|+_{S_z}\rangle|^2 = \frac{1}{2} \tag{3.16}$$

3.3 Dynamics

In a mechanical theory, the equations of motion (or *dynamics*) describe how a state evolves with time. In classical Newtonian mechanics, this is given by Newton's law of motion

$$\vec{F} = m\vec{a}. \tag{3.17}$$

These dynamics are *unitary*, meaning that given the final state of a physical process, the corresponding initial state is recovered by applying the dynamics with time reversed. The dynamics can be represented by a one-to-one map from initial to final states.

The projection postulate describes one type of dynamics, which apply only during measurement. When applied, all information about the initial state is lost as the state instantaneously becomes an eigenstate of the measured variable. The map from initial to final states is not one-to-one; “collapse dynamics” are non-unitary.

Quantum theory postulates an another type of dynamics that is analogous to Newton’s law of motion. These dynamics are unitary, and apply at all times (not just during measurement).

POSTULATE 6 *TODO write Schrödinger equation*

4 Measurement

TODO: chapter preview

State collapse requires extra assumption, ambiguous defs lead to interepretation issues, arrow of time

4.1 Issues with State Collapse

The projection postulate introduces foundational assumptions to describe the measurement process. The principle of Occam's razor says that, in general, a theory is strengthened by making as few assumptions as possible. In classical mechanics, there are no foundational assumptions made to describe measurement; this motivates the pursuit to describe quantum measurement without using the projection postulate.

Furthermore, the projection postulate relies on ambiguous definitions. State collapse occurs upon "interaction with a classical measuring apparatus", yet there is no specification of what makes a system classical. Classical systems are not described by the theory, yet they play a fundamental role in the measurement process.

TODO: describe interpretational issues

Because the measurement process cannot be reversed, state collapse injects time asymmetry into the foundations of quantum mechanics. TODO: discuss arrow of time.

The issues with interpretation of state collapse and non-unitary dynamics in general are indicators that collapse dynamics are formulated with ignorance of some underlying process. To begin describing this process, we discard the projection postulate and describe measurement using dynamics permitted by the Schrödinger equation.

Describing measurement as a unitary process is desirable for multiple reasons:

- less fundamental assumptions
- Humans and measurement apparatuses do not play a special role indescribale by the theory
- interpretational issues are circumvented
- With dynamics symmetric in time, the emergence of the "arrow of time" can be studied
- TODO: describe cosmology benefit.

Fortunately, such a description is possible using the von Neumann measurement scheme.

4.2 von Neumann Measurement Scheme

In the discussion of Stern-Gerlach experiments, the position of the electron played an implicit role in measurement. In exemplifying use of the projection posulate TODO, we define a measurement result as the localization of the electron in the spin-up or spin-down regions. The primary measurement is that of position, which is used to imply the spin state, yet the position system is never formalized.

Our goal is to formalize the correlation of position and spin eigenstates observed in Stern-Gerlach experiments. We start by representing the electron with a composite spin-position system,

$$\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_{\mathcal{X}} \tag{4.1}$$

4.2.1 Experiment 1

We revisit the example of one Stern-Gerlach measurement of spin along the z axis. We name position states of interest; $|\varnothing_{\mathcal{X}}\rangle$ represents the localization at the beginning of the analyzer (which we call the "ready" position), while $|+_{\mathcal{X}}\rangle$ and $|-_{\mathcal{X}}\rangle$ represent localization at the spin-up and spin-down outputs, respectively (reference TODO REF FIG). Numerous papers detail these position states using Gaussian wave packets TODO REF, but for simplicity we leave these abstracted. We will say that $\{|+_{\mathcal{X}}\rangle, |-_{\mathcal{X}}\rangle, |\varnothing_{\mathcal{X}}\rangle\}$ are mutually orthonormal.

We introduce an operator that correlates these position states with spin eigenstates in explicit form:

$$\begin{aligned} U(t_1, t_0) = P_+^{S_z} \otimes \Big(&|+_{\mathcal{X}}\rangle \langle\varnothing_{\mathcal{X}}| \;+\; |\varnothing_{\mathcal{X}}\rangle \langle +_{\mathcal{X}}| \;+\; I_{\mathcal{X}} \;-\; P_+^{\mathcal{X}} \;-\; P_\varnothing^{\mathcal{X}} \Big) \\ + P_-^{S_z} \otimes \Big(&|-_{\mathcal{X}}\rangle \langle\varnothing_{\mathcal{X}}| \;+\; |\varnothing_{\mathcal{X}}\rangle \langle -_{\mathcal{X}}| \;+\; I_{\mathcal{X}} \;-\; P_-^{\mathcal{X}} \;-\; P_\varnothing^{\mathcal{X}} \Big) \end{aligned} \tag{4.2}$$

Focusing on the $H_{\mathcal{X}}$ component, the first term maps TODO.

Starting with a general spin state, the final state is

$$\begin{aligned} U(t_1, t_0)\, |\psi\rangle &= U(t_1, t_0)\, (|\psi_s\rangle \otimes |\varnothing_{\mathcal{X}}\rangle) \\ &= P_+^{S_z} |\psi_s\rangle \otimes |+_{\mathcal{X}}\rangle \;+\; P_-^{S_z} |\psi_s\rangle \otimes |-_{\mathcal{X}}\rangle \end{aligned} \tag{4.3}$$

At the instant measurement begins t_0, the position state is $|\varnothing_{\mathcal{X}}\rangle$ as the electron enters the magnetic field. At the instant measurement ends t_1, the position state is either $|+_{\mathcal{X}}\rangle$ or $|-_{\mathcal{X}}\rangle$, realized with spin-up and spin-down spin states respectively. Notice that the final sum does not contain any terms representing incorrect correlations between spin and position states (such as

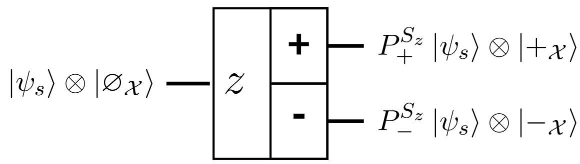

FIGURE 4.1 The Stern-Gerlach experiment as described by the von Neumann measurement scheme. Each measurement outcome corresponds to a term in the time-evolved state (Equation 4.3). Notice that the measurement interaction results in a branching structure, represented here as a tree graph with the apparatus as a node.

$P_+^{S_z}|\psi_s\rangle \otimes |-_\mathcal{X}\rangle$). Consequently, the final state cannot be written as the tensor product of a state in $\mathcal{H}_s$ and a state in $\mathcal{H}_\mathcal{X}$ (as the inital state was). This is the definition of *entanglement*; the von Neumann measurement scheme describes the measurement process as entanglement. $U(t_1, t_0)$ is *the* unitary operator accomplishing the desired correlation, as evident by $UU^\dagger = I$.

The von Neumann scheme is usually written as a linear map:

$$U(t_1, t_0):$$
$$|\psi\rangle = \left(\sum_n P_n^{S_z}|\psi_s\rangle\right) \otimes |\varnothing_\mathcal{X}\rangle \mapsto \sum_n \left(P_n^{S_z}|\psi_s\rangle \otimes |n_\mathcal{X}\rangle\right) \tag{4.4}$$

where $n = +, -$.

Notice that the initial state is a single tensor product, while the final state is a sum of tensor products. The coherence initially present only in the spin state is extended to the composite spin-momentum system. This process is represented schematically in TOD ref; the initial state branches into two distinct outcomes, each represented by a term in the final state.

4.3 Preferred Basis Problem

By describing measurement as a unitary process, we have eliminated many aspects of the quantum measurement problem. TODO cite specific examples. However, there are two overarching problems with interpretation of the final state of the von Neumann measurement scheme. These are called the *preferred basis problem* and the *problem of outcomes* in literature [**?**] [**?**]. The problem of outcomes remains an open research question, and potential solutions lay outside the scope of this thesis. The preferred basis problem can be solved using the tools developed so far, so we show how it may be solved by including the apparatus in our measurement model.

The preferred basis problem arises from the ability to write the final state in Equation 4.4 using different bases:

$$|\psi'\rangle = \sum_n \left(P_n^{S_z} |\psi_s\rangle \otimes |n_{\mathcal{X}}\rangle\right) = \sum_n \left(P_n^{S_z'} |\psi_s\rangle \otimes |n_{\mathcal{X}}'\rangle\right) = \ldots \tag{4.5}$$

Such a case is exemplified using Experiment 1.

4.3.1 Experiment 1

Consider setting the initial spin state to spin-up in the x direction:

$$|\psi_s\rangle = |+_{S_x}\rangle = \frac{|+_{S_z}\rangle + |-_{S_z}\rangle}{\sqrt{2}} \tag{4.6}$$

The final state by Equation 4.2 is

$$|\psi'\rangle = \frac{|+_{S_z}\rangle \otimes |+_{\mathcal{X}}\rangle + |-_{S_z}\rangle \otimes |-_{\mathcal{X}}\rangle}{\sqrt{2}} \tag{4.7}$$

Similar to the S_x eigenstates, we define orthonormal position states

$$\begin{aligned} |+_{\mathcal{X}_x}\rangle &= \frac{|+_{\mathcal{X}}\rangle + |-_{\mathcal{X}}\rangle}{\sqrt{2}} \\ |-_{\mathcal{X}_x}\rangle &= \frac{|+_{\mathcal{X}}\rangle - |-_{\mathcal{X}}\rangle}{\sqrt{2}} \end{aligned} \tag{4.8}$$

so the final state can be written

$$\begin{aligned} |\psi'\rangle &= \frac{\left(\frac{|+_{S_x}\rangle+|-_{S_x}\rangle}{\sqrt{2}} \otimes \frac{|+_{\mathcal{X}_x}\rangle+|-_{\mathcal{X}_x}\rangle}{\sqrt{2}}\right) + \left(\frac{|+_{S_x}\rangle-|-_{S_x}\rangle}{\sqrt{2}} \otimes \frac{|+_{\mathcal{X}_x}\rangle-|-_{\mathcal{X}_x}\rangle}{\sqrt{2}}\right)}{\sqrt{2}} \\ |\psi'\rangle &= \frac{|+_{S_x}\rangle \otimes |+_{\mathcal{X}_x}\rangle + |-_{S_x}\rangle \otimes |-_{\mathcal{X}_x}\rangle}{\sqrt{2}} \end{aligned} \tag{4.9}$$

Equation 4.9 matches Equation 4.7 in form; it appears that the measurement process of spin along the z axis has entangled orthonormal position states with spin states along the x axis. If we regard such an entanglement as the measurement process, the von Neumann measurement scheme violates the princple of complementarity by simultaneously measuring S_z and S_x. We know the experimental setup was configured to measure S_z, but nothing in the theory singles out S_z as the *preferred basis*.

Returning to the general problem Equation 4.5, we note that $\{|n_s\rangle\}$ and $\{|n_{\mathcal{X}}\rangle\}$ are orthognal sets, so $|\psi'\rangle$ is a biorthongal system. The biorthogonal decomposition theorem states that alternate bases $\{|n_s'\rangle\}$ and $\{|n_{\mathcal{X}}'\rangle\}$ exist when $\langle n_s|\psi_s\rangle$ are not all distinct [1]. Limiting this condition to normalized states of spin-$\frac{1}{2}$ systems, states with coefficients $\langle n_s|\psi_s\rangle = \frac{1}{\sqrt{2}}$ for both $n = +$ and $n = -$ are subject to the preferred basis problem as shown. Since every state in $\mathcal{H}$ can be expressed in this form by expansion in the pertinent basis, the system is always subject to the preferred basis problem.

4.4 Inselection

To solve this problem, we need only to look back to the original phrasing of the von Neumann measurement scheme, where measurement is described as the entanglement of a microscopic system with a macroscopic measuring apparatus [2]. Many descriptions of non-unitary Stern-Gerlach measurement equate the electron position system with the apparatus itself [3]. This is a reasonable abstraction, as the position of the electron is used to "read off" the result of the measurement. However, this approach is misleading, because it conflates two distinct physical systems; the apparatus, and the position system belonging to the electron. By labeling the position system as the "apparatus", the degree of freedom corresponding to the actual apparatus is effectively ignored.

The preferred basis is most precisely solved by formalizing the spin-apparatus entanglement just as we did with spin and position. Newton's third law asserts that the force exerted on the electron by the apparatus magnet is paired with a force exerted on the magnet by the electron. The effect of torque exerted on the magnet is represented by the state of the apparatus. Similar to the position states, we define "up", "down", and "ready" apparatus states $\{|+_a\rangle, |-_a\rangle, |\varnothing_a\rangle\}$ and assert orthonormality. An entangled spin-position-appartus system takes the form

$$|\psi'\rangle = \sum_n \left(P_n^{S_z}|\psi_s\rangle \otimes |n_{\mathcal{X}}\rangle \otimes |n_a\rangle\right) \tag{4.10}$$

after measurement. While systems in the form Equation 4.4 do not generally have unique decompositions, systems with three or more components do by the triorthongal decomposition theorem, so long as all three components are expanded in orthongal sets [1]. Using this result, we cannot write the final state in another basis:

$$|\psi'\rangle = \sum_n \left(P_n^{S_z}|\psi_s\rangle \otimes |n_{\mathcal{X}}\rangle \otimes |n_a\rangle\right) \neq \sum_n \left(P_n^{S_z{}'}|\psi_s\rangle \otimes |n_{\mathcal{X}}{}'\rangle \otimes |n_a{}'\rangle\right) \tag{4.11}$$

Since there is the only possible way to write $|\psi'\rangle$, the preferred basis has been chosen by including interaction with the apparatus. Solving the preferred basis problem by formalizing the system-apparatus interaction is called *interaction induced superselection* or *inselection* [**?**].

4.5 Einselection

A more established and compatible approach uses the environment to select the preferred basis. This approach is called *environment induced superselection* or *einselection* [**?**]. The environment consists of every imaginable system other than the electron; because this necessarily includes the apparatus, inclusion of the environment selects a preferred basis.

The apparatus state is entangled with the spin state during measurement, and then returns to the ready state after measurement. The electron-apparatus system has no memory of its prior states; a given state only describes the system at one instant in time. However, the electron-apparatus system is in constant interaction with a multitude of other quantum systems surrounding it, the composition of which we call the environment. The environment continuously records *which-state information* about the electron and apparatus [**?**]. When the electron is realized with spin-up or spin-down, there are causal effects in the environment that encode the history of the electron. We can think of the environment as constantly interacting with our system to establish "facts of the universe".

We now formalize the spin-position-environment interaction, similar to how the spin-position correlation implied in the projection postulate was formalized. We name $|\varnothing_\epsilon\rangle$ representing the environment's recording of the apparatus in the ready state only, and $|\pm_\epsilon\rangle$ representing the environment's recording of the apparatus in spin-up or spin-down. Representing classically distinct outcomes, we assert orthonormality

$$\langle i_\epsilon | j_\epsilon \rangle = \delta_{i,j} \tag{4.12}$$

for $i, j = \varnothing, +, -$. Idealizing the environment as a perfect record keeper, the dynamics must map

$$U(t_1, t_0) : \quad |\psi\rangle = \left(\sum_n P_n^{S_z} |\psi_s\rangle \right) \otimes |\varnothing_\mathcal{X}\rangle \otimes |\varnothing_\epsilon\rangle \mapsto \sum_n \left(P_n^{S_z} |\psi_s\rangle \otimes |n_\mathcal{X}\rangle \otimes |n_\epsilon\rangle \right) \tag{4.13}$$

where $n = +, -$.

4.5.1 Experiment 1

Our system is now composed of spin, position, and environment systems $H = H_s \otimes H_\mathcal{X} \otimes H_\epsilon$. The unitary operator satisfying Equation 4.13 is

$$U(t_1, t_0) = P_+^{S_z} \otimes \Big(|+_\mathcal{X}\rangle \langle \varnothing_\mathcal{X}| \; + \; |\varnothing_\mathcal{X}\rangle \langle +_\mathcal{X}| \; + \; |-_\mathcal{X}\rangle \langle -_\mathcal{X}| \; + \; I_\mathcal{X} \; - \; P_+^\mathcal{X} \; - \; P_\varnothing^\mathcal{X} \Big) \tag{4.14}$$

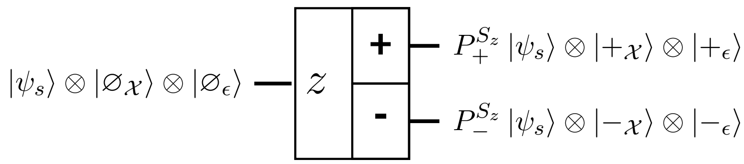

FIGURE 4.2 The most complete description of Experiment 1 presented, including position and environment degres of freedom. The seemingly redundant correlation of both position and environment states to spin states validifies the abstraction of the apparatus as the position system. However, formalizing the interaction with the apparatus, and more generally the environment, resolves the preferred basis problem and provides a description of record keeping.

$$
\begin{aligned}
&\otimes \Big(|+_\epsilon\rangle \langle \varnothing_\epsilon| \,+\, |\varnothing_\epsilon\rangle \langle +_\epsilon| \,+\, |-_\epsilon\rangle \langle -_\epsilon| \,+\, I_\epsilon \,-\, P_+^\epsilon \,-\, P_\varnothing^\epsilon \Big) \\
+ P_-^{S_z} &\otimes \Big(|-_\mathcal{X}\rangle \langle \varnothing_\mathcal{X}| \,+\, |\varnothing_\mathcal{X}\rangle \langle -_\mathcal{X}| \,+\, |+_\mathcal{X}\rangle \langle +_\mathcal{X}| \,+\, I_\mathcal{X} \,-\, P_-^\mathcal{X} \,-\, P_\varnothing^\mathcal{X} \Big) \\
&\otimes \Big(|-_\epsilon\rangle \langle \varnothing_\epsilon| \,+\, |\varnothing_\epsilon\rangle \langle -_\epsilon| \,+\, |+_\epsilon\rangle \langle +_\epsilon| \,+\, I_\epsilon \,-\, P_-^\epsilon \,-\, P_\varnothing^\epsilon \Big)
\end{aligned}
$$

To shorten this expression, we define the "entanglement operator"

$$
E^i_{\pm,\varnothing} = |\pm_i\rangle \langle \varnothing_i| \,+\, |\varnothing_i\rangle \langle \pm_i| \,+\, |\mp_i\rangle \langle \mp_i| + I_i - P^i_\pm - P^i_\mp \tag{4.15}
$$

Note that E is Hermitian $\left(E^\dagger = E\right)$ and unitary $\left(E^\dagger E = I\right)$. The unitary operator is now

$$
\begin{aligned}
U(t_1, t_0) = P_+^{S_z} &\otimes E^\mathcal{X}_{+,\varnothing} \otimes E^\epsilon_{+,\varnothing} \\
+ P_-^{S_z} &\otimes E^\mathcal{X}_{-,\varnothing} \otimes E^\epsilon_{-,\varnothing}
\end{aligned} \tag{4.16}
$$

For a general initial spin state, this produces final state

$$
|\psi'\rangle = P_+^{S_z} |\psi_s\rangle \otimes |+_\mathcal{X}\rangle \otimes |+_\epsilon\rangle \,+\, P_-^{S_z} |\psi_s\rangle \otimes |-_\mathcal{X}\rangle \otimes |-_\epsilon\rangle \tag{4.17}
$$

This experiment is visualized schematically in Figure 4.2.

4.6 Consecutive Measurements

In subsection 3.2.4, we applied the projection postulate succesively to account for consecutive measurements. When measurement was conditional on a prior result, we applied the postulate only to states routed to the second analyzer. The von Neumann measurement scheme can also be applied succesively, and is used on a term-by-term basis to account for conditional measurement.

4.6.1 Experiment 2

Now the environment must record the state of two apparatuses. To simplify this, we seperate the components of the environment responsible for recording each apparatus

$$H_\epsilon = H_{\epsilon_1} \otimes H_{\epsilon_2} \tag{4.18}$$

so that the complete Hilbert space is

$$\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_\mathcal{X} \otimes \mathcal{H}_{\epsilon_1} \otimes \mathcal{H}_{\epsilon_2} \tag{4.19}$$

We also name new position states for the second apparatus, so that position states of interest are now $\{|\varnothing^1_\mathcal{X}\rangle, |+^1_\mathcal{X}\rangle, |-^1_\mathcal{X}\rangle, |\varnothing^2_\mathcal{X}\rangle, |+^2_\mathcal{X}\rangle, |-^2_\mathcal{X}\rangle\}$. These states are visualized in TODO FIG.

The dynamics are unchanged from Equation 4.16 for the first measurement, with the identity acting on the second environment system to leave it unaffected:

$$\begin{aligned} U(t_1, t_0) = {} & P^{S_z}_+ \otimes E^\mathcal{X}_{+^1,\varnothing^1} \otimes E^{\epsilon_1}_{+,\varnothing} \otimes I_{\epsilon_2} \\ & + P^{S_z}_- \otimes E^\mathcal{X}_{-^1,\varnothing^1} \otimes E^{\epsilon_1}_{-,\varnothing} \otimes I_{\epsilon_2} \end{aligned}$$

For the second measurement, we begin by selecting z spin-down states and leave them unaffected

$$U(t_2, t_1) = P^{S_z}_- \otimes I_\mathcal{X} \otimes I_{\epsilon_1} \otimes I_{\epsilon_2} \tag{4.20}$$

Then, we select z spin-up states and perform the measurement scheme again, since these are routed to the second analyzer.

$$\begin{aligned} U(t_2, t_1) = {} & P^{S_x}_+ P^{S_z}_+ \otimes E^\mathcal{X}_{+^2,+^1} \otimes I_{\epsilon_1} \otimes E^{\epsilon_2}_{+,\varnothing} \\ & + P^{S_x}_- P^{S_z}_+ \otimes E^\mathcal{X}_{+^2,-^1} \otimes I_{\epsilon_1} \otimes E^{\epsilon_2}_{-,\varnothing} \\ & + P^{S_z}_- \otimes I_\mathcal{X} \otimes I_{\epsilon_1} \otimes I_{\epsilon_2} \end{aligned} \tag{4.21}$$

The ϵ_2 entanglement operator functions just as the ϵ_1 operator did for the first measurement. The position entanglement operator is more subtle. Immediately after the first measurement, we expect the up and down position state to be mapped to the ready state of the second analyzer. Immediately after that, the ready state is mapped to up and down position states as the second measurement takes place. Considering both of these mappings as the second measurement, we directly map up/down position states from the first analyzer to up/down position states of the second.

Because the environment is used as the third degree of freedom, we do not have to worry about

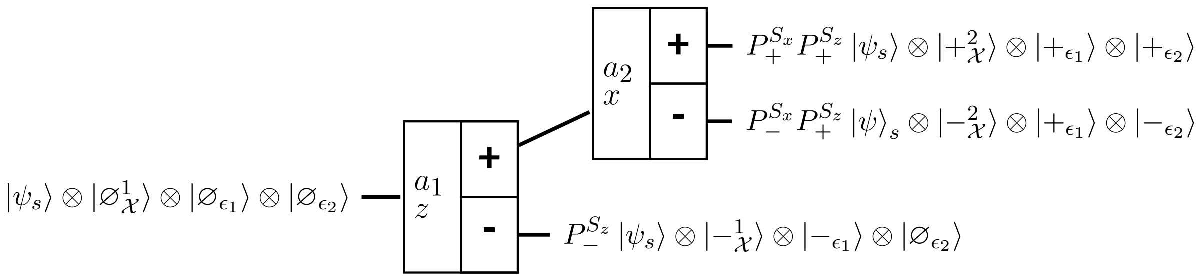

FIGURE 4.3 Schematic of consecutive measurement with spin, position and environment degrees of freedom. Notice the temporary encoding of measurement results in position states, and persistent encoding of measurement results in environment states.

reversing the spin-apparatus entanglement after measurement. The role of the environment is to keep a persistent record, so we let it be.

5 Consistent Histories

The probability postulate makes predictions about the results of the standard description of measurement. We can no longer frame the calculation of probabilities within this context, as we have described measurement as a unitary process and discarded the projection postulate. This motivates the search for a method of assigning probability distributions to sets of unitary outcomes in general.

The *consistent histories* approach extends the probability postulate to make predictions about more general *quantum histories* rather than measurement results. This is accomplished by modifying both the formalism and interpretation of quantum mechanics; hence, consistent histories is called an ”approach‘ to quantum mechanics.

First, we rearticulate some of the foundational mathematics and reasoning originally proposed by Robert Griffiths in 1984 TODO CITE. Then, we develop the approach further to make predictions about Stern-Gerlach experiments; this is done using the perspective of Murray Gell-Mann and James Hartle, who independently published the same ideas in 199TODO under the name “decoherent histories”.

5.1 Events and Histories

In probability theory, a sample space consists of an exhaustive set of mutually exclusive outcomes, called *events*. In quantum mechanics, the sample space for a physical variable is found by decomposing the identity in that variable’s basis. Each term in the decomposition represents a *quantum event*. For example, the sample space of S_z consists of the terms in

$$I = \sum_i P_i^{S_z} = P_+^{S_z} + P_-^{S_z} \tag{5.1}$$

Events exist within the context of a particular sample space. Consequently, we can use logical reasoning to make conjuctive or negative statements about events. For example, the event for “spin is either up or down” is the sum of spin-up and spin-down events, $P_+^z + P_-^z = I$. Note that asserting that the system is either spin-up or spin-down is the conjunction of all outcomes in the sample space, so it is equivalent to asserting nothing at all.

A *quantum history* is a set of events at sequential times. As finite sets, histories necessarily ignore an infinite amount of insignificant events.

5.1.1 Experiment 1

The histories for measuring spin-up and spin-down are

$$h_{\pm} = \left(\left(P_{\psi_{S_z}} \otimes P_{\varnothing}^{\mathcal{X}} \otimes P_{\varnothing}^{\epsilon}\right), \left(P_{\pm}^{S_z} \otimes P_{\pm}^{\mathcal{X}} \otimes P_{\pm}^{\epsilon}\right)\right) \tag{5.2}$$
$$h_{\pm} = (P_{\varnothing}, P_{\pm_z})$$

naming the events in the second line to simplify future expressions.

The first event describes the system at time t_0 before measurement. It asserts that the spin system is in the initial state $|\psi_s\rangle$, that the position system is in the ready state $|\varnothing_{\mathcal{X}}\rangle$, and that the environment has not yet encoded a result $|\varnothing_{\epsilon}\rangle$. The second event describes the system at time t_1 after measurement. It asserts that the spin system is either up or down, $|\psi_{\pm}\rangle$, that the position system is in the up or down state $|\pm\mathcal{X}\rangle$, and that the environment has encoded a result or up or down $|\varnothing_{\pm}\rangle$. It is the occurence of spin-up measurement h_+ and spin-down measurement h_- for which we make predictions.

5.2 Class Operators

To facilitate the calculation of probabilitys, we seek to write an operator that takes an initial state through the events of a given history.

5.2.1 Experiment 2

To find such an operator, we follow a path from the initial state to some final outcome in TODO REF FIGURE, operating on the initial state with unitary dynamics and event projectors along the way. We call the resulting state the *branch wavefunction* for that history, labeled $|\psi_h\rangle$.

Returning to Experiment 2, we are interested in events at times t_0 (before the first measurement), t_1 (after the first measurement / before the second measurement), and t_2 (after the sceond measurement). The history for measuring spin-up at the z analyzer followed by measuring spin-down at the x analyzer is

$$h_{+_z,-_x} = (P_{\varnothing}, P_{+_z}, P_{-_x}) \tag{5.3}$$

First, we assert that the state initially occurs with the first event in the history

$$|\psi_h\rangle = P_{\varnothing}\ |\psi\rangle \tag{5.4}$$

Then, we assert that after evolution from t_0 to t_1, the event for measuring spin-up occurs

$$|\psi_h\rangle = P_+^{S_z}\, U(t_1,t_0)\, P_\varnothing\ |\psi\rangle \tag{5.5}$$

Finally, we assert that after evolution from t_1 to t_2, the event for measuring spin-down occurs

$$|\psi_h\rangle = P_-^{S_x}\, U(t_2,t_1)\, P_+^{S_z}\, U(t_1,t_0)\, P_\varnothing\ |\psi\rangle \tag{5.6}$$

Writing all unitary operators with time starting at t_0,

$$\begin{aligned} |\psi_h\rangle &= P_-^{S_x}\, U(t_2,t_0)\, U^\dagger(t_1,t_0)\, P_+^{S_z}\, U(t_1,t_0)\, P_\varnothing\ |\psi\rangle \\ &= \left(P_-^{S_x}\, U(t_2,t_0)\right)\left(U^\dagger(t_1,t_0)\, P_+^{S_z}\, U(t_1,t_0)\right)(P_\varnothing)\ |\psi\rangle \\ &= U(t_2,t_0)\left(U^\dagger(t_2,t_0)\, P_-^{S_x}\, U(t_2,t_0)\right)\left(U^\dagger(t_1,t_0)\, P_+^{S_z}\, U(t_1,t_0)\right)(P_\varnothing)\ |\psi\rangle \end{aligned} \tag{5.7}$$

we recognize the Heisenberg picture event operators $P(t) = U^\dagger(t,t_0)\ P\ U(t,t_0)$. In terms of Heisenberg projectors,

$$\begin{aligned} |\psi_h\rangle &= U(t_2,t_0)\, P_-^{S_x}(t_2)\, P_+^{S_z}(t_1)\, P_\varnothing(t_0)\ |\psi\rangle \\ |\psi_h\rangle &= U(t_2,t_0) C_h^\dagger\, |\psi\rangle \end{aligned} \tag{5.8}$$

C_h is called the *class operator* for history h. Conventionally, this operator is written with projectors appearing in the same left-to-right reading order as the order of events in the history. By calling our operator $C_h^\dagger$, we invert this order so that the event projectors operate on the initial state in the correct order.

In general, the class operator for a set of n histories is defined by

$$C_h^\dagger = P_n(t_n)\, P_{n-1}(t_{n-1})\ ...\ P_0(t_0) \tag{5.9}$$

which corresponds to the branch wavefunction

$$|\psi_h\rangle = U(t_n,t_0) C_h^\dagger\, |\psi\rangle \tag{5.10}$$

5.3 Extending the Probability Postulate

The standard probability postulate is written as the inner product of the system state $|\psi\rangle$ and an eigenstate of a physical variable $|a_n\rangle$. It can instead be written in terms of the projection operator for $|a_n\rangle$ by expanding the complex square.

$$\begin{aligned}\mathcal{P}(n) &= |\,\langle a_n|\psi\rangle\,|^2 \\ &= \langle\psi|a_n\rangle\,\langle a_n|\psi\rangle \\ &= \langle\psi|\,P^a_n\,|\psi\rangle\end{aligned} \tag{5.11}$$

To extend this postulate to make predictions about histories, we replace measurement result projector P^a_n with the history projector $C^\dagger_h$

$$\mathcal{P}(h_n) = \langle\psi|\,C^\dagger_h\,|\psi\rangle \tag{5.12}$$

5.3.1 Experiment 1

The class operator for measuring spin-up or spin-down is

$$C^\dagger_\pm = P^{S_z}_\pm(t_1)\;P_\varnothing(t_0) \tag{5.13}$$

so the extended Born Rule assigns probability

$$\begin{aligned}\mathcal{P}(\pm) &= \langle\psi|\,C^\dagger_\pm\,|\psi\rangle \\ &= \langle\psi|\,P^{S_z}_\pm(t_1)\;P_\varnothing(t_0)\,|\psi\rangle\end{aligned} \tag{5.14}$$

$P_\varnothing(t_0)$ confirms that $|\psi\rangle$ is the correct initial state: so

$$\mathcal{P}(\pm) = \langle\psi|\,P^{S_z}_\pm(t_1)\,|\psi\rangle \tag{5.15}$$

The Heisenberg projector at t_1 is

$$P^{S_z}_\pm(t_1) = U^\dagger(t_1,t_0)P^{S_z}_\pm U(t_1,t_0) \tag{5.16}$$

so

$$\mathcal{P}(\pm) = \langle\psi|\,U^\dagger(t_1,t_0)P^{S_z}_\pm U(t_1,t_0)\,|\psi\rangle \tag{5.17}$$

Using the dynamics found in TODO REF, we find

$$\begin{aligned}U(t_1,t_0)\,|\psi\rangle &= P^{S_z}_+\,|\psi_s\rangle\otimes|+_\mathcal{X}\rangle\otimes|+_\epsilon\rangle + P^{S_z}_-\,|\psi_s\rangle\otimes|-_\mathcal{X}\rangle\otimes|-_\epsilon\rangle \\ \langle\psi|\,U^\dagger(t_1,t_0) &= \langle\psi_s|\,P^{S_z}_+\otimes\langle+_\mathcal{X}|\otimes\langle+_\epsilon| + \langle\psi_s|\,P^{S_z}_-\otimes\langle-_\mathcal{X}|\otimes\langle-_\epsilon|\end{aligned} \tag{5.18}$$

Applying the event projector,

$$P_{\pm}^{S_z} U(t_1, t_0) |\psi\rangle = P_{\pm}^{S_z} |\psi_s\rangle \otimes |\pm_{\mathcal{X}}\rangle \otimes |\pm_\epsilon\rangle \tag{5.19}$$

so

$$\begin{aligned}
\mathcal{P}(\pm) &= \Big(\langle\psi| U^\dagger(t_1, t_0)\Big) \Big(P_{\pm}^{S_z} |\psi_s\rangle \otimes |\pm_{\mathcal{X}}\rangle \otimes |\pm_\epsilon\rangle\Big) \\
&= \Big(\langle\psi_s| P_{\pm}^{S_z} \otimes \langle\pm_{\mathcal{X}}| \otimes \langle\pm_\epsilon| \pm \langle\psi_s| P_{-}^{S_z} \otimes \langle -_{\mathcal{X}}| \otimes \langle -_\epsilon|\Big) \Big(P_{\pm}^{S_z} |\psi_s\rangle \otimes |\pm_{\mathcal{X}}\rangle \otimes |\pm_\epsilon\rangle\Big) \\
&= \langle\psi_s| P_{\pm}^{S_z} |\psi_s\rangle \langle\pm_{\mathcal{X}}|\pm_{\mathcal{X}}\rangle \langle\pm_\epsilon|\pm_\epsilon\rangle \\
&= \langle\psi_s| P_{\pm}^{S_z} |\psi_s\rangle
\end{aligned} \tag{5.20}$$

We have reproduced the prediction of the standard Born Rule TODO ref.

5.4 Consistency Conditions

In standard quantum mechanics, the third postulate defines the subset of states corresponding to "measurement results", and the fourth postulate makes predictions about these states only. Now that we have extended the Born Rule to make predictions about histories, we need to be careful about the context in which the predictions are made.

5.4.1 Experiment 1

We saw that the extended Born Rule reproduces the probabilities of the standard Born Rule. However, the extension goes on to make predictions about other outcomes. For example, we could ask about the probability that spin-up in the x direction is measured using the history

$$h_{+_x} = (P_\varnothing, P_{+_x}) \tag{5.21}$$

which results in

$$\begin{aligned}
\mathcal{P}(h_{+_x}) &= \langle\psi_h| C_{+_x} |\psi_h\rangle \\
&= \langle\psi_s| P_{+}^{S_x} |\psi_s\rangle
\end{aligned} \tag{5.22}$$

which is non-zero in general. We could ask an infinite amount of similar questions, since the spin Hilbert space includes states representing spin-up along every direction in space TODO REF STATE SPACE SECTION. Consequently, our probabilities no longer sum to unity, which is not consistent with probaility theory. The cause of this inconsistency is that P_{+_z} and P_{+_x} belong to incompatible sample spaces, so we are comparing predictions made within different contexts.

In TODO REF 5.1, we defined events as elements of some specific sample space. The extended Born Rule assigns probabilities to sequences of events *within the context of their sample spaces*. The idea of consistency conditions is to explicitly state the context in which preidctions are made.

To establish such a context, we construct a sample space of histories by finding a set of histories that are mutually exclusive and exhaustive:

$$\begin{aligned} S = \{h\} : \\ \sum_{h \in S} C_h = I \\ C_h^\dagger C_h' = \delta_{h,h'} C_h \end{aligned} \tag{5.23}$$

$$\tag{5.24}$$

A set of histories S satisfying these conditions a *family* of histories. To find a family for this experiment, we start with the spin-up and spin-down histories:

$$S = \{(P_\varnothing, P_{+_z}), (P_\varnothing, P_{-_x})\} \tag{5.25}$$

Though the probabilities of these histories sum to one, the set is not yet complete:

$$\begin{aligned} \sum_{h \in S} C_h &= P_{\psi_s} P_+^{S_z} \otimes P_\varnothing^{\mathcal{X}} \otimes P_\varnothing^{\epsilon} + P_{\psi_s} P_-^{S_z} \otimes P_\varnothing^{\mathcal{X}} \otimes P_\varnothing^{\epsilon} \\ &= P_{\psi_s} \otimes P_\varnothing^{\mathcal{X}} \otimes P_\varnothing^{\epsilon} \neq I \end{aligned} \tag{5.26}$$

Finding the class operator that completes the set,

$$C_{I-\varnothing} = I - \left(P_{\psi_s} \otimes P_\varnothing^{\mathcal{X}} \otimes P_\varnothing^{\epsilon}\right) \tag{5.27}$$

This is the class operator for any system not starting in the initial spin state, not starting in the ready position, or not starting with a ready environment. In other words, $C_{I-\varnothing}$ represents histories that start in some state other than the initial state. Even though these histories never occur, they must be included to complete the family; the prediction of non-occurence of $I - \varnothing$ is part of the context.

Due to the position system's continuous sample space, there are an infinite number of histories distinct from the initial state with zero chance of occurence. Calling one such history $h^i_{I-\varnothing}$,

$$S = \{\{P_\varnothing, P_+^{S_z}\}, \{P_\varnothing, P_-^{S_z}\}\} \cup \{h^i_{I-\varnothing}\} \tag{5.28}$$

Now that we have an exhaustive set, we show that its histories are mutually exclusive:

$$\begin{aligned}
C_+^\dagger C_- &= \left(P_{+_z}(t_1)P_\varnothing(t_0)\right)\left(P_\varnothing(t_0)P_{-_z}(t_1)\right) \\
&= P_{+_z}(t_1)P_{-_z}(t_1) \\
&= U^\dagger(t_1,t_0)P_{+_z}U(t_1,t_0)U^\dagger(t_1,t_0)P_{-_z}U(t_1,t_0) \\
&= U^\dagger(t_1,t_0)P_{+_z}P_{-_z}U(t_1,t_0) \\
&= U^\dagger(t_1,t_0)\left(P_+^{S_z}P_-^{S_z}\otimes P_+^{\mathcal{X}}P_-^{\mathcal{X}}\otimes P_+^{\epsilon}P_-^{\epsilon}\right)U(t_1,t_0) \\
&= 0
\end{aligned} \tag{5.29}$$

and

$$\begin{aligned}
C_\pm^\dagger C_{I-\varnothing} &= \left(P_{+_z}(t_1)P_\varnothing(t_0)\right)\left((I-P_\varnothing)\right) \\
&= P_{+_z}(t_1)\left(P_\varnothing - P_\varnothing\right) \\
&= 0
\end{aligned} \tag{5.30}$$

The set S is shown to be a consistent family of histories, so the predictions of REF are made within this context.

5.5 Experiment 2

Using the same reasoning, we find a consistent family for the experiment shown in TODO REF FIG

$$\left\{h_{+_z,+_x} = \left(P_\varnothing, P_{+_z}, P_{+_x}\right), h_{+_z,-_x} = \left(P_\varnothing, P_{+_z}, P_{-_x}\right), h_{-_z} = \left(P_\varnothing, P_{-_z}\right), h_{I-\varnothing} = \left(I-P_\varnothing\right)\right\} \tag{5.31}$$

Notice that histories do not need to specify the same amount of events to form a consistent family.

Using the extended Born Rule, we calculate the probability of measuring spin up in both analyzers

$$\begin{aligned}
\mathcal{P}(+_z,+_x) &= \langle\psi|\,C_{+_z,+_x}^\dagger\,|\psi\rangle \\
&= \langle\psi|\,P_{+_x}(t_2)P_{+_z}(t_1)P_{\psi_s}\,|\psi\rangle \\
&= \langle\psi|\,U^\dagger(t_2,t_0)P_{+_x}U(t_2,t_0)U^\dagger(t_1,t_0)P_{+_z}U(t_1,t_0)\,|\psi\rangle \\
&= \langle\psi|\,U^\dagger(t_2,t_0)P_{+_x}U(t_2,t_1)P_{+_z}U(t_1,t_0)\,|\psi\rangle
\end{aligned} \tag{5.32}$$

Using dynamics found in TODO ref,

$$
\begin{aligned}
U(t_2,t_0) &= U(t_2,t_1)U(t_1,t_0) \\
&= P_+^{S_x}P_+^{S_z} \otimes E^{\mathcal{X}}_{+^2,+^1}E^{\mathcal{X}}_{+^1,\varnothing^1} \otimes E^{\epsilon_1}_{+,\varnothing} \otimes E^{\epsilon_2}_{+,\varnothing} \\
&\quad + P_-^{S_x}P_+^{S_z} \otimes E^{\mathcal{X}}_{+^2,-^1}E^{\mathcal{X}}_{+^1,\varnothing^1} \otimes E^{\epsilon_1}_{+,\varnothing} \otimes E^{\epsilon_2}_{-,\varnothing} \\
&\quad + P_-^{S_z} \otimes I_{\mathcal{X}} \otimes I_{\epsilon_1} \otimes I_{\epsilon_2}
\end{aligned} \tag{5.33}
$$

The event operators select the spin-up/spin-up branch

$$
P_{+_x}U(t_2,t_1)P_{+_z} = P_+^{S_x}P_+^{S_z} \otimes E^{\mathcal{X}}_{+^2,+^1}E^{\mathcal{X}}_{+^1,\varnothing^1} \otimes I_{\epsilon_1} \otimes E^{\epsilon_2}_{+,\varnothing} \tag{5.34}
$$

so

$$
\begin{aligned}
\mathcal{P}(+_z,+_x) &= \langle\psi| U^\dagger(t_2,t_0)P_{+_x}U(t_2,t_1)P_{+_z}U(t_1,t_0)|\psi\rangle \\
&= \langle\psi| U^\dagger(t_2,t_0)\left(P_+^{S_x}P_+^{S_z} \otimes E^{\mathcal{X}}_{+^2,+^1}E^{\mathcal{X}}_{+^1,\varnothing^1} \otimes I_{\epsilon_1} \otimes E^{\epsilon_2}_{+,\varnothing}\right)U(t_1,t_0)|\psi\rangle \\
&= \langle\psi| U^\dagger(t_2,t_0)\left(P_+^{S_x}P_+^{S_z} \otimes E^{\mathcal{X}}_{+^2,+^1}E^{\mathcal{X}}_{+^1,\varnothing^1}E^{\mathcal{X}}_{+^1,\varnothing^1} \otimes E^{\epsilon_1}_{+,\varnothing} \otimes E^{\epsilon_2}_{+,\varnothing}\right)|\psi\rangle \\
&= \langle\psi| U^\dagger(t_2,t_0)\left(P_+^{S_x}P_+^{S_z} \otimes E^{\mathcal{X}}_{+^2,+^1} \otimes E^{\epsilon_1}_{+,\varnothing} \otimes E^{\epsilon_2}_{+,\varnothing}\right)|\psi\rangle \\
&= \langle\psi| \left(P_+^{S_z}P_+^{S_x}P_+^{S_z} \otimes E^{\mathcal{X}}_{+^2,\varnothing^2}E^{\mathcal{X}}_{+^2,\varnothing^2} \otimes E^{\epsilon_1}_{+,\varnothing}E^{\epsilon_1}_{+,\varnothing} \otimes E^{\epsilon_2}_{+,\varnothing}E^{\epsilon_2}_{+,\varnothing}\right)|\psi\rangle \\
&= \langle\psi| \left(P_+^{S_z}P_+^{S_x}P_+^{S_z} \otimes I_{\mathcal{X}} \otimes I_{\epsilon_1} \otimes I_{\epsilon_2}\right)|\psi\rangle \\
&= \langle\psi_s| P_+^{S_z}P_+^{S_x}P_+^{S_z}|\psi_s\rangle
\end{aligned} \tag{5.35}
$$

Similarly, the probabilities for all other possible outcomes are

$$
\begin{aligned}
\mathcal{P}(+_z,-_x) &= \langle\psi_s| P_+^{S_z}P_-^{S_x}P_+^{S_z}|\psi_s\rangle \\
\mathcal{P}(-_z) &= \langle\psi_s| P_-^{S_z}|\psi_s\rangle
\end{aligned} \tag{5.36}
$$

5.6 Decoherence

TODO: exemplify how tracing out degrees of freedom results in mixed state, discuss what happens if environment is not a perfect record keeper.

It is important to note that the vast majority of which state information is unaccessible for all practical purposes;

If one understood the nature of all systems interacting with the electron, how well would they be able to answer questions about the history of the electron? The existence of which path

6 Simulation

TODO: discuss structural and algorithmic changes resulting from rewriting spins simulation

7 Conclusion

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Schrödinger Picture

Gell-Man/Hartle/Craig's work informs (and is informed by) cosmological applications, assumes less about the environment (allowing for PFBP discussion)

A history, as a set of events, specifies possible system states at multiple instances of time. Formally, this is no different than specifying possible states of a composite system consisting of a copy of $\mathcal{H}$ for each instant in time [4]. This motivates the definition of a *history Hilbert space*. Once again, the tensor product is employed to create a composite system; this time, the entire spin-pointer system is considered at different points in time. The history Hilbert space representing $|\psi\rangle$ at times $(t_0, t_1, t_2, ..., t_f)$ is

$$\mathcal{H}_h = \mathcal{H}_{t_0} \odot \mathcal{H}_{t_1} \odot \mathcal{H}_{t_2} \odot ... \odot \mathcal{H}_{t_f} \tag{1}$$

where $\odot$ is the ordinary tensor product, but denotes that the same quantum system $\mathcal{H}$ is considered at different times.

In this history Hilbert space, the state representing the system at times $(t_0, t_1, t_2, ..., t_f)$ is

$$|\psi_h\rangle = |\psi_{t_0}\rangle \odot |\psi_{t_1}\rangle \odot |\psi_{t_2}\rangle \odot ... \odot |\psi_{t_f}\rangle \tag{2}$$

where each component $|\psi_{t_i}\rangle$ is determined by the unitary dynamics experienced by the system up until that point, $U(t_i, t_0)$. In other words,

$$|\psi_h\rangle = |\psi_{t_0}\rangle \odot U(t_1, t_0)\,|\psi_{t_0}\rangle \odot U(t_2, t_0)\,|\psi_{t_0}\rangle \odot ... \odot U(t_f, t_0)\,|\psi_{t_0}\rangle \tag{3}$$

In this history Hilbert space, a history is represented by the tensor product of its events. That is, history $n = (P_0, P_1, P_2, ..., P_f)$ is represented by $P^h_n = P_0 \odot P_1 \odot P_2 \odot ... \odot P_f$.

.0.1 Extending the Probability Postulate

The standard probability postulate TODO REF is written as the inner product of the system state $|\psi\rangle$ and an eigenstate of a physical variable $|a_n\rangle$. It can instead be written in terms of the projection operator for $|a_n\rangle$ by expanding the complex square.

$$\begin{aligned} \mathcal{P}(n) &= |\langle a_n|\psi\rangle|^2 \\ &= \langle\psi|a_n\rangle\langle a_n|\psi\rangle \\ &= \langle\psi| P^a_n |\psi\rangle \end{aligned} \tag{4}$$

To extend this postulate to make predictions about histories, we replace the system $|\psi\rangle$ with the history system $|\psi_h\rangle$ and the measurement projector P^a_n with the history projector P^h_n.

$$\mathcal{P}(h_n) = \langle\psi_h| P^h_n |\psi_h\rangle \tag{5}$$

.0.1.1 Example 1

As shown in TODO REF, the measurement of an initial spin state $|\psi_S\rangle$ results in

$$\begin{aligned} |\psi_1\rangle &= U(t_1, t_0)|\psi_0\rangle \\ &= P^{S_z}_+ |\psi_S\rangle \otimes |+_{\mathcal{X}}\rangle \otimes |+_A\rangle \; + \; P^{S_z}_- |\psi_S\rangle \otimes |-_{\mathcal{X}}\rangle \otimes |-_A\rangle \end{aligned} \tag{6}$$

In the history Hilbert space, the state representing the system before and after measurement is

$$\begin{aligned} |\psi_h\rangle &= |\psi_0\rangle \odot |\psi_1\rangle \\ &= (|\psi_S\rangle \otimes |\varnothing_{\mathcal{X}}\rangle \otimes |\varnothing_A\rangle) \odot \left(P^{S_z}_+ |\psi_S\rangle \otimes |+_{\mathcal{X}}\rangle \otimes |+_A\rangle \; + \; P^{S_z}_- |\psi_S\rangle \otimes |-_{\mathcal{X}}\rangle \otimes |-_A\rangle\right) \end{aligned} \tag{7}$$

The history for measuring spin-up is composed of the projector for an initial spin state with a ready position and apparatus, and the projector for an up spin state with an up position and apparatus:

$$\begin{aligned} P^h_+ &= P_\varnothing \odot P_+ \\ &= \left(P^{S_z}_{\psi_S} \otimes P^{\mathcal{X}}_\varnothing \otimes P^A_\varnothing\right) \odot \left(P^{S_z}_+ \otimes P^{\mathcal{X}}_+ \otimes P^A_+\right) \end{aligned} \tag{8}$$

Using the new probaility postulate, the probability of measuring spin-up is

$$\begin{aligned}
\mathcal{P}(h_+) &= \langle\psi_h| P_+^h |\psi_h\rangle \\
&= \langle\psi_h| P_\varnothing \odot P_+ |\psi_h\rangle \\
&= \langle\psi_h| (|\psi_S\rangle \otimes |\varnothing_\mathcal{X}\rangle \otimes |\varnothing_A\rangle) \odot \left(P_+^{S_z} |\psi_S\rangle \otimes |+_\mathcal{X}\rangle \otimes |+_A\rangle\right) \\
&= \langle\psi_S| P_+^{S_z} |\psi_S\rangle
\end{aligned} \tag{9}$$

and we recover the prediction of the standard Born Rule.

.0.2 Consistency Conditions

The third postulate TODO REF defines the subset of states corresponding to "measurement results", and the fourth postulate makes predictions about these states only. Now that we have extended the Born Rule to make predictions about histories, we need to be careful about the context in which the predictions are made, as it is no longer postulated for us.

The solution is to use *consistency conditions* to determine the sets of histories that are consistent with probability theory. The consistency conditions require that histories represent exhaustive and mutually exclusive outcomes:

$$P_i^{h\dagger} P_j^h = \delta_{i,j} P_i^h \tag{10}$$

$$\sum_i P_i^h = I_h \tag{11}$$

A set of histories satisfying these conditions is called a *consistent family* of histories. Once a consistent family is specified, we can use the extended Born Rule to calculate probabilities within that context. For our example, $\{P_+^h, P_-^h, P_0^h\}$ is a consistent family where

$$\begin{aligned}
P_+^h &= P_\varnothing \odot P_+ \\
P_-^h &= P_\varnothing \odot P_- \\
P_0^h &= I_h - P_+^h - P_-^h
\end{aligned} \tag{12}$$

P_0^h represents any history distinct from P_+^h and P_-^h. Showing this,

$$\begin{aligned}
P_0^{h\dagger} P_\pm^h &= (I_h - P_+^h - P_-^h) P_\pm^h \\
&= P_\pm^h - P_\pm^h \\
&= 0
\end{aligned} \tag{13}$$

Furthermore, P_+^h and P_-^h are distinct since

$$P_+^{\mathcal{X}\dagger} P_-^{\mathcal{X}} = 0 \tag{14}$$

so all histories in this set are mutually exclusive.

Showing that the set is exhaustive,

$$\begin{aligned} P_+^h + P_-^h + P_0^h &= P_+^h + P_-^h + \left(I_h - P_+^h - P_-^h\right) \\ &= I_h \end{aligned} \tag{15}$$

Now that the consistency of the family is confirmed, we can use TODO REF to find probabilities for each history. Notice that P_0^h is included to make the set exhaustive; even though its probability of occurence is 0, its inclusion in the family enables such a prediction.

.0.3 Interpretation

TODO: describe how environment is implied, records every component of system at all times