Double Jointed Pendulum Theory

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This document presents the mathematical derivation for computing the motion of a double jointed pendulum using Lagrangian Mechanics. The model pendulum has no friction, and therefore conserves energy and will run forever.

DERIVATION

Background

Lagrangian Mechanics is an alternative formulation of classical mechanics. While it provides entirely equivalent results, it allows one to use arbitrary coordinates (called generalised coordinates). It can be particularly useful for systems which have constraints, such as our double jointed pendulum which possess fixed lengths between the pivots. Instead of Newtons second law

$$\mathbf{F} = m\mathbf{a},$$

force $=$ *mass* $\times$ *acceleration*, Lagrangian mechanics is based on finding an extreme of the *Lagrangian* (also called the *action*)

$$L = T - V$$

where T is the kinetic energy and V is the potential energy. One then needs to solve the equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} \tag{1}$$

where q_j are the set of generalised coordinates that specify the system. For our double pendulum there are 4 degrees of freedom, the two angles θ_1 and θ_2 which specify the position of the pendulum and their velocities $\dot{\theta}_1$ and $\dot{\theta}_2$. These two angles are illustrated in Fig. 1.

For further background on Lagrangian mechanics the Wikipedia page is a good place to start: `https://en.wikipedia.org/wiki/Lagrangian_mechanics`

The Energy

For our pendulum the potential energy is solely due to gravity. We may start with Cartesian coordinates, x_1 and y_1 for the position of m_1 and x_2 and y_2 for the position of m_2. The potential energy is then

$$V = gm_1y_1 + gm_2y_2 + c_g$$

where g is the acceleration due to gravity, $g \simeq 9.8$ on Earth, and c_g is some arbitrary constant we are free to specify. The transformations between our coordinates are given by

$$\begin{aligned} x_1 &= L_1 \sin\theta_1 \\ y_1 &= -L_1 \cos\theta_1 \\ x_2 &= L_1 \sin\theta_1 + L_2 \sin\theta_2 \\ y_2 &= -L_1 \cos\theta_1 - L_2 \cos\theta_2 \end{aligned}$$

which allows us to write the potential energy as

$$\begin{aligned} V = {} & (m_1 + m_2)gL_1\left(1 - \cos\theta_1\right) \\ & + m_2 g L_2 \left(1 - \cos\theta_2\right). \end{aligned}$$

The kinetic energy is given in terms of the velocities

$$\begin{aligned} T &= \tfrac{1}{2}m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \tfrac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right) \\ &= \tfrac{1}{2}\left(m_1 + m_2\right)\dot{\theta}_1^2 L_1^2 + \tfrac{1}{2}m_2\dot{\theta}_2^2 L_2^2 \\ &\quad + m_2\dot{\theta}_1\dot{\theta}_2 L_1 L_2 \cos\left(\theta_1 - \theta_2\right), \end{aligned}$$

and now we have all we need to specify the Lagrangian L.

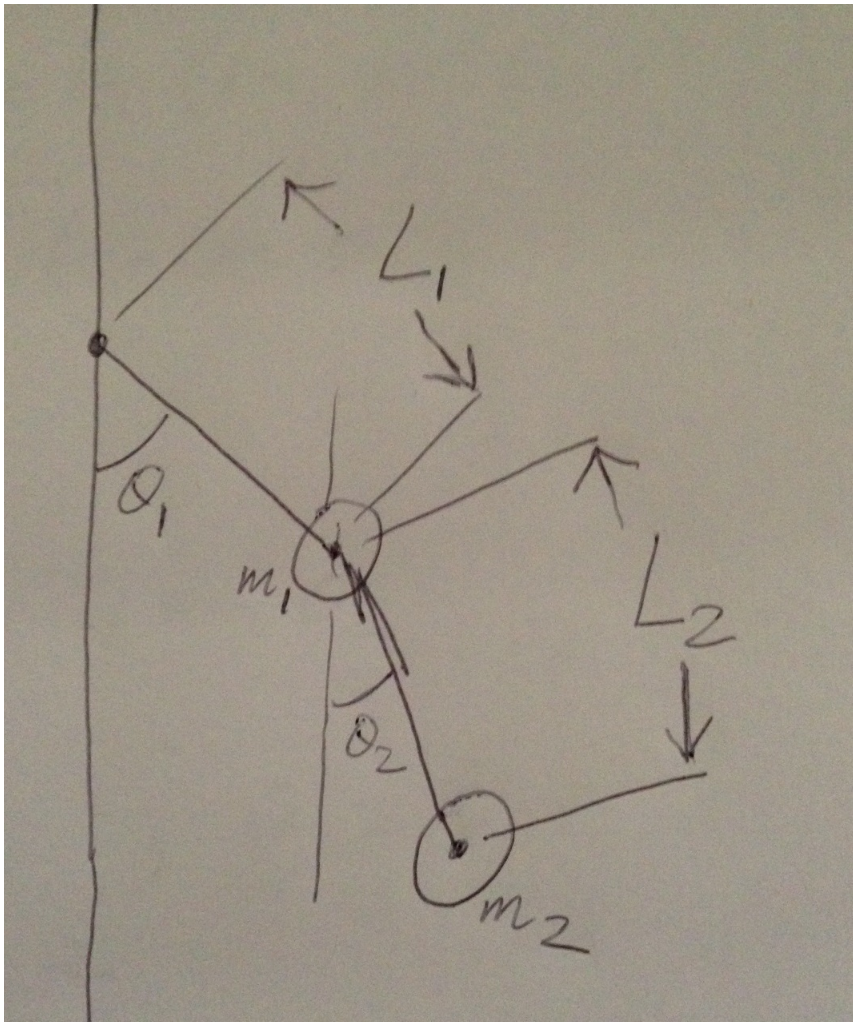

Figure 1. The coordinates for the double jointed pendulum.

The Result

Upon applying Eq. 1 to the Lagrangian we obtain a pair of simultaneous equations

$$\ddot{\theta}_1 + \frac{m_2}{m_1 + m_2} \ddot{\theta}_2 \frac{L_2}{L_1} \cos(\theta_1 - \theta_2)$$
$$+ \frac{m_2}{m_1 + m_2} \dot{\theta}_2^2 \frac{L_2}{L_1} \sin(\theta_1 - \theta_2) = -\frac{g}{L_1} \sin(\theta_1)$$

and

$$\ddot{\theta}_2 + \ddot{\theta}_1 \frac{L_1}{L_2} \cos(\theta_1 - \theta_2)$$
$$-\dot{\theta}_1^2 \frac{L_2}{L_1} \sin(\theta_1 - \theta_2) = -\frac{g}{L_2} \sin(\theta_2) \, .$$

This pair of equations need to be solved at every time step to obtain values for $\ddot{\theta}_1$ and $\ddot{\theta}_2$. We can then solve the system of first order ordinary differential equations given by

$$\frac{d}{dt}\theta_1 = \dot{\theta}_1$$
$$\frac{d}{dt}\dot{\theta}_1 = \ddot{\theta}_1$$
$$\frac{d}{dt}\theta_2 = \dot{\theta}_2$$
$$\frac{d}{dt}\dot{\theta}_2 = \ddot{\theta}_2$$

for our chosen set of generalised coordinates θ_1, θ_2, $\dot{\theta}_1$, $\dot{\theta}_2$ which unambiguously specify the position and the velocity of our pendulum.