Chapter 1. Characteristics of Time Series

September 12, 2018

Outline

Stochastic Process

Measure of Dependence

Stationarity

Joint Stationarity

Cross-Correlation Function

Basics

- A **time series** is a stochastic process, i.e., a set $\{x_t(\omega) = x(\omega, t) : t = 0, \pm 1, \pm 2, \pm 3, \cdots\}$.

- An **observed time series** is a sample realization or a sample trajectory/path of a certain stochastic process, i.e., a set $\{x_t : t = 0, \pm 1, \pm 2, \pm 3, \cdots\}$ (fixed ω).

- The complete probabilistic structure of a stochastic process is determined by the set of distributions of all finite collections of the x_t's.

- Rather than studying the joint distributions of the x_t's, we look at statistics of the stochastic process: *means, variances, and covariances.*

Expectation and Variance

- Let X, Y, Z be continuous rv's.

  1. $E(h(X)) = \int h(x)f(x)dx = \mu_{h(X)}, \qquad F(x) = \int_{-\infty}^{x} f(u)du.$
  2. $E(aX + bY + c) = aE(X) + bE(Y) + c.$

- Variance:

  1. $\sigma_X^2 = var(X) = cov(X, X) = E((X - \mu_x)^2) \geq 0.$
  2.
  3.

Covariance and Correlation

- Covariance:

1. $cov(X, Y) = cov(Y, X) = E[(X - \mu_x)(Y - \mu_y)] = E(XY) - \mu_x\mu_y$.
2.
3. $cov(aX + bY, Z) = acov(X, Z) + bcov(Y, Z)$.
4.

Covariance and Correlation

- The correlation $\rho = corr(X, Y) = \frac{cov(X,Y)}{\sqrt{var(X)var(Y)}}$ satisfies:

    1. $-1 \leq corr(X, Y) = corr(Y, X) \leq 1$.
    2.
    3. If $Y = a \pm bX$ then

- If rv's X and Y are *independent* then
  $cov(X, Y) = 0, \quad \rho = 0, \quad var(X + Y) = var(X) + var(Y)$.

Mean and Autocovariance

- Suppose that x_t has the **probability density function (pdf)** $f_t(x)$. Then the **mean** function is given by

$$\mu_t = E(x_t) = \int_{\mathbb{R}} y \cdot f_t(y)\, dy, \qquad t = 0, \pm 1, \pm 2, \cdots$$

- The **autocovariance** function, $\gamma_{t,s}$, is defined as

$$\gamma_{t,s} = cov(x_t, x_s) = E[(x_t - \mu_t)(x_s - \mu_s)] \qquad t, s = 0, \pm 1, \pm 2, \cdots$$

- If $s = t \pm h$ then

$$\gamma_{t,s} \;=\; \gamma_{t,t\pm h} \;=\; cov\{x_t, x_{t\pm h}\}$$

- $h = t - s$ is called the `lag`.
- In particular, $\gamma_{t,t} = cov(x_t, x_t) = \gamma_0 =$ **variance** of x_t.

Autocorelation

- The **autocorrelation** function, $\rho_{t,s}$, is given by

$$\rho_{t,s} = corr(x_t, x_s) = \frac{cov(x_t, x_s)}{\sqrt{var(x_t) \cdot var(x_s)}} = \frac{\gamma_{t,s}}{\sqrt{\gamma_{t,t} \cdot \gamma_{s,s}}}.$$

- If $\rho_{t,s} = 0$, we say that x_t and x_s are **uncorrelated**.
- If $s = t \pm h$ then we write

$$\rho_{t,s} = \rho_{t,t \pm h}$$

- We usually consider $h \geq 0$.

Example 1: White Noise Process

- Let $w_1, w_2, \cdots$ be a *white noise* process. Then

$$E(w_t) = \mu_w = \mu \text{ and } \; var(w_t) = \sigma_w^2.$$

- For $s = t$ ($h = 0$),
- For $s \neq t$ ($h \neq 0$),

- Thus, the autocovariance function is

$$\gamma_{t,s} = \begin{cases} \sigma_w^2, & \text{for } \; |t - s = h| = 0 \\ 0, & \text{for } \; |t - s = h| \neq 0. \end{cases}$$

Example 1: White Noise Process

- For $t = s$ $(h = 0)$, the autocorrelation function is

$$\rho_{t,s} = \rho_w(0) = \rho_0 = corr(w_t, w_s) = corr(w_s, w_s) = \frac{\gamma_{t,t}}{\sqrt{\gamma_{t,t}\gamma_{t,t}}} = 1.$$

- For $t \neq s$ $(h \neq 0)$, the autocorrelation function is

$$\rho_{t,s} = corr(w_t, w_s) = \frac{\gamma_{t,s}}{\sqrt{\gamma_{t,t}\gamma_{s,s}}} = 0.$$

- Hence,

$$\rho_{t,s} = \begin{cases} 1, & \text{for } |t - s = h| = 0 \\ 0, & \text{for } |t - s = h| \neq 0 \end{cases} \implies \rho_h = \begin{cases} 1, & \text{for } |h| = 0 \\ 0, & \text{for } |h| \neq 0. \end{cases}$$

- A *white noise* process is not that interesting. White noise processes are used to capture/model the *random component* or the *inherent background noise* of time series data.

Estimation of Correlation

- We estimate the autocovariance function $\gamma_x(h), h = t - s$ by its sample analog, i.e.,

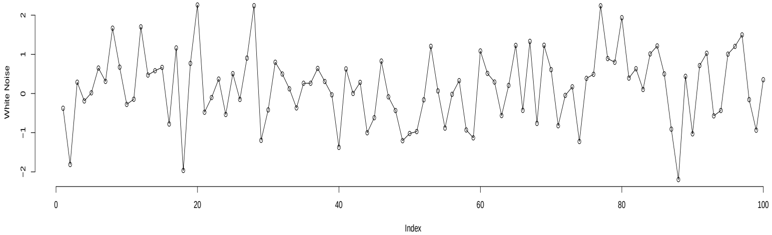
$$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-h} (x_{t+h} - \bar{x})(x_t - \bar{x}).$$

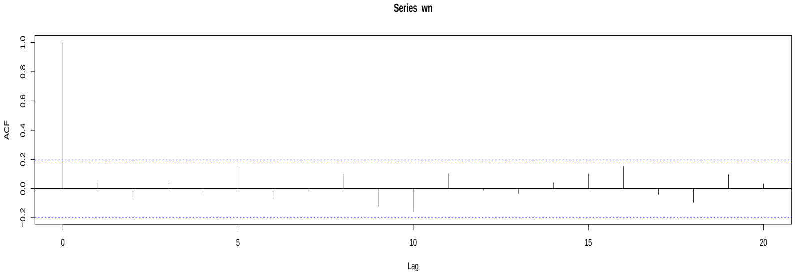
  and define $\hat{\gamma}(-h) = \hat{\gamma}(h)$ for any $h = 0, 1, \ldots, n-1$.
- The plot of $\rho(h)$ vs. h is known as the **correlogram**.
- The estimator of the autocorrelation function is

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}.$$

- In R, use `acf()`.

Example 1: White Noise Process





Random Walk

- A **simple random walk or random walk without drift** is defined as:

$$\begin{aligned} x_1 &= w_1, \quad x_2 = w_1 + w_2, \\ &\vdots \\ x_t &= w_1 + w_2 + \cdots + w_t \implies x_t = x_{t-1} + w_t \implies x_t = \sum_{j=1}^{t} w_j. \end{aligned}$$

- If the w_t's are interpreted as the sizes of the steps taken (forward or backward) along a number line, then x_t is the position of the random walker at time t.
- A special case of an **autoregressive** TS model.

Example 2: Simple Random Walk

- Let $x_t = \sum_{j=1}^{t} w_j$. Then

$$\begin{aligned} \mu_t &= E(x_t) = E(\sum_{j=1}^{t} w_j) = \sum_{j=1}^{t} E(w_j) = 0, \\ var(x_t) &= var(\sum_{j=1}^{t} w_j) \stackrel{\text{ind}}{=} \sum_{j=1}^{t} var(w_j) = t\sigma_w^2. \end{aligned}$$

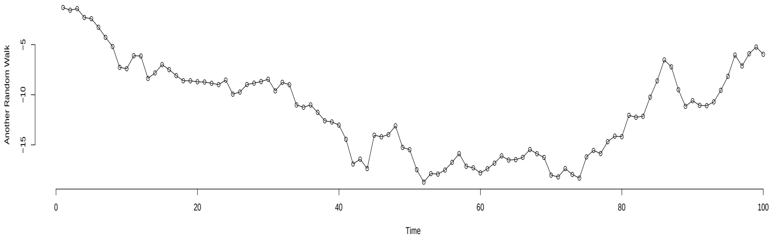
- Suppose $1 \le s \le t$, then

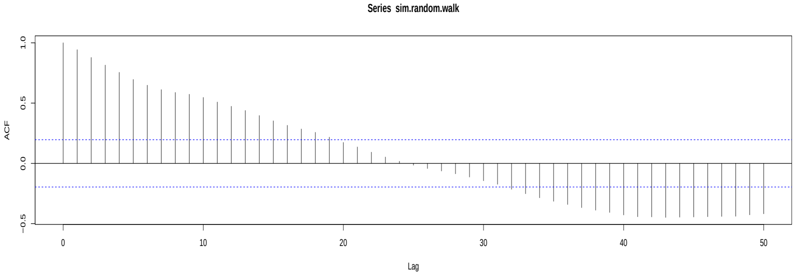
$$\begin{aligned} \gamma_{t,s} &= cov(x_t, x_s) = cov(\sum_{i=1}^{t} w_i, \sum_{j=1}^{s} w_j) = s\sigma_w^2, \\ \rho_{t,s} &= corr(x_t, x_s) = \frac{\gamma_{t,s}}{\sqrt{\gamma_{t,t}\gamma_{s,s}}} = \sqrt{\frac{s}{t}}. \end{aligned}$$

- In general,

$$\gamma_{t,s} = \min\{s,t\}\,\sigma_w^2, \qquad \rho_{t,s} = \sqrt{\frac{\min\{s,t\}}{\max\{s,t\}}}.$$

Example 2: Simple Random Walk





Example 3: Moving Average

- Suppose that
$$x_t = \frac{w_t + w_{t-1} + w_{t-2}}{3}.$$
Then
$$\mu_t = E(x_t) = E\left(\frac{w_t + w_{t-1} + w_{t-2}}{3}\right) = 0.$$
- Also,
$$var(x_t) = var\left(\frac{w_t + w_{t-1} + w_{t-2}}{3}\right) = \frac{\sigma_w^2}{3}$$
- For $t = s$ ($h = 0$),
$$\gamma_{t,s} = \gamma_{t,t} = \frac{\sigma_w^2}{3}.$$

Example 3: Moving Average

- For $s = t-1$ ($h = 1$),

$$\begin{aligned}\gamma_{t,t-1} &= cov\{\frac{w_t + w_{t-1} + w_{t-2}}{3}, \frac{w_{t-1} + w_{t-2} + w_{t-3}}{3}\} \\ &= \frac{cov\{w_{t-1}, w_{t-1}\} + cov\{w_{t-2}, w_{t-2}\}}{9} \\ &= \frac{2\sigma_w^2}{9},\end{aligned}$$

- For $s = t-2$ ($h = 2$),

$$\begin{aligned}\gamma_{t,t-2} &= cov\{\frac{w_t + w_{t-1} + w_{t-2}}{3}, \frac{w_{t-2} + w_{t-3} + w_{t-4}}{3}\} \\ &= \frac{cov\{w_{t-2}, w_{t-2}\}}{9} = \frac{\sigma_w^2}{9},\end{aligned}$$

- For $s < t-2$ ($h > 2$), $\gamma_{t,s} = 0$
  as there are no more common white noise.

Example 3: Moving Average

- Hence,

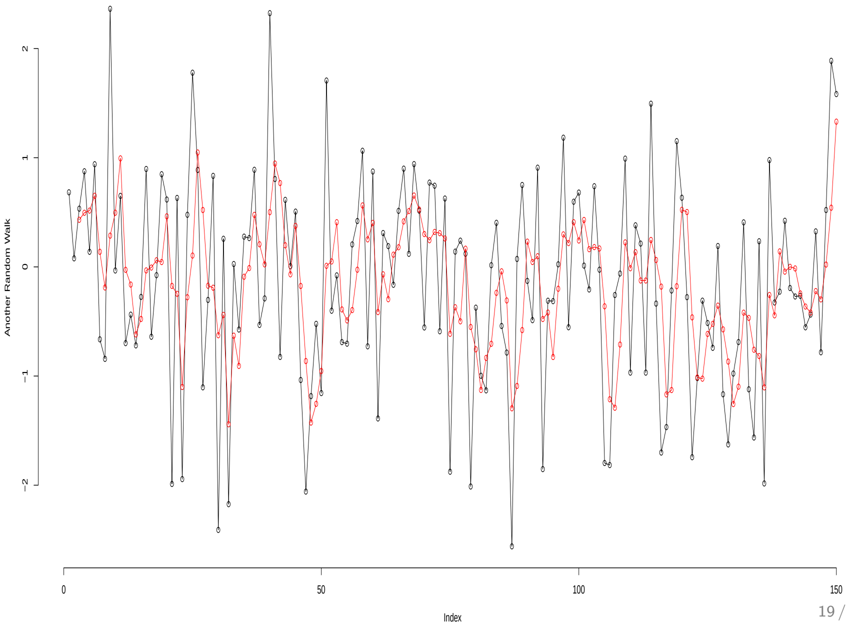
$$\gamma_{t,s} = \gamma_h = \begin{cases} \sigma_w^2/3, & \text{for } h = 0 \\ 2\sigma_w^2/9, & \text{for } h = \pm 1 \\ \sigma_w^2/9, & \text{for } h = \pm 2 \\ 0, & \text{for } |h| > 2 \end{cases}$$

- and

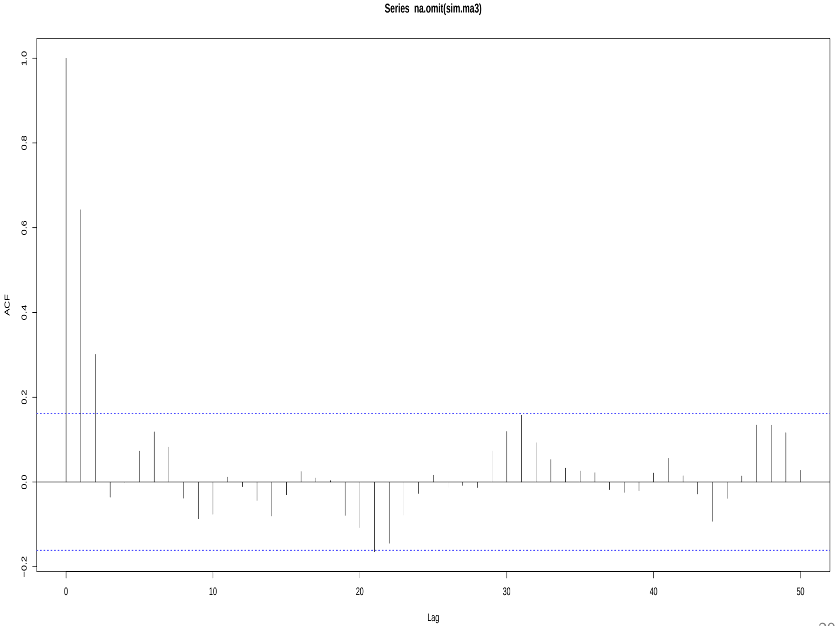
$$\rho_{t,s} = \rho_h = \begin{cases} 1, & \text{for } h = 0 \\ 2/3, & \text{for } h = \pm 1 \\ 1/3, & \text{for } h = \pm 2 \\ 0, & \text{for } |h| > 2 \end{cases}$$

- In this example, the values of $\gamma_{t,t-h}$ and $\rho_{t,t-h}$ depend only on h (but not on t).

Example 3: Moving Average

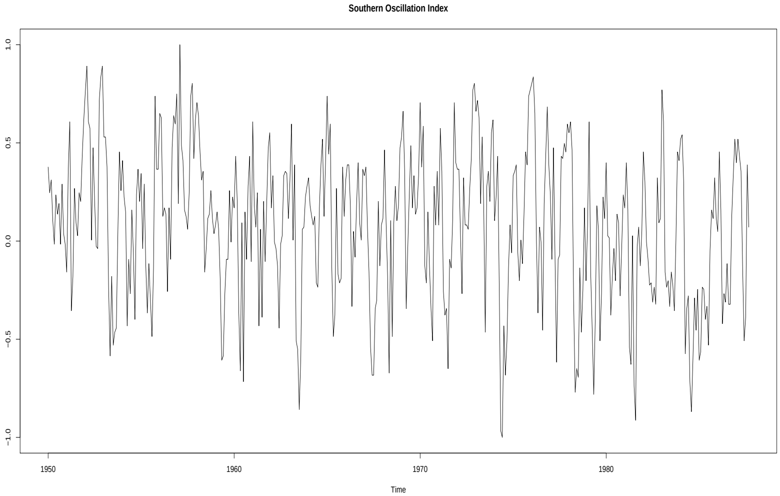


Example 3: Moving Average



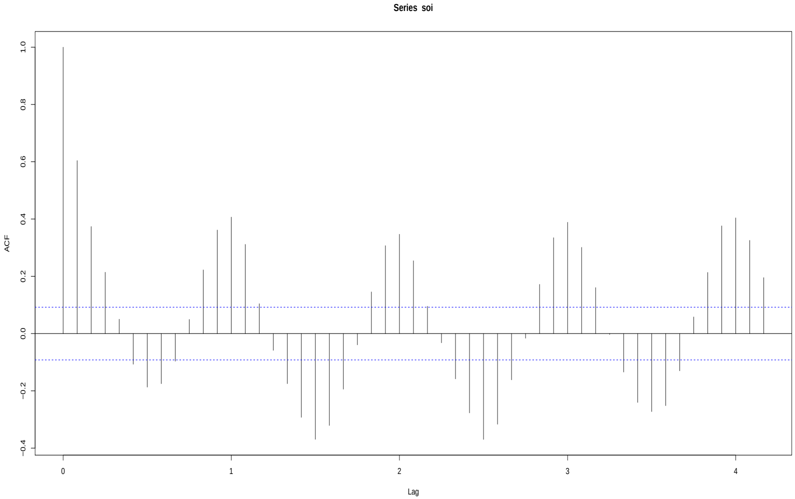
Estimation of Correlation

- **E.g. El Niño and Fish Population:** The monthly **Southern Oscillation Index (SOI)** of new fish from the Pacific Environmental Fisheries Group ($1957-1987, n=453$).



Estimation of Correlation

- **E.g. El Niño and Fish Population:** The monthly **SOI**'s first seven ACF values: `1, 0.604, 0.374 , 0.214 , 0.050 ,-0.107, -0.187.`



Estimation of Correlation

- **Testing for White Noise**: Any *white noise* x_t with the finite fourth moment (that clearly includes Gaussian noise) has the sample autocorrelation function (for each fixed h),

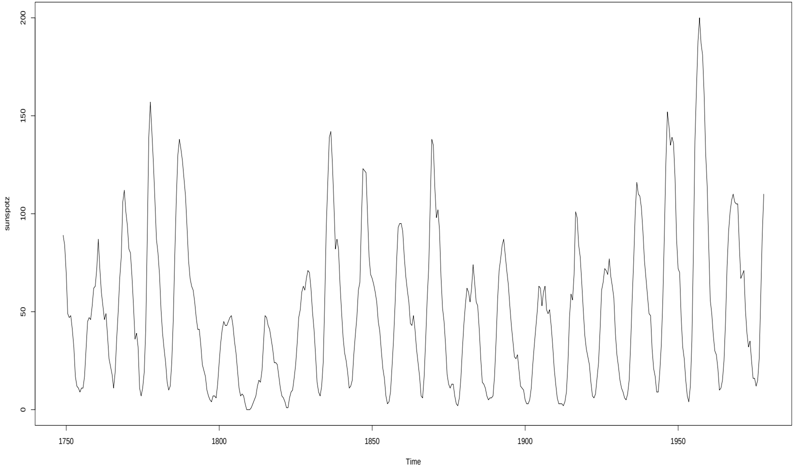
$$\sqrt{n}\hat{\rho}(h) \sim N\left(0, 1\right).$$

- A large-sample test for $H_o : \rho_x(h) = 0$ vs $H_a : \rho_x(h) \neq 0$ at level α is to reject H_o(in favor of H_a) if the p-value $2P(Z > |\sqrt{n}\hat{\rho}(h)|) \leq \alpha$ or if $|\sqrt{n}\hat{\rho}(h)| > z_{\alpha/2}$.

- The **p-value** is the probability of observing a value of the test statistic which is at least as extreme as what was already observed, if the H_o is true. Use multiplicity correction methods in testing several lag h's (e.g., Bonferroni, etc).

Estimation of Correlation

- **E.g. Sunspot Data:** Monthly count of sunspots ($n = 459$). These are highly correlated with solar storms (intense bursts of magnetic energy, disrupting communications systems on Earth) and others.



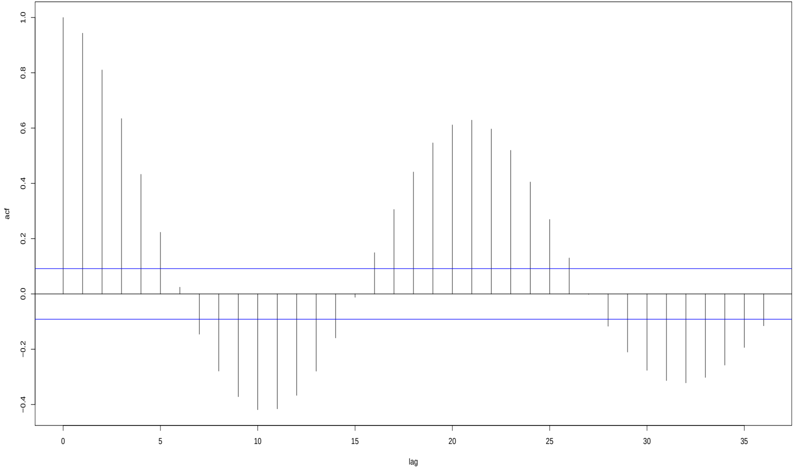
Estimation of Correlation

- **E.g. Sunspot Data:**

```
> eg2 = acf(sunspotz,9)
> mat = cbind(0:9, eg2$acf)
> mat
      [,1]         [,2]
 [1,]    0  1.000000000
 [2,]    1  0.943268450
 [3,]    2  0.810296900
 [4,]    3  0.634430886
 [5,]    4  0.432717891
 [6,]    5  0.223065311
 [7,]    6  0.024495760
 [8,]    7 -0.145762754
 [9,]    8 -0.279109772
[10,]    9 -0.371856111
```

Estimation of Correlation

- **E.g. Sunspot Data:** Or add the 95% limits: $\pm 1.96 \frac{1}{\sqrt{459}}$



Estimation of Correlation

- **E.g. Sunspot Data:** Or calculate the p-values.

```
> pv=2*pnorm( sqrt(459)*abs(mat[,2]),lower.tail = FALSE )
> cbind(0:36, pv)
                   pv
 [1,]  0 7.934308e-102
 [2,]  1  8.182851e-91
 [3,]  2  1.656312e-67
 [4,]  3  4.453133e-42
 [5,]  4  1.849776e-20
 [6,]  5  1.761562e-06
 [7,]  6  5.997193e-01
 [8,]  7  1.791012e-03
 [9,]  8  2.235114e-09
[10,]  9  1.628999e-15
[11,] 10  2.833295e-19
```

Strict Stationarity

- **Def'n.** *A process* $\{x_t\}$ *is* **strictly stationary** *if the joint distribution of* $x_{t_1}, x_{t_2}, \cdots, x_{t_n}$ *is the same as the joint distribution of* $x_{t_1-h}, x_{t_2-h}, \cdots, x_{t_n-h}$ *for all choices of time points* $t_1, t_2, \cdots, t_n$ *and all choices of time lag h, i.e.,*

$$F_{x_{t_1}, x_{t_2}, \cdots, x_{t_n}} = F_{x_{t_1-h}, x_{t_2-h}, \cdots, x_{t_n-h}}.$$

- If x_{t_i}'s are independent then $F_{x_{t_1}, x_{t_2}, \cdots, x_{t_n}} = \prod_{j=1}^{n} F_{x_{t_j}}$ – easier.
- Under *strict stationarity*, the x_t's are (marginally, $n = 1$) identically distributed. In particular,

$$E(x_t) = E(x_s), \quad var(x_t) = var(x_s). \qquad \text{(free of } t\text{)} \qquad (1)$$

- For a (strictly) stationary process, the covariance $\gamma_{t,s}$ depends only on the time difference $t - s = h$.

Weak Stationarity

- **Def'n.** *A stochastic process* $\{x_t\}$ *is said to be* **weakly or second-order stationary** *if it satisfies 1 and 2 below for all indices* t *and* s*, i.e.,*

  1. *All* $\{x_t\}$*'s have constant mean (independent of* t*), i.e.,* $\mu_t = E(x_t) =$ *constant and*
  2. *the covariance between any two observations* $\gamma_{t,t-h}$ *depends only on lag* h*.*

- We use the term "*stationary process*," to mean that the process is *weakly stationary*.

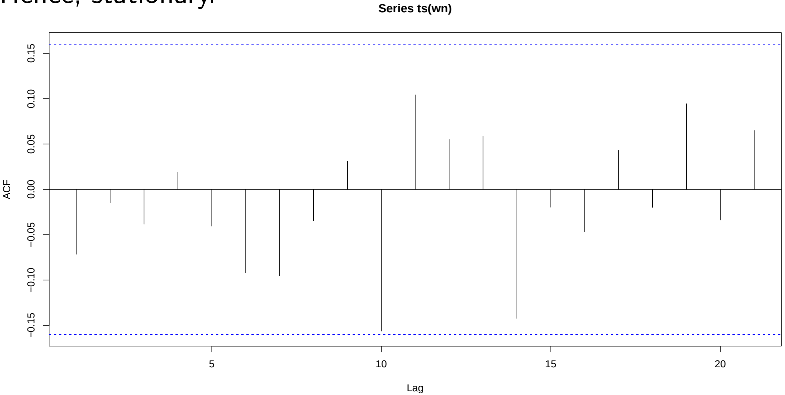
- Strict stationarity $\Longrightarrow$ weak stationarity. The converse is not true, in general, i.e., weak stationarity $\not\Longrightarrow$ strict stationarity.

Examples

- **White Noise:** That is $\{w_t\}$ consists of iid rv's with $E(w_t) = \mu_w = 0$ and $var(w_t) = \sigma_w^2$ (both constant and free of t). The acf is

$$\gamma_h = cov\{w_t, w_{t-h}\} = \begin{cases} \sigma_w^2, & h = 0 \\ 0, & h \neq 0. \end{cases}$$

Hence, stationary.

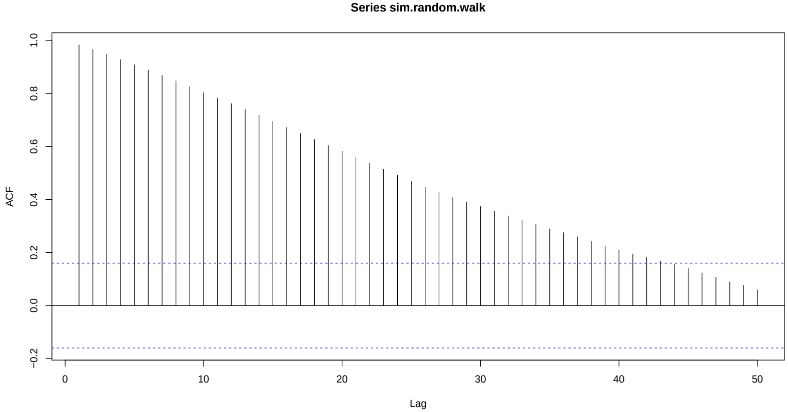


Examples

- **Simple Random Walk:** $x_t = x_{t-1} + w_t$. We know that $E(x_t) = 0$ (constant and free of t). However,

$$\gamma_h = cov\{x_t, x_{t-h}\} = (t - h)\sigma_w^2.$$

Therefore, nonstationary.

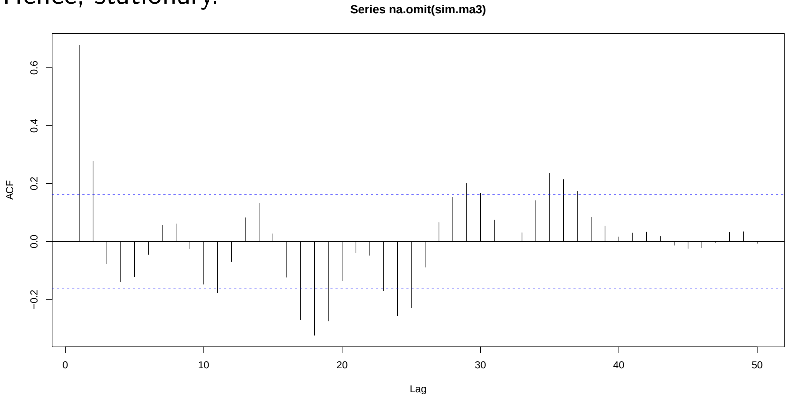


Examples

- **Moving Average:** $x_t = \frac{w_t + w_{t-1} + w_{t-2}}{3}$. We know that $E(x_t) = 0$ and

$$\gamma_h = \begin{cases} \sigma_w^2/3, & \text{for } h = 0 \\ 2\sigma_w^2/9, & \text{for } h = 1 \\ \sigma_w^2/9, & \text{for } h = 2 \\ 0, & \text{for } h > 2 \end{cases}$$

Hence, stationary.

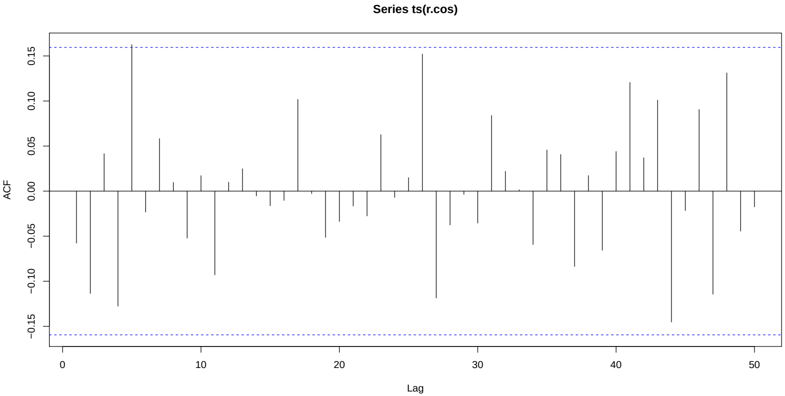


Examples

- **Random Cosine Wave Process:** Let

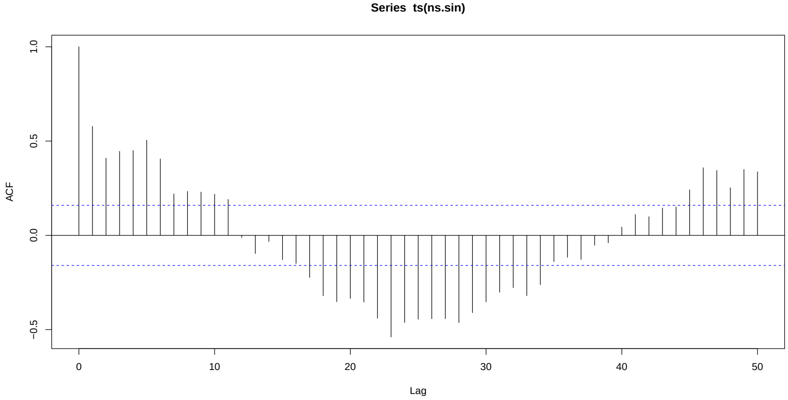
$$x_t = \cos[2\pi(\frac{t}{12} + \Phi)],$$

where $\Phi \stackrel{d}{=} U(0,1)$. It is shown on pp 18-19 (CaC) that this process is surprisingly stationary.



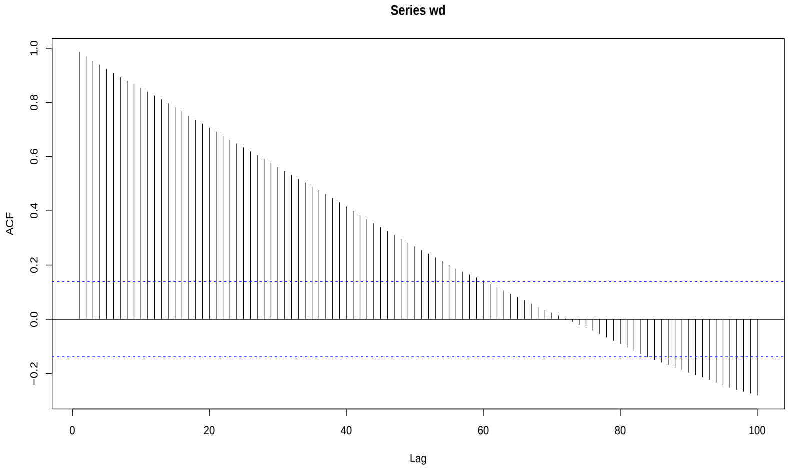
Examples

- **Sinusoidal Process:** $x_t = 5\sin[\,2\pi(1/52)t + .7\pi\,] + w_t$, where $\{w_t\}$ is white noise process with mean zero and $var(w_t) = \sigma_w^2$. We see that $E(x_t) = a\sin(2\pi wt + \phi)$, where a is the *amplitude*, w is the *frequency* of oscillation, and ϕ is the *phase shift*; hence, non-stationary.



Examples

- **Random Walk with Drift:** Let $x_t = .2t + \sum_{j=1}^{t} w_j$. Then $E(x_t) = \delta t$, which is dependent on t, thus, not stationary.



Stationary vs Non-Stationary

- For a stationary series, $\hat{\rho}(h)$ commonly exhibits **short memory**: a few large values of $\hat{\rho}(h)$ for small h are followed by even smaller ones.

- For a non-stationary series, $\rho(h)$ usually *tapers off very slowly* as h increases.

- Several operations can be carried out to transform a non-stationary series to stationary and thus allows us to employ the methods of this course.

Crosscorrelations and Crosscovariances

- Suppose two (or more) time series $\{x_t\}$ and $\{y_t\}$ are available and we want to study relationships between two time series, possibly to predict (forecast) one from the other.

- The **cross-covariance** function between the two series x_t and y_t is

$$\gamma_{xy}(s,t) = cov(x_s, y_t) = E\left[(x_s - \mu_{xs})(y_t - \mu_{yt})\right].$$

- The cross-correlation function is

$$\rho_{xy}(s,t) = \frac{\gamma_{xy}(s,t)}{\sqrt{\gamma_x(s,s)\gamma_y(t,t)}}.$$

- In R, use `ccf()`.

Crosscorrelations and Crosscovariances

- **Def'n.** *Two series x_t and y_t are* **jointly stationary** *if each one of them is stationary, and their cross-covariance function*

$$\gamma_{xy}(h) = cov(x_{t+h}, y_t)$$

is a function of the lag h only.

- The cross-correlation function of the jointly stationary time series x_t and y_t is

$$\rho_{xy}(h) = \frac{\gamma_{xy}(h)}{\sqrt{\gamma_x(0)\gamma_y(0)}}.$$

- The cross-covariance function is *not symmetric*: $\rho_{xy}(h) \neq \rho_{xy}(-h)$.

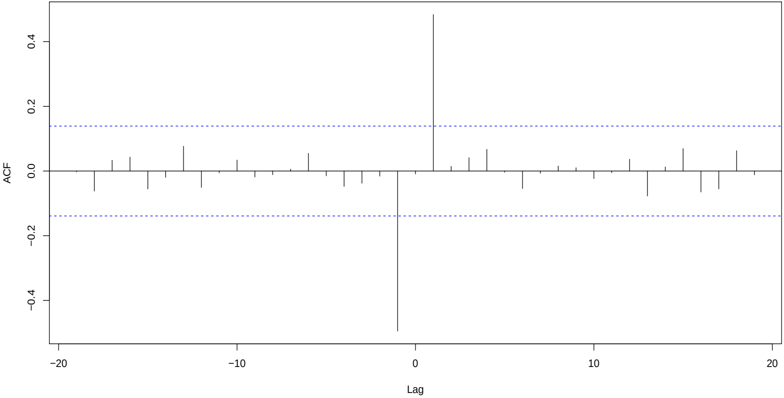
- However,

$$\rho_{xy}(h) = \rho_{yx}(-h).$$

Crosscorrelations and Crosscovariances

- **E.g.** Consider $x_t = w_t + w_{t-1}$ and $y_t = w_t - w_{t-1}$. Verify that

$$\rho_{xy}(h) = \begin{cases} 0, h = 0 \\ \frac{1}{2}, h = 1 \\ -\frac{1}{2}, h = -1 \\ 0, |h| \geq 2. \end{cases}$$



Crosscorrelations and Crosscovariances

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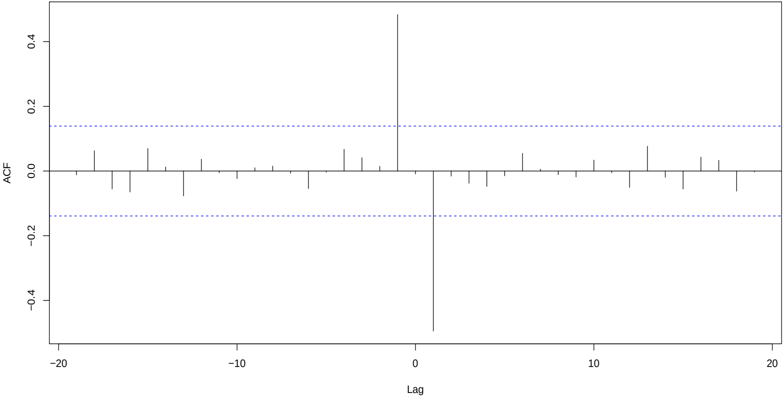


Crosscorrelations and Crosscovariances

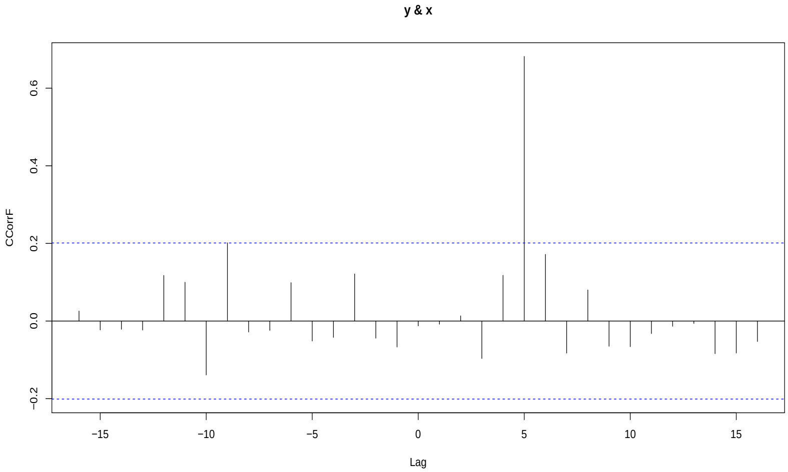
- **E.g.** Let x_t and y_t be related as

$$y_t = Ax_{t-l} + w_t$$

- We say that x_t **leads** y_t if $l > 0$ and **lags** y_t if $l < 0$.

Crosscorrelations and Crosscovariances

- **Illustration in R using** `ccf()`**:** Let $l = 5$ and $y_t = x_{t-5} + w_t$. Generate `x=rnorm(100); y=lag(x,-5) + rnorm(100)` and using `ccf(y,x, ...)` gives



- Note that peaks at positive lags indicate x **leads** y.

Cross-Correlation

- Estimator for cross-covariance function is

$$\hat{\gamma}_{xy}(h) = \frac{1}{n}\sum_{t=1}^{n-h}(x_{t+h} - \bar{x})(y_t - \bar{y})$$

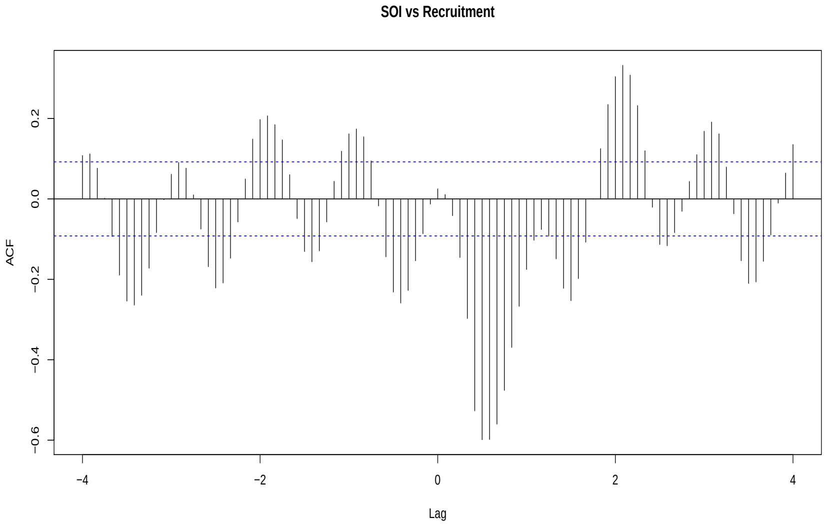
  with $\hat{\gamma}_{xy}(-h) = \hat{\gamma}_{yx}(h)$.

- Estimator for cross-correlation function is

$$\hat{\rho}_{xy}(h) = \frac{\hat{\gamma}_{xy}(h)}{\sqrt{\hat{\gamma}_x(0)\hat{\gamma}_y(0)}}.$$

Cross-Correlation

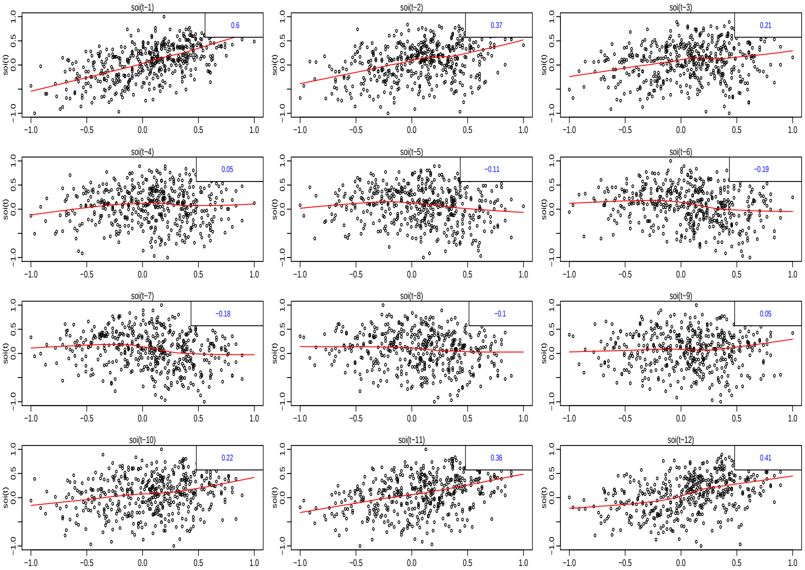
- **SOI (x) and Recruitment (y) Correlation Analysis**: Simultaneous monthly readings.



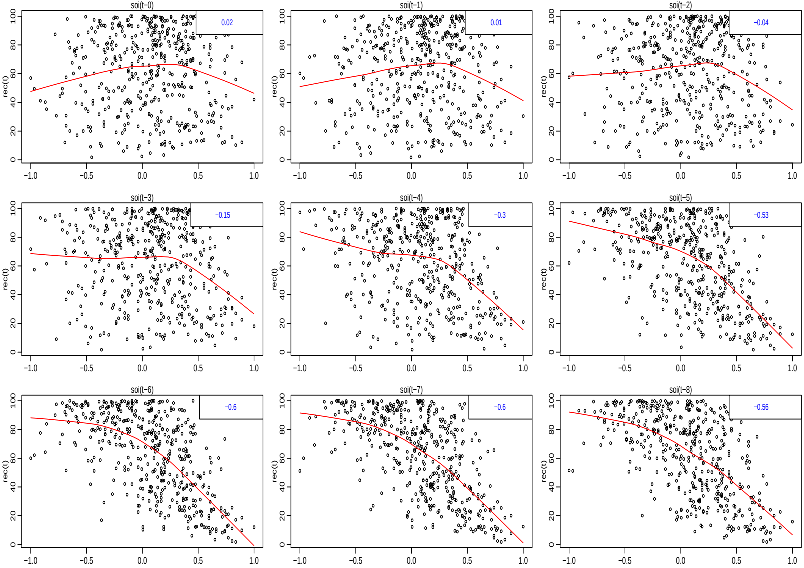
Cross-Correlation

- **SOI (x) and Recruitment (y) Correlation Analysis**: Obvious negative peak at $h = 6$.
- Meaning, SOI measured at time $t - 6$ is correlated with Recruitment measured at time t. Hence, SOI leads Recruitment by six months with most CCF values in the preceding/after six months being significant.
- CCF also suggests regressing Recruitment y_t with the lagged SOI x_{t-l}, particularly $l = 6$.
- This indicates that an increase in SOI is likely to lead to a decrease in Recruitment about 6 months later. And, a decrease in SOI is associated with a likely increase in Recruiment 6 months later.

Cross-Correlation: EDA for SOI



Cross-Correlation: EDA for SOI vs REC



Summary

- Review of basic properties of mean, variance, covariance, and correlation of random variables.
- Definitions of mean functions, autocovariance functions, autocorrelation functions of a stochastic process.
- Stationarity in strict and weak forms.
- Examples of stationary and non-stationary processes.
- Cross-correlation analyses for two time series and estimation.

Lab for Today

1. For SOI data, calculcate the 95% bounds for the autocorrelation.

2. Verify your results in 1 by superimposing those bounds on the corresponding correlogram.

3. Consider two data sets: `oil` and `gas` from `astsa` package. Are any of these datasets stationary? Why or why not?

4. Apply the transformation $\nabla \log x_t$, where $\nabla y_t = y_t - y_{t-1}$ to both data sets. Are any of these datasets stationary? Why or why not?

5. Produce a cross-correlation plot for the transformed data.

Note: At most 5 pages all in all.